Mathematical and Computational Methods in Photonics

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Mathematics for photonics

- Control, manipulate, reshape, guide, focus electromagnetic waves at subwavelength length scales (beyond the resolution limit).
- Direct, inverse, and optimal design problems for electromagnetic wave propagation in complex and resonant media.
- Build mathematical frameworks and develop effective numerical algorithms for photonic applications.
- Partial differential equations, spectral analysis, integral equations, computational techniques, and multi-scale analysis.
Resonances for plasmonic nanoparticles

- **Key to super-resolution:** push the resolution limit by reducing the focal spot size; confine light to a length scale significantly smaller than half the wavelength.

- **Resolution:** smallest detail that can be resolved.
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- Mathematical and computational tools:
  - Diffraction gratings;
  - Photonic crystals;
  - Plasmonic resonant nanoparticles;
  - Metamaterials and metasurfaces.
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- **Diffraction gratings:**
  - Scattering by periodic structures: dominated by diffraction; small features of the structure → small number of propagating modes (other modes are evanescent).
  - Spectroscopic, telecommunications and laser applications.
  - Design problem: grating profile that give rise to a specified diffraction pattern.
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- **Photonic crystals** (also known as photonic band-gap materials):
  - Periodic dielectric structures that have a band gap that forbids propagation of a certain frequency range of light.
  - Band gap calculations: high-contrast materials, periodicity of the same order as the wavelength; efficient numerical schemes.
  - Control light and produce effects that are impossible with conventional optics.
  - Resonant cavities: making point defects in a photonic crystal → light can be localized, trapped in the defect. The frequency, symmetry, and other properties of the defect mode can be easily tuned to anything desired.
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- **Plasmonic nanoparticles:**
  - Subwavelength resonance: quasi-static regime.
  - Scattering and absorption enhancement.
  - Superresolution: single particle imaging.
  - Nanoantenna, concentrate light at subwavelength scale.
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- **Metamaterials and metasurfaces:**
  - Negative material parameters.
  - Electromagnetic invisibility and cloaking: make a target invisible when probed by electromagnetic waves:
    - Interior cloaking: scattering cancellations techniques.
    - Exterior cloaking by anomalous resonances.
  - Subwavelength band gap materials: microstructure periodicity smaller than the wavelength.
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- Metamaterials and metasurfaces:
  - Microstructured materials.
  - Building block microstructure: subwavelength resonator.
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- Effective medium theory:
  - High contrast materials: for some range of frequencies.
  - Superresolution and superfocusing of electromagnetic waves.
- Unify the mathematical theory of superresolution, photonic bandgap materials, metamaterials, and cloaking.
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• Near-field optics:
  • Interaction between the plasmonic probe and the sample.
  • Superresolution imaging of the sample.
  • Mechanism $\rightarrow$ quantitative imaging.
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- **Spectral analysis and integral equation formulations.**
- **Green’s functions** (free space, periodic, quasi-periodic, ...) \(\rightarrow\) **eigenvalue problems reduced to characteristic value problems (nonlinear eigenvalue problems).**

- **Gohberg-Sigal theory:**
  - Generalization of **Rouché** theorem for operator valued function.
  - **Sensitivity analysis** (change in the shape, material parameters, environment, ...) of diffraction pattern, band gaps, resonance for plasmonic nanoparticles, ...
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- "for their transformative contributions to the field of nano-optics that have broken long-held beliefs about the limitations of the resolution limits of optical microscopy and imaging."
  - "for the discovery of the extraordinary transmission of light through sub-wavelength apertures."
  - "for ground-breaking developments that have led to fluorescence microscopy with nanometre scale resolution, opening up nanoscale imaging to biological applications."
  - "for developing the theory underlying new optical nanoscale materials with unprecedented properties, such as the negative index of refraction, allowing for the formation of perfect lenses."
Mathematics for photonics

- **Phononics:**
  - Sound / light.
  - Elasticity equations / Maxwell’s equations.
  - Subwavelength resonances: Helmholtz resonator, Minnaert bubble / plasmonic nanoparticle.

- Similar physical mechanisms and mathematical and computational frameworks to those in photonics:
  - Scattering enhancement by subwavelength acoustic resonators.
  - Phononic crystals.
  - Acoustic metamaterials and metasurfaces, subwavelength phononic band gap materials.
  - High contrast acoustic materials, superresolution and superfocusing for acoustic waves.
Mathematics for photonics

- **Gohberg-Sigal** theory:
  
  - **Argument principle**: $\mathcal{V} \subset \mathbb{C}$: bounded domain with smooth boundary $\partial \mathcal{V}$ positively oriented; $f(z)$: meromorphic function in a neighborhood of $\mathcal{V}$; $P$ and $N$: the number of poles and zeros of $f$ in $\mathcal{V}$, counted with their multiplicities. If $f$ has no poles and never vanishes on $\partial \mathcal{V}$, then
  
  $$\frac{1}{2\pi i} \int_{\partial \mathcal{V}} \frac{f'(z)}{f(z)} \, dz = N - P.$$  

  - **Rouché’s theorem**: $f(z)$ and $g(z)$: holomorphic in a neighborhood of $\overline{\mathcal{V}}$. If $|f(z)| > |g(z)|$ for all $z \in \partial \mathcal{V}$, then $f$ and $f + g$ have the same number of zeros in $\mathcal{V}$. 
• $L(B, B')$: linear bounded operators from $B$ into $B'$ (Banach spaces).

• $U(z_0)$: set of all operator-valued functions in $L(B, B')$ which are holomorphic in some neighborhood of $z_0$, except possibly at $z_0$.

• $z_0$ characteristic value of $A(z) \in U(z_0)$ if there exists a vector-valued function $\phi(z)$ with values in $B$ such that
  • $\phi(z)$: holomorphic at $z_0$ and $\phi(z_0) \neq 0$,
  • $A(z)\phi(z)$: holomorphic at $z_0$ and vanishes at this point.
  • $\phi(z)$: root function of $A(z)$ associated with the characteristic value $z_0$. 
Mathematics for photonics

• Generalized argument principle:

\[ \mathcal{M}(A(z); \partial V) = \frac{1}{2\pi i} \text{tr} \int_{\partial V} A^{-1}(z) \frac{d}{dz} A(z) dz. \]

• \( \mathcal{M}(A(z); \partial V) \): number of characteristic values of \( A(z) \) in \( V \), counted with their multiplicities, minus the number of poles of \( A(z) \) in \( V \), counted with their multiplicities.

