Finite Element Models for Eddy Current Effects in Windings: Application to Superconductive Rutherford Cable

Herbert De Gersem\textsuperscript{1}$\leftrightarrow$\textsuperscript{2}, Thomas Weiland\textsuperscript{1}

\textsuperscript{1}Technische Universität Darmstadt (Germany), Computational Electromagnetics Laboratory
\textsuperscript{2}Katholieke Universiteit Leuven (Belgium)
Dep. ESAT, Div. ELECTA

MACSI-NET-13 & 2: “Electromagnetics in Telecommunication”
Workshop “Coupled problems / model reduction”
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1. Introduction: field-circuit coupling
2. Voltage/current-driven branches
3. Circuit description
   + consistency check
   + partial loop/cutset transformations
   + topological changes
4. Field-circuit coupling
   + examples
5. Algebraic system properties
   + selection of iterative solution techniques
6. Superconductive Rutherford cable
   + adjacency eddy currents
   + cross-over eddy currents
7. Conclusions
field-circuit coupling

FE/FIT model
- geometrical details
- ferromagnetic saturation (non-linear!!)
- (motional) eddy currents

circuit
- external sources/loads, (e.g. power electronic equipment)
- parts outside the FE model (e.g. end windings/rings)
- representing (linear) parts for which an equivalent circuit suffices (e.g. homopolar shaft flux)
Quasistatic formulation

\[ \nabla \times (\nabla \times \mathbf{A}) - \sigma \nabla \times (\nabla \times \mathbf{A}) + j \omega \sigma \mathbf{A} + \sigma \frac{\partial \mathbf{A}}{\partial t} = -\sigma \nabla V \]

\[ -\frac{\partial}{\partial x} \left( v \frac{\partial A_z}{\partial x} \right) - \frac{\partial}{\partial y} \left( v \frac{\partial A_z}{\partial y} \right) + \sigma v_x \frac{\partial A_z}{\partial x} + \sigma v_y \frac{\partial A_z}{\partial y} + j \omega \sigma A_z + \sigma \frac{\partial A_z}{\partial t} = \frac{\sigma}{\ell_z} V \]

discretisation

\[ A_z = \sum_j A_z, j N_j (x, y) \]

\[ [k_{ij} + m_{ij} + l_{ij}] A_z, j = [f_i] \]

large & sparse system of equations
Solid (massive) conductor

- **magnetic equation**
\[
\nabla \times (\mu \nabla \times A) + \sigma \frac{\partial A}{\partial t} = \frac{\sigma}{\ell_z} \Delta V_{sol}
\]

- **total current**
\[
I_{sol} = \int J d\Omega
\]
\[
I_{sol} = \int \frac{\sigma}{\ell_z} d\Omega \Delta V_{sol} - \int \sigma \frac{\partial A}{\partial t} d\Omega
\]

- equivalent scheme

“voltage-driven” conductor
no circuit equation if voltage drop is known
no fill-in in the FE matrix part if the voltage drop is an independent degree of freedom
Stranded (filamentary) conductor

- magnetic equation
  \[ \nabla \times (\nu \nabla \times \mathbf{A}) = \frac{N_{str}}{\Delta_{str}} \mathbf{I}_{str} \]

- total voltage
  \[ \Delta V_{str} = \frac{N_{str} \ell_z}{\Delta_{str}} \int_{\Omega_{str}} (-\nabla V) d\Omega \]
  \[ \Delta V_{str} = R_{str} I_{str} + \frac{N_{str} \ell_z}{\Delta_{str}} \int_{\Omega_{str}} \frac{\partial \mathbf{A}}{\partial t} d\Omega \]

- equivalent scheme

“current-driven” conductor

no circuit equation if current is known
no fill-in in the FE matrix part if the current
is an independent degree of freedom
**Coupling requirements**

\[
\begin{bmatrix}
K & B^T \\
B & C
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
f \\
g
\end{bmatrix}
\]

(compacted) modified nodal analysis (compacted) loop analysis hybrid analysis

- **unknowns**: nodal voltages (+a few currents) loop currents (+a few voltages) twig voltages link currents

- **keep the FE matrix part unchanged**
  - sparsity
  - preconditioners (multigrid)
  - possible benefits thanks to structured grids (FIT)

- **preserve symmetry**
  - Krylov subspace solvers for symmetric systems (CG, MINRES, QMR)
  - storage

- **preserve positive definiteness**
  - solvers (CG)
  - preconditioners (IC)
1. Trace a tree through the circuit

starting from the circuit node n0#, the twigs are selected in the order
1. voltage source 10 V
2. resistor 1 Ω
3. resistor 3 Ω

Priority list

highest priority, preferably twig

- voltage sources
- solid conductors (coupled)
- capacitors (largest capacitance first)
- resistors (largest conductance first)
- inductors (smallest inductance first)
- stranded conductors (coupled)
- current sources

smallest priority, preferably link
2. **Determine fundamental cutsets and fundamental loops**

A **fundamental cutset** is formed by 1 twig and the unique set of links completing the set of branches which would upon removal result in two disconnected circuit parts.

