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Exponential convergence in H^1 of hp-FEM for Gevrey regularity with isotropic singularities*

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Abstract For functions $u \in H^1(\Omega)$ in a bounded polytope $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$ which are Gevrey regular in $\overline{\Omega}\backslash\mathscr{S}$ with point singularities concentrated at a set $\mathscr{S} \subset \overline{\Omega}$ consisting of a finite number of points in $\overline{\Omega}$, we prove exponential rates of convergence of hp-version continuous Galerkin finite element methods on families of regular, simplicial meshes in Ω. The simplicial meshes are geometrically refined towards $\mathscr{S}$ but are otherwise unstructured.

1 Introduction

Many nonlinear PDEs admit solutions which are smooth in a bounded, physical domain $\Omega \subset \mathbb{R}$, but exhibit isolated point singularities at a set $\mathscr{S} \subset \overline{\Omega}$. We mention only nonlinear Schrödinger equations with self-focusing, density functional models in electron structure calculations (eg. [14, 3, 6, 7] and the references there), nonlinear parabolic PDEs with critical growth (eg. [25, 19] and the references there, or continuum models of crystalline solids with isolated point defects (eg. [21] and the references there).

We prove an exponential convergence result for C^0-conforming hp-FEM on regular, simplicial mesh families with *isotropic, geometric refinement* towards the singular point(s) $c \in \mathscr{S}$. These meshes are in addition required to be shape-regular. This type of mesh arises for example in adaptive bisection-tree refinements. Specifically, for singular solutions $u \in H^1(\Omega)$ in the bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$ which belong, in addition, to a countably normed space with non-homogeneous, radial weights as introduced, for example, in [4, 10], and with Gevrey-regular growth

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of derivatives in $\overline{\Omega}\backslash\mathscr{S}$, we construct a sequence $\{I_N^{hp}\}_N$ of continuous, piecewise polynomial (quasi-)interpolation operators on sequences of regular, simplicial partitions that are geometrically refined towards $\mathscr{S}$ with exponential convergence in $H^1(\Omega)$: for a bounded domain $\Omega \subset \mathbb{R}^d$ and for functions $u \in H^1(\Omega) \cap \mathscr{G}^\delta(\Omega)$, a class of δ-Gevrey-regular functions in $\overline{\Omega}\backslash\mathscr{S}$ (to be defined in (8) below), there exist constants $b, C > 0$ which depend on Ω and on u, such that

$$\|u - I^{hp}u\|_{H^1(\Omega)} \leq C \begin{cases} \exp(-bN^{\frac{1}{1+\delta d}}) & \delta \geq 1, \\ \Gamma(N^{\frac{1}{1+\delta d}})^{-b(1-\delta)} & 0 < \delta < 1 \,. \end{cases} \tag{1}$$

Here, $d = 2,3$ denotes the space dimension and N denotes the number of degrees of freedom in the hp-FE approximation, $\Gamma()$ denotes the Gamma function and $\delta > 0$ denotes the Gevrey regularity parameter. Note that $\delta = 1$ corresponds to functions which are analytic in $\overline{\Omega}\backslash\mathscr{S}$.

The rate (1) coincides, in the cases $d = 1,2$ and for analytic solutions, i.e. when $\delta = 1$, with the exponential convergence rate bounds obtained in [15, 16] for corner singularities on structured geometric meshes (consisting of axiparallel quadrilaterals with inserted triangles to remove irregular nodes). In space dimension $d = 3$, (1) generalizes the hp-approximations in [26, Sec. 5.2.2] in the case of vertex singularities, for meshes of axiparallel hexahedra to unstructured, tetrahedral meshes with geometric refinement towards $\mathscr{S}$. For $0 < \delta < 1$ the rate is super-exponential, thus benefiting from the higher regularity of u.

The structure of the note is as follows: in Section 2, we introduce a model problem, the geometric assumptions on the singularities, and precise the analytic regularity in countably normed, weighted Sobolev spaces with radial weight functions. In Section 3, we introduce the hp-version FEM; we specify in particular the assumptions on the simplicial, geometric meshes, on the elemental polynomial degrees, and on the definition of the hp FE spaces. Section 4 contains a proof of the exponential convergence bound in $H^1(\Omega)$ on regular, simplicial geometric mesh families.

2 Analytic Regularity

Analytic regularity is characterized in countably normed weighted Sobolev spaces which have been introduced and used in exponential convergence estimates in a number of references; we only mention [15, 16, 2, 17, 18, 10] and the references there. Here, we denote by $\mathscr{S} \subset \overline{\Omega}$ the set of singular points c; we consider solutions $u \in H^1(\Omega)$ which are smooth in $\overline{\Omega}\backslash\mathscr{S}$ so that the singular support of u coincides with $\mathscr{S}$. We work under the following separation assumption on $\mathscr{S}$.

$$\text{The singular set } \mathscr{S} \text{ consist of a finite number of isolated points } c \in \overline{\Omega}. \tag{2}$$

Assumption (2) implies $\varepsilon(\Omega,\mathscr{S}) := \min\{\mathrm{dist}(c,c') : c,c' \in \mathscr{S}, c \neq c'\} > 0$, and allows to partition the set Ω into $|\mathscr{S}|$ many disjoint neighborhoods ω_c of the singularities $c \in \mathscr{S}$. We set denote $\Omega_0 := \Omega \backslash \overline{\bigcup_{c\in\mathscr{S}} \omega_c}$.

2.1 Weighted Sobolev Spaces. Gevrey Classes

We characterize analytic regularity of singular solutions by weighted Sobolev spaces. To define these, we introduce distance functions:

$$r_c(x) = \mathrm{dist}(x,c)\ , \qquad x \in \Omega\ , \quad c \in \mathscr{S}\ . \tag{3}$$

With $c \in \mathscr{S}$ we collect all singular exponents $\beta_c \in \mathbb{R}$ in the "multi-exponent"

$$\underline{\beta} = \{\beta_c : c \in \mathscr{S}\} \in \mathbb{R}^{|\mathscr{S}|}\ . \tag{4}$$

We assume for $d = 3$ ($\underline{\beta} > s$ and $\underline{\beta} \pm s$ being understood componentwise for $s \in \mathbb{R}$)

$$\underline{b} := -1 - \underline{\beta} \in (0,1/2)\ ,\ \text{ie.}\ -3/2\ < \underline{\beta} <\ -1\ . \tag{5}$$

For $d = 2$, we assume for some $\varepsilon > 0$ that

$$\underline{b} := -1 - \underline{\beta} \in (0,\varepsilon)\ ,\ \text{ie.}\ -1-\varepsilon\ < \underline{\beta} <\ -1\ . \tag{6}$$

Consider the *inhomogeneous, weighted semi-norms* $|u|_{N^k_{\underline{\beta}}(\Omega)}$ given by (cp. [10, Definition 6.2 and Equation (6.9)], [2] and [17]),

$$|u|^2_{N^k_{\underline{\beta}}(\Omega)} = |u|^2_{H^k(\Omega_0)} + \sum_{c\in\mathscr{S}} \sum_{\substack{\alpha\in\mathbb{N}_0^d \\ |\alpha|=k}} \left\| r_c^{\max\{\beta_c+|\alpha|,0\}} \mathsf{D}^\alpha u \right\|^2_{L^2(\omega_c)}\ ,\ \ k \in \mathbb{N}_0\ . \tag{7}$$

We define the *inhomogeneous weighted norm* $\|u\|_{N^m_{\underline{\beta}}(\Omega)}$ by $\|u\|^2_{N^m_{\underline{\beta}}(\Omega)} = \sum_{k=0}^m |u|^2_{N^k_{\underline{\beta}}(\Omega)}$. Here, $|u|_{H^m(\Omega_0)}$ signifies the Hilbertian Sobolev semi-norm of integer order m on Ω_0, and D^α denotes the weak partial derivative of order $\alpha \in \mathbb{N}_0^d$. The space $N^m_{\underline{\beta}}(\Omega)$ is the weighted Sobolev space obtained as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{N^m_{\underline{\beta}}(\Omega)}$.

Remark 1. (i) Under (5), for $\Omega \subset \mathbb{R}^3$ holds $N^2_{\underline{\beta}}(\Omega) \subset H^{1+\theta}(\Omega)$ for some $\theta > 1/2$: choose $\theta(\underline{\beta}) = 1 - \beta_m - \varepsilon$ in [17, Thm. 3.5] with $\beta_m := -1 - \beta_c \in (0,1/2)$, and $0 < \varepsilon < 1/2 - \beta_m = 3/2 + \beta_c$.

(ii) In dimension $d = 2$, ie. for $\Omega \subset \mathbb{R}^2$, we find under the assumption (5) that $N^2_{\underline{\beta}}(\Omega) \subset H^{1+\theta}(\Omega)$ for some $\theta > 0$, so that for $d = 2$ holds $N^2_{\underline{\beta}}(\Omega) \subset C^0(\overline{\Omega})$ with continuous embedding.

(iii) The spaces $N^m_{\underline{\beta}}(\Omega)$ are closely related to the nonhomogeneous, weighted spaces of type $J^m_\gamma(\Omega)$ which arise in connection with the Mellin transformation of elliptic problems in conical domains. We refer to [9] for a definition and properties of the spaces $J^m_\gamma(\Omega)$.