• Generalized Rouché’s theorem:

\[ \mathcal{M}(A(z); \partial V) = \mathcal{M}(A(z) + S(z); \partial V). \]

• \( S(z) \): finitely meromorphic in \( V \) and continuous on \( \partial V \) s.t.

\[ \| A^{-1}(z)S(z) \|_{\mathcal{L}(\mathcal{B}, \mathcal{B})} < 1, \quad z \in \partial V. \]

• Finitely meromorphic operator: coefficients of the principal part of its Laurent expansion are operators of finite rank.
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• $0 = \mu_1 < \mu_2 \leq \ldots$: eigenvalues of $-\Delta$ in $\Omega$ with Neumann conditions,

\[
\begin{cases}
\Delta u + \mu u = 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

• $(u_j)_{j \geq 1}$: orthonormal basis of $L^2(\Omega)$ of normalized eigenvectors.

• $\omega = \sqrt{\mu}$; $S_\Omega^\omega, D_\Omega^\omega, K_\Omega^\omega$: single- and double-layer potentials and Neumann-Poincaré operator associated with the outgoing fundamental solution $G_\omega(x, z)$ to the Helmholtz operator $\Delta + \omega^2$:

\[
G_\omega(x, z) := \begin{cases}
-\frac{i}{4} H_0^{(1)}(\omega|x-z|), & d = 2, \\
\frac{e^{i|x-z|}}{4\pi|x-z|}, & d = 3.
\end{cases}
\]

• $H_0^{(1)}$: Hankel function of the first kind of order 0.
Mathematics for photonics

• Sommerfeld radiation condition: $|x| \to +\infty$,

$$\frac{x}{|x|} \cdot \nabla G_\omega(x, z) - i\omega G_\omega(x, z) = \begin{cases} O(|x|^{-3/2}), & d = 2, \\ O(|x|^{-2}), & d = 3. \end{cases}$$

• Layer potentials: $\varphi \in L^2(\partial \Omega)$,

$$S_\Omega^{\omega}[\varphi](x) = \int_{\partial \Omega} G_\omega(x, y)\varphi(y) \, d\sigma(y), \quad x \in \mathbb{R}^d,$$

$$D_\Omega^{\omega}[\varphi](x) = \int_{\partial \Omega} \frac{\partial G_\omega(x, y)}{\partial \nu(y)} \varphi(y) \, d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial \Omega,$$

$$K_\Omega^{\omega}[\varphi](x) = \text{p.v.} \int_{\partial \Omega} \frac{\partial G_\omega(x, y)}{\partial \nu(y)} \varphi(y) \, d\sigma(y).$$

• $\sqrt{\mu_j}$: characteristic value of $\omega \mapsto (1/2)I - K_\Omega^{\omega}$.

• Muller’s method: compute zeros of $\omega \mapsto 1/(((1/2)I - K_\Omega^{\omega})^{-1}[\varphi], \psi)$ for fixed $\varphi$ and $\psi$. 
Mathematics for photonics

- \(D\) conductive particle inside \(\Omega\), \(D = \varepsilon B + z\); \(k \neq 1\): conductivity parameter; \(\varepsilon\): characteristic size; \(d\): space dimension.

- **Characteristic values** of the operator-valued function \(A_\varepsilon(\omega)\):

\[
\omega \mapsto A_\varepsilon(\omega) := \begin{pmatrix}
\frac{1}{2}I - \mathcal{K}_\Omega & -S_D^\omega & 0 \\
D_\Omega^\omega & S_D^\omega & -S_D^{\frac{\omega}{\sqrt{k}}}
\end{pmatrix}.
\]

- **Generalized argument principle**:

\[
\omega_\varepsilon - \omega_0 = \frac{1}{2\pi i} \text{tr} \int_{\partial \Omega_0} (\omega - \omega_0)A_\varepsilon(\omega)^{-1} \frac{d}{d\omega} A_\varepsilon(\omega) d\omega.
\]
Mathematics for photonics

- **Eigenvalue expansion:**
  \[
  \mu_j^\varepsilon - \mu_j = \varepsilon^d \nabla u_j(z) \cdot M \nabla u_j(z) + o(\varepsilon^d).
  \]

- **Polarization tensor** \(M = (m_{ll'}) := \)
  \[
  m_{ll'} = (k - 1) \int_{\partial B} \psi_l \frac{\partial x_{l'}}{\partial \nu} \, d\sigma.
  \]

\[
\begin{cases}
  \nabla \cdot (1 + (k - 1)\chi(B)) \nabla \psi_l &= 0 \quad \text{in } \mathbb{R}^d, \\
  \psi_l(x) - x_l &= O(|x|^{1-d}) \quad \text{as } |x| \to +\infty.
\end{cases}
\]

- **Eigenfunction expansion** in \(\Omega:
  \[
  u_j^\varepsilon(x) = u_j(z) + \varepsilon \sum_{l=1}^{d} \partial_l u_j(z) \psi_l \left(\frac{x - z}{\varepsilon}\right) + o(\varepsilon).
  \]

- **\(u_j^\varepsilon\): normalized eigenfunction associated with \(\mu_j^\varepsilon.\)
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- Photonic crystals:
  - Floquet transform:
    \[
    \mathcal{U}[f](x, \alpha) = \sum_{n \in \mathbb{Z}^d} f(x - n) e^{i\alpha \cdot n}.
    \]

- \(f(x)\): function decaying sufficiently fast.
- \(\mathcal{U}\): analogue of the Fourier transform for the periodic case.
- \(\alpha \in \text{Brillouin zone } \mathbb{R}^d/(2\pi \mathbb{Z}^d)\): quasi-momentum (analogue of the dual variable in the Fourier transform).
- Expansion of a periodic operator \(L\) in \(L^2(\mathbb{R}^d)\) into a direct integral of operators:
  \[
  L = \int_{\mathbb{R}^d/(2\pi \mathbb{Z}^d)} L(\alpha) \, d\alpha.
  \]

- \(L(\alpha)[f] = \mathcal{U}[L[f]]\).
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• **Spectral theorem** for a self-adjoint operator:

\[ \sigma(L) = \bigcup_{\alpha \in \mathbb{R}^d/(2\pi\mathbb{Z}^d)} \sigma(L(\alpha)), \]

• \( \sigma(L) \): spectrum of \( L \).