Property: \( \text{priority(twig)} \geq \text{priority(branch)}, \quad \forall \text{branch} \in \text{fundamental cutset} \)

A **fundamental loop** consists of 1 link and the unique path through the tree closing the loop.

Property: \( \text{priority(link)} \leq \text{priority(branch)}, \quad \forall \text{branch} \in \text{fundamental loop} \)

The orientation of the fundamental cutset/loop is determined by the orientation of the corresponding twig/link.
3. Construct the fundamental cutset and fundamental loop matrices

fundamental cutset matrix

\[ D = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

fundamental loop matrix

\[ B = \begin{bmatrix} -1 & 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \]

remark: \( B_{\text{ln}, \text{tw}} = -D_{\text{tw}, \text{ln}}^T \)
4. Partition the fundamental incidence matrices

- twv  twigs at which the voltage is known (voltage sources)
- two  twigs at which an unknown voltage is assigned ("free twigs")
- twu  eliminated twigs ("eliminated twigs") (not in this example)
- lnu  eliminated links ("eliminated links") (not in this example)
- lno  links at which an unknown current is assigned ("free links")
- ini  links at which the current is known (current sources)
5. Write impedance/admittance matrices and voltage/current vectors

1. admittance matrix for the free twigs
   \[ Y_{\text{two}} = \begin{bmatrix} 1/1 & \_ \_ \\ \_ \_ & 1/3 \end{bmatrix} \]

2. impedance matrix for the free links
   \[ Z_{\text{lno}} = [4] \]

3. voltage vector for the voltage sources
   \[ v_{\text{twv}} = [10] \]

4. current vector for the current sources
   \[ i_{\text{lmi}} = [2] \]
6. Write system of equations

intuitive approach:

1. write the Kirchhoff current law for each fundamental cutset associated with a free twig

2. write the Kirchhoff voltage law for each fundamental loop associated with a free link

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 0.333 & 1 \\
-1 & 1 & -4
\end{bmatrix}
\begin{bmatrix}
v_b \\
v_c \\
i_d
\end{bmatrix}
= \begin{bmatrix} 0 \\ -2 \\ -10 \end{bmatrix}
\]

\[
\begin{bmatrix}
Y_{\text{two}} & D_{\text{two,ino}} \\
-B_{\text{ino,two}} & -Z_{\text{ino}}
\end{bmatrix}
\begin{bmatrix}
v_{\text{two}} \\
i_{\text{ino}}
\end{bmatrix}
= \begin{bmatrix}
-D_{\text{two,ini}}i_{\text{ini}} \\
B_{\text{ino,twv}}v_{\text{twv}}
\end{bmatrix}
\]

remark: symmetric because \( B_{\text{ino,two}} = -D_{\text{two,ino}}^T \)
7. Solve system of equations & propagate the circuit solution

Circuit diagram:

Twig currents:
\[
i_{\text{two}} = -D_{\text{two},\text{lno}}i_{\text{lno}} - D_{\text{two},\text{lni}}i_{\text{lni}}
\]
\[
i_{\text{twv}} = -D_{\text{twv},\text{lno}}i_{\text{lno}} - D_{\text{twv},\text{lni}}i_{\text{lni}}
\]

Link voltages:
\[
v_{\text{lno}} = -B_{\text{lno},\text{two}}v_{\text{two}} - B_{\text{lno},\text{twv}}v_{\text{twv}}
\]
\[
v_{\text{lni}} = -B_{\text{lni},\text{two}}v_{\text{two}} - B_{\text{lni},\text{twv}}v_{\text{twv}}
\]

Solution:
\[
\begin{bmatrix}
v_b \\ v_c \\ i_d
\end{bmatrix} =
\begin{bmatrix}
0.5 \\ -7.5 \\ 0.5
\end{bmatrix}
\]
Particularities

1. Distinct circuit parts

2. Dangling nodes
   - A branch to a dangling node is always a twig.
   - Associated fundamental cutset only contains the twig.

