Lecture 10: Viscoelastic modulus reconstruction and full-field OCT elastography

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Elastography

- **Elastography**: quantitative visualization of the mechanical properties of human tissues by using the relation between the wave propagation velocity and the mechanical properties of the tissues.
- Mechanical properties of tissue include the shear modulus, shear viscosity, and compression modulus.
- Quantification of the tissue shear modulus in vivo can provide evidence of the manifestation of tissue diseases.
- Image reconstruction methods for tissue viscoelasticity imaging.
- Recover the distribution of the shear modulus ($\mu$) and shear viscosity ($\eta$) from the internal measurement of the time-harmonic mechanical displacement field $u$ produced by the application of an external time harmonic excitation at frequency $\omega/2\pi$ in the range $50 \sim 200$Hz through the surface of the subject.
- Modeling soft tissue as being linearly viscoelastic and nearly incompressible.
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- **Displacement:** \( \Re(u(x)e^{i\omega t}) \),

\[
\nabla \cdot \left( (\mu + i\omega\eta)(\nabla u + \nabla u^T) \right) + \nabla((\lambda + i\omega\eta\lambda)\nabla \cdot u) + \rho\omega^2 u = 0;
\]

\( \rho \): density of the medium, \( \nabla u^T \): transpose of \( \nabla u \), \( \lambda \): compression modulus and \( \eta \lambda \): compression viscosity.

- **Algebraic inversion** method: For any non-zero constant vector \( a \),

\[
\mu + i\omega\eta = -\frac{\rho\omega^2(a \cdot u)}{\nabla \cdot \nabla(a \cdot u)}.\]

- **Strong assumptions** of \( \nabla(\mu + i\omega\eta) \approx 0 \) (local homogeneity) and \( (\lambda + i\omega\eta\lambda)\nabla \cdot u \approx 0 \) (negligible pressure).

- **Algebraic formula:** ignores reflection effects of the propagating wave due to abrupt changes of \( \mu + i\omega\eta \), so that the method cannot measure any change of \( \mu + i\omega\eta \) in the direction of \( a \).
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- **Boundary conditions**: $\Gamma_D$ and $\Gamma_N$ s.t. $\overline{\Gamma_D \cup \Gamma_N} = \partial \Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$,
  
  \[ u = g \quad \text{on} \ \Gamma_D, \]
  
  \[ 2(\mu + i\omega\eta)\nabla^s u \nu + (\lambda + i\omega\eta\lambda)(\nabla \cdot u)\nu = 0 \quad \text{on} \ \Gamma_N. \]

- Soft tissues: nearly incompressible.

- $\lambda \approx \infty \Rightarrow \nabla \cdot u \approx 0$.

- Internal pressure $p = \lim_{\lambda \to +\infty} \lambda \nabla \cdot u$.

- **Quasi-incompressible viscoelasticity model**:
  
  \[
  \begin{cases}
  2\nabla \cdot ((\mu + i\omega\eta)\nabla^s u) + \nabla p + \rho\omega^2 u = 0 & \text{in} \ \Omega, \\
  \nabla \cdot u = 0 & \text{in} \ \Omega, \\
  u = g & \text{on} \ \Gamma_D, \\
  2(\mu + i\omega\eta)\nabla^s u \nu + p\nu = 0 & \text{on} \ \Gamma_N.
  \end{cases}
  \]

- If $\Gamma_D = \partial \Omega$ ($\Gamma_N = \emptyset$), then $g$ should satisfy the compatibility condition
  
  \[ \int_{\partial \Omega} g \cdot \nu \ ds = 0. \]

- $u^{(m)} = u^{(m)}[\mu_*, \eta_*]$: displacement measured in $\Omega$; $\mu_*$ and $\eta_*$: true distributions of shear elasticity and viscosity.