• \( L \): elliptic \( \rightarrow \) \( L(\alpha) \): compact resolvents \( \rightarrow \) discrete spectra \( (\mu_l(\alpha))_l \),

\[ \sigma(L) = \left[ \min_{\alpha} \mu_l(\alpha), \max_{\alpha} \mu_l(\alpha) \right]. \]
Mathematics for photonics

- **Gohberg-Sigal** theory:
  - Sensitivity analysis of band gaps with respect to changes of the coefficients of $L$.
  - Analysis of photonic crystal cavities: defect mode inside the band gap.
Resonances for plasmonic nanoparticles

- **Gold nano-particles**: accumulate selectively in tumor cells; bio-compatible; reduced toxicity.
- Detection: localized enhancement in radiation dose (strong scattering).
- Ablation: localized damage (strong absorption).
- Functionalization: targeted drugs.

M.A. El-Sayed et al.
Resonances for plasmonic nanoparticles

- Mechanisms of scattering and absorption enhancements and supreresolution using plasmonic nanoparticles.
- Spectral properties of Neumann-Poincaré operator.
Resonances for plasmonic nanoparticles

• \(D\): nanoparticle in \(\mathbb{R}^d\), \(d = 2, 3\); \(C^{1,\alpha}\) boundary \(\partial D\), \(\alpha > 0\).

• \(\varepsilon_c(\omega)\): complex permittivity of \(D\); \(\varepsilon_m > 0\): permittivity of the background medium;

• Permittivity contrast: \(\lambda(\omega) = (\varepsilon_c(\omega) + \varepsilon_m)/(2(\varepsilon_c(\omega) - \varepsilon_m))\).

• **Causality** ⇒ Kramer-Kröning relations (Hilbert transform),

\[\varepsilon_c(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega):\]

\[
\varepsilon'(\omega) - \varepsilon_\infty = -\frac{2}{\pi} \text{p.v.} \int_0^{+\infty} \frac{s\varepsilon''(s)}{s^2 - \omega^2} \, ds,
\]

\[
\varepsilon''(\omega) = \frac{2\omega}{\pi} \text{p.v.} \int_0^{+\infty} \frac{\varepsilon'(s) - \varepsilon_\infty}{s^2 - \omega^2} \, ds.
\]

• **Drude** model for the dielectric permittivity \(\varepsilon_c(\omega)\):

\[\varepsilon_c(\omega) = \varepsilon_\infty(1 - \frac{\omega_p^2}{\omega^2 + i\tau\omega}), \quad \varepsilon'(\omega) \leq 0 \quad \text{for} \quad \omega \leq \omega_p.\]

\(\omega_p, \tau\): positive constants.
Resonances for plasmonic nanoparticles

- **Fundamental solution** to the Laplacian:

\[ G(x) := \begin{cases} 
\frac{1}{2\pi} \ln |x|, & d = 2, \\
- \frac{1}{4\pi} |x|^{2-d}, & d = 3;
\end{cases} \]

- **Single-layer potential**:

\[ S_D[\varphi](x) := \int_{\partial D} G(x - y)\varphi(y) \, ds(y), \quad x \in \mathbb{R}^d. \]

- **Neumann-Poincaré operator** \( K_D^* \):

\[ K_D^*[\varphi](x) := \int_{\partial D} \frac{\partial G}{\partial \nu(x)} (x - y)\varphi(y) \, ds(y) , \quad x \in \partial D. \]

\( \nu \): normal to \( \partial D \).

- **\( K_D^* \): compact operator** on \( L^2(\partial D) \),

\[ \frac{|\langle x - y, \nu(x) \rangle|}{|x - y|^d} \leq \frac{C}{|x - y|^{d-1-\alpha}}, \quad x, y \in \partial D. \]

- Spectrum of \( K_D^* \) lies in \( (-\frac{1}{2}, \frac{1}{2}] \) (Kellog).
Resonances for plasmonic nanoparticles

- $K_D^*$ self-adjoint on $L^2(\partial D)$ if and only if $D$ is a disk or a ball.
- Symmetrization technique for Neumann-Poincaré operator $K_D^*$:
  - Calderón’s identity: $K_D S_D = S_D K_D^*$;
  - In three dimensions, $K_D^*$: self-adjoint in the Hilbert space $\mathcal{H}^*(\partial D) = H^{-\frac{1}{2}}(\partial D)$ equipped with
    $$(u, v)_{\mathcal{H}^*} = -(u, S_D[v])_{-\frac{1}{2}, \frac{1}{2}}$$
    $(\cdot, \cdot)_{-\frac{1}{2}, \frac{1}{2}}$: duality pairing between $H^{-\frac{1}{2}}(\partial D)$ and $H^{\frac{1}{2}}(\partial D)$.
  - In two dimensions: $\exists! \tilde{\varphi}_0$ s.t. $S_D[\tilde{\varphi}_0] = \text{constant on } \partial D$ and $(\tilde{\varphi}_0, 1)_{-\frac{1}{2}, \frac{1}{2}} = 1$. $S_D \to \tilde{S}_D$:
    $$\tilde{S}_D[\varphi] = \begin{cases} S_D[\varphi] & \text{if } (\varphi, 1)_{-\frac{1}{2}, \frac{1}{2}} = 0, \\ -1 & \text{if } \varphi = \tilde{\varphi}_0. \end{cases}$$
Resonances for plasmonic nanoparticles

- Symmetrization technique for Neumann-Poincaré operator $\mathcal{K}_D^*$:
  - Spectrum $\sigma(\mathcal{K}_D^*)$ discrete in $]-1/2, 1/2[$;
  - Ellipse: $\pm \frac{1}{2}(\frac{a-b}{a+b})^j$, elliptic harmonics ($a$, $b$: long and short axis).
  - Ball: $\frac{1}{2(2j+1)}$, spherical harmonics.
  - Twin property in two dimensions;
  - $(\lambda_j, \varphi_j)$, $j = 0, 1, 2, \ldots$: eigenvalue and normalized eigenfunction pair of $\mathcal{K}_D^*$ in $\mathcal{H}^*(\partial D)$; $\lambda_j \in (-\frac{1}{2}, \frac{1}{2}]$ and $\lambda_j \to 0$ as $j \to \infty$;
  - $\varphi_0$: eigenfunction associated to $1/2$ ($\tilde{\varphi}_0$ multiple of $\varphi_0$);
  - Spectral decomposition formula in $H^{-1/2}(\partial D)$,

$$\mathcal{K}_D^*[\psi] = \sum_{j=0}^{\infty} \lambda_j (\psi, \varphi_j)_{\mathcal{H}^*} \varphi_j.$$
Resonances for plasmonic nanoparticles

