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ANALYTIC REGULARITY FOR THE NAVIER-STOKES EQUATIONS IN POLYGONS WITH MIXED BOUNDARY CONDITIONS

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Abstract. We prove weighted analytic regularity of Leray-Hopf variational solutions for the stationary, incompressible Navier-Stokes Equations (NSE) in plane polygonal domains, subject to analytic body forces.

We admit mixed boundary conditions which may change type at each vertex, under the assumption that homogeneous Dirichlet ("no-slip") boundary conditions are prescribed on at least one side at each vertex of the domain. The weighted analytic regularity results are established in Hilbertian Kondrat'ev spaces with homogeneous corner weights. The proofs rely on a priori estimates for the corresponding linearized boundary value problem in sectors in corner-weighted Sobolev spaces and on an induction argument for the weighted norm estimates on the quadratic nonlinear term in the NSE, in a polar frame.

1. Introduction.

The regularity properties of viscous, incompressible flow governed by the incompressible Navier-Stokes Equations (NSE) have attracted considerable attention since their introduction. We mention only the intense research in recent years around the Onsager conjecture and on the boundedness of the velocity field of Leray solutions in three space dimensions.

Regularity results of weak, Leray-Hopf solutions in Sobolev and Besov scales in domains are at the core of the numerical analysis of the NSE. The stationary NSE, being for large values of the viscosity parameter, a perturbation of its linearization, the Stokes Equation, is an elliptic system in the sense of Agmon-Douglis-Nirenberg and affords analytic regularity in interior points of domains, for analytic forcing [20], see also [16]. This local analyticity of the velocity and the pressure extend to analytic parts of the boundary.

However, it is also classical that in the vicinity of corner points (in space dimension $d = 2$) and near edges and vertices (for polyhedra in space dimension $d = 3$), analyticity is lost, even if all other data of the stationary NSE is analytic. See, e.g., [5, 8, 21, 19, 4] and the references there. The reason is the appearance of corner singularities (in space dimension $d = 2$) and of corner- and edge-singularities (in polyhedra in space dimension $d = 3$). While singular solutions of the Stokes equation are well known to encode physically relevant effects (see, e.g., [18, 19]), they do obstruct large elliptic regularity shifts in standard (Besov or Triebel-Lizorkin) scales of function spaces and, consequently, high convergence rates of numerical discretizations. This has initiated the investigation of regularity of solutions in the presence of non smooth boundaries. One, in a sense, minimally regular situation is the assumption of mere Lipschitz regularity of the boundary. For the mixed boundary conditions of interest here, some regularity of velocity and pressure of Leray solutions in Sobolev spaces have been obtained in [6]. In the mentioned polygonal and polyhedral domains, it has been known for some time that the velocity fields of stationary solutions for the incompressible NSE in plane, polygonal domains allow higher regularity in so-called corner-weighted Sobolev spaces. Here, weight functions which vanish in the corners of the polygon to a suitable power compensate for the loss of regularity in the vicinity of the corner. The corresponding Mellin calculus goes back to [12]. See, e.g., [8, 21] and the references there. In [17], an authoritative account of these results, also for NSE in polyhedra, has been given. The results in [17] provide regularity shifts in weighted spaces of finite order. To prove weighted, analytic regularity for velocity field $\boldsymbol{u}$ and the pressure field p in P of the stationary, incompressible NSE in polygons is the purpose of the present paper.

Specifically, in a bounded polygon $P \subset \mathbb{R}^2$ whose boundary ∂P consists of a finite number n of straight sides, we consider the analytic regularity of solutions of the viscous, incompressible Navier-Stokes equations. Extending and revisiting our work [15] which addressed homogeneous Dirichlet ("no-slip") boundary conditions, we consider here the NSE in plane polygonal domains P with mixed boundary

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conditions, where now also slip and so-called "open" boundary parts are admitted. These conditions arise in numerous configurations in engineering and the sciences. The specific geometric condition which limits generality in the present paper is that we assume throughout the present paper that so-called mixed boundary conditions are imposed on ∂P. This is to say that at least one edge at each vertex of P will carry homogeneous Dirichlet boundary conditions. With this constraint (i.e., at least one "no-slip edge" abutting each corner) the elliptic regularity can be developed still in homogeneous corner-weighted spaces, i.e., in the the functional setting of [15]. Fully general BCs will require extension to weighted Kondrat'ev function spaces with non-homogeneous weights. Furthermore, our present proof of the weighted analytic regularity requires a proof technique which differs from the approach used in [15]. As the corresponding analysis for plane, linearized elasticity in [10], it is based on regularity results for the linearization (the Stokes problem) in a sector built on the Agranovich-Vishik theory of complex-parametric operator pencils which was already used in [9] and [10] to obtain a priori estimates and shift theorems in corner-weighted spaces. The present paper provides a proof of weighted analytic regularity for the velocity $\boldsymbol{u}$ and the pressure field p of the stationary, incompressible Navier-Stokes equations in a polygon P, subject to mixed boundary conditions on the sides of P. It is distinct from the argument in our previous work [15] even for pure Dirichlet boundary conditions. In [15], a bootstrapping argument based on local, Caccioppoli estimates on balls and scaling was proposed. Furthermore, the proof proposed in [15] was incomplete; the gap is closed by the argument in the present paper, which provides, in the case of homogeneous Dirichlet (so-called "no-slip") boundary conditions the weighted analytic regularity result in [15] which was used in [22] to prove exponential rates of convergence of a certain hp-DGFEM discretization of the stationary NSE in polygons.

Analytic regularity results for solutions in corner-weighted Kondrat'ev-Sobolev spaces imply, as is well-known, exponential convergence rate bounds for numerical approximations by so-called hp-Finite Element Methods and also by model order reduction methods. We refer to [22] and to the references there for recent results on exponential convergence for the Navier-Stokes equations, for discontinuous Galerkin discretizations, and also to the discussion in [15, Section 2.2] for exponential rates for certain model order reduction approaches to the NSE in P.

1.1. Contributions. We establish weighted, analytic regularity results for Leray-Hopf solutions of the NSE in bounded, connected polygonal domains $P \subset \mathbb{R}^2$ with finitely many, straight sides. We generalize the analytic regularity results stated in [15] from the pure Dirichlet (also referred to as "no-slip") boundary conditions as studied in [15] to the case of mixed boundary conditions at any two sides of P which meet at one common vertex of ∂P. As in [15] we work under a small data hypothesis, ensuring in particular the uniqueness of weak solutions. We also develop the regularity theory based on a priori estimates of solutions for a linearization, the Stokes problem, in weighted, Hilbertian Sobolev spaces in a sector. The result contains the analytic regularity result in [15] as a special case, and its proof proceeds in a way that is fundamentally different from [15]. As mentioned, it is based on a regularity analysis in corner-weighted spaces and a novel bootstrapping argument in the quadratic nonlinearity in weighted Kondrat'ev spaces. As in [9, 10], the weighted a priori estimates for the velocity field and the bounds on the quadratic nonlinearity near corners $\mathfrak{c}$ are obtained for the projection of the velocity components in a polar frame centered at $\mathfrak{c}$, rather than for their Cartesian components.

The main result of the present paper is stated in Theorem 4.8. Specifically, under the small data hypothesis and the stated assumptions on the boundary conditions (see Assumption 1 for details), we show that there exist $A > 0$ and $\gamma \in (0, 1)$ such that the Leray-Hopf solutions $(\boldsymbol{u}, p)$ to the NSE satisfy, for all $j, k \in \{0, 1, \dots\}$, and for any corner $\mathfrak{c}$ of P

$$\left\| \left(\prod_{\mathfrak{c}\in\mathfrak{C}} |\cdot - \mathfrak{c}|^{i+j-\gamma} \right) \partial_{x_1}^{j} \partial_{x_2}^{k} \boldsymbol{u} \right\|_{L^2(P)} \leq A^{j+k+1} (j+k)!,$$

and

$$\left\| \left(\prod_{\mathfrak{c}\in\mathfrak{C}} |\cdot - \mathfrak{c}|^{i+j-\gamma-1} \right) \partial_{x_1}^{j} \partial_{x_2}^{k} p \right\|_{L^2(P)} \leq A^{j+k+1}(j+k)!.$$

1.2. Layout. As is well-known (e.g. [13] and the references there) the analysis of point singularities near corners of solutions of elliptic PDEs is based on polar coordinates centered at the corner. For elliptic systems of PDEs such as those of interest here, as in [10, 9] in addition we require projections of Cartesian components of the vector-valued solutions to a polar frame. In Section 1.3, we collect the corresponding notation for partial derivatives and solution fields. Section 2 presents strong formulations of the boundary value problems under consideration, detailing in particular also the boundary operators. Furthermore, the corner-weighted, Kondrat'ev spaces that appear in the statement of the analytic regularity shifts are introduced. Section 3 then presents a key technical step for the subsequent analytic regularity proof: a priori estimates in corner-weighted Sobolev norms in a sector for the linearized Stokes boundary value problem are recapitulated, from [9]. Importantly, they hold for several combinations of boundary conditions on the sides of the sector, and for the velocity field in a polar coordinate frame. With this in hand, Section 4 addresses the proof of the principal analytic regularity result for the NSE, Theorem 4.8, which is also the main result of the present paper. The key novel step in its proof is an inductive bootstrap argument for the quadratic nonlinear term in the NSE, in corner-weighted spaces and for the velocity field in a polar frame at each corner of P. This is developed in Section 4.1.

1.3. Notation. We define $\mathbb{N} = \{1, 2, \dots\}$ as the set of positive natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We refer to tuples $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ as multi-indices and we write $|\alpha| = \alpha_1 + \alpha_2$. For $k \in \mathbb{N}_0$, we write

$$\sum_{|\alpha|\leq k} = \sum_{\alpha\in\mathbb{N}_0^2 : |\alpha|\leq k} .$$

Given Cartesian coordinates (x_1, x_2) and polar coordinates (r, ϑ), whose origin will be clear from the context, we denote derivatives as $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$ and $\mathcal{D}^\alpha = \partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2}$.

For any vector field $\boldsymbol{u}$ with components in Cartesian coordinates

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

we denote its polar coordinate frame projection as

$$\overline{\boldsymbol{u}} := \begin{pmatrix} u_r \\ u_\vartheta \end{pmatrix} = A\boldsymbol{u} \; . \tag{1.1}$$

where

$$A := \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \tag{1.2}$$

denotes the transformation matrix. Here and throughout, vector-valued quantities such as $\boldsymbol{u}$ shall be understood as column vectors, with $\boldsymbol{u}^\top$ denoting the transpose vector, which accordingly denotes a row vectors. The symbol L_{St} shall denote the Stokes operator, with various super- and subscripts indicating Cartesian or polar coordinates and frame, i.e. we write $\overline{L_{\mathrm{St}}}$ for its projection onto polar coordinates acting on the corresponding velocity components.

All quantities which occur in this paper are real-valued. The overline symbol which will indicate polar-coordinate representation of vectors is therefore non-ambiguous.

We denote with an underline n-dimensional tuples $\underline{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and suppose arithmetic operations and inequalities such as $\underline{\gamma} < \underline{\beta}$ are understood component-wise: e.g., $\underline{\beta}+k = (\beta_1+k, \dots, \beta_n+k)$ for all $k \in \mathbb{N}$; furthermore, we indicate, e.g., $\underline{\beta} > 0$ if $\beta_i > 0$ for all $i \in \{1, \dots, n\}$.

Finally, for $a \in \mathbb{R}$, we denote its nonnegative real part as $[a]_+ = \max(0, a)$.

For summability index $1 \le r \le \infty$, the usual Lebesgue spaces in P shall be denoted by $L^r(P)$, with norm defined also for vector fields $\boldsymbol{v} : P \to \mathbb{R}^2$ as $\|\boldsymbol{v}\|^r_{L^r(P)} = \int_P \|\boldsymbol{v}\|^r_{\ell^r}$. We denote the usual Sobolev spaces of differentiation order $s > 0$ by $W^{s,r}(P)$; we write $H^s(P)$ in the Hilbertian case $r = 2$.

2. The Navier-Stokes equations and functional setting. After the introduction of the polygonal domain in Section 2.1, in Section 2.2 we state the strong form of the boundary value problems, and of the boundary operators, in Cartesian coordinates. Section 2.3 is devoted to the saddle point variational form of the boundary value problems of interest. It also reviews statements on existence and uniqueness of weak solutions, under the small data hypothesis. In Section 2.5 we introduce the corner-weighted spaces on which the weighted analytic regularity results will be based.

2.1. Geometry of the domain. Let P be a polygon with straight sides and $n \geq 3$ corners $\mathfrak{C} = \{\mathfrak{c}_1, \dots, \mathfrak{c}_n\}$. Let Γ_D, Γ_N, and Γ_G be a disjoint partition of the boundary $\Gamma = \partial P$ of P comprising each of $n_D \geq 1$, $n_N \geq 0$ and $n_G \geq 0$ many sides of P, respectively, with $n = n_D + n_N + n_G$. We denote by $\boldsymbol{n} : \Gamma \to \mathbb{R}^2$ the exterior unit normal vector to P, defined almost everywhere on Γ, which belongs to $[L^\infty(\Gamma)]^2$, and by $\boldsymbol{t} : \Gamma \to \mathbb{R}^2$ correspondingly the unit tangent vector to Γ, pointing in counterclockwise tangential direction.

2.2. The Navier-Stokes boundary value problems. We assume that a kinematic viscosity $\nu > 0$ is given. For a velocity field $\boldsymbol{u} : P \to \mathbb{R}^2$ and a scalar $p : P \to \mathbb{R}$, define

$$\varepsilon(\boldsymbol{u}) \coloneqq \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top\right), \qquad \sigma(\boldsymbol{u}, p) \coloneqq 2\nu\varepsilon(\boldsymbol{u}) - p\,\mathrm{Id}_2,$$

where Id_2 is the 2×2 identity matrix.

