Lecture 2. High Frequency Behaviour of Formulations of Time-Harmonic Scattering

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Zurich Summer School, August 2006

Focus of Today's Lecture

For

$$\Delta u + k^2 u = 0,$$

and boundary or finite element methods for its solution:

1. **How does conditioning depend on k (and the geometry)?**

2. **How can we remove or reduce this dependence?**

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What is conditioning? For a linear system

$$Ax = b$$

the condition number is

$$\mathrm{cond}\, A := \|A\|\,\|A^{-1}\| \text{ where } \|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Large condition numbers associated with:

- slow convergence of iterative solution methods;
- magnification of effects of errors, e.g. in entries of A.

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and boundary or finite element methods for its solution:

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2. **How can we remove or reduce this dependence?**

What is conditioning? For an operator equation

$$Ax = b$$

($A : X \to Y$ a continuous linear operator, $x \in X$, $b \in Y$) the condition number is

$$\mathrm{cond}\, A := \|A\|_{X\to Y}\, \|A^{-1}\|_{Y\to X} \text{ where } \|A\|_{X\to Y} := \sup_{0\neq x\in X} \frac{\|Ax\|_Y}{\|x\|_X}.$$

For

$$\Delta u + k^2 u = 0,$$

and boundary or finite element methods for its solution:

1. **How does conditioning depend on k (and the geometry)?**
2. **How can we remove or reduce this dependence?**

What is conditioning? For the variational equation: find $u \in X$ such that

$$a(u, v) = f(v), \quad v \in Y,$$

(X and Y Hilbert spaces, $a : X \times Y \to \mathbb{C}$ a continuous sesquilinear form) a relevant **condition number** is that of the associated operator $A : X \to Y'$, defined by

$$Au(v) = a(u, v), \quad u \in X, \ v \in Y.$$

Since (see Melenk notes, Theorem 3 (Babuška-Brezzi), Hiptmair, 'Fundamental Concepts', §2),

$$\begin{aligned} \|A\|_{X\to Y'} &= M, \quad M := \sup_{0\neq u\in X,\, v\in Y} \frac{|a(u,v)|}{\|u\|_X \, \|v\|_Y}, \\ \|A^{-1}\|_{Y'\to X} &= \gamma^{-1}, \quad \gamma := \inf_{0\neq u\in X} \sup_{0,\neq v\in Y} \frac{|a(u,v)|}{\|u\|_X \, \|v\|_Y}, \end{aligned}$$

M often called the **norm** of a and γ its **inf-sup constant**, it holds that

$$\operatorname{cond} A = \frac{M}{\gamma}.$$

Precise Focus of Today's Lecture

For

$$\Delta u + k^2 u = 0,$$

and integral equation or domain methods for its solution:

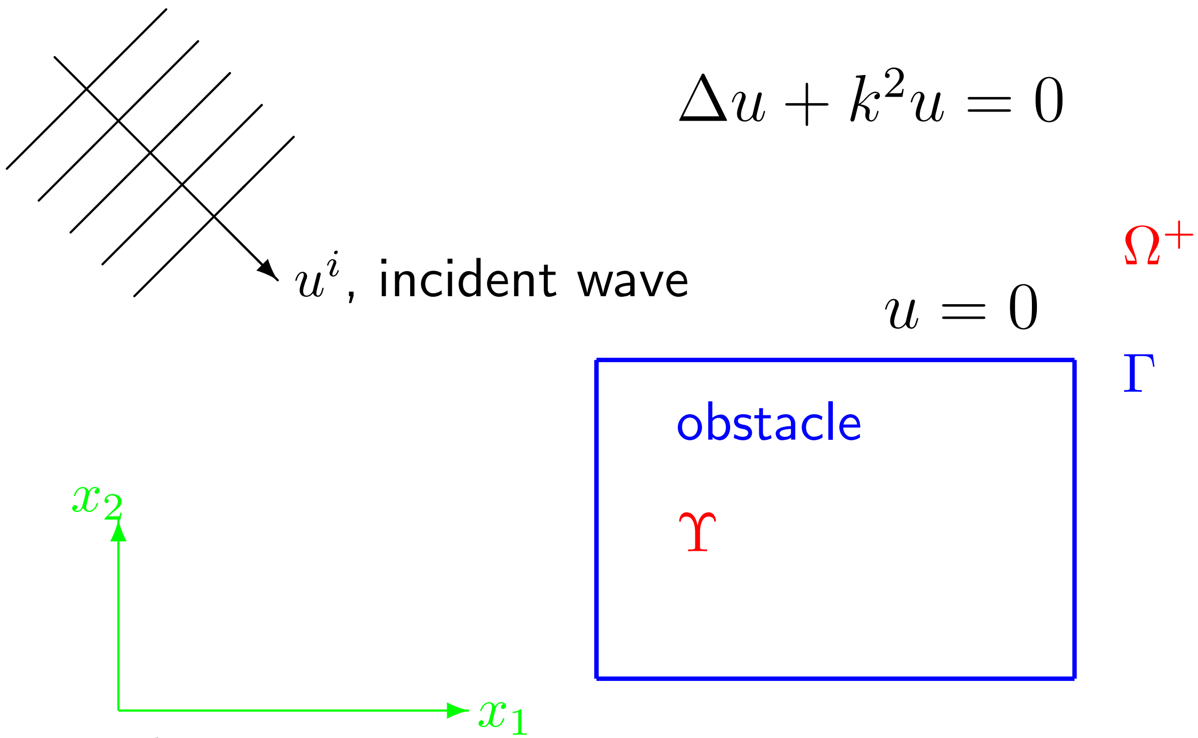
1. How does conditioning depend on k (and the geometry)?

