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ANALYSIS OF A MONTE-CARLO NYSTROM METHOD*

FLORIAN FEPPON† AND HABIB AMMARI‡

Abstract. This paper considers a Monte-Carlo Nystrom method for solving integral equations of the second kind, whereby the values $(z(y_i))_{1\le i\le N}$ of the solution z at a set of N random and independent points $(y_i)_{1\le i\le N}$ are approximated by the solution $(z_{N,i})_{1\le i\le N}$ of a discrete, N-dimensional linear system obtained by replacing the integral with the empirical average over the samples $(y_i)_{1\le i\le N}$. Under the unique assumption that the integral equation admits a unique solution $z(y)$, we prove the invertibility of the linear system for sufficiently large N with probability one, and the convergence of the solution $(z_{N,i})_{1\le i\le N}$ towards the point values $(z(y_i))_{1\le i\le N}$ in a mean-square sense at a rate $O(N^{-\frac{1}{2}})$. For particular choices of kernels, the discrete linear system arises as the Foldy-Lax approximation for the scattered field generated by a system of N sources emitting waves at the points $(y_i)_{1\le i\le N}$. In this context, our result can equivalently be considered as a proof of the well-posedness of the Foldy-Lax approximation for systems of N point scatterers, and of its convergence as $N\to+\infty$ in a mean-square sense to the solution of a Lippmann-Schwinger equation characterizing the effective medium. The convergence of Monte-Carlo solutions at the rate $O(N^{-1/2})$ is numerically illustrated on 1D examples and for solving a 2D Lippmann-Schwinger equation.

Key words. Monte-Carlo method, Nystrom method, Foldy-Lax approximation, point scatterers, effective medium.

AMS subject classifications. 65R20, 65C05, 47B80, 78M40.

1. Introduction.

Let $\Omega\subset\mathbb{R}^d$ be a bounded Lipschitz domain of dimension $d\in\mathbb{N}$. This paper is concerned with the stability and convergence analysis of a Monte-Carlo Nystrom method for approximating the solution of integral equations of the second kind of the form

$$z(y)+\int_\Omega k(y,y')z(y')\rho(y')\mathrm{d}y'=f(y),\qquad y\in\Omega,\tag{1.1}$$

where $z\in L^2(\Omega,\mathbb{C})$ is the unknown, $\rho\in L^\infty(\Omega,\mathbb{R}^+)$ is a probability distribution (satisfying $\rho\ge 0$ in Ω and $\int_\Omega\rho(y')\mathrm{d}y'=1$), $k\in L^\infty(\Omega,L^2(\Omega,\mathbb{C}))$ is a square integrable kernel and $f\in L^2(\Omega,\mathbb{C})$ is a square integrable right-hand side: more precisely we assume

$$||k||^2_{L^\infty(L^2(\Omega))}:=\sup_{y'\in\Omega}\int_\Omega|k(y,y')|^2\mathrm{d}y<+\infty,\ ||f||_{L^2(\Omega)}:=\int_\Omega|f(y)|^2\mathrm{d}y<+\infty.\tag{1.2}$$

Let $(y_i)_{1\le i\le N}$ be a set of N points drawn independently from the distribution $\rho(y')\mathrm{d}y'$ in the domain Ω. We consider the approximation of (1.1) by the N dimensional linear system

$$z_{N,i}+\frac{1}{N}\sum_{j\ne i}k(y_i,y_j)z_{N,j}=f(y_i),\qquad 1\le i\le N,\tag{1.3}$$

where the integral of (1.1) has been replaced with the empirical average. Assuming that (1.1) is well-posed, it is a natural question to ask whether the linear system

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(1.3) admits a unique solution, and if there is a sort of convergence of $(z_{N,i})_{1\leq i\leq N}$ towards the vector $(z(y_i))_{1\leq i\leq N}$. The goal of this paper is to provide a quantitative and positive answer to this problem: our main result is given in Proposition 3.6 below, where we prove without further assumption that there exists an event $\mathcal{H}_{N_0}$ (specified in (3.5) below) satisfying $\mathbb{P}(\mathcal{H}_{N_0})\to 1$ as $N_0\to+\infty$ such that the linear system (1.3) is well-posed for any $N\geq N_0$ when $\mathcal{H}_{N_0}$ is realized. Moreover, we prove that there exists a constant $C>0$ independent of N such that

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|z_{N,i}-z(y_i)|^2\,\middle|\,\mathcal{H}_{N_0}\right]^{\frac{1}{2}}\leq CN^{-\frac{1}{2}}. \tag{1.4}$$

We also obtain in the meantime the convergence of the Nystrom interpolant

$$z_N(y):=f(y)-\frac{1}{N}\sum_{i=1}^{N}k(\cdot,y_i)z_{N,i} \tag{1.5}$$

towards the function z solution to (1.1) in the following mean-square sense:

$$\mathbb{E}[||z_N-z||^2_{L^2(\Omega)}|\mathcal{H}_{N_0}]^{\frac{1}{2}}\leq CN^{-\frac{1}{2}}. \tag{1.6}$$

If the mean-square error rate of $N^{-\frac{1}{2}}$ is to be expected for such Monte-Carlo method, the analysis of (1.3) is not completely standard because the variables $(z_{N,i})_{1\leq i\leq N}$ depend on the joint distribution of the full set of points $(y_i)_{1\leq i\leq N}$. In particular, these are not independent random variables and the correlations $\mathbb{E}[\langle z_{N,i}-z(y_i),z_{N,j}-z(y_j)\rangle]$ do not vanish in (1.4) and (1.6).

The convergence rate $O(N^{-\frac{1}{2}})$ may not seem competitive when compared to standard (deterministic) Nystrom methods for two or three-dimensional domains Ω which are known to converge at the same rate as the quadrature rule considered for the numerical integration in (1.1); see e.g. [4, 24]. However, if is independent of the dimension d, which may prove beneficial if one wish to solve (1.1) in large dimensions. We further note that the system (1.3) is rather easy to implement in any dimension and does not require a particular treatment of the singularity such as product integration in deterministic Nystrom methods (see e.g. section 11.5 in [4]). Actually, Esmaeili et al. recently proposed a convergence analysis of a variant of the scheme (1.3) involving radial kernel functions [16]. The authors still rely on a Monte-Carlo approximation for estimating the integral of (1.1); the main difference with our analysis lies in the fact that they assume the kernel k to be continuous, which can be limiting for practical applications where k is singular.

There is further an important physical motivation for studying the convergence of the solution of the linear system (1.3) to the one of the integral equation (1.1), which arises in its connexion with the Foldy-Lax approximation [18, 27, 29] used to understand multiple scattering of waves. For instance, if Ω is a domain containing N tiny acoustic obstacles located at the points $(y_i)_{1\leq i\leq N}$ and illuminated with an incoming sound wave f, the Foldy-Lax approximation assumes that the scattered wave u_s can be approximated by the contribution of N point sources emitting sound waves with intensity $z_{i,N}$:

$$u_s(y)\simeq-\frac{1}{N}\sum_{i=1}^{N}z_{i,N}\Gamma^k(y-y_i), \tag{1.7}$$

see e.g. [12] for a justification and the references therein. In (1.7), Γ^k is the fundamental solution to the Helmoltz equation with wave number $k \in \mathbb{R}$, e.g. $\Gamma^k(x) = -\frac{e^{ik|x|}}{4\pi|x|}$ in three dimensions, and we choose to normalize the amplitudes $(z_{i,N})_{1\leq i\leq N}$ without loss of generality by the factor $-\frac{1}{N}$ so as to emphasize the connexion with (1.3). The scattered intensity $z_{i,N}$ at the point y_i is determined by assuming that it is the sum of the intensity of the source wave f received at the location y_i and of the contributions $(z_{j,N}\Gamma^k(y_i - y_j)/N)_{1\leq j\neq i\leq N}$ of the waves scattered from the other obstacles at the points $(y_j)_{1\leq j\neq i\leq N}$. This yields the linear system (1.3) with the kernel $k(y, y') := \Gamma^k(y - y')$:

$$z_{i,N} + \frac{1}{N}\sum_{j\neq i}\Gamma^k(y_i - y_j)z_{j,N} = f(y_i), \quad 1 \leq i \leq N. \tag{1.8}$$

In this context, the result of Proposition 3.6 states that the scattered intensity $(z_{i,N})$ converges to the solution of the Lippmann-Schwinger equation

$$z(y) + \int_\Omega \Gamma^k(y - y')z(y')\mathrm{d}y' = f(y), \qquad y \in \Omega, \tag{1.9}$$

in a mean-square sense at the rate $O(N^{-\frac{1}{2}})$. Let us note that linear systems analogous to (1.8) occur in many applications (such as in the classical Nystrom method for solving (1.9)) and can be solved efficiently with the Fast Multipole Method (FMM) from Greengard and Rohklin [21] or some alternatives such as the Efficient Bessel Decomposition [6]. For instance, the FMM was used in [34, 19] to speed up the computation of matrix-vector products by iterative linear solvers, or by [25] for computing the wave scattered by a collection of large number of acoustic obstacles.

The Foldy-Lax approximation arises in various works concerned with the understanding of heat diffusion or wave propagation in heterogeneous media [17, 23, 3, 30, 14, 11, 13, 2, 28], where the integral equation (1.9) characterizes the effective medium. In [17, 23, 30] the convergence of the intensities $(z_{N,i})$ of the point scatterers towards the continuous field $z(y)$ is obtained under smallness assumptions on the integral kernel $k(y, y')$ which allows to obtain the well-posedness of (1.3) by treating it as a perturbation of the equation $(z_{i,N})_{1\leq i\leq N} = (f(y_i))_{1\leq i\leq N}$. In [3], quantitative convergence estimates are derived by assuming several strong ergodicity conditions on the distribution of points $(y_i)_{1\leq i\leq N}$ which can be difficult to realize with independent random samples, e.g. Assumptions 2.3 to 2.5 in this reference; see also [20] for a discussion on their limiting aspects. Challa et al. [14, 13] followed Maz'ya et al. [30, 31] where the well-posedness of some variants of (1.8) is proved for arbitrary distributions $(y_i)_{1\leq i\leq N}$ by assuming the geometric condition $\cos(k|y_i - y_j|) \geq 0$ for $1 \leq i \neq j \leq N$, which is realized if Ω has a small diameter. As one can expect, the proofs depend very much on the properties of the kernel k and on very technical assumptions made on the distribution of points.