With this notation, we consider in P, the stationary, incompressible Navier-Stokes equations

$$\begin{aligned} -\nabla\cdot\sigma(\boldsymbol{u},p) + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= \boldsymbol{f} \quad \text{in } P\,, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } P\,, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_D, \\ \sigma(\boldsymbol{u},p)\boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \Gamma_N\,, \\ (\sigma(\boldsymbol{u},p)\boldsymbol{n})\cdot\boldsymbol{t} = 0 \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_G. \end{aligned} \tag{2.1}$$

Remark 2.1. From the identity

$$2\nabla\cdot\varepsilon(\boldsymbol{u}) = \Delta\boldsymbol{u} + \nabla(\nabla\cdot\boldsymbol{u}), \tag{2.2}$$

the boundary value problem (2.1) is equivalent to

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \quad \text{in } P\,, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } P\,, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_D\,, \\ \sigma(\boldsymbol{u},p)\boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \Gamma_N\,, \\ (\sigma(\boldsymbol{u},p)\boldsymbol{n})\cdot\boldsymbol{t} = 0 \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_G. \end{aligned} \tag{2.3}$$

2.3. Variational Formulation. Weak solution of the NSE (2.1) in the sense of Leray-Hopf satisfy the NSE (2.1) in variational form. To state it, we introduce standard Sobolev spaces in P. Throughout the remainder of this article, we shall work under

Assumption 1. The boundary value problems (2.1), (2.3) satisfy the following conditions.

1. P is a bounded, connected polygon with a finite number of straight sides, and Lipschitz boundary $\Gamma = \partial P$.
2. For each corner $\mathfrak{c} \in \mathfrak{C}$, at least one of the two sides of P meeting in $\mathfrak{c}$ is a Dirichlet side with no-slip BCs.
3. All interior opening angles at corners of P are in $(0, 2\pi)$. In particular, slit domains which correspond to opening angle 2π are excluded.

Assumption 1 implies that the Dirichlet case considered in [15] is a special case of the present setting. Furthermore, since Item 2 implies that $n_D \geq 1$, it also ensures that the linearization of the Navier-Stokes equations, i.e., the Stokes problem, admits unique variational velocity field solutions $\boldsymbol{u}$, possibly with pressure p unique up to constants if $\Gamma = \Gamma_D$.

We denote henceforth the space of velocity fields of variational solutions to the Navier-Stokes equations (2.1) as

$$\boldsymbol{W} = \left\{ \boldsymbol{v} \in [H^1(P)]^2 : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_G \right\}. \tag{2.4}$$

We denote by $\boldsymbol{W}^* \subset [H^{-1}(P)]^2$ its dual, with identification of $L^2(P)^2 \simeq [L^2(P)^2]^*$. We also define $Q = L^2(P)$ if $|\Gamma_D| < |\Gamma|$ (i.e., if not the entire boundary is a Dirichlet boundary) and set $Q = L^2_0(P) := L^2(P)/\mathbb{R}$ in the case that $\Gamma = \Gamma_D$.

We are interested in variational solutions $(\boldsymbol{u}, p)$ of (2.1). To state the corresponding variational formulation, we introduce the usual bi- and trilinear forms:

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) &:= 2\nu \int_P \sum_{i,j=1}^{2} [\varepsilon(\boldsymbol{v})]_{ij} [\varepsilon(\boldsymbol{v})]_{ij} d\boldsymbol{x} \,, \\ B(\boldsymbol{u}, p) &:= -\int_P p \nabla \cdot \boldsymbol{u} d\boldsymbol{x} \,, \\ O(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) &:= \int_P ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}) \cdot \boldsymbol{v} d\boldsymbol{x} \,. \end{aligned} \tag{2.5}$$

With these forms, we state the variational formulation of (2.1): find $(\boldsymbol{u}, p) \in \boldsymbol{W} \times Q$ such that

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) + O(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + B(\boldsymbol{v}, p) &= \int_P \boldsymbol{f} \cdot \boldsymbol{v} d\boldsymbol{x} \,, \\ B(\boldsymbol{u}, q) &= 0 \,, \end{aligned} \tag{2.6}$$

for all $\boldsymbol{v} \in \boldsymbol{W}$ and all $q \in Q$.

2.4. Existence and uniqueness of solutions to the NS equations.

We introduce the coercivity constant of the viscous (diffusion) term

$$C_{\text{coer}} := \inf_{\substack{\boldsymbol{v} \in \boldsymbol{W} \\ \|\boldsymbol{v}\|_{H^1(P)} = 1}} 2 \int_P \sum_{i,j=1}^{2} [\varepsilon(\boldsymbol{v})]_{ij} [\varepsilon(\boldsymbol{v})]_{ij}$$

and the continuity constant for the trilinear transport term

$$C_{\text{cont}} := \sup_{\substack{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{W} \\ \|\boldsymbol{u}\|_{H^1(P)} = \|\boldsymbol{v}\|_{H^1(P)} = \|\boldsymbol{w}\|_{H^1(P)} = 1}} \int_P ((\boldsymbol{u} \cdot \nabla)\boldsymbol{w}) \cdot \boldsymbol{w}.$$

We denote a ball of bounded functions in $\boldsymbol{W}$

$$\mathbf{M} := \left\{ \boldsymbol{v} \in \boldsymbol{W} : \|\boldsymbol{v}\|_{H^1(P)} \leq \frac{C_{\text{coer}} \nu}{2 C_{\text{cont}}} \right\} .$$

The following existence and uniqueness result is classical, see e.g. [21, Theorem 3.2].

Theorem 2.2. Suppose that Assumption 1 holds and assume that $\|\boldsymbol{f}\|_{\boldsymbol{W}^*} \leq \frac{C_{\mathrm{coer}}^2 \nu^2}{4C_{\mathrm{cont}}}$. There exists a solution $(\boldsymbol{u}, p) \in \boldsymbol{W} \times L^2(P)$ to (2.1) with right hand side $\boldsymbol{f}$. The velocity field $\boldsymbol{u}$ is unique in $\mathbf{M}$.

Remark 2.3. From Assumption 1, Item 1, and from known regularity results for the Navier-Stokes equations in Lipschitz domains (e.g. [6]) it follows that the velocity field $\boldsymbol{u}$ has Cartesian components which are continuous in $\overline{P}$ and thus in particular in a vicinity of each corner $\mathfrak{c} \in \mathfrak{C}$ of P.

From Assumption 1, Item 2, it follows that for all corners $\mathfrak{c} \in \partial P$, the weak solution $\boldsymbol{u}$ satisfies $\boldsymbol{u}(\mathfrak{c}) = \mathbf{0}$.

As we assumed above $n_D \geq 1$, there is always at least one side of P where homogeneous Dirichlet ("no-slip") BCs are imposed.

2.5. Functional setting. For $x \in P$ and for $i \in \{1, \dots, n\}$, let $r_i(x) \coloneqq \mathrm{dist}(x, \mathfrak{c}_i)$. We define the corner weight function

$$\Phi_{\underline{\beta}}(x) \coloneqq \prod_{i=1}^{n} r_i^{\beta_i}(x).$$

We next introduce the corner-weighted function spaces to be used for the regularity analysis. As the notation used in the literature dealing with weighted Sobolev spaces is not always uniform, we introduce slightly different definitions of the spaces and discuss how they relate for the range of weight exponents that is relevant to the present work.

2.5.1. Corner-weighted function spaces in P. In the polygon P, for $j, k \in \mathbb{N}_0$ and $\underline{\gamma} \in \mathbb{R}^n$, we introduce homogeneous corner-weighted seminorms and associated norms given by

$$|v|^2_{\mathcal{K}^j_{\underline{\gamma}}(P)} \coloneqq \sum_{|\alpha|=j} \|\Phi_{|\alpha|-\underline{\gamma}} \partial^\alpha v\|^2_{L^2(P)}, \qquad \|v\|^2_{\mathcal{K}^k_{\underline{\gamma}}(P)} \coloneqq \sum_{j=0}^{k} |v|^2_{\mathcal{K}^j_{\underline{\gamma}}(P)}. \tag{2.7}$$

Furthermore, we also require non-homogeneous, corner-weighted Sobolev norms. They are, for $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ with $k > \ell$, and $\underline{\beta} \in \mathbb{R}^n$ given by

$$\|v\|^2_{H^{k,\ell}_{\underline{\beta}}(P)} \coloneqq \|v\|^2_{H^{\ell-1}(P)} + \sum_{\ell \leq |\alpha| \leq k} \|\Phi_{\underline{\beta}+|\alpha|-\ell} \partial^\alpha v\|^2_{L^2(P)}, \tag{2.8}$$

with the convection that the first term gets dropped when $\ell = 0$. We therefore define the homogeneous, corner-weighted Sobolev spaces $\mathcal{K}^k_{\underline{\gamma}}(P)$ and the non-homogeneous, corner-weighted Sobolev spaces $H^{k,\ell}_{\underline{\beta}}(P)$ as the spaces of, respectively, weakly differentiable functions with bounded $\mathcal{K}^k_{\underline{\gamma}}(P)$ and $H^{k,\ell}_{\underline{\beta}}(P)$ norms. Finally, we introduce weighted analytic classes

$$B^{\ell}_{\underline{\beta}}(P) \coloneqq \Big\{ v \in \bigcap_{k \geq \ell} H^{k,\ell}_{\underline{\beta}}(P) : \exists C, A > 0 \text{ s. t. } \|\Phi_{\underline{\beta}+|\alpha|-\ell} \partial^\alpha v\|_{L^2(P)} \leq C A^{|\alpha|-\ell} (|\alpha| - \ell)!,\ \forall |\alpha| \geq \ell \Big\}, \tag{2.9}$$

and

$$\mathcal{K}^{\varpi}_{\underline{\gamma}}(P) \coloneqq \Big\{ v \in \bigcap_{k \in \mathbb{N}_0} \mathcal{K}^k_{\underline{\gamma}}(P) : \exists C, A > 0 \text{ s. t. } \|\Phi_{|\alpha|-\underline{\gamma}} \partial^\alpha v\|_{L^2(P)} \leq C A^k k!,\ \forall k \in \mathbb{N}_0 \Big\}. \tag{2.10}$$

The aforementioned weighted analytic classes are defined in terms of two constants $C > 0$ and $A > 0$. Evidently, the constant $C > 0$ quantifies the size of a function in terms of linear scaling of norms, whereas the constant $A > 0$ relates to the size of the domain of analyticity.

2.5.2. Corner-weighted spaces in sectors. We shall require function spaces in plane sectors $Q_{\delta,\omega}(\mathfrak{c})$ of opening $\omega \in (0, 2\pi)$, radius $\delta \in (0, \infty]$ and with vertex $\mathfrak{c} \in \mathbb{R}^2$, defined as

$$Q_{\delta,\omega}(\mathfrak{c}) = \left\{x \in \mathbb{R}^2 : r(x, \mathfrak{c}) \in (0, \delta),\, \vartheta(x) \in (0, \omega)\right\}.$$

We do not indicate the dependence on the vertex $\mathfrak{c}$ when this is clear from the context.

For all $k \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, we introduce the (homogeneous) corner-weighted, Hilbertian Kondratev space $W^k_\beta(Q_{\delta,\omega})$ of functions v in $Q_{\delta,\omega}(\mathfrak{c})$ with bounded norm given by

$$\|v\|^2_{W^k_\beta(Q_{\delta,\omega})} = \sum_{|\alpha|\leq k} \|r^{\beta-k+\alpha_1}\mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega})}. \tag{2.11}$$

Here, $\mathcal{D}^\alpha = \partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2}$ denotes the partial derivative of order $\alpha \in \mathbb{N}_0$ in polar coordinates. We write $L_\beta = W^0_\beta$. For $k, \ell \in \mathbb{N}_0$ with $k \geq \ell$ and for $\beta \in \mathbb{R}$, $\mathcal{H}^{k,l}_\beta(Q_{\delta,\omega})$ denote the space of functions with finite norm

$$\|v\|^2_{\mathcal{H}^{k,\ell}_\beta(Q_{\delta,\omega})} := \|v\|^2_{H^{\ell-1}(Q_{\delta,\omega})} + \sum_{|\alpha|\geq \ell} \|r^{\alpha_1+\beta-\ell}\mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega})},$$

where the first term is dropped if $\ell = 0$. For $\ell \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, the corner-weighted analytic class with weak derivatives in polar coordinates is given by

$$\mathcal{B}^\ell_\beta(Q_{\delta,\omega}) = \left\{ v \in \bigcap_{k=\ell}^{\infty} \mathcal{H}^{k,\ell}_\beta(Q_{\delta,\omega}) : \exists C, A > 0 \text{ s. t. } \|r^{\alpha_1+\beta-\ell}\mathcal{D}^\alpha v\|_{L^2(Q_{\delta,\omega})} \leq CA^{|\alpha|-\ell}(|\alpha|-\ell)!,\ \forall |\alpha| \geq \ell \right\}. \tag{2.12}$$

In the sector $Q_{\delta,\omega}(\mathfrak{c})$, the definition of the spaces $H^{k,\ell}_\beta(Q_{\delta,\omega}(\mathfrak{c}))$ and $B^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))$ follows from (2.9) by replacing $\Phi_{\underline{\beta}+|\alpha|-\ell}$ in (2.8) and (2.9) with $r(\cdot, \mathfrak{c})^{\beta+|\alpha|-\ell}$. Similarly, the corner-weighted spaces $\mathcal{K}^k_\gamma(Q_{\delta,\omega}(\mathfrak{c}))$ and $\mathcal{K}^\varpi_\gamma(Q_{\delta,\omega}(\mathfrak{c}))$ can be defined by replacing $\Phi_{|\alpha|-\underline{\gamma}}$ in (2.7) and (2.10) with $r(\cdot, \mathfrak{c})^{|\alpha|-\gamma}$.

