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Discrete Regular Decompositions of Tetrahedral Discrete 1-Forms

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Abstract. For a piecewise polynomial finite element space $\mathcal{W}^1_{p,\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ built on a mesh $\mathcal{T}$ of a Lipschitz domain $\Omega \subset \mathbb{R}^3$ and with vanishing tangential trace on $\Gamma_D \subset \partial\Omega$, a discrete regular decomposition is a *stable* splitting of elements of $\mathcal{W}^1_{p,\Gamma_D}(\mathcal{T})$ into (i) piecewise polynomial continuous vector fields on Ω, vanishing on Γ_D, (ii) gradients of piecewise polynomial continuous scalar finite element functions, and (iii) a "small" remainder. Such decompositions have turned out to be a key tool in the numerical analysis of "edge" finite element methods for variational problems in $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ that commonly occur in computational electromagnetics.

We show the existence of such decompositions for Nédélec's tetrahedral edge element spaces of any polynomial degree with stability depending only on Ω, Γ_D, and the shape regularity of the mesh. Our decompositions also respect homogeneous boundary conditions on a part of the boundary of Ω. Key tools for our construction are continuous regular decompositions, boundary-aware local co-chain projections, projection-based interpolation, and quasi-interpolation with low regularity requirements.

Keywords. Regular decomposition, edge elements, hp-FEM, polynomial extension, projection-based interpolation, quasi-interpolation.

AMS classification. 65N30.

1 Introduction

We study an important aspect of the theory of finite element subspaces of $\boldsymbol{H}(\mathbf{curl}, \Omega)$, $\Omega \subset \mathbb{R}^3$ a bounded domain whose properties will be specified below. We restrict ourselves to spaces introduced as spaces of discrete 1-forms on simplicial meshes in finite element exterior calculus (FEEC). They are also known as edge elements and their pivotal role in the Galerkin discretization of electromagnetic boundary value problem is no longer a moot point.

The starting point are stable decompositions of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ into vetorfields with components in $H^1(\Omega)$ and gradients, which have been developed as powerful tools in the theory of function spaces [8, 11, 17, 18]. We refer to them as *regular decompositions*. In Section 2 we are going to present a particular instance. It later turned out that discrete counterparts of regular decompositions of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ are similarly useful in the numerical analysis of edge element schemes. We are going to survey a few applications and give references in Section 1.5.

Section 3 will be devoted to proving a discrete regular decomposition theorem for lowest order tetrahedral edge elements, also known as Whitney-1-forms. Compared to what was known previously, we establish enhanced stability properties also in $L^2(\Omega)$. We owe these stronger results to the use of so-called local commuting co-chain projections pioneered by Falk and Winther [27, 28]. A tailored version of those will be introduced and examined in Section 3.1.

Subsequently, in Section 4 we tackle tetrahedral discrete 1-forms of higher (uniform) polynomial degree p. For them we can establish p-uniformly stable discrete regular decompositions, with weaker stability properties than those achievable for Whitney 1-forms, though. The key tool are commuting local projection based interpolation operators presented in Section 4.1 combined with a p-stable quasi-interpolation borrowed from [47].

The focus of this work is on numerical analysis techniques required to establish existence and properties of discrete regular decompositions. In detail we gather, review, assemble, and, sometimes, extend theoretical results from the finite element literature, with the intention of conveying the guiding ideas and tricks underlying the proofs. The actual use of regular decompositions will be addressed only briefly in Section 1.5.

1.1 Geometric Setting

Since subtle geometric arguments will play a major role for parts of the theory, we have to give a precise characterization of the geometric setting: We let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected *Lipschitz polyhedron*. Its boundary $\Gamma := \partial\Omega$, is partitioned according to $\Gamma = \Gamma_D \cup \Sigma \cup \Gamma_N$, with relatively open sets Γ_D and Γ_N. We assume that this provides a *piecewise* C^1 *dissection* of Γ in the sense of [31, Definition 2.2]. Sloppily speaking, this means that Σ is the union of closed curves that are piecewise C^1. Actually, we demand that Σ consists of disjoint closed polygons.

We triangulate Ω with a simplicial mesh $\mathcal{T}$, which will be identified with its set of tetrahedral elements: $\mathcal{T} = \{T\}$. We assume that the partitioning of the boundary Γ is resolved by the mesh. We endow edges and faces of $\mathcal{T}$ with intrinsic orientations, see Section 3.1.1

Assumption 1.1. Both $\overline{\Gamma}_D$ and $\overline{\Gamma}_N$ are unions of closed faces of elements of $\mathcal{T}$.

We write h_T for the local mesh size, that is, the diameter of $T \in \mathcal{T}$, and r_T for the radius of the largest ball contained in T. These numbers enter the global *shape* *regularity measure* $\rho(\mathcal{T})$ of the mesh defined as [15], [49, Sect. II.4],

$$\rho(\mathcal{T}) := \max\{{}^{h_T}\!/_{r_T},\ T \in \mathcal{T}\}\ . \tag{1.1}$$

The symbol h will also denote a function $h \in L^\infty(\Omega)$ with $h(\mathbf{x}) := h_T$ for $\mathbf{x} \in T$, $T \in \mathcal{T}$.

1.2 Notations and Function Spaces

We adhere to the de-facto standard notations for function spaces in the numerical analysis literature [36, Sect. 2.4]. In particular, we write $H^s(D)$, $s \in \mathbb{R}$, for the Sobolev (Hilbert) space of order s on the domain D, see [50, Ch. 3]. It is endowed with the usual norm $\|\cdot\|_{s,D}$, and the semi-norm $|\cdot|_{s,D}$. We write $H^s_\Sigma(D)$, $s > \frac{1}{2}$, for the subspace with zero boundary conditions imposed on $\Sigma \subset \partial D$. Bold typeface distiguishes (spaces of) vector valued functions, e.g., $\boldsymbol{H}^s_\Sigma(D)$. The notations $\boldsymbol{H}_\Sigma(\mathbf{curl}, D)$ and $\boldsymbol{H}_\Sigma(\mathrm{div}, D)$ stand for spaces of vector fields with rotation and divergence, respectively, in $\boldsymbol{L}^2(D)$, and zero tangential/normal trace on $\Sigma \subset \partial D$. The associated norms read $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},D)}$ and $\|\cdot\|_{\boldsymbol{H}(\mathrm{div},D)}$.

1.3 Tetrahedral Discrete Differential Forms

Discrete differential forms provide finite element spaces of differential forms. They are studied in the new field of Finite Element Exterior Calculus (FEEC) using tools from the calculus of differential forms [34, 4, 5]. In this article we stick to the classical calculus of vector analysis, because all developments are set in 3D Euclidean space. Yet, the differential forms background has inspired our notations: integer superscripts label spaces and operators related to differential forms of a particular degree.

We restrict ourselves to the so-called first family of simplicial discrete differential forms. It comprises the following $\mathcal{T}$-piecewise polynomial finite element spaces.

① Discrete 0-forms, continuous Lagrangian finite elements:

$$\begin{aligned} \mathcal{W}^0_{p,\Gamma_D}(\mathcal{T}) &:= \{v \in H^1_{\Gamma_D}(\Omega),\ v_{|T} \in \mathcal{W}^0_p(T) \quad \forall T \in \mathcal{T}\}\ , \\ \mathcal{W}^0_p(T) &:= \mathcal{P}_{p+1}(\mathbb{R}^3)\ , \end{aligned}$$

② Discrete 1-forms, Nedéléc's first family of $\mathbf{curl}$-conforming elements (“edge elements”):

$$\begin{aligned} \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T}) &:= \{\mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega),\ \mathbf{v}_{|T} \in \boldsymbol{\mathcal{W}}^1_p(T) \quad \forall T \in \mathcal{T}\}\ , \\ \boldsymbol{\mathcal{W}}^1_p(T) &:= \{\boldsymbol{x} \mapsto \mathbf{p}(\boldsymbol{x}) + \mathbf{q}(\boldsymbol{x}) \times \boldsymbol{x},\ \mathbf{p}, \mathbf{q} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)\}\ , \end{aligned}$$

③ Discrete 2-forms, div-conforming Raviart-Thomas finite elements (“face elements”):

$$\begin{aligned} \boldsymbol{\mathcal{W}}^2_{p,\Gamma_D}(\mathcal{T}) &:= \{\mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathrm{div}, \Omega),\ \mathbf{v}_{|T} \in \boldsymbol{\mathcal{W}}^2_p(T) \quad \forall T \in \mathcal{T}\}\ , \\ \boldsymbol{\mathcal{W}}^2_p(T) &:= \{\boldsymbol{x} \mapsto \mathbf{p}(\boldsymbol{x}) + q(\boldsymbol{x})\boldsymbol{x},\ \mathbf{p} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3),\ q \in \mathcal{P}_p(\mathbb{R}^3)\}\ , \end{aligned}$$

④ Discrete 3-forms, discontinuous piecewise polynomials:

$$\mathcal{W}^3_p(\mathcal{T}) := \{v \in L^2(\Omega),\ v_{|T} \in \mathcal{W}^3_p(T) \quad \forall T \in \mathcal{T}\}\ ,$$
$$\mathcal{W}^3_p(T) := \mathcal{P}_p(\mathbb{R}^3)\ .$$

Here $p \in \mathbb{N}$ stands for the polynomial degree and $\mathcal{P}_p(\mathbb{R}^3)$/$\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)$ for the spaces of polynomials/polynomials vector fields of degree $\leq p$ in three variables. Dropping the Γ_D subscript indicates that no boundary conditions are enforced. Notice that our notations above differ from what is adopted in the seminal work [4] on FEEC, where the authors write $\mathcal{P}^-_p\Lambda^\ell(\mathcal{T})$ instead of $\mathcal{W}^\ell_p(\mathcal{T})$.

First-order differential operators related to the exterior derivative connect these spaces to a discrete de Rham complex:

$$\mathcal{K}_{\Gamma_D}(\Omega) \xrightarrow{\mathrm{Id}} \mathcal{W}^0_{p,\Gamma_D}(\mathcal{T}) \xrightarrow{\mathbf{grad}} \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T}) \xrightarrow{\mathbf{curl}} \boldsymbol{\mathcal{W}}^2_{p,\Gamma_D}(\mathcal{T}) \xrightarrow{\mathrm{div}} \mathcal{W}^3_p(\mathcal{T}) \xrightarrow{0} \{0\}. \tag{1.2}$$

Here the space of constants is given by

$$\mathcal{K}_{\Gamma_D}(\Omega) := \{v \in H^1_{\Gamma_D}(\Omega)\colon\ v_{|\Omega} = \mathrm{const}\} = \begin{cases} \mathrm{span}\{1\} & \text{if } \Gamma_D = \emptyset, \\ \{0\} & \text{otherwise.} \end{cases} \tag{1.3}$$

In the complex (1.2) the range of an operator is contained in the kernel of the subsequent operator.

In the lowest-order case ($p = 0$) the elements of $\mathcal{W}^\ell_{0,\Gamma_D}(\mathcal{T})$ are called Whitney forms. In the sections devoted to these spaces, we are going to replace the subscript $p = 0$ with h and write $\mathcal{W}^\ell_{h,\Gamma_D}(\mathcal{T}) := \mathcal{W}^\ell_{0,\Gamma_D}(\mathcal{T})$.

Finally, we need spaces of vectorial continuous Lagrangian finite element functions,

$$\boldsymbol{\mathcal{V}}^0_{p,\Gamma_D}(\mathcal{T}) := [\mathcal{W}^0_{p,\Gamma_D}(\mathcal{T})]^3, \qquad \boldsymbol{\mathcal{V}}^0_{h,\Gamma_D}(\mathcal{T}) := [\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})]^3\ . \tag{1.4}$$

1.4 Main Results

Our main theorem about the discrete regular decomposition of the spaces of Whitney 1-forms ("edge elements") involves a *local* projection operator $\boldsymbol{R}^1_D\colon \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ that respects the homogeneous boundary conditions. This operator and a related one will be constructed in Section 3.1.6 below, together with several stability estimates.

Theorem 1.2 (Stable discrete regular decomposition for Whitney-1-forms in 3D). *For every discrete 1-form of the lowest-order first family $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$, there exists a continuous and piecewise linear vector field $\mathbf{z}_h \in \boldsymbol{\mathcal{V}}^0_{h,\Gamma_D}(\mathcal{T}) = [\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})]^3$, a continuous and piecewise linear scalar function $\varphi_h \in \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$, and a remainder*

$\widetilde{\mathbf{v}}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$*, all depending linearly on* $\mathbf{v}_h$*, providing the* discrete regular decomposition

$$\mathbf{v}_h = \boldsymbol{R}^1_D \mathbf{z}_h + \widetilde{\mathbf{v}}_h + \mathbf{grad}\,\varphi_h \ ,$$

and satisfying the norm estimates

$$\|\mathbf{z}_h\|_{0,\Omega} \leq C\,\|\mathbf{v}_h\|_{0,\Omega} \ , \qquad |\mathbf{z}_h|_{1,\Omega} \leq C\left(\frac{1}{d}\,\|\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}\right) \ , \tag{1.5}$$

$$|\varphi_h|_{1,\Omega} \leq C\,\|\mathbf{v}_h\|_{0,\Omega} \ , \tag{1.6}$$

$$\|\widetilde{\mathbf{v}}_h\|_{0,\Omega} \leq C\,\|\mathbf{v}_h\|_{0,\Omega} \ , \quad \left\|h^{-1}\widetilde{\mathbf{v}}_h\right\|_{0,\Omega} \leq C\left(\frac{1}{d}\,\|\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}\right) \ , \tag{1.7}$$

with $d = \mathrm{diam}(\Omega)$ *and constants* $C > 0$ depending only *on the shape of* Ω*,* Γ_D*, and the shape regularity measure* $\rho(\mathcal{T})$*.*

Similar but weaker results are stated in [39, Lemma 5.1] and [41, Lemma 5.1]. These estimates did not bound the $\boldsymbol{L}^2(\Omega)$-norm of $\mathbf{z}_h$ by the $\boldsymbol{L}^2(\Omega)$-norm of $\mathbf{v}_h$. The proof of Theorem 1.2 is given in Section 3 and it will demonstrate the substantial additional effort required to establish stability in $\boldsymbol{L}^2(\Omega)$.

The next result presents a "p-version" counterpart of Theorem 1.2, because it targets spaces of discrete 1-forms with *arbitrary* polynomial degree p with a focus on p-uniform stability estimates.

Theorem 1.3 (Discrete regular decomposition for discrete 1-forms). *For every discrete 1-form of the first family* $\mathbf{v}_p \in \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T})$*,* $p \in \mathbb{N}_0$*, there exists a continuous vector field* $\mathbf{z}_p \in \boldsymbol{\mathcal{V}}^0_{p,\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}^1_{\Gamma_D}(\Omega)$*,* $\mathcal{T}$*-piecewise polynomial of degree* $\leq p+1$*, a continuous,* $\mathcal{T}$*-piecewise polynomial scalar function* $\varphi_p \in \mathcal{W}^0_{p,\Gamma_D}(\mathcal{T})$*, and a remainder* $\widetilde{\mathbf{v}}_p \in \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T})$*,*

(I) all depending linearly on $\mathbf{v}_p$*,*

(II) satisfying the norm estimates

$$\|\mathbf{z}_p\|_{0,\Omega} \leq C\,\|\mathbf{v}_p\|_{0,\Omega} \ , \quad |\mathbf{z}_p|_{1,\Omega} \leq C\left(\frac{1}{d}\,\|\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega}\right) \ , \tag{1.8}$$

$$|\varphi_p|_{1,\Omega} \leq C\left(\|\mathbf{v}_p\|_{0,\Omega} + \max_{T\in\mathcal{T}}\left\{(1+\log(p+1))^{3/2}\frac{h_T}{p}\right\}\|\mathbf{curl}\,\mathbf{v}_p\|_{0,\Omega}\right) \ , \tag{1.9}$$

$$\left(\sum_{T\in\mathcal{T}}\left\|\frac{p+1}{h_T}\widetilde{\mathbf{v}}_p\right\|^2_{0,T}\right)^{1/2} \leq C(1+\log(p+1))^{3/2}\left(\frac{1}{d}\,\|\mathbf{v}_h\|_{0,\Omega} + \|\mathbf{curl}\,\mathbf{v}_p\|_{0,\Omega}\right) \ , \tag{1.10}$$

with $d := \mathrm{diam}(\Omega)$ *and constants* $C > 0$ depending only *on the shape of* Ω*,* Γ_D*, and the shape regularity measure* $\rho(\mathcal{T})$*,* *(III) and providing the* discrete regular decomposition

$$\mathbf{v}_p = \Pi_p^1 \mathbf{z}_p + \widetilde{\mathbf{v}}_p + \mathbf{grad}\,\varphi_p\ ,$$

where $\Pi_p^1 : \boldsymbol{\mathcal{V}}^0_{p,\Gamma_D}(\mathcal{T}) \to \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T})$ *is a strictly local linear interpolation operator.*

This result has no precursor in the literature. Its obvious shortcoming is the restriction to a uniform polynomial degree p. More desirable would be a version admitting variable polynomial degree and, thus, encompassing finite element spaces created by hp-refinement, see [2]. However, there is a single technical obstacle that has prevented us from admitting variable p, refer to Theorem 4.17.