- Inverse problem: reconstruct the distribution of $\mu$ and $\eta$ from $u^{(m)}$. 
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- For a fixed $\epsilon > 0$; $\Omega' := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > \epsilon \}$ and $\mathcal{E} := \Omega \setminus \overline{\Omega'}$.
- $\mu$ and $\eta \in \tilde{S} := \{ (\mu_0, \eta_0) + (\phi_1, \phi_2) \mid (\phi_1, \phi_2) \in S \}; \mu_0$ and $\eta_0$: positive constants.
- $\tilde{S} = (\mu_0, \eta_0) + S$:
  \[
  S := \{ (\phi_1, \phi_2) \in H^2_0(\Omega) \times H^2_0(\Omega) : \ c_1 < \phi_1 + \mu_0 < c_2, \\
  c_1 < \phi_2 + \eta_0 < c_2, \| \phi_j \|_{W^{2,2}(\Omega)} \leq c_3, \ \text{supp} \phi_j \subset \Omega' \text{ for } j = 1, 2 \};
  \]
  $c_1, c_2, c_3$: positive constants.
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- Optimal control algorithm.
- Discrepancy functional:

\[ J[\mu, \eta] = \frac{1}{2} \int_{\Omega} |u[\mu, \eta] - u^{(m)}|^2 \, dx. \]

- Fréchet derivatives of \( J[\mu, \eta] \) with respect to \( \mu \) and \( \eta \).
- Assume that \( \delta_\mu \) and \( \delta_\eta \): small perturbations of \( \mu \) and \( \eta \), respectively, by regarding \( \frac{\delta_\mu + i\omega\delta_\eta}{\mu + i\omega\eta} \approx 0 \).
- \( u_0 := u[\mu, \eta] \), \( p_0 := \) the pressure corresponding to \( u_0 \) and \( p_0 + p_1 := \) the pressure corresponding to \( u[\mu + \delta_\mu, \eta + \delta_\eta] \);

\[ \delta u := u[\mu + \delta_\mu, \eta + \delta_\eta] - u_0. \]

- \[
2 \nabla \cdot ((\mu + i\omega\eta) \nabla^s \delta u) + \nabla p_1 + \rho \omega^2 \delta u = -2 \nabla \cdot ((\delta_\mu + i\omega\delta_\eta) \nabla^s u_0) \\
- 2 \nabla \cdot ((\delta_\mu + i\omega\delta_\eta) \nabla^s \delta u) \quad \text{in} \quad \Omega.
\]
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- $u_1$: solution of

$$\begin{cases}
2\nabla \cdot ((\mu + i\omega \eta)\nabla^s u_1) + \nabla p_1 + \rho \omega^2 u_1 = \\
-2\nabla \cdot ((\delta \mu + i\omega \delta \eta)\nabla^s u_0) & \text{in } \Omega, \\
\nabla \cdot u_1 = 0 & \text{in } \Omega, \\
u_1 = 0 & \text{on } \Gamma_D, \\
2(\mu + i\omega \eta)\nabla^s u_1 \nu + p_1 \nu = 0 & \text{on } \Gamma_N.
\end{cases}$$

- For $(\delta \mu + \mu, \delta \eta + \eta) \in \tilde{S}$,

$$\mathbb{R} \int_\Omega u_1(u_0 - u^{(m)}) \, dx = \mathbb{R} \int_\Omega 2(\delta \mu + i\omega \delta \eta)\nabla^s u_0 : \nabla^s \bar{v} \, dx.$$

- Fréchet derivatives of $J[\mu, \eta]$ with respect to $\mu$ and $\eta$:

$$\frac{\partial}{\partial \mu} J[\mu, \eta] = \mathbb{R} \left[ 2 \nabla^s u_0 : \nabla^s \bar{v} \right], \quad \frac{\partial}{\partial \eta} J[\mu, \eta] = \mathbb{R} \left[ 2(i\omega \nabla^s u_0) : \nabla^s \bar{v} \right];$$
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• $\nu$ solution of the adjoint problem:

$$\begin{cases}
2\nabla \cdot \left( (\mu - i\omega\eta)\nabla^s \nu \right) + \nabla q + \rho\omega^2 \nu = (u_0 - u^{(m)}) & \text{in } \Omega, \\
\nabla \cdot \nu = 0 & \text{in } \Omega, \\
\nu = 0 & \text{on } \Gamma_D, \\
2(\mu - i\omega\eta)\nabla^s \nu \nu + q\nu = 0 & \text{on } \Gamma_N.
\end{cases}$$

• $J[\mu, \eta]$: Fréchet differentiable for $(\mu, \eta) \in \tilde{S}$.