In this article we justify the well-posedness of (1.3) for a general kernel k and the convergence of the sequence $(z_N)_{N\in\mathbb{N}}$ in the context of random and independent distributions of points $(y_i)_{1\leq i\leq N}$ under the *minimal* condition that the continuous limit model (1.1) is well-posed (assumption (H1) below). Our proof adapts arguments used in the convergence analysis of classical Nystrom methods [4, 24] and outlines as follows. We start by reformulating (1.3) as the finite range functional equation

$$z_N(y) + \frac{1}{N}\sum_{j=1}^{N} k(y, y_j)z_N(y_j) = f(y), \quad \forall y \in \Omega \tag{1.10}$$

where one sets $k(y,y)=0$ on the diagonal. Classically, the invertibility of the problems (1.3) and (1.10) are equivalent and it holds $z_N(y_i)=z_{N,i}$ for $1\leq i\leq N$. Equation (1.10) can be reformulated as

$$\left(\mathrm{I}+\frac{1}{N}\sum_{i=1}^{N}A_i\right)z_N=f, \tag{1.11}$$

where I is the identity operator and $(A_i)_{1\leq i\leq N}$ are independent realizations of the operator valued random variable

$$\begin{array}{rcccl} A_i & : & L^2(\Omega,\mathbb{C}) & \to & L^2(\Omega,\mathbb{C}) \\ & & z & \mapsto & k(\cdot,y_i)z(y_i). \end{array} \tag{1.12}$$

Note that despite (1.12) considers point-wise values $z(y_i)$ of square integrable functions $z\in L^2(\Omega,\mathbb{C})$, the random operators $(A_i)_{1\leq i\leq N}$ are well-defined because (1.12) makes sense for almost any $y_i\in\Omega$; this subtlety is clarified in section 2 below. We then prove in Proposition 2.7 the convergence

$$\frac{1}{N}\sum_{i=1}^{N}A_i\to\mathbb{E}[A], \tag{1.13}$$

where $\mathbb{E}[A]$ is the expectation of any single instance $A\equiv A_i$ of the random operators $(A_i)_{1\leq i\leq N}$:

$$\mathbb{E}[A]\,:\,z\mapsto\int_\Omega k(\cdot,y)z(y)\rho(y)\mathrm{d}y.$$

The convergence (1.13) holds in the operator norm. This allows to obtain the invertibility of (1.3) and the convergence of the resolvent:

$$\left(\lambda\mathrm{I}-\frac{1}{N}\sum_{i=1}^{N}A_i\right)^{-1}\to(\lambda\mathrm{I}-\mathbb{E}[A])^{-1} \tag{1.14}$$

for any λ sufficiently close to -1. Finally, (1.14) imposes some control on the spectrum of $\frac{1}{N}\sum_{i=1}^{N}A_i$ which enables to prove that the linear system (1.3) is well-conditioned (Proposition 3.5) and to obtain the point-wise error bound (1.4) (Proposition 3.6).

The paper is organized in three parts. Section 2 introduces a simple theory of bounded random operators of $L^2(\Omega,\mathbb{C})\to L^2(\Omega,\mathbb{C})$ in which the law of large number (1.13) and the convergence (1.14) hold. This framework is then applied to the particular case of the operators (1.12) in section 3 in order to prove the well-posedness of (1.3) and the error bounds (1.4) and (1.6). The last section 4 illustrates the above results and the predicted convergence rate of order $O(N^{-\frac{1}{2}})$ on numerical 1D and 2D examples.

Before we proceed, let us note that the analysis proposed to this paper can be extended easily to many variants of (1.1). For instance, the result of Proposition 3.6 holds true if the domain Ω is replaced with a codimension one surface in (1.1). Similar results would also extend for first kind integral equations.

2. Bounded random operators $L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$. There is a well established literature on random operators on Banach spaces where one can prove variant of the law of large numbers (1.13) in very general and abstract settings, with some applications in the field of random integral equations [10, 22, 33]. Here, we rather consider a simple and generic probability framework which is sufficient for the purpose of the convergence analysis of the discrete linear system (1.3) towards the second kind integral equation (1.1).

DEFINITION 2.1. *We say that a mapping $A : \Omega \times L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ is a random operator $L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ if:*
(i) $\phi \mapsto A(y,\phi)$ is a linear operator $L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ for almost any $y \in \Omega$;
(ii) $(x,y) \mapsto A(y,\phi(x))$ is a measurable function of $\Omega \times \Omega$ for any $\phi \in L^2(\Omega,\mathbb{C})$.

Note that in our context, the arguments of random operators $L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ are *deterministic* square integrable functions $\phi \in L^2(\Omega,\mathbb{C})$. For simplicity, we denote by $A\phi$ the mapping $y \mapsto A(y,\phi(\cdot))$ and we think of $A\phi$ as a random field of $L^2(\Omega,\mathbb{C})$ and of A as an operator valued random operator. With a slight abuse of notation, we may write $A : L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$, even if A is strictly speaking a mapping $\Omega \times L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$.

If $A : L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ is *any* continuous linear operator, we denote by $|||A|||$ the operator norm

$$|||A||| := \inf_{\phi \in L^2(\Omega,\mathbb{C})} \frac{||A\phi||_{L^2(\Omega)}}{||\phi||_{L^2(\Omega)}}.$$

In case A is a random operator $L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$, the quantity $|||A|||$ is a real random variable. For our applications we consider the class of random operators for which $|||A|||$ is square integrable, which is a sufficient condition for the existence of the expectation $\mathbb{E}[A]$ as a deterministic operator $L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$.

DEFINITION 2.2. *A random operator $A : L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ is said to be* bounded *if $\mathbb{E}[|||A|||^2]^{\frac{1}{2}} < +\infty$, or in other words if there exists a constant $C > 0$ such that*

$$\forall \phi \in L^2(\Omega,\mathbb{C}),\ \mathbb{E}[||A\phi||^2_{L^2(\Omega)}]^{\frac{1}{2}} \le C||\phi||_{L^2(\Omega)}. \tag{2.1}$$

DEFINITION 2.3. *Let $A : L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ be a bounded random operator. The (deterministic) operator defined for any $\phi \in L^2(\Omega,\mathbb{C})$ by the formula:*

$$\mathbb{E}[A]\phi := \mathbb{E}[A\phi] \tag{2.2}$$

determines an operator $\mathbb{E}[A] : L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C})$ and is called the expected value *of A. Furthermore, the following bounds hold true:*

$$|||\mathbb{E}[A]||| \le \mathbb{E}[|||A|||^2]^{\frac{1}{2}} \tag{2.3}$$

$$\mathbb{E}[|||A - \mathbb{E}[A]|||^2] \le \mathbb{E}[|||A|||^2]^{\frac{1}{2}}. \tag{2.4}$$

Proof. It is sufficient to prove (2.3) in order to show that $\mathbb{E}[A]$ is an operator of $L^2(\Omega,\mathbb{C})$. For $\phi \in L^2(\Omega,\mathbb{C})$, Jensen's inequality implies

$$\begin{aligned}\int_\Omega |\mathbb{E}[A\phi](x)|^2 \mathrm{d}x &= \int_\Omega \left| \int_\Omega [A(y)\phi](x)\rho(y)\mathrm{d}y \right|^2 \mathrm{d}x \le \int_\Omega \int_\Omega |[A(y)\phi](x)|^2 \rho(y)\mathrm{d}y\mathrm{d}x \\ &\le \int_\Omega ||A(y)\phi||^2_{L^2(\Omega)} \rho(y)\mathrm{d}y = \mathbb{E}[||A\phi||^2] \le \mathbb{E}[|||A|||^2]\, ||\phi||^2_{L^2(\Omega)}.\end{aligned}$$

The bound (2.4) is then obtained by observing that for any $\phi \in L^2(\Omega, \mathbb{C})$,

$$\mathbb{E}[||(A-\mathbb{E}[A])\phi||^2] = \mathbb{E}[||A\phi||^2_{L^2(\Omega)}] - ||\mathbb{E}[A]\phi||^2_{L^2(\Omega)} \leq \mathbb{E}[||A\phi||^2_{L^2(\Omega)}] \leq \mathbb{E}[|||A|||^2]||\phi||^2_{L^2(\Omega)}. \quad \square \tag{2.5}$$

In order to prove a law of large numbers result of the type of (1.13), we consider the following definition of independent operator valued random variables.

Definition 2.4. *Let $(A_i)_{i\in\mathbb{N}}$ be a family of bounded random operators*

$$A_i : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C}).$$

The operators $(A_i)_{i\in\mathbb{N}}$ are said to be mutually independent if for any $i \neq j$ and any $\phi, \psi \in L^2(\Omega, \mathbb{C})$, it holds

$$\mathbb{E}[\langle A_i\phi, A_j\psi\rangle] = \langle \mathbb{E}[A_i]\phi, \mathbb{E}[A_j]\psi\rangle. \tag{2.6}$$

Remark 2.5. This definition of independence is rather weak, but sufficient for our purpose. A stronger definition could be to require the identity

$$\mathbb{E}[\langle f(A_i)\phi, g(A_j)\psi\rangle] = \langle \mathbb{E}[f(A_i)]\phi, \mathbb{E}[g(A_j)]\psi\rangle$$

for any functions f and g such that $f(A_i)$ and $g(A_j)$ can be defined by mean of the functional Riesz-Dunford's calculus [9, 26].

Lemma 2.6. *Let $(y_i)_{i\in\mathbb{N}}$ be a sequence of independent realizations of the distribution $\rho(x)\mathrm{d}x$. If $A : \Omega \times L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ is a random operator, then $(A(y_i, \cdot))_{i\in\mathbb{N}}$ are independent realizations of the random operator A.*

Proof. The fact that $(y_i)_{i\in\mathbb{N}}$ is a sequence of independent random real variables means strictly speaking that each random variable $y_i : \Omega \to \Omega$ is the identity mapping and that

$$\mathbb{E}[\psi(y_i, y_j)] := \int_\Omega \int_\Omega \psi(y, y')\rho(y)\rho(y')\mathrm{d}y\mathrm{d}y' \tag{2.7}$$

for any $i \neq j$ and any integrable multivariate function $\psi : \Omega \times \Omega \to \mathbb{C}$. Then $A_i \equiv A(y_i, \cdot)$ is defined as the composition of A with y_i; it is of course a realization of the random operator A. Then by using (2.7), we obtain the independence (2.6):

$$\begin{aligned}\mathbb{E}[\langle A_i\phi, A_j\psi\rangle] &= \int_\Omega \int_\Omega \int_\Omega A_i(y, \phi)(x) A_j(y', \psi)(x)\rho(y)\rho(y')\mathrm{d}y\mathrm{d}y'\mathrm{d}x \\ &= \int_\Omega \left(\int_\Omega A_i(y, \phi)(x)\rho(y)\mathrm{d}y\right)\left(\int_\Omega A_j(y', \psi)(x)\rho(y')\mathrm{d}y'\right)\mathrm{d}x \\ &= \langle \mathbb{E}[A_i]\phi, \mathbb{E}[A_j]\psi\rangle. \quad \square\end{aligned}$$

We have now all the ingredients for stating a version of the law of large number in the present context of bounded random operators.