Another class of results on discrete regular decompositions beyond the scope of the above two theorems addresses stability estimates with non-constant positive weight functions entering the norms. Currently (2017) this is an area of active research and first results for piecewise constant weight functions are reported in [46, 44, 45].

1.5 Applications

The discrete regular decompositions of the kind provided by Theorem 1.2 have turned out to a powerful tool for numerical analysis of various aspects of edge finite element methods. We emphasize their role as *theoretical tool*, because there is not a single algorithm, which relies on the actual computation of the finite element functions comprising a discrete regular decomposition. The following, probably incomplete, list mentions a few pieces of research in numerical analysis, where h-version discrete regular decompositions played a pivotal role:

- Analysis of geometric multigrid methods for $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$-elliptic variational problems discretized by means of edge elements [41, 35, 62]: Here discrete regular decompositions allow to harness results on the stability of multilevel nodal decompositions of $\boldsymbol{\mathcal{V}}^0_1(\mathcal{T})$.
- Convergence theory of domain decomposition methods for discrete $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$-elliptic variational problems [55, 25, 24, 46, 42, 43, 45]: In the same vein as multigrid theory, these approaches manage to exploit results for Lagrangian finite elements and $H^1(\Omega)$-elliptic variational problems.
- Foundation of nodal auxiliary space preconditioners [40, 39, 48]: the stable discrete regular decomposition directly spawn s a subspace correction method for discrete $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$-elliptic variational problems whose key step amounts to the solution of scalar elliptic boundary value problems.
- Analysis of geometric auxiliary space methods for edge elements [38].
- Reliability estimates for residual based local error estimators for $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$-elliptic variational problems [23, 13, 58].

2 Continuous Regular Decomposition

It goes without saying that all results about discrete regular decompositions have their roots in stability properties of continuous regular decompositions of the function space $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$. Now we state and prove the corresponding key estimates. For ease of presentation we set $\operatorname{diam}(\Omega)=1$ throughout the remainder of this manuscript. Simple scaling arguments will then produce the more general estimates of Theorem 1.2 and Theorem 1.3.

The following result can essentially be found in [41, 32], except that we also assert extra L^2-stability. Note that there are neither restrictions on the topology of Ω nor on the connectedness of the Dirichlet boundary Γ_D. A more general version of the theory will be published in a forthcoming manuscript [56].

Theorem 2.1 (Boundary aware regular decomposition). *For each $\mathbf{v}\in\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$ there exists a vector field $\mathbf{z}\in\boldsymbol{H}^1_{\Gamma_D}(\Omega)$ and a scalar function $\varphi\in H^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathbf{v}$ such that*

$$\mathbf{v}=\mathbf{z}+\mathbf{grad}\,\varphi,$$

and

$$\|\mathbf{z}\|_{0,\Omega}\leq C\ \|\mathbf{v}\|_{0,\Omega}\ ,\qquad |\mathbf{z}|_{1,\Omega}\leq C\ \|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\ , \tag{2.1}$$

$$\|\varphi\|_{1,\Omega}\leq C\ \|\mathbf{v}\|_{0,\Omega}\ , \tag{2.2}$$

with constants independent of $\mathbf{v}$.

For the proof, we need a few auxiliary results that will be provided in the next three sections.

2.1 Collars and Bulges

Under the assumptions on Ω made in Section 1.1, [31, Lemma 4.4] guarantees the existence of an open Lipschitz neighborhood Ω_Γ ("Lipschitz collar") of $\Gamma:=\partial\Omega$ and of a smooth vector field $\widetilde{\boldsymbol{n}}\in C^\infty(\mathbb{R}^3,\mathbb{R}^3)$ with $\|\widetilde{\boldsymbol{n}}\|\equiv 1$ on Ω_Γ that is *transversal* to Γ:

$$\exists\kappa>0:\quad \widetilde{\boldsymbol{n}}(\boldsymbol{x})\cdot\boldsymbol{n}(\boldsymbol{x})\geq\kappa\quad\text{for almost all } \boldsymbol{x}\in\Gamma\ . \tag{2.3}$$

Extrusion of Γ_D by the local flow induced by $\widetilde{\boldsymbol{n}}$ spawns the "bulge" $\Upsilon_D\subset\Omega_\Gamma\setminus\Omega$, see Fig. 1. We recall the properties of bulge domains from [31, Section 2].

Theorem 2.2 (Bulge-augmented domain). *There exists a* Lipschitz domain $\Upsilon_D\subset\mathbb{R}^3\setminus\overline{\Omega}$*, such that $\overline{\Upsilon}_D\cap\overline{\Omega}=\Gamma_D$, $\Omega^e:=\Upsilon_D\cup\Gamma_D\cup\Omega$ is Lipschitz,* $\operatorname{diam}(\Omega^e)\leq 2$*, and* $\overline{\Upsilon}_D\subset\Omega_\Gamma$.

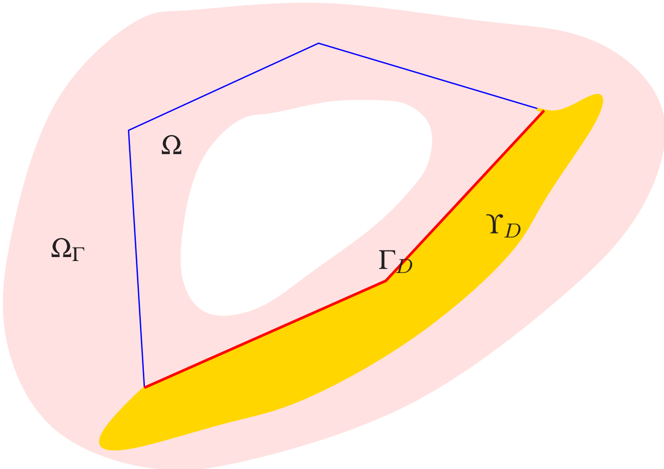

Figure 1. Collar domain Ω_Γ (pink) and bulge domain Υ_D (gold)

Remark 2.3. If Γ_D has several components Γ_k, $k = 1, \ldots, N$, then each of them gives rise to a separate bulge Υ_k with $\overline{\Upsilon}_k \cap \overline{\Omega} = \Gamma_k$, and the individual bulges have positive distance from each other. This is a consequence of our assumptions on Γ and has to be kept in mind though we are not going to mention this fact explicitly in the sequel.

2.2 Extension operators

Lemma 2.4 ([60]). *Let $\mathcal{D}$ be a bounded Lipschitz domain with* $\operatorname{diam}(\mathcal{D}) = 1$*. Then there exists a bounded linear extension operator $E_{\mathcal{D}} \colon L^2(\mathcal{D}) \to L^2(\mathbb{R}^3)$ such that for $k \in \mathbb{N}_0$,*

$$\|E_{\mathcal{D}} v\|_{k,\mathcal{D}} \leq C \|v\|_{k,\mathcal{D}} \qquad \forall v \in H^k(\mathcal{D}), \tag{2.4}$$

with C depending only on $\mathcal{D}$ and k. Moreover, $E_{\mathcal{D}} v$ has compact support in $\mathbb{R}^3$.

We apply this fundamental result to the bulge domain Υ_D introduced in Section 2.1.

Corollary 2.5. *There exists an extension operator $E^{(2)}_{\Upsilon_D} \colon L^2(\Upsilon_D) \to L^2(\mathbb{R}^3)$ such that for $k \in \mathbb{N}_0$,*

$$\|E^{(2)}_{\Upsilon_D} v\|_{k,\mathbb{R}^3} \leq C\, \|v\|_{k,\Upsilon_D} \qquad \forall v \in H^k(\Upsilon_D), \tag{2.5}$$

where the constant C depends on Ω, Υ_D, and k.

Lemma 2.6. *For a Lipschitz domain $\mathcal{D}$ with* $\operatorname{diam}(\mathcal{D}) = 1$ *there exists a bounded linear extension operator $E^{\mathbf{curl}}_{\mathcal{D}} \colon \boldsymbol{L}^2(\mathcal{D}) \to \boldsymbol{L}^2(\mathbb{R}^3)$ such that, with constants depending only* *on $\mathcal{D}$,*

$$\|E_{\mathcal{D}}^{\mathbf{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} \leq C\,\|\mathbf{v}\|_{0,\mathcal{D}} \qquad \forall \mathbf{v}\in \boldsymbol{L}^2(\mathcal{D}),$$
$$\|E_{\mathcal{D}}^{\mathbf{curl}}\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)} \leq C\,\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\mathcal{D})} \qquad \forall \mathbf{v}\in \boldsymbol{H}(\mathbf{curl},\mathcal{D}).$$

Moreover, $E_{\mathcal{D}}^{\mathbf{curl}}\mathbf{v}$ has compact support in $\mathbb{R}^3$.

Proof. Since $\mathcal{D}$ is (strong) Lipschitz, it is also *weak* Lipschitz, and so the Lipschitz collar is locally the image of the unit cube under a bi-Lipschitz mapping such that the exterior is mapped to the upper halfspace [49, Sect. VII.1]. On the cube, we define the extension of $\mathbf{w}(x_1,x_2,x_3)$ as $\mathrm{diag}(1,1,-1)\mathbf{w}(x_1,x_2,-x_3)$. Mapping back to the collar and using a partition of unity, one obtains the desired result, since the bi-Lipschitz mapping preserves the $\mathbf{curl}$-operator. □

We note that a similar result with higher order $\mathbf{curl}$-derivatives (but not with the pure L^2-stability) has been shown in [37].

2.3 A Fourier-based Projection

The next lemma builds on similar results from [3, Lemma 3.5], [36, Lemma 2.5], and[37, Lemma 5.1].

Lemma 2.7. *There exists a bounded linear operator $\mathsf{L}_{\mathbf{curl}}\colon \boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)\to \boldsymbol{H}^1(\mathbb{R}^3)$ such that for all $\mathbf{v}\in \boldsymbol{H}(\mathbf{curl},\Omega)$*

(L_1) $\mathbf{curl}\,\mathsf{L}_{\mathbf{curl}}\mathbf{v} = \mathbf{curl}\ \mathbf{v}$,

(L_2) $\mathrm{div}\,\mathsf{L}_{\mathbf{curl}}\mathbf{v} = 0$,

(L_3) $\|\mathsf{L}_{\mathbf{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\mathbf{v}\|_{0,\mathbb{R}^3}$ *and* $\|(I-\mathsf{L}_{\mathbf{curl}})\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\mathbf{v}\|_{0,\mathbb{R}^3}$,

(L_4) $\|\boldsymbol{\nabla}\mathsf{L}_{\mathbf{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} \leq \|\,\mathbf{curl}\,\mathbf{v}\|_{0,\mathbb{R}^3}$,

(L_5) $\mathsf{L}_{\mathbf{curl}}^2\mathbf{v} = \mathsf{L}_{\mathbf{curl}}\mathbf{v}$, *i.e., $\mathsf{L}_{\mathbf{curl}}$ is a projection.*

In the statement (L_4), $\boldsymbol{\nabla}$ applied to a vector field yields the Jacobian.

Proof. The proof is classical; see e.g., [29, Ch. I, Theorem 3.4] and [55, Lemma 2.1]. Let $\widehat{\mathbf{v}}(\boldsymbol{\xi}) := (\mathcal{F}\mathbf{v})(\boldsymbol{\xi}) := \int_{\mathbb{R}^3} e^{-2\pi\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}}\mathbf{v}(\mathbf{x})\,d\mathbf{x}$ denote the (component-wise) Fourier transform of $\mathbf{v}\in \boldsymbol{L}^2(\mathbb{R}^3)$. Recall that $\partial_k\mathbf{v}$, $\mathbf{curl}\,\mathbf{v}$, $\mathrm{div}\ \mathbf{v}$ correspond to $2\pi\mathrm{i}\boldsymbol{\xi}_k\widehat{\mathbf{v}}$, $2\pi\mathrm{i}\boldsymbol{\xi}\times\widehat{\mathbf{v}}$, and $2\pi\mathrm{i}\boldsymbol{\xi}\cdot\widehat{\mathbf{v}}$, respectively. We set

$$\mathsf{L}_{\mathbf{curl}}\mathbf{v} := \mathcal{F}^{-1}\widehat{\mathbf{w}}, \qquad \text{with } \widehat{\mathbf{w}}(\boldsymbol{\xi}) := -|\boldsymbol{\xi}|^{-2}(\boldsymbol{\xi}\times\boldsymbol{\xi}\times\widehat{\mathbf{v}}(\boldsymbol{\xi})).$$

Elementary properties of $\widehat{\mathbf{w}}\in \boldsymbol{L}^2(\mathbb{R}^3)$ yield most of the assertions: (L_1) from $2\pi\boldsymbol{\xi}\times\widehat{\mathbf{w}} = 2\pi\boldsymbol{\xi}\times\widehat{\mathbf{v}}$. ($\mathrm{L}_2$) from $2\pi\boldsymbol{\xi}\cdot\widehat{\mathbf{w}} = 0$. ($\mathrm{L}_3$) from $|\widehat{\mathbf{w}}| \leq |\widehat{\mathbf{v}}|$, because, due to

Plancherel's theorem,

$$\left.\begin{aligned} \|\mathsf{L}_{\mathbf{curl}}\mathbf{v}\|_{0,\mathbb{R}^3} &= \|\widehat{\mathbf{w}}\|_{0,\mathbb{R}^3} \\ \|(I-\mathsf{L}_{\mathbf{curl}})\mathbf{v}\|_{0,\mathbb{R}^3} &= \|\underbrace{\widehat{\mathbf{v}} + |\boldsymbol{\xi}|^{-2}\boldsymbol{\xi}\times\boldsymbol{\xi}\times\widehat{\mathbf{v}}}_{=|\boldsymbol{\xi}|^{-2}(\widehat{\mathbf{v}}\cdot\boldsymbol{\xi})\boldsymbol{\xi}}\|_{0,\mathbb{R}^3} \end{aligned}\right\} \le \|\widehat{\mathbf{v}}\|_{0,\mathbb{R}^3} = \|\mathbf{v}\|_{0,\mathbb{R}^3}.$$

(L_4) is obtained as follows

$$\|\boldsymbol{\nabla}\mathsf{L}_{\mathbf{curl}}\mathbf{v}\|^2_{0,\mathbb{R}^3} = \sum_{k=1}^{3} \|2\pi \mathrm{i}\xi_k\widehat{\mathbf{w}}\|^2_{0,\mathbb{R}^3} \le \sum_{k=1}^{3}\left\|\frac{|\xi_k|^2}{|\boldsymbol{\xi}|^2}2\pi\mathrm{i}\boldsymbol{\xi}\times\widehat{\mathbf{v}}\right\|^2_{0,\mathbb{R}^3} \le \|\,\mathbf{curl}\,\mathbf{v}\|^2_{0,\mathbb{R}^3}.$$

The last estimate shows that indeed $\mathsf{L}_{\mathbf{curl}}\mathbf{v} \in \boldsymbol{H}^1(\mathbb{R}^3)$. (L_5) is checked easily. □

2.4 Proof of Theorem 2.1

We follow the proof as in [41, Thm. 5.9] and establish the L^2-stability using the ideas from [55, Lemma 2.2]. Let $\mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$ be arbitrary but fixed.