2.5.3. Relation between corner-weighted spaces. In this section we collect results on imbeddings between some of the corner-weighted spaces we introduced. They are of independent interest, and will be required at various stages in the ensuing proofs of the analytic regularity shifts.

We postpone all proofs, for ease of reading, to Appendix A. The following implication between polar frame velocity components $\overline{\boldsymbol{u}}$ in (1.1) and Cartesian components $\boldsymbol{u}$ holds.

Lemma 2.4. For all $0 < \delta \leq 1$, $\omega \in (0, 2\pi)$, $\mathfrak{c} \in \mathbb{R}^2$, $\ell \in \{0, 1, 2\}$, and $\beta \in (0, 1)$, if $\overline{\boldsymbol{u}} \in \mathcal{B}^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))$ and $\overline{\boldsymbol{u}}(\mathfrak{c}) = 0$ when $\ell = 2$, then there holds $\boldsymbol{u} \in B^\ell_\beta(Q_{\delta,\omega})$.

The reverse implication, in the case $\ell = 0$, is treated in the following statement.

Lemma 2.5. For all $0 < \delta \leq 1$, $\omega \in (0, 2\pi)$, $\mathfrak{c} \in \mathbb{R}^2$, and $\beta \in (0, 1)$, if $\boldsymbol{v} \in [B^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))]^2$ then $\overline{\boldsymbol{v}} \in \mathcal{B}^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.

The following two lemmas about equivalence and imbedding between weighted spaces will be used later. For the proof of the first lemma see [2, Theorem 1.1, Theorem 2.1, Lemma A.2], and for the proof of the second lemma see [2, Lemma 1.1, Lemma A.1, Lemma A.2] and [3, Section 2].

Lemma 2.6. Let $0 < \delta \leq 1$, $\omega \in (0, 2\pi)$, $\beta \in (0, 1)$, $\mathfrak{c} \in \mathbb{R}^2$. Then the following equivalence relations hold for any $\ell \in \{0, 1, 2\}$ and $\mathbb{N}_0 \ni k \geq \ell$:

1. $v \in H^{k,\ell}_\beta(Q_{\delta,\omega}(\mathfrak{c})) \Longleftrightarrow v \in \mathcal{H}^{k,\ell}_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.

2. $v \in B^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c})) \iff v \in \mathcal{B}^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.
3. $v \in H^{1,1}_\beta(Q_{\delta,\omega}(\mathfrak{c})) \iff v \in W^1_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.

Lemma 2.7. Let $0 < \delta \le 1$, $\omega \in (0, 2\pi)$, $\beta \in (0, 1)$, $\mathfrak{c} \in \mathbb{R}^2$. Then the following imbedding relations hold:

1. $W^2_\beta(Q_{\delta,\omega}(\mathfrak{c})) \subset H^{2,2}_\beta(Q_{\delta,\omega}(\mathfrak{c})) \subset C^0(\overline{Q_{\delta,\omega}(\mathfrak{c})})$.
2. If $v \in H^{2,2}_\beta(Q_{\delta,\omega}(\mathfrak{c}))$ and $v(\mathfrak{c}) = 0$, then $v \in W^2_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.

The following lemma asserts that functions that belong to corner-weighted Kondrat'ev spaces with non-homogeneous weights for a certain range of indices, with the additional requirement of the function vanishing at the vertex for second order spaces, also belong to the corresponding spaces with homogeneous weights. We refer to [13, Section 7.1] for an in-depth presentation.

Lemma 2.8. Let $0 < \delta \le 1$, $\omega \in (0, 2\pi)$, $\beta \in (0, 1)$, $\mathfrak{c} \in \mathbb{R}^2$, $k \in \{1, 2\}$, and $v \in H^{k,k}_\beta(Q_{\delta,\omega}(\mathfrak{c}))$. Let furthermore $v(\mathfrak{c}) = 0$ when $k = 2$. Then, $v \in \mathcal{K}^k_{k-\beta}(Q_{\delta,\omega}(\mathfrak{c}))$.

3. The Stokes equation in a sector. Consider, for $\mathfrak{c} \in \partial P$, $\delta \in (0, 1)$ and $\omega \in (0, 2\pi)$, the sector $Q_{\delta,\omega}(\mathfrak{c})$. Denote by $\Gamma_1 := \{x \in \mathbb{R}^2 : r(x, \mathfrak{c}) \in (0, \delta),\ \vartheta(x) = 0\}$ and $\Gamma_2 := \{x \in \mathbb{R}^2 : r(x, \mathfrak{c}) \in (0, \delta),\ \vartheta(x) = \omega\}$ the two edges meeting at $\mathfrak{c}$. Let also $\Gamma_\delta = \Gamma_1 \cup \Gamma_2$. As all the results in this section are independent of $\mathfrak{c}$, we omit the dependence of the sector in the notation and write $Q_{\delta,\omega} = Q_{\delta,\omega}(\mathfrak{c})$.

We consider variational solutions to the Stokes problem in $Q_{\delta,\omega}$

$$L^\sigma_{\mathrm{St}}(\boldsymbol{u}, p) = \begin{pmatrix} \boldsymbol{f} \\ h \end{pmatrix} \quad \text{in } Q_{\delta,\omega}\,, \qquad B(\nabla, \boldsymbol{u}, p) = \boldsymbol{0} \quad \text{on } \Gamma_\delta\,.$$

This reads in components

$$\begin{aligned} -\nabla \cdot \sigma(\boldsymbol{u}, p) &= \boldsymbol{f} && \text{in } Q_{\delta,\omega} \\ \nabla \cdot \boldsymbol{u} &= h && \text{in } Q_{\delta,\omega} \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma^S_D \\ \sigma(\boldsymbol{u}, p)\boldsymbol{n} &= \boldsymbol{g} && \text{on } \Gamma^S_N \\ (\sigma(\boldsymbol{u}, p)\boldsymbol{n}) \cdot \boldsymbol{t} = 0 \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma^S_G, \end{aligned} \tag{3.1}$$

where $\Gamma^S_D, \Gamma^S_N, \Gamma^S_G \in \{\varnothing, \Gamma_1, \Gamma_2\}$ are pairwise disjoint and such that $\Gamma^S_D \cup \Gamma^S_N \cup \Gamma^S_G = \Gamma_\delta$. We observe that in (3.1) we did not include inhomogeneous boundary data on Γ^S_G, as this is the physical case of the "no-slip" BCs. We also observe that the nonzero boundary data $\boldsymbol{g}$ on Γ^S_N will appear in the analytic regularity shift argument in the proof of Lemma 4.7.

For the Stokes problem in $Q_{\delta,\omega}$, the following regularity result is a slight extension of [9, Theorem 5.2]. The proof is along the lines of that of the cited theorem, by localizing $\boldsymbol{u}$ and p near each corner $\mathfrak{c}$ and solving a Stokes problem in a corresponding infinite sector; for a detailed development, see [11, Lemma 5.1.1].

Theorem 3.1. Let $\omega \in (0, 2\pi)$ and $\beta_f \in (0, 1)$. There exists a constant $\beta \in (\beta_f, 1)$ such that, for all $\delta > 0$, there exists a constant $C_{\mathrm{sec}} > 0$ such that for all $\boldsymbol{f} \in L_{\beta_f}(Q_{\delta,\omega})$ and $(\boldsymbol{u}, p)$ satisfying (3.1) in $Q_{\delta,\omega}$ and with right hand side $(\boldsymbol{f}, 0)$,

$$\begin{aligned} &\|\overline{\boldsymbol{u}}\|_{W^2_\beta(Q_{\delta/2,\omega})} + \|p\|_{W^1_\beta(Q_{\delta/2,\omega})} \\ &\qquad \le C_{\mathrm{sec}} \left(\|\overline{\boldsymbol{f}}\|_{L_\beta(Q_{\delta,\omega})} + \|\boldsymbol{u}\|_{H^1(Q_{\delta,\omega}\setminus Q_{\delta/2,\omega})} + \|p\|_{L^2(Q_{\delta,\omega}\setminus Q_{\delta/2,\omega})} + \|\overline{\boldsymbol{g}}\|_{W^{1/2}_\beta(\Gamma^S_N)} \right) \end{aligned} \tag{3.2}$$

Remark 3.2. By relation (2.2), if $(\boldsymbol{u}, p) \in [W^2_\beta(Q_{\delta,\omega})]^2 \times W^1_\beta(Q_{\delta,\omega})$ is a solution of

$$L^\Delta_{\mathrm{St}}(\boldsymbol{u}, p) = \begin{pmatrix} \boldsymbol{f} + \nu\nabla h \\ h \end{pmatrix} \quad \text{in } Q_{\delta,\omega}\,, \qquad B(\boldsymbol{u}, p) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{g} \\ \boldsymbol{0} \end{pmatrix} \quad \text{on } \Gamma^S_D \times \Gamma^S_N \times \Gamma^S_G,$$

or, in components,

$$
\begin{aligned}
-\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f} + \nu\nabla h \quad \text{in } Q_{\delta,\omega}\\
\nabla\cdot\boldsymbol{u} &= h \quad \text{in } Q_{\delta,\omega}\\
\boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_D^S\\
\sigma(\boldsymbol{u},p)\boldsymbol{n} &= \boldsymbol{g} \quad \text{on } \Gamma_N^S\\
(\sigma(\boldsymbol{u},p)\boldsymbol{n})\cdot\boldsymbol{t} = 0 \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_G^S,
\end{aligned} \tag{3.3}
$$

then it is also a solution of (3.1). Estimate (3.2) therefore also holds for solutions of (3.3) when $h = 0$.

4. Analytic regularity of solutions to the NS equations. We now prove our main result, i.e., the weighted analyticity of solutions to the Navier-Stokes equations (2.1). First, we will devote our attention to the nonlinear transport term, as treating this term is, obviously, the main difficulty with respect to the analysis of the Stokes problem.

4.1. Estimate of the nonlinear term. We start by rewriting the quadratic nonlinearity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ in polar coordinates and projecting its Cartesian components into the polar frame as in (1.1). We note here that the gradient operator in Cartesian coordinates is projected to a polar-frame by (cf. (1.1))

$$
\nabla = A^{-1}\begin{pmatrix}\partial_r\\ r^{-1}\partial_\vartheta\end{pmatrix}. \tag{4.1}
$$

Lemma 4.1. The following equality holds:

$$
\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}} = \begin{pmatrix} u_r\partial_r u_r + \frac{1}{r}(u_\vartheta\partial_\vartheta u_r - u_\vartheta^2)\\ u_r\partial_r u_\vartheta + \frac{1}{r}(u_\vartheta\partial_\vartheta u_\vartheta + u_r u_\vartheta)\end{pmatrix}. \tag{4.2}
$$

Proof. We have

$$
\begin{aligned}
\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}} &= A\left(\left(\overline{\boldsymbol{u}}\cdot(A^{-\top}A^{-1}\begin{pmatrix}\partial_r\\ r^{-1}\partial_\vartheta\end{pmatrix})\right)A^{-1}\overline{\boldsymbol{u}}\right)\\
&= A\left(\left(\overline{\boldsymbol{u}}\cdot\begin{pmatrix}\partial_r\\ r^{-1}\partial_\vartheta\end{pmatrix}\right)A^{-1}\overline{\boldsymbol{u}}\right)\\
&= A\Bigg[\begin{pmatrix}\cos\vartheta u_r\partial_r u_r - \sin\vartheta u_r\partial_r u_\vartheta\\ \sin\vartheta u_r\partial_r u_r + \cos\vartheta u_r\partial_r u_\vartheta\end{pmatrix}\\
&\qquad + \frac{1}{r}\begin{pmatrix}\cos\vartheta u_\vartheta\partial_\vartheta u_r - \sin\vartheta u_\vartheta u_r - \sin\vartheta u_\vartheta\partial_\vartheta u_\vartheta - \cos\vartheta u_\vartheta^2\\ \sin\vartheta u_\vartheta\partial_\vartheta u_r + \cos\vartheta u_\vartheta u_r + \cos\vartheta u_\vartheta\partial_\vartheta u_\vartheta - \sin\vartheta u_\vartheta^2\end{pmatrix}\Bigg]\\
&= \begin{pmatrix} u_r\partial_r u_r + \frac{1}{r}(u_\vartheta\partial_\vartheta u_r - u_\vartheta^2)\\ u_r\partial_r u_\vartheta + \frac{1}{r}(u_\vartheta\partial_\vartheta u_\vartheta + u_r u_\vartheta)\end{pmatrix}.
\end{aligned}
$$

□

In order to treat the individual nonlinear terms arising from the polar representation of the transport term of the Navier-Stokes equation obtained above, we need a technical result on weighted interpolation estimates in plane sectors. This is the following statement, the polar version of [15, Lemma 1.10].