2. How can we remove or reduce this dependence?

Estimating $\|A\|$ and $\|A^{-1}\|$ when A is an integral operator, and **norm** and **inf-sup** constants of sesquilinear forms.

The Scattering Problem in $\mathbb{R}^d$ ($d = 2$ or 3)

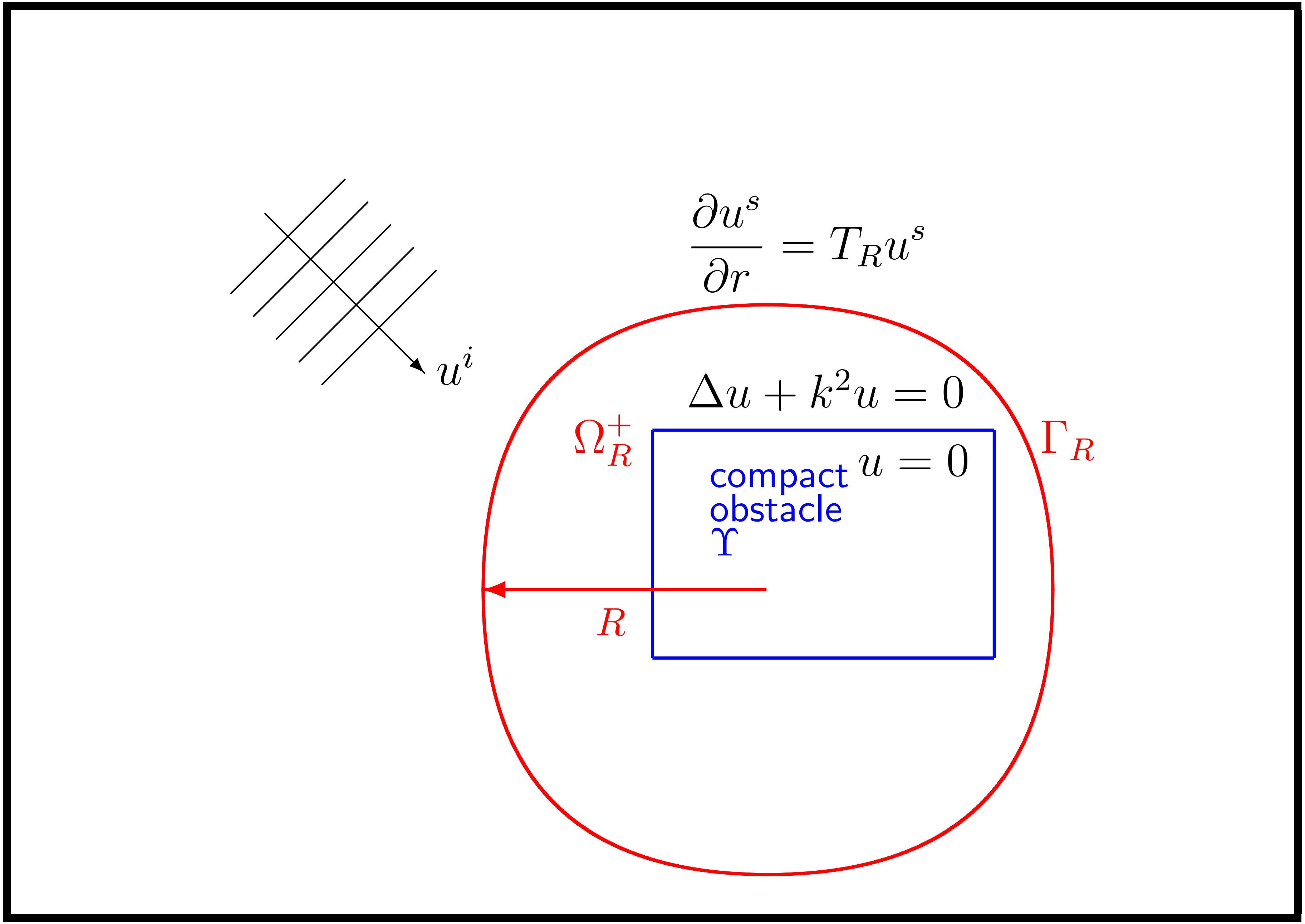
$$\Delta u + k^2 u = 0$$



We seek $u \in H_0^{1,\mathrm{loc}}(\Omega^+) \cap C^2(\Omega^+)$ which satisfies the Sommerfeld radiation condition $\dfrac{\partial u}{\partial r} - iku = o\left(r^{-(d-1)/2}\right)$ as $r = |x| \to \infty$.