Step 1: We extend $\mathbf{v}$ by zero to a function in $\boldsymbol{H}(\mathbf{curl},\Omega^e)$, where Ω^e is the extended domain from Sect. 2.1 and then to $\widetilde{\mathbf{v}} \in \boldsymbol{H}(\mathbf{curl},\mathbb{R}^3)$ using $E^{\mathbf{curl}}_{\Omega^e}$. We observe $\widetilde{\mathbf{v}}_{|\Upsilon_D} \equiv 0$ and that Lemma 2.6 implies

$$\|\widetilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \;\le\; C\,\|\mathbf{v}\|_{0,\Omega} \quad,\quad \|\,\mathbf{curl}\,\widetilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \;\le\; C\,\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,. \tag{2.6}$$

Step 2: Let $B \supseteq \Omega^e$ be a ball such that $1 \le \mathrm{diam}(B) \le 2$ and define

$$\mathbf{w} := (\mathsf{L}_{\mathbf{curl}}\widetilde{\mathbf{v}})_{|B}.$$

Due to (L_1) of Lemma 2.7, $\mathbf{curl}\,\mathbf{w} = \mathbf{curl}\,\widetilde{\mathbf{v}}$ in B. Since B is simply connected, there exists a scalar potential $\psi \in H^1(B)$ with zero average $\int_B \psi\, d\mathbf{x} = 0$ such that

$$\widetilde{\mathbf{v}} = \mathbf{w} + \mathbf{grad}\,\psi.$$

Lemma 2.7 together with (2.6) implies

$$\begin{aligned} \|\mathbf{w}\|_{0,B} = \|\mathsf{L}_{\mathbf{curl}}\widetilde{\mathbf{v}}\|_{0,B} \le \|\widetilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \le C\,\|\mathbf{v}\|_{0,\Omega}\,,\\ \|\,\mathbf{grad}\,\psi\|_{0,B} = \|(I-\mathsf{L}_{\mathbf{curl}})\widetilde{\mathbf{v}}\|_{0,B} \le \|\widetilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \le C\,\|\mathbf{v}\|_{0,\Omega}\,,\\ \|\boldsymbol{\nabla}\mathbf{w}\|_{0,B} \le \|\,\mathbf{curl}\,\widetilde{\mathbf{v}}\|_{0,\mathbb{R}^3} \le C\,\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,,\\ \|\psi\|_{0,B} \le C\,\|\,\mathbf{grad}\,\psi\|_{0,B} \le C\,\|\mathbf{v}\|_{0,\Omega}\,, \end{aligned} \tag{2.7}$$

where in the last estimate we have used Poincaré's inequality on the convex ball B [6].
Step 3: Since

$$0 = \mathbf{w} + \mathbf{grad}\,\psi \quad \text{in } \Upsilon_D\,,$$

we conclude that $\psi_{|\Upsilon_D} \in H^2(\Upsilon_D)$. We define $\widetilde{\psi} := (E^{(2)}_{\Upsilon_D}\psi)_{|B} \in H^2(B)$. From Corollary 2.5, we obtain

$$\begin{aligned} \|\widetilde{\psi}\|_{0,B} &\leq C\,\|\psi\|_{0,\Upsilon_D} \leq C\,\|\mathbf{v}\|_{0,\Omega}\,, \\ \|\,\mathbf{grad}\,\widetilde{\psi}\|_{0,B} &\leq C\,\|\psi\|_{1,\Upsilon_D} \leq C\,\|\,\mathbf{grad}\,\psi\|_{0,B} \leq C\,\|\mathbf{v}\|_{0,\Omega}\,, \\ \|\boldsymbol{\nabla}\,\mathbf{grad}\,\widetilde{\psi}\|_{0,B} &\leq C\,\big(\|\underbrace{\boldsymbol{\nabla}\,\mathbf{grad}\,\psi}_{=-\boldsymbol{\nabla}\mathbf{w}}\|^2_{0,\Upsilon_D} + \|\psi\|^2_{1,\Upsilon_D}\big)^{1/2} \leq C\,\|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,, \end{aligned} \tag{2.8}$$

where $\boldsymbol{\nabla}\,\mathbf{grad}$ indicates the Hessian.

Step 4: In B, it holds that

$$\widetilde{\mathbf{v}} = \mathbf{w} + \mathbf{grad}\,\psi = \underbrace{\mathbf{w} + \mathbf{grad}\,\widetilde{\psi}}_{=:\mathbf{z}\in\boldsymbol{H}^1} + \mathbf{grad}(\underbrace{\psi - \widetilde{\psi}}_{=:\varphi\in H^1}).$$

It is easy to see that $\varphi = 0$ in Υ_D and so $\varphi \in H^1_{\Gamma_D}(\Omega)$. Correspondingly, $\mathbf{grad}\,\varphi = 0$ and $\widetilde{\mathbf{v}} = 0$ in Υ_D, and so $\mathbf{z} \in \boldsymbol{H}^1_{\Gamma_D}(\Omega)$. Combining (2.7) and (2.8) yields the desired esimates for $\mathbf{z}$ and p.

3 Discrete Regular Decomposition: Lowest-Order Case

Now we tackle the proof of Theorem 1.2. We employ an extended version of the local projectors invented by Falk and Winther in [27], see also [28]. Our extension is aimed at enforcing compliance with the boundary conditions on Γ_D and the sophisticated technical details will be elaborated in Section 3.1. With this tool at our disposal, the proof of Theorem 1.2 can be done in a few simple steps as we are going to demonstrate in Section 3.2.

3.1 Local Bounded Boundary-Aware Co-Chain Projections

In this section, we construct two sets of operators parallel to developments in [27], from where we have also borrowed a good deal of the notations. The first one are modified Clément type operators $M^0_D\colon L^2(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$ and $\boldsymbol{M}^1_D\colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ that commute with the gradient on $H^1_{\Gamma_D}(\Omega)$:

$$\begin{array}{ccc} H^1_{\Gamma_D}(\Omega) & \stackrel{\mathbf{grad}}{\longrightarrow} & \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega) \\ \downarrow M^0_D & & \downarrow \boldsymbol{M}^1_D \\ \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}) & \stackrel{\mathbf{grad}}{\longrightarrow} & \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T}) \end{array} \tag{3.1}$$

The operators feature also some of the local stability and approximation properties of the classical Clément quasi-interpolant [16], see below. The second class of operators are so-called bounded co-chain projections, originally introduced by Falk and Winther

[27]. The operators are defined on the spaces of the de Rham complex, they are projections onto spaces of discrete differential forms, commute with the exterior derivative, and are locally defined. Here, we modify two of these operators, in the sequel called R_D^0 and $\boldsymbol{R}_D^1$ such that they additionally respect homogeneous boundary conditions. We have the commuting diagram

$$\begin{array}{ccc} H^1_{\Gamma_D}(\Omega) & \stackrel{\mathbf{grad}}{\longrightarrow} & \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega) \\ \downarrow R_D^0 & & \downarrow \boldsymbol{R}_D^1 \\ \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}) & \stackrel{\mathbf{grad}}{\longrightarrow} & \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T}), \end{array} \tag{3.2}$$

where opposed to (3.1), the operators are projectors.

3.1.1 Notation and assumptions

We need a little more notation for the subsequent construction. Let $\mathcal{V}$, $\mathcal{E}$, and $\mathcal{F}$ denote the set of vertices, edges, and faces (respectively) of the mesh $\mathcal{T}$. We also introduce the sets $\mathcal{V}_f := \{\mathsf{v} \in \mathcal{V} \colon \mathsf{v} \notin \overline{\Gamma_D}\}$, $\mathcal{E}_f := \{\mathsf{e} \in \mathcal{E} \colon \mathsf{e} \not\subset \overline{\Gamma_D}\}$, and $\mathcal{F}_f := \{\mathsf{f} \in \mathcal{F} \colon \mathsf{f} \not\subset \overline{\Gamma_D}\}$ of "free" vertices, edges, and faces, respectively. Let φ_v denote the nodal vertex basis function fulfilling $\varphi_\mathsf{v}(\mathsf{v}') = \delta_{\mathsf{vv}'}$ for $\mathsf{v}, \mathsf{v}' \in \mathcal{V}$. Edges and faces have to be oriented: For an edge $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2]$ with endpoints $\mathsf{e}_1, \mathsf{e}_2 \in \mathcal{V}$, the orientation is given by the unit tangent $\boldsymbol{\tau}_\mathsf{e} := (\mathsf{e}_2 - \mathsf{e}_1)/|\mathsf{e}_2 - \mathsf{e}_1|$. The orientation of a face $\mathsf{f} \in \mathcal{F}$ is provided by the unit normal $\mathbf{n}_\mathsf{f}$. By $\boldsymbol{\psi}_\mathsf{e} \in \boldsymbol{\mathcal{W}}_h^1(\mathcal{T})$ and $\boldsymbol{\zeta}_\mathsf{f} \in \boldsymbol{\mathcal{W}}_h^2(\mathcal{T})$ we denote the Nédélec edge and face basis functions, fulfilling $\int_{\mathsf{e}'} \boldsymbol{\psi}_\mathsf{e} \cdot \boldsymbol{\tau}_\mathsf{e}\, ds = \delta_{\mathsf{ee}'}$ for $\mathsf{e}, \mathsf{e}' \in \mathcal{E}$ and $\int_{\mathsf{f}'} \boldsymbol{\zeta}_\mathsf{f} \cdot \mathbf{n}_{\mathsf{f}'}\, ds = \delta_{\mathsf{ff}'}$ for $\mathsf{f}, \mathsf{f}' \in \mathcal{F}$. We find that

$$\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T}) = \mathrm{span}\{\varphi_\mathsf{v}\}_{\mathsf{v}\in\mathcal{V}_f}\,,$$
$$\boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T}) = \mathrm{span}\{\boldsymbol{\psi}_\mathsf{e}\}_{\mathsf{e}\in\mathcal{E}_f}\,,$$
$$\boldsymbol{\mathcal{W}}^2_{h,\Gamma_D}(\mathcal{T}) = \mathrm{span}\{\boldsymbol{\zeta}_\mathsf{f}\}_{\mathsf{f}\in\mathcal{F}_f}\,.$$

Finally, for a vertex $\mathsf{v} \in \mathcal{V}$, its node patch ω_v is defined by

$$\omega_\mathsf{v} := \bigcup_{T\in\mathcal{T}\colon \mathsf{v}\in\overline{T}} \overline{T}.$$

For an edge $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2] \in \mathcal{E}$ and a triangular face $\mathsf{f} = [\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_2] \in \mathcal{F}$, the corresponding patches are given by

$$\omega_\mathsf{e} = \omega_{\mathsf{e}_1} \cup \omega_{\mathsf{e}_2}\,, \qquad \omega_\mathsf{f} = \omega_{\mathsf{f}_1} \cup \omega_{\mathsf{f}_2} \cup \omega_{\mathsf{f}_3}\,.$$

See Fig. 2 for a sketch of two edge patches. Finally, the element patch corresponding to an element $T \in \mathcal{T}$ is given by

$$\omega_T = \bigcup_{\mathsf{v}\in\mathcal{V}\cap\overline{T}} \omega_\mathsf{v}\,.$$

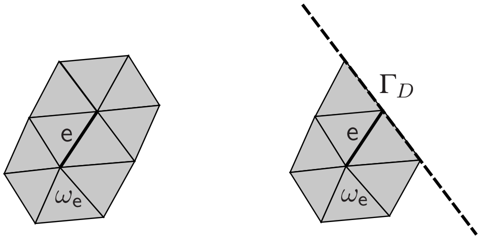

Figure 2. Sketch of edge patches.

For a patch $\omega \subset \Omega$ of elements of $\mathcal{T}$, we will frequently use the space $H^1_{\Gamma_D}(\omega) := \{u \in H^1(\omega)\colon\ u_{|\Gamma_D} = 0\}$. If $\text{meas}_2(\partial\omega \cap \Gamma_D) = 0$ the functions in this space fulfill no boundary condition.

The following, technical assumption is fulfilled for standard meshes.

Assumption 3.1. For each vertex $\mathsf{v} \in \mathcal{V}$, edge $\mathsf{e} \in \mathcal{E}$, and face $\mathsf{f} \in \mathcal{F}$, the vertex patch ω_v, edge patch ω_e, and face patch ω_f, respectively, is simply connected and has a simply connected boundary.

The following results will be helpful in the development of our theory later on.

Lemma 3.2. *Let* $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2] \in \mathcal{E}_f$ *with* $\mathsf{e}_1 \in \overline{\Gamma_D}$ *(or* $\mathsf{e}_2 \in \overline{\Gamma_D}$*). Then there exists a face* $\mathsf{f} \subset \partial\omega_\mathsf{e} \cap \Gamma_D$ *with* $\mathsf{e}_1 \in \bar{\mathsf{f}}$ *(or* $\mathsf{e}_2 \in \bar{\mathsf{f}}$*, respectively).*

Proof. Suppose that $\mathsf{e}_1 \in \overline{\Gamma_D}$. Then there exists a face $\mathsf{f} \subset \Gamma_D$ with $\mathsf{e}_1 \subset \bar{\mathsf{f}}$. Since $\omega_\mathsf{e} \supset \omega_{\mathsf{e}_1}$, there is an element $T \subset \omega_\mathsf{e}$ such that f is a face of T, and moreover, $\mathsf{f} \subset \partial\omega_\mathsf{e}$. □

We will use a couple of times that

$$\begin{aligned} \text{diam}(\omega_\mathsf{v}) &\le C\, h_T\,, & \text{diam}(\omega_\mathsf{v})^{-1} &\le C\, h_T^{-1} & \forall \mathsf{v} \in \mathcal{V} \cap \overline{T},\\ \text{diam}(\omega_\mathsf{e}) &\le C\, h_T\,, & \text{diam}(\omega_\mathsf{e})^{-1} &\le C\, h_T^{-1} & \forall \mathsf{e} \in \mathcal{E} \cap \overline{T},\\ h_\mathsf{e} := \text{diam}(\mathsf{e}) &\le C\, h_T\,, & h_\mathsf{e}^{-1} &\le C\, h_T^{-1} & \forall \mathsf{e} \in \mathcal{E} \cap \overline{T}, \end{aligned}$$

with a (generic) constant C only depending on the shape regularity of $\mathcal{T}$. Furthermore, we need the following discrete estimates:

Lemma 3.3. *For any element* $T \in \mathcal{T}$ *and any vertex* $\mathsf{v} \subset \overline{T}$*,*

$$\begin{aligned} |u_h(\mathsf{v})| &\le C\, h_T^{-3/2}\, \|u_h\|_{0,T} \qquad \forall u_h \in \mathcal{W}_h^0(T),\\ \|\,\mathbf{grad}\,\varphi_\mathsf{v}\|_{0,T} &\le\ C\, h_T^{1/2}\,. \end{aligned}$$

Moreover, for every edge $\mathsf{e} \subset \overline{T}$,

$$\Big| \int_{\mathsf{e}} \mathbf{w}_h \cdot \tau_{\mathsf{e}} \, ds \Big| \leq C\, h_T^{-1/2} \, \|\mathbf{w}_h\|_{0,T} \qquad \forall \mathbf{w}_h \in \boldsymbol{\mathcal{W}}_h^1(T),$$

$$\|\boldsymbol{\psi}_{\mathsf{e}}\|_{0,T} \leq C\, h_T^{1/2} \, .$$

Proof. The proof is carried out using standard techniques from finite elements, transformation to the reference element, and an eigenvalue analysis of the reference element mass matrix. □

3.1.2 Locally exact sequences and Poincaré-Friedrichs type inequalities

Let ω be a patch of elements which is simply connected with simply connected boundary, and let $\gamma \subset \partial\omega$ be a simply connected surface that is a union of faces of elements; the cases $\gamma = \emptyset$ and $\gamma = \partial\omega$ are admitted. Then the local sequence

$$\mathcal{K}_\gamma(\omega) \xrightarrow{\text{id}} \mathcal{W}^0_{h,\gamma}(\omega) \xrightarrow{\mathbf{grad}} \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega) \xrightarrow{\mathbf{curl}} \boldsymbol{\mathcal{W}}^2_{h,\gamma}(\omega) \xrightarrow{\text{div}} \mathcal{W}^3_{h,\gamma}(\omega) \xrightarrow{0} \{0\} \tag{3.3}$$

is *exact*, i.e., the range of an operator is equal to the kernel of the subsequent operator [4, 5]. Above, $\mathcal{K}_\gamma(\omega)$ is the space of constants if $\gamma = \emptyset$ and $\mathcal{K}_\gamma(\omega) = \{0\}$ otherwise, and

$$\mathcal{W}^3_{h,\gamma}(\omega) = \begin{cases} \{v \in \mathcal{W}^3_h(\omega) \colon \int_\omega v \, dx = 0, \} & \text{if } \gamma = \partial\omega \\ \mathcal{W}^3_h(\omega) & \text{otherwise.} \end{cases}$$

We have the classical Poincaré inequality

$$\|u - \overline{u}^\omega\|_{0,\omega} \leq C\, h_\omega \, \|\,\mathbf{grad}\, u\|_{0,\omega} \qquad \forall u \in H^1(\omega) \tag{3.4}$$

and the Friedrichs inequality

$$\|u\|_{0,\omega} \leq C\, h_\omega \, \|\,\mathbf{grad}\, u\|_{0,\omega} \qquad \forall u \in H^1_\gamma(\omega), \qquad \text{if } \text{meas}_2(\gamma) > 0. \tag{3.5}$$

where $h_\omega := \text{diam}(\omega)$ and the constants C depend only on the shape regularity of the mesh $\mathcal{T}$, for a proof see, e.g., [57]. We can write these inequalities in a more abstract way by introducing the $L^2(\omega)$-orthogonal projector $\Pi^0_{\omega,\gamma} \colon H^1_\gamma(\omega) \to \mathcal{K}_\gamma(\omega) = \ker(\mathbf{grad}_{|H^1_\gamma(\omega)})$:

$$\|u - \Pi^0_{\omega,\gamma} u\|_{0,\omega} \leq C\, h_\omega \| \,\mathbf{grad}\, u\|_{0,\omega} \quad \forall u \in H^1_\gamma(\omega) \, . \tag{3.6}$$