Lemma 4.2. Let $\delta,\omega\in\mathbb{R}$ such that $0<\delta\le 1$ and $\omega\in(0,2\pi)$. For all $\tilde{\beta}_2,\tilde{\beta}_1\in\mathbb{R}$ such that $\tilde{\beta}_2\ge\tilde{\beta}_1+1/2$, there exists a constant $C_{\text{int}}>0$ such that, for all $\alpha\in\mathbb{N}_0^2$ and all functions φ such that

$$
\max_{|\eta|\le 1}\|r^{\tilde{\beta}_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|_{L^2(Q_{\delta,\omega})}<\infty,
$$

the following bound holds:

$$\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^4(Q_{\delta,\omega})} \leq C_{\mathrm{int}}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(Q_{\delta,\omega})} \times \left(\sum_{|\eta|\leq 1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|^{1/2}_{L^2(Q_{\delta,\omega})} + \alpha_1^{1/2}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(Q_{\delta,\omega})}\right).$$

Proof. We set $\delta = 1$. Consider the dyadic partition given by the sets

$$S^j := \left\{x \in Q_{1,\omega} : 2^{-j-1} < r(x) < 2^{-j}\right\}, \qquad j \in \mathbb{N}_0,$$

and denote the linear maps $\Psi_j : S^j \to S^0$. Denote $\widehat\varphi_j := \varphi\circ\Psi_j^{-1} : S^0 \to \mathbb{R}$ and write $\widehat{\mathcal{D}}^\alpha$ for derivation with respect to polar coordinates (r, ϑ) in S^0. Then, by scaling, for any $q \in [1, \infty)$,

$$\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^q(S^j)} = 2^{-j(\tilde\beta_2+2/q)}\|r^{\tilde\beta_2+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat\varphi\|_{L^q(S^0)}. \tag{4.3}$$

Furthermore, the following interpolation inequality holds in S^0: there exists $C_0 > 0$ such that

$$\|v\|_{L^4(S^0)} \leq C_0\|v\|^{1/2}_{H^1(S^0)}\|v\|^{1/2}_{L^2(S^0)} \tag{4.4}$$

holds for all $v \in H^1(S^0)$. Since in addition by (4.1) holds $\nabla = B\bar\nabla$, there also holds, for all $v \in H^1(S^0)$,

$$\|v\|^2_{H^1(S^0)} \leq 16\left(\|v\|^2_{L^2(S^0)} + \|\partial_r v\|^2_{L^2(S^0)} + \|\partial_\vartheta v\|^2_{L^2(S^0)}\right). \tag{4.5}$$

Combining (4.4) and (4.5) and choosing $v = r^{\alpha_1}\mathcal{D}^\alpha\varphi$ gives

$$\begin{aligned}
&\|r^{\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^4(S^0)}\\
&\quad\leq 2C_0\|r^{\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(S^0)}\left(\sum_{|\eta|\leq1}\|\mathcal{D}^\eta(r^{\alpha_1}\mathcal{D}^\alpha\varphi)\|^2_{L^2(S^0)}\right)^{1/4}\\
&\quad\leq 2C_0\|r^{\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(S^0)}\left(\sum_{|\eta|\leq1}\|r^{\alpha_1}\mathcal{D}^{\alpha+\eta}\varphi\|^2_{L^2(S^0)} + \alpha_1^2\|r^{\alpha_1-1}\mathcal{D}^\alpha\varphi\|^2_{L^2(S^0)}\right)^{1/4}
\end{aligned}$$

Therefore, using the bound $2^{-[a]_+} \leq r(x)^a \leq 2^{[a]_+}$ valid for all $x \in S^0$ and all $a \in \mathbb{R}$,

$$\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^4(S^0)} \leq 2^{[\tilde\beta_2]_++[\tilde\beta_1]_++1/2}2C_0\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(S^0)} \times \left(\sum_{|\eta|\leq1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|^2_{L^2(S^0)} + \alpha_1^2\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^2_{L^2(S^0)}\right)^{1/4}.$$

We denote $C_1 := 2^{[\tilde\beta_2]_+ + [\tilde\beta_1]_+ + 1/2} 2C_0$. Using this last inequality and (4.3) twice,

$$
\begin{aligned}
&\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^4(S^j)} \\
&\quad\le 2^{-j(\tilde\beta_2+1/2)}\|r^{\tilde\beta_2+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat\varphi\|_{L^4(S^0)} \\
&\quad\le 2^{-j(\tilde\beta_2+1/2)}C_1\|r^{\tilde\beta_1+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat\varphi_j\|_{L^2(S^0)}^{1/2} \\
&\qquad\quad\times\left(\sum_{|\eta|\le1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\widehat{\mathcal{D}}^{\alpha+\eta}\widehat\varphi_j\|^2_{L^2(S^0)}+\alpha_1^2\|r^{\tilde\beta_1+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat\varphi_j\|^2_{L^2(S^0)}\right)^{1/4} \\
&\quad\le C_1 2^{-j(\tilde\beta_2-\tilde\beta_1-1/2)}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(S^j)} \\
&\qquad\quad\times\left(\sum_{|\eta|\le1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|^2_{L^2(S^j)}+\alpha_1^2\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^2_{L^2(S^j)}\right)^{1/4}.
\end{aligned}
$$

Since $\tilde\beta_2-\tilde\beta_1-1/2\ge0$, we can conclude that

$$
\begin{aligned}
\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|^4_{L^4(S^j)} &\le C_1^4\left(\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^2_{L^2(S^j)}\right) \\
&\times\left(\sum_{|\eta|\le1}\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|^2_{L^2(S^j)}+\alpha_1^2\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^2_{L^2(S^j)}\right).
\end{aligned}
$$

Taking the fourth root of both sides of the inequality above concludes the proof for the case $\delta = 1$. The general case $\delta\in(0,1]$ follows by scaling (with constant C_{int} depending on δ). ∎

Using the interpolation result obtained above, we can estimate, under a regularity assumption on $\boldsymbol{u}$, the individual terms appearing in (4.2). This is done in the following Lemma 4.3 and Corollary 4.4.

Lemma 4.3. Let $\beta\in(0,1)$, $0<\delta\le1$, $\omega\in(0,2\pi)$. There exists $C_{\mathrm{d}}>0$ such that for all $u\in W^2_\beta(Q_{\delta,\omega})$ such that $\|u\|_{W^2_\beta(Q_{\delta,\omega})}\le1$ and such that there exists $A_u, E_u>1$ and $k\in\mathbb{N}$ such that

$$\|r^{\beta+\alpha_1-2}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta,\omega})}\le A_u^{|\alpha|-2}E_u^{\alpha_2}(|\alpha|-2)!,\quad\forall\alpha\in\mathbb{N}_0^2: 2\le|\alpha|\le k+1, \tag{4.6}$$

and for all $\alpha,\eta\in\mathbb{N}_0^2$ such that $|\eta|\le1$ and $|\alpha|\le k-|\eta|$,

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(r^{\eta_1}\mathcal{D}^\eta u)\|_{L^4(Q_{\delta,\omega})}\le C_{\mathrm{d}}(|\alpha|+1)^{1/2}A_u^{[|\alpha|+|\eta|-3/2]_+}E_u^{\alpha_2+\eta_2+1/2}[|\alpha|+|\eta|-2]_+!. \tag{4.7}$$

Proof. We start by proving the case $|\eta|=0$. Applying Lemma 4.2 with $\tilde\beta_2=\beta/2-1$ and $\tilde\beta_1=\beta-2$ (note that $\beta\in(0,1)$ implies $\tilde\beta_2>\tilde\beta_1+1/2$), for all $|\alpha|\le k$,

$$
\begin{aligned}
&\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})}\le C_{\mathrm{int}}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha u\|^{1/2}_{L^2(Q_{\delta,\omega})} \\
&\qquad\times\left(\sum_{|\eta|\le1}\|r^{\beta-2+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}u\|^{1/2}_{L^2(Q_{\delta,\omega})}+\alpha_1^{1/2}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha u\|^{1/2}_{L^2(Q_{\delta,\omega})}\right).
\end{aligned} \tag{4.8}
$$

When $|\alpha| \geq 2$, using (4.6), we have

$$\begin{aligned}
&\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} \\
&\quad \leq C_{\mathrm{int}} A_u^{|\alpha|-3/2} E_u^{\alpha_2+1/2}(2(|\alpha|-1)!^{1/2} + (1+\alpha_1^{1/2})(|\alpha|-2)!^{1/2})(|\alpha|-2)!^{1/2} \\
&\quad \leq C_{\mathrm{int}} A_u^{|\alpha|-3/2} E_u^{\alpha_2+1/2}(2(|\alpha|-1)^{1/2} + 1 + \alpha_1^{1/2})(|\alpha|-2)! \\
&\quad \leq C_{\mathrm{int}} A_u^{|\alpha|-3/2} E_u^{\alpha_2+1/2} 4|\alpha|^{1/2}(|\alpha|-2)!.
\end{aligned}$$

If $|\alpha| \leq 1$, instead, it follows from $\|u\|_{W^2_\beta(Q_{\delta,\omega})} \leq 1$ and (4.8) that

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} \leq C_{\mathrm{int}}(2+\alpha_1^{1/2}) \leq 4C_{\mathrm{int}}.$$

This proves (4.7) for $|\eta| = 0$, i.e., that for all $|\alpha| \leq k$,

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} \leq 4C_{\mathrm{int}} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}(|\alpha|+1)^{1/2}[|\alpha|-2]_+!. \tag{4.9}$$

Consider now the case $|\eta| = 1$. We have

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(r^{\eta_1}\mathcal{D}^\eta u)\|_{L^4(Q_{\delta,\omega})} \leq \|r^{\beta/2-1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}u\|_{L^4(Q_{\delta,\omega})} + \alpha_1\eta_1\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})}.$$

For all $|\alpha| \leq k-1$, we can apply (4.9) to the two terms at the right hand side above:

$$\begin{aligned}
\alpha_1\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} &\leq 4C_{\mathrm{int}} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}(|\alpha|+1)^{1/2}\alpha_1[|\alpha|-2]_+! \\
&\leq 4C_{\mathrm{int}} A_u^{[|\alpha|-1/2]_+} E_u^{\alpha_2+\eta_2+1/2}(|\alpha|+1)^{1/2} 2[|\alpha|-1]_+!,
\end{aligned}$$

and

$$\|r^{\beta/2-1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}u\|_{L^4(Q_{\delta,\omega})} \leq 4C_{\mathrm{int}} A_u^{[|\alpha|-1/2]_+} E_u^{\alpha_2+\eta_2+1/2}(|\alpha|+1)^{1/2}[|\alpha|-1]_+!.$$

Hence, for all $|\alpha| \leq k-1$ and all $|\eta| = 1$,

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(r^{\eta_1}\mathcal{D}^\eta u)\|_{L^4(Q_{\delta,\omega})} \leq 12C_{\mathrm{int}} A_u^{[|\alpha|-1/2]_+} E_u^{\alpha_2+\eta_2+1/2}(|\alpha|+1)^{1/2}[|\alpha|-1]_+!,$$

which concludes the proof, with $C_{\mathrm{d}} = 12C_{\mathrm{int}}$. □

Corollary 4.4. Let $\beta \in (0,1)$, $0 < \delta \leq 1$, $\omega \in (0, 2\pi)$, and let $u \in W^2_\beta(Q_{\delta,\omega})$ such that $\|u\|_{W^2_\beta(Q_{\delta,\omega})} \leq 1$. Suppose that there exists $A_u, E_u > 1$ and $k \in \mathbb{N}$ such that

$$\|r^{\beta+\alpha_1-2}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta,\omega})} \leq A_u^{|\alpha|-2} E_u^{\alpha_2}(|\alpha|-2)!, \quad \forall \alpha \in \mathbb{N}^2_0 : 2 \leq |\alpha| \leq k+1.$$

Then, for all $\alpha \in \mathbb{N}^2_0$ such that $|\alpha| \leq k$

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(ru)\|_{L^4(Q_{\delta,\omega})} \leq 4C_{\mathrm{d}}(|\alpha|+1)^{1/2} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}[|\alpha|-2]_+!. \tag{4.10}$$

Proof. We start from the bound

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(ru)\|_{L^4(Q_{\delta,\omega})} \leq \|r^{\beta/2+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} + \alpha_1\|r^{\beta/2-1+\alpha_1}\mathcal{D}^{(\alpha_1-1,\alpha_2)}u\|_{L^4(Q_{\delta,\omega})},$$

where the second term is absent if $\alpha_1 = 0$. From Lemma 4.3, it follows that

$$\|r^{\beta/2+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} \leq \delta C_{\mathrm{d}}(|\alpha|+1)^{1/2} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}[|\alpha|-2]_+!$$

and that (when $\alpha_1 \geq 1$)

$$\begin{aligned}
&\alpha_1 \|r^{\beta/2-1+\alpha_1}\mathcal{D}^{(\alpha_1-1,\alpha_2)}u\|_{L^4(Q_{\delta,\omega})} \\
&\quad\leq \delta\alpha_1|\alpha|^{1/2}A_u^{[|\alpha|-5/2]_+}E_u^{\alpha_2+1/2}[|\alpha|-3]_+! \\
&\quad\leq \max_{j\in\mathbb{N}}\left(\frac{j^{3/2}}{(j+1)^{1/2}\max(j-2,1)}\right)(|\alpha|+1)^{1/2}A_u^{[|\alpha|-3/2]_+}E_u^{\alpha_2+1/2}[|\alpha|-2]_+! \\
&\quad\leq \frac{3}{2}\sqrt{3}(|\alpha|+1)^{1/2}A_u^{[|\alpha|-3/2]_+}E_u^{\alpha_2+1/2}[|\alpha|-2]_+!
\end{aligned}$$

Equation (4.10) follows from the above, bounding $1+\frac{3}{2}\sqrt{3}\leq 4$ for ease of notation. □

We are now in position to estimate the weighted norms of the nonlinear term in the sector $Q_{\delta,\omega}(\mathfrak{c})$, under the assumptions of analytic bounds on the weighted norms of $\boldsymbol{u}$. Initially, we do this under the assumption that $\overline{\boldsymbol{u}} \in W^2_\beta(Q_{\delta,\omega}(\mathfrak{c}))^2$ (which implies that $\boldsymbol{u}$ vanishes at the vertex of the sector) in Lemma 4.5.

Lemma 4.5 (Weighted analytic estimates for the quadratic nonlinearity in polar frame). Assume that $\beta\in(0,1)$, $0<\delta\leq 1$, $\omega\in(0,2\pi)$.