• $u^i$: incident plane wave; **Helmholtz equation:**

\[
\begin{align*}
\nabla \cdot \left( \varepsilon_m \chi(\mathbb{R}^d \setminus \bar{D}) + \varepsilon_c(\omega) \chi(\bar{D}) \right) \nabla u + \omega^2 u &= 0, \\

u^s := u - u^i \text{ satisfies the outgoing radiation condition.}
\end{align*}
\]

• **Uniform small volume expansion** with respect to the contrast:

\[
D = z + \delta B, \quad \delta \to 0, \quad |x - z| \gg 2\pi/k_m,
\]

\[
u^s = - M(\lambda(\omega), D) \nabla_z G_{k_m}(x - z) \cdot \nabla u^i(z) + O\left(\frac{\delta^{d+1}}{\text{dist}(\lambda(\omega), \sigma(K^*_D))}\right).
\]

• $G_{k_m}$: outgoing fundamental solution to $\Delta + k^2_m$; $k_m := \omega/\sqrt{\varepsilon_m}$;

• Polarization tensor:

\[
M(\lambda(\omega), D) := \int_{\partial D} x(\lambda(\omega) I - \mathcal{K}_D^*)^{-1} [\nu](x) \, ds(x).
\]

• **Scaling and translation properties:** $M(\lambda(\omega), z + \delta B) = \delta^d M(\lambda(\omega), B)$. 
Resonances for plasmonic nanoparticles

Representation by equivalent ellipses and ellipsoids:

- Nanoparticle’s permittivity: $\varepsilon_c(\omega) = \varepsilon'(\omega) + i\varepsilon''(\omega)$.
- $\varepsilon'(\omega) > 0$ and $\varepsilon''(\omega) = 0$: canonical representation; equivalent ellipse or ellipsoid with the same polarization tensor.
- Plasmonic nanoparticles: non Hermitian case.
- $\Im M(\lambda(\omega), D)$: equivalent frequency depending ellipse or ellipsoid with the same imaginary part of the polarization tensor.
Resonances for plasmonic nanoparticles

- Spectral decomposition: \((l, m)\)-entry

\[
M_{l,m}(\lambda(\omega), D) = \sum_{j=1}^{\infty} \frac{(\nu_m, \varphi_j)_{\mathcal{H}^*}(\nu_l, \varphi_j)_{\mathcal{H}^*}}{(1/2 - \lambda_j)(\lambda(\omega) - \lambda_j)}.
\]

- \((\nu_m, \varphi_0)_{\mathcal{H}^*} = 0; \varphi_0\): eigenfunction of \(K_D^*\) associated to 1/2.

- Quasi-static far-field approximation: \(\delta \to 0\),

\[
u^s = -\delta^d M(\lambda(\omega), B) \nabla_z G_{km}(x - z) \cdot \nabla u^i(z) + O(\frac{\delta^{d+1}}{\text{dist}(\lambda(\omega), \sigma(K_D^*))}).
\]

- Quasi-static plasmonic resonance: \(\text{dist}(\lambda(\omega), \sigma(K_D^*))\) minimal \((\Re \varepsilon_c(\omega) < 0)\).
Resonances for plasmonic nanoparticles

- \( M(\lambda(\omega), B) = \left( \frac{\varepsilon_c(\omega)}{\varepsilon_m} - 1 \right) \int_B \nabla v(y) dy \):
  \[
  \begin{cases}
  \nabla \cdot \left( \varepsilon_m \chi(\mathbb{R}^d \setminus \bar{B}) + \varepsilon_c(\omega) \chi(\bar{B}) \right) \nabla v = 0, \\
  v(y) - y \to 0, \quad |y| \to +\infty.
  \end{cases}
  \]

- Corrector \( v \):
  \[
  v(y) = y + S_B(\lambda(\omega)I - \mathcal{K}_D^*)^{-1} [v](y), \quad y \in \mathbb{R}^d.
  \]

- Inner expansion: \( \delta \to 0, \ |x - z| = O(\delta) \),
  \[
  u(x) = u^i(z) + \delta v(\frac{x - z}{\delta}) \cdot \nabla u^i(z) + O\left( \frac{\delta^2}{\text{dist}(\lambda(\omega), \sigma(\mathcal{K}_D^*))} \right).
  \]

- Monitoring of temperature elevation due to nanoparticle heating:
  \[
  \begin{cases}
  \rho C \frac{\partial T}{\partial t} - \nabla \cdot \tau \nabla T = \frac{\omega}{2\pi} \Im(\varepsilon_c(\omega)) |u|^2 \chi(D), \\
  T|_{t=0} = 0.
  \end{cases}
  \]

\( \rho \): mass density; \( C \): thermal capacity; \( \tau \): thermal conductivity.
Resonances for plasmonic nanoparticles

• Scattering amplitude:

\[ u^s(x) = -ie^{-\frac{\pi i}{4}} \frac{e^{ik_m|x|}}{\sqrt{8\pi k_m|x|}} A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') + o(|x|^{-\frac{1}{2}}), \]

\(|x| \rightarrow \infty; \theta, \theta': \text{incident and scattered directions.}\)

• Scattering cross-section:

\[ Q^s[D, \varepsilon_c, \varepsilon_m, \omega](\theta') := \int_0^{2\pi} \left| A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') \right|^2 d\theta. \]

• Enhancement of the absorption and scattering cross-sections \(Q^a\) and \(Q^s\) at plasmonic resonances:

\[ Q^a + Q^s(= \text{extinction cross-section } Q^e) \propto \Im \text{Trace}(M(\lambda(\omega), D)); \]

\[ Q^s \propto |\text{Trace}(M(\lambda(\omega), D))|^2. \]
Resonances for plasmonic nanoparticles

Norm of the polarization tensor for a circular inclusion.
Resonances for plasmonic nanoparticles

Norm of the polarization tensor for an elliptic inclusion.
Resonances for plasmonic nanoparticles
Resonances for plasmonic nanoparticles

Norm of the polarization tensor for a flower-shaped inclusion.
Resonances for plasmonic nanoparticles