For the other spaces in (3.3), let

$$\boldsymbol{\Pi}^1_{h,\omega,\gamma} \colon \boldsymbol{H}(\mathbf{curl}, \omega) \to \mathbf{grad}\, \mathcal{W}^0_{h,\gamma}(\omega), \quad \boldsymbol{\Pi}^2_{h,\omega,\gamma} \colon \boldsymbol{H}(\text{div}, \omega) \to \mathbf{curl}\, \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega)$$

denote the $\boldsymbol{L}^2(\omega)$-orthogonal projectors onto $\mathbf{grad}\,\mathcal{W}^0_{h,\gamma}(\omega) = \ker(\mathbf{curl}_{|\boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega)})$ and $\mathbf{curl}\,\boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega) = \ker(\mathrm{div}_{|\boldsymbol{\mathcal{W}}^2_{h,\gamma}(\omega)})$, respectively. Then the following discrete Poincaré/Friedrichs type inequalities hold:

$$\|\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}\|_{0,\omega} \leq C\, h_\omega \,\|\,\mathbf{curl}\;\mathbf{w}\|_{0,\omega} \qquad \forall \mathbf{w} \in \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega), \tag{3.7}$$

$$\|\mathbf{q} - \mathbf{\Pi}^2_{h,\omega,\gamma}\mathbf{q}\|_{0,\omega} \leq C\, h_\omega \,\|\,\mathrm{div}\;\mathbf{q}\|_{0,\omega} \qquad \forall \mathbf{q} \in \boldsymbol{\mathcal{W}}^2_{h,\gamma}(\omega), \tag{3.8}$$

where the constant C depends only on the shape regularity of $\mathcal{T}$. These important results can be shown by transformation to a few number of reference patches. From the L^2-projection property, we obtain that

$$\|\mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}\|_{0,\omega} \leq \|\mathbf{w}\|_{0,\omega} \qquad \forall \mathbf{w} \in \boldsymbol{H}(\mathbf{curl},\omega), \tag{3.9}$$

$$\|\mathbf{\Pi}^2_{h,\omega,\gamma}\mathbf{q}\|_{0,\omega} \leq \|\mathbf{q}\|_{0,\omega} \qquad \forall \mathbf{q} \in \boldsymbol{H}(\mathrm{div},\omega). \tag{3.10}$$

3.1.3 Modified Clément operators

We define $M^0_D\colon L^2(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$ by

$$M^0_D u := \sum_{\mathsf{v}\in\mathcal{V}_f} \overline{u}^{\omega_\mathsf{v}}\, \varphi_\mathsf{v}\,, \tag{3.11}$$

where $\overline{e}^{\omega_\mathsf{v}} := \frac{1}{|\omega_\mathsf{v}|}\int_{\omega_\mathsf{v}} u\,dx$ is the mean value of u over ω_v. As a simple but useful property,

$$(M^0_D u)(\mathsf{v}) = \begin{cases} \overline{u}^{\omega_\mathsf{v}} & \text{if } \mathsf{v} \in \mathcal{V}_f \\ 0 & \text{otherwise,} \end{cases} \tag{3.12}$$

i.e., the operator respects the homogeneous boundary conditions. Next, we define $\boldsymbol{M}^1_D\colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ by

$$\boldsymbol{M}^1_D \mathbf{w} := \sum_{\mathsf{e}\in\mathcal{E}_f} \int_{\omega_\mathsf{e}} \mathbf{w}\cdot\mathbf{z}^1_\mathsf{e}\,dx\,\boldsymbol{\psi}_\mathsf{e}\,, \tag{3.13}$$

where the weight function $\mathbf{z}^1_\mathsf{e} \in \boldsymbol{H}(\mathrm{div},\omega_\mathsf{e})$ is yet to be constructed. Beforehand, we define for $\mathsf{e} = [\mathsf{e}_1,\mathsf{e}_2]$ the piecewise constant function

$$y^0_\mathsf{e} := \sum_{\mathsf{v}\in\overline{\mathsf{e}}\cap\mathcal{V}_f} \sigma^\mathsf{v}_\mathsf{e} \frac{1}{|\omega_\mathsf{v}|}\chi_{\omega_\mathsf{v}} = \begin{cases} \frac{1}{|\omega_{\mathsf{e}_2}|}\chi_{\omega_{\mathsf{e}_2}} - \frac{1}{|\omega_{\mathsf{e}_1}|}\chi_{\omega_{\mathsf{e}_1}} & \text{if } \mathsf{e}_1 \notin \overline{\Gamma_D}, \mathsf{e}_2 \notin \overline{\Gamma_D}\,, \\ \frac{1}{|\omega_{\mathsf{e}_2}|}\chi_{\omega_{\mathsf{e}_2}} & \text{if } \mathsf{e}_1 \in \overline{\Gamma_D}\,, \\ -\frac{1}{|\omega_{\mathsf{e}_1}|}\chi_{\omega_{\mathsf{e}_1}} & \text{if } \mathsf{e}_2 \in \overline{\Gamma_D}\,. \end{cases} \tag{3.14}$$

Above, $\chi_{\omega_{\mathsf{v}}}$ is the characteristic function, $\sigma_{\mathsf{e}}^{\mathsf{v}} = -1$ if v is the starting point of e and $\sigma_{\mathsf{e}}^{\mathsf{v}} = +1$ if v is its endpoint (i.e., $\sigma_{\mathsf{e}}^{\mathsf{v}}$ is an entry of the edge-vertex incidence matrix). It is seen easily that $y_{\mathsf{e}}^0 \in \mathcal{W}_h^3(\omega_{\mathsf{e}})$ and that

$$\mathsf{e}_1 \notin \overline{\Gamma_D} \text{ and } \mathsf{e}_2 \notin \overline{\Gamma_D} \qquad \Longrightarrow \qquad \int_{\omega_{\mathsf{e}}} y_{\mathsf{e}}^0 \, dx = 0. \tag{3.15}$$

We require that

$$-\operatorname{div} \mathbf{z}_{\mathsf{e}}^1 = y_{\mathsf{e}}^0 \qquad \text{in } \omega_{\mathsf{e}}\,, \tag{3.16}$$

$$\mathbf{z}_{\mathsf{e}}^1 \cdot \mathbf{n} = 0 \qquad \text{on } \partial\omega_{\mathsf{e}} \setminus \gamma_{\mathsf{e}}\,, \tag{3.17}$$

where γ_{e} is constructed as follows:

(i) If $\mathsf{e}_1 \notin \overline{\Gamma_D}$ and $\mathsf{e}_2 \notin \overline{\Gamma_D}$, we set $\gamma_{\mathsf{e}} := \emptyset$.

(ii) If one of the endpoints of e, say e_1, lies on $\overline{\Gamma_D}$, then we set $\gamma_{\mathsf{e}} := \mathsf{f}$, where f is the triangular face from Lemma 3.2 such that $\mathsf{f} \subset \partial\omega_{\mathsf{e}} \cap \Gamma_D$ and $\mathsf{e}_1 \subset \overline{\mathsf{f}}$. See Fig. 3 for an illustration.

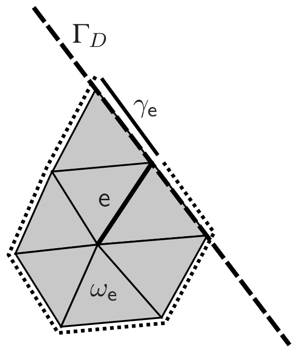

Figure 3. Sketch of an edge patch ω_{e} and the surface γ_{e} for the case that one of the endpoints of the edge lies on the Dirichlet boundary Γ_D. The weight function $\mathbf{z}_{\mathsf{e}}^1$ has vanishing normal component on γ_{e}^c (dotted line).

From the construction of γ_{e} and from (3.15) we can conclude that

$$y_{\mathsf{e}}^0 \in \mathcal{W}_{h,\gamma_{\mathsf{e}}^c}^3(\omega_{\mathsf{e}}), \tag{3.18}$$

where $\gamma_{\mathsf{e}}^c := \partial\omega_c \backslash \gamma_{\mathsf{e}}$. In particular, for the case $\gamma_{\mathsf{e}} = \emptyset$, (3.15) serves as a compatibility condition for (3.16)–(3.17) due to Gauss' theorem. In order to fix $\mathbf{z}_{\mathsf{e}}^1$ uniquely, we require two additional properties:

$$\mathbf{z}_{\mathsf{e}}^1 \in \boldsymbol{\mathcal{W}}_{h,\gamma_{\mathsf{e}}^c}^2(\omega_{\mathsf{e}}), \tag{3.19}$$

$$\int_{\omega_{\mathsf{e}}} \mathbf{z}_{\mathsf{e}}^1 \cdot \mathbf{curl}\, \mathbf{w}_h \, dx = 0 \qquad \forall \mathbf{w}_h \in \boldsymbol{\mathcal{W}}_{h,\gamma_{\mathsf{e}}^c}^1(\omega_{\mathsf{e}}). \tag{3.20}$$

Recall that due to Assumption 3.1, ω_{e} is simply connected with simply connected boundary. Therefore, since γ_{e} is either empty or a triangular face, the complementary surface γ_{e}^c is simply connected. Therefore, the sequence (3.3) with $\omega \mapsto \omega_{\mathsf{e}}$ and $\gamma \mapsto \gamma_{\mathsf{e}}^c$ is exact, and it follows that the weight function $\mathbf{z}_{\mathsf{e}}^1$ indeed exists and is unique.

From (3.16)–(3.17), we can conclude that

$$\int_{\omega_{\mathsf{e}}} \mathbf{grad}\, q \cdot \mathbf{z}_{\mathsf{e}}^1 \, dx = \int_{\omega_{\mathsf{e}}} q\, y_{\mathsf{e}}^0 \, dx \qquad \forall q \in \begin{cases} H^1(\omega_{\mathsf{e}}) & \text{if } \mathsf{e}_1 \notin \overline{\Gamma_D} \text{ and } \mathsf{e}_2 \notin \overline{\Gamma_D}\,, \\ H^1_{\gamma_{\mathsf{e}}}(\omega_{\mathsf{e}}) & \text{if } \mathsf{e}_1 \in \overline{\Gamma_D} \text{ or } \mathsf{e}_2 \in \overline{\Gamma_D}\,. \end{cases} \tag{3.21}$$

Lemma 3.4. *For all $u \in H^1_{\Gamma_D}(\Omega)$, we have the commuting property*

$$\boldsymbol{M}_D^1 \, \mathbf{grad}\, u = \mathbf{grad}\, M_D^0 u.$$

Moreover, for an edge $\mathsf{e} \in \mathcal{E}_f$ with $\mathsf{e}_1 \in \overline{\Gamma_D}$ or $\mathsf{e}_2 \in \overline{\Gamma_D}$ and for $u_h \in \mathcal{W}^0_{h,\gamma_{\mathsf{e}}}(\omega_{\mathsf{e}})$,

$$\int_{\mathsf{e}} (\boldsymbol{M}_D^1 \, \mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}} \, ds = \int_{\mathsf{e}} (\mathbf{grad}\, M_D^0 u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}} \, ds,$$

where the two expressions are well-defined.

Proof. For the first part of the proof, we just consider $u \in H^1(\Omega)$. By construction, both $\boldsymbol{M}_D^1 \, \mathbf{grad}\, u$ and $\mathbf{grad}\, M^0 u$ belong to $\boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$, even for a non-trivial topology of Ω, Γ_D. Therefore, in order to show the first identity, it suffices to check all the edge integrals on $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2] \in \mathcal{E}_f$:

$$\begin{aligned} &\int_{\mathsf{e}} (\mathbf{grad}\, M^0 u) \cdot \boldsymbol{\tau}_{\mathsf{e}} \, ds \\ &\quad = \sum_{v \in \mathcal{V}^f} \overline{u}^{\omega_v} \int_{\mathsf{e}} \mathbf{grad}\, \varphi_v \cdot \boldsymbol{\tau}_{\mathsf{e}} \, ds = \begin{cases} \overline{u}^{\omega_{\mathsf{e}_2}} - \overline{u}^{\omega_{\mathsf{e}_1}} & \text{if } \mathsf{e}_1 \notin \overline{\Gamma_D}, \mathsf{e}_2 \notin \overline{\Gamma_D}\,, \\ \overline{u}^{\omega_{\mathsf{e}_2}} & \text{if } \mathsf{e}_1 \in \overline{\Gamma_D}, \\ -\overline{u}^{\omega_{\mathsf{e}_1}} & \text{if } \mathsf{e}_2 \in \overline{\Gamma_D}\,. \end{cases} \end{aligned}$$

Since

$$\overline{u}^{\omega_{\mathsf{e}_i}} = \int_{\omega_{\mathsf{e}}} u\, \chi_{\omega_{\mathsf{e}_i}} \, dx,$$

we can conclude from (3.14) that

$$\int_{\mathsf{e}} (\mathbf{grad}\, M_D^0 u) \cdot \boldsymbol{\tau}_{\mathsf{e}} \, ds = \int_{\omega_{\mathsf{e}}} u\, y_{\mathsf{e}}^0 \, dx \qquad \forall u \in H^1(\Omega). \tag{3.22}$$

We now show the first identity and assume that $u \in H^1_{\Gamma_D}(\Omega)$. Consequently, $u_{|\omega_{\mathsf{e}}} \in H^1_{\Gamma_D}(\omega_{\mathsf{e}})$, in particular $u_{|\omega_{\mathsf{e}}} \in H^1_{\gamma_{\mathsf{e}}}(\omega_{\mathsf{e}})$, and so (3.21) and the definition (3.13) of $\boldsymbol{M}_D^1$

imply

$$\int_{\mathsf{e}} (\mathbf{grad}\, M_D^0 u) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\omega_{\mathsf{e}}} \mathbf{grad}\, u \cdot \mathbf{z}_{\mathsf{e}}^1\, dx$$
$$= \int_{\mathsf{e}} \int_{\omega_{\mathsf{e}}} \mathbf{grad}\, u \cdot \mathbf{z}_{\mathsf{e}}^1\, dx\, \boldsymbol{\psi}_{\mathsf{e}} \cdot\ \boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\mathsf{e}} (\boldsymbol{M}_D^1\, \mathbf{grad}\, u) \cdot \tau_{\mathsf{e}}\, ds.$$

The second identity follows by the same arguments and the locality of M_D^0, $\boldsymbol{M}_D^1$. □

Lemma 3.5. *For all $u \in L^2(\Omega)$ and $T \in \mathcal{T}$,*

$$\|M_D^0 u\|_{0,T} \leq C \|u\|_{0,\omega_T}\,.$$

Proof. From the definition of M_D^0 we derive

$$\|M_D^0 u\|_{0,T} \leq \sum_{\mathsf{v} \in \mathcal{V}_f \cap \overline{T}} |\overline{u}^{\omega_{\mathsf{v}}}|\, \|\varphi_{\mathsf{v}}\|_{0,T}\,.$$

Cauchy's inequality yields $|\overline{u}^{\omega_{\mathsf{v}}}| \leq |\omega_{\mathsf{v}}|^{-1/2} \|u\|_{0,\omega_{\mathsf{v}}}$ and standard FE arguments show that $|\omega_{\mathsf{v}}| \geq c\, h_T^3$ and $\|\varphi_{\mathsf{v}}\|_{0,\omega_T} \leq C\, h_T^{3/2}$. □

For the approximation property of M_D^0, we need another construction. For elements T where

$$\partial\omega_T \cap \Gamma_D \neq \emptyset \quad \text{but} \quad \text{meas}_2(\partial\omega_T \cap \Gamma_D) = 0,$$

we define a *slightly enlarged* element patch $\widetilde{\omega}_T \supset \omega_T$ such that

$$\text{meas}_2(\partial\widetilde{\omega}_T \cap \Gamma_D) > 0 \qquad \text{and} \qquad \text{diam}(\widetilde{\omega}_T) \leq C\, h_T\,, \tag{3.23}$$

with a uniform constant C depending only on the shape regularity of $\mathcal{T}$, see Fig. 4 for an illustration. For all other elements, we simply set $\widetilde{\omega}_T = \omega_T$.