Then, there exists $C_{\mathrm{t}}>0$ such that for all $\boldsymbol{w}: Q_{\delta,\omega}\to\mathbb{R}^2$, all $k\in\mathbb{N}$, and all $A_w, E_w\geq 1$ such that $\|\overline{\boldsymbol{w}}\|_{W^2_\beta(Q_{\delta,\omega})}\leq 1$ and

$$\begin{cases}\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha w_r\|_{L^2(Q_{\delta,\omega})}\leq A_w^{|\alpha|-2}E_w^{\alpha_2}(|\alpha|-2)! \\ \|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha w_\vartheta\|_{L^2(Q_{\delta,\omega})}\leq A_w^{|\alpha|-2}E_w^{\alpha_2}(|\alpha|-2)!,\end{cases}\qquad \text{for all } 2\leq|\alpha|\leq k+1,$$

the following inequality holds:

$$\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^2\overline{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}})\|_{L^2(Q_{\delta,\omega})}\leq C_{\mathrm{t}}A_w^{|\alpha|-1}E_w^{\alpha_2+2}|\alpha|!,\qquad \forall\alpha\in\mathbb{N}_0^2: 1\leq|\alpha|\leq k.$$

Proof. By Lemma 2.7, the bound $\|\overline{\boldsymbol{w}}\|_{W^2_\beta(Q_{\delta,\omega})}\leq 1$ implies $\overline{\boldsymbol{w}}\in[C^0(\overline{Q_{\delta,\omega}})]^2$ and thus $\|\overline{\boldsymbol{w}}\|_{L^\infty(Q_{\delta,\omega})}<+\infty$.

Next, we recall from Lemma 4.1 that

$$r^2\overline{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}}=\begin{pmatrix}r^2w_r\partial_r w_r+r(w_\vartheta\partial_\vartheta w_r-w_\vartheta^2)\\ r^2w_r\partial_r w_\vartheta+r(w_\vartheta\partial_\vartheta w_\vartheta+w_rw_\vartheta)\end{pmatrix}. \tag{4.11}$$

We will estimate the individual terms.

Estimate of rw_ϑ^2 and rw_rw_ϑ. Let $v\in\{w_r,w_\vartheta\}$. From (4.10), Lemma 4.3 and Corollary 4.4 it follows that

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta v)\|_{L^2(Q_{\delta,\omega})} \\
&\quad\leq\sum_{j=0}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}\|r^{\eta_1+\beta/2-1}\mathcal{D}^\eta(rv)\|_{L^4(Q_{\delta,\omega})}\|r^{\alpha_1-\eta_1+\beta/2-1}\mathcal{D}^{\alpha-\eta}w_\vartheta\|_{L^4(Q_{\delta,\omega})} \\
&\quad\leq\sum_{j=0}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}4C_{\mathrm{d}}^2(|\eta|+1)^{1/2}A_w^{[|\eta|-3/2]_+}E_w^{\eta_2+1/2}[|\eta|-2]_+! \\
&\qquad\qquad\times(|\alpha|-|\eta|+1)^{1/2}A_w^{[|\alpha|-|\eta|-3/2]_+}E_w^{\alpha_2-\eta_2+1/2}[|\alpha|-|\eta|-2]_+! \\
&\quad\leq 4C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1} \\
&\qquad\qquad\times\sum_{j=0}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}j!(|\alpha|-j)!\frac{(j+1)^{1/2}(|\alpha|-j+1)^{1/2}}{\max(j(j-1),1)\max((|\alpha|-j)(|\alpha|-j-1),1)}.
\end{aligned}$$

Here we have used $[|\eta|-3/2]_+ + [|\alpha|-|\eta|-3/2]_+ \leq [|\alpha|-3/2]_+$ for all $\eta \leq \alpha$.

Now, for all $j \in \mathbb{N}_0$,

$$\frac{(j+1)^{1/2}}{\max(j(j-1),1)} = \frac{(j+1)^{1/2}\max(j,1)^{1/2}}{\max(j-1,1)}\frac{1}{\max(j,1)^{3/2}} \leq \sqrt{6}\frac{1}{\max(j,1)^{3/2}}.$$

In addition,

$$\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta} = \binom{|\alpha|}{j}.$$

Therefore,

$$\begin{aligned}&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta v))\|_{L^2(Q_{\delta,\omega})}\\
&\quad\leq 24C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1}\sum_{j=0}^{|\alpha|}j!(|\alpha|-j)!\frac{1}{\max(j,1)^{3/2}\max(|\alpha|-j,1)^{3/2}}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}.\\
&\quad\leq 24C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1}|\alpha|!\sum_{j=0}^{|\alpha|}\frac{1}{\max(j,1)^{3/2}\max(|\alpha|-j,1)^{3/2}}.\end{aligned}$$

We have, by the Cauchy-Schwarz inequality,

$$\sum_{j=0}^{|\alpha|}\frac{1}{\max(j,1)^{3/2}\max(|\alpha|-j,1)^{3/2}} \leq \sum_{j=0}^{|\alpha|}\frac{1}{\max(j,1)^{3}} \leq 1+\zeta(3) \leq \frac{5}{2}.$$

We conclude that

$$\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta^2))\|_{L^2(Q_{\delta,\omega})} \leq 60C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1}|\alpha|! \tag{4.12}$$

and

$$\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta w_r))\|_{L^2(Q_{\delta,\omega})} \leq 60C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1}|\alpha|!. \tag{4.13}$$

Estimate of the remaining terms. Let $v, w \in \{w_r, w_\vartheta\}$ and let $\xi \in \mathbb{N}_0^2$ such that $|\xi| = 1$. We have

$$\begin{aligned}&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{1+\xi_1}w\mathcal{D}^\xi v))\|_{L^2(Q_{\delta,\omega})}\\
&\quad\leq \sum_{j=1}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}\|r^{\eta_1+\beta/2-1}\mathcal{D}^\eta(rw)\|_{L^4(Q_{\delta,\omega})}\|r^{\alpha_1-\eta_1+\beta/2-1}\mathcal{D}^{\alpha-\eta}(r^{\xi_1}\mathcal{D}^\xi v)\|_{L^4(Q_{\delta,\omega})}\\
&\qquad+\|r^{\alpha_1+\beta-1}w\mathcal{D}^\alpha(r^{\xi_1}\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}\\
&\quad= (I)+(II).\end{aligned} \tag{4.14}$$

We bound the sum in term (I) by similar techniques as above, using Lemma 4.3 and Corollary 4.4:

$$\begin{aligned}(I) &\leq \sum_{j=1}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}4C_{\mathrm{d}}^2(|\eta|+1)^{1/2}A_w^{[|\eta|-3/2]_+}E_w^{\eta_2+1/2}[|\eta|-2]_+!\\
&\qquad\times(|\alpha|-|\eta|+1)^{1/2}A_w^{[|\alpha|-|\eta|-1/2]_+}E_w^{\alpha_2-\eta_2+\xi_2+1/2}[|\alpha|-|\eta|-1]_+!\\
&\leq 4C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1+\xi_2}\sum_{j=1}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}j!(|\alpha|-j)!\frac{(j+1)^{1/2}(|\alpha|-j+1)^{1/2}}{\max(j(j-1),1)\max(|\alpha|-j,1)},\end{aligned}$$

where we have used that

$$[|\eta| - 3/2]_+ + [|\alpha| - |\eta| - 1/2]_+ \leq [|\alpha| - 3/2]_+, \qquad \forall \eta \leq \alpha : |\eta| \geq 1.$$

Since

$$\frac{(j+1)^{1/2}}{\max(j,1)} = \frac{(j+1)^{1/2}}{\max(j,1)^{1/2}} \frac{1}{\max(j,1)^{1/2}} \leq \sqrt{2} \frac{1}{\max(j,1)^{1/2}},$$

and using Hölder's inequality, we obtain

$$\begin{aligned}
(I) &\leq 8C_{\mathrm{d}}^2 A_w^{[|\alpha|-3/2]_+} E_w^{\alpha_2+\xi_2+1} |\alpha|! \sum_{j=1}^{|\alpha|} \frac{1}{\max(j-1,1)\max(j,1)^{1/2}\max(|\alpha|-j,1)^{1/2}} \\
&\leq 8C_{\mathrm{d}}^2 A_w^{[|\alpha|-3/2]_+} E_w^{\alpha_2+\xi_2+1} |\alpha|! \sum_{j=1}^{|\alpha|} \frac{1}{\max(j-1,1)^{3/2}\max(|\alpha|-j,1)^{1/2}} \\
&\leq 8C_{\mathrm{d}}^2 A_w^{[|\alpha|-3/2]_+} E_w^{\alpha_2+\xi_2+1} |\alpha|! \left(1 + \sum_{j=1}^{|\alpha|-1} j^{-2}\right)^{3/4} \left(1 + \sum_{j=1}^{|\alpha|-1} j^{-2}\right)^{1/4} \\
&\leq 24C_{\mathrm{d}}^2 A_w^{[|\alpha|-3/2]_+} E_w^{\alpha_2+\xi_2+1} |\alpha|!,
\end{aligned} \tag{4.15}$$

where we have used $1 + \zeta(2) \leq 3$. We now estimate term (II) in (4.14). Remark that

$$(II) \leq \|rw\|_{L^\infty(Q_{\delta,\omega})} \|r^{\alpha_1+\beta-2} \mathcal{D}^\alpha (r^{\xi_1} \mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}. \tag{4.16}$$

In addition,

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2} \mathcal{D}^\alpha (r^{\xi_1} \mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})} \\
&\quad \leq \|r^{\alpha_1+\xi_1+\beta-2} \mathcal{D}^{\alpha+\xi} v\|_{L^2(Q_{\delta,\omega})} + \alpha_1 \xi_1 \|r^{\alpha_1+\beta-2} \mathcal{D}^\alpha v\|_{L^2(Q_{\delta,\omega})} \\
&\quad \leq A_w^{|\alpha|-1} E_w^{\alpha_2+\xi_2} (|\alpha|-1)! + \xi_1 |\alpha| A_w^{|\alpha|-1} E_w^{\alpha_2} [|\alpha|-2]_+! \\
&\quad \leq 3A_w^{|\alpha|-1} E_w^{\alpha_2+\xi_2} (|\alpha|-1)!.
\end{aligned}$$

Hence, from (4.16),

$$(II) \leq 3\delta \|\overline{\boldsymbol{w}}\|_{L^\infty(Q_{\delta,\omega})} A_w^{|\alpha|-1} E_w^{\alpha_2+\xi_2} (|\alpha|-1)!. \tag{4.17}$$

It follows from (4.14), (4.15), and (4.17) that, for any $v, w \in \{w_r, w_\vartheta\}$ and any multi-index ξ such that $|\xi| = 1$,

$$\|r^{\alpha_1+\beta-2} \mathcal{D}^\alpha (r^{1+\xi_1} w \mathcal{D}^\xi v))\|_{L^2(Q_{\delta,\omega})} \leq (24C_{\mathrm{d}}^2 + 3\|\overline{\boldsymbol{w}}\|_{L^\infty(Q_{\delta,\omega})}) A_w^{|\alpha|-1} E_w^{\alpha_2+1+\xi_2} |\alpha|!. \tag{4.18}$$

The combination of the formulation (4.11) and of the bounds (4.12), (4.13), and (4.18) concludes the proof, with

□

$$C_{\mathrm{t}} = 6 \max\left(60C_{\mathrm{d}}^2, 24C_{\mathrm{d}}^2 + 3\|\overline{\boldsymbol{w}}\|_{L^\infty(Q_{\delta,\omega})}\right).$$

4.2. Analytic regularity in the polygon P. We can now prove the main result of this paper. With analyticity in the interior and up to edges of P being classical, we concentrate on the sectors near the corners of the domain. We define for $\delta \in (0,1)$,

$$S^i_\delta \coloneqq Q_{\delta,\omega_i}(\mathfrak{c}_i), \qquad i = 1,\dots,n. \tag{4.19}$$

We prepare the bootstrapping argument required for establishing analytic regularity by proving the regularity of the solution $(\boldsymbol{u},p)$ in the weighted spaces $[W^2_\beta(S^i_\delta)]^2 \times W^1_\beta(S^i_\delta)$.

Lemma 4.6. Let $\underline{\beta}_f \in (0,1)^n$ and $\boldsymbol{f} \in [L_{\underline{\beta}_f}(P)]^2 \cap \boldsymbol{W}^*$ such that $\|\boldsymbol{f}\|_{\boldsymbol{W}^*} \le \frac{C^2_{\mathrm{coer}}\nu^2}{4C_{\mathrm{cont}}}$. Suppose that Assumption 1 holds. Let $(\boldsymbol{u},p)$ be the solution to (2.1) with right hand side $\boldsymbol{f}$.