Recall from Yesterday ...

a standard weak formulation in Ω_R^+, that part of Ω^+ inside a ball of radius R, with the exact Dirichlet to Neumann map on the sphere Γ_R truncating the domain.



$\frac{\partial u^s}{\partial r} = T_R u^s$
u^i
$\Delta u + k^2 u = 0$
Ω_R^+
Γ_R
compact
obstacle
Υ
$u = 0$
R

Let V_R denote the closure of $\{v|_{\Omega_R^+} : v \in C_0^\infty(\Omega^+)\} \subset H^1(\Omega_R^+)$ in the norm of $H^1(\Omega_R^+)$.

u satisfies the scattering problem if and only if the restriction of u to Ω_R^+ satisfies a variational problem of the form: find $u \in V_R$ such that

$$a(u, v) = f(v), \quad v \in V_R.$$

The functional f depends on the incident field. $a(\cdot,\cdot)$ is the sesquilinear form on $V_R \times V_R$ defined by

$$a(u, v) := \int_{\Omega_R^+} (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v})\, dx - \int_{\Gamma_R} \gamma \bar{v} T_R \gamma u \, ds,$$

where $\gamma : V_R \to H^{1/2}(\Gamma_R)$ is the usual trace operator.

Summary of Known Results

With Markus's norm, i.e. $\|u\|_{V_R}^2 = \int_{\Omega_R^+}(|\nabla u|^2 + k^2|u|^2)\,dx$, ...

1. An **upper bound** on the **inf-sup constant** $\gamma := \inf_{\|u\|_{V_R}=1} \sup_{\|v\|_{V_R}=1} |a(u,v)|$, that [a]

$$\gamma \leq \frac{C_1}{kR} + \frac{C_2}{k^2 R^2}.$$

2. That, if the scatterer Υ is starlike (i.e. $x \in \Upsilon \Rightarrow \theta x \in \Upsilon$, for $0 \leq \theta \leq 1$), then the **lower bound** holds that

$$\frac{1}{5 + 4\sqrt{2}\,kR} \leq \gamma.$$

[a] This upper bound also holds for Markus Melenk's example on his page 11, yesterday.

3. An example where Υ is not starlike (two parallel plates) for which

$$\gamma \leq \frac{C}{k^2 R^2}.$$

for an unbounded sequence of (nearly resonant) wavenumbers k.

Details: see the blackboard and
`www.rdg.ac.uk/~sms03snc/monk_bounded_submitted.pdf` and the references therein.

Lemma 2.1 Suppose $w \in V_R \cap H^2(\Omega_R^+)$ is such that $\gamma w = \gamma \nabla w = 0$ and w is non-zero. Then the inf-sup constant γ is bounded above by

$$\gamma \leq \frac{C_1}{kR} + \frac{C_2}{k^2 R^2},$$

where $C_1 := 2R \left\| \frac{\partial w}{\partial x_1} \right\|_2 / \|w\|_2$, $C_2 := R^2 \|\Delta w\|_2 / \|w\|_2$ and $C_1 \geq 2\sqrt{2} \approx 2.83$.

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For Markus's interior impedance/Robin problem (p.11 of his notes), the same bound holds if his bounded domain Ω contains a ball of radius R, with

$$C_1 = 2\sqrt{24 + 3d}/3 \approx 3.7$$

Where do the lower bounds on the inf-sup constant come from?

I.e. the lower bound I just showed or the lower bound Markus showed yesterday.

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The main ingredients are:

1. A rephrasing of Markus's Theorem 3 gives us the following general result:

If there exists $C > 0$ such that, for every $u \in V_R$ and $f \in V_R'$ satisfying

$$a(u, v) = f(v), \quad v \in V_R,$$

it holds that

$$\|u\|_{V_R} \le C\|f\|_{V_R'}, \quad (*)$$

then

$$\gamma \ge C^{-1}.$$

1. A rephrasing of Markus's Theorem 3 gives us the following general result:

If there exists $C > 0$ such that, for every $u \in V_R$ and $f \in V_R'$ satisfying

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it holds that

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then $\gamma \geq C^{-1}$.

2. If there exists $\tilde{C} > 0$ such that, for every $u \in V_R$ and $g \in L^2(\Omega_R^+)$ satisfying

$$a(u,v) = -(g,v)_2 := -\int_{\Omega_R^+} g\bar{v}\,dx, \quad v \in V_R,$$

it holds that

$$\|u\|_{V_R} \leq k^{-1}\tilde{C}\|g\|_{L^2(\Omega_R^+)},$$

then $(*)$ holds with $C = 1 + 2\tilde{C}$.

3. To establish this last bound, Green's theorem and a Rellich(-Payne-Weinberger-Nečas) type identity.

Such identities, useful for obtaining explicit a prioiri bounds and regularity estimates for strongly elliptic systems, follow from the divergence theorem, and date back to Rellich (1943).