With $N^k_{\underline{\beta}}(\Omega)$ as defined in (7), for $\delta > 0$ we define the δ-*Gevrey regular class of solutions with point singularities at* $\mathscr{S}$ by

$$\mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega) = \left\{ u \in \bigcap_{k\geq 0} N^k_{\underline{\beta}}(\Omega) \,:\, \exists C_u > 0 \text{ s.t. } |u|_{N^k_{\underline{\beta}}(\Omega)} \leq C_u^{k+1}(k!)^\delta \ \forall k \in \mathbb{N}_0 \right\}. \tag{8}$$

We mention that in the case $\delta = 1$, the norm in the definition (8) of the Gevrey class $\mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ coincides with the norm for the analytic class $\mathscr{B}_{\underline{\beta}}(\mathscr{S};\Omega)$ introduced in [10, Definition 6.9–6.11] in three space dimensions. In two space dimensions, it equals the weighted analytic classes introduced in [16, 17]. All ensuing approximation results in particular apply for this analytic solution class, as has been indicated in [27, 28]. Naturally, the present construction parallels earlier constructions in particular cases; for example, the polynomial trace lifting in Section 4.2.6 is identical to the analytic case in [28].

2.2 Examples of Boundary Value Problems with Gevrey-regular solutions

Large classes of linear and nonlinear elliptic boundary value and eigenvalue problems with analytic input data admit solutions in the analytic class $\mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ with $\delta = 1$. We refer to, e.g., [4, 10, 18] and also [14] for electron structure models, [18, 2, 10] for elliptic problems in polyhedral domains, and [23] and the references there for nonlinear Schrödinger eigenvalue problems.

2.2.1 Linear Elliptic boundary value problems in Polygons

In space dimension $d = 2$, let Ω denote a polygon with straight sides. Consider the model Dirichlet boundary value problem

$$-\nabla \cdot (A(x)\nabla u) = f \quad \text{in} \quad \Omega \ , u|_{\partial\Omega} = 0\,. \tag{9}$$

In (9), we assume that $A(x) = (a_{ij}(x))_{1\leq i,j\leq 2}$ and $f(x)$ are analytic in $\overline{\Omega}$ and that the matrix function $x \mapsto A(x) \in \mathbb{R}^{2\times 2}_{sym}$ is uniformly positive definite: there exists $\alpha > 0$ such that for every $\xi \in \mathbb{R}^2$ holds

$$\operatorname*{ess\,inf}_{x\in\Omega} \xi^\top A(x)\xi \geq \alpha|\xi|^2\,.$$

The unique, weak solution $u \in V = H_0^1(\Omega)$ of (9) exists by the Lax-Milgram Lemma, and satisfies the weak form of (9): find

$$u \in V \qquad a(u,v) = (f,v) \quad \forall v \in V \ . \tag{10}$$

For a closed subspace $V_N \subset V$, approximate solutions $u_N \in V_N$ of (10) are obtained by Galerkin projection: find

$$u_N \in V_N \qquad a(u_N,v) = (f,v) \quad \forall v \in V_N \ . \tag{11}$$

The approximate solutions u_N exist, are unique and quasioptimal:

$$\|u - u_N\|_V \leq C \inf_{v \in V_N} \|u - v\|_V \ . \tag{12}$$

Convergence rates of sequences $\{u_N\}_N$ of approximate solutions thus depend on a) the choice of V_N and b) on the solution regularity.

For problem (10), it has been shown in [16] that the solution $u \in \mathscr{G}^1_\delta(\Omega)$. For $\{V_N\}_N$ being a sequence of so-called *hp-FE spaces* (to be defined in the next section), we recover from (12) and (1) (with $\delta = 1$ and $d = 2$) the exponential convergence rate $\exp(-b\sqrt[3]{N})$ already obtained in [27]. Gevrey regularity in conical domains for Gevrey-regular data A and f for solutions u of (10) was first obtained in [4].

2.2.2 Three-dimensional problems

For the analog of (9) in polyhedral domains Ω, the regularity classes $\mathscr{G}^\delta_{\underline{\beta}}(\Omega)$ are *not* adequate, as even for analytic data A and f, the solutions are locally analytic in Ω, but exhibit apart from corner singularities also so-called *edge-singularities*. Their precise mathematical description mandates more sophisticated function spaces (see, e.g., [17, 10] and the references there and [26] for exponential convergence results for *hp*-FEM.

However, in large classes of applications, solutions are Gevrey regular with point singularities only. We mention only the source problem (10) in domains Ω which exhibit isolated vertices, e.g. conical domains with a smooth (analytic) base, such as circular cones with apex c.

Another important class of problems arises from mathematical models of quantum chemistry (see, e.g., [6, 7] and the references there). For instance, consider the nonlinear Schrödinger EVP: find $\lambda \in \mathbb{R}$ and $0 \neq u \in H^1(\mathbb{R}^3)$ such that

$$Lu = -\Delta u + Vu + |u|u = \lambda u \qquad \text{in} \quad \mathbb{R}^3 \ . \tag{13}$$

Here, for analytic potentials V which become singular at a finite set $\mathscr{S} \subset \mathbb{R}^3$ of isolated points, eigenfunctions u belong to $\mathscr{G}^1_\beta(\Omega)$ for compact sets $\Omega \subset \mathbb{R}^3$ containing $\mathscr{S}$ in their interior, see [22] and [23, Thm. 7]. Quasioptimality in $H^1(\Omega)$ of Galerkin-FEM for the EVP (13) can be found, for example, in [6, 7].

3 hp-Finite Element spaces

The hierarchies of FE spaces which underlie the *hp*-FEM are based on two key ingredients: (i) *geometric mesh families* $\mathfrak{M}_{\kappa,\sigma} = \{\mathscr{M}^{(\ell)}\}_{\ell\geq 0}$ and (ii) *simultaneous refinement of meshes and polynomial degree distributions*. They also exhibit (iii) a layer-structure among the Finite Elements $T \in \mathscr{M}^{(\ell)}$ which we describe next.

3.1 Geometric Mesh Families $\mathfrak{M}_{\kappa,\sigma}$

For two parameters $0 < \kappa, \sigma < 1$, we consider in the bounded domain Ω *geometric mesh families* $\mathfrak{M}_{\kappa,\sigma} = \{\mathscr{M}^{(\ell)}\}_{\ell\geq 1}$ of geometric meshes $\mathscr{M}^{(\ell)} \in \mathfrak{M}_{\kappa,\sigma}$. The meshes $\mathscr{M} \in \mathfrak{M}_{\kappa,\sigma}$ are regular partitions of the polyhedron Ω into a finite number of open simplices (triangles in space dimension $d = 2$, tetrahedra in space dimension $d = 3$) $T \in \mathscr{M}^{(\ell)}$. Here, regular means that for every $\mathscr{M} \in \mathfrak{M}_{\kappa,\sigma}$, the intersections of closures of any two distinct $T, T' \in \mathscr{M}$ are either empty, a vertex v, an entire edge e or an entire face f. We assume the family $\mathfrak{M}_\sigma$ to be *uniformly κ-shape regular*: for a simplex $T \in \mathscr{M}^{(\ell)}$, we denote by $h_T = \mathrm{diam}(T)$ its diameter and by $\rho_T = \sup\{\rho > 0 | B_\rho \subset T\}$, the radius of the largest ball B_ρ that can be inscribed into T. For a regular, simplicial mesh $\mathscr{M}$, the (nondimensional) shape parameter $\kappa(\mathscr{M}) = \max\{h_T/\rho_T | T \in \mathscr{M}\}$ is well defined. A collection $\{\mathscr{M}^{(\ell)}\}_{\ell\geq 1}$ of regular, simplicial meshes is called *κ-shape regular*, if $\sup_{\ell\geq 1} \kappa(\mathscr{M}^{(\ell)}) \leq \kappa < \infty$.

Each simplex $T \in \mathscr{M}_\ell$ is the affine image of the reference simplex, i.e, it is defined by $\widehat{T} := \{\hat{x} \in \mathbb{R}^3 : \hat{x}_i > 0, \sum_{i=1}^d \hat{x}_i < 1\}$, under the affine element map F_T, ie.

$$T = F_T(\widehat{T}), \quad T \ni x = F_T(\hat{x}) = B_T\hat{x} + b_T, \ \hat{x} \in \widehat{T}\ . \tag{14}$$

For a regular, simplicial triangulation $\mathscr{M}$ of Ω with $\kappa(\mathscr{M}) < \infty$, the affine element maps are nondegenerate: the jacobians $B_T = DF_T$ in (14) are nonsingular, and $\|B_T\|_F \leq \kappa(\mathscr{M})$, see, eg., [5, Sec. II]. The reference simplex $\widehat{T}$ is contained in the unit cube $\widehat{K} = (0,1)^d$; with each $T \in \mathscr{M}$, we associate a parallelepiped via $K_T = F_T(\widehat{K})$ and *assume* that $K_T \subset \Omega$.