- Quasi-plasmonic resonances for multiple particles: $D_1$ and $D_2$: $C^{1,\alpha}$-bounded domains; $\text{dist}(D_1, D_2) > 0$; $\nu^{(1)}$ and $\nu^{(2)}$: outward normal vectors at $\partial D_1$ and $\partial D_2$.
- Neumann-Poincaré operator $K^*_{D_1 \cup D_2}$ associated with $D_1 \cup D_2$:

  $$K^*_{D_1 \cup D_2} := \left( \begin{array}{cc} K^*_{D_1} & \frac{\partial}{\partial \nu^{(1)}} S_{D_2} \\ \frac{\partial}{\partial \nu^{(2)}} S_{D_1} & K^*_{D_2} \end{array} \right).$$

- Symmetrization of $K^*_{D_1 \cup D_2}$.
- Behavior of the eigenvalues of $K^*_{D_1 \cup D_2}$ as $\text{dist}(D_1, D_2) \to 0$. 
Resonances for plasmonic nanoparticles

Norm of the polarization tensor for two disks for various separating distances.
Resonances for plasmonic nanoparticles

- **Algebraic domains:** finite number of quasi-static plasmonic resonances:
  \[
  \#\{j : (\nu_l, \varphi_j)_{\mathcal{H}^*} \neq 0\} : \text{finite}.
  \]

- **Algebraic domains:** zero level sets of polynomials; dense in Hausdorff metric among all planar domains.

- **Blow-up of the polarization tensor** for finite number of eigenvalues of the Neumann-Poincaré operator:
  \[
  M_{l,m}(\lambda(\omega), D) = \sum_{j=1}^{\infty} \frac{(\nu_m, \varphi_j)_{\mathcal{H}^*}(\nu_l, \varphi_j)_{\mathcal{H}^*}}{(1/2 - \lambda_j)(\lambda(\omega) - \lambda_j)}.
  \]

- **Two nearly touching disks:** infinite number of quasi-static plasmonic resonances.
  \[
  \lambda_j = \pm \frac{1}{2} e^{-2|j|\xi}, \xi = \sinh^{-1}\left(\sqrt{\frac{\delta}{r}}(1 + \frac{\delta}{4r})\right);
  \]

- **r:** radius of the disks; **δ:** separating distance.

- **Separating distance δ:** estimated from the first plasmonic resonance (associated to \(\lambda_1\)).
Resonances for plasmonic nanoparticles

- **Singular** nature of the interaction between nearly touching plasmonic nanoparticles.
- Applications in nanosensing (beyond the resolution limit).
- **Blow-up** of $\nabla u$ between the disks at plasmonic resonances:
  \[
  \nabla u \propto \frac{r}{\Im(\lambda(\omega)) \delta} e^{-2|j|\xi}.
  \]
- Accurate scheme for computing the field distribution between an arbitrary number of nearly touching plasmonic nanospheres: **transformation optics + method of image charges**.
Resonances for plasmonic nanoparticles

- \((m, l)\)-entry of the polarization tensor \(M\):

\[
M_{l,m}(\lambda(\omega), D) = \sum_{j=1}^{\infty} \frac{\alpha_{l,m}^{(j)}}{\lambda(\omega) - \lambda_j},
\]

\[
\alpha_{l,m}^{(j)} := \frac{(\nu_m, \varphi_j)^{\mathcal{H}^*}(\nu_l, \varphi_j)^{\mathcal{H}^*}}{(1/2 - \lambda_j)}, \quad \alpha_{l,l}^{(j)} \geq 0, \quad j \geq 1.
\]

- **Sum rules** for the polarization tensor:

\[
\sum_{j=1}^{\infty} \alpha_{l,m}^{(j)} = \delta_{l,m} |D|; \quad \sum_{j=1}^{\infty} \lambda_i \sum_{l=1}^{d} \alpha_{l,l}^{(j)} = \frac{(d-2)}{2} |D|.
\]

\[
\sum_{j=1}^{\infty} \lambda_j^2 \sum_{l=1}^{d} \alpha_{l,l}^{(j)} = \frac{(d-4)}{4} |D| + \sum_{l=1}^{d} \int_D |\nabla S_D[\nu_l]|^2 \, dx.
\]

- \(f\) holomorphic function in an open set \(U \subset \mathbb{C}\) containing \(\sigma(K_D^*)\):

\[
f(K_D^*) = \sum_{j=1}^{\infty} f(\lambda_j)(\cdot, \varphi_j)^{\mathcal{H}^*} \varphi_j.
\]
Resonances for plasmonic nanoparticles

- **Upper bound** for the averaged extinction cross-section $Q^e_m$ of a randomly oriented nanoparticle:

$$\left| \Im(\text{Trace}(M(\lambda, D))) \right| \leq \frac{d|\lambda''||D|}{\lambda''^2 + 4\lambda'^2}$$

$$+ \frac{1}{|\lambda''|(\lambda''^2 + 4\lambda'^2)} \left( d\lambda'^2|D| + \frac{(d-4)}{4}|D| \right)$$

$$+ \sum_{l=1}^{d} \int_{D} |\nabla S_D[\nu_l]|^2 \, dx + 2\lambda' \frac{(d-2)}{2}|D|) + O\left( \frac{\lambda''^2}{4\lambda'^2 + \lambda''^2} \right).$$

$\lambda' = \Re \lambda, \lambda'' = \Im \lambda.$
Resonances for plasmonic nanoparticles

Averaged extinction

Wavelength of the incoming plane wave

- Bound
- $a/b = 2$
- $a/b = 4$
Resonances for plasmonic nanoparticles

Hadamard’s formula for $\mathcal{K}_D^*$:

- $\partial D$: class $C^2$; $\partial D := \{ x = X(t), t \in [a, b] \}$.
- $\Psi_\eta : \partial D \mapsto \partial D_\eta := \{ x + \eta h(t) \nu(x) \}; \: \Psi_\eta$: diffeomorphism.
- Hadamard’s formula for $\mathcal{K}_D^*$:

$$||\mathcal{K}_D^*[\tilde{\phi}] \circ \Psi_\eta - \mathcal{K}_D^*[\phi] - \eta \mathcal{K}_D^{(1)}[\phi]||_{L^2(\partial D)} \leq C \eta^2 ||\phi||_{L^2(\partial D)},$$

$C$: depends only on $||X||_{C^2}$ and $||h||_{C^1}; \: \phi := \tilde{\phi} \circ \Psi_\eta$.

- $\mathcal{K}_D^{(1)}$: explicit kernel.
- Hadamard’s formula for the eigenvalues of $\mathcal{K}_D^*$.
- Shape derivative of plasmonic resonances for nanoparticles.
- Generalization to 3D.
Resonances for plasmonic nanoparticles

- $\mathcal{K}_D^*$: scale invariant $\Rightarrow$ Quasi-static plasmonic resonances: size independent.