Proposition 2.7. *Let $(A_i)_{i\in\mathbb{N}}$ be a family of independent realizations of a given bounded random operator $A : L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$. Then as $N \to +\infty$,*

$$\frac{1}{N}\sum_{i=1}^N A_i \longrightarrow \mathbb{E}[A],$$

where the convergence holds at the rate $O(N^{-\frac{1}{2}})$ in the following mean-square sense:

$$\mathbb{E}\left[\left|\left|\left|\frac{1}{N}\sum_{i=1}^{N}A_i-\mathbb{E}[A]\right|\right|\right|^2\right]^{\frac{1}{2}}\leq\frac{\mathbb{E}[|||A-\mathbb{E}[A]|||^2]^{\frac{1}{2}}}{\sqrt{N}}\ \textit{for any } N\in\mathbb{N}.$$

Proof. The independence of the random operators implies that for $j\neq i$ and any $\phi\in L^2(\Omega,\mathbb{C})$:

$$\mathbb{E}[\langle(A_i-\mathbb{E}[A])\phi,(A_j-\mathbb{E}[A])\phi\rangle]=0.$$

Then, for any $\phi\in L^2(\Omega)$,

$$\mathbb{E}\left[\left|\left|\left(\frac{1}{N}\sum_{i=1}^{N}A_i-\mathbb{E}[A]\right)\phi\right|\right|^2_{L^2(\Omega)}\right]=\frac{1}{N^2}\mathbb{E}\left[\left|\left|\sum_{i=1}^{N}(A_i-\mathbb{E}[A])\phi\right|\right|^2_{L^2(\Omega)}\right]$$
$$=\frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}[||(A_i-\mathbb{E}[A])\phi||^2_{L^2(\Omega)}]=\frac{1}{N}\mathbb{E}[||(A-\mathbb{E}[A])\phi||^2_{L^2(\Omega)}].$$

The result follows. ∎

We conclude this section with a useful convergence result for the resolvent sets of the operator $\frac{1}{N}\sum_{i=1}^{N}A_i$. This statement turns out to be essential for establishing the point-wise estimate (1.4) in the next section. In what follows we denote by $\rho(A)$, $\sigma(A)$ and $\mathcal{R}_\lambda(A)$ respectively the resolvent set, the spectrum, and the resolvent of a bounded linear operator $A:L^2(\Omega)\to L^2(\Omega)$:

$$\rho(A):=\{\lambda\in\mathbb{C}\,|\,(\lambda\mathrm{I}-A)^{-1}:L^2(\Omega,\mathbb{C})\to L^2(\Omega,\mathbb{C})\text{ exists and is bounded}\},$$

$$\sigma(A):=\mathbb{C}\backslash\rho(A).$$

$$\mathcal{R}_\lambda(A):=(\lambda\mathrm{I}-A)^{-1},\quad\lambda\in\rho(A).$$

If A is a bounded random operator, $\rho(A)$ and $\sigma(A)$ are random sets and $\mathcal{R}_\lambda(A)$ is a bounded random operator.

The following result shows the convergence of both the resolvent set of $\frac{1}{N}\sum_{i=1}^{N}A_i$ towards the resolvent set of $\mathbb{E}[A]$ and the convergence of the respective resolvent operators.

Proposition 2.8. *Let A be a bounded random operator and $(A_i)_{i\in\mathbb{N}}$ be a sequence of independent realizations of A. Consider $\omega\subset\rho(\mathbb{E}(A))$ an open subset of the resolvent set of $\mathbb{E}[A]$. Then with probability one, ω is a subset of the resolvent set of $\frac{1}{N}\sum_{i=1}^{N}A_i$ for N large enough:*

$$\exists N_0\in\mathbb{N},\forall N\geq N_0,\quad\omega\subset\rho\left(\frac{1}{N}\sum_{i=1}^{N}A_i\right).\tag{2.8}$$

More precisely, (2.8) is satisfied as soon as the event

$$\mathcal{H}_{N_0}=\left\{\forall N\geq N_0,\quad\sup_{\lambda\in\omega}|||\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A]|||<\frac{1}{3}\right\}\tag{2.9}$$

is realized, and it holds $\mathbb{P}(\mathcal{H}_{N_0})\to 1$ as $N_0\to+\infty$. Moreover, for any $\lambda\in\omega$ and conditionally to $\mathcal{H}_{N_0}$, the following bound holds true for N large enough:

$$\mathbb{E}\left[\left|\left|\left|\mathcal{R}_\lambda\left(\frac{1}{N}\sum_{i=1}^N A_i\right)-\mathcal{R}_\lambda\left(\mathbb{E}[A]\right)\right|\right|\right|^2\middle|\mathcal{H}_{N_0}\right]^{\frac12} \leq 2N^{-\frac12}|||\mathcal{R}_\lambda(\mathbb{E}[A])|||^2\mathbb{E}[|||A-\mathbb{E}[A]|||^2]^{\frac12}. \tag{2.10}$$

Proof. Let $\lambda\in\omega$. Denote $X=\frac{1}{N}\sum_{i=1}^N A_i$. One can write

$$\lambda\mathrm{I}-X=\lambda\mathrm{I}-\mathbb{E}[A]+(\mathbb{E}[A]-X)=(\lambda\mathrm{I}-\mathbb{E}[A])(\mathrm{I}+\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A])). \tag{2.11}$$

From Proposition 2.7 we know that

$$\mathbb{E}[|||X-\mathbb{E}[A]|||^2]^{\frac12}\leq\frac{\mathbb{E}[|||A-\mathbb{E}[A]|||^2]^{\frac12}}{\sqrt{N}}\to 0 \text{ as } N\to+\infty. \tag{2.12}$$

Since the L^2 convergence implies the almost sure convergence, it holds

$$|||\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A])|||\leq\sup_{\lambda\in\omega}|||\mathcal{R}_\lambda(\mathbb{E}[A])|||\;|||X-\mathbb{E}[A]|||\xrightarrow{N\to+\infty}0 \text{ a.e.},$$

where we recall that the resolvent $\mathcal{R}_\lambda(\mathbb{E}[A])$ is holomorphic in λ for the existence of the supremum [9]. This almost sure convergence implies in turn the convergence in probability $\mathbb{P}(\mathcal{H}_{N_0})\to 1$ as $N\to+\infty$.

The event $\mathcal{H}_{N_0}$ entails the invertibility of $\mathrm{I}+\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A])$ and then of $\lambda\mathrm{I}-X$ due to (2.11); more explicitly the inverse of $\lambda\mathrm{I}-X$ is given by

$$(\lambda I-X)^{-1}=(\mathrm{I}+\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A]))^{-1}\mathcal{R}_\lambda(\mathbb{E}[A]),$$

where the prefactor can be expressed as a convergent Neumann series in the space of bounded (deterministic) operators $L^2(\Omega,\mathbb{C})\to L^2(\Omega,\mathbb{C})$:

$$(\mathrm{I}+\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A]))^{-1}=\sum_{p=0}^{+\infty}(-1)^p[\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A])]^p. \tag{2.13}$$

This implies $\lambda\in\rho(X)$. Then (2.13) yields the following estimate when $\mathcal{H}_{N_0}$ is satisfied with $N\geq N_0$:

$$\begin{aligned}|||\mathcal{R}_\lambda(X)-\mathcal{R}_\lambda(\mathbb{E}[A])||| &= \frac{|||\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A])\mathcal{R}_\lambda(\mathbb{E}[A])|||}{1-|||\mathcal{R}_\lambda(\mathbb{E}[A])(X-\mathbb{E}[A])|||}\\ &\leq\frac32|||\mathcal{R}_\lambda(\mathbb{E}[A])|||^2|||X-\mathbb{E}[A]|||.\end{aligned}$$

The result of (2.10) follows by applying the expectation and using the upper bound

$$\mathbb{E}[|||X-\mathbb{E}[A]|||^2|\mathcal{H}_{N_0}]^{\frac12}=\frac{\mathbb{E}[|||X-\mathbb{E}[A]|||^2 1_{\mathcal{H}_{N_0}}]^{\frac12}}{\mathbb{P}(\mathcal{H}_{N_0})}\leq\frac43\mathbb{E}[|||X-\mathbb{E}[A]|||^2]^{\frac12}$$

which holds for N large enough since $\mathbb{P}(\mathcal{H}_{N_0})\to 1$. Finally, (2.8) holds with probability one because this event has a probability larger than $\mathbb{P}(\cup_{N_0\geq N}\mathcal{H}_{N_0})=1$. ∎

3. Convergence analysis of the Monte-Carlo Nystrom method. We now apply the results of the previous section to the rank one operators $(A_i)_{1\leq i\leq N}$ of (1.12), in order to prove the convergences (1.4) and (1.6) of the solution of the linear system (1.3) to the one of the integral equation (1.1). We start by verifying that these operators satisfy the defining axioms of section 2.

LEMMA 3.1. *Let A be the random operator defined by*

$$A \; : \; \Omega \times L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C}), \quad (y,z) \mapsto k(\cdot,y)z(y). \tag{3.1}$$

Then A is a bounded random operator and

$$\mathbb{E}[|||A|||^2]^{\frac{1}{2}} \le ||\rho||^{\frac{1}{2}}_{L^\infty(\Omega)}||k||_{L^\infty(L^2(\Omega))}, \tag{3.2}$$

where we recall (1.2) for the definition of $||k||_{L^\infty(L^2(\Omega))}$. The expectation of A is the integral operator

$$\mathbb{E}[A] \; : \; L^2(\Omega,\mathbb{C}) \to L^2(\Omega,\mathbb{C}), \quad z \mapsto \int_\Omega k(\cdot,y)z(y)\rho(y)\mathrm{d}y. \tag{3.3}$$

Proof. It is enough to prove (3.2). For any $\phi \in L^2(\Omega,\mathbb{C})$, we have

$$\begin{aligned}\mathbb{E}[||A\phi||^2_{L^2(\Omega)}] &= \int_\Omega \left(\int_\Omega |k(y,y')|^2|\phi(y')|^2\mathrm{d}y\right)\rho(y')\mathrm{d}y' \\ &\le \sup_{y'\in\Omega}\int_\Omega |k(y,y')|^2\mathrm{d}y||\rho||_{L^\infty(\Omega)}||\phi||^2_{L^2(\Omega)}.\end{aligned}$$

∎

In what follows, we consider independent realizations $(A_i)_{i\in\mathbb{N}}$ of the operator A. We assume that

(H1) $\mathrm{I} + \mathbb{E}[A]$ is an invertible Fredholm operator

which holds if and only if $\mathrm{I} + \mathbb{E}[A]$ is injective [32]. In that case, $-1 \in \rho(\mathbb{E}[A])$ and (1.1) admits a unique solution. Since the resolvent set $\rho(\mathbb{E}[A])$ is an open subset of the complex plane, there exists $\varepsilon > 0$ such that

$$B(-1,\varepsilon) \subset \rho(\mathbb{E}[A]). \tag{3.4}$$

Applying Proposition 2.8 with $\omega := B(-1,\varepsilon)$ yields immediately the following result.