3. Self-loops
   - A self-loop is always a link.
   - The associated fundamental loop only contains the link.
Consistency check (1)

1. Fundamental loop consisting of voltage sources

Problem: a voltage source is necessarily selected as link

Treatment: check the Kirchhoff voltage law in the associated fundamental loop

\[ e.g. \, v0 - v1 + v2 = 0 \, ?? \]

if valid

omit the voltage source link

if not valid

the circuit has no solution
Consistency check (2)

2. Fundamental cutset consisting of current sources

Problem: a current source is necessarily selected as twig

Treatment: check the Kirchhoff current law in the associated fundamental cutset
  e.g. $i_0 + i_1 - i_2 = 0$ ??
  if valid
    replace the current source twig by a short-circuit connection
  if not valid
    the circuit has no solution
1. Stranded conductor being selected as twig

Problem: a stranded conductor (current-driven branch) is necessarily selected as twig

Property: \( \text{priority(twig)} \geq \text{priority(branch)} \), \( \forall \text{branch} \in \text{associated fundamental cutset} \)

Treatment: apply the Kirchhoff current law in the associated fundamental cutset to express the stranded-conductor current in terms of link currents
e.g. \( \text{istr} = \text{i1} + \text{i2} \)

\~ small and independent Schur complements
Partial transformation (2)

2. Solid conductor being selected as link

Problem: a solid conductor (voltage-driven branch) is necessarily selected as link

Property: $\text{priority(link)} \leq \text{priority(branch)}$, $\forall \text{branch} \in \text{associated fundamental loop}$

Treatment: apply the Kirchhoff voltage law in the associated fundamental loop to express the solid-conductor voltage in terms of twig voltage

\[ \text{e.g. } \text{vsol} = \text{v1} + \text{v2} \]

~ small and independent Schur complements
1. Switching elements closes (switch, diode, thyristor, …)

Topological changes (1)

Problem: the priority of a branch increases during (transient) simulation

\[
\begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Treatment: consider associated fundamental loop and possibly change link/twig-mode with the branch with the lowest priority

\[
\begin{bmatrix} G_1 & 1 \\ 1 & -R_2 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix}
\]
Topological changes (2)

1. Switching element opens (switch, diode, thyristor, …)

Problem: the priority of a branch decreases during (transient) simulation

\[
\begin{bmatrix}
G_1 & 1 \\
1 & -R_2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
i_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
U
\end{bmatrix}
\]

Treatment: consider associated fundamental cutset and possibly change link/twig-mode with the branch with the highest priority

\[
\begin{bmatrix}
G_1 & 0 \\
0 & G_2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]
Field-circuit coupling (3)

KCL applied to the fundamental cutsets

\[ i_t + D_t i = 0 \]

KVL applied to the fundamental loops

\[ v_t + B_t v = 0 \]

magnetodynamic PDE

\[ \nabla \times (v \nabla \times A) + j \omega \sigma A = -\sigma \nabla V \]

FE equations:
\[
\begin{bmatrix}
    k_{ij} + j \omega l_{ij} & q_{it} & -p_{il} \\
    q_{sj} & \chi G_m & \chi D \\
    -p_{kj} & -\chi B & -\chi R_m
\end{bmatrix}
\begin{bmatrix}
    A_{zj} \\
    v_t \\
    i_l
\end{bmatrix}
= \begin{bmatrix} 0 \\ \chi I_{app} \\ -\chi V_{app} \end{bmatrix}
\]

cutset equations:

loop equations:

\[ i_t = G_m v_t - j \omega q_{mj} A_{zj} \]

\[ v_l = R_m i_l + j \omega p_{mj} A_{zj} \]

twig branch relations

\[ B = -D^T \]

symmetrisation factor \( \chi \)

symmetric !
Algebraic solution (1)

**Coupled system matrix**
- symmetric/indefinite

**FE/FIT part**
- sparse (no fill-in)
- changes due to non-linearities

**Circuit part**
- dense
- changes due to non-linearities & switching

**Coupling blocks**
- dense
- does not change

#positive-eigenvalues = #FE/FIT-dofs + #cutset-equations

#negative-eigenvalues = #loop-equations
### Algebraic solution (2)