3.2 Local Polynomial Spaces

For $T \in \mathscr{M}$ the local polynomial approximation space $\mathbb{P}^p(T) = \mathrm{span}\{x^\alpha : |\alpha| \leq p\}$ is the linear space of all multivariate polynomials on $T \in \mathscr{M}$ whose total degree does not exceed p. The space $\mathbb{P}^p(T)$ is invariant under the affine mapping F_T, i.e. $u \in \mathbb{P}^p(T)$ if and only if $\hat{u} := u \circ F_T \in \mathbb{P}^p(\widehat{T})$. On parallelepipeds K, $\mathbb{Q}^p(K)$ is the affine image of $\mathbb{Q}^p(\widehat{K})$, $\widehat{K} = \widehat{I}^d$ with $\widehat{I} = (0,1)$,

$$\mathbb{Q}^p(\widehat{K}) = \operatorname{span}\{\widehat{x}^\alpha \,:\, 0 \leq \alpha_i \leq p,\ 1 \leq i \leq d\}\ . \tag{15}$$

For each parallelepiped K_T associated with a tetrahedron $T \in \mathscr{M}$ (resp. a triangle if $\Omega \subset \mathbb{R}^2$), with associated affine element mapping $F_T : \widehat{K} \to K_T$ and polynomial degree $p \geq 0$, we set

$$\mathbb{Q}^p(K_T) = \left\{ v \in L^2(K_T) : (v|_{K_T} \circ F_T) \in \mathbb{Q}^p(\widehat{K}) \right\}\ . \tag{16}$$

For polynomial degree $p \geq 1$, and for a family of regular, simplicial triangulations $\mathscr{M}^{(\ell)} \in \mathfrak{M}_{\kappa,\sigma}$ of Ω, we introduce finite element spaces of continuous, piecewise polynomial functions of total degree p on $\mathscr{M}^{(\ell)}$, i.e.

$$S^p(\mathscr{M}^{(\ell)}) = \left\{ u \in H^1(\Omega) \,:\, u|_T \in \mathbb{P}^p(T),\, T \in \mathscr{M}^{(\ell)} \right\}\ . \tag{17}$$

Typically, *hp*-FEMs are obtained when the level ℓ of geometric mesh refinement is tied to the polynomial degree p.

3.3 Mesh layers

A key ingredient in exponential convergence proofs of *hp*-FEM is *geometric mesh refinement* towards the set $\mathscr{S}$ of singularities. For a parameter $0 < \sigma < 1$, we call a regular, simplicial mesh family $\mathfrak{M}_{\kappa,\sigma} = \{\mathscr{M}^{(\ell)}\}_{\ell \geq 1}$ *σ-geometrically refined towards* $\mathscr{S} \subset \Omega$ if there exists $0 < \sigma < 1$ such that for every $T \in \mathscr{M}^{(\ell)} : \overline{T} \cap \mathscr{S} = \emptyset$, $\ell = 1, 2, \ldots$ holds

$$0 < \sigma < \rho(T;\mathscr{S}) := \frac{\operatorname{diam}(T)}{\operatorname{dist}(T,\mathscr{S})} < \frac{1}{\sigma}\ . \tag{18}$$

We tag members of a σ-geometric family $\mathfrak{M}_{\kappa,\sigma}$ by a subscript σ, i.e. we write $\mathscr{M}_\sigma^{(\ell)}$.

Proposition 1. *For $\mathscr{S} \subset \overline{\Omega}$, and for some $0 < \sigma < 1$ and $\kappa > 1$, consider a regular, nested and σ-geometrically refined (towards $\mathscr{S}$) κ-shape regular simplicial mesh family $\mathfrak{M}_{\kappa,\sigma}$ in Ω.*

Then, all elements $T \in \mathscr{M}_\sigma^{(\ell)}$ for every $\ell \geq 1$, can be grouped in mesh-layers*: there exists a partition*

$$\bigcup_{\ell \geq 1} \mathscr{M}_\sigma^{(\ell)} = \mathfrak{L}_1 \,\dot{\cup}\, \mathfrak{L}_2 \,\dot{\cup}\, \ldots . \tag{19}$$

and a constant $c(\mathfrak{M}_{\kappa,\sigma}) \geq 1$ with

$$\forall k \geq 1 : \qquad \#(\mathfrak{L}_k) \leq c(\mathfrak{M}_{\kappa,\sigma}) \tag{20}$$

and such that, for every $T \in \mathfrak{L}_k$ and every $k \geq 1$,

$$0 < \frac{1}{c(\mathfrak{M}_{\kappa,\sigma})} \leq \frac{\mathrm{diam}(T)}{\sigma^k} \leq c(\mathfrak{M}_{\kappa,\sigma}) \; . \tag{21}$$

Proof. The proof is by induction over ℓ.

Based on Proposition 1, we may assume that $\mathscr{M}_\sigma^{(\ell)}$ consists of $O(\ell)$ layers. Then, for ℓ sufficiently large, and for any constant $c_{\mathfrak{T}}(\kappa) > 0$ which is independent of ℓ, every mesh $\mathscr{M}_\sigma^{(\ell)} \in \mathfrak{M}_{\kappa,\sigma}$ may be partitioned into

$$\mathscr{M}_\sigma^{(\ell)} = \mathfrak{O}_\sigma^{(\ell)} \,\dot\cup\, \mathfrak{T}_\sigma^{(\ell)} \; , \tag{22}$$

where

$$\mathfrak{O}_\sigma^{(\ell)} := \mathfrak{O}_\sigma^{(\ell-1)} \,\dot\cup\, \mathfrak{L}_{\ell-1} = \mathfrak{L}_1 \,\dot\cup\, \mathfrak{L}_2 \,\dot\cup \ldots \dot\cup\, \mathfrak{L}_{\ell-1} \; ,$$

and there exists $c_{\mathfrak{T}} > 0$ being independent of ℓ such that for all ℓ holds

$$\mathscr{S} \subset \bigcup_{T \in \mathfrak{T}_\sigma^{(\ell)}} \overline{T} \; , \qquad \mathrm{dist}(\mathscr{S}, \mathfrak{O}^{(\ell)}) \geq c_{\mathfrak{T}} \sigma^\ell \; . \tag{23}$$

The terminal layers $\mathfrak{T}_\sigma^{(\ell)} \subset \mathscr{M}_\sigma^{(\ell)}$ in (22) satisfy the following properties.

Proposition 2. *There exists a constant $c_{\mathfrak{T}}(\kappa,\sigma) > 0$ such that for every $\mathscr{M}_\sigma^{(\ell)} \in \mathfrak{M}_{\kappa,\sigma}$, the set $\mathfrak{T}_\sigma^{(\ell)}$ has the following properties: for all $\ell \geq 1$ holds*

(i) $\#(\mathfrak{T}_\sigma^{(\ell)}) \leq c_{\mathfrak{T}}(\kappa,\sigma)$,
(ii) $\forall c \in \mathscr{S} : \; |\mathfrak{T}_\sigma^{(\ell)} \cap \omega_c| \leq c_{\mathfrak{T}}(\kappa,\sigma)\sigma^{d\ell}$,
(iii) $\forall T \in \mathfrak{T}_\sigma^{(\ell)} : \; h_T \leq c_{\mathfrak{T}}(\kappa,\sigma)\sigma^\ell$.

Proof. Assertion (i) follows from (20). Property (23) implies that for every $T \in \mathfrak{T}_\sigma^{(\ell)}$, $\mathrm{dist}(T, \mathscr{S}) \leq c_{\mathfrak{T}}(\kappa,\sigma)\sigma^\ell$. This implies, with the shape regularity of T, that for every $T \in \mathfrak{T}_\sigma^{(\ell)}$ holds $|T| \leq c_{\mathfrak{T}}(\kappa,\sigma)\sigma^{d\ell}$. This, in turn, implies assertion (ii). Since $\mathfrak{T}_\sigma^{(\ell)}$ is just the terminal layer $\mathfrak{L}_\ell$, Proposition 1 implies (i) and (iii).

Remark 2. (i) We *do not assume* that the singular supports $c \in \mathscr{S}$ comprise nodes of some triangulation $\mathscr{M}^{(\ell)} \in \mathfrak{M}_{\kappa,\sigma}$. This implies, in particular, that the ensuing exponential convergence proofs remain valid for "nearly coalescing" singular supports $c, c' \in \mathscr{S}$: for $c, c' \in \mathscr{S}$ such that $\mathrm{dist}(c,c') < \sigma^p$, both c and c' are contained in the terminal layers $\mathfrak{T}_\sigma^{(\ell)}$. There, a low-order quasi interpolant of Clément (resp. Scott-Zhang) type is used, see Section 4.2.8 ahead. The constants in the exponential convergence bound are uniform in w.r.t. to $\mathrm{dist}(c,c')$. (ii) Due to Prop. 2, item (iii), geometric mesh refinement implies that $\mathrm{dist}(c,c')$ is resolved with geometric refinements with $\ell \geq O(|\log(\mathrm{dist}(c,c'))|)$ many mesh layers.

4 Exponential Convergence

4.1 Statement of the Exponential Convergence Result

Theorem 1. *Suppose given a weight vector $\underline{\beta}$ as in* (5) *in a bounded polytope $\Omega \subset \mathbb{R}^d$, $d = 2,3$, with plane sides resp. faces.*

Then, for every sequence $\mathfrak{M}_{\kappa,\sigma}(\mathscr{S})$ of nested, regular simplicial meshes in Ω which are σ-geometrically refined towards $\mathscr{S}$ and which are κ shape-regular, there exist continuous projectors $\Pi^p_{\kappa,\sigma} : N^2_{-1-\underline{\beta}}(\Omega) \to S^p(\mathscr{M}^{(\ell)}_\sigma)$ with $\ell \simeq p^{1/\delta}$ and, for every $u \in \mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega))$ there exist constants $b, C > 0$ (depending on κ, C_u, d_u in (8) *and on σ) such that there holds the error bound*

$$\left\| u - \Pi^p_{\kappa,\sigma} u \right\|_{H^1(\Omega)} \le C \begin{cases} \exp(-bN^{\frac{1}{1+\delta d}}) & \delta \ge 1, \\ \left(\Gamma\left(N^{\frac{1}{1+\delta d}}\right)\right)^{-b(1-\delta)} & 0 < \delta < 1. \end{cases} \tag{24}$$

Here,

$$N = \dim(S^p(\mathscr{M}^{(\ell)}_\sigma)) = O(\ell p^d) = O(p^{d+1/\delta}).$$

If, moreover, $u|_{\partial\Omega} = 0$, then $(\Pi^p_{\kappa,\sigma} u)|_{\partial\Omega} = 0$ and (24) *holds.*