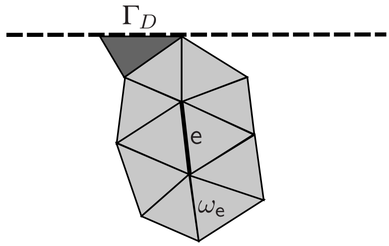

Figure 4. Sketch of construction of enlarged element patch $\widetilde{\omega}_{\mathsf{e}}$. Light grey area: original patch ω_{e}. Dark grey area: element that is added in order to obtain $\widetilde{\omega}_{\mathsf{e}}$.

Lemma 3.6. *For all* $u \in H^1_{\Gamma_D}(\Omega)$ *and* $T \in \mathcal{T}$,

$$\|u - M_D^0 u\|_{0,T} \leq C\, h_T \, \|\,\mathbf{grad}\, u\|_{0,\widetilde{\omega}_T}$$

Proof. Let T be such that $\text{meas}_2(\partial\widetilde{\omega}_T \cap \Gamma_D) = 0$, which implies that $\partial\omega_T \cap \Gamma_D = \emptyset$, and so all vertices on $\overline{\omega}_T$ are in $\mathcal{V}_f$. Due to the partition of unity property of the vertex basis functions,

$$(M_D^0 c)_{|\omega_T} = c \qquad \text{for any constant } c.$$

Hence,

$$u - M_D^0 u = u - \overline{u}^{\omega_\mathsf{e}} + M_D^0(u - \overline{u}^{\omega_\mathsf{e}}).$$

From the triangle inequality and the L^2-estimate from Lemma 3.5, we obtain

$$\|u - M_D^0 u\|_{0,T} \leq C\|u - \overline{u}^{\omega_\mathsf{e}}\|_{0,\omega_T} \leq C h_T \, \|\,\mathbf{grad}\, u\|_{0,\omega_T} \,,$$

where in the last step, we have used Poincaré's inequality (3.4). Finally, let T be such that $\text{meas}_2(\partial\widetilde{\omega}_T \cap \Gamma_D) > 0$. We apply Lemma 3.5 directly, leading to

$$\|u - M_D^0 u\|_{0,T} \leq C\|u\|_{0,\omega_T} \leq C\|u\|_{0,\widetilde{\omega}_T} \,.$$

Since u vanishes on $\partial\widetilde{\omega}_T \cap \Gamma_D$ by assumption, Friedrichs' inequality (3.5) yields the desired bound. □

The stability of M_D^0 in the H^1-semi norm will be a consequence of Lemma 3.8 below. The L^2-stability of $\boldsymbol{M}_D^1$ involves the particular choice of the weight function $\mathbf{z}_\mathsf{e}^1$ and needs the following auxiliary estimate:

Lemma 3.7. *Let the weight function* $\mathbf{z}_\mathsf{e}^1$ *be defined by* (3.16), (3.17), (3.19), *and* (3.20). *Then*

$$\|\mathbf{z}_\mathsf{e}^1\|_{0,\omega_\mathsf{e}} \leq C h_\mathsf{e}^{-1/2}.$$

Proof. The orthogonality condition (3.20) implies that $\mathbf{\Pi}^2_{h,\omega_\mathsf{e},\gamma_\mathsf{e}} \mathbf{z}_\mathsf{e}^1 = 0$, and so the discrete Poincaré-Friedrichs type inequality (3.8) implies

$$\|\mathbf{z}_\mathsf{e}^1\|_{0,\omega_\mathsf{e}} \leq C\, h_\mathsf{e} \, \|\,\text{div}\ \mathbf{z}_\mathsf{e}^1\|_{0,\omega_\mathsf{e}} = C\, h_\mathsf{e} \, \|y_\mathsf{e}^0\|_{0,\omega_\mathsf{e}} \,,$$

where we have used (3.16). From the definition (3.14) of y_e^0, we see that $\|y_\mathsf{e}^0\|_{0,\omega_\mathsf{e}} \leq C\, h_\mathsf{e}^{3/2} h_\mathsf{e}^{-3} = C\, h_\mathsf{e}^{-3/2}$. □

Lemma 3.8. *For all* $\mathbf{w} \in \boldsymbol{L}^2(\Omega)$ *and* $T \in \mathcal{T}$,

$$\|\boldsymbol{M}_D^1 \mathbf{w}\|_{0,T} \leq C\, \|\mathbf{w}\|_{0,\omega_T} \,.$$

Proof.

$$\|\boldsymbol{M}_D^1 \mathbf{w}\|_{0,T} \le \sum_{\mathsf{e}\in\mathcal{E}_f\cap\overline{T}} \left|\int_{\omega_\mathsf{e}} \mathbf{w}\cdot\mathbf{z}_\mathsf{e}^1\,dx\right| \|\boldsymbol{\psi}_\mathsf{e}\|_{0,\omega_\mathsf{e}} \le \sum_{\mathsf{e}\in\mathcal{E}_f\cap\overline{T}} \|\mathbf{w}\|_{0,\omega_\mathsf{e}} \|\mathbf{z}_\mathsf{e}^1\|_{0,\omega_\mathsf{e}} \|\boldsymbol{\psi}_\mathsf{e}\|_{0,\omega_\mathsf{e}} .$$

From Lemma 3.3, $\|\boldsymbol{\psi}_\mathsf{e}\|_{0,\omega_\mathsf{e}} \le C\,h_\mathsf{e}^{-1}h_\mathsf{e}^{3/2}$. The proof is concluded by applying Lemma 3.7. □

Corollary 3.9. *For all $u \in H^1_{\Gamma_D}(\Omega)$ and $T \in \mathcal{T}$,*

$$\|\,\mathbf{grad}\, M_D^0 u\|_{0,T} \le C\,\|\,\mathbf{grad}\, u\|_{0,\omega_T} .$$

Proof. Due to Lemma 3.4, $\mathbf{grad}\, M_D^0 u = \boldsymbol{M}_D^1\,\mathbf{grad}\, u$ for all $u \in H^1_{\Gamma_D}(\Omega)$, so the statement follows from Lemma 3.8. □

3.1.4 Auxiliary projectors on local patches

Let ω be a simply connected patch of a few elements with simply connected boundary and $\gamma \subset \partial\omega$ a simply connected union of faces such that the exact sequence property (3.3) holds; the cases $\gamma = \emptyset$, $\gamma = \partial\omega$ are admitted. We define $Q^0_{\omega,\gamma}\colon H^1(\omega) \to \mathcal{W}^0_{h,\gamma}(\omega)$ by

$$\int_\omega Q^0_{\omega,\gamma} u\,dx = \int_\omega u\,dx \qquad \text{if } \gamma = \emptyset, \tag{3.24}$$

$$\int_\omega \mathbf{grad}(Q^0_{\omega,\gamma} u)\cdot\mathbf{grad}\, p_h\,dx = \int_\omega \mathbf{grad}\, u\cdot\mathbf{grad}\, p_h\,dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega). \tag{3.25}$$

and $\boldsymbol{Q}^1_{\omega,\gamma}\colon \boldsymbol{H}(\mathbf{curl},\omega) \to \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega)$ by

$$\int_\omega \boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\cdot\mathbf{grad}\, p_h\,dx = \int_\omega \mathbf{w}\cdot\mathbf{grad}\, p_h\,dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega), \tag{3.26}$$

$$\int_\omega \mathbf{curl}(\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w})\cdot\mathbf{curl}\ \mathbf{q}_h\,dx = \int_\omega \mathbf{curl}\ \mathbf{w}\cdot\mathbf{curl}\ \mathbf{q}_h\,dx \qquad \forall \mathbf{q}_h \in \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega). \tag{3.27}$$

Obviously,

$$\boldsymbol{Q}^1_{\omega,\gamma}\,\mathbf{grad}\, u = \mathbf{grad}\, Q^0_{\omega,\gamma} u \qquad \forall u \in H^1(\omega), \tag{3.28}$$

$$Q^0_{\omega,\gamma} u_h = u_h \qquad \forall u_h \in \mathcal{W}^0_{h,\gamma}(\omega), \tag{3.29}$$

$$\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}_h = \mathbf{w}_h \qquad \forall \mathbf{w}_h \in \boldsymbol{\mathcal{W}}^1_{h,\gamma}(\omega). \tag{3.30}$$

Finally, we define a *lifting operator* $Q^1_{\omega,\gamma,-}\colon \boldsymbol{H}(\mathbf{curl},\omega) \to \mathcal{W}^0_{h,\gamma}(\omega)$ by

$$\int_\omega Q^1_{\omega,\gamma,-}\mathbf{w}\,dx = 0 \qquad \text{if } \gamma = \emptyset, \tag{3.31}$$

$$\int_\omega \mathbf{grad}(Q^1_{\omega,\gamma,-}\mathbf{w}) \cdot \mathbf{grad}\, p_h\,dx = \int_\omega \mathbf{w} \cdot \mathbf{grad}\, p_h\,dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega). \tag{3.32}$$

Summarizing, we have

$$Q^0_{\omega,\gamma}u = \begin{cases} \overline{u}^\omega + Q^1_{\omega,\gamma,-}\,\mathbf{grad}\,u & \text{if } \gamma = \emptyset, \\ Q^1_{\omega,\gamma,-}\,\mathbf{grad}\,u & \text{otherwise.} \end{cases} \tag{3.33}$$

Lemma 3.10. *For* $u \in H^1(\omega)$*,*

$$\begin{aligned} \|\,\mathbf{grad}\,Q^0_{\omega,\gamma}u\|_{0,\omega} &\le \|\,\mathbf{grad}\,u\|_{0,\omega} \\ \|Q^0_{\omega,\gamma}u\|_{0,\omega} &\le \|u\|_{0,\omega} + C\ \mathrm{diam}(\omega)\,\|\,\mathbf{grad}\,u\|_{0,\omega}\,. \end{aligned}$$

Proof. The first estimate follows immediately from (3.25) by setting $p_h = Q^0_{\omega,\gamma}u$ and applying Cauchy's inequality. For the second estimate we treat two cases:

- If $\mathrm{meas}_2(\gamma) = 0$ then the mean value property (3.24) implies

$$Q^0_{\omega,\gamma}u = Q^0_{\omega,\gamma}(u - \overline{u}^\omega) + \overline{u}^\omega$$

  and the first term has vanishing mean over ω. From the triangle inequality, Cauchy-Schwarz, and Poincaré's inequality (3.4), we obtain

$$\begin{aligned} \|Q^0_{\omega,\gamma}u\|_{0,\omega} &\le \|Q^0_{\omega,\gamma}(u - \overline{u}^\omega)\|_{0,\omega} + \|\overline{u}^\omega\|_{0,\omega} \\ &\le C\ \mathrm{diam}(\omega)\|\,\mathbf{grad}\,Q^0_{\omega,\gamma}u\|_{0,\omega} + \|u\|_{0,\omega}\,. \end{aligned}$$

- If $\mathrm{meas}_2(\gamma) > 0$, we obtain from Friedrichs' inequality (3.5) that

$$\|Q^0_{\omega,\gamma}u\|_{0,\omega} \le C\ \mathrm{diam}(\omega)\|\,\mathbf{grad}\,Q^0_{\omega,\gamma}u\|_{0,\omega}\,.$$

In both cases, employing the first estimate concludes the proof. □

Lemma 3.11. *For* $\mathbf{w} \in \boldsymbol{H}(\mathbf{curl},\omega)$*,*

$$\begin{aligned} \|\,\mathbf{curl}\ \boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\|_{0,\omega} &\le \|\,\mathbf{curl}\ \mathbf{w}\|_{0,\omega} \\ \|\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\|_{0,\omega} &\le \|\mathbf{w}\|_{0,\omega} + C\ \mathrm{diam}(\omega)\|\,\mathbf{curl}\ \mathbf{w}\|_{0,\omega}\,. \end{aligned}$$

Proof. The first estimate follows immediately from (3.27) by setting $\mathbf{q}_h = \mathbf{Q}^1_{\omega,\gamma}\mathbf{w}$ and applying Cauchy's inequality. For the second estimate recall the projection operator $\mathbf{\Pi}^1_{h,\omega,\gamma}\colon \boldsymbol{H}(\mathbf{curl},\omega) \to \mathbf{grad}\,\mathcal{W}^0_{h,\gamma}(\omega)$, from Sect. 3.1.2, which has the property

$$\int_\omega \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}\cdot \mathbf{grad}\, p_h\, dx = \int_\omega \mathbf{w}\cdot \mathbf{grad}\, p_h\, dx \qquad \forall p_h \in \mathcal{W}^0_{h,\gamma}(\omega). \tag{3.34}$$

Since $\mathbf{\Pi}^1_{h,\omega,\gamma}$, $\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{\Pi}^1_{h,\omega,\gamma}$, and $\mathbf{\Pi}^1_{h,\omega,\gamma}\boldsymbol{Q}^1_{\omega,\gamma}$ have the same range, we can conclude from (3.26) and (3.34) that

$$\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w} = \mathbf{\Pi}^1_{h,\omega,\gamma}\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w} = \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}.$$

Therefore

$$\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w} = \boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}) + \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}$$

and

$$\mathbf{\Pi}^1_{h,\omega,\gamma}\boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}) = 0.$$

Hence, the discrete Poincaré-Friedrichs type inequality (3.7) together with the L^2-stability (3.9) of $\mathbf{\Pi}^1_{h,\omega,\gamma}$ yields

$$\begin{aligned}\|\boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}\|_{0,\omega} &\le \|\boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w})\|_{0,\omega} + \|\mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w}\|_{0,\omega}\\ &\le C\ \mathrm{diam}(\omega)\| \underbrace{\mathbf{curl}\ \boldsymbol{Q}^1_{\omega,\gamma}(\mathbf{w} - \mathbf{\Pi}^1_{h,\omega,\gamma}\mathbf{w})}_{\mathbf{curl}\ \boldsymbol{Q}^1_{\omega,\gamma}\mathbf{w}} \|_{0,\omega} + \|\mathbf{w}\|_{0,\omega}\,.\end{aligned}$$

Employing the first estimate once again concludes the proof. □

Finally, we need stability estimates for the lifting operator $Q^1_{\omega,\gamma,-}$:

Lemma 3.12. *For any* $\mathbf{w} \in \boldsymbol{H}(\mathbf{curl},\omega)$*,*

$$\begin{aligned}\|\,\mathbf{grad}\, Q^1_{\omega,\gamma,-}\mathbf{w}\|_{0,\omega} &\le \|\mathbf{w}\|_{0,\omega}\,,\\ \|Q^1_{\omega,\gamma,-}\mathbf{w}\|_{0,\omega} &\le C\ \mathrm{diam}(\omega)\,\|\mathbf{w}\|_{0,\omega}\,.\end{aligned}$$

Proof. Choosing $p_h := Q^1_{\omega,\gamma,-}\mathbf{w}$ in (3.32) applying Cauchy-Schwarz we find that

$$\|\,\mathbf{grad}\, Q^1_{\omega,\gamma,-}\mathbf{w}\|^2_{0,\omega} = \int_\omega \mathbf{w}\cdot\mathbf{grad}(Q^1_{\omega,\gamma,-}\mathbf{w})\, dx \le \|\mathbf{w}\|_{0,\omega}\,\|Q^1_{\omega,\gamma,-}\mathbf{w}\|_{0,\omega}\,,$$

which implies the first inequality. For $\gamma = \emptyset$, the second inequality follows from the first one by Poincaré's inequality (3.4) because $Q^1_{\omega,\emptyset,-}\mathbf{w}$ has vanishing mean over ω. If $\mathrm{meas}_2(\gamma) > 0$, then we can use Friedrichs' inequality (3.5) to obtain the same result. □

3.1.5 The auxiliary operators S_D^0 and S_D^1

For $\mathsf{v} \in \mathcal{V}_f$, we set

$$Q_{\mathsf{v}}^0 := Q_{\omega_{\mathsf{v}},\emptyset}^0\,, \qquad \boldsymbol{Q}_{\mathsf{v}}^1 := \boldsymbol{Q}_{\omega_{\mathsf{v}},\emptyset}^1\,, \qquad Q_{\mathsf{v},-}^1 := Q_{\omega_{\mathsf{v}},\emptyset,-}^1\,. \tag{3.35}$$

We define $S_D^0\colon H^1(\Omega) \to \mathcal{W}_{h,\Gamma_D}^0(\mathcal{T})$ by

$$S_D^0 u := M_D^0 u, \tag{3.36}$$

and $\boldsymbol{S}_D^1\colon \boldsymbol{H}(\mathbf{curl},\Omega) \to \boldsymbol{\mathcal{W}}_{h,\Gamma_D}^1(\mathcal{T})$ by

$$\boldsymbol{S}_D^1 \mathbf{w} := \boldsymbol{M}_D^1 \mathbf{w} + \sum_{\mathsf{v}\in\mathcal{V}_f} \big(Q_{\mathsf{v},-}^1 \mathbf{w}\big)(\mathsf{v})\ \mathbf{grad}\,\varphi_{\mathsf{v}}\,. \tag{3.37}$$

Remark 3.13. Following the original paper by Falk and Winther [27], the operator $\boldsymbol{S}_D^1$ should be defined by

$$\boldsymbol{S}_D^1 \mathbf{w} := \boldsymbol{M}_D^1 \mathbf{w} + \sum_{\mathsf{v}\in\mathcal{V}_f} \big[(I - S_D^0) Q_{\mathsf{v},-}^1 \mathbf{w}\big](\mathsf{v})\ \mathbf{grad}\,\varphi_{\mathsf{v}} \tag{3.38}$$

and one needs to argue firstly that the expression $\big[(I - S_D^0) Q_{\mathsf{v},-}^1 \mathbf{w}\big](\mathsf{v})$ is well-defined. Indeed, for $\mathsf{v} \in \mathcal{V}_f$,

$$\big[(I - \underbrace{S_D^0}_{M_D^0}) Q_{\mathsf{v},-}^1 \mathbf{w}\big](\mathsf{v}) \;=\; (Q_{\omega_{\mathsf{v}},\emptyset,-}^1 \mathbf{w})(\mathsf{v}) - \underbrace{\overline{Q_{\omega_{\mathsf{v}},\emptyset,-}^1 \mathbf{w}}^{\omega_{\mathsf{v}}}}_{=0},$$

which is also the reason for the simplified definition (3.37) compared to (3.38).