• As $\delta_\mu, \delta_\eta \to 0$,

$$\left| J[\mu + \delta_\mu, \eta + \delta_\eta] - J[\mu, \eta] - \mathbb{R} \int_\Omega u_1 (u_0 - u^{(m)}) dx \right|$$

$$= O \left( (\|\delta_\mu\|_{W^{2,2}(\Omega)} + \|\delta_\eta\|_{W^{2,2}(\Omega)})^2 \right).$$
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- Gradient descent scheme:
  - Let $n = 0$. Start with an initial guess of shear modulus $\mu^{(0)}$ and shear viscosity $\eta^{(0)}$.
  - For $n = 0, 1, \ldots$, compute $u_0^{(n)}$ by solving the forward problem with $\mu$ and $\eta$ replaced by $\mu^{(n)}$ and $\eta^{(n)}$, respectively. Compute $v^{(n)}$ by solving the adjoint problem with $\mu, \eta, u_0$ replaced by $\mu^{(n)}, \eta^{(n)}, u_0^{(n)}$, respectively.
  - For $n = 0, 1, \ldots$, compute the Fréchet derivatives $\frac{\partial J}{\partial \mu}[\mu^{(n)}, \eta^{(n)}]$ and $\frac{\partial J}{\partial \eta}[\mu^{(n)}, \eta^{(n)}]$.
  - Update $\mu$ and $\eta$ as follows:

\[
\begin{align*}
\mu^{(n+1)} &= \mu^{(n)} - \delta \frac{\partial J}{\partial \mu}[\mu^{(n)}, \eta^{(n)}], \\
\eta^{(n+1)} &= \eta^{(n)} - \delta \frac{\partial J}{\partial \eta}[\mu^{(n)}, \eta^{(n)}],
\end{align*}
\]

$\delta$: step size.

- Repeat Steps 2, 3, and 4 until $||\mu^{(n+1)} - \mu^{(n)}|| \leq \epsilon$ and $||\eta^{(n+1)} - \eta^{(n)}|| \leq \epsilon$ for a given $\epsilon > 0$. 
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• Initial guess:
  • Ignore the pressure term:

\[
2\nabla \cdot (\mu + i\omega \eta) \nabla^s u^\diamond + \rho \omega^2 u^\diamond = 0 \quad \text{in } \Omega,
\]

• Helmholtz decomposition:

\[
(\mu + i\omega \eta) \nabla^s u^\diamond = \nabla f + \nabla \times W \quad \text{with } \nabla \cdot W = 0,
\]

• \(f\) and \(W\): vector and matrix, respectively.

\[
\mu + i\omega \eta = \frac{\nabla f : \nabla^s \bar{u}^\diamond}{|\nabla^s u^\diamond|^2} + \frac{\nabla \times W : \nabla^s \bar{u}^\diamond}{|\nabla^s u^\diamond|^2}.
\]

\[
\Delta f = -\frac{1}{2} \rho \omega^2 u^\diamond \quad \text{in } \Omega.
\]

\[
\Delta W = \nabla \times ((\mu + i\omega \eta) \nabla^s u^\diamond) \quad \text{in } \Omega.
\]
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• $\tilde{f}$:
\[
\begin{aligned}
\Delta \tilde{f} &= -\frac{1}{2} \rho \omega^2 u^{(m)} \quad \text{in } \Omega, \\
\nabla \tilde{f} \cdot \nu &= (\mu_0 + i \omega \eta_0) \nabla^s u^{(m)} \cdot \nu \quad \text{on } \partial \Omega.
\end{aligned}
\]