4.2 Proof

The proof of the *approximation result* Theorem 1 is based on constructing the projectors $\Pi^p_{\kappa,\sigma}$; our construction will proceed in several steps and we detail it for $d = 3$, the case $d = 2$ being a (minor) modification. First, we review from [26, Section 5] a family of univariate *hp*-projections with error bounds which are explicit in the polynomial degree as well as in the regularity of the functions to be approximated. A corresponding family of polynomial projectors on the unit cube $\widehat{K} = (0,1)^3$ with analogous consistency error bounds is then obtained as in [26, Section 5] by tensorization and scaling. We shall use these bounds for a tetrahedron $T \in \mathfrak{O}^{(\ell)}_\sigma \subset \mathscr{M}^{(\ell)}_\sigma \in \mathfrak{M}_{\kappa,\sigma}$ as follows. By Proposition 1, $T \in \mathfrak{L}_k$ for some $1 \le k \le \ell - 1$. The (up to orientation) unique parallelepiped $K_T = F_T(\widehat{K})$ associated with $T \in \mathfrak{L}_k$ has the same scaling properties as T, in particular (21) also holds for K_T. For u belonging to the analytic class (8) with weight vector satisfying (5), $u \in C^0(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \backslash \mathscr{S})$. For $T \in \mathfrak{O}^{(\ell)}_\sigma$, the pullback $\widehat{u}_T = u|_{K_T} \circ F_T$ satisfies on $\widehat{K}$ the same analytic derivative bounds as $u|_T \circ F_T$ on $\widehat{T}$ (with possibly larger constant C_u, depending on κ, but independent of ℓ and of T). The tensorized *hp* interpolation operator from [26] on $\widehat{K}$ is therefore well-defined and allows to construct a polynomial approximation $\widehat{u}^p_T \in \mathbb{Q}^p(\widehat{K})$ with analytic consistency error bounds on $\widehat{K}$; since $\widehat{T} \subset \widehat{K}$, and since $\mathbb{Q}^p(\widehat{T}) \subset \mathbb{P}^{pd}(\widehat{T})$, the pushforwards of the restrictions $\widehat{u}^p_T|_{\widehat{T}}$ under the affine mapping $F_T : \widehat{T} \to T$ will be local polynomial approximations of degree pd with exponential convergence estimates in $H^1(T)$. Moreover, since the tensorized interpolant is nodally exact in the vertices of $\widehat{K}$, and since the set of vertices of $\widehat{T}$ is a subset of the set of vertices of $\widehat{K}$, *the pushforwards of $\widehat{u}_T^p|_{\widehat{T}}$ under F_T are nodally exact in the vertices of T.*

For elements $T \in \mathfrak{T}_\sigma^{(\ell)}$, we only require a first order approximation property, as the geometric refinement guarantees the necessary convergence rate. We can not use nodal interpolation as functions $u \in \mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ may not be bounded near a singularity $c \in \mathscr{S}$. Thus, we construct a quasi interpolation operator on elements in the terminal layers $T \in \mathfrak{T}_\sigma^{(\ell)}$ that interpolates at those vertices of T which are not in $\mathscr{S}$.

By the continuity of $u \in \mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ on $\Omega\backslash\mathscr{S}$, the resulting global, piecewise polynomial approximation is nodally exact in all vertices of $\mathscr{M}_\sigma^{(\ell)}$ except those which coincide with singularities $c \in \mathscr{S}$. Particularly, the resulting piecewise polynomial hp-approximation is globally continuous at all vertices of $\mathscr{M}_\sigma^{(\ell)}$. However, it still has polynomial jump discontinuities across edges and (in space dimension $d=3$) faces of $T \in \mathscr{M}_\sigma^{(\ell)}$ which we remove by *polynomial trace liftings*, preserving the exponential convergence estimates.

4.2.1 Univariate *hp*-Projectors and *hp* Error Bounds

Let $I=(-1,1)$ be the unit interval. For any $k \geq 1$, we write $H^k(I)$ for the usual Sobolev space endowed with norm $\|u\|_{H^k(I)}$. For $q \geq 0$, we denote by $\widehat{\pi}_{q,0} : L^2(I) \to \mathbb{P}^q(I)$ the $L^2(I)$-projection. The following C^{k-1}-conforming and univariate projector has been constructed in [11, Section 8].

Lemma 1. *For any $p,k \in \mathbb{N}$ with $p \geq 2k-1$, there is a projector $\widehat{\pi}_{p,k} : H^k(I) \to \mathbb{P}^p(I)$ that satisfies $(\widehat{\pi}_{p,k}u)^{(k)} = \widehat{\pi}_{p-k,0}(u^{(k)})$, and $(\widehat{\pi}_{p,k})^{(j)}u(\pm 1) := u^{(j)}(\pm 1)$, for any $j=0,\ldots,k-1$.*

Moreover, there holds:

(i) For every $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that

$$\forall u \in H^k(I)\ , \forall p \geq 2k-1 : \quad \|\widehat{\pi}_{p,k}u\|_{H^k(I)} \leq C_k \|u\|_{H^k(I)}\ . \tag{25}$$

(ii)For integers $p,k \in \mathbb{N}$ with $p \geq 2k-1$, $\kappa = p-k+1$ and for $u \in H^{k+s}(I)$ with any $k \leq s \leq \kappa$ there holds the error bound

$$\|(u-\widehat{\pi}_{p,k}u)^{(j)}\|^2_{L^2(I)} \leq \frac{(\kappa-s)!}{(\kappa+s)!}\|u^{(k+s)}\|^2_{L^2(I)}, \qquad j=0,1,\ldots,k. \tag{26}$$

We refer to [11, Proposition 8.4] and [11, Theorem 8.3], respectively, for proofs, and further references.

4.2.2 Tensor projector on the unit cube

Based on the univariate projectors $\widehat{\pi}_{p,k}$, we constructed in [26] polynomial projection operators on $I^d = (0,1)^d$ by a) translation and scaling of the projectors $\widehat{\pi}_{p,k}$ to $(0,1)$ and b) by tensorization, as follows: for integers $k \geq 0$ and $d > 1$, we define

$$H^k_{mix}(I^d) = H^k(I) \underbrace{\otimes \cdots \otimes}_{d-\text{times}} H^k(I), \tag{27}$$

where $\otimes$ denotes the tensor-product of separable Hilbert spaces. These spaces are isomorphic to Bochner spaces, ie. $H^k_{mix}(I^d) \simeq H^k(I; H^k_{mix}(I^{d-1})) \simeq H^k_{mix}(I^{d-1}; H^k(I))$. In I^d of dimension $d > 1$ and for $p \geq 2k-1$, we define the projector

$$\widehat{\Pi}^d_{p,k} = \bigotimes_{i=1}^{d} \widehat{\pi}^{(i)}_{p,k} : \;\; H^k_{mix}(I^d) \to \mathbb{Q}^p(I^d) \tag{28}$$

where $\widehat{\pi}^{(i)}_{p,k}$ denotes the univariate projector in Lemma 1, applied in coordinate $1 \leq i \leq d$. For $d, k \geq 1$ there exists a constant $C_{k,d} > 0$ such that for all $p \geq 2k-1$ there holds the stability bound

$$\|\widehat{\Pi}^d_{p,k} v\|_{H^k_{mix}(I^d)} \leq C_{k,d} \|v\|_{H^k_{mix}(I^d)} \tag{29}$$

and

$$\left\| v - \widehat{\Pi}^d_{p,k} v \right\|_{H^k_{mix}(I^d)} \leq C_{k,d} \sum_{i=1}^{d} \|v - \widehat{\pi}^{(i)}_{p,k} v\|_{H^k(I; H^k_{mix}(I^{d-1}))} \,. \tag{30}$$

We choose throughout what follows $k = 2$ as in [26], and obtain from (30), (26)

Proposition 3. *[26] Assume that the polynomial degree $p \geq 5$. Then, for any integers $3 \leq s \leq p$, and for $v \in H^{s+5}(\widehat{K})$, there holds*

$$\|v - \widehat{\Pi}^3_{p,2} v\|^2_{H^2_{mix}(\widehat{K})} \lesssim \Psi_{p-1,s-1} \sum_{m=s}^{s+5} |v|^2_{m,\widehat{K}} \tag{31}$$

where the constant implied in $\lesssim$ is independent of s and of p, and where

$$\Psi_{q,r} = 2^{2(r+3)} \frac{\Gamma(q+1-r)}{\Gamma(q+1+r)}, \qquad 0 \leq r \leq q \,. \tag{32}$$

Moreover, $\widehat{\Pi}^3_{p,2} v$ is nodally exact in the vertices of $\widehat{K} = (0,1)^3$:

$$(\widehat{\Pi}^3_{p,2} v)(x_1, x_2, x_3) = v(x_1, x_2, x_3) \quad \forall x_i \in \{0,1\} \,. \tag{33}$$

4.2.3 Transformation Formula

For $u \in H^k(\Omega)$, and for a simplex $T \in \mathfrak{O}^{(\ell)}_\sigma$, $\hat{u}_T = u|_T \circ F_T \in H^k(\widehat{T})$ for every $k \geq 0$. Quantitative bounds on derivatives under affine transformations F_T in (14) are provided by the *transformation formula* (eg. [5, Section II.6.6]).