COROLLARY 3.2. *Assume (H1). The event*

$$\mathcal{H}_{N_0} := \left\{\forall N \ge N_0, \sup_{\lambda\in B(-1,\varepsilon)}\left|\left|\left|\mathcal{R}_\lambda(\mathbb{E}[A])\left(\frac{1}{N}\sum_{i=1}^N A_i - \mathbb{E}[A]\right)\right|\right|\right| < \frac{1}{3}\right\} \tag{3.5}$$

holds with probability $\mathbb{P}(\mathcal{H}_{N_0})$ converging to one as $N_0 \to +\infty$. Furthermore, the following properties hold when $\mathcal{H}_{N_0}$ is realized:

1. *the ball $B(-1,\varepsilon)$ belongs to the resolvent set of $\frac{1}{N}\sum_{i=1}^N A_i$ for $N \ge N_0$;*
2. *in particular, the linear system (1.3) admits a unique solution $(z_{N,i})_{1\le i\le N}$ for $N \ge N_0$;*
3. *the Nystrom interpolant (1.5) converges to the solution $z \in L^2(\Omega,\mathbb{C})$ of the integral equation (1.1) in the following mean-square sense: for N large enough,*

$$\begin{aligned}&\mathbb{E}[||z_N - z||^2_{L^2(\Omega)}|\mathcal{H}_{N_0}]^{\frac{1}{2}} \\ &\quad \le 2N^{-\frac{1}{2}}||\rho||^{\frac{1}{2}}_{L^\infty(\Omega)}||k||_{L^\infty(L^2(\Omega))}|||(\mathrm{I}+\mathbb{E}[A])^{-1}|||^2||f||_{L^2(\Omega)}.\end{aligned} \tag{3.6}$$

Proof. From the equivalence between (1.3) and (1.11), the system (1.11) is invertible as soon as $\mathcal{H}_{N_0}$ is satisfied. Using then the result of Proposition 2.8 with $\lambda = -1$ and (2.4), we obtain the bound

$$\mathbb{E}\left[\left|\left|\left|\mathcal{R}_{-1}\left(\frac{1}{N}\sum_{i=1}^N A_i\right) - \mathcal{R}_{-1}(\mathbb{E}[A])\right|\right|\right|^2 \middle| \mathcal{H}_{N_0}\right]^{\frac{1}{2}} \tag{3.7}$$
$$\leq 2N^{-\frac{1}{2}}|||\mathcal{R}_{-1}(\mathbb{E}[A])|||^2\mathbb{E}[|||A-\mathbb{E}[A]|||^2]^{\frac{1}{2}} \leq 2N^{-\frac{1}{2}}|||\mathcal{R}_{-1}(\mathbb{E}[A])|||^2\mathbb{E}[|||A|||^2]^{\frac{1}{2}}$$
$$\leq 2N^{-\frac{1}{2}}|||\mathcal{R}_{-1}(\mathbb{E}[A])|||^2||\rho||^{\frac{1}{2}}_{L^\infty(\Omega)}||k||_{L^\infty(L^2(\Omega))}.$$

The estimate (3.6) follows since

$$z_N = \mathcal{R}_{-1}\left(\frac{1}{N}\sum_{i=1}^N A_i\right)f, \quad z = \mathcal{R}_{-1}(\mathbb{E}[A])f.$$ ∎

The remainder of this section establishes point-wise estimates for comparing the solution $(z_{N,i})_{1\leq i\leq N}$ of the linear system (1.3) to the values $(z(y_i))_{1\leq i\leq N}$ of the integral equation (1.1). We state two different convergence results expressed in terms of two different weighted quadratic norms. The first one is given in Proposition 3.3 below and is simply obtained by expressing directly $\mathbb{E}[||z_N - z||^2_{L^2(\Omega)}|\mathcal{H}_{N_0}]^{\frac{1}{2}}$ in terms of the values $(z_{N,i})_{1\leq i\leq N}$; however this yields a mean-square error measured with respect to a non-standard Hermitian product. The second result is the bound (1.4) claimed in the introduction, which is stated with the standard Hermitian product of $\mathbb{C}^N$. Its proof requires more subtle arguments and is stated in Proposition 3.6 thereafter.

PROPOSITION 3.3. *Assume (H1). For N large enough and conditionally to the event $\mathcal{H}_{N_0}$ of (3.5), the following mean-square estimate holds between the solution $(z_{N,i})_{1\leq i\leq N}$ of the linear system (1.3) and the point values $(z(y_i))_{1\leq i\leq N}$ of the integral equation (1.1):*

$$\mathbb{E}\left[\frac{1}{N^2}\sum_{1\leq i,j\leq N} K_{ij}(z_{N,i}-z(y_i))\overline{(z_{N,j}-z(y_j))}\middle| \mathcal{H}_{N_0}\right]^{\frac{1}{2}} \tag{3.8}$$
$$\leq N^{-\frac{1}{2}}|||(\mathrm{I}+\mathbb{E}[A])^{-1}|||(1+2|||(\mathrm{I}+\mathbb{E}[A])^{-1}|||)||\rho||^{\frac{1}{2}}_{L^\infty(\Omega)}||k||_{L^\infty(L^2(\Omega))}||f||_{L^2(\Omega)},$$

where $(K_{ij})_{1\leq i,j\leq N}\in\mathbb{C}^{N\times N}$ is the non-negative Hermitian matrix defined by

$$K_{ij} := \int_\Omega k(y,y_i)\overline{k(y,y_j)}\mathrm{d}y.$$

Proof. Denote by r_N the random function

$$r_N := \frac{1}{N}\sum_{i=1}^N A_i z - \mathbb{E}[A]z = \frac{1}{N}\sum_{i=1}^N k(y,y_i)z(y_i) - \int_\Omega k(y,y')z(y')\mathrm{d}y'.$$

The result of Proposition 2.7, (2.4) and (3.2) imply that

$$\mathbb{E}[||r_N||^2_{L^2(\Omega)}]^{\frac{1}{2}} \leq N^{-\frac{1}{2}}\mathbb{E}[|||A-\mathbb{E}[A]|||^2]^{\frac{1}{2}}||z||_{L^2(\Omega)} \tag{3.9}$$

$$\leq N^{-\frac{1}{2}}||\rho||_{L^\infty(\Omega)}^{\frac{1}{2}}||k||_{L^\infty(L^2(\Omega))}|||(\mathrm{I}+\mathbb{E}[A])^{-1}|||\,||f||_{L^2(\Omega)}.$$

By subtracting (1.1) from (1.11) and using the triangle inequality, we obtain

$$\begin{aligned}\mathbb{E}[||z-z_N||_{L^2(\Omega)}^2]^{\frac{1}{2}} &= \mathbb{E}\left[\left|\left|\mathbb{E}[A]z-\frac{1}{N}\sum_{i=1}^N A_i z_N\right|\right|_{L^2(\Omega)}^2\right]^{\frac{1}{2}}\\ &\geq \mathbb{E}\left[\left|\left|\frac{1}{N}\sum_{i=1}^N A_i(z_N-z)\right|\right|_{L^2(\Omega)}^2\right]^{\frac{1}{2}} - \mathbb{E}[||r_N||_{L^2(\Omega)}^2]^{\frac{1}{2}}.\end{aligned}$$

The result follows by using Proposition 3.6 and (3.6), remarking that

$$\begin{aligned}\mathbb{E}\left[\left|\left|\frac{1}{N}\sum_{i=1}^N A_i(z_N-z)\right|\right|^2\right] &= \frac{1}{N^2}\mathbb{E}\left[\sum_{i,j=1}^N \langle A_i(z_N-z), A_j(z_N-z)\rangle\right]\\ &= \frac{1}{N^2}\mathbb{E}\left[\sum_{i,j=1}^N \int_\Omega k(y,y_i)(z_N(y_i)-z(y_i))\overline{k(y,y_j)(z_N(y_j)-z(y_j))}\mathrm{d}y\right]\\ &= \frac{1}{N^2}\mathbb{E}\left[\sum_{1\leq i,j\leq N} K_{ij}(z_{N,i}-z(y_i))(\overline{(z_{N,j}-z(y_j)})\right]. \quad \square\end{aligned} \tag{3.10}$$

The estimate of Proposition 3.3 is obtained as a rather straightforward consequence of (3.6), but the norm associated with the matrix $(K_{ij})_{1\leq i,j\leq N}$ is not standard. In what follows, we prove the point-wise estimate (1.4) expressed in the standard quadratic norm, as well as a bound on the inverse of the matrix associated to the linear system (1.3). The proof is based on the following result from Bandtlow [8] which bounds the norm of the resolvent of a possibly nonnormal Hilbert-Schmidt operator in terms of the distance to the spectrum $\sigma(A)$. In our context, we apply this result in the space of complex matrices $A \equiv (A_{ij})_{1\leq i,j\leq N} \in \mathbb{C}^{N\times N}$ equipped with the spectral norm

$$|||A|||_2 := \sup_{z\in\mathbb{C}^N\setminus\{0\}} \frac{|Az|_2}{|z|_2} \text{ with } |z|_2 := \left(\sum_{i=1}^N |z_i|^2\right)^{\frac{1}{2}}.$$

PROPOSITION 3.4. *Let $A \in \mathbb{C}^{N\times N}$. For any $\lambda \in \rho(A)$, the following inequality holds:*

$$|||\mathcal{R}_\lambda(A)|||_2 \leq \frac{1}{d(\lambda,\sigma(A))}\exp\left(\frac{1}{2}\left(\frac{\mathrm{Tr}(\overline{A^T}A)}{d(\lambda,\sigma(A))}+1\right)\right), \tag{3.11}$$

where $d(\lambda,\sigma(A))$ is the distance of λ to the spectrum of A:

$$d(\lambda,\sigma(A)) := \inf_{\mu\in\sigma(A)} |\lambda-\mu|.$$

Proof. See Theorem 4.1 in [8]. $\square$

This proposition applied to the matrix $(\mathrm{I}+A_N)$ associated to the linear system (1.3) yields the following result.