• $W_1$:
\[
\begin{aligned}
\Delta W_1 &= \nabla \times \left( \frac{\nabla \tilde{f} : \nabla^s \bar{u}^{(m)}}{|\nabla^s \bar{u}^{(m)}|^2} \nabla^s u^{(m)} \right) \quad \text{in } \Omega, \\
W_1 &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

• $W_2$:
\[
\begin{aligned}
\Delta W_2 &= \nabla \times \left( -\frac{\rho \omega^2 (a \cdot u^{(m)})}{\nabla \cdot (a \cdot u^{(m)})} \nabla^s u^{(m)} \right) \quad \text{in } \Omega, \\
W_2 &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

$a$: any nonzero vector.
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- Initial guess formula:

\[
\mu^{(0)} + i\omega\eta^{(0)} = \frac{\nabla f : \nabla^s \tilde{u}^{(m)}}{|\nabla^s u^{(m)}|^2} + \frac{\nabla \times (W_1 + W_2) : \nabla^s \tilde{u}^{(m)}}{2|\nabla^s u^{(m)}|^2}.
\]
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- **Local reconstruction:**
  - $\Omega_{\text{loc}} \subset \Omega$; Localized minimization problem:
    \[
    J_{\text{loc}}[\mu, \eta] = \frac{1}{2} \int_{\Omega_{\text{loc}}} |u_{\text{loc}}[\mu, \eta] - u^{(m)}|^2 \, dx;
    \]
  - $u_{\text{loc}}[\mu, \eta]$:
    \[
    \begin{aligned}
    2 \nabla \cdot \left( (\mu + i \omega \eta) \nabla s u \right) + \nabla p + \rho \omega^2 u &= 0 & \text{in } \Omega_{\text{loc}}, \\
    \nabla \cdot u &= 0 & \text{in } \Omega_{\text{loc}}, \\
    u &= u^{(m)} & \text{on } \partial \Omega_{\text{loc}}.
    \end{aligned}
    \]
  - **Adjoint problem:**
    \[
    \begin{aligned}
    2 \nabla \cdot \left( (\mu - i \omega \eta) \nabla s v \right) + \nabla q + \rho \omega^2 v &= u_{\text{loc}} - u^{(m)} & \text{in } \Omega_{\text{loc}}, \\
    \nabla \cdot v &= 0 & \text{in } \Omega_{\text{loc}}, \\
    v &= 0 & \text{on } \partial \Omega_{\text{loc}}.
    \end{aligned}
    \]
Full-field optical coherence elastography

- Full-field optical coherence tomography (OCT): optical image with sub-cellular resolution.
- Apply a load on the sample.
- **OCTE**: Use a set of optical images before and after mechanical solicitation to reconstruct the shear modulus distribution inside the sample.
- Map of mechanical properties: added as a supplementary contrast mechanism to enhance specificity.
Full-field optical coherence elastography
Full-field optical coherence elastography
Full-field optical coherence elastography

- Reconstruct the shear modulus $\mu$ from $\varepsilon$ and $\varepsilon_u$.
- $\varepsilon(x) = \varepsilon_u(x + u(x))$;
- Displacement field $u$:

$$\begin{cases} 
\nabla \cdot \left( \mu (\nabla u + \nabla u^T) \right) + \nabla p = 0 \quad \text{in} \quad \Omega, \\
\nabla \cdot u = 0 \quad \text{in} \quad \Omega, \\
u = f \quad \text{on} \quad \partial\Omega.
\end{cases}$$
Full-field optical coherence elastography

- $\text{BV}(\Omega)$: subspace of $L^1(\Omega)$ of all the functions $f$ whose weak derivative $Df$: a finite Radon measure:

- $f$ s.t.

  $$\int_{\Omega} f \nabla \cdot F \leq C \sup_{x \in \Omega} |F|, \quad \forall F \in C_0^1(\Omega)^d$$

  for some positive constant $C$ with $C_0^1(\Omega)$: set of compactly supported $C^1$ functions.
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• Derivative of a function $f \in \text{BV}(\Omega)$ can be decomposed as

$$Df = \nabla f \mathcal{H}^d + [f] \nu_S \mathcal{H}_S^{d-1} + D_c f;$$

• $\mathcal{H}^d$: Lebesgue measure on $\Omega$, $\mathcal{H}_S^{d-1}$: surface Hausdorff measure on a rectifiable surface $S$, $\nu_S$: normal vector defined a.e. on $S$;