Unlike M_D^0, $\boldsymbol{M}_D^1$, the operators S_D^0 and $\boldsymbol{S}_D^1$ do not commute and they are not projections either. The key property of $\boldsymbol{S}_D^1$ is the following one:

Lemma 3.14. *For all* $\mathsf{e} = [\mathsf{e}_1, \mathsf{e}_2] \in \mathcal{E}$ *and for all* $u_h \in \mathcal{W}_{h,\Gamma_D}^0(\mathcal{T})$,

$$\int_{\mathsf{e}} (\boldsymbol{S}_D^1\, \mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\mathsf{e}} \mathbf{grad}\, u_h \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds.$$

The same identity holds for a particular edge e *if* u_h *is only given in* $\mathcal{W}_{h,\gamma_{\mathsf{e}}}^0(\omega_{\mathsf{e}})$.

Proof. For edges e on the Dirichlet boundary Γ_D, both integrals evaluate to zero. Let us therefore consider $\mathsf{e} \in \mathcal{E}_f$ and $u_h \in \mathcal{W}_h^0(\omega_{\mathsf{e}})$. We will specify boundary conditions for u_h later on. Insertion of the definition of $\boldsymbol{S}_D^1$ into the left-hand side yields

$$\begin{aligned}
&\int_{\mathsf{e}} (\boldsymbol{S}_D^1 \,\mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds \\
&\quad = \int_{\mathsf{e}} (\boldsymbol{M}_D^1 \,\mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds + \int_{\mathsf{e}} \sum_{\mathsf{v}\in\mathcal{V}_f} \big(Q_{\mathsf{v},-}^1 \,\mathbf{grad}\, u_h\big)(\mathsf{v})\, \mathbf{grad}\, \varphi_{\mathsf{v}} \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds \\
&\quad = \int_{\mathsf{e}} (\boldsymbol{M}_D^1 \,\mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds + \sum_{\mathsf{v}\in\bar{\mathsf{e}}\cap\mathcal{V}_f} \sigma_{\mathsf{v}}^{\mathsf{e}} \big(Q_{\mathsf{v},-}^1 \,\mathbf{grad}\, u_h\big)(\mathsf{v}),
\end{aligned}$$

where $\sigma_{\mathsf{e}_2}^{\mathsf{e}} = +1$ and $\sigma_{\mathsf{e}_1}^{\mathsf{e}} = -1$. Apparently, these expressions are well-defined although u_h is only given in $\mathcal{W}_h^0(\omega_{\mathsf{e}})$. Identity (3.33) and the projection property (3.29) of Q_{v}^0 yield

$$Q_{\mathsf{v},-}^1 \,\mathbf{grad}\, u_h = Q_{\mathsf{v}}^0 u_h - \overline{u_h}^{\omega_{\mathsf{v}}} = u_h - \overline{u_h}^{\omega_{\mathsf{v}}} .$$

Therefore,

$$\big(Q_{\mathsf{v},-}^1 \,\mathbf{grad}\, u_h\big)(\mathsf{v}) = u_h(\mathsf{v}) - \overline{u_h}^{\omega_{\mathsf{v}}} .$$

Substitution in the earlier formula yields (still for arbitrary $u_h \in \mathcal{W}_h^0(\omega_{\mathsf{e}})$)

$$\begin{aligned}
&\int_{\mathsf{e}} (\boldsymbol{S}_D^1 \,\mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds \\
&\quad = \int_{\mathsf{e}} (\boldsymbol{M}_D^1 \,\mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds + \sum_{\mathsf{v}\in\bar{\mathsf{e}}\cap\mathcal{V}_f} \sigma_{\mathsf{v}}^{\mathsf{e}} \big(u_h(\mathsf{v}) - \overline{u_h}^{\omega_{\mathsf{v}}}\big) \\
&\quad = \underbrace{\int_{\mathsf{e}} (\boldsymbol{M}_D^1 \,\mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds}_{\text{(I)}} - \underbrace{\sum_{\mathsf{v}\in\bar{\mathsf{e}}\cap\mathcal{V}_f} \sigma_{\mathsf{v}}^{\mathsf{e}} \overline{u_h}^{\omega_{\mathsf{v}}}}_{\text{(II)}} + \underbrace{\sum_{\mathsf{v}\in\bar{\mathsf{e}}\cap\mathcal{V}_f} u_h(\mathsf{v})}_{=\int_{\mathsf{e}} \mathbf{grad}\, u_h \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds} .
\end{aligned}$$

For the remainder of the proof, we treat two cases:

- If $u_h \in \mathcal{W}_{h,\Gamma_D}^0(\mathcal{T})$ then we obtain from the first commuting property of Lemma 3.4 and Identity (3.22) in its proof that

$$\text{(I)} = \int_{\mathsf{e}} (\boldsymbol{M}_D^1 \,\mathbf{grad}\, u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\mathsf{e}} \mathbf{grad}(M_D^0 u_h) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\omega_{\mathsf{e}}} u_h\, y_{\mathsf{e}}^0\, dx = \text{(II)}.$$

- If $u_h \in \mathcal{W}_{h,\gamma_{\mathsf{e}}}^0(\omega_{\mathsf{e}})$, then the second commuting property of Lemma 3.4 and Identity (3.22) in its proof imply the same formula.

□

Next, we provide a stability estimate for $\boldsymbol{S}_D^1$.

Lemma 3.15. *For all* $\mathbf{w} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ *and* $T \in \mathcal{T}$,

$$\|\boldsymbol{S}_D^1 \mathbf{w}\|_{0,T} \leq C \|\mathbf{w}\|_{0,\omega_T}$$

Proof. The definition of $\boldsymbol{S}_D^1$ and the triangle inequality imply

$$\|\boldsymbol{S}_D^1 \mathbf{w}\|_{0,T} \leq \|\boldsymbol{M}_D^1 \mathbf{w}\|_{0,T} + \sum_{\mathsf{v} \in \mathcal{V}_f \cap \overline{T}} \left|(Q_{\mathsf{v},-}^1 \mathbf{w})(\mathsf{v})\right| \|\, \mathbf{grad}\, \varphi_{\mathsf{v}}\|_{0,T}\,.$$

The first term can be estimated from above by $C\,\|\mathbf{w}\|_{0,\omega_T}$, cf. Lemma 3.8. Using Lemma 3.3 we can now estimate the second term:

$$\sum_{\mathsf{v} \in \mathcal{V}_f \cap \overline{T}} \left|(Q_{\mathsf{v},-}^1 \mathbf{w})(\mathsf{v})\right| \|\, \mathbf{grad}\, \varphi_{\mathsf{v}}\|_{0,T} \leq C \sum_{\mathsf{v} \in \mathcal{V}_f \cap \overline{T}} h_T^{-1} \, \|Q_{\mathsf{v},-}^1 \mathbf{w}\|_{0,T}\,.$$

Recall that $\overline{Q_{\mathsf{v},-}^1 \mathbf{w}}^{\omega_{\mathsf{v}}} = 0$, so Poincaré's inequality (3.4) implies

$$h_T^{-1} \|Q_{\mathsf{v},-}^1 \mathbf{w}\|_{0,T} \leq C \, \|\, \mathbf{grad}\, Q_{\mathsf{v},-}^1 \mathbf{w}\|_{0,\omega_{\mathsf{v}}} \leq C \, \|\mathbf{w}\|_{0,\omega_{\mathsf{v}}}\,,$$

where in the last step, we have used Lemma 3.12. Combination of the above yields

$$\sum_{\mathsf{v} \in \mathcal{V}_f \cap \overline{T}} \left|(Q_{\mathsf{v},-}^1 \mathbf{w})(\mathsf{v})\right| \|\, \mathbf{grad}\, \varphi_{\mathsf{v}}\|_{0,T} \leq C \, \|\mathbf{w}\|_{0,\omega_T}\,.$$

Combination of the estimates for the first and second term concludes the proof. □

In addition to the previous lemma, we need another local estimate for $\boldsymbol{S}_D^1$:

Lemma 3.16. *For all* $\mathbf{w} \in \boldsymbol{H}(\mathbf{curl}, \omega_{\mathsf{e}})$,

$$\Big| \int_{\mathsf{e}} (\boldsymbol{S}_D^1 \mathbf{w}) \cdot \tau_{\mathsf{e}} \, ds \Big| \leq C \, h_{\mathsf{e}}^{-1/2} \, \|\mathbf{w}\|_{0,\omega_{\mathsf{e}}}\,.$$

Proof. From the definition of $\boldsymbol{S}_D^1$, we see that

$$\begin{aligned} &\Big| \int_{\mathsf{e}} (\boldsymbol{S}_D^1 \mathbf{w}) \cdot \tau_{\mathsf{e}} \, ds \Big| \\ &\qquad \leq \Big| \int_{\mathsf{e}} (\boldsymbol{M}_D^1 \mathbf{w}) \cdot \tau_{\mathsf{e}} \, ds \Big| + \sum_{\mathsf{v} \in \mathcal{V}_f \cap \overline{\mathsf{e}}} \left|(Q_{\mathsf{v},-}^1 \mathbf{w})(\mathsf{v})\right| \Big| \underbrace{\int_{\mathsf{e}} \mathbf{grad}\, \varphi_{\mathsf{v}} \cdot \tau_{\mathsf{e}} \, ds}_{=\pm 1} \Big|. \end{aligned}$$

From the definition of $\boldsymbol{M}_D^1$ we easily conclude from Lemma 3.7 that

$$\Big| \int_{\mathsf{e}} (\boldsymbol{M}_D^1 \mathbf{w}) \cdot \tau_{\mathsf{e}} \, ds \Big| \leq \Big| \int_{\omega_{\mathsf{e}}} \mathbf{w} \cdot \mathbf{z}_{\mathsf{e}}^1 \, dx \Big| \leq C \, h_{\mathsf{e}}^{-1/2} \, \|\mathbf{w}\|_{0,\omega_{\mathsf{e}}}\,.$$

Due to Lemma 3.3 and Lemma 3.12,

$$\left|(Q_{\mathsf{v},-}^1 \mathbf{w})(\mathsf{v})\right| \leq C \, h_T^{-3/2} \, \|Q_{\mathsf{v},-}^1 \mathbf{w}\|_{0,T} \leq C \, h_T^{-1/2} \, \|\mathbf{w}\|_{0,\omega_{\mathsf{v}}}\,.$$

Summation over the above estimates yields the desired result. □

3.1.6 The bounded co-chain projectors

Recall that we defined, for $\mathsf{v} \in \mathcal{V}_f$,

$$Q^0_{\mathsf{v}} := Q^0_{\omega_{\mathsf{v}},\emptyset}\,, \qquad \boldsymbol{Q}^1_{\mathsf{v}} := \boldsymbol{Q}^1_{\omega_{\mathsf{v}},\emptyset}\,, \qquad Q^0_{\mathsf{v},-} := Q^0_{\omega_{\mathsf{v}},\emptyset,-}\,.$$

In addition, for $\mathsf{e} \in \mathcal{E}_f$, we set

$$Q^0_{\mathsf{e}} := Q^0_{\omega_{\mathsf{e}},\gamma_{\mathsf{e}}}\,, \qquad \boldsymbol{Q}^1_{\mathsf{e}} := \boldsymbol{Q}^1_{\omega_{\mathsf{e}},\gamma_{\mathsf{e}}}\,, \qquad Q^1_{\mathsf{e},-} := Q^1_{\omega_{\mathsf{e}},\gamma_{\mathsf{e}},-}\,, \tag{3.39}$$

where γ_{e} is constructed as in Sect. 3.1.3 when specifying the weight function $\mathbf{z}^1_{\mathsf{e}}$. Recall that $\gamma_{\mathsf{e}} = \emptyset$ for the case that $\mathsf{e}_1 \notin \overline{\Gamma_D}$ and $\mathsf{e}_2 \notin \overline{\Gamma_D}$.

Based on these operators, we define $R^0_D \colon H^1(\Omega) \to \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$ by

$$R^0_D u = S^0_D u + \sum_{\mathsf{v}\in\mathcal{V}_f} \big[(I - S^0_D) Q^0_{\mathsf{v}} u\big](\mathsf{v})\,\varphi_{\mathsf{v}}$$

and $\boldsymbol{R}^1_D \colon \boldsymbol{H}(\mathbf{curl},\Omega) \to \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$ by

$$\boldsymbol{R}^1_D \mathbf{w} = \boldsymbol{S}^1_D \mathbf{w} + \sum_{\mathsf{e}\in\mathcal{E}_f} \int_{\mathsf{e}} \big[(I - \boldsymbol{S}^1_D)\boldsymbol{Q}^1_{\mathsf{e}} \mathbf{w}\big] \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds\, \boldsymbol{\psi}_{\mathsf{e}}\,.$$

Before we continue, we have to argue that the two operators are well-defined. For R^0_D, observe that

$$\big[(I - S^0_D) Q^0_{\mathsf{v}} u\big](\mathsf{v}) = \big[(I - M^0_D) Q^0_{\mathsf{v}} u\big](\mathsf{v}).$$

Since for any $p \in H^1(\Omega)$, the value $(M^0_D p)(\mathsf{v})$ depends only on $p_{|\omega_{\mathsf{v}}}$, the expression above is valid. For $\boldsymbol{R}^1_D$, recall that for $\bar{\mathbf{w}} \in \boldsymbol{H}(\mathbf{curl},\Omega)$,

$$\boldsymbol{S}^1_D \bar{\mathbf{w}} = \boldsymbol{M}^1_D \bar{\mathbf{w}} + \sum_{\mathsf{v}\in\mathcal{V}_f} (Q^1_{\mathsf{v},-}\bar{\mathbf{w}})(\mathsf{v})\ \mathbf{grad}\,\varphi_{\mathsf{v}}\,.$$

From the definition of $\boldsymbol{M}^1_D$, we see that $\int_{\mathsf{e}} (\boldsymbol{M}^1_D \bar{\mathbf{w}}) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds$ depends only on $\bar{\mathbf{w}}_{|\omega_{\mathsf{e}}}$. Since $(Q^1_{\mathsf{v},-}\bar{\mathbf{w}})(\mathsf{v})$ depends only on $\bar{\mathbf{w}}_{|\omega_{\mathsf{v}}}$, we can conclude altogether that $\int_{\mathsf{e}} (\boldsymbol{S}^1_D \bar{\mathbf{w}}) \cdot \boldsymbol{\tau}_{\mathsf{e}}\, ds$ only depends on $\bar{\mathbf{w}}_{|\omega_{\mathsf{e}}}$. Setting (formally) $\bar{\mathbf{w}} = \boldsymbol{Q}^1_{\mathsf{e}} \mathbf{w}$ shows that $\boldsymbol{R}^1_D$ is well-defined.

As a next step, we show the projection property of R^0_D and $\boldsymbol{R}^1_D$.