PROPOSITION 3.5. *Assume (H1) and denote by $(A_N)_{1\leq i,j\leq N}$ the random matrix defined by*

$$A_{N,ij}=\begin{cases}\dfrac{1}{N}k(y_i,y_j) \text{ if } i\neq j,\\ 0 \text{ if } i=j.\end{cases}$$

Then with probability one, there exists $N_0\in\mathbb{N}$ such that the matrix $\mathrm{I}+A_N$ is invertible for any $N\geq N_0$ and

$$\forall N\geq N_0, |||(I+A_N)^{-1}|||_2\leq C(\varepsilon,\rho,k,\Omega), \tag{3.12}$$

where the constant $C(\varepsilon,\rho,k,\Omega)$ independent of N can be chosen as

$$C(\varepsilon,\rho,k,\Omega):=\frac{1}{\varepsilon}\exp\left(\varepsilon^{-1}||\rho||^2_{L^\infty(\Omega)}|\Omega|\,||k||^2_{L^\infty(L^2(\Omega))}+\frac{1}{2}\right).$$

Proof. Clearly, the matrix A_N and the operator $\frac{1}{N}\sum_{1\leq i\leq N}A_i$ have the same spectrum. According to the point 1. of Corollary 3.2 and with probability one, there exists $N_0\in\mathbb{N}$ such that $d(-1,\sigma(A_N))>\varepsilon$ for any $N\geq N_0$. Furthermore, we find that

$$\mathrm{Tr}(\overline{A_N^T}A_N)=\frac{1}{N^2}\sum_{1\leq i\neq j\leq N}|k(y_i,y_j)|^2.$$

By the strong law of large numbers and the independence of the points $(y_i)_{1\leq i\leq N}$, we have with probability one:

$$\mathrm{Tr}(\overline{A_N^T}A_N)\to\int_\Omega\int_\Omega|k(y,y')|^2\rho(y)\rho(y')\mathrm{d}y\mathrm{d}y'\leq||\rho||^2_{L^\infty(\Omega)}|\Omega|||k||^2_{L^\infty(L^2(\Omega))}.$$

Therefore, for almost any realization $(y_i)_{1\leq i\leq N}$, there exists $N_0\in\mathbb{N}$ such that

$$\forall N\geq N_0,\quad \mathrm{Tr}(\overline{A_N^T}A_N)\leq 2||\rho||^2_{L^\infty(\Omega)}|\Omega|\,||k||^2_{L^\infty(L^2(\Omega))}.$$

The result follows by combining this bound with the resolvent estimate (3.11) with $\lambda=-1$:

$$|||(\mathrm{I}+A_N)^{-1}|||_2\leq\frac{1}{d(-1,\sigma(A_N))}\exp\left(\frac{1}{2}\frac{\mathrm{Tr}(\overline{A_N^T}A_N)}{d(-1,\sigma(A_N))}+\frac{1}{2}\right).$$ ∎

We can now state a point-wise convergence result in the discrete L^2 norm.

PROPOSITION 3.6. *Let $\mathcal{H}_{N_0}$ be the event of (3.5) which satisfies $\mathbb{P}(\mathcal{H}_{N_0})\to 1$ as $N_0\to+\infty$. When $\mathcal{H}_{N_0}$ is realized, (1.3) admits a unique solution $(z_{i,N})_{1\leq i\leq N}$ which converges to the vector $(z(y_i))_{1\leq i\leq N}$ at the rate $O(N^{-\frac{1}{2}})$ in the following mean-square sense:*

$$\begin{aligned}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N|z_{i,N}-z(y_i)|^2\,|\,\mathcal{H}_{N_0}\right]^{\frac{1}{2}} \\ \leq N^{-\frac{1}{2}}C(\varepsilon,\rho,k,\Omega)||k||_{L^\infty(L^2(\Omega))}||\rho||_{L^\infty(\Omega)}|||(\mathrm{I}+\mathbb{E}[A])^{-1}|||\,||f||_{L^2(\Omega)}.\end{aligned} \tag{3.13}$$

Proof. Let us denote by $r_N = (r_{N,i})_{1\le i\le N}$ the random vector

$$\begin{aligned} r_{N,i} :&= \frac{1}{N}\sum_{1\le j\ne i\le N} k(y_i,y_j)z(y_j) - \int_\Omega k(y_i,y')z(y')\rho(y')\mathrm{d}y' \\ &= \frac{1}{N}\sum_{1\le j\ne i\le N} (X_{ij} - \mathbb{E}[X_{ij}|y_i]), \end{aligned}$$

where $X_{ij} := k(y_i,y_j)z(y_j)$ and $\mathbb{E}[\cdot|y_i]$ denotes the conditional expectation with respect to y_i. Due to the independence of the variables $(y_i)_{1\le i\le N}$, we have the conditional mean-square estimate

$$\mathbb{E}[|r_{N,i}|^2|y_i] = \frac{1}{N}\mathbb{E}[|X_{ij} - \mathbb{E}[X_{ij}|y_i]|^2|y_i] \le \frac{1}{N}\mathbb{E}[|X_{ij}|^2|y_i].$$

This entails that the vector r_N satisfies the mean-square estimate

$$\begin{aligned} \mathbb{E}\left[\frac{1}{N}|r_N|_2^2\right] &= \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}[\mathbb{E}[|X_{ij}|^2|y_i]] \\ &\le \frac{1}{N}\int_\Omega\int_\Omega |k(y,y')z(y')|^2\rho(y)\rho(y')\mathrm{d}y\mathrm{d}y' \\ &\le \frac{1}{N}||k||^2_{L^\infty(L^2(\Omega))}||\rho||^2_{L^\infty(\Omega)}||z||^2_{L^2(\Omega)}. \end{aligned} \tag{3.14}$$

Observing that

$$z(y_i) + \frac{1}{N}\sum_{j\ne i} k(y_i,y_j)z(y_j) = f(y_i) + \frac{1}{N}\sum_{j\ne i}k(y_i,y_j)z(y_j) - \int_\Omega k(y_i,y_j)z(y')\rho(y')\mathrm{d}y'$$

we find by subtracting (1.3) that the vector $v_N := (v_{N,i})_{1\le i\le N}$ defined by $v_{N,i} := z_{N,i} - z(y_i)$ satisfies

$$(\mathrm{I} + A_N)v_N = -r_N.$$

Therefore, we obtain when the event $\mathcal{H}_{N_0}$ is satisfied that

$$\forall N \ge N_0, \quad |v_N|_2 \le |||(\mathrm{I}+A_N)^{-1}|||_2\, |r_N|_2 \le C(\varepsilon,\rho,k,\Omega)|r_N|_2,$$

where $C(\varepsilon,\rho,k,\Omega)$ is the constant of (3.12). Finally, applying the conditional expectation and using (3.14) yields

$$\begin{aligned} &\mathbb{E}\left[\frac{1}{N}|v_N|_2^2\,|\,\mathcal{H}_{N_0}\right]^{\frac{1}{2}} \\ &\qquad \le N^{-\frac{1}{2}}C(\varepsilon,\rho,k,\Omega)||k||_{L^\infty(L^2(\Omega))}||\rho||_{L^\infty(\Omega)}|||(\mathrm{I}+\mathbb{E}[A])^{-1}|||\,||f||_{L^2(\Omega)}. \quad \square \end{aligned} \tag{3.15}$$

4. Numerical examples. In the next subsections, we illustrate the previous results on a few 1D and 2D examples. We solve both the linear system (1.3) and the integral equation (1.1) with a standard Nystrom method, and we experimentally verify the convergence rate $O(N^{-\frac{1}{2}})$ claimed in Proposition 3.6.

4.1. Numerical 1D examples. We start by illustrating the procedure on the one dimensional square integrable kernel

$$k(y,y') := |y-y'|^{-\alpha}$$

with $0<\alpha<\frac{1}{2}$. We consider the integral equation (1.1) on the interval $\Omega=(0,1)$:

$$z(y)+\int_0^1 k(y,y')z(y')\mathrm{d}y' = f(y), \qquad y\in(0,1), \tag{4.1}$$

and its Monte-Carlo approximation by the solution to the linear system (1.3). Of course (4.1) has a unique solution because k is a positive kernel.

4.1.1. Numerical methodology. In order to estimate $z(y)$ accurately, we solve (4.1) with the classical Nystrom method [4, 24] on a regular grid with $N+1$ points $(y_i)_{0\le i\le N}$ with $y_i = i/N$ and $N=100$. We use the integration scheme

$$z_i + \sum_{j=0}^{N-1}\int_{y_j}^{y_{j+1}} k(y_i,y')z(y')\mathrm{d}y' = f(y_i), \tag{4.2}$$

where every integral is approximated by the trapezoidal rule off the diagonal, and by exact integration of the singularity on the diagonal:

$$\int_{y_j}^{y_{j+1}} k(y_i,y')z(y')\mathrm{d}y' \simeq \begin{cases} \dfrac{1}{2N}(k(y_i,y_{j+1})z_{j+1}+k(y_i,y_j)z_j) \text{ if } j\notin\{i,i-1\}, \\ z_i\displaystyle\int_0^{\frac{1}{N}} |t|^{-\alpha}\mathrm{d}t \text{ if } j=i, \\ z_i\displaystyle\int_0^{\frac{1}{N}} |t|^{-\alpha}\mathrm{d}t \text{ if } j=i-1, \end{cases}$$

where an analytical integration yields

$$\int_0^{\frac{1}{N}} |t|^{-\alpha}\mathrm{d}t = \frac{1}{(1-\alpha)N^{1-\alpha}}.$$

Substituting these approximations into (4.2) yields a linear system of the form

$$\sum_{j=0}^{N} K_{ij}z_j = f(y_i), \qquad 0\le i\le N,$$

whose vector solution $(z_i)_{0\le i\le N}$ is an accurate estimation of the values $(z(y_i))_{0\le i\le N}$ of the analytic solution to (4.1).