See Chapter 5 of Nečas (1967) or McLean (2000). Our particular version of the identity is essentially that from the PhD of Melenk (1995).

Lemma 2.2. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and that $v \in H^2(G)$. Then, for every $k \geq 0$, where $g := \Delta v + k^2 v$ and the unit normal vector n is directed into Ω, it holds that

$$\int_\Omega \left(|\nabla v|^2 - k^2|v|^2 + g\bar{v}\right)\, dx = -\int_{\partial\Omega} \bar{v}\,\frac{\partial v}{\partial n}\, ds$$

and

$$\int_\Omega \left((2-d)|\nabla v|^2 + dk^2|v|^2 + 2\Re\left(gx \cdot \nabla \bar{v}\right)\right)\, dx =$$

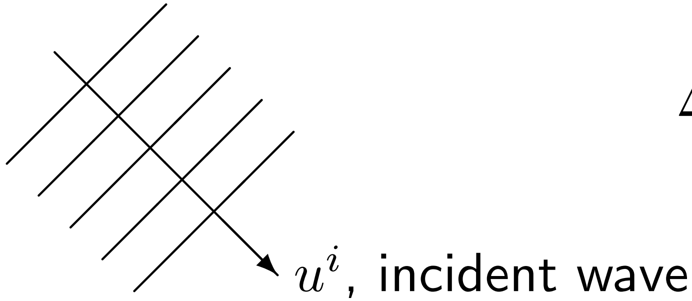
$$-\int_{\partial\Omega}\left(x \cdot n\left(k^2|v|^2 + \left|\frac{\partial v}{\partial n}\right|^2 - |\nabla_T v|^2\right) + 2\Re\left(x \cdot \nabla_T \bar{v}\,\frac{\partial v}{\partial n}\right)\right)\, ds.$$

4. To complete the proof for the scattering problem, a subtle property of radiating solutions of the Helmholtz equation, that, if v is radiating and Γ_R is the boundary of the sphere of radius R, then

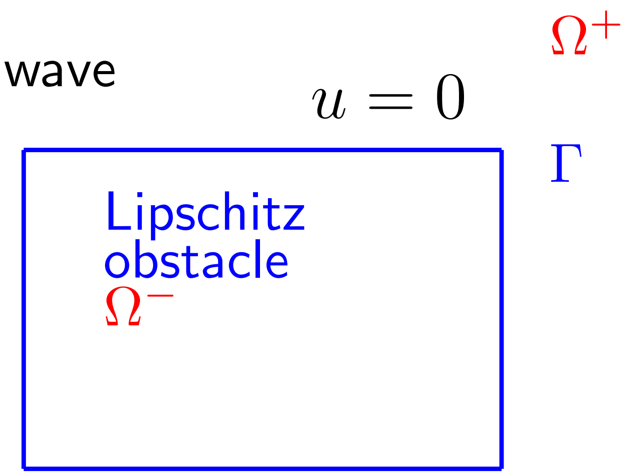
$$\Re \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r}\, ds + R \int_{\Gamma_R} \left(k^2 |v|^2 + \left| \frac{\partial v}{\partial r} \right|^2 - |\nabla_T v|^2 \right) ds \le 2kR\, \Im \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r}\, ds.$$

Proof. Expand everything in Bessel functions and use the monotonicity property that $|H_\nu^{(1)}(z)|^2$ is decreasing for $\nu \ge 0$, $|H_\nu^{(1)}(z)|^2 z$ for $\nu \ge 1/2$. (Cf. proof of Lemma 1.13 yesterday.)

The Standard 2nd Kind Integral Equations When the Domain is Lipschitz
(Brakhage-Werner and its adjoint)



$$\Delta u + k^2 u = 0$$

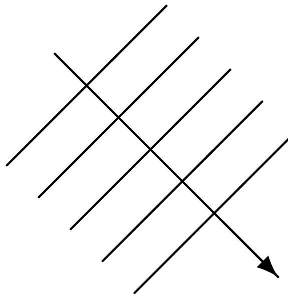


By Green's representation theorem (Hiptmair notes, Theorem 2.1.5, as in Ralf notes, γ_D^+ and γ_N^+ are Dirchlet, Neumann trace operators),

$$u(x) = u^i(x) - \int_\Gamma G(x,y)\gamma_N^+ u(y) ds(y), \quad x \in \Omega^+,$$

where $\gamma_N^+ u \in H^{-1/2}(\Gamma)$ and

$$G(x,y) := \tfrac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \text{ (2D)}, \qquad := \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|} \text{ (3D)}.$$



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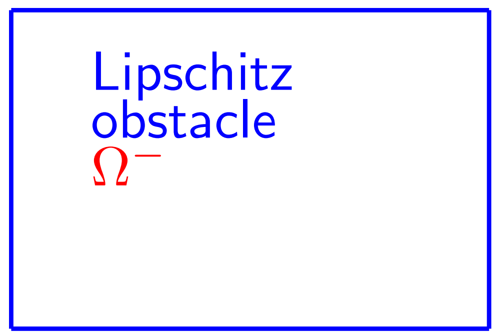