Then, there exists $\underline{\beta} = (\beta_1,\dots,\beta_n) \in (0,1)^n$ such that for all $0<\delta\le 1$ with $\delta < \frac{1}{4}\min_{i,j}|\mathfrak{c}_j - \mathfrak{c}_i|$ holds

$$(\boldsymbol{u},p) \in [W^2_{\beta_i}(S^i_\delta)]^2 \times W^1_{\beta_i}(S^i_\delta), \qquad \forall i \in \{1,\dots,n\}.$$

Proof. For all $s \in (1,2)$ and for $t = (1/s - 1/2)^{-1}$,

$$\|\boldsymbol{f}\|_{L^s(P)} \le \|\Phi_{-\underline{\beta}_f}\|_{L^t(P)} \|\Phi_{\underline{\beta}_f}\boldsymbol{f}\|_{L^2(P)}.$$

Therefore $\boldsymbol{f} \in [L_{\underline{\beta}_f}(P)]^2$ implies

$$\boldsymbol{f} \in [L^s(P)]^2, \qquad \forall s \in \left(1, \frac{2}{1+\max\underline{\beta}_f}\right).$$

In addition, $\boldsymbol{u} \in [H^1(P)]^2$ implies by Sobolev imbedding $\boldsymbol{u} \in [L^t(P)]^2$ for all $t \in [1,\infty)$. By Hőlder's inequality, choosing $t \in [1,\infty)$ and $s = (1/2+1/t)^{-1}$,

$$\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{L^s(P)} \le \|\boldsymbol{u}\|_{L^t(P)}\|\nabla\boldsymbol{u}\|_{L^2(P)} < \infty$$

which implies $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in [L^s(P)]^2$, for all $s \in [1,2)$. It follows from [21, Corollary 4.2] that there exists $q>1$ such that $(\boldsymbol{u},p) \in [W^{2,q}(P)]^2 \times W^{1,q}(P)$. This in turn implies, by Sobolev imbedding, $\boldsymbol{u} \in [L^\infty(P)]^2$ hence $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in [L^2(P)]^2$. We can conclude by applying Theorem 3.1 to each corner sector: for each $i \in \{1,\dots,n\}$, there exists $\beta \in ((\underline{\beta}_f)_i, 1)$ such that

$$\|\overline{\boldsymbol{u}}\|_{W^2_\beta(S^i_\delta)} + \|p\|_{W^1_\beta(S^i_\delta)} \le C_{\mathrm{sec}}\left(\|\overline{\boldsymbol{f}}\|_{L_\beta(S^i_\delta)} + \|\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}}\|_{L_\beta(S^i_\delta)} + \|\boldsymbol{u}\|_{H^1(P)} + \|p\|_{L^2(P)}\right).$$

Now, since $\boldsymbol{f} \in [L_{\underline{\beta}_f}(P)]^2$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in [L^2(P)]^2$, it holds that $\overline{\boldsymbol{f}} \in [L_\beta(S^i_\delta)]^2$ and $\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}} \in [L_\beta(S^i_\delta)]^2$; hence, the right hand side of the inequality above is bounded. Using [10, Corollary 4.2] to bound the norm of the Cartesian version of the flux concludes the proof. ∎

We prove weighted analytic estimates in each corner sector.

Lemma 4.7. Let $\underline{\beta}_f \in (0,1)^n$ and $\boldsymbol{f} \in [B^0_{\underline{\beta}_f}(P)]^2 \cap \boldsymbol{W}^*$ such that $\|\boldsymbol{f}\|_{\boldsymbol{W}^*} \le \frac{C^2_{\mathrm{coer}}\nu^2}{4C_{\mathrm{cont}}}$. Suppose that Assumption 1 holds and let $(\boldsymbol{u},p)$ be the solution to (2.1) with right hand side $\boldsymbol{f}$.

Then, there exists $\underline{\beta} \in (0,1)^n$, with $\underline{\beta} \ge \underline{\beta}_f$, $\delta \in (0,1]$, and $A_u, E_u > 0$ such that, for all $i \in \{1,\dots,n\}$,

$$\|r_i^{\beta_i+\alpha_1-2}\mathcal{D}^\alpha\boldsymbol{u}\|_{L^2(S^i_{\delta/2})} \le A_u^{|\alpha|-2}E_u^{[\alpha_2-4/3]_+}(|\alpha|-2)!, \qquad \forall\alpha\in\mathbb{N}^2_0 : |\alpha|\ge 2,$$

and

$$\|r_i^{\beta_i+\alpha_1-1}\mathcal{D}^\alpha p\|_{L^2(S^i_{\delta/2})} \le A_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!, \qquad \forall\alpha\in\mathbb{N}^2_0 : |\alpha|\ge 1.$$

Proof. Choose $\underline{\beta} = (\beta_1, \dots, \beta_n) \in (0,1)^n$, $\beta_i \geq (\underline{\beta}_f)_i$, such that, for all $i \in \{1, \dots, n\}$, Theorem 3.1 holds in $Q_{1,\omega_i}(\mathfrak{c}_i)$ with $\beta = \beta_i$. Fix $0 < \delta_P \leq 1$ such that $\delta_P < \frac{1}{4}\min_{i,j} |\mathfrak{c}_j - \mathfrak{c}_i|$ and such that

$$\|\overline{\boldsymbol{u}}\|_{W^2_{\beta_i}(S^i_{\delta_P})} \leq 1, \qquad \|p\|_{W^1_{\beta_i}(S^i_{\delta_P})} \leq 1, \qquad \forall i \in \{1, \dots, n\}. \tag{4.20}$$

Note that this is possible thanks to Lemma 4.6. The proof proceeds by induction, in each of the corner sectors. Fix $i \in \{1, \dots, n\}$. We write $r(x) \coloneqq r_i(x) = |x - \mathfrak{c}_i|$ for compactness.

Before setting up the inductive bootstrap argument, we rewrite the NSE in polar coordinates and rearrange the equations in the sector $S^i_{\delta_P}$ as

$$\overline{L^{\Delta}_{\mathrm{St}}}(\overline{\boldsymbol{u}}, p) = \begin{pmatrix} A(\boldsymbol{f} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) \\ 0 \end{pmatrix} \quad \text{in } S^i_{\delta_P}, \tag{4.21a}$$

$$\overline{B}(\overline{\boldsymbol{u}}, p) = \boldsymbol{0} \quad \text{on } \Gamma_\delta, \tag{4.21b}$$

This set of equations has the following specific form:

$$-\frac{1}{r^2}\begin{pmatrix} \nu((r\partial_r)^2 + \partial_\vartheta^2 - 1) & -2\nu\partial_\vartheta \\ 2\nu\partial_\vartheta & \nu((r\partial_r)^2 + \partial_\vartheta^2 - 1) \end{pmatrix}\begin{pmatrix} u_r \\ u_\vartheta \end{pmatrix} + \frac{1}{r}\begin{pmatrix} r\partial_r \\ \partial_\vartheta \end{pmatrix} p = \overline{\boldsymbol{f}} - \overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}} \text{ in } S^i_{\delta_P}, \tag{4.22}$$

$$\frac{1}{r}\left((r\partial_r + 1)\, u_r + \partial_\vartheta u_\vartheta\right) = 0 \text{ in } S^i_{\delta_P}, \tag{4.23}$$

$$\overline{\boldsymbol{u}} = \boldsymbol{0} \text{ on } \partial S^i_{\delta_P} \cap \Gamma_D. \tag{4.24}$$

On $\partial S^i_{\delta_P} \cap \Gamma_N$ and $\partial S^i_{\delta_P} \cap \Gamma_G$, respectively, there holds

$$\begin{pmatrix} \nu(r^{-1}\partial_\vartheta u_r + \partial_r u_\vartheta - r^{-1}u_\vartheta) \\ -p + 2\nu r^{-1}(\partial_\vartheta u_\vartheta + u_r)) \end{pmatrix} = \boldsymbol{0} \tag{4.25}$$

and

$$\begin{pmatrix} u_\vartheta \\ \nu(\partial_r u_\vartheta + \frac{1}{r}\partial_\vartheta u_r - \frac{1}{r}u_\vartheta) \end{pmatrix} = \boldsymbol{0}\,. \tag{4.26}$$

See Appendix B for details of the derivation.

The analyticity of $\boldsymbol{u}$ and p in $\overline{P \setminus \left(\bigcup_{i=1}^n S^i_{\delta_P/2}\right)}$ and the analyticity assumption on $\boldsymbol{f}$, i.e., $\boldsymbol{f} \in [B^0_{\underline{\beta}_f}(P)]^2 \subset [B^0_{\underline{\beta}}(P)]^2$ (whence $\overline{\boldsymbol{f}} \in [\mathcal{B}^0_{\beta_i}(S^i_{\delta_P})]^2$ by Lemma 2.5), imply that there exists $A_1 > 0$ such that, for all $|\alpha| \geq 1$,

$$\|r^{\beta_i + \alpha_1 - 2}\mathcal{D}^\alpha(r^2\overline{\boldsymbol{f}})\|_{L^2(S^i_{\delta_P})} \leq A_1^{|\alpha|}|\alpha|!, \tag{4.27a}$$

$$\|r^{\beta_i + \alpha_1 - 2}\mathcal{D}^\alpha(r^2\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})\|_{L^2(S^i_{\delta_P} \setminus S^i_{\delta_P/2})} \leq A_1^{|\alpha|}|\alpha|!, \tag{4.27b}$$

$$\|r^{\beta_i + \alpha_1 - 1}\mathcal{D}^\alpha p\|_{L^2(S^i_{\delta_P} \setminus S^i_{\delta_P/2})} \leq A_1^{|\alpha|-1}(|\alpha| - 1)!, \tag{4.27c}$$

and, for all $k \in \mathbb{N}$,

$$\|r^k\partial_r^k\overline{\boldsymbol{u}}\|_{H^1(S^i_{\delta_P} \setminus S^i_{\delta_P/2})} \leq A_1^k k!. \tag{4.27d}$$

Define the constants

$$E_u = \max\left(2, 8\left(1 + \frac{1}{\nu}\right)^{3/2}, (8\nu)^{3/2}\right), \tag{4.28a}$$

and

$$A_u = \max\left(22C_{\mathrm{sec}}A_1, 2C_{\mathrm{sec}}(C_{\mathrm{t}}+9)E_u^2, \frac{4}{\nu}A_1, 4\left(\frac{1}{\nu}(C_{\mathrm{t}}+2)+4\right)E_u^{4/3},\right.$$
$$\left. 4A_1, 4(C_{\mathrm{t}}+1+3\nu)E_u, 2\right). \tag{4.28b}$$

We now formulate our induction assumption.

Induction assumption. For $\hat{k} \in \mathbb{N}$ and $k_2 \in \mathbb{N}$ with $k_2 \leq \hat{k}$, we say $H_{\hat{k},k_2}$ holds if

$$\begin{aligned} \|r_i^{\beta_i+\alpha_1-2}\mathcal{D}^\alpha u_r\|_{L^2(S^i_{\delta_P/2})} &\leq A_u^{|\alpha|-2}E_u^{[\alpha_2-4/3]_+}(|\alpha|-2)!, \\ \|r_i^{\beta_i+\alpha_1-2}\mathcal{D}^\alpha u_\vartheta\|_{L^2(S^i_{\delta_P/2})} &\leq A_u^{|\alpha|-2}E_u^{[\alpha_2-4/3]_+}(|\alpha|-2)!, \end{aligned} \qquad \forall \alpha \in \mathbb{N}_0^2 : \begin{cases} 2 \leq |\alpha| \leq \hat{k}+1, \\ \alpha_2 \leq k_2+1, \end{cases} \tag{4.29a}$$

and

$$\|r_i^{\beta_i+\alpha_1-1}\mathcal{D}^\alpha p\|_{L^2(S^i_{\delta_P/2})} \leq A_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!, \qquad \forall \alpha \in \mathbb{N}_0^2 : \begin{cases} 1 \leq |\alpha| \leq \hat{k}, \\ \alpha_2 \leq k_2, \end{cases} \tag{4.29b}$$

where A_u and E_u are the constants in (4.28b) and (4.28a).

Strategy of the proof. The proof of the statement will be composed of two main steps. $H_{1,1}$ holds due to Lemma 4.6 and to (4.20). We will show that, for all $k \in \mathbb{N}$,

$$H_{k,k} \implies H_{k+1,1}. \tag{4.30}$$

Then, in the following step, we will show that, for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}$ such that $j \leq k$,

$$H_{k,k} \text{ and } H_{k+1,j} \implies H_{k+1,j+1}. \tag{4.31}$$

Combining (4.30) and (4.31), we obtain that

$$H_{k,k} \implies H_{k+1,k+1}, \tag{4.32}$$

We infer from (4.32) that $H_{k,k}$ is verified for all $k \in \mathbb{N}$. This will conclude the proof.

Step 1: proof of (4.30). We fix $k \in \mathbb{N}$ and suppose that $H_{k,k}$ holds. Define

$$\overline{\boldsymbol{v}} := r^k\partial_r^k\overline{\boldsymbol{u}}, \qquad q := r^k\partial_r^k p. \tag{4.33}$$

Then, for all $|\eta| \leq 2$,

$$r^{\eta_1}\mathcal{D}^\eta\overline{\boldsymbol{v}} = r^k\partial_r^k(r^{\eta_1}\mathcal{D}^\eta\overline{\boldsymbol{u}}) \tag{4.34}$$

and

$$\mathcal{D}^\eta q = r^{k-2}\partial_r^k(r^{\eta_1+1}\mathcal{D}^\eta p) - kr^{k-1}\partial_r^{k-1}\mathcal{D}^\eta p - \eta_1 k(k-1)r^{k-2}\partial_r^{k-1}p. \tag{4.35}$$

Furthermore, multiplying (4.23) by r and differentiating by ∂_r^k we obtain

$$(r\partial_r + (k+1))\partial_r^k u_r + \partial_r^k\partial_\vartheta u_\vartheta = 0,$$

hence

$$0 = r^{k-1}(r\partial_r + (k+1))\partial_r^k u_r + r^{k-1}\partial_\vartheta\partial_r^k u_\vartheta = \frac{1}{r}\left((r\partial_r+1)v_r + \partial_\vartheta v_\vartheta\right) \tag{4.36}$$

From (4.34), (4.35), and (4.36), it follows that the pair $(\overline{\boldsymbol{v}}, q)$ as defined in (4.33) satisfies, with $\overline{L^{\Delta}_{\mathrm{St}}}$ and $\overline{B}$ in polar frame and acting on the velocity field $\bar{u}$ in polar frame as defined in (4.21a) and (4.21b) formally the Stokes boundary value problem

$$
\begin{aligned}
&\overline{L^{\Delta}_{\mathrm{St}}}(\overline{\boldsymbol{v}}, q) = \begin{pmatrix} \widetilde{\boldsymbol{f}} \\ 0 \end{pmatrix}, \quad \text{in } S^i_{\delta_P},\\
&\overline{B}(\overline{\boldsymbol{v}}, q) = \begin{pmatrix} \boldsymbol{0} \\ \widetilde{\boldsymbol{g}} \\ \boldsymbol{0} \end{pmatrix}, \quad \text{on } (\partial S^i_{\delta_P} \cap \Gamma_D) \times (\partial S^i_{\delta_P} \cap \Gamma_N) \times (\partial S^i_{\delta_P} \cap \Gamma_G),.
\end{aligned}
\tag{4.37}
$$