Lemma 3.17. *For all $u_h \in \mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$,*

$$R^0_D u_h = u_h\,.$$

Proof. Since both expressions are in $\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})$, it suffices to check the values at each free vertex $\mathsf{v} \in \mathcal{V}_f$:

$$(R^0_D u_h)(\mathsf{v}) = (S^0_D u_h)(\mathsf{v}) + \big[(I - S^0_D)\underbrace{Q^0_{\mathsf{v}} u_h}_{=u_{h|\omega_{\mathsf{v}}}}\big](\mathsf{v}) = u_h(\mathsf{v}),$$

where we have used (3.29). □

Lemma 3.18. *For all* $\mathbf{w}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$*,*

$$\boldsymbol{R}^1_D \mathbf{w}_h = \mathbf{w}_h \,.$$

Proof. Since both expressions are in $\boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$, it suffices to check the integrals over each free edge $\mathsf{e} \in \mathcal{E}_f$:

$$\int_{\mathsf{e}} (\boldsymbol{R}^1_D \mathbf{w}_h)\cdot\boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\mathsf{e}} (\boldsymbol{S}^1_D \mathbf{w}_h)\cdot\boldsymbol{\tau}_{\mathsf{e}}\, ds + \int_{\mathsf{e}} \big[(I-\boldsymbol{S}^1_D)\underbrace{\boldsymbol{Q}^1_{\mathsf{e}}\mathbf{w}_h}_{=\mathbf{w}_{h|\omega_{\mathsf{e}}}}\big]\cdot\boldsymbol{\tau}_{\mathsf{e}}\, ds = \int_{\mathsf{e}} \mathbf{w}_h\cdot\boldsymbol{\tau}_{\mathsf{e}}\, ds,$$

where we have used (3.30). □

The following lemma shows the commuting property of R^0_D, $\boldsymbol{R}^1_D$.

Lemma 3.19. *For all* $u \in H^1_{\Gamma_D}(\Omega)$*,*

$$\boldsymbol{R}^1_D\, \mathbf{grad}\, u = \mathbf{grad}\, R^0_D u.$$

Proof. Let $u \in H^1_{\Gamma_D}(\Omega)$. Firstly, using the definition of R^0_D, $S^0_D = M^0_D$, and Lemma 3.4 we obtain

$$\mathbf{grad}\, R^0_D u \;=\; \underbrace{\mathbf{grad}\, M^0_D u}_{=\boldsymbol{M}^1_D\,\mathbf{grad}\, u} + \sum_{\mathsf{v}\in\mathcal{V}_f} \underbrace{\big[(I-M^0_D)Q^0_{\mathsf{v}}u\big](\mathsf{v})}_{=\big[(I-M^0_D)Q^1_{\mathsf{v},-}\,\mathbf{grad}\, u\big](\mathsf{v})}\; \mathbf{grad}\,\varphi_{\mathsf{v}} \;=\; \boldsymbol{S}^1_D\,\mathbf{grad}\, u,$$

where in the last steps we have used that M^0_D preserves constants on each of the patches ω_{v}, $\mathsf{v} \in \mathcal{V}_f$ as well as representation (3.38) of $\boldsymbol{S}^1_D$. Secondly, by the commuting property (3.28) of the operators Q^0_{e}, $\boldsymbol{Q}^1_{\mathsf{e}}$,

$$\boldsymbol{R}^1_D\,\mathbf{grad}\, u - \boldsymbol{S}^1_D\,\mathbf{grad}\, u \;=\; \sum_{\mathsf{e}\in\mathcal{E}_f} \int_{\mathsf{e}} \big[(I-\boldsymbol{S}^1_D)\underbrace{\boldsymbol{Q}^1_{\mathsf{e}}\,\mathbf{grad}\, u}_{\mathbf{grad}\, Q^0_{\mathsf{e}}u}\big]\cdot\boldsymbol{\tau}_{\mathsf{e}}\, ds\, \boldsymbol{\psi}_{\mathsf{e}}\,.$$

Recall, for any $\mathsf{e} \in \mathcal{E}_f$, that $Q^0_{\mathsf{e}}u \in \mathcal{W}^0_{h,\gamma_{\mathsf{e}}}(\omega_{\mathsf{e}})$, see (3.39). Therefore, we can apply Lemma 3.14 and obtain that

$$\boldsymbol{R}^1_D\,\mathbf{grad}\, u - \boldsymbol{S}^1_D\,\mathbf{grad}\, u = 0.$$

To summarize,

$$\mathbf{grad}\, R^0_D u = \boldsymbol{S}^1_D\,\mathbf{grad}\, u = \boldsymbol{R}^1_D\,\mathbf{grad}\, u.$$ □

In the following, we show stability estimates for R^0_D, $\boldsymbol{R}^1_D$.

Lemma 3.20. *For all* $u \in H^1(\Omega)$ *and* $T \in \mathcal{T}$*,*

$$\|R^0_D u\|_{0,T} \le C\big(\|u\|_{0,\omega_T} + h_T\,\|\,\mathbf{grad}\, u\|_{0,\omega_T}\big).$$

Proof. Following the definition of R_D^0 we obtain from the triangle inequality that

$$\|R_D^0 u\|_{0,T} \le \|M_D^0 u\|_{0,T} + \sum_{\mathsf{v}\in\mathcal{V}_f\cap\overline{T}} \big|[(I - M_D^0)Q_{\mathsf{v}}^0 u](\mathsf{v})\big|\,\|\varphi_{\mathsf{v}}\|_{0,T}\,.$$

The first term is bounded by $C\,\|u\|_{0,\omega_T}$, cf. Lemma 3.5. We bound the second term step by step. Let $\mathsf{v}\in\mathcal{V}_f\cap\overline{T}$. Using the definitions of M_D^0 and Q_{v}^0, we find that

$$(M_D^0 Q_{\mathsf{v}}^0 u)(\mathsf{v}) = \overline{Q_{\mathsf{v}}^0 u}^{\omega_{\mathsf{v}}} = \overline{u}^{\omega_{\mathsf{v}}}\,,$$

and so, together with Lemma 3.3, we obtain

$$\begin{aligned}\big|[(I - M_D^0)Q_{\mathsf{v}}^0 u](\mathsf{v})\big| &\le \big|(Q_{\mathsf{v}}^0 u)(\mathsf{v})\big| + \big|(M_D^0 Q_{\mathsf{v}}^0 u)(\mathsf{v})\big| \\ &\le C\,h_T^{-3/2}\|Q_{\mathsf{v}}^0 u\|_{0,T} + \big|\overline{u}^{\omega_{\mathsf{v}}}\big|.\end{aligned}$$

Due to Lemma 3.10,

$$\|Q_{\mathsf{v}}^0 u\|_{0,T} \le \|u\|_{0,\omega_{\mathsf{v}}} + C\,h_T\,\|\mathbf{grad}\,u\|_{0,\omega_{\mathsf{v}}},$$

and with the Cauchy-Schwarz inequality,

$$\big|\overline{u}^{\omega_{\mathsf{v}}}\big| = \frac{1}{|\omega_{\mathsf{v}}|}\Big|\int_{\omega_{\mathsf{v}}} u\,dx\Big| \le \frac{1}{|\omega_{\mathsf{v}}|^{1/2}}\|u\|_{\omega_{\mathsf{v}}} \le C\,h_T^{-3/2}\,\|u\|_{\omega_{\mathsf{v}}}\,.$$

Combining all the estimate from above, we can conclude that

$$\big|[(I - M_D^0)Q_{\mathsf{v}}^0 u](\mathsf{v})\big| \le C\,h_T^{-3/2}\big(\|u\|_{0,\omega_{\mathsf{v}}} + h_T\,\|\mathbf{grad}\,u\|_{0,\omega_{\mathsf{v}}}\big).$$

Since $\|\varphi_{\mathsf{v}}\|_{0,T} \le C\,h_T^{3/2}$, we obtain the following bound for the second term:

$$\sum_{\mathsf{v}\in\mathcal{V}_f\cap\overline{T}} \big|[(I - M_D^0)Q_{\mathsf{v}}^0 u](\mathsf{v})\big|\,\|\varphi_{\mathsf{v}}\|_{0,T} \le C\,\big(\|u\|_{0,\omega_T} + h_T\,\|\mathbf{grad}\,u\|_{0,\omega_T}\big),$$

which concludes the proof. □

Lemma 3.21. *For all* $\mathbf{w}\in\boldsymbol{H}(\mathbf{curl},\Omega)$ *and* $T\in\mathcal{T}$,

$$\|\boldsymbol{R}_D^1\mathbf{w}\|_{0,T} \le C\big(\|\mathbf{w}\|_{0,\omega_T} + h_T\,\|\,\mathbf{curl}\ \mathbf{w}\|_{0,\omega_T}\big).$$

Proof. Following the definition of $\boldsymbol{R}_D^1$, we find that

$$\|\boldsymbol{R}_D^1\mathbf{w}\|_{0,T} \le \|\boldsymbol{S}_D^1\mathbf{w}\|_{0,T} + \sum_{\mathsf{e}\in\mathcal{E}_f\cap\overline{T}} \Big|\int_{\mathsf{e}}[(I - \boldsymbol{S}_D^1)\boldsymbol{Q}_{\mathsf{e}}^1\mathbf{w}]\cdot\tau_{\mathsf{e}}\,ds\Big|\,\|\boldsymbol{\psi}_{\mathsf{e}}\|_{0,T}$$

The first term can be bounded by $C\,\|\mathbf{w}\|_{0,\omega_T}$, cf. Lemma 3.15. The second term is bounded step by step. Let $\mathsf{e} \in \mathcal{E}_f \cap \overline{T}$. Then due to Lemma 3.3, Lemma 3.16, and Lemma 3.11,

$$\begin{aligned}\Big|\int_{\mathsf{e}}[(I-\boldsymbol{S}_D^1)\boldsymbol{Q}_{\mathsf{e}}^1\mathbf{w}]\cdot\tau_{\mathsf{e}}\,ds\Big| &\le \Big|\int_{\mathsf{e}}(\boldsymbol{Q}_{\mathsf{e}}^1\mathbf{w})\cdot\tau_{\mathsf{e}}\,ds\Big| + \Big|\int_{\mathsf{e}}(\boldsymbol{S}_D^1\boldsymbol{Q}_{\mathsf{e}}^1\mathbf{w})\cdot\tau_{\mathsf{e}}\,ds\Big| \\ &\le C\,h_T^{-1/2}\,\|\boldsymbol{Q}_{\mathsf{e}}^1\mathbf{w}\|_{0,T} + Ch_T^{-1/2}\,\|\boldsymbol{Q}_{\mathsf{e}}^1\mathbf{w}\|_{0,T} \\ &\le C\,h_T^{-1/2}\big(\|\mathbf{w}\|_{0,\omega_{\mathsf{e}}} + h_T\,\|\,\mathbf{curl}\;\mathbf{w}\|_{0,\omega_{\mathsf{e}}}\big).\end{aligned}$$

Since $\|\boldsymbol{\psi}_{\mathsf{e}}\| \le C\,h_T^{1/2}$ (Lemma 3.3), summation over the free edges of T and incorporating the estimate for $\boldsymbol{S}_D^1\mathbf{w}$ yields

$$\begin{aligned}\|\boldsymbol{R}_D^1\mathbf{w}\|_{0,T} &\le C\,\|\mathbf{w}\|_{0,\omega_T} + \sum_{\mathsf{e}\in\mathcal{E}_f\cap\overline{T}} C\,\big(\|\mathbf{w}\|_{0,\omega_{\mathsf{e}}} + h_T\,\|\,\mathbf{curl}\;\mathbf{w}\|_{0,\omega_{\mathsf{e}}}\big) \\ &\le C\,\big(\|\mathbf{w}\|_{0,\omega_T} + h_T\,\|\,\mathbf{curl}\;\mathbf{w}\|_{0,\omega_T}\big),\end{aligned}$$

which concludes the proof. □

Corollary 3.22. *For all $u \in H^1_{\Gamma_D}(\Omega)$ and $T \in \mathcal{T}$,*

$$|R_D^0 u|_{1,T} \le C\,|u|_{1,\omega_T}\,.$$

Proof. The statement follows immediately from Lemma 3.19 and Lemma 3.21. □

3.2 Proof of Theorem 1.2

Throughout the proof we assume that $\operatorname{diam}(\Omega) = 1$, because the general case then follows by a simple scaling argument. Given $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{h,\Gamma_D}(\mathcal{T})$, we apply the continuous regular decomposition from Theorem 2.1, so

$$\mathbf{v}_h = \mathbf{z} + \mathbf{grad}\,\varphi$$

with $\mathbf{z} \in \boldsymbol{H}^1_{\Gamma_D}(\Omega)$, $\varphi \in H^1_{\Gamma_D}(\Omega)$ depending linearly on $\mathbf{v}_h$, and

$$\|\varphi\|_{1,\Omega} \le C\,\|\mathbf{v}_h\|_{0,\Omega}, \tag{3.40}$$

$$\|\mathbf{z}\|_{0,\Omega} \le C\,\|\mathbf{v}_h\|_{0,\Omega}, \tag{3.41}$$

$$\|\mathbf{z}\|_{1,\Omega} \le C\,\|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\,. \tag{3.42}$$

Recall the projection operators R_D^0 and $\boldsymbol{R}_1^D$ from Sect. 3.1.6 and the modified Clément operator M_D^0 from Sect. 3.1.3. Let $\boldsymbol{M}_D^0\colon \boldsymbol{L}^2(\Omega) \to \boldsymbol{\mathcal{V}}^0_{h,\Gamma_D}(\mathcal{T}) = \big(\mathcal{W}^0_{h,\Gamma_D}(\mathcal{T})\big)^3$

denote the corresponding vector-valued operator (defined component-wise). Due to the projection property Lemma 3.18, $\boldsymbol{R}_D^1 \mathbf{v}_h = \mathbf{v}_h$, and so

$$\mathbf{v}_h = \boldsymbol{R}_D^1 \mathbf{z} + \underbrace{\boldsymbol{R}_D^1 \operatorname{\mathbf{grad}} \varphi}_{= \operatorname{\mathbf{grad}} R_D^0 \varphi} = \boldsymbol{R}_D^1 \underbrace{\boldsymbol{M}_D^0 \mathbf{z}}_{=: \mathbf{z}_h} + \underbrace{\boldsymbol{R}_D^1 (I - \boldsymbol{M}_D^0) \mathbf{z}}_{=: \widetilde{\mathbf{v}}_h} + \operatorname{\mathbf{grad}} \underbrace{R_D^0 \varphi}_{=: \varphi_h} . \tag{3.43}$$

From Lemma 3.5, Lemma 3.6, and Corollary 3.9 we obtain

$$\|\boldsymbol{M}_D^0 \mathbf{z}\|_{0,T} \le C \, \|\mathbf{z}\|_{0,\omega_T} , \tag{3.44}$$

$$|\boldsymbol{M}_D^0 \mathbf{z}|_{1,T} \le C \, |\mathbf{z}|_{1,\omega_T} , \tag{3.45}$$

$$\|(I - \boldsymbol{M}_D^0) \mathbf{z}\|_{0,T} \le C \, h_T \, |\mathbf{z}|_{1,\widetilde{\omega}_T} , \tag{3.46}$$

where $\widetilde{\omega}_T$ is the possibly enlarged element patch, see (3.23).