Lemma 2. *Let $G \subset \mathbb{R}^d$, $d \geq 2$, denote a bounded polyhedron which is affine equivalent to $\widehat{G}$ via* (14)*, ie. $G = F_T(\widehat{G})$. For $v \in H^k(G)$ and for any $k \in \mathbb{N}$, the pullback $\hat{v}_T := v|_G \circ F_T$ satisfies with $|v|^2_{m,T} = \sum_{|\alpha|=m} \|D^\alpha v\|^2_{L^2(G)}$ and with the Frobeniusnorm $\|B_T\|_F$ of the matrix B_T in* (14) *the bound*

$$|\hat{v}|_{m,\widehat{G}} \leq d^m \|B_T\|^m_F |\det(B_T)|^{-1/2} |v|_{m,G} \ . \tag{34}$$

4.2.4 Element Interpolants

For any simplex $T \in \mathfrak{O}^{(\ell)}_\sigma$, the function $u \in \mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ the polynomial approximation of $u|_T$, $u \in \mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ is obtained by applying Proposition 3 to $\hat{u}_T := u|_{K_T} \circ F_T$:

$$\forall T \in \mathfrak{O}^{(\ell)}_\sigma : \quad u^p_T := \left(\widehat{\Pi}^3_{p,2}(u|_{K_T} \circ F_T)\right)|_{\widehat{T}} \circ F^{(-1)}_T \ . \tag{35}$$

With u^p_T as in (35) we define the *hp-base interpolant* $\tilde{I}^p$ in $\mathfrak{O}^{(\ell)}_\sigma$ by

$$\forall T \in \mathfrak{O}^{(\ell)}_\sigma \subset \mathscr{M}^{(\ell)}_\sigma : \qquad (\tilde{I}^p u)|_T := u^p_T \ . \tag{36}$$

The bound (23) *with $c_{\mathfrak{T}} > 0$ sufficiently large, independent of ℓ* ensures that there exists $c(\kappa,\sigma) > 0$ such that the associated K_T satisfies

$$\forall \ell \in \mathbb{N} \ \forall T \in \mathfrak{O}^{(\ell)}_\sigma : \qquad \mathrm{dist}(K_T,\mathscr{S})/\mathrm{diam}(K_T) \geq 1/c \ . \tag{37}$$

4.2.5 Exponential Convergence in Broken Sobolev Norms

Proposition 4. *For $u \in \mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ with* (5)*, there are $b, C > 0$ (depending on u) such that for every $p \geq 1$ and for $\tilde{I}^p$ from* (36) *holds with $\ell \geq 1$*

$$\|u - \tilde{I}^p u\|_{H^1(\mathfrak{O}^{(\ell)}_\sigma)} \leq C \begin{cases} \exp(-bp^{1/\delta}) & \delta \geq 1, \\ (p!)^{b(\delta-1)} & \delta < 1. \end{cases} \tag{38}$$

Here $C > 0$ depends on u and σ, but is independent of p, and $H^1(\mathfrak{O}^{(\ell)}_\sigma)$ denotes the broken H^1 space over $\mathfrak{O}^{(\ell)}$, with corresponding norm.

Proof. Since $\mathscr{S}$ consists of finitely many singular points c, by localization and superposition, we may assume wlog. $\mathscr{S} = \{c\}$ and denote by $\beta = \beta_c > -2$. For

$1 \le k \le \ell < p$, consider a simplex $T \in \mathfrak{L}_k \cap \omega_c \subset \mathscr{M}_\sigma^{(\ell)}$ and the associated parallelepiped $K_T = F_T(\widehat{K}) \supset T$. It satisfies

$$0 < \sigma/c(\mathfrak{M}_{\kappa,\sigma}) < r_c(x)|_{K_T}/\sigma^k < c(\mathfrak{M}_{\kappa,\sigma})/\sigma \ , \quad x \in K_T \ .$$

By assumption, $K_T \subset \Omega$ and, by (23), $\mathrm{dist}(K_T, \mathscr{S}) \ge c_{\mathfrak{T}}\sigma^k$. Then, for $u \in \mathscr{G}^\delta_{\underline{\beta}}(\mathscr{S};\Omega)$ and for this $T \in \mathfrak{L}_k$, $\hat{u}_T := u|_{K_T} \circ F_T$ is smooth in $\overline{\widehat{K}}$ and satisfies, by (34) with $G = K_T$ and $\widehat{G} = \widehat{K}$,

$$\forall m \in \mathbb{N} : \quad |\hat{u}_T|_{m,\widehat{K}} \le d^m \|B_T\|_F^m |\det(B_T)|^{-1/2} |u|_{m,K_T} \ .$$

We obtain for $|u|_{m,K_T}$ using (18) and (21)

$$\begin{aligned} |u|^2_{m,K_T} &= \|D^m u\|^2_{L^2(K_T)} \lesssim \|r_c^{\beta+m}\sigma^{-k(\beta+m)} D^m u\|^2_{L^2(K_T)} \\ &\le \sigma^{-2k(\beta+m)} \|r_c^{\beta+m} D^m u\|^2_{L^2(K_T)} \le \sigma^{-2k(\beta+m)} C_u^{2(m+1)} (m!)^{2\delta} \ . \end{aligned}$$

We define $u_T^p \in \mathbb{Q}^p(T) \subset \mathbb{P}^{pd}(T)$ as in (35). From (31), for every integer $3 \le s \le p$ and with $\Psi_{q,r}$ as in (32) and for $j = 0,1,2$,

$$\|D^j(\hat{u} - \hat{u}_T^p)\|^2_{L^2(\widehat{T})} \le \|D^j(\hat{u} - \hat{u}_T^p)\|^2_{L^2(\widehat{K})} \le \Psi_{p-1,s-1} \sum_{m=s}^{s+5} |\widehat{u}_T|^2_{m,\widehat{K}} \ .$$

Using the κ-shape regularity of $T \in \mathfrak{L}_k \subset \mathscr{M}_\sigma^{(k)} \in \mathfrak{M}_{\kappa,\sigma}$, we find $h_T \lesssim \|B_T\|_F \le \kappa h_T$ (eg. [5, (Chap. II, (6.9)]) and, by (21) and (34), that $h_T \simeq \kappa\sigma^k$ so that for every $m \in \mathbb{N}$

$$|\widehat{u}_T|^2_{m,\widehat{K}} \le \frac{(\kappa d \sigma^k)^{2m}}{|\det(B_T)|} |u|^2_{m,K_T} \le \frac{(\kappa d \sigma^k)^{2m}}{|\det(B_T)|} \sigma^{-2k(\beta+m)} C_u^{2(m+1)} (m!)^{2\delta} \ .$$

We obtain for $j = 0,1,2$ the bound

$$\|\widehat{D}^j(\hat{u} - \hat{u}_T^p)\|^2_{L^2(\widehat{T})} \le \Psi_{p-1,s-1} \sum_{m=s}^{s+5} \frac{(\kappa d \sigma^k)^{2m}}{|\det(B_T)|} \sigma^{-2k(\beta+m)} C_u^{2(m+1)} (m!)^{2\delta} \ .$$

Transporting to $T = F_T(\widehat{T}) \in \mathfrak{L}_k$, we find for $\beta_c = -1 - b_c$ and $j = 0,1,2$.

$$\begin{aligned} \|D^j(u - u_T^p)\|^2_{L^2(T)} &\lesssim \Psi_{p-1,s-1} \sum_{m=s}^{s+5} (\kappa d \sigma^k)^{2(m-j)} \sigma^{-2k(\beta+m)} C_u^{2(m+1)} (m!)^{2\delta} \\ &\lesssim \Psi_{p-1,s-1} (\kappa d C_u)^{2s} \sigma^{2k(1+b_c-j)} \Gamma(s+6)^{2\delta} \ . \end{aligned} \tag{39}$$

For $T \in \mathfrak{O}^{(\ell)}$, we define the piecewise polynomial interpolant $\tilde{I}^p u|_T$ by (35). Then $\tilde{I}^p u$ coincides with u in the vertices of all $T \in \mathfrak{O}^{(\ell)}$ and is in particular continuous in these vertices; it is, however, in general discontinuous across edges and faces.

Using the finite cardinality (20), and summing the bound (39) with $j = 0,1$ over layers $\mathfrak{L}_1, ..., \mathfrak{L}_{\ell-1}$, we obtain with $\bar{C} := C_u \kappa d$ and $\beta_c = -1 - b_c$, $0 < b_c < 1$

$$\begin{aligned}\|u-\tilde{I}^p u\|_{H^1(\mathfrak{O}^{(p)})} &\leq C(\kappa,\sigma)\Psi_{p-1,s-1}\bar{C}^{2s}\Gamma(s+6)^{2\delta}\sum_{k=1}^{\ell-1}\sigma^{2kb_c} \\ &= C(\kappa,\sigma)\Psi_{p-1,s-1}\bar{C}^{2s}\Gamma(s+6)^{2\delta}\frac{\sigma^{2b_c}}{1-\sigma^{2b_c}}\,.\end{aligned} \tag{40}$$

We have for $s<p$ with the recursion formula $\Gamma(z+1)=z\Gamma(z)$ that

$$\frac{\Gamma(p-s+1)\Gamma(s+6)^{2\delta}}{\Gamma(p+s-1)} \lesssim (p-s)^{-2s}s^{2\delta s} \tag{41}$$

Choosing $p=cs^{\delta}$ with $c>1$ to be selected (this ensures $s<p$ in the upper bound (41)) we obtain in the case $\delta\geq 1$

$$\frac{\Gamma(p-s+1)\Gamma(s+6)^{2\delta}}{\Gamma(p+s-1)} \lesssim \Big(\frac{s^{\delta}}{cs^{\delta}-s}\Big)^{2s}.$$

Choosing $c=2\bar{C}+1>1$ this implies for $s\gtrsim 1$ sufficiently large the bound

$$\Psi_{p-1,s-1}\bar{C}^{2s}\Gamma(s+6)^{2\delta} \lesssim \Big(\frac{2\bar{C}s^{\delta}}{cs^{\delta}-s}\Big)^{2s} \lesssim \Big(\frac{2\bar{C}}{2\bar{C}+1}\Big)^{2c^{-1/\delta}p^{1/\delta}} \tag{42}$$

where the constant hidden in $\lesssim$ is independent of the polynomial degree p.