Here, $\widetilde{\boldsymbol{f}}$ and (assuming that $\partial S^i_{\delta_P} \cap \Gamma_N \neq \varnothing$) $\widetilde{\boldsymbol{g}}$ are defined by

$$
\widetilde{\boldsymbol{f}} = r^{k-2}\partial_r^k(r^2(\overline{\boldsymbol{f}} - \overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})) - kr^{k-2}\begin{pmatrix} r\partial_r^k p + (k-1)\partial_r^{k-1}p \\ \partial_r^{k-1}\partial_\vartheta p \end{pmatrix}, \qquad \widetilde{\boldsymbol{g}} = \begin{pmatrix} 0 \\ kr^{k-1}\partial_r^{k-1}p \end{pmatrix}.
\tag{4.38}
$$

Using (4.27a), Lemma 4.5 with $\boldsymbol{w} = \boldsymbol{u}$, the inductive hypothesis $H_{k,k}$, and the fact that for all $v \in L^2(S^i_{\delta_P})$

$$
\|v\|_{L^2(S^i_{\delta_P})} \le \|v\|_{L^2(S^i_{\delta_P/2})} + \|v\|_{L^2(S^i_{\delta_P}\setminus S^i_{\delta_P/2})},
$$

we find from (4.38)

$$
\begin{aligned}
\|\widetilde{\boldsymbol{f}}\|_{L_{\beta_i}(S^i_{\delta_P})} &\le \|r^{\beta_i+k-2}\partial_r^k(r^2\overline{\boldsymbol{f}})\|_{L^2(S^i_{\delta_P})} + \|r^{\beta_i+k-2}\partial_r^k(r^2\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})\|_{L^2(S^i_{\delta_P})}\\
&\quad + k\|r^{\beta_i+k-1}\partial_r^k p\|_{L^2(S^i_{\delta_P})} + k(k-1)\|r^{\beta_i+k-2}\partial_r^{k-1}p\|_{L^2(S^i_{\delta_P})}\\
&\quad + k\|r^{\beta_i+k-2}\partial_r^{k-1}\partial_\vartheta p\|_{L^2(S^i_{\delta_P})}\\
&\le A_1^k k! + \left(C_{\mathrm{t}}A_u^{k-1}E_u^2 + A_1^k\right)k! + k\left(A_u^{k-1} + A_1^{k-1}\right)(k-1)!\\
&\quad + k(k-1)\left(A_u^{k-2} + A_1^{k-2}\right)(k-2)! + k\left(A_u^{k-1}E_u + A_1^{k-1}\right)\\
&\le \left(5A_1^k + (C_{\mathrm{t}}+3)A_u^{k-1}E_u^2\right)k!.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\|\widetilde{\boldsymbol{g}}\|_{W^{1/2}_\beta(\partial S^i_{\delta_P}\cap\Gamma_N)} &\le k\|r^{k-1}\partial_r^{k-1}p\|_{W^1_\beta(S^i_{\delta_P})}\\
&\le k\Big(\|r^{k-2+\beta}\partial_r^{k-1}p\|_{L^2(S^i_{\delta_P})} + \|r^{k-2+\beta}\partial_r^{k-1}\partial_\vartheta p\|_{L^2(S^i_{\delta_P})} + \|r^{k-1+\beta}\partial_r^k p\|_{L^2(S^i_{\delta_P})}\\
&\qquad + (k-1)\|r^{k-2+\beta}\partial_r^{k-1}p\|_{L^2(S^i_{\delta_P})}\Big)\\
&\le 4k\left(A_1^{k-1} + A_u^{k-1}E_u\right)(k-1)!\\
&\le 4\left(A_1^{k-1} + A_u^{k-1}E_u\right)k!.
\end{aligned}
$$

It follows from (4.37), Theorem 3.1, (4.27d), (4.27c), and the two inequalities above that

$$
\begin{aligned}
&\|\overline{\boldsymbol{v}}\|_{W^2_{\beta_i}(S^i_{\delta_P/2})} + \|q\|_{W^1_{\beta_i}(S^i_{\delta_P/2})}\\
&\le C_{\mathrm{sec}}\left(\|\widetilde{\boldsymbol{f}}\|_{L_{\beta_i}(S^i_{\delta_P})} + \|\overline{\boldsymbol{v}}\|_{H^1(S^i_{\delta_P}\setminus S^i_{\delta_P/2})} + \|q\|_{L^2(S^i_{\delta_P}\setminus S^i_{\delta_P/2})} + \|\widetilde{\boldsymbol{g}}\|_{W^{1/2}_\beta(\partial S^i_{\delta_P}\cap\Gamma_N)}\right)\\
&\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+7)A_u^{k-1}E_u^2\right)k!.
\end{aligned}
\tag{4.39}
$$

Now, for all $|\eta| = 2$,

$$
\mathcal{D}^\eta\overline{\boldsymbol{v}} = r^k\partial_r^k\mathcal{D}^\eta\overline{\boldsymbol{u}} + \eta_1 kr^{k-1}\partial_r^{k+\eta_1-1}\partial_\vartheta^{\eta_2}\overline{\boldsymbol{u}} + [\eta_1 - 1]_+k(k-1)r^{k-2}\partial_r^k\overline{\boldsymbol{u}}.
$$

Therefore, for all $|\eta| = 2$,

$$
\begin{aligned}
&\|r^{\beta_i+k+\eta_1-2}\partial_r^k\mathcal{D}^\eta\overline{\boldsymbol{u}}\|_{L^2(S^i_{\delta_P/2})}\\
&\qquad\le \|\overline{\boldsymbol{v}}\|_{W^2_{\beta_i}(S^i_{\delta_P/2})} + \eta_1 k\|r^{\beta_i+k+\eta_1-3}\partial_r^{k+\eta_1-1}\partial_\vartheta^{\eta_2}\overline{\boldsymbol{u}}\|_{L^2(S^i_{\delta_P/2})} + k(k-1)\|r^{\beta_i+k-2}\partial_r^k\overline{\boldsymbol{u}}\|_{L^2(S^i_{\delta_P/2})}\\
&\qquad\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+7)A_u^{k-1}E_u^2\right)k! + 2kA_u^{k-1}(k-1)! + k(k-1)A_u^{k-2}(k-2)!\\
&\qquad\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+9)A_u^{k-1}E_u^2\right)k!.
\end{aligned}
$$

For all $|\eta| = 1$,

$$\mathcal{D}^\eta q = r^k\partial_r^k\mathcal{D}^\eta q + \eta_1 k r^{k-1}\partial_r^k p,$$

hence

$$
\begin{aligned}
\|r^{\beta_i+k+\eta_1-1}\partial_r^k\mathcal{D}^\eta p\|_{L^2(S^i_{\delta_P/2})} &\le \|q\|_{W^1_{\beta_i}(S^i_{\delta_P/2})} + k\|r^{\beta_i+k-1}\partial_r^k p\|_{L^2(S^i_{\delta_P/2})}\\
&\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+7)A_u^{k-1}E_u^2\right)k! + kA_u^{k-1}(k-1)!\\
&\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+8)A_u^{k-1}E_u^2\right)k!.
\end{aligned}
$$

From (4.28b) it follows that for every $k\in\mathbb{N}$

$$\max_{|\eta|=2}\|r^{\beta_i+k+\eta_1-2}\partial_r^k\mathcal{D}^\eta\overline{\boldsymbol{u}}\|_{L^2(S^i_{\delta_P/2})} \le A_u^k k!,\quad \max_{|\eta|=1}\|r^{\beta_i+k+\eta_1-1}\partial_r^k\mathcal{D}^\eta p\|_{L^2(S^i_{\delta_P/2})} \le A_u^k k!,$$

i.e., that $H_{k+1,1}$ holds. We have shown implication (4.30).

Step 2: proof of (4.31). We now fix $j\in\{1,\dots,k\}$ and we assume that $H_{k,k}$ and $H_{k+1,j}$ hold true. Multiply (4.23) by r and differentiate by $\partial_r^{k-j}\partial_\vartheta^{j+1}$ to obtain

$$r\partial_r^{k+1-j}\partial_\vartheta^{j+1}u_r + (k+1-j)\partial_r^{k-j}\partial_\vartheta^{j+1}u_r + \partial_r^{k-j}\partial_\vartheta^{j+2}u_\vartheta = 0.$$

Therefore, using $H_{k+1,j}$,

$$
\begin{aligned}
&\|r^{\beta_i+k-j-2}\partial_r^{k-j}\partial_\vartheta^{j+2}u_\vartheta\|_{L^2(S^i_{\delta_P/2})}\\
&\qquad\le \|r^{\beta_i+k-j-1}\partial_r^{k+1-j}\partial_\vartheta^{j+1}u_r\|_{L^2(S^i_{\delta_P/2})} + k\|r^{\beta_i+k-j-2}\partial_r^{k-j}\partial_\vartheta^{j+1}u_r\|_{L^2(S^i_{\delta_P/2})}\\
&\qquad\le A_u^kE_u^{j-1/3}k! + kA_u^{k-1}E_u^{j-1/3}(k-1)!\\
&\qquad= 2A_u^kE_u^{j-1/3}k!\\
&\qquad\le A_u^kE_u^{j+2/3}k!.
\end{aligned}
\tag{4.40}
$$

This proves the estimate for u_ϑ.

To prove the bound on u_r, multiply the first equation in (4.22) by r^2 and differentiate by $\partial_r^{k-j}\partial_\vartheta^j$, to obtain

$$
\begin{aligned}
\nu\partial_r^{k-j}\partial_\vartheta^{j+2}u_r = &-\nu\left(r^2\partial_r^2 + (2(k-j)+1)r\partial_r + (k-j)^2 - 1\right)\partial_r^{k-j}\partial_\vartheta^j u_r - 2\nu\partial_r^{k-j}\partial_\vartheta^{j+1}u_\vartheta\\
&+ (r^2\partial_r^2 + 2(k-j)r\partial_r + (k-j)(k-j-1))\partial_r^{k-j-1}\partial_\vartheta^j p\\
&- \partial_r^{k-j}\partial_\vartheta^j\left(r^2(\overline{\boldsymbol{f}} - \overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})_r\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|r^{\beta_i+k-j-2}\partial_r^{k-j}\partial_\vartheta^{j+2}u_r\|_{L^2(S^i_{\delta_P/2})} \\
&\quad\leq \Big(A_u^2k! + 2kA_u(k-1)! + k(k-2)(k-2)!\Big)A_u^{k-2}E_u^{[j-4/3]_+} + 2A_u^{k-1}E_u^{j-1/3}(k-1)! \\
&\qquad + \frac{1}{\nu}\Big(A_u^kk! + 2(k-1)A_u^{k-1}(k-1)! + (k-1)(k-2)A_u^{k-2}(k-2)!\Big)E_u^j \\
&\qquad + \frac{1}{\nu}A_1^kk! + \frac{1}{\nu}C_{\mathrm{t}}A_u^{k-1}E_u^{j+2}k! \\
&\quad\leq \left(\frac{1}{\nu}A_1^k + \left(1+\frac{1}{\nu}\right)A_u^kE_u^j + \left(\frac{1}{\nu}(C_{\mathrm{t}}+2)+4\right)A_u^{k-1}E_u^{j+2} + \left(1+\frac{1}{\nu}\right)A_u^{k-2}E_u^j\right)k!. \\
&\quad\leq A_u^kE_u^{j+2/3}k!
\end{aligned}
\tag{4.41}
$$

This provides the estimate for u_r.

Last, consider the second equation of (4.22): multiplying by r^2 and differentiating by $\partial_r^{k-j}\partial_\vartheta^j$ we obtain

$$
\begin{aligned}
r\partial_r^{k-j}\partial_\vartheta^{j+1}p &= \nu\left(r^2\partial_r^2 + (2(k-j)+1)r\partial_r + (k-j)^2 - 1 + \partial_\vartheta^2\right)\partial_r^{k-j}\partial_\vartheta^ju_\vartheta \\
&\quad + 2\nu\partial_r^{k-j}\partial_\vartheta^{j+1}u_r - (k-j)\partial_r^{k-j-1}\partial_\vartheta^{j+1}p + \partial_r^{k-j}\partial_\vartheta^j\left(r^2(\overline{\boldsymbol{f}} - \overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})_\vartheta\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|r^{\beta_i+k-j-1}\partial_r^{k-j}\partial_\vartheta^{j+1}p\|_{L^2(S^i_{\delta_P/2})} \\
&\quad\leq \nu\Big(A_u^2k! + 2kA_u(k-1)! + k(k-2)(k-2)!\Big)A_u^{k-2}E_u^{[j-4/3]_+} \\
&\qquad + \nu A_u^kE_u^{j+1/3}k! + 2\nu A_u^{k-1}E_u^{j-1/3}(k-1)! + (k-1)A_u^{k-2}E_u^{j+1}(k-2)! \\
&\qquad + A_1^kk! + C_{\mathrm{t}}A_u^{k-1}E_u^{j+2}k! \\
&\quad\leq \left(A_1^k + 2\nu A_u^kE_u^{j+1/3} + (C_{\mathrm{t}}+1+3\nu)A_u^{k-1}E_u^{j+2} + A_u^{k-2}E_u^{j+1}\right)k! \\
&\quad\leq A_u^kE_u^{j+1}k!.
\end{aligned}
\tag{4.42}
$$

Then, the estimates in (4.40), (4.41), and (4.42) imply that $H_{k+1,j+1}$ holds true. By the strategy outlined above, this shows implication (4.32) and concludes the proof. ∎

Combining the estimates in each sector with classical results on the analyticity of the solution in the interior of the domain and on regular parts of the boundary, we obtain the weighted analyticity of solutions to the Navier-Stokes equations, stated in the following theorem.