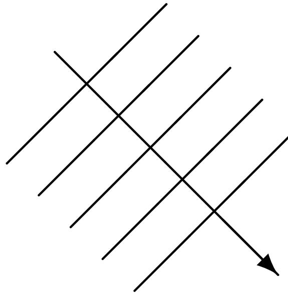
By Green's representation theorem,

$$u(x) = u^i(x) - \int_\Gamma G(x,y)\gamma_N^+ u(y)ds(y), \quad x \in \Omega^+,$$

where $\gamma_N^+ u \in H^{-1/2}(\Gamma)$, in operator form

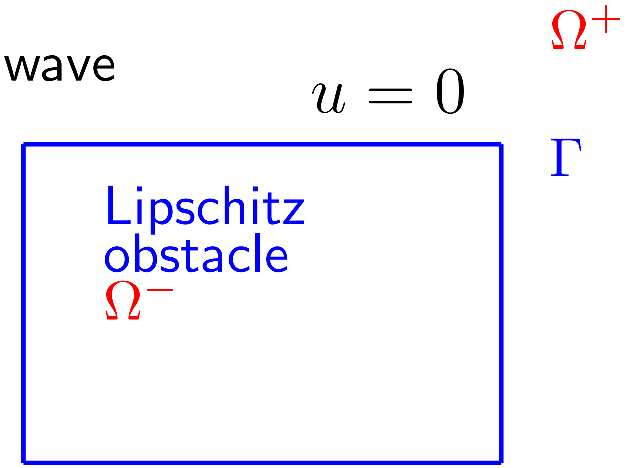
$$u = u^i - \Psi_{\rm SL}\gamma_N^+ u$$

where $\Psi_{\rm SL} : H^{-1/2}(\Gamma) \to H^{1,\rm loc}(\mathbb{R}^N)$ and is continuous (Ralf, (2.1.5)).



u^i, incident wave

$$\Delta u + k^2 u = 0$$



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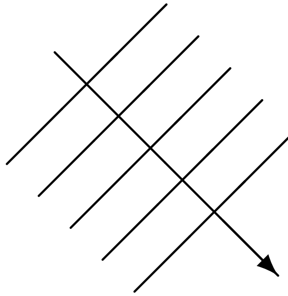
where $\gamma_N^+ u \in H^{-1/2}(\Gamma)$, in operator form

$$u = u^i - \Psi_{\mathrm{SL}}\gamma_N^+ u$$

$$\Rightarrow 0 = \gamma_D^+ u^i - \gamma_D^+ \Psi_{\mathrm{SL}}\gamma_N^+ u, \quad \gamma_N^+ u = \gamma_N^+ u^i - \gamma_N^+ \Psi_{\mathrm{SL}}\gamma_N^+ u.$$



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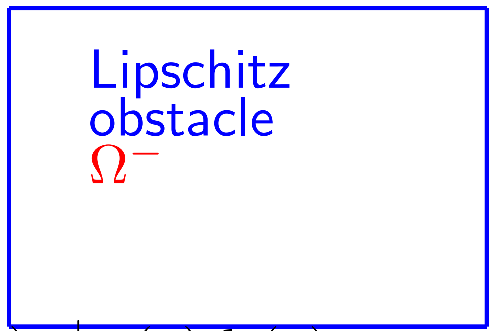


By Green's representation theorem,

$$u(x) = u^i(x) - \int_\Gamma G(x,y)\gamma_N^+ u(y)\,ds(y), \quad x \in \Omega^+,$$