In the case $0<\delta<1$, we choose $s=p/2$ in (41) and obtain

$$\frac{\Gamma(p-s+1)\Gamma(s+6)^{2\delta}}{\Gamma(p+s-1)} \lesssim s^{-(1-\delta)2s} \lesssim (p!)^{-b(1-\delta)}$$

for some $0<b<1$. Inserting this bound into (40) completes the proof. □

4.2.6 Polynomial Trace Lifting in $\mathfrak{O}_{\sigma}^{(p)}$

By the nodal exactness (33), the hp base interpolant $\tilde{I}^p$ constructed in (36) of Proposition 4 is exact, and hence continuous in vertices of simplices $T\in\mathfrak{O}_{\sigma}^{(\ell)}$, but has in general discontinuities across interelement edges $E\in\mathscr{E}_T$ of simplices $T\in\mathfrak{O}_{\sigma}^{(\ell)}$ (in dimensions $d=2,3$) and across interelement faces $F\in\mathscr{F}_T$ of simplices $T\in\mathfrak{O}_{\sigma}^{(\ell)}$ (in dimension $d=3$). The jumps of interpolant across edges and faces are polynomial, i.e. $[\![\tilde{I}^p]\!]_E$ and $[\![\tilde{I}^p]\!]_F$.

For each $T\in\mathfrak{O}_{\sigma}^{(\ell)}$, the nodal exactness (33) of the base hp-interpolant $\tilde{I}^p u$ implies for each $E\in\mathscr{E}_T$ that $[\![\tilde{I}^p u]\!]_E\in\mathbb{P}_0^{pd}(E):=(\mathbb{P}^{pd}\cap H_0^1)(E)$, $d=2,3$, and, for $d=3$ and each $F\in\mathscr{F}_T$, $[\![\tilde{I}^p u]\!]_F\in\mathbb{P}^{pd}(F)$. We build a continuous, piecewise polynomial interpolant by successively lifting these polynomial trace jumps of $\tilde{I}^p$ while retaining its consistency, in particular the analytic estimates (31).

First, we lift jumps on interelement edges $E \in \mathscr{E}_T$ and, second, in dimension $d = 3$ also for all interelement faces $F \in \mathscr{F}_T$, for every $T \in \mathfrak{O}^{(\ell)}_\sigma$. Since $T \in \mathfrak{O}^{(\ell)}_\sigma \subset \mathscr{M}^{(\ell)}_\sigma \in \mathfrak{M}_{\kappa,\sigma}$ is κ shape-regular, so are all $F \in \mathscr{F}_T$. For $E \in \mathscr{E}_T$, let $F_E \in \mathscr{F}_T$ denote any face in $\mathscr{F}_T$ with $E \subset \partial F$.

We recapitulate from [24, Lemma 15, Thm. 1] the required lifting and the stability estimates. Consider the reference simplex $\widehat{T} \subset \mathbb{R}^d$, $d = 2,3$. Given a piecewise polynomial function $\hat{g}_p$ of degree p on each $\widehat{F} \in \mathscr{F}_{\widehat{T}}$ that is continuous on $\partial\widehat{T}$, in [24, Lemma 15, Thm. 1], a polynomial trace lifting $\hat{v}_p = \mathscr{L}_{\widehat{T},\widehat{\partial T}}(\hat{g}_p) \in \mathbb{P}^p(\widehat{T})$ is constructed which satisfies on the reference simplex $\widehat{T}$ in space dimension $d = 2,3$ the bound $\|\hat{v}_p\|_{H^1(\widehat{T})} \le \widehat{C}\|\hat{g}_p\|_{H^{1/2}(\partial\widehat{T})}$ (with $\hat{C} > 0$ independent of p). As $H^{1/2}(\widehat{T}) = (L^2(\widehat{T}), H^1(\widehat{T}))_{1/2}$, we have the interpolation inequality $\|\hat{g}_p\|_{H^{1/2}(\partial\widehat{T})} \le \widehat{C}\|\hat{g}_p\|^{1/2}_{L^2(\partial\widehat{T})}\|\hat{g}_p\|^{1/2}_{H^1(\partial\widehat{T})}$. With the polynomial inverse inequality (see, e.g., [29]) on each face $\widehat{F} \subset \partial\widehat{T}$ we get (with a possibly different constant $\hat{C} > 0$ which is independent of p)

$$\|\hat{v}_p\|_{H^1(\hat{T})} \le \widehat{C}p\|\hat{g}_p\|_{L^2(\partial\hat{T})} \,. \tag{43}$$

Squaring this and scaling $\widehat{T}$ to $T = F_T(\hat{T}) \in \mathfrak{O}^{(p)}_\sigma$ we find

$$\|\mathscr{L}_{T,\partial T}(g_p)\|^2_{L^2(T)} + h_T^2\|D^1\mathscr{L}_{T,\partial T}(g_p)\|^2_{L^2(T)} \le C(\kappa)p^2 h_T\|g_p\|^2_{L^2(\partial T)} \,. \tag{44}$$

Iterating (43) twice, from $\hat{E} \subset \partial\hat{F}$ to $\hat{F} \subset \partial\hat{T}$ to $\hat{T}$, we obtain for $\widehat{g}_p \in \mathbb{P}^p_0(\widehat{E})$ a polynomial edge lifting $\widehat{\mathscr{L}_{\widehat{T},\widehat{E}}}(\widehat{g}_p) \in \mathbb{P}^p(\widehat{T})$ on the reference simplex $\widehat{T} \subset \mathbb{R}^3$ with

$$\|\hat{\mathscr{L}}_{\hat{T},\hat{E}}(\hat{g}_p)\|_{H^1(\hat{T})} \le \widehat{C}p^2\|\hat{g}_p\|_{L^2(\hat{E})} \,. \tag{45}$$

Squaring (45) and scaling to $T = F_T(\hat{T}) \in \mathfrak{O}^{(p)}_\sigma$ yields for $g_p \in \mathbb{P}^p_0(E)$ on $E \in \mathscr{E}_T$

$$h_T^{-2}\|\mathscr{L}_{T,E}(g_p)\|^2_{L^2(T)} + \|D^1\mathscr{L}_{T,E}(g_p)\|^2_{L^2(T)} \le C(\kappa)p^4\|g_p\|^2_{L^2(E)} \,. \tag{46}$$

Let now $d = 3$ and let $F, F' \in \mathscr{F}_T$ be two distinct faces which share edge $\bar{E} = \overline{F} \cap \overline{F'}$. Using (43) in dimension $d = 2$ and scaled to T, we lift $g_p = [\![\tilde{I}^p u]\!]_E \in \mathbb{P}^{pd}_0(E)$ twice, once into F and once into F', resulting in a $v_p \in C^0(\overline{F \cup F'})$, $v_p \in \mathbb{P}^{pd}(F) \cup \mathbb{P}^{pd}(F')$, and $v_p\,|_{\partial\overline{F\cup F'}} = 0$ which satisfies (44) with F in place of T. We may therefore extend this continuous, piecewise polynomial function v_p from $\overline{F \cup F'}$ by zero to a function $\tilde{v}_p \in C^0(\partial T)$ which is, on each $F \in \mathscr{F}_T$, a polynomial of total degree at most pd. There exists a lifting $\mathscr{L}_{T,F}(\tilde{v}_p) \in \mathbb{P}^{pd}(T)$ such that for each $F \in \mathscr{F}_T$ we have $\mathscr{L}_{T,F}(\tilde{v}_p)\,|_F = v_p\,|_F$ on $F \in \mathscr{F}_E$, $(\mathscr{L}_{T,F}(\tilde{v}_p)\,|_F)\,|_E \equiv g_p$ on E and such that (46) holds. For each edge E in $\mathfrak{O}^{(p)}_\sigma$, we lift the polynomial jump in this way into all $T \in \mathfrak{O}^{(p)}_\sigma$ for which $E \in \mathscr{E}_T$ by the *edge-lifting operator*

$$\mathscr{L}_E(g_p) := \sum_{T:E \in \mathscr{E}_T} \mathscr{L}_{T,E}(g_p) \,. \tag{47}$$

By κ shape regularity, $\#\{T \in \mathfrak{O}_\sigma^{(p)} : E \in \mathscr{E}_T\}$ is bounded independently of p and of the particular edge E by an absolute constant depending only on κ. With $\tilde{I}^p$ in (36), we define

$$\check{I}^p u := \tilde{I}^p u - \sum_E \mathscr{L}_E([\![\tilde{I}^p u]\!]_E) . \tag{48}$$

Then, $\check{I}^p u$ is continuous across edges $E \in \mathscr{E}_T$ for every $T \in \mathfrak{O}_\sigma^{(p)}$, and $[\![\check{I}^p u]\!]_F \in \mathbb{P}_0^{pd}(F) := (\mathbb{P}^{pd} \cap H_0^1)(F)$ for all $F \in \mathscr{F}_T$.

