Lecture 9: Acousto-electric imaging

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Acousto-electric imaging

- Acousto-electric effect:
  - Acoustic pressure: $p(x,t) = p_0 b(x) a(t)$; p_0: amplitude; b: beam pattern; a: ultrasound waveform.
  - Acousto-electric effect:

$$\Delta\sigma = \eta\sigma p; \quad \eta : \text{ interaction constant.}$$

- Acousto-electric imaging:
  - Change of conductivity induces a change of the boundary voltage measurements.
  - Scan the sample, record the boundary variations, and determine the conductivity distribution.

Acousto-electric imaging

- Acousto-electric imaging: mathematical and numerical framework.
- u the voltage potential induced by a current g in the absence of acoustic perturbations:

$$\begin{cases} \nabla_x \cdot (\sigma(x)\nabla_x u) = 0 \text{ in } \Omega \,, \\ \sigma(x)\dfrac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega \,. \end{cases}$$

- Suppose σ bounded from below and above and known in a neighborhood of the boundary $\partial\Omega$: $\sigma = \sigma_\star$; Set $\Omega' \subset \Omega$ where σ is unknown.

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- Use of focalized ultrasonic waves with D as a focal spot $\rightarrow$

$$\sigma_\delta(x) \quad = \quad \sigma(x)\Big[1+\chi(D)(x)\,(\nu(x)-1)\Big],$$

with $\nu(x)=\eta p(x)$: known.

- u_δ induced by g in the presence of acoustic perturbations localized in the focal spot $D := z+\delta B$:

$$\begin{cases} \nabla_x\cdot(\sigma_\delta(x)\nabla_x u_\delta(x))=0 \text{ in } \Omega, \\ \qquad \sigma(x)\dfrac{\partial u_\delta}{\partial \nu}=g \text{ on } \partial\Omega. \end{cases}$$

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- Suppose the focal spot D to be a disk and $u \in W^{2,\infty}(D)$. Then,

$$\int_{\partial\Omega} (u_\delta - u) g \, d\sigma = |\nabla u(z)|^2 \int_D \sigma(x) \frac{(\nu(x)-1)^2}{\nu(x)+1} dx + O(|D|^{1+\beta}),$$

  - $O(|D|^{1+\beta}) \leq C|D|^{1+\beta} ||\nabla u||_{L^\infty(D)} |\nabla^2 u||_{L^\infty(D)}$ with C: independent of D and u.
  - β: depends only on Ω', ν, $\sup_\Omega \sigma$, $\min_\Omega \sigma$.

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- Suppose $\sigma \in \mathcal{C}^{0,\alpha}(D)$, $0 \leq \alpha \leq 2\beta \leq 1$. Then

$$\begin{aligned} \mathcal{E}(z) &:= \left(\int_D \frac{(\nu(x)-1)^2}{\nu(x)+1} dx \right)^{-1} \int_{\partial\Omega} (u_\delta - u) g \, d\sigma \\ &= \sigma(z) \, |\nabla u(z)|^2 + O(|D|^{\alpha/2}). \end{aligned}$$

- $\mathcal{E}(z)$: electrical energy density; known function from the boundary measurements.

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- Substitute σ by $\mathcal{E}/\left|\nabla u\right|^2$.
- Nonlinear PDE (the 0–Laplacian)

$$\begin{cases} \nabla_x \cdot \left(\dfrac{\mathcal{E}}{\left|\nabla u\right|^2} \nabla u \right) = 0 & \text{in } \Omega\,, \\ \dfrac{\mathcal{E}}{\left|\nabla u\right|^2} \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega\,. \end{cases}$$

- g such that u has no critical point inside Ω.
- Choose two currents g_1 and g_2 s.t. $\nabla u_1 \times \nabla u_2 \neq 0$ for all $x \in \Omega$.

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- Substitution algorithm.
- Polarized measurements:

$$\mathcal{E}_{ij} := \int_{\partial\Omega} (u_\delta^{(j)} - u^{(j)}) g_i \, d\sigma, \quad i,j = 1,2.$$

- $$\begin{cases} \nabla_x \cdot \left(\dfrac{\mathcal{E}_{jj}}{|\nabla u^{(j)}|^2} \nabla u^{(j)} \right) = 0 & \text{in } \Omega\,, \\ \dfrac{\mathcal{E}_{jj}}{|\nabla u^{(j)}|^2} \dfrac{\partial u^{(j)}}{\partial \nu} = g_j & \text{on } \partial\Omega\,. \end{cases}$$

- Proper set of measurements: (g_1, g_2) s.t.

$$|\nabla u^{(j)}| > 0; (\mathcal{E}_{ij})_{i,j} : \text{ invertible in } \Omega.$$