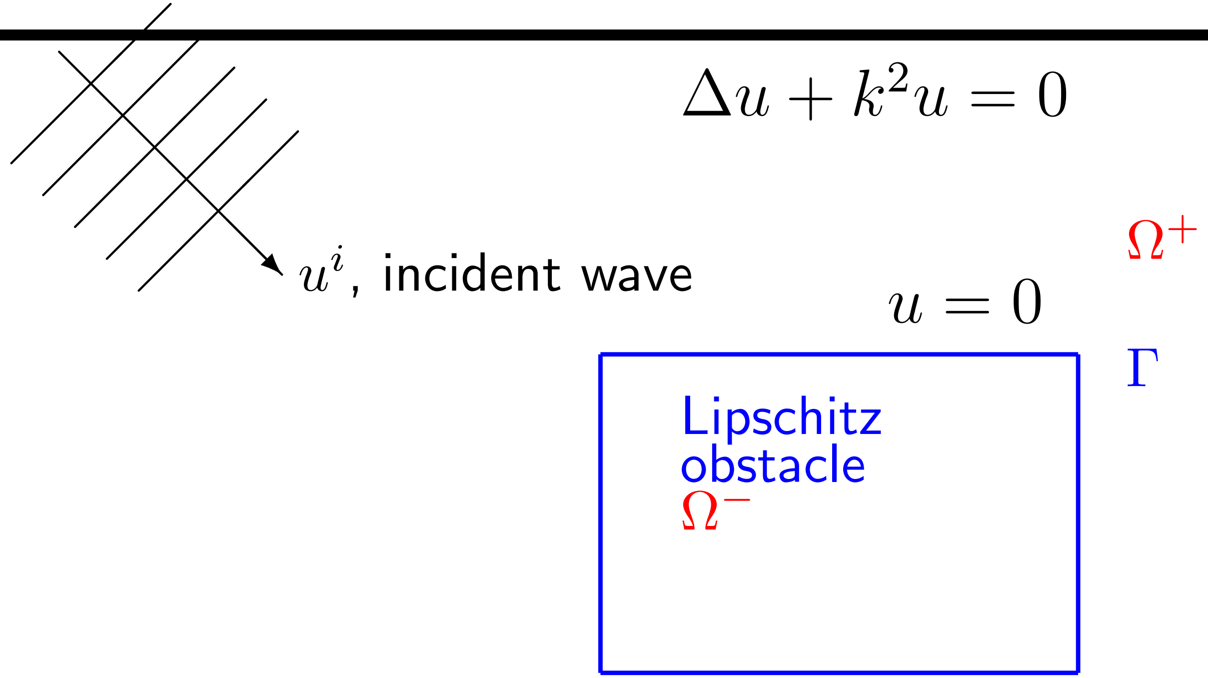
where $\gamma_N^+ u \in H^{-1/2}(\Gamma)$, in operator form

$$u = u^i - \Psi_{\mathrm{SL}}\gamma_N^+ u$$

$$\Rightarrow V\gamma_N^+ u = 2\gamma_D^+ u^i, \quad \gamma_N^+ u + K'\gamma_N^+ u = 2\gamma_N^+ u^i,$$

where $\quad V := 2\gamma_D^+ \Psi_{\mathrm{SL}}, \quad K' := (\gamma_N^+ + \gamma_N^-)\Psi_{\mathrm{SL}}.$

(Ralf Defn 2.1.7, but N.B. my $V = 2\times$ Ralf V, etc.)



$$V\gamma_N^+ u = 2\gamma_D^+ u^i, \quad \gamma_N^+ u + K'\gamma_N^+ u = 2\gamma_N^+ u^i,$$

with $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ given by

$$V := 2\gamma_D^+ \Psi_{\mathrm{SL}}, \quad K' := (\gamma_N^+ + \gamma_N^-)\Psi_{\mathrm{SL}},$$

explicitly, for $\varphi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$,

$$V\varphi(x) = 2\int_\Gamma G(x,y)\varphi(y)\, ds(y), \quad K'\varphi(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(x)}\varphi(y) ds(y).$$

$$V\gamma_N^+ u = 2\gamma_D^+ u^i, \quad \gamma_N^+ u + K'\gamma_N^+ u = 2\gamma_N^+ u^i,$$
$$\Rightarrow A'\gamma_N^+ u = f,$$

where

$$\boxed{A' := I + K' - \mathrm{i}\eta V,}$$

I is the identity operator, $\eta \in \mathbb{R}$ the **coupling parameter**, $f := 2\gamma_N^+ u^i - 2\mathrm{i}\eta\gamma_D^+ u^i$, and, for $\varphi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$,

$$V\varphi(x) = 2\int_\Gamma G(x,y)\varphi(y)\,ds(y), \quad K'\varphi(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(x)}\varphi(y)ds(y).$$

Alternatively ..., following Brakhage & Werner (1965) we find that an ansatz for u^s as a combined single and double-layer potential, with density φ and **coupling parameter** $\eta \in \mathbb{R}$ satisfies the scattering problem iff

$$A\varphi = -2\gamma_D^+ u^i,$$

where

$$\boxed{A := I + K - \mathrm{i}\eta V,}$$

and, for $\varphi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$,

$$V\varphi(x) = 2\int_\Gamma G(x,y)\varphi(y)\,ds(y), \quad K\varphi(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(y)}\varphi(y)ds(y).$$

N.B., where $(\phi, \psi) := \displaystyle\int_\Gamma \phi\psi ds$,

$$(A\phi, \psi) = (\phi, A'\psi), \quad \phi, \psi \in C^\infty(\Gamma).$$

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Mapping Properties. (Follows from Ralf, Thm 2.1.9)

$$A' : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \quad A : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$$

and these mappings are bounded, for $|s| \leq 1/2$.
(See Costabel (1988), McLean (2000), Meyer & Coifmann (2000).)

$$A' := I + K' - \mathrm{i}\eta V, \quad A := I + K - \mathrm{i}\eta V$$

where I is the identity operator, $\eta \in \mathbb{R}$ the coupling parameter, and, for $\varphi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$,

$$V\varphi(x) = 2\int_\Gamma G(x,y)\varphi(y)\, ds(y), \quad K'\varphi(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(x)}\varphi(y) ds(y),$$

$$K\varphi(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(y)}\varphi(y) ds(y).$$

Injectivity. (Ralf, Thm 2.1.16)
If $\eta \neq 0$, $A' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is injective.
(See C-W & Langdon, preprint, but same standard argument as for smooth boundaries, see e.g. Colton & Kress (1983).)