- Analytic formula for the first-order correction to quasi-static plasmonic resonances in terms of the particle's characteristic size $\delta$:

M.A. El-Sayed et al.
Resonances for plasmonic nanoparticles

- **Helmholtz equation:**
  \[
  \nabla \cdot \left( \varepsilon_m \chi(\mathbb{R}^d \setminus \bar{D}) + \varepsilon_c(\omega) \chi(\bar{D}) \right) \nabla \varphi + \omega^2 \varphi = 0,
  \]

  \[
  \varphi^s := \varphi - \varphi^i \text{ satisfies the outgoing radiation condition.}
  \]

  \(\varphi^i: \text{ incident plane wave; } k_m := \omega \sqrt{\varepsilon_m}, k_c := \omega \sqrt{\varepsilon_c(\omega)}.\)

- **Integral formulation on } \partial D:*
  \[
  \begin{align*}
  S^{k_c}_D[\phi] - S^{k_m}_D[\psi] &= \varphi^i, \\
  \varepsilon_c \left( \frac{1}{2} - (\mathcal{K}^{k_c}_D)^* \right) [\phi] - \varepsilon_m \left( \frac{1}{2} + (\mathcal{K}^{k_m}_D)^* \right) [\psi] &= \varepsilon_m \varphi^i / \partial \nu.
  \end{align*}
  \]

- **Operator-Valued function \( \delta \mapsto A_\delta(\omega) \in \mathcal{L}(\mathcal{H}^*(\partial B), \mathcal{H}^*(\partial B)):\)**
  \[
  A_\delta(\omega) = (\lambda(\omega) I - \mathcal{K}_B^*) + (\omega \delta)^2 A_1(\omega) + \mathcal{O}((\omega \delta)^3).
  \]

- **Quasi-static limit:**
  \[
  A_0(\omega)[\psi] = \sum_{j=0}^{\infty} \tau_j(\omega)(\psi, \varphi_j)_{\mathcal{H}^*}^* \varphi_j, \quad \tau_j(\omega) := \frac{1}{2} \left( \varepsilon_m + \varepsilon_c(\omega) \right) - \left( \varepsilon_c(\omega) - \varepsilon_m \right) \lambda_j.
  \]
Resonances for plasmonic nanoparticles

• Shift in the plasmonic resonance:

$$\arg \min_\omega \left| \frac{1}{2} (\varepsilon_m + \varepsilon_c(\omega)) - (\varepsilon_c(\omega) - \varepsilon_m) \lambda_j + (\omega \delta)^2 \tau_{j,1} \right|$$

• $$\tau_{j,1} := (A_1(\omega)[\varphi_j], \varphi_j)_{\mathcal{H}^*}$$.

• Gohberg-Sigal theory.
Resonances for plasmonic nanoparticles

- **Full Maxwell’s equations:**

\[
\begin{aligned}
\nabla \times \nabla \times E - \omega^2 \left( \varepsilon_m \chi(\mathbb{R}^d \setminus \bar{D}) + \varepsilon_c(\omega) \chi(\bar{D}) \right) E &= 0, \\
E^s := E - E^i &\text{ satisfies the outgoing radiation condition.}
\end{aligned}
\]

- **Small-volume expansion:**

\[
E^s(x) = -\delta^3 \omega^2 G_{km}(x, z) M(\lambda(\omega), B) E^i(z) + O\left( \frac{\delta^4}{\text{dist}(\lambda(\omega), \sigma(K_D^*))} \right)
\]

- **\(G_{km}\):** fundamental (outgoing) solution to Maxwell’s equations in free space.

- **Shift** in the plasmonic resonances due to the finite size of the nanoparticle.
Resonances for plasmonic nanoparticles

• Integral formulation:

$$\begin{pmatrix}
I + M_D^{k_c} - M_D^{k_m} & \mathcal{L}_D^{k_c} - \mathcal{L}_D^{k_m} \\
\mathcal{L}_D^{k_c} - \mathcal{L}_D^{k_m} & \frac{1}{2}(k_c^2 + k_m^2)I + k_c^2 M_D^{k_c} - k_m^2 M_D^{k_m}
\end{pmatrix}$$

• Integral operators:

$$M_D^{k} \left[ \varphi \right] : H_T^{-\frac{1}{2}}(\text{div}, \partial D) \rightarrow H_T^{-\frac{1}{2}}(\text{div}, \partial D) \quad \text{(compact)}$$

$$\varphi \mapsto -\int_{\partial D} \nu(x) \times \nabla_x \times G_k(x, y) \varphi(y) ds(y);$$

$$\mathcal{L}_D^{k} \left[ \varphi \right] : H_T^{-\frac{1}{2}}(\text{div}, \partial D) \rightarrow H_T^{-\frac{1}{2}}(\text{div}, \partial D)$$

$$\varphi \mapsto \nu(x) \times \left( k^2 S_D[k] \varphi(x) + \nabla S_D[k] \nabla \partial D \cdot \varphi \right)(x).$$

• Key identities: $M_D^{k=0} [\text{curl}_{\partial D} \varphi] = \text{curl}_{\partial D} \mathcal{K}_D \left[ \varphi \right], \quad \forall \varphi \in H^{\frac{1}{2}}(\partial D)$,

$$M_D^{k=0} [\nabla_{\partial D} \varphi] = -\nabla_{\partial D} \Delta_{\partial D}^{-1} \mathcal{K}_D^* \left[ \Delta_{\partial D} \varphi \right] + \mathcal{R}_D \left[ \varphi \right],$$

$$\mathcal{R}_D = -\text{curl}_{\partial D} \Delta_{\partial D}^{-1} \text{curl}_{\partial D} M_D \nabla_{\partial D}, \quad \forall \varphi \in H^{\frac{3}{2}}(\partial D).$$
Resonances for plasmonic nanoparticles

• Quasi-static approximation:

\[ \tilde{M}_B = \begin{pmatrix} -\Delta^{-1}_{\partial B} K_B^* \Delta_{\partial B} & 0 \\ \mathcal{R}_B & K_B \end{pmatrix}. \]

• \( H(\partial B) := H^3_0(\partial B) \times H^{\frac{1}{2}}(\partial B) \), equipped with the inner product

\[ (u, v)_{H(\partial B)} = (\Delta_{\partial B} u^{(1)}, \Delta_{\partial B} v^{(1)})_{\mathcal{H}^*} + (u^{(2)}, v^{(2)})_{\mathcal{H}}, \]

\[ (u, v)_{\mathcal{H}^*} := -(u, S_D[v])_{-\frac{1}{2}, \frac{1}{2}}, \quad (u, v)_{\mathcal{H}} = -(S_D^{-1}[u], v)_{-\frac{1}{2}, \frac{1}{2}}. \]

• The spectrum \( \sigma(\tilde{M}_B) = \sigma(-K_B^*) \cup \sigma(K_B^*) \setminus \{-\frac{1}{2}\} \) in \( H(\partial B) \).