<table>
<thead>
<tr>
<th>Solver</th>
<th>Preconditioner</th>
</tr>
</thead>
<tbody>
<tr>
<td>MINRES/QMR</td>
<td>SSOR/AMG</td>
</tr>
<tr>
<td>QMR</td>
<td>SSOR</td>
</tr>
<tr>
<td>MINRES</td>
<td>BJac (AMG,LU)</td>
</tr>
<tr>
<td>CG(LU)</td>
<td>? SSOR/AMG</td>
</tr>
<tr>
<td>? DD</td>
<td>? DD</td>
</tr>
</tbody>
</table>

**Schur complement 1**

\[
\left( K - B^T C^{-1} B \right) x = f - B^T C^{-1} g
\]

**Schur complement 2**

\[
\left( C - B K^{-1} B^T \right) y = g - B K^{-1} f
\]

& Domenico Lahaye: Algebraic multigrid for field-circuit coupled systems
Algebraic solution (3)

Algebraic multigrid for field-circuit coupled systems
(Dr. Domenico Lahaye, CRS4, Cagliari, Italy)

fine grid system:

\[
\begin{bmatrix}
K & B^T \\
B & C
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
f \\
g
\end{bmatrix}
\]

prolongation:
\[ P_{H \rightarrow h} \]

restriction:
\[ R_{h \rightarrow H} = P_{H \rightarrow h}^T \]

coarse grid system:

\[
\begin{bmatrix}
P_{H \rightarrow h}^T \\
I
\end{bmatrix}
\begin{bmatrix}
K_H & B_H^T \\
B_H & C
\end{bmatrix}
\begin{bmatrix}
P_{H \rightarrow h} \end{bmatrix}
\begin{bmatrix}
x_H \\
y_H
\end{bmatrix}
= \begin{bmatrix}
P_{H \rightarrow h}^T \\
I
\end{bmatrix}
\begin{bmatrix}
f \\
g
\end{bmatrix}
\]

smoother: block-Jacobi with
1. Gauss-Seidel for the FE/FIT unknowns
2. no smoothing or an exact solution for the circuit unknowns
Algebraic solution (4)

Performance of an indefinite and definite preconditioner

![Graph showing the performance of SSORQMR and MINRES algorithms over iteration steps. The graph plots error on a logarithmic scale against iteration steps. The error decreases significantly with increasing iteration steps for both algorithms.]
Algebraic solution (5)

Preconditioning of the Schur-complement system

![Graph showing iteration steps vs. error for different methods: AMGCG, SSORCG, CG.](image)
### Algebraic solution (6)

Iteration counts and computation times of the iterative system solution for 1 time step of a transient simulation.

<table>
<thead>
<tr>
<th>Solver</th>
<th>Iteration steps</th>
<th>Computation time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MINRES</td>
<td>1997</td>
<td>59.80</td>
</tr>
<tr>
<td>QMR</td>
<td>1330</td>
<td>40.23</td>
</tr>
<tr>
<td>GMRES</td>
<td>1031</td>
<td>159.71</td>
</tr>
<tr>
<td>SSOR MINRES</td>
<td>476</td>
<td>11.03</td>
</tr>
<tr>
<td>QMR</td>
<td>320</td>
<td>7.41</td>
</tr>
<tr>
<td>GMRES</td>
<td>312</td>
<td>38.13</td>
</tr>
<tr>
<td>JAC (SSOR, LU)GMRES</td>
<td>325</td>
<td>53.91</td>
</tr>
<tr>
<td>JAC (AMG, LU)QMR</td>
<td>245</td>
<td>5.23</td>
</tr>
<tr>
<td>CG*</td>
<td>2376</td>
<td>30.06</td>
</tr>
<tr>
<td>SSOR CG*</td>
<td>1105</td>
<td>17.32</td>
</tr>
<tr>
<td>AMG CG*</td>
<td>199</td>
<td>9.28</td>
</tr>
</tbody>
</table>

The iteration steps and computation times are for 1 time step of a transient simulation. The solutions are categorized into:

- **Without preconditioning**
- **Non-symmetric ↔ Symmetric solver**
- **Block preconditioning**
- **Schur complement**

For indefinite ↔ definite preconditioning, see Domenico Lahaye: Algebraic multigrid for field-circuit coupled systems.
Time-harmonic simulation of a three-phase induction machine

- three-phase voltage excitation
- end-winding resistances & inductances
- end-rind resistances & inductances
Examples (2)

1. magnetic field + magnetic circuit
2. magnetic field + analytical model + electric circuit
3. electrokinetic field + magnetic circuit + electric circuit
More general conductor systems?