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- Linearisation:
  - σ: small perturbation of conductivity profile σ_0: $\sigma = \sigma_0 + \delta\sigma$.
  - $u_0^{(j)}$ and $u^{(j)} = u_0^{(j)} + \delta u^{(j)}$: potentials corresponding to σ_0 and σ with the same Neumann boundary data g_j.
  - $\delta u^{(j)}$ satisfies $\nabla \cdot \left(\sigma \nabla \delta u^{(j)}\right) = -\nabla \cdot (\delta\sigma \nabla u_0^{(j)})$ in Ω with the homogeneous Dirichlet boundary condition.
  - $$\begin{aligned} \mathcal{E}_{jj} &= (\sigma_0 + \delta\sigma)|\nabla(u_0^{(j)} + \delta u^{(j)})|^2 \\ &\approx \sigma_0|\nabla u_0^{(j)}|^2 + \delta\sigma|\nabla u_0^{(j)}|^2 + 2\sigma_0 \nabla u_0^{(j)} \cdot \nabla \delta u^{(j)} \ ; \end{aligned}$$
  - Neglecting $\delta\sigma \nabla u_0^{(j)} \cdot \nabla \delta u^{(j)}$ and $\delta\sigma|\nabla \delta u^{(j)}|^2 \Rightarrow$

$$\delta\sigma \approx \frac{\mathcal{E}_{jj}}{\left|\nabla u_0^{(j)}\right|^2} - \sigma_0 - 2\sigma_0 \frac{\nabla \delta u^{(j)} \cdot \nabla u_0^{(j)}}{\left|\nabla u_0^{(j)}\right|^2} \ .$$

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- Substitution algorithm.
  - Start from an initial guess for the conductivity σ;
  - Solve the corresponding Dirichlet conductivity problem

$$\begin{cases} \nabla \cdot (\sigma \nabla u_0) = 0 & \text{in } \Omega \,, \\ u_0 = \psi & \text{on } \partial\Omega \,. \end{cases}$$

  - ψ: Dirichlet data measured as a response to the current $g = g_1$ in absence of elastic deformation;
  - Define the discrepancy between the data and the guessed solution by

$$\epsilon_0 := \frac{\mathcal{E}_{11}}{|\nabla u_0|^2} - \sigma \,.$$

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- Introduce the corrector, δu, computed as the solution to

$$\begin{cases} \nabla \cdot (\sigma \nabla \delta u) = -\nabla \cdot (\varepsilon_0 \nabla u_0) & \text{in } \Omega \,, \\ \delta u = 0 & \text{on } \partial\Omega \,; \end{cases}$$

- Update the conductivity

$$\sigma := \frac{\mathcal{E}_{11} - 2\sigma \nabla \delta u \cdot \nabla u_0}{|\nabla u_0|^2} \,.$$

- Iteratively update the conductivity, alternating directions of currents (with $g = g_2$ and $\mathcal{E}_{11}$ replaced with $\mathcal{E}_{22}$).

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- Optimal control algorithm
- (g_1, g_2): proper set of measurements.
- σ and $\tilde{\sigma}$ be two $\mathcal{C}^1$-conductivities with $\sigma(x_0) = \tilde{\sigma}(x_0)$ for some $x_0 \in \overline{\Omega}$,

$$\|\log\sigma - \log\widetilde{\sigma}\|_{W^{1,\infty}(\Omega)} \leq C\|\mathcal{E}[\sigma] - \mathcal{E}[\tilde{\sigma}]\|_{W^{1,\infty}(\Omega)}\,.$$

- $\sigma \mapsto \mathcal{E}[\sigma]$: Fréchet differentiable.
- $\Rightarrow$

$$\mathrm{Ker}(d\mathcal{E}[\sigma])|_{H_0^{1,\infty}(\Omega)} = \{0\}\,,$$

provided that $\sigma > c_0 > 0$ for some constant c_0.

- For all $h \in W_0^{1,\infty}(\Omega)$,

$$\begin{aligned}
\|d\mathcal{E}[\sigma](h)\|_{W_0^{1,\infty}(\Omega)} &= \lim_{t\to 0}\frac{\|\mathcal{E}[\sigma+th]-\mathcal{E}[\sigma]\|_{W_0^{1,\infty}(\Omega)}}{|t|}\\
&\geq \frac{1}{C}\lim_{t\to 0}\frac{\|\log(\sigma+th)-\log\sigma\|_{W_0^{1,\infty}(\Omega)}}{|t|}\\
&= \frac{1}{C\sigma}\|h\|_{W_0^{1,\infty}(\Omega)}\,.
\end{aligned}$$

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- Admissible set of conductivities: open subset of $W^{1,\infty}(\Omega)$

$$A = \{\sigma \in W^{1,2}(\Omega) : c_0 < \sigma < C_0, |\nabla\sigma| < C_1\}\,.$$