We then draw M times a sample of N random points $(y_i^p)_{1\le i\le N}$ for $1\le p\le M$ uniformly and independently in the interval $(0,1)$, and we solve M times the linear system

$$z_{N,i}^p + \frac{1}{N}\sum_{j\neq i} k(y_i^p,y_j^p)z_{N,j}^p = f(y_i^p), \qquad 1\le i\le N. \tag{4.3}$$

We obtain as such M independent realizations of the random vector $(z_{N,i})$ solution to (1.3). We then estimate the mean-square error of (3.13) by computing an empirical average based on the M realizations with $M = 100$:

$$\texttt{MSE} := \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|z_{N,i} - z(y_i)|^2\right]^{\frac{1}{2}} \simeq \sqrt{\frac{1}{MN}\sum_{p=1}^{M}\sum_{i=1}^{N}|z^p_{N,i} - z(y^p_i)|^2}. \tag{4.4}$$

For our numerical applications, we set $\alpha = 2/5 = 0.4$ and we solve (4.4) for three different right-hand sides:

- *Case 1:* $f(y) = 1$
- *Case 2:* $f(y) = (1 - y)y$
- *Case 3:* $f(y) = \sin(6\pi y)$.

In each of the three cases, the system is solved for several values of N lying between 50 and 4,000. We estimate the convergence rate by using a least-squares interpolation of the logarithm of the mean-square error $\log_{10}(\texttt{MSE})$ with respect to $\log_{10}(N)$.

We then plot a few realizations of the Monte-Carlo solution $(z^p_{N,i})_{1\le i\le N}$ and of the Nystrom interpolant

$$z^p_N(y) := f(y) - \frac{1}{N}\sum_{i=1}^{N} k(y, y^p_i) z^p_{N,i} \tag{4.5}$$

to allow for the comparison with the solution $z(y)$ to (4.2). Finally, we numerically estimate the expectation $\mathbb{E}[z_N]$ of the Nystrom interpolants from the empirical average

$$\mathbb{E}[z_N] \simeq \frac{1}{M}\sum_{p=1}^{M} z^p_N \tag{4.6}$$

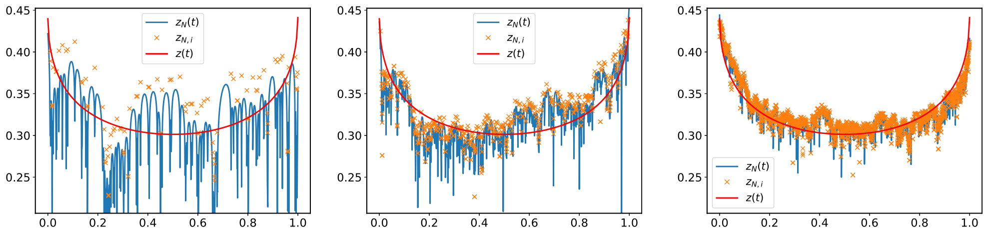
and we verify that $\mathbb{E}[z_N]$ matches closely the solution z, as it can be expected from the result of Corollary 3.2.

4.1.2. Case 1: constant right-hand side. We apply the previous methodology to the constant right-hand side $f(y) = 1$. Samples of the Monte-Carlo solution $(z^p_{N,i})$ to (4.3) and of the Nystrom interpolant z_N of (4.5) are plotted for three different values of N and compared to the solution $z(y)$ of (4.1) on Figure 1.

The mean-square error MSE of (4.4) is then plotted on Figure 2 in logarithm scale, which allows to estimate a convergence rate of order $O(N^{-0.42})$ close to the predicted value $-1/2$ in Proposition 3.6. Finally, the empirical mean $\mathbb{E}[z_N]$ of the Nystrom interpolant is computed for three values of N on Figure 3, which enables one to visually verify the convergence of the Monte-Carlo solution toward the solution to the integral equation (4.1).

For this example, we see that quite a few isolated values of the Monte-Carlo solution $z^p_{N,i}$ remain distant from the analytical solution $z(y_i)$, although one can still verify the convergence of the mean-square error as $O(N^{-\frac{1}{2}})$.

4.1.3. Case 2: quadratic right-hand side. We now apply the methodology of subsection 4.1.1 for solving the equation (4.1) with the right-hand side $f(y) = y(y-1)$. We proceed as in the previous case. Sample solutions of the Monte-Carlo solution $(z^p_{N,i})$ to (4.3) and of the Nystrom interpolant z_N of (4.5) are plotted and compared to the solution $z(y)$ of (4.1) on Figure 4.



(a) $N = 100$ (b) $N = 500$ (c) $N = 2,000$

Fig. 1: Plots of one realization of the Monte-Carlo solution $(z^p_{N,i})$ to (4.3) (*orange crosses*) and of the corresponding Nystrom interpolant $z^p_N(y)$ of (4.5) (*in blue*) for the right hand-side $f(y) = 1$ of subsection 4.1.2. The red line depicts the solution $z(y)$ to (4.1) solved with the standard Nystrom method.

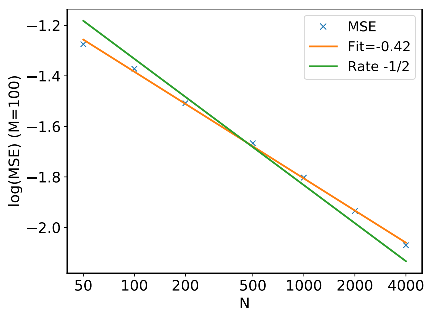


Fig. 2: Plot of the mean-square error `MSE` of (4.4) estimated for various values of N in logarithm scale for the case 1 of subsection 4.1.2. Using a least-squares regression, we find a convergence rate `Fit` $= -0.42$.

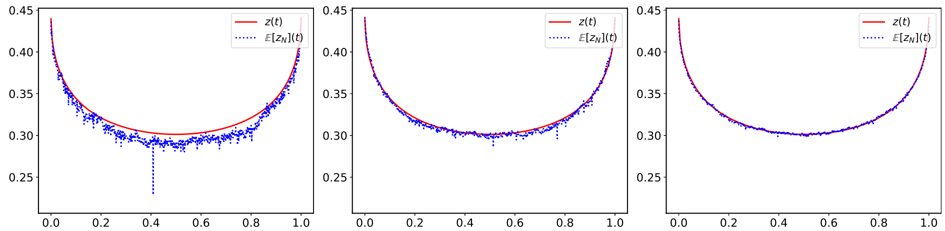
The mean-square error `MSE` of (4.4) is plotted on Figure 5 in logarithm scale, which allows to estimate a convergence rate of order $O(N^{-0.44})$. Finally, the empirical mean $\mathbb{E}[z_N]$ of the Nystrom interpolant is displayed on Figure 6 for three values of N.

For this example, the convergence seem to be faster than in the previous case since Figure 4 presents fewer values of $z^p_{N,i}$ lying exceptionally far from their limit $z(y^p_i)$. In fact, Figure 5 shows that convergence remains of order $O(N^{-\frac{1}{2}})$ as predicted in Proposition 3.6, however with a smaller multiplicative constant.

4.1.4. Periodic right-hand side. Finally, we consider the periodic right-hand side given by $f(y) = \sin(6\pi y)$. Sample solutions of the Monte-Carlo solution $(z^p_{N,i})$ to (4.3) and of the Nystrom interpolant z_N of (4.5) are plotted and compared to the solution $z(y)$ of (4.1) on Figure 7.

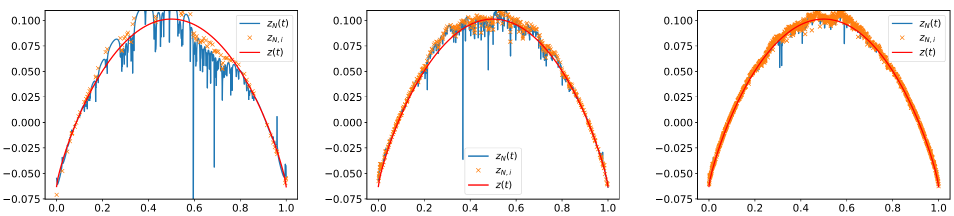
The mean-square error `MSE` of (4.4) is then plotted on Figure 8 in logarithm scale, which allows to estimate a convergence rate of order $O(N^{-0.45})$. Finally, the empirical mean $\mathbb{E}[z_N]$ of the Nystrom interpolant is displayed on Figure 9 for three values of N.

This final example shows that the Monte-Carlo solution $(z^p_{N,i})$ lies close to the analytic solution $z(y^p_i)$ even for moderate values of N: only a few outliers are visible on Figure 7. As in the previous example, Figure 8 shows that convergence remains of order $O(N^{-\frac{1}{2}})$ as predicted in Proposition 3.6, with multiplicative constant similar



(a) $N = 100$ (b) $N = 500$ (c) $N = 2,000$

Fig. 3: Plots of the empirical average (4.6) of the Nystrom interpolant $\mathbb{E}[z_N]$ (*in blue dotted line*) for the case 1 of subsection 4.1.2, compared to the analytical solution $z(t)$ estimated by solving (4.1) with the standard Nystrom method (*in red*).



(a) $N = 100$ (b) $N = 500$ (c) $N = 2,000$

Fig. 4: Plots of one realization of the Monte-Carlo solution $(z^p_{N,i})$ to (4.3) (*orange crosses*) and of the corresponding Nystrom interpolant $z^p_N(y)$ of (4.5) (*in blue*) for the right hand-side $f(y) = y(1-y)$ of subsection 4.1.3. The red line depicts the solution $z(y)$ to (4.1) solved with the standard Nystrom method.

to the one of the case 1 with constant right-hand side of subsection 4.1.2.

4.2. Numerical 2D example : a Lippmann-Schwinger equation. We now illustrate the results of Proposition 3.6 on a more challenging 2D example. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded two-dimensional domain. We consider the Lippmann-Schwinger equation

$$\begin{cases} (\Delta + k^2 n_\Omega) z = 0 \text{ in } \mathbb{R}^2, \\ (\partial_r - ik)(z - u_{in}) = O(|x|^{-2}) \text{ as } r \to +\infty, \end{cases} \tag{4.7}$$

whose solution z is the scattered field produced by an incident wave u_{in} propagating through a heterogeneous material with refractive index $n_\Omega(x)$ given by

$$n_\Omega(x) = \begin{cases} m \text{ if } x \in \Omega, \\ 1 \text{ if } x \in \mathbb{R}^2 \backslash \Omega, \end{cases}$$

where $m > 0$ is the index of the acoustic obstacle Ω. Assuming u_{in} solves the Helmoltz equation with wave number k, i.e. $(\Delta + k^2) u_{in} = 0$, z can be found as the unique solution to the Lippmann-Schwinger equation

$$z(y) + (m-1)k^2 \int_\Omega \Gamma^k(y - y') z(y') \mathrm{d}y' = u_{in}(y), \quad y \in \Omega, \tag{4.8}$$

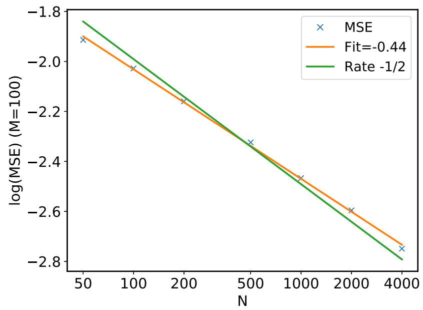
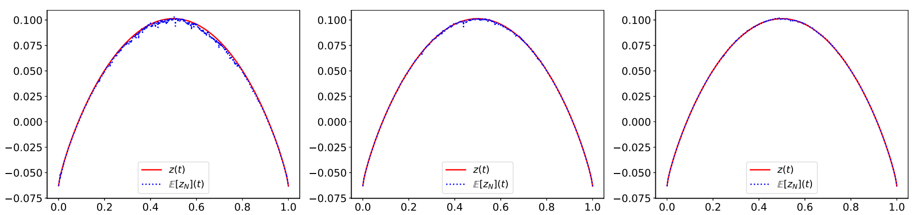


Fig. 5: Plot of the mean-square error `MSE` of (4.4) estimated for various values of N in logarithm scale for the case 2 of subsection 4.1.3. Using a least-squares regression, we find a convergence rate `Fit` $= -0.44$.