• $\nabla f \in L^1(\Omega)$: smooth derivative of $f$, $[f] \in L^1(S, \mathcal{H}_S^{d-1})$: jump of $f$ across $S$ and $D_c f$: vector measure supported on a set of Hausdorff dimension less than $(d - 1)$. 
Full-field optical coherence elastography

- $\text{SBV}(\Omega)$: subspace of $\text{BV}(\Omega)$ of all the functions $f$ satisfying $D_c f = 0$.
- For any $1 \leq p \leq +\infty$,

$$\text{SBV}^p(\Omega) = \left\{ f \in \text{SBV}(\Omega) \cap L^p(\Omega), \nabla f \in L^p(\Omega)^d \right\}.$$
Full-field optical coherence elastography

• $u^*$: true displacement; $\widetilde{\varepsilon}$: measured deformed optical:

$$\widetilde{\varepsilon} = \varepsilon \circ (I + u^*)^{-1}.$$  

• Optical image: discontinuous.
• Optimal control algorithm:

$$I(u) = \frac{1}{2} \int_\Omega |\widetilde{\varepsilon} \circ (I + u) - \varepsilon|^2 \, dx.$$  

• $I$ has a nonempty subgradient.
• $\xi \in \partial I$:

$$\xi : h \mapsto \int_\Omega [\widetilde{\varepsilon}(x + u) - \varepsilon(x)]h(x) \cdot D\widetilde{\varepsilon} \circ (I + u)(x) \, dx.$$  

Mathematics of super-resolution biomedical imaging

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Full-field optical coherence elastography

- Nondifferentiable functional $u \mapsto I(u)$ has a nonempty subgradient if there exists $\xi$ s.t.
  \[ I(u + h) - I(u) \geq (\xi, h) \]
  holds for $||h||$ small enough;
- $\xi \in \partial I$ with $\partial I$: subgradient of $I$.
- In order to minimize $I$, it is sufficient to find one $\xi \in \partial I$. 
Full-field optical coherence elastography

Initial guess:

- Detect the surface of jumps of the optical image (edge detection algorithm).
- Local recovery by linearization: \( \text{data} = \varepsilon - \varepsilon u (\approx \nabla \varepsilon \cdot u) \)

\[
J_x(u) = \int_{\Omega} |\nabla \varepsilon(y) \cdot u - \text{data}(y)|^2 w_\delta(|x - y|) \, dy.
\]

- \( w_\delta = \frac{1}{\delta^d} w \left( \frac{\cdot}{\delta} \right) \); \( w \): a mollifier supported on \([-1, 1]\).
- Least-squares solution:

\[
u^T = \left( \int_{\Omega} w_\delta(|x - y|) \nabla \varepsilon(y) \nabla \varepsilon^T(y) \, dy \right)^{-1} \int_{x + \delta B} \text{data} w_\delta(|x - y|) \nabla \varepsilon(y) \, dy.
\]

- If all vectors \( \nabla \varepsilon \) in \( \{ y : w_\delta(|y - x|) \neq 0 \} \) not collinear, then

\[
\int_{\Omega} w_\delta(|x - y|) \nabla \varepsilon(y) \nabla \varepsilon^T(y) \, dy \quad \text{invertible.}
\]

- Resolution = variation of \( \varepsilon \).
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Optical image $\varepsilon$ of the medium:
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Averaging kernel $w_\delta$:

Conditioning of the matrix $w_\delta \star \nabla \varepsilon \nabla \varepsilon^T$:
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Displacement field and its reconstruction.
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Shear modulus distribution $\mu$

Reconstructed shear modulus distribution $\mu_{\text{rec}}$

Shear modulus reconstruction.