• Only \( \sigma(K_B^*) \) can be excited in the quasi-static approximation.
Scattering coefficients

- Scattering coefficients: cloaking structures and dictionary matching approach for inverse scattering.
- Mechanism underlying plasmonic resonances in terms of the scattering coefficients corresponding to the nanoparticle.
- Scattering coefficients of order $\pm 1$: only scattering coefficients inducing the scattering-cross section enhancement.
Scattering coefficients

- **Helmholtz** equation:

\[
\begin{cases}
\nabla \cdot \left( \varepsilon_m \chi(\mathbb{R}^d \setminus \bar{D}) + \varepsilon_c(\omega) \chi(D) \right) \nabla u + \omega^2 u = 0, \\
u^s := u - u^i \text{ satisfies the outgoing radiation condition.}
\end{cases}
\]

- **Scattering coefficients**:

\[
W_{mn}(D, \varepsilon_c, \varepsilon_m, \omega) = \int_{\partial D} \psi_m(y) J_n(\omega |y|) e^{-i\theta y} ds(y).
\]

- \(\psi_m\): electric current density on \(\partial D\) induced by the **cylindrical wave** \(J_m(\omega |x|) e^{im\theta_x}\).

- \(J_n\): **Bessel** function.
Scattering coefficients

Properties of the scattering coefficients:

• $W_{mn}$ decays rapidly:

$$|W_{mn}| \leq \frac{O(\omega|m|+|n|)}{\min |\tau_j(\omega)|} \frac{C|m|+|n|}{|m||m||n||n|}, \quad m, n \in \mathbb{Z},$$

$C$: independent of $\omega$; $\tau_j = \frac{1}{2}(\varepsilon_m + \varepsilon_c(\omega)) - (\varepsilon_c(\omega) - \varepsilon_m)\lambda_j$.

• For any $z \in \mathbb{R}^2$, $\theta \in [0, 2\pi)$, $s > 0$,

$$W_{mn}(D^z) = \sum_{m',n' \in \mathbb{Z}} J_{n'}(\omega |z|) J_{m'}(\omega |z|) e^{i(m'-n')\theta} W_{m-m',n-n'}(D),$$

$$W_{mn}(D^\theta) = e^{i(m-n)\theta} W_{mn}(D),$$

$$W_{mn}(D^s, \omega) = W_{mn}(D, s\omega).$$
Scattering coefficients

- **Scattering amplitude:**

  \[
  u^s(x) = -ie^{-\frac{\pi i}{4}} \frac{e^{ik_m|x|}}{\sqrt{8\pi k_m|x|}} A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') + o(|x|^{-\frac{1}{2}}),
  \]

  \(|x| \to \infty\); \(\theta, \theta'\): incident and scattered directions.

- **Graf’s formula:**

  \[
  A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') = \sum_{n,m \in \mathbb{Z}} (-i)^n i^m e^{in\theta'} W_{nm}(D, \varepsilon_c, \varepsilon_m, \omega) e^{-im\theta}.
  \]

- **Scattering cross-section:**

  \[
  Q^s[D, \varepsilon_c, \varepsilon_m, \omega](\theta') := \int_0^{2\pi} \left| A_\infty[D, \varepsilon_c, \varepsilon_m, \omega](\theta, \theta') \right|^2 d\theta.
  \]
Cloaking: scattering coefficient cancellation

- **Cloaking**: make a target invisible when probed by electromagnetic waves.
- **Scattering coefficient cancellation technique**:
  - Small layered object with vanishing first-order scattering coefficients.
  - Transformation optics:
    \[
    (F_\rho)_*[\phi](y) = \frac{DF_\rho(x)\phi(x)DF_\rho(x)^t}{\det(DF_\rho(x))}, \quad x = F_\rho^{-1}(y).
    \]
  - Change of variables \(F_\rho\) sends the annulus \([\rho, 2\rho]\) onto a fixed annulus.
- **Scattering coefficients vanishing structures of order \(N\)**:
  \[
  Q^s\left[D, (F_\rho)_*(\varepsilon \circ \Psi_{1/\rho}), \varepsilon_m, \omega\right](\theta') = o(\rho^{4N}), \quad \Psi_{1/\rho}(x) = (1/\rho)x.
  \]
  \(\rho\): size of the small object; \(N\): number of layers.
- **Anisotropic permittivity distribution**.
- **Invisibility at \(\omega \Rightarrow\)** invisibility at all frequencies \(\leq \omega\).
Cloaking: scattering coefficient cancellation

Cancellation of the scattered field and the scattering cross-section: 4 orders of magnitude (with wavelength of order 1, $\rho = 10^{-1}$, and $N = 1$).
Cloaking: anomalous resonance

- $\Omega$: bounded domain in $\mathbb{R}^2$; $D \subset \Omega$. $\Omega$ and $D$ of class $C^{1,\mu}$, $0 < \mu < 1$. For a given loss parameter $\delta > 0$, the permittivity distribution in $\mathbb{R}^2$ is given by

$$\varepsilon_\delta = \begin{cases} 
    1 & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\
    -1 + i\delta & \text{in } \Omega \setminus \overline{D}, \\
    1 & \text{in } D.
\end{cases}$$

- Configuration (plasmonic structure): core with permittivity $1$ coated by the shell $\Omega \setminus \overline{D}$ with permittivity $-1 + i\delta$. 
Dictionary matching approach

Dictionary matching approach:

- Form an image from the echo due to targets.
- Identify and classify the target, knowing by advance that it belongs to a learned dictionary of shapes.
  - Extract the features from the data.
  - Construct invariants with respect to rigid transformations and scaling.
- Compare the invariants with precomputed ones for the dictionary.
Dictionary matching approach