(a) $N = 100$ (b) $N = 500$ (c) $N = 2,000$

Fig. 6: Plots of the empirical average (4.6) of the Nystrom interpolant $\mathbb{E}[z_N]$ (*in blue dotted line*) for the case 2 of subsection 4.1.2, compared to the analytical solution $z(t)$ estimated by solving (4.1) with the standard Nystrom method (*in red*).

where Γ^k is the (outgoing) fundamental solution to the two-dimensional Helmoltz equation given by

$$\Gamma^k(y - y') = -\frac{i}{4} H_0^{(1)}(k|y - y'|),$$

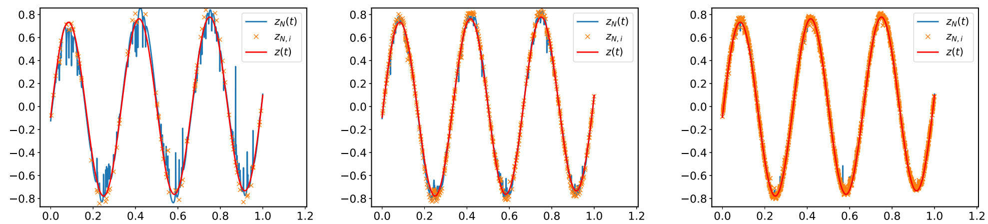
with $H_0^{(1)}$ being the first Hankel function of the first kind [32]. It is known that the integral equation (4.8) admits a unique solution $z \in \mathcal{C}^0(\Omega)$, see e.g. [15, 24]. Once the integral equation (4.8) has been solved, the identity (4.8) determines an extension $y \mapsto z(y)$ on the whole space $\mathbb{R}^2$ and the resulting function is the solution to the original the scattering problem (4.7).

For our numerical application, $\Omega = \{y \in \mathbb{R}^2 \,|\, |y| < 1\}$ is the unit disk and we choose u_{in} to be an incident plane wave propagating in the horizontal direction:

$$f(y) := e^{iky_1}, \qquad y = (y_1, y_2) \in \Omega.$$

The value of the wave number and of the refractive index in the acoustic medium are respectively set to $k = 5$ and $m = 10$.

4.2.1. Accurate evaluation of the scattered field with the Volume Integral Equation Method. We first compute an accurate numerical approximation of $z(y)$ in order to obtain a reference solution for estimating the numerical error associated with Monte-Carlo solutions. We solve (4.8) with the Volume Integral Equation



(a) $N = 100$ (b) $N = 500$ (c) $N = 2,000$

Fig. 7: Plots of one realization of the Monte-Carlo solution $(z^p_{N,i})$ to (4.3) (*orange crosses*) and of the corresponding Nystrom interpolant $z^p_N(y)$ of (4.5) (*in blue*) for the right hand-side $f(y) = \sin(6\pi y)$ of subsection 4.1.4. The red line depicts the solution $z(y)$ to (4.1) solved with the standard Nystrom method.

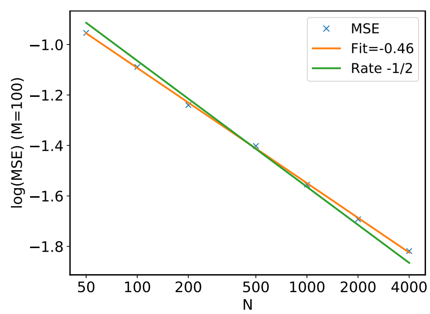


Fig. 8: Plot of the mean-square error `MSE` of (4.4) estimated for various values of N in logarithm scale for the case 3 of subsection 4.1.4. Using a least-squares regression, we find a convergence rate `Fit` $= -0.46$.

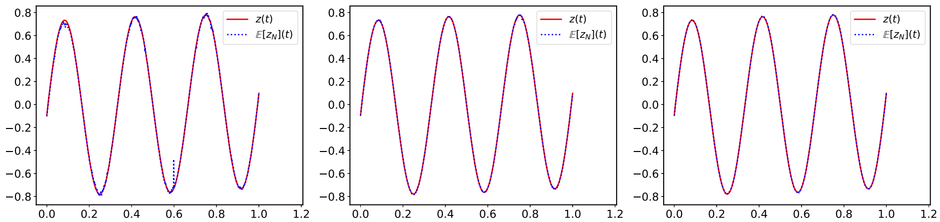
Method by using $\mathbb{P}_1$-Lagrange finite elements on a triangular mesh $\mathcal{T}$ with $N_v = 1084$ vertices $(\widehat{y}_i)_{1\leq i\leq N_v}$ (represented on Figure 10a). Our implementation is written in MATLAB and relies on the open-source library GYPSILAB [1, 5].

The solution $z(y)$ computed in the disk Ω is displayed on Figure 11a. For visualisation purposes, we also plot on Figure 11b its extension to a surrounding disk Ω' centered at $(1,0)$ and of radius 3. The domain Ω' surrounding Ω is represented on Figure 10b.

4.2.2. Monte-Carlo approximations. We draw M times N independent samples $(y^p_i)_{1\leq i\leq N}$ with $1 \leq p \leq M$ from the uniform distribution in the disk Ω. These samples are obtained from their polar coordinates $(r^p_i, \theta^p_i)_{1\leq i\leq p}$ drawn independently from the distributions $2r\mathrm{d}r$ and $\frac{1}{2\pi}\mathrm{d}\theta$ in the cartesian product $(0,1)\times(0,2\pi)$. The values (r^p_i) are themselves obtained as square roots $\sqrt{U^p_i}$ of random variables U_i uniformly and independently distributed in the interval $(0,1)$.

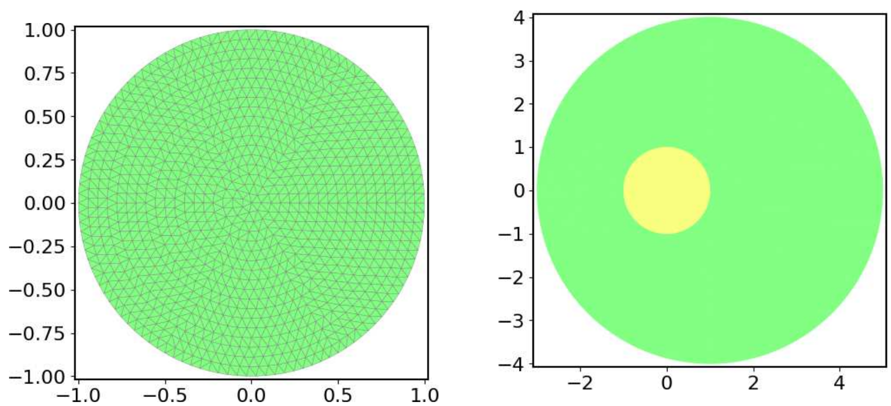
We then compute $M = 100$ Monte-Carlo approximations $(z^p_{N,i})_{1\leq i\leq N}$ of (4.8) by solving the following M linear systems for $1 \leq p \leq M$:

$$z^p_{N,i} + \frac{1}{N}|\Omega|(m-1)k^2 \sum_{j\neq i} \Gamma^k(y^p_i - y^p_j) z^p_{N,j} = u_{in}(y^p_i), \qquad 1 \leq i \leq N. \tag{4.9}$$



(a) $N = 100$ (b) $N = 500$ (c) $N = 2,000$

Fig. 9: Plots of the empirical average (4.6) of the Nystrom interpolant $\mathbb{E}[z_N]$ (*in blue dotted line*) for the case 3 of subsection 4.1.4, compared to the analytical solution $z(t)$ estimated by solving (4.1) with the standard Nystrom method (*in red*).



(a) The discretization mesh $\mathcal{T}$ considered for the acoustic obstacle Ω (the unit disk).

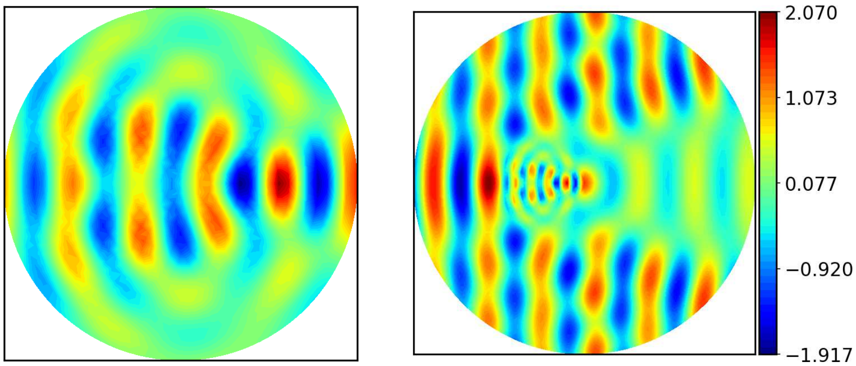
(b) The surrounding disk Ω' (the disk centered at $(1,0)$ of radius 4, in green) containing the accoustic obstacle Ω (in yellow).

Fig. 10: Setting of the exterior acoustic problem (4.7): mesh of the circular acoustic obstacle Ω and portion of the exterior domain Ω' for the visualisation of the solution outside Ω.