- For $j, l = 1, 2$, the map $\sigma \mapsto \mathcal{E}_{jl}[\sigma]$: Fréchet differentiable.
- For $h \in W_0^{1,2}(\Omega)$ s.t. $\sigma + h \in A$,

$$d\mathcal{E}_{jl}[\sigma]h = h\nabla u^{(j)} \cdot \nabla u^{(l)} + \sigma\left[\nabla u^{(j)} \cdot \nabla v^{(l)} + \nabla u^{(l)} \cdot \nabla v^{(j)}\right];$$

$v^{(j)}$: solution of

$$\begin{cases} \nabla \cdot \sigma\nabla v^{(j)} = -\nabla \cdot h\nabla u^{(j)} & \text{in } \Omega\,, \\ \sigma\dfrac{\partial v^{(j)}}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \\ \displaystyle\int_{\partial\Omega} v^{(j)} = 0\,. \end{cases}$$

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- Minimization problem:

$$\min_{\sigma \in A} J[\sigma] := \frac{1}{2} \sum_{j,l=1}^{2} \int_{\Omega} \left| \mathcal{E}_{jl}[\sigma] - \mathcal{E}_{jl}^{(m)} \right|^2 dx \,,$$

  $\mathcal{E}_{jl}^{(m)}$: measurements.

- Fréchet derivative of $J[\sigma]$:

$$dJ[\sigma] = \frac{1}{2} \sum_{j,l=1}^{2} (\mathcal{E}_{jl}[\sigma] - \mathcal{E}_{jl}{}^{(m)}) \nabla u^{(j)} \cdot \nabla u^{(l)} + \sum_{j,l=1}^{2} \nabla u^{(j)} \cdot \nabla p^{(j,l)} \,;$$

- $p^{(j,l)}$: solution of the adjoint problem

$$\begin{cases} \nabla \cdot \sigma \nabla p^{(j,l)} &= -\nabla \cdot (\mathcal{E}_{jl}[\sigma] - \mathcal{E}_{jl}^{(m)}) \sigma \nabla u^{(l)} & \text{in } \Omega \,, \\ \sigma \frac{\partial p^{(j,l)}}{\partial \nu} &= 0 & \text{on } \partial\Omega \,, \\ \int_{\partial\Omega} p^{(j,l)} = 0 \,. \end{cases}$$

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- For $\sigma \in A$,
  - $h \mapsto \nabla v^{(j)} : W_0^{1,2}(\Omega) \to L^2(\Omega)$: compact.
  - $d\mathcal{E}_{jl}$ takes the form $I + \text{compact}$, up to a multiplication by a continuous function.
- Proof:
  - $h \in H_0^1(\Omega)$. $v^{(j)}$ satisfies

$$\begin{aligned} -\sigma \Delta v^{(j)} &= \nabla\sigma \cdot \nabla v^{(j)} + \nabla \cdot h \nabla u^{(j)}\,, \\ &= \nabla\sigma \cdot \nabla v^{(j)} + h\Delta u^{(j)} + \nabla h \cdot \nabla u^{(j)}\,, \\ &= \nabla\sigma \cdot \nabla v^{(j)} - h\frac{\nabla\sigma \cdot \nabla u^{(j)}}{\sigma} + \nabla h \cdot \nabla u^{(j)} \in L^2(\Omega)\,. \end{aligned}$$

  - $\Rightarrow v^{(j)} \in H^2(\Omega)$ and its H^2-norm: bounded by $\|h\|_{H_0^1(\Omega)}$.
  - Compact embeddings

$$H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^2(\Omega)\,.$$

  - $\mathcal{E}[\sigma]^{-1} d\mathcal{E}[\sigma] = I + \text{compact operator}.$

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- $\Rightarrow$

$$\|d\mathcal{E}[\sigma]\|_{\mathcal{L}(H_0^1(\Omega),L^2(\Omega))} \geq C\,.$$

- Convergence of the Landweber iteration scheme:
  - Assume that $\sigma^{(0)}$: good initial guess for σ_*.
  - As $n \to +\infty$, the sequence

$$\sigma^{(n+1)} = T\sigma^{(n)} - \eta d\mathcal{E}^*[T\sigma^{(n)}](\mathcal{E}[\sigma^{(n)}] - \mathcal{E}^{(m)})$$

converges to σ_*; T: Hilbert projection of $H^1(\Omega)$ onto A; σ_*: true conductivity distribution, η: step size;
$\mathcal{E}^{(m)} = (\mathcal{E}_{jl}^{(m)})_{j,l=1,2}$.

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Reconstruct the conductivity distribution knowing the internal energies:

- Linearized versions of the nonlinear (zero-Laplacian) PDE problems.
- Optimal control approach: minimize over the conductivity the discrepancy between the computed and reconstructed internal energies.
- Optimal control approach: more efficient approach specially with incomplete internal measurements of the internal energy densities.
- Resolution of order the size of the focal spot + stability (wrt measurement noise).
- Exact inversion formulas: derivatives of the data $\Rightarrow$ used only to obtain a good initial guess.