- **Feature extraction:**
  - Extract $\mathbf{W}$ by solving a least-squares method
    \[
    \mathbf{W} = \arg \min_{\mathbf{W}} \| \mathbf{L}(\mathbf{W}) - \mathbf{V} \|.
    \]
  - $\mathbf{L}$ is ill-conditioned ($\mathbf{W}$ decays rapidly).
  - **Maximum resolving order $K$:**
    \[
    K^{K+1/2} = C(\omega)\text{SNR}.
    \]
  - Form a **multi-frequency shape descriptor.**
  - Match in a **multi-frequency dictionary.**
Dictionary matching approach

Shape descriptor matching in a multi-frequency dictionary.
Resonances for plasmonic nanoparticles

- Asymptotic expansion of the scattering amplitude:

\[ A_\infty \left( \frac{x}{|x|}, d \right) = \frac{x}{|x|^t} W_1 d + O(\omega^2), \]

where:
- \( d \): incident direction;
- \( x/|x| \): observation direction;

\[ W_1 = \begin{pmatrix} W_{-11} + W_{1-1} - 2W_{11} & i(W_{1-1} - W_{-11}) \\ i(W_{1-1} - W_{-11}) & -W_{-11} - W_{1-1} - 2W_{11} \end{pmatrix}. \]

- Blow up of the scattering coefficients:

\[ W_{\pm 1 \pm 1} = \pm \pm k_m^2 \frac{4}{4} \left( \varphi_j, |x|e^{\pm i\theta x} \right)_{-\frac{1}{2}, \frac{1}{2}} \frac{e^{\pm i\theta y}, \varphi_j}{\lambda - \lambda_j} \mathcal{H}^* + O(1). \]
Super-resolution

- Super-resolution for plasmonic nanoparticles:
  - Subwavelength resonators;
  - High contrast: effective medium theory;
  - Single nanoparticle imaging.
Super-resolution

- **Resolution**: determined by the behavior of the imaginary part of the Green function. Helmholtz-Kirchhoff identity:

\[
\Im m G_{km}(x, x_0) = k_m \int_{|y| = R} G_{km}(y, x_0) G_{km}(x, y) ds(y), \quad R \to +\infty.
\]

- **The sharper** is \( \Im m G_{km} \), the better is the resolution.
- **Local resonant media** used to make shape peaks of \( \Im m G_{km} \).
- **Mechanism of super-resolution in resonant media**:
  - Interaction of the point source \( x_0 \) with the resonant structure excites high-modes.
  - Resonant modes encode the information about the point source and can propagate into the far-field.
  - **Super-resolution**: only limited by the resonant structure and the signal-to-noise ratio in the data.
Super-resolution

- System of weakly coupled plasmonic nanoparticles.
- Size of the nanoparticle $\delta \ll$ wavelength $2\pi/k_m$; distance between the nanoparticles of order one.
- $\Im G^\delta = \Im G^k_m +$ exhibits subwavelength peak with width of order one.
- Break the resolution limit.

S. Nicosia & C. Ciraci, Cover, Science 2012
Super-resolution

- **Subwavelength resonator:**

- **Asymptotic expansion of the Green function** ($\delta$: size of the resonator openings; $z_j$: center of aperture for $j$th resonator; $J$: number of resonators; $\omega = O(\sqrt{\delta})$):

  $$\Im m G^\delta(x, x_0, \omega) \approx \frac{\sin \omega |x - x_0|}{2\pi |x - x_0|} + \sqrt{\delta} \sum_{j=1}^{J} \frac{c_j}{|x - z_j| |x_0 - z_j|}.$$
Super-resolution

• Effective medium theory:

\[ \varepsilon_{\text{eff}}(\omega) = \varepsilon_m(I + f M(\lambda(\omega), B))(1 - \frac{f}{3}M(\lambda(\omega), B))^{-1} + O\left(\frac{f^{8/3}}{\text{dist}(\lambda(\omega), \sigma(K_D^*))^2}\right). \]

• \( f \): volume fraction; \( B \): rescaled particle.

• \( \varepsilon_{\text{eff}}(\omega) \): anisotropic.

• Validity of the effective medium theory:

\[ f \ll \text{dist}(\lambda(\omega), \sigma(K_D^*))^{3/5}. \]
Super-resolution

- High contrast effective medium at plasmonic resonances:
  \[ \nabla \times \nabla \times E - \omega^2 \left( \varepsilon_m \chi(\mathbb{R}^d \setminus \bar{\Omega}) + \varepsilon_{\text{eff}}(\omega) \chi(\bar{\Omega}) \right) E = 0. \]

- \( E|_{\Omega} \mapsto \int_{\Omega} (\varepsilon_{\text{eff}}(\omega) - \varepsilon_m)E(y)G_{km}(x, y) \, dy, \quad x \in \Omega. \)

- Mixing of resonant modes: intrinsic nature of non-hermitian systems.

- Subwavelength resonance modes excited \( \Rightarrow \) dominate over the other ones in the expansion of the Green function.

- Imaginary part of the Green function may have sharper peak than the one of \( G \) due to the excited sub-wavelength resonant modes.

- Subwavelength modes: determine the superresolution.
Super-resolution

- Single nanoparticle imaging:
  \[
  \max_{z^S} I(z^S, \omega)
  \]

- \(I(z^S, \omega)\): imaging functional; \(z^S\): search point.
- Resolution: limited only by the signal-to-noise-ratio.
- Cross-correlation techniques: robustness with respect to medium noise.

![Medium without the reflector](image1)

![V shg](image2)
Plan

• Part I: Mathematical and computational tools
  • Gohberg-Sigal theory
  • Layer potentials, Green’s functions (free space, grating, quasi-periodic), integral formulations, Helmholtz-Kirchhoff identities, scattering coefficients, Floquet theory, Muller’s method, Ewald’s method for grating and quasi-periodic Green’s functions.

• Part II: Diffraction gratings and photonic crystals
  • Diffraction gratings: radiation condition, existence and uniqueness of a solution, optimal design problem.
  • Photonic crystals: sensitivity of band gaps, analysis of photonic crystal cavities.
Plan

- Part III: *Subwavelength resonators and superresolution*
  - Plasmonic nanoparticles.
  - Scattering and absorption enhancement.
  - Resolution enhancement.
  - Superresolution in high contrast media.
  - Effective medium theory for subwavelength resonators.
  - Near-field optics.

- Part IV: *Metamaterials, metasurfaces, and subwavelength photonic crystals*
  - Metamaterials and cloaking.
  - Metasurfaces with superabsorption effect: layers of periodically distributed plasmonic nanoparticles.
  - Subwavelength photonic crystals.