We next lift, for each face $F \in \mathscr{F}_T$, the face jump $[\![\check{I}^p u]\!]_F \in \mathbb{P}_0^{pd}(F)$ by extending first by zero to all other faces $F' \in \mathscr{F}_T \backslash \{F\}$, then lift polynomially by referring to [24, Theorem 1]. By construction, this lifting $\mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F) \in \mathbb{P}^p(T)$ will vanish on all $F' \in \mathscr{F}_T : F' \neq F$. For each face F, we repeat this lifting at most twice for $T, T' \in \mathfrak{O}_\sigma^{(p)}$ such that $F \in \mathscr{F}_T \cap \mathscr{F}_{T'}$. We define the continuous interpolant

$$\begin{aligned} I^p u &:= \check{I}^p u - \sum_{F \in \mathscr{F}_T : T \in \mathfrak{O}_\sigma^{(p)}} \mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F) \\ &= \tilde{I}^p u - \sum_{E \in \mathscr{E}_T : T \in \mathfrak{O}_\sigma^{(p)}} \mathscr{L}_E([\![\tilde{I}^p u]\!]_E) - \sum_{F \in \mathscr{F}_T : T \in \mathfrak{O}_\sigma^{(p)}} \mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F) . \end{aligned} \tag{49}$$

To verify exponential convergence in submesh $\mathfrak{O}_\sigma^{(\ell)}$, we estimate in (49)

$$\begin{aligned} \|u - I^p u\|_{H^1(\mathfrak{O}_\sigma^{(\ell)})} \leq \|u - \tilde{I}^p u\|_{H^1(\mathfrak{O}_\sigma^{(\ell)})} &+ \Big\| \sum_{E \in \mathscr{E}_T : T \in \mathfrak{O}_\sigma^{(\ell)}} \mathscr{L}_E([\![\tilde{I}^p u]\!]_E) \Big\|_{H^1(\mathfrak{O}_\sigma^{(\ell)})} \\ &+ \Big\| \sum_{F \in \mathscr{F}_T : T \in \mathfrak{O}_\sigma^{(\ell)}} \mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F) \Big\|_{H^1(\mathfrak{O}_\sigma^{(\ell)})} . \end{aligned} \tag{50}$$

The first term was bound in Prop. 4. We bound the second term.

For $T \in \mathfrak{O}_\sigma^{(p)}$, we write, using $[\![u]\!]_E = 0$ for $E \in \mathscr{E}_T$

$$\begin{aligned} & h_T^{-2} \|\mathscr{L}_{T,E}([\![\tilde{I}^p u]\!]_E)\|_{L^2(T)}^2 + \|D^1 \mathscr{L}_{T,E}([\![\tilde{I}^p u]\!]_E)\|_{L^2(T)}^2 \\ \leq \quad & C(\kappa) p^4 \|[\![\tilde{I}^p u]\!]_E\|_{L^2(E)}^2 = C(\kappa) p^4 \|[\![u - \tilde{I}^p u]\!]_E\|_{L^2(E)}^2 . \end{aligned} \tag{51}$$

The multiplicative trace inequality implies for a κ-shape regular simplex $T \subset \mathbb{R}^d$ with diameter h_T that for every $F \in \mathscr{F}_T$ and for every $\varphi \in H^1(T)$ holds

$$\|\varphi|_F\|_{L^2(F)}^2 \leq C(\kappa) \left(h_T^{-1} \|\varphi\|_{L^2(T)}^2 + h_T \|D^1 \varphi\|_{L^2(T)}^2 \right) . \tag{52}$$

Iterating this for $T \in \mathfrak{O}_\sigma^{(\ell)}$ from $E \in \mathscr{E}_T$ to $F \in \mathscr{F}_T$ gives, for $\varphi \in H^2(T)$,

$$\|\varphi|_E\|_{L^2(E)}^2 \lesssim h_T^{-2} \|\varphi\|_{L^2(T)}^2 + \|D^1 \varphi\|_{L^2(T)}^2 + h_T^2 \|D^2 \varphi\|_{L^2(T)}^2 \tag{53}$$

where the implied constant depends only on κ.

Using (53) with $\varphi = (u - \tilde{I}^p u)|_T = u|_T - u_T^p \in H^2(T)$ for $T \in \mathfrak{O}_\sigma^{(\ell)}$ in (51) gives

$$h_T^{-2} \|\mathscr{L}_{T,E}([\![\tilde{I}^p u]\!]_E)\|^2_{L^2(T)} + \|D^1 \mathscr{L}_{T,E}([\![\tilde{I}^p u]\!]_E)\|^2_{L^2(T)} \lesssim p^4 \sum_{j=0}^{2} h_T^{2(j-1)} \|D^j (u - u_T^p)\|^2_{L^2(T)} .$$

Using (39) and that $h_T \sim \sigma^k$ for $T \in \mathfrak{L}_k$ we obtain

$$\|\mathscr{L}_{T,E}([\![\tilde{I}^p u]\!]_E)\|^2_{H^1(T)} \lesssim p^4 \Psi_{p-1,s-1} (\kappa d C_u)^{2s} \Gamma(s+6)^{2\delta} \sigma^{2kb_c} . \tag{54}$$

Finally, we bound the third term in (50), ie. $\|\mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F)\|_{H^1(T)}$ for $F \in \mathscr{F}_T$. Since $\mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F) = 0$ on $\partial T \backslash F$, by the Poincaré inequality in $\{v \in H^1(T) : v|_{\partial T \backslash F} = 0\}$ it suffices to bound $\|D^1 \mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F)\|_{L^2(T)}$. Since $[\![u]\!]_F = 0$, using (48) we obtain

$$h_T^{-1} \|\mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F)\|_{L^2(T)} \lesssim \|D^1 \mathscr{L}_{T,F}([\![\check{I}^p u]\!]_F)\|_{L^2(T)} = \|D^1 \mathscr{L}_{T,F}([\![u - \check{I}^p u]\!]_F)\|_{L^2(T)} .$$

We estimate further, using the stability of the lifting $\mathscr{L}_{T,F}$ and (52),

$$\begin{aligned} \|D^1 \mathscr{L}_{T,F}([\![u - \check{I}^p u]\!]_F)\|^2_{L^2(T)} &\lesssim p^2 \|u - \check{I}^p u\|^2_{L^2(F)} \\ &\lesssim p^2 (h_T^{-1} \|u - \check{I}^p u\|^2_{L^2(T)} + h_T \|D^1 (u - \check{I}^p u)\|^2_{L^2(T)}) . \end{aligned} \tag{55}$$

Recalling (48), we bound for $j = 0, 1$

$$\begin{aligned} \|D^j (u - \check{I}^p u)\|^2_{L^2(T)} &= \|D^j (u - \tilde{I}^p u + \textstyle\sum_E \mathscr{L}_{T,E}([\![\tilde{I}^p u]\!]_E))\|^2_{L^2(T)} \\ &\lesssim \|D^j (u - \tilde{I}^p u)\|^2_{L^2(T)} + \textstyle\sum_E \|D^j (\mathscr{L}_{T,E}([\![\tilde{I}^p u]\!]_E))\|^2_{L^2(T)} . \end{aligned}$$

We use (39) for the first term, and (54) for the second term to conclude for $j = 0, 1$

$$\|D^j (u - \check{I}^p u)\|^2_{L^2(T)} \lesssim p^4 \Psi_{p-1,s-1} (\kappa d C_u)^{2s} \Gamma(s+6)^{2\delta} \sigma^{2k(1+b_c-j)} .$$

Using again that $T \in \mathfrak{L}_k$ satisfies $h_T \sim \sigma^k$, we insert into (55) and arrive at

$$\|D^1 \mathscr{L}_{T,F}([\![u - \check{I}^p u]\!]_F)\|^2_{L^2(T)} \lesssim p^6 \Psi_{p-1,s-1} (\kappa d C_u)^{2s} \Gamma(s+6)^{2\delta} \sigma^{2kb_c} .$$

Inserting this and the bound (54) into (50), we obtain for $\|u - I^p U\|_{H^1(\mathfrak{O}_\sigma^{(\ell)})}$ exactly once more the bound (40) (with a slightly higher power of p). Absorbing the polynomial factor into the exponential, we conclude the exponential error bounds from Proposition 4, i.e.,

$$\|u - I^p u\|_{H^1(\mathfrak{O}_\sigma^{(\ell)})} \leq C \begin{cases} \exp(-b p^{1/\delta}) & \delta \geq 1, \\ (p!)^{b(\delta-1)} & \delta < 1. \end{cases} \tag{56}$$

also for the resulting continuous *hp*-interpolant $I^p u$ defined in (49) in $\mathfrak{O}_\sigma^{(p)}$ using again (42).

4.2.7 Enforcement of homogeneous Dirichlet boundary conditions

The preceding polynomial trace liftings allow to obtain interpolation operators $\{I_N^{hp}\}_N$ which preserve homogeneous Dirichlet boundary conditions on $\partial\Omega$. For simplicity, we discuss this only for the case of global homogeneous Dirichlet boundary conditions, i.e., for $u|_{\partial\Omega} = 0$ (the argument being local, i.e., element-by-element, allows to treat homogeneous Dirichlet boundary conditions also on a proper subset $\Gamma_D \subset \partial\Omega$, as long as $\overline{\Gamma_D}$ coincides with the closure of a set of boundary faces). In space dimension $d = 3$, for $T \in \mathfrak{O}_\sigma^{(\ell)}$ with $F \in \mathscr{F}_T$ satisfying $F \subset \partial\Omega$, it holds $u|_F = 0$. Hence, on T we may adjust the (nodally exact) hp (quasi-)interpolant $(\tilde{I}^p u)|_T$ by lifting its trace $(\tilde{I}^p u)|_F = -(u - \check{I}^p u)|_F$ on the boundary face $F \in \mathscr{F}_T \cap \partial\Omega$ exactly as in (49), in particular preserving the exponential convergence bound (56). In space dimension $d = 2$, a corresponding polynomial edge-lifting can like wise be applied. In space dimension $d = 1$, Ω is a bounded interval on $\mathbb{R}$. The nodal exactness of the hp (quasi-)interpolant $(\tilde{I}^p u)$ implies that is satisfies the zero Dirichlet boundary conditions, so that trace lifting is not necessary in space dimension $d = 1$.