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Invertibility. If $\eta \neq 0$, then

$$A' : H^{s-1/2}(\Gamma) \to H^{s-1/2}(\Gamma), \quad A : H^{s+1/2}(\Gamma) \to H^{s+1/2}(\Gamma)$$

are bijections, for $|s| \leq 1/2$.
(See C-W & Langdon, preprint: follows since A is Fredholm of index zero on $H^1(\Gamma)$ and $L^2(\Gamma)$; Verchota (1985), Elschner (1992).)

$$\boxed{A' := I + K' - \mathrm{i}\eta V, \quad A := I + K - \mathrm{i}\eta V}$$

where I is the identity operator, $\eta \in \mathbb{R}$ the coupling parameter, and, for $\varphi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$,

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Coercivity. (Ralf, Lemmma 2.1.17) A is coercive (elliptic + compact) as an operator on $H^{1/2}(\Gamma)$ (and A' as an operator on $H^{-1/2}(\Gamma)$), in fact in the 3D case (with the right choice of norm) $\frac{1}{2}A = I - (\frac{1}{2}I - K)$ and $\frac{1}{2}I - K$ is a contraction when $k = 0$.
(Corollary of results in Steinbach & Wendland (2001).)
(Ralf, p. 33, not great for discretization as inner products non-local.)

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where I is the identity operator, $\eta \in \mathbb{R}$ the coupling parameter, and, for $\varphi \in L^2(\Gamma)$ and (almost all) $x \in \Gamma$,

$$V\varphi(x) = 2\int_\Gamma G(x,y)\varphi(y)\, ds(y), \quad K'\varphi(x) = 2\int_\Gamma \frac{\partial G(x,y)}{\partial n(x)}\varphi(y)ds(y),$$

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Wave Number Dependence. But how do $\|A\|$ and $\|A^{-1}\|$ depend on k, especially as $k \to \infty$, and how should we choose η?

Theorem. (See Dominguez, Graham, and Smyshlyaev, preprint, and cf. Buffa & Sauter, to appear SISC.)

If Γ is a circle, and $\eta = k$, then, for all sufficiently large k, A is elliptic on $L^2(\Gamma)$, precisely

$$\Re(A\phi, \bar{\phi}) \geq \frac{1}{2}\|\phi\|_2^2,$$

so that $\|A^{-1}\|_2 \leq 2$. Further $\|A\|_2 = O(k^{1/3})$ as $k \to \infty$.

Proof. Explicit calculation of spectrum of A (this dates back to Kress and Spassov 1983), and clever estimates of Bessel functions uniform in argument and order.

N.B. In the circle case $A = A'$.

N.B. Suggests variational formulation in $L^2(\Gamma)$ attractive and natural!? (cf. Ralf, p.33)

Let $n(x)$ denote the outward unit normal at $x \in \Gamma$, and

$$R_0 := \max_{x \in \Gamma} |x|, \quad \delta_- := \mathrm{ess.} \inf_{x \in \Gamma} \; x \cdot n(x).$$

Theorem. (C-W & Monk, preprint.) If Ω^- is a polyhedron which is starlike with respect to the origin (i.e. $\delta_- > 0$), or a more general piecewise smooth, Lipschitz, starlike domain, $\eta = k$ and $kR_0 \geq 1$, then

$$\|A^{-1}\|_2 = \|A'^{-1}\|_2 \quad \leq \quad \frac{1}{2}\left(1 + 13\theta + 4\theta^2\right),$$

where $\theta := R_0/\delta_-$.

Examples.
Circle/sphere: $\theta = 1$, $\|A^{-1}\|_2 = \|A'^{-1}\|_2 \leq 9$.
Cube: $\theta = \sqrt{3}$, $\|A^{-1}\|_2 = \|A'^{-1}\|_2 \leq 18$.

The Main Ingredients in the Proof

1. Green's theorem and a Rellich(-Payne-Weinberger-Nečas) type identity.

   Such identities, useful for obtaining explicit a priori bounds and regularity estimates for strongly elliptic systems, follow from the divergence theorem, and date back to Rellich (1943).

   See Chapter 5 of Nečas (1967) or McLean (2000). Our particular version of the identity is essentially that from the PhD of Melenk (1995).