Theorem 4.8. Let $\underline{\beta}_f \in (0,1)^n$ and $\boldsymbol{f} \in [B^0_{\underline{\beta}_f}(P)]^2 \cap \boldsymbol{W}^*$ such that $\|\boldsymbol{f}\|_{\boldsymbol{W}^*} \leq \frac{C_{\mathrm{coer}}^2\nu^2}{4C_{\mathrm{cont}}}$. Suppose that Assumption 1 holds and let $(\boldsymbol{u}, p)$ be the solution to (2.6) with right hand side $\boldsymbol{f}$.

Then, there exists $\underline{\beta} \in (0,1)^n$, with $\underline{\beta} \geq \underline{\beta}_f$ such that

$$
(\boldsymbol{u}, p) \in [\mathcal{K}^{\varpi}_{2-\underline{\beta}}(P)]^2 \times \mathcal{K}^{\varpi}_{1-\underline{\beta}}(P)
$$

Proof. The analyticity of weak solutions $(\boldsymbol{u}, p)$ in the interior and up to analytic parts of the boundary is classical, see, e.g., [16, 7].

It therefore remains to prove weighted analytic regularity near corners of P. To this end, we apply Lemma 4.7 to obtain non-overlapping sectors $S^i_{\delta_P}$ such that

$$
(\overline{\boldsymbol{u}}, p) \in [\mathcal{B}^2_{\beta_i}(S^i_{\delta_P/2})]^2 \times \mathcal{B}^1_{\beta_i}(S^i_{\delta_P/2}), \qquad \forall i \in \{1,\dots,n\}.
$$

Furthermore, $\overline{\boldsymbol{u}}(\mathfrak{c}) = \mathbf{0}$ for all corners $\mathfrak{c} \in \partial P$ due to Assumption 1, Item 2. This implies, by Lemma 2.4,

$$(\boldsymbol{u}, p) \in [B^2_{\beta_i}(S^i_{\delta_P/2})]^2 \times B^1_{\beta_i}(S^i_{\delta_P/2}), \qquad \forall i \in \{1, \dots, n\}.$$

Therefore, in particular, $\boldsymbol{u} \in [H^{2,2}_{\beta}(S^i_{\delta_P/2})]^2$ and $p \in H^{1,1}_{\beta}(S^i_{\delta_P/2})$. By Lemma 2.8, we obtain that $\boldsymbol{u} \in [\mathcal{K}^2_{2-\beta}(S^i_{\delta_P/2})]^2$ and that $p \in \mathcal{K}^1_{1-\beta}(S^i_{\delta_P/2})$. By definition, $B^{\ell}_{\beta}(S^i_{\delta_P/2}) \cap \mathcal{K}^{\ell}_{\ell-\beta}(S^i_{\delta_P/2}) = \mathcal{K}^{\varpi}_{\ell-\beta}(S^i_{\delta_P/2})$ and this concludes the proof. ∎

We remark that the argument in the proof, in particular, provides also

$$(\boldsymbol{u}, p) \in [B^2_{\underline{\beta}}(P)]^2 \times B^1_{\underline{\beta}}(P) \ .$$

5. Conclusion and Discussion.

We have shown analytic regularity of Leray-Hopf solutions of the stationary, viscous and incompressible Navier-Stokes equations in polygonal domains P, subject to sufficiently small and analytic in $\overline{P}$ forcing. Our result holds under Assumption 1, which implies that for each corner point of P, at least of the sides at that point has homogeneous Dirichlet ("no-slip") BCs. We proved analytic regularity in scales of corner-weighted, Kondrat'ev spaces with homogeneous weight functions. The present setting of mixed BCs covers most of the example of interest in applications, such as, e.g., channel flow with homogeneous Neumann condition at the outflow boundary. With the argument in [15] containing a gap, in the particular case of homogeneous Dirichlet ("no-slip") boundary conditions on all of ∂P the present result implies that the result in [22] stands under the assumptions stated in [22]. The analytic regularity in homogeneous weighted spaces implies, as explained in the discussion in [22, Section 5], corresponding bounds on n-widths of solution sets which, in turn, imply exponential convergence or reduced basis and of Model Order Reduction methods. Corresponding remarks imply also in the present, more general situation, and we do not spell them out here. The present results also imply, along the lines of [22] (where only the case of no-slip BCs on all of ∂P was considered), exponential rates of convergence of hp-approximations. Details on the exponential convergence rate bounds for further discretizations in the case of the presently considered mixed boundary conditions shall be elaborated elsewhere. Likewise, analytic regularity of flows for the remaining combinations of boundary conditions not covered in the present analysis, can be established in a similar fashion. To that end, however, a different functional setting of corner-weighted Kondrat'ev spaces with nonhomogeneous weights is necessary. The details, also of the bootstrapping analysis for the nonlinearity, will be developed elsewhere.

Appendix A. Proofs of Section 2.5.3.

Proof of Lemma 2.4. The third item of Lemma 2.6 and the second item of Lemma 2.7 give that for any $\ell \in \{0, 1, 2\}$ there exists a constant $A_0 > 1$ such that for any $\alpha \in \mathbb{N}^2_0$,

$$\|r^{\beta+\alpha_1-\ell}\mathcal{D}^{\alpha}\overline{\boldsymbol{u}}\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \leq A_0^{|\alpha|+1}|\alpha|!.$$

Then we have

$$\|r^{\beta-\ell}\boldsymbol{u}\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \leq 4\|r^{\beta-\ell}\overline{\boldsymbol{u}}\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))},$$

and for all $|\alpha| \geq 1$,

$$\begin{aligned}
\|r^{\beta+\alpha_1-\ell}\mathcal{D}^{\alpha}u_1\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} &\leq \sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial_{\vartheta}^j \cos\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\beta+\alpha_1-\ell}\partial_r^{\alpha_1}\partial_{\vartheta}^{\alpha_2-j}u_r\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \\
&\quad + \sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial_{\vartheta}^j \sin\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\beta+\alpha_1-\ell}\partial_r^{\alpha_1}\partial_{\vartheta}^{\alpha_2-j}u_{\vartheta}\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \\
&\leq 2A_0^{|\alpha|+1}|\alpha|!\sum_{j=0}^{\alpha_2}A_0^{-j}\binom{\alpha_2}{j} \leq 2(2A_0)^{|\alpha|+1}|\alpha|!.
\end{aligned}$$

A similar estimate holds for u_2. By the above results and using the third item of Lemma 2.6 and the first item of Lemma 2.7 we have $\boldsymbol{u} \in [\mathcal{B}^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))]^2$, which, by the second item of Lemma 2.6, is equivalent to $\boldsymbol{u} \in [B^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))]^2$. ∎

Proof of Lemma 2.5From $\boldsymbol{v} \in [B^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))]^2$ it follows that $\boldsymbol{v} \in [\mathcal{B}^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))]^2$ by [2, Theorem 1.1]. Then, there exists $A_0 > 1$ such that, for all $|\alpha| \geq 1$,

$$\begin{aligned}\|r^{\alpha_1+\beta}\mathcal{D}^\alpha v_r\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} &\leq \sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial_\vartheta^j \cos\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\alpha_1+\beta}\partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2-j}v_1\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))}\\ &\quad+\sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial_\vartheta^j \sin\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\alpha_1+\beta}\partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2-j}v_2\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))}\\ &\leq 2A_0^{|\alpha|}|\alpha|!\sum_{j=0}^{\alpha_2}A_0^{-j}\binom{\alpha_2}{j} \leq 2(2A_0)^{|\alpha|}|\alpha|!.\end{aligned}$$

The estimate for v_ϑ follows by the same argument. ∎

Proof of Lemma 2.8Lemma 2.7 implies that $v \in W^k_\beta(Q_{\delta,\omega}(\mathfrak{c}))$. Elementary calculus yields

$$\begin{aligned}\partial_{x_1} &= \cos\vartheta\partial_r - \frac{\sin\vartheta}{r}\partial_\vartheta,\\ \partial_{x_2} &= \sin\vartheta\partial_r + \frac{\cos\vartheta}{r}\partial_\vartheta,\\ \partial^2_{x_1} &= \cos^2\vartheta\partial_r^2 + \frac{2\cos\vartheta\sin\vartheta}{r^2}\partial_\vartheta + \frac{\sin^2\vartheta}{r}\partial_r - \frac{2\cos\vartheta\sin\vartheta}{r}\partial_{r\vartheta} + \frac{\sin^2\vartheta}{r^2}\partial^2_\vartheta,\\ \partial^2_{x_2} &= \sin^2\vartheta\partial_r^2 - \frac{2\cos\vartheta\sin\vartheta}{r^2}\partial_\vartheta + \frac{\cos^2\vartheta}{r}\partial_r + \frac{2\cos\vartheta\sin\vartheta}{r}\partial_{r\vartheta} + \frac{\cos^2\vartheta}{r^2}\partial^2_\vartheta,\\ \partial_{x_1}\partial_{x_2} &= \cos\vartheta\sin\vartheta\partial_r^2 + \frac{\sin^2\vartheta-\cos^2\vartheta}{r^2}\partial_\vartheta + \frac{\cos^2\vartheta-\sin^2\vartheta}{r}\partial_{r\vartheta} - \frac{\sin\vartheta\cos\vartheta}{r}\partial_r - \frac{\sin\vartheta\cos\vartheta}{r^2}\partial^2_\vartheta.\end{aligned}$$

Therefore there exists $C > 0$ ($C = 7$ when $k = 2$ and $C = 2$ when $k = 1$ will suffice) such that for any $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \leq k$,

$$\|r^{\beta-k+|\alpha|}\partial^\alpha v\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \leq C\left(\sum_{|\alpha|\leq k}\|r^{\beta-k+\alpha_1}\mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega}(\mathfrak{c}))}\right)^{1/2} = C\|v\|_{W^k_\beta(Q_{\delta,\omega}(\mathfrak{c}))}.$$

By definition, it follows that $v \in \mathcal{K}^k_{k-\beta}(Q_{\delta,\omega}(\mathfrak{c}))$. ∎

Appendix B. Stokes operator in polar coordinates. In this appendix we provide the elementary calculations to verify (4.22)-(4.26), which describe the NSE with boundary conditions in polar coordinates and polar components. We recall the representation of the NSE in the Cartesian reference frame

$$L^\Delta_{\mathrm{St}}(\boldsymbol{u},p) = \begin{pmatrix}\boldsymbol{f}-(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\\ 0\end{pmatrix} \quad \text{in } S^i_{\delta_P}, \tag{B.1}$$

$$B(\boldsymbol{u},p) = \boldsymbol{0} \quad \text{on } \Gamma_\delta. \tag{B.2}$$

The vector Laplacian in a polar reference frame reads [1, Equation (3.151)]

$$\overline{\Delta\boldsymbol{u}} = \frac{1}{r^2}\begin{pmatrix}(r\partial_r)^2+\partial_\vartheta^2-1 & -2\partial_\vartheta\\ 2\partial_\vartheta & (r\partial_r)^2+\partial_\vartheta^2-1\end{pmatrix}\overline{\boldsymbol{u}}$$

and [14, Equation (II.4.C3)]

$$\overline{\nabla p} = \begin{pmatrix} \partial_r p \\ r^{-1}\partial_\vartheta p \end{pmatrix}.$$

The divergence of $\overline{\boldsymbol{u}}$ is [14, Equation (II.4.C5)] $\nabla \cdot \overline{\boldsymbol{u}} = \frac{1}{r}\left((r\partial_r + 1)\, u_r + \partial_\vartheta u_\vartheta\right)$, whence (4.22) and (4.23).

Regarding the boundary conditions (B.2), we start from the expression of the stress tensor in polar coordinates and polar frame, see [14, Equation (II.4.C9)],

$$\overline{\varepsilon(\boldsymbol{u})} = \begin{pmatrix} \partial_r u_r & \frac{1}{2}(\partial_r u_\vartheta + r^{-1}(\partial_\vartheta u_r - u_\vartheta)) \\ \frac{1}{2}(\partial_r u_\vartheta + r^{-1}(\partial_\vartheta u_r - u_\vartheta)) & r^{-1}(\partial_\vartheta u_\vartheta + u_r) \end{pmatrix} \tag{B.3}$$

hence the stress tensor in a polar reference frame reads

$$\overline{\sigma(\boldsymbol{u}, p)} = 2\nu\overline{\varepsilon(\boldsymbol{u})} - p\,\mathrm{Id}_2 = \nu \begin{pmatrix} 2\partial_r u_r & \partial_r u_\vartheta + r^{-1}(\partial_\vartheta u_r - u_\vartheta) \\ \partial_r u_\vartheta + r^{-1}(\partial_\vartheta u_r - u_\vartheta) & 2r^{-1}(\partial_\vartheta u_\vartheta + u_r) \end{pmatrix} - p\,\mathrm{Id}_2\,. \tag{B.4}$$

We have furthermore

$$\overline{\boldsymbol{n}} = \pm\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \overline{\boldsymbol{t}} = \mp\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where the sign depends on the side of the sector being considered. Then, by matrix-vector multiplication,

$$\overline{\sigma(\boldsymbol{u}, p)\boldsymbol{n}} = \pm\nu\begin{pmatrix} \partial_r u_\vartheta + r^{-1}(\partial_\vartheta u_r - u_\vartheta) \\ 2r^{-1}(\partial_\vartheta u_\vartheta + u_r) - p \end{pmatrix}$$

and consequently

$$(\sigma(\boldsymbol{u}, p)\boldsymbol{n}) \cdot \boldsymbol{t} = \overline{\sigma(\boldsymbol{u}, p)\boldsymbol{n}} \cdot \overline{\boldsymbol{t}} = -\partial_r u_\vartheta - \frac{1}{r}(\partial_\vartheta u_r - u_\vartheta).$$

Finally, it follows from the definition that $\boldsymbol{u} \cdot \boldsymbol{n} = \overline{\boldsymbol{u}} \cdot \overline{\boldsymbol{n}} = \pm u_\vartheta$.

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