4.2.8 Approximation in $\mathfrak{T}_\sigma^{(\ell)}$

Exponential consistency errors for error contributions of the hp-interpolant from the terminal layer will be obtained essentially by a Bramble-Hilbert style scaling ("h-version FEM") argument combined with the exponentially small meshwidth of elements $T \in \mathfrak{T}_\sigma^{(\ell)}$ (see Proposition 2, items (ii), (iii)).

Under (5), for $\Omega \subset \mathbb{R}^3$ holds $N^2_{\underline{\beta}}(\Omega) \subset H^{1+\theta}(\Omega)$ for some $\theta > (d-2)/2$, $d = 2,3$, by [17, Thm. 3.5]. Specifically, $\theta(\underline{\beta}) = 1 - \beta_m - \varepsilon$ (cp. [17, Thm. 3.5] with $\beta_m := -1 - \beta_c \in (0, 1/2)$, and $0 < \varepsilon < 1/2 - \beta_m = 3/2 + \beta_c$). For $\Omega \subset \mathbb{R}^2$, $N^2_{\underline{\beta}}(\Omega) \subset H^{1+\theta}(\Omega)$ for some $\theta > 0$ (cp. [16]). which implies $\theta = 2 + \beta_c - \varepsilon > 1/2$). From Proposition 2 items (1)-(3), the collections $\mathfrak{T}_c := \{T \in \mathfrak{T}_\sigma^{(\ell)} : T \in \omega_c\}$, $c \in \mathscr{S}$ have uniformly bounded (w.r. to ℓ) cardinality and shape regularity. We construct the approximation by use of a Scott-Zhang quasi-interpolating projection operator $J_c \colon H^1(\bigcup \mathfrak{T}_c) \to S^1(\mathfrak{T}_c)$. This operator is constructed by choosing faces (edges) F_z for each vertex of $\mathfrak{T}_c$ via

$$J_c(v) := \sum_{z \text{ vertex of } \mathfrak{T}_c} \phi_z \int_{F_z} \phi_z^\star v \, dx,$$

where $\phi_z \in S^1(\mathfrak{T}_c)$ denotes the hat function associated with the vertex z and $\phi_z^\star \in S^1(F_z)$ denotes the unique dual basis function associated with ϕ_z. We define $\Gamma_c := \partial(\bigcup \mathfrak{T}_c) \cap \Omega$ (the interface of $\mathfrak{T}_c$ and $\mathfrak{O}_\sigma^{(\ell)}$) and choose $F_z \subseteq \Gamma_c$ whenever $z \in \Gamma_c$. The definition of J_c implies that $J_c(\cdot)|_{\Gamma_c} \colon L^2(\Gamma_c) \to S^1(\mathfrak{T}_c|_{\Gamma_C})$ is well-defined. The result [1, Lemma 3] shows that $J_c(\cdot)|_{\Gamma_c}$ is a $H^{1/2}$-stable projection (with constants depending only on shape regularity of $\mathfrak{T}_c$) and hence is quasi-optimal in the sense

$$\|u - J_c u\|_{H^{1/2}(\Gamma_c)} \lesssim \min_{v_p \in S^1(\mathfrak{T}_c)} \|u - v_p\|_{H^{1/2}(\Gamma_c)} \leq \|u - I^p u\|_{H^{1/2}(\Gamma_c)}. \tag{57}$$

We construct the approximation $u_c \in S^1(\mathfrak{T}_c^{(\ell)})$ by setting $u_c = J_c u$ on vertices in $\bigcup \mathfrak{T}_c \setminus \Gamma_c$ and $u_c = I^p u$ on the remaining vertices in Γ_c. The estimate (57) allows us to bound the difference

$$\begin{aligned}\|D^1(u_c - J_c u)\|_{L^2(\bigcup \mathfrak{T}_c^{(\ell)})} &\lesssim \|I^p u - J_c u\|_{H^{1/2}(\Gamma_c)} \lesssim \|u - I^p u\|_{H^{1/2}(\Gamma_c)} \\ &\lesssim \|u - I^p u\|_{H^1(\mathfrak{L}_{\ell-1})}.\end{aligned}$$

This leads to

$$\begin{aligned}\|D^1(u - u_c)\|_{L^2(\bigcup \mathfrak{T}_c^{(\ell)})} &\lesssim \|D^1(u - J_c u)\|_{L^2(\bigcup \mathfrak{T}_c)} + \|u - I^p u\|_{H^1(\mathfrak{L}_{\ell-1})} \\ &\lesssim \operatorname{diam}(\bigcup \mathfrak{T}_c)^{\theta} \|u\|_{H^{1+\theta}(\bigcup \mathfrak{T}_c^{(\ell)})} + \|u - I^p u\|_{H^1(\mathfrak{L}_{\ell-1})}.\end{aligned}$$

By Proposition 2, items (ii) and (iii) it holds that $\operatorname{diam}(\bigcup \mathfrak{T}_c) \simeq \sigma^{\ell}$, we obtain with (56) and $u_{\mathscr{S}} := \sum_{c \in \mathscr{S}} u_c$

$$\|u - u_{\mathscr{S}}\|_{H^1(\bigcup \mathfrak{T}_\sigma^{(\ell)})} \leq c(\kappa, \sigma)\sigma^{\theta\ell} + C\exp(-bp^{1/\delta}). \tag{58}$$

Combining this and (56) and applying a bounded (uniformly w.r.t. to p by Prop. 2, item (1)) number of further polynomial edge- and face liftings at the interface of $\mathfrak{O}_\sigma^{(\ell)}$ and $\mathfrak{T}_\sigma^{(\ell)}$ (note that the combination of $I^p u$ and $u_{\mathscr{S}}$ is continuous at the vertices of $\mathscr{M}_\sigma^{(\ell)}$) completes the construction of I^{hp} in (1). Choosing $p \simeq \ell^{\delta}$ concludes the proof for $\delta \geq 1$.

For $0 < \delta < 1$, we additionally use the fact that $\sigma^{\theta p^{1/\delta}} \lesssim (p!)^{b(\delta-1)}$ with some $b(\theta) > 0$ that is indepedent of p by Stirling's approximation. □

5 Concluding Remarks

We have proved the exponential convergence rate (24) for continuous *hp*-FE approximations of κ shape-regular, simplicial meshes with geometric refinement to analytic functions with isolated point singularities at a finite set $\mathscr{S}$ in a bounded domain $D \subset \mathbb{R}^d$, $d = 1, 2, 3$. Apart from κ-shape regularity and σ-geometric mesh refinement the proof did not assume further structural assumptions on the triangulations. In particular, simplicial partitions which are obtained by successive bisection tree refinement in the course of adaptive subdivisions are admissible. The approximation results imply the exponential convergence rate $\exp(-b\sqrt[3]{N})$ for second order, elliptic PDEs in polygons $D \subset \mathbb{R}^2$ (where $\mathscr{S}$ denotes the set of corners of D) where solutions belong to the analytic class (i.e., where $\delta = 1$) which are considered, for example, in [2, 11, 18, 27, 28]. Theorem 1 also implies the exponential convergence rate $\exp(-b\sqrt[4]{N})$ for hp-approximations of electron densities in DFT, due to the quasioptimality of Galerkin approximations shown, for example, in [3, 6] and the references there. In this application, $\mathscr{S}$ denotes the set of nuclei, whose centers $c \in \mathscr{S}$ are assumed known. Furthermore, the extension of [27, 28] to Gevrey-regular solutions is essential in this case, as analyticity of electron densities can not be expected, generally, in the presence of empirical potential functions constructed, for example, from smooth partitions of unity.

Also, unlike other approaches such as plane waves, hp-approximations do not, apriori, impose any specific functional form of the electron densities. Due to the locality of approximation and the separation (2) of the points $c \in \mathscr{S}$, we may apply Theorem 1 in each neighborhood ω_c, $c \in \mathscr{S}$, implying that the total number of degrees of freedom to achieve accuracy $\varepsilon > 0$ in the norm $H^1(D)$ scales as $O(\#(\mathscr{S})|\log\varepsilon|^4)$, ie. *linear scaling* in the number $\#(\mathscr{S})$ of nuclei and *polylogarithmic scaling* in the target accuracy ε. This is analogous to what is reported recently for discontinuous Galerkin discretizations in [20], where Proposition 4 can be used a starting point of proof of an exponential convergence result on tetrahedral meshes; for geometric meshes of hexahedra, analogous results can be found in [26, Sec. 5.2.2]. Exponentially convergent quadrature algorithms for the (singular) electron-pair integrals are available in [8]. The results in the present note are confined to space dimension $d \leq 3$. The approach generalizes, however, directly to hp-approximations of point singularities in any dimension d with exponential rate; we remark that I_N^{hp} in the terminal layers $\mathfrak{T}_\sigma^{(\ell)}$ of the geometric meshes $\mathfrak{M}_\sigma$ introduced in Section 4.2.8 were built from low-order *quasi-interpolants* of Scott-Zhang type, which do not require continuity of u near $\mathscr{S}$. Likewise, the exponential convergence rate bound (1) will remain true for *linear polynomial degree vectors* and, more generally, for degree vectors of bounded variation as introduced in [26]. Also, our construction of I_N^{hp} was based on a-priori knowledge of the singular support $\mathscr{S}$. In case $\mathscr{S}$ is not known a-priori, *adaptive hp-approximations* have to be used. The details will be reported elsewhere.

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