Lemma 2.2. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and that $v \in H^2(G)$. Then, for every $k \geq 0$, where $g := \Delta v + k^2 v$ and the unit normal vector n is directed into Ω, it holds that

$$\int_\Omega \left(|\nabla v|^2 - k^2|v|^2 + g\bar{v}\right)\, dx = -\int_{\partial\Omega} \bar{v}\, \frac{\partial v}{\partial n}\, ds$$

and

$$\int_\Omega \left((2-d)|\nabla v|^2 + dk^2|v|^2 + 2\Re\left(gx\cdot\nabla\bar{v}\right)\right)\, dx =$$

$$-\int_{\partial\Omega}\left(x\cdot n\left(k^2|v|^2 + \left|\frac{\partial v}{\partial n}\right|^2 - |\nabla_T v|^2\right) + 2\Re\left(x\cdot\nabla_T\bar{v}\,\frac{\partial v}{\partial n}\right)\right)\, ds.$$

Corollary 2.3. Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain and that $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $\Delta v + k^2 v = 0$ in Ω. Then, where the unit normal vector n is directed into Ω, it holds that

$$\int_\Omega \left(|\nabla v|^2 - k^2|v|^2\right)\, dx = -\int_{\partial\Omega} \bar{v}\, \frac{\partial v}{\partial n}\, ds \tag{1}$$

and

$$\int_\Omega \left((2-d)|\nabla v|^2 + dk^2|v|^2\right)\, dx =$$

$$-\int_{\partial\Omega} \left(x\cdot n \left(k^2|v|^2 + \left|\frac{\partial v}{\partial n}\right|^2 - |\nabla_T v|^2 \right) + 2\Re \left(x \cdot \nabla_T \bar{v}\, \frac{\partial v}{\partial n} \right) \right) ds.$$

N.B. This is applied in Ω^- and in Ω_R^+ to $v := \Psi_{\mathrm{SL}}\varphi$, where $\varphi = {A'}^{-1}\psi$, in order to bound $(\gamma_N^+ - \gamma_N^-)v = \varphi$ in terms of ψ, starting from

$$\varphi = {A'}^{-1}\psi \Rightarrow \gamma_N^- v - \mathrm{i}\eta\gamma_D^- v = \frac{1}{2}\psi.$$

2. Once again, the property of radiating solutions of the Helmholtz equation, that, if v is radiating and Γ_R is the boundary of the sphere of radius R, then

$$\Re \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r} \, ds + R \int_{\Gamma_R} \left(k^2 |v|^2 + \left| \frac{\partial v}{\partial r} \right|^2 - |\nabla_T v|^2 \right) \, ds \leq 2kR \, \Im \int_{\Gamma_R} \bar{v} \frac{\partial v}{\partial r} \, ds.$$

Summary

1. Discussed wave number explicit lower and upper bounds on the inf-sup constant for the weak formulation in Ω_R^+[a] of the Dirchlet scattering problem.

2. Presented results on invertibility of the standard combined single- and double-layer boundary integral equation formulations for this problem, in the case of a Lipschitz domain, including the Brakhage-Werner (1965) formulation $A\varphi = -2\gamma_D u^i$ where $A := I + K - \mathrm{i}\eta V$.

3. Showed that, if Ω^- is piecewise smooth, Lipschitz and starlike, then $\|A^{-1}\|_2 \leq C$, with an explicit formula for C as a function of the geometry and η/k.

[a] That part of Ω^+ inside a ball of radius R.

Further Reading on Wave-Number-Explicit Estimates

A hybrid numerical-asymptotic boundary integral method for high-frequency acoustic scattering.
Dominguez, Graham, Smyshlyaev, University of Bath preprint, which builds on . . .

Schnelle Summationsverfahren zur numerischen Lösung von Integralgleichungen für Streuprobleme im $\mathbb{R}^3$.
Giebermann, PhD, Karlsruhe, 1997.

On Generalized Finite Element Methods.
Melenk, PhD, Maryland, 1995.

An elliptic regularity coefficient estimate for a problem arising from a frequncy domain treatment of waves.
Feng & Sheen, *Trans. Amer. Math. Soc.*, 1994.

Sharp regularity coefficient estimates for complex-valued acoustic and

elastic Helmholtz equations.
Cummings and Feng, *Math. Models Methods Appl. Sci.*, 2006.

Wave-number-explicit bounds in time-harmonic scattering.
C-W & Monk 2006, preprint.

A well-posed integral equation formulation for 3D rough surface scattering.
C-W, Heinemeyer & Potthast, *Proc. R. Soc. Lond. A*, 2006.

Existence, uniqueness and variational methods for scattering by unbounded rough surfaces.
C-W & Monk, *SIAM J. Math. Anal.*, 2005.

The mathematics of scattering by unbounded, rough, inhomogeneous layers.
C-W, Monk & Thomas *J. Comp. Appl. Math.* 2006

For copies of my stuff: `www.reading.ac.uk/~sms03snc`

Four Open Problems

1. Sharp estimates on $\|A\|_2$ as $k \to \infty$. This is much harder, see the harmonic analysis literature on oscillatory integral operators (Stein, Phong). (A crude bound that $\|A\|_2 \leq \max(\|A\|_\infty, \|A'\|_\infty) = O(k^{(d-1)/2})$ is straightforward, but it seems, from the circle/sphere, that $\|A\|_2 = O(k^{1/3})$.)

2. Bounds on $\|A^{-1}\|_2$ and lower bounds on the inf-sup constant for the weak problem in Ω_R^+ when the scatterer is not starlike.

3. Any wave-number-explicit bounds in the discrete case.

4. Preconditioners/new formulations which remove this k-dependence – and proofs!