The numerical solution of the system (4.9) requires a priori to inverse a dense matrix, which can be potentially challenging for large values of N with direct methods. In order to solve (4.9) in reasonable computational time, we rely on the Efficient Bessel Decomposition (EBD) algorithm of Averseng [6]. This algorithm allows to evaluate N convolution products

$$\left(\sum_{j \neq i} \Gamma^k (y_i^p - y_j^p) z_{N,j}^p \right)_{1 \leq i \leq N}$$

with a single offline pass of complexity strictly better than $O(N^2)$, and online passes of quasilinear complexity for each new argument $(z_{N,i}^p)_{1 \leq i \leq N}$. Although this algorithm is strictly speaking suboptimal compared to the Fast Multipole Method [21], it achieves comparable performances in practice and is rather simple to use and to implement. Our application relies on the open-source EBD toolbox [7] directly available in Gypsilab.



(a) Plot of the solution z in the interior domain Ω.

(b) Plot of the solution z in the exterior domain Ω'.

Fig. 11: Numerical estimation of the scattered field z obtained by solving (4.8) with the Volume Integral Equation Method on the mesh $\mathcal{T}$.

4.2.3. Numerical results. We solve $M = 100$ times the linear system (4.9) for various values of N between 500 and 40,000. Samples of corresponding independent distributions of random points $(y_i^p)_{1\leq i\leq N}$ in the unit disk are shown for $N = 500$, $N = 1,000$ and $N = 5,000$ on Figure 12.

Once the solution $(z_i^p)_{1\leq i\leq N}$ to the linear system (4.9) has been computed, interpolated values $(\widehat{z}_i^p)_{1\leq i\leq N_v}$ are estimated at the vertices $(\widehat{y}_i)_{1\leq i\leq N_v}$ of the discretization mesh $\mathcal{T}$ (Figure 10a) thanks to a Delaunay based piecewise linear interpolation[1]. Monte-Carlo solutions thus obtained are displayed on Figure 13 for several values of N. In order to help the reader to better visualize the convergence of the Monte-Carlo samples $(z_i^p)_{1\leq i\leq N}$ towards the vectors $(z(y_i^p)_{1\leq i\leq N}$, we also represent on Figure 14 the estimated averaged of the Monte-Carlo solutions at the vertices $(\widehat{y}_i)_{1\leq i\leq N_v}$:

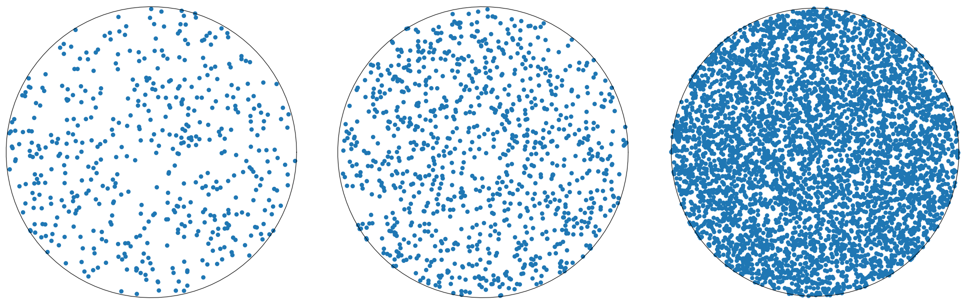
$$\mathbb{E}[(\widehat{z}_i^p)_{1\leq i\leq N_v}] \simeq \left(\frac{1}{M}\sum_{p=1}^{M}\widehat{z}_i^p\right)_{1\leq i\leq N_v}. \tag{4.10}$$

Comparing the plots of Figure 14 to the one of Figure 11a allows to appraise the convergence of the average of the Monte-Carlo solutions towards the solution z of the Lippmann-Schwinger equation (4.8).

We then represent individual samples $(z_i^p)_{1\leq i\leq N}$ interpolated on the mesh $\mathcal{T}$ on Figure 13. Qualitatively, the almost-sure convergence of individual samples towards their limit $z(y)$ starts to be visible only for N greater or equal to 20,000.

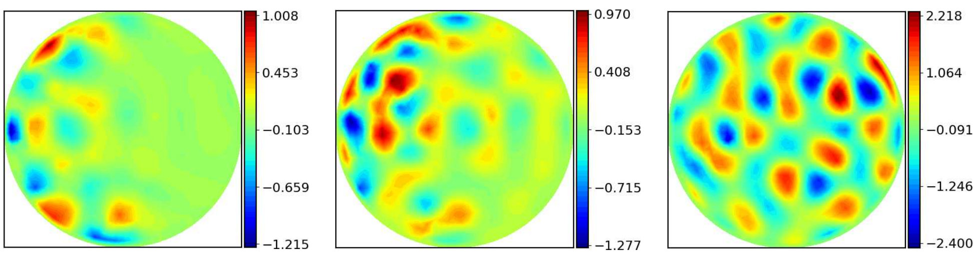
Finally, the mean-square error `MSE` is evaluated by using the estimator (4.4) for several values of N, where the values of the solution $z(y_i^p)$ are estimated at sample points $(y_i^p)_{1\leq i\leq N}$ from its $\mathbb{P}_1$-Lagrange approximation on the triangulated mesh $\mathcal{T}$. We plot on Figure 15 the logarithm of the mean-square error as a function of $\log_{10}(N)$ obtained for $N \in \{5,000; 7,500; 15,000; 20,000; 40,000\}$. Using a least-squares regression, we observe numerically a convergence rate of order $O(N^{-0.56})$ which is in agreement with the prediction $O(N^{-1/2})$ of Proposition 3.6.

[1]This is achieved by using the function `griddata` of MATLAB.

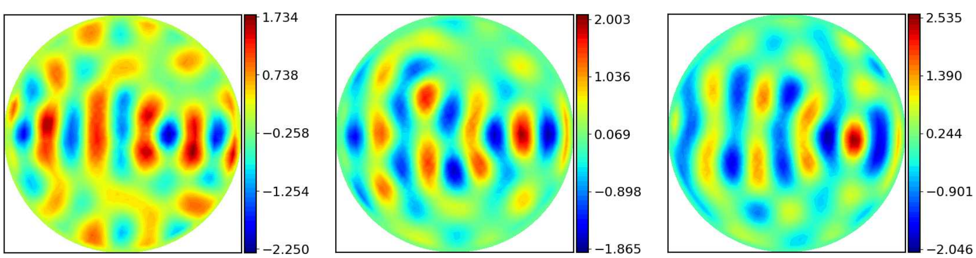


(a) $N = 500$ (b) $N = 1,000$ (c) $N = 5,000$

Fig. 12: Samples of N random points drawn randomly and independently from the uniform distribution in the unit disk.



(a) $N = 500$ (b) $N = 1,000$ (c) $N = 5,000$



(d) $N = 10,000$ (e) $N = 20,000$ (f) $N = 40,000$

Fig. 13: Plots of Monte-Carlo solutions $(z_i^p)_{1\leq i\leq N}$ obtained by solving the linear system (4.9) for several values of N. The visualisation is obtained by using interpolated values on the triangle mesh $\mathcal{T}$.

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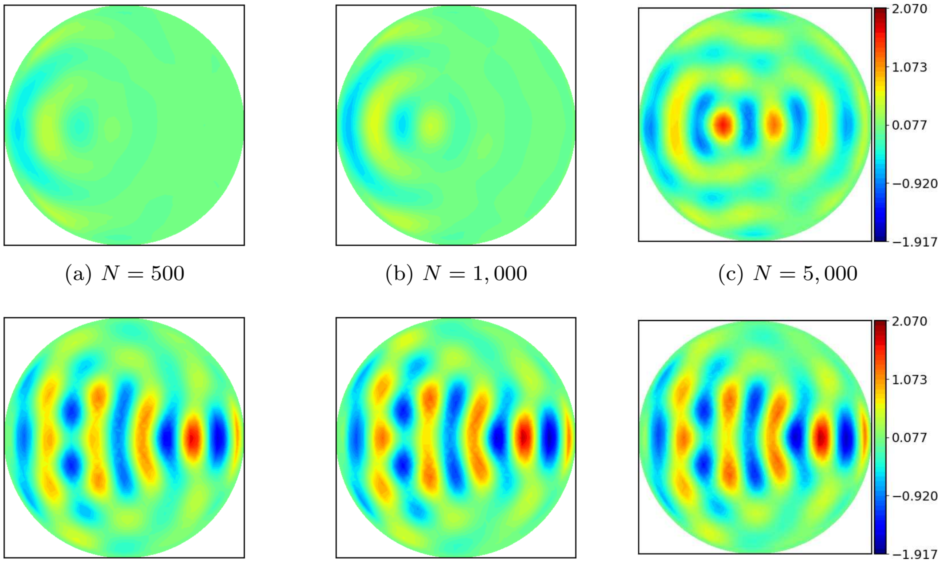

(a) $N = 500$ (b) $N = 1,000$ (c) $N = 5,000$

(d) $N = 10,000$ (e) $N = 20,000$ (f) $N = 40,000$

Fig. 14: Plots of the average $\mathbb{E}[(\widehat{z}_i^p)]$ of the Monte-Carlo solutions $(z_i^p)_{1\leq i\leq N}$ obtained at the vertices of the mesh $\mathcal{T}$ from the estimator (4.10). This plot allows to appraise the convergence towards the solution $z(t)$ to the Lippmann-Schwinger equation (4.8) represented on Figure 11a.

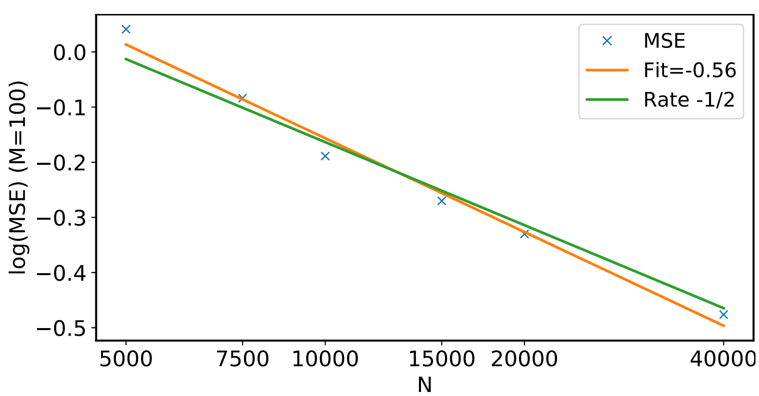

Fig. 15: Plot of the mean-square error `MSE` of (4.4) estimated for various values of N in logarithm scale for the 2D example of subsection 4.2. Using a least-squares regression, we find a convergence rate $\texttt{Fit} = -0.56$ in agreement with the prediction $O(N^{-\frac{1}{2}})$ of Proposition 3.6.

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