\[ \delta = \sqrt{\frac{1}{\pi f \sigma \mu}} \]  

Skin depth
Foil windings

foil winding transformer

current distribution

voltage distribution

foil

insulation
Foil conductor model

\[ \Delta V(x) = \sum_{q=1}^{n_f} \Delta V_q M_q(x) \]

- Foil shape functions
- Foil mesh
- Magnetic finite element mesh
- Foil winding geometry

Wrong skin effect
Large mesh

50 foils

De Gersem, Hameyer, CEFC2000
Dular, Geuzaine, Compumag2001
Foil winding transformer

- Voltage distribution
- Current distribution

Considerable saving
Superconductive magnets (1)

fast ramped magnets

\[
B(t) = 1 \text{T/s}
\]

yoke: eddy currents & hysteresis

superconductive filaments: persistent & coupling currents

Rutherford cable: cable eddy currents/magnetisation

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Superconductive magnets (2)
Rutherford Cable (1)

- Strand
- Copper wire
- Superconductive filament
- Twisted strands
- Adjacency loop
- Cross-over loop
perpendicular flux
\( B_p(r, \theta) \)

parallel flux
\( B_\ell(r, \theta) \)

keystoning:
\[
J_{s,z} = \frac{N_{\text{turns}} I_{\text{app}}}{(r_2 - r_1)^2} \left( \frac{r_2 - r}{2b_1} + \frac{r - r_1}{2b_2} \right)
\]
Magnetic flux density

\[ B_p(r, \theta) \]

parallel magnetic field

perpendicular magnetic field

\[ B_r(\theta) \]
Adjacency eddy current (1)

1. additional discretisation for unknown electric field $E_{z,q}$:

$$E_z(x, y) = \sum_q E_{z,q} M_q(x, y)$$

2. adjacency eddy current density:

$$J_{pa,z}(r, \theta) = \sigma_{pa} E_{z,q} - \sigma_{pa} \frac{\partial A_z}{\partial t}$$

due to perpendicular magnetic field

3. netto current through $\Omega_q = 0$

$$I_{z,q} = \int_{\Omega_q} J_{pa,z}(r, \theta) \, d\Omega = 0$$

additional constraint!
Adjacency eddy current (2)

**additional load term for magnetic FE model**

\[ g_{pa} = M_{pa} \frac{\partial u}{\partial t} - Z_{pa} e_{pa} \]

**additional constraint**

\[ -Z_{pa}^T \frac{\partial u}{\partial t} + G_{pa} e_{pa} = 0 \]

**degrees of freedom for** \( E_{z,q} \)

\[
\begin{bmatrix}
M_{pa} & 0 \\
Z_{pa}^T & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial t}
\end{bmatrix}
\begin{bmatrix}
u \\
e_{pa}
\end{bmatrix}
+ 
\begin{bmatrix}
K & Z_{pa} \\
0 & G_{pa}
\end{bmatrix}
\begin{bmatrix}
u \\
e_{pa}
\end{bmatrix}
= 
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

\[ M_{pa,ij} = \int_{\Omega} \sigma_{pa} N_i(x,y) N_j(x,y) \, d\Omega \]

\[ Z_{pa,iq} = \int_{\Omega} \sigma_{pa} N_i(x,y) M_q(x,y) \, d\Omega \]

\[ G_{pa,pq} = \int_{\Omega} \sigma_{pa} M_p(x,y) M_q(x,y) \, d\Omega \]
shape functions \( M_q(x, y) \) related (but not necessarily equal) to the zones of current redistribution due to parallel magnetic field.
Adjacency eddy current

adjacency eddy currents due to perpendicular magnetic field

$B_p(r, \theta)$

$J_{pa,z}$
Cross-over eddy current (1)

1. coupled flux

\[ \phi_p (\theta) = \ell_z (A_z (r_2, \theta) - A_z (r_1, \theta)) \]

2. magnetisation

\[ \phi_{pc} (\theta) = \tau_{pc} \frac{\partial \phi_p (\theta)}{\partial t} \]

(time constant)
Cross-over eddy currents due to perpendicular magnetic field

$J_{pc,z}$

$B_p(r, \theta)$

Cross-over magnetisation
Conclusions

• hybrid method for circuit analysis
  convenient field-circuit modelling

• solution of the coupled algebraic systems of equations

• specialised conductor models
  complicated eddy current phenomena

• applications to
  ♦ electrical machine
  ♦ foil winding transformer
  ♦ superconductive magnets, Rutherford cable