Due to the mapping properties of R_D^0 and $\boldsymbol{R}_D^1$, we obtain that

$$\mathbf{v}_h = \boldsymbol{R}_D^1 \mathbf{z}_h + \widetilde{\mathbf{v}}_h + \operatorname{\mathbf{grad}} \varphi_h$$

with

$$\mathbf{z}_h \in \boldsymbol{\mathcal{V}}_{h,\Gamma_D}^0(\mathcal{T}), \qquad \widetilde{\mathbf{v}}_h \in \boldsymbol{\mathcal{W}}_{h,\Gamma_D}^1(\mathcal{T}), \qquad \varphi_h \in \mathcal{W}_{h,\Gamma_D}^0(\mathcal{T}).$$

Combining (3.41), (3.42), (3.44), and (3.45) imply the following estimates for $\mathbf{z}_h$:

$$\|\mathbf{z}_h\|_{0,\Omega} = \|\boldsymbol{M}_D^0 \mathbf{z}\|_{0,\Omega} \le C \, \|\mathbf{z}\|_{0,\Omega} \le C \, \|\mathbf{v}_h\|_{0,\Omega} ,$$
$$|\mathbf{z}_h|_{1,\Omega} = |\boldsymbol{M}_D^0 \mathbf{z}|_{1,\Omega} \le C \, |\mathbf{z}|_{1,\Omega} \le C \, \|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} .$$

From Lemma 3.21 and an inverse inequality, we conclude

$$\begin{aligned} \|\boldsymbol{R}_D^1 \mathbf{z}_h\|_{0,\Omega}^2 &= \sum_{T \in \mathcal{T}} \|\boldsymbol{R}_D^1 \mathbf{z}_h\|_{0,T}^2 \\ &\le C \sum_{T \in \mathcal{T}} \Big(\|\mathbf{z}_h\|_{0,\omega_T}^2 + h_T^2 \underbrace{\|\operatorname{\mathbf{curl}} \mathbf{z}_h\|_{0,\omega_T}^2}_{\le |\mathbf{z}_h|_{1,\omega_T}^2} \Big) \; \le \; C \, \|\mathbf{z}_h\|_{0,\Omega}^2 . \end{aligned}$$

Our next term to be considered is $\widetilde{\mathbf{v}}_h$. Lemma 3.21, (3.46), and (3.45) yield

$$\begin{aligned} \|h^{-1} \widetilde{\mathbf{v}}_h\|_{0,\Omega}^2 &= \sum_{T \in \mathcal{T}} h_T^{-2} \|\boldsymbol{R}_D^1 (I - \boldsymbol{M}_D^0) \mathbf{z}\|_{0,T}^2 \\ &\le C \sum_{T \in \mathcal{T}} h_T^{-2} \Big(\underbrace{\|(I - \boldsymbol{M}_D^0) \mathbf{z}\|_{0,\omega_T}^2}_{\le C h_T^2 |\mathbf{z}|_{1,\widetilde{\omega}_T}^2} + h_T^2 \underbrace{|(I - \boldsymbol{M}_D^0) \mathbf{z}|_{1,\omega_T}^2}_{\le C \, |\mathbf{z}|_{1,\omega_T}^2} \Big) \\ &\le C \, |\mathbf{z}|_{1,\Omega}^2 \; \le \; C \, \|\mathbf{v}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} . \end{aligned}$$

For the same vector field without the scaling factor, we obtain from Lemma 3.21

$$\begin{aligned}\|\widetilde{\mathbf{v}}_h\|_{0,\Omega} &\leq C \sum_{T\in\mathcal{T}} \big(\|(I-\boldsymbol{M}_D^0)\mathbf{z}\|_{0,\omega_T} + h_T\,\|\,\mathbf{curl}(I-\boldsymbol{M}_D^0\mathbf{z}\|_{0,\omega_T}\big)\\ &\leq C \sum_{T\in\mathcal{T}} \big(\|\mathbf{z}\|_{0,\omega_T} + \|\boldsymbol{M}_D^0\mathbf{z}\|_{0,\omega_T} + h_T\,\|\,\mathbf{curl}\;\mathbf{z}\|_{0,\omega_T} + h_T\|\,\mathbf{curl}\;\boldsymbol{M}_D^0\mathbf{z}\|_{0,\omega_T}\big).\end{aligned}$$

Since $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_h$, local inverse inequalities imply

$$h_T\,\|\,\mathbf{curl}\,\mathbf{z}\|_{0,\omega_T} \leq C\,\|\mathbf{v}_h\|_{0,\omega_T}\,,$$
$$h_T\,\|\,\mathbf{curl}\,\boldsymbol{M}_D^0\mathbf{z}\|_{0,\omega_T} \leq C\,\|\boldsymbol{M}_D^0\mathbf{z}\|_{0,\omega_T}\,.$$

Together with (3.44) and (3.41), we find that

$$\|\widetilde{\mathbf{v}}_h\|_{0,T} \leq C\left(\|\mathbf{z}\|_{0,\Omega} + \|\mathbf{v}_h\|_{0,\Omega}\right) \leq C\,\|\mathbf{v}_h\|_{0,\Omega}\,.$$

Finally, we consider the scalar potential. From Corollary 3.22 and (3.40) we obtain

$$|\varphi_h|_{1,\Omega} = |R_D^0\varphi|_{1,\Omega} \leq C\,|\varphi|_{1,\Omega} \leq C\,\|\mathbf{v}_h\|_{0,\Omega}\,.$$

For an estimate in the L^2-norm, we use the (global) Friedrichs (for $\Gamma_D \neq \emptyset$) or Poincaré inequality

$$\|\varphi_h\|_{0,\Omega} \leq C\,|\varphi_h|_{1,\Omega} \leq C\,\|\mathbf{v}_h\|_{0,\Omega}\,,$$

(recall that $\mathrm{diam}(\Omega) = 1$). This implies an overall estimate in the full H^1-norm and concludes the proof of Theorem 1.2.

4 Discrete Regular Decomposition: p-Version

Now we aim to establish existence and stability of discrete regular decompositions of the finite element space $\mathcal{W}^1_{\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$ for arbitrary polynomial degree $p \in \mathbb{N}_0$. The final result has already been stated in Theorem 1.3. The key objective is to ensure that stability holds *uniformly in* p, in addition to independence of the local mesh width of $\mathcal{T}$, of course. Thus, in this section, we use the symbols $\lesssim$, $\gtrsim$, and $\eqsim$ to express one- and two-sided inequalities up to constants that may depend only on Ω, Γ_D, and the shape regularity measure $\rho(\mathcal{T})$ of the mesh as defined in (1.1); the constants must not depend on p!

The proof of Theorem 1.3 given in this section runs structurally parallel to that of Theorem 1.2 as presented in Section 3.2. There are substantial differences in the two main ingredients, the commuting projector and quasi-interpolation operator,

(I) For want of p-stable local commuting co-chain projections generalizing the construction of Section 3.1, we have to resort to an alternative tool: commuting projection-based interpolation operators, whose details will be explained in Section 4.1.

(II) The modified Clement operator M_D^0 will be replaced with smoothed interpolation, which will be elaborated in Section 4.2.

4.1 Projection-Based Interpolation

Projection based interpolation supplies perfectly local projectors onto the local spaces of discrete differential form that commute with the differential operators $\mathbf{grad}$, $\mathbf{curl}$, div, respectively. Locality also extends to the values on the facets (vertices, edges, faces) of tetrahedra, which makes it possible to assemble the local operators into projectors onto $\mathcal{W}^l_{\Gamma_D}(\mathcal{T})$.

The design of these operators is an intricate multi-stage procedure and we follow [36, Sect. 3.5]. Their main algebraic properties are stated in Lemmata 4.5, 4.5, and 4.7. Even more demanding is the proof of p-uniform approximation properties, which was accomplished in [20]. We recall the result only for 0-forms, that is, scalar functions, in Theorem 4.10, since it will be instrumental for getting the special interpolation error estimate of Theorem 4.16. Its proof will also hinge on a special stable lifting operator from [19] that we recall in the next section.

All considerations in this section are purely local. Therefore, in the beginning we single out an arbitrary tetrahedron $T \in \mathcal{T}$. All constants in estimates may only depend on its shape regularity measure $\rho(T) := {}^{h_T}/{}_{r_T}$.

4.1.1 Tool: Smoothed Poincaré Lifting

Let $D \subset \mathbb{R}^3$ stand for a bounded domain that is star-shaped with respect to a subdomain $B \subset D$, that is,

$$\forall \boldsymbol{a} \in B,\, \boldsymbol{x} \in D: \quad \{t\boldsymbol{a} + (1-t)\boldsymbol{x},\ 0 < t < 1\} \subset D\ . \tag{4.1}$$

Definition 4.1. The *Poincaré lifting* $\mathsf{R}_{\boldsymbol{a}} : \boldsymbol{C}^0(\overline{\Omega}) \mapsto \boldsymbol{C}^0(\overline{\Omega})$, $\boldsymbol{a} \in B$, is defined as

$$\mathsf{R}_{\boldsymbol{a}}(\mathbf{u})(\boldsymbol{x}) := \int_0^1 t\mathbf{u}(\boldsymbol{x} + t(\boldsymbol{x} - \boldsymbol{a}))\,\mathrm{d}t \times (\boldsymbol{x} - \boldsymbol{a})\ , \quad \boldsymbol{x} \in D\ , \tag{4.2}$$

where $\times$ designates the cross product of two vectors in $\mathbb{R}^3$.

This is a special case of the generalized path integral formula for differential forms, which is instrumental in proving the exactness of closed forms on star-shaped domains, the so-called "Poincaré lemma", see [12, Sect. 2.13].

The linear mapping $\mathsf{R}_{\boldsymbol{a}}$ provides a right inverse of the $\mathbf{curl}$-operator on divergence-free vector fields, see [30, Prop. 2.1] for the simple proof, and [12, Sect. 2.13] for a general proof based on differential forms.

Lemma 4.2. *If* $\operatorname{div}\mathbf{u} = 0$*, then, for any* $\boldsymbol{a} \in B$*,* $\mathbf{curl}\,\mathsf{R}_{\boldsymbol{a}}\mathbf{u} = \mathbf{u}$ *for all* $\mathbf{u} \in \boldsymbol{C}^1(\overline{D})$.

Unfortunately, the mapping $\mathsf{R}_{\boldsymbol{a}}$ cannot be extended to a continuous mapping $\boldsymbol{L}^2(D) \mapsto \boldsymbol{H}^1(D)$, *cf.* [30, Thm. 2.1]. As discovered in the breakthrough paper [19] based on earlier work of Bogovskiĭ [10], it takes a smoothed version to accomplish this: we introduce the *smoothed Poincaré lifting* [1]

$$\mathsf{R}(\mathbf{u}) := \int_B \Phi(\boldsymbol{a})\mathsf{R}_{\boldsymbol{a}}(\mathbf{u})\,\mathrm{d}\boldsymbol{a}\ , \tag{4.3}$$

where

$$\Phi \in C^\infty(\mathbb{R}^3)\ , \quad \operatorname{supp}\Phi \subset B\ , \quad \int_B \Phi(\boldsymbol{a})\,\mathrm{d}\boldsymbol{a} = 1\ . \tag{4.4}$$

The substitution

$$\boldsymbol{y} := \boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}) \quad , \quad \tau := \frac{1}{1-t}\ , \tag{4.5}$$

transforms the integral (4.4) into

$$\begin{aligned} \mathsf{R}(\mathbf{u})(\boldsymbol{x}) &= \int\limits_{\mathbb{R}^3}\int\limits_1^\infty \tau(1-\tau)\mathbf{u}(\boldsymbol{y}) \times (\boldsymbol{x}-\boldsymbol{y})\Phi(\boldsymbol{y}+\tau(\boldsymbol{y}-\boldsymbol{x}))\,\mathrm{d}\tau\mathrm{d}\boldsymbol{y} \\ &= \int_{\mathbb{R}^3} \mathbf{k}(\boldsymbol{x}, \boldsymbol{y}-\boldsymbol{x}) \times \mathbf{u}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}\ , \end{aligned} \tag{4.6}$$

that is, R is a convolution-type integral operator with kernel

$$\begin{aligned} \mathbf{k}(\boldsymbol{x}, \boldsymbol{z}) &= \int_1^\infty \tau(1+\tau)\Phi(\boldsymbol{x}+\tau\boldsymbol{z})\boldsymbol{z}\,\mathrm{d}\tau \\ &= \frac{\boldsymbol{z}}{|\boldsymbol{z}|^2}\int_1^\infty \zeta\Phi(\boldsymbol{x}+\zeta\frac{\boldsymbol{z}}{|\boldsymbol{z}|})\,\mathrm{d}\zeta + \frac{\boldsymbol{z}}{|\boldsymbol{z}|^3}\int_1^\infty \zeta^2\Phi(\boldsymbol{x}+\zeta\frac{\boldsymbol{z}}{|\boldsymbol{z}|})\,\mathrm{d}\zeta\ . \end{aligned} \tag{4.7}$$

The kernel can be bounded by $|\mathbf{k}(\boldsymbol{x}, \boldsymbol{z})| \le K(\boldsymbol{x})|\boldsymbol{z}|^{-2}$, where $K \in C^\infty(\mathbb{R}^3)$ depends only on Φ and is locally uniformly bounded. As a consequence, (4.6) exists as an improper integral.

The intricate but elementary analysis of [19, Sect. 3.3] further shows, that $\mathbf{k}$ belongs to the Hörmander symbol class $S^{-1}_{1,0}(\mathbb{R}^3)$, see [61, Ch. 7]. Invoking the theory of pseudo-differential operators [61, Prop. 5.5] we obtain the following following continuity result, which is a special case of [19, Cor. 3.4].

Theorem 4.3. *The mapping* R *can be extended to a* continuous *linear operator* $\boldsymbol{L}^2(D) \mapsto \boldsymbol{H}^1(D)$*, which is still denoted by* R*. It satisfies*

$$\mathbf{curl}\,\mathsf{R}\mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \boldsymbol{H}(\mathrm{div}, D),\ \mathrm{div}\,\mathbf{u} = 0\ . \tag{4.8}$$

The smoothed Poincaré lifting shares this continuity property with many other mappings, see [36, Sect. 2.4]. Yet, it enjoys another essential feature, which is immediate from its definition (4.2): R maps polynomials of degree p to other polynomials of degree $\le p+1$. The next section will highlight the significance of this observation.

[1] The dependence of R on Φ is dropped from the notation.

4.1.2 $\mathcal{W}_p^1(\mathcal{T})$: A Local View

According to [34, Sect. 3], for any $T \in \mathcal{T}$, $\boldsymbol{a} \in T$, we can obtain the local space of discrete 1-forms of the first family as

$$\mathcal{W}_p^1(T) = \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3) + \mathsf{R}_{\boldsymbol{a}}\big(\boldsymbol{\mathcal{P}}_p(\mathrm{div}\, 0, \mathbb{R}^3)\big) \ . \tag{4.9}$$

Independence of $\boldsymbol{a}$ is discussed in [34, Sect. 3]. The representation (4.9) can be established by dimensional arguments: from the formula (4.2) for the Poincaré lifting we immediately see that $\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3) + \mathsf{R}_{\boldsymbol{a}}(\boldsymbol{\mathcal{P}}_p(\mathbb{R}^3)) \subset \mathcal{W}_p^1(T)$. In addition, from [54, Lemma 4] and [34, Thm. 6, case $l = 1$, $n = 3$] we learn that the dimensions of both spaces agree and are equal to

$$\dim \mathcal{W}_p^1(T) = \tfrac{1}{2}(1+p)(3+p)(4+p) \ . \tag{4.10}$$

As a consequence, the two finite dimensional spaces must agree.

For the remainder of this section, which focuses on local spaces, we single out a tetrahedron $T \in \mathcal{T}$. On T we can introduce a smoothed Poincaré lifting R_T according to (4.3) with $B = T$ and a suitable $\Phi \in C_0^\infty(T)$ complying with (4.4). An immediate consequence of (4.9) is that

$$\mathsf{R}_T\big(\{\mathbf{v} \in \boldsymbol{\mathcal{P}}_p(\mathbb{R}^3) : \ \mathrm{div}\, \mathbf{v} = 0\}\big) \subset \mathcal{W}_p^1(T) \ . \tag{4.11}$$

We introduce the notation $\mathcal{F}_m(T)$ for the set of all m-dimensional facets of T, $m = 0, 1, 2, 3$. Hence, $\mathcal{F}_0(T)$ contains the vertices of T, $\mathcal{F}_1(T)$ the edges, $\mathcal{F}_2(T)$ the faces, and $\mathcal{F}_3(T) = \{T\}$. Moreover, for some $F \in \mathcal{F}_m(T)$, $m = 1, 2, 3$, $\mathcal{P}_p(F)$ denotes the space of m-variate polynomials of total degree $\leq p$ in a local coordinate system of the facet F, and $\boldsymbol{\mathcal{P}}_p(F)$ will designate corresponding tangential polynomial vector fields. Further, we write

$$\mathcal{W}_p^1(e) = \mathcal{W}_p^1(T) \cdot \boldsymbol{t}_e \ , \quad \boldsymbol{t}_e \text{ the unit tangent vector of } e, \ e \in \mathcal{F}_1(T) \ , \tag{4.12}$$

$$\mathcal{W}_p^1(f) = \mathcal{W}_p^1(T) \times \boldsymbol{n}_f \ , \quad \boldsymbol{n}_f \text{ the unit normal vector of } f, \ f \in \mathcal{F}_2(T) \ , \tag{4.13}$$

for the tangential traces of local edge element vector fields onto edges and faces. Simple vector analytic manipulations permit us to deduce from (4.9) that

$$\mathcal{W}_p^1(e) = \mathcal{P}_p(e) \ , \quad e \in \mathcal{F}_1(T) \ , \tag{4.14}$$

$$\mathcal{W}_p^1(f) = \boldsymbol{\mathcal{P}}_p(f) + \mathsf{R}_{\boldsymbol{a}}^{2D}(\mathcal{P}_p(f)) \ , \quad \boldsymbol{a} \in f \ , \quad f \in \mathcal{F}_2(T) \ , \tag{4.15}$$

where the projection $\mathsf{R}_{\boldsymbol{a}}^{2D}$ of the Poincaré lifting in the plane reads

$$\mathsf{R}_{\boldsymbol{a}}^{2D}(u)(\boldsymbol{x}) := \int_0^1 t u(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})](\boldsymbol{x} - \boldsymbol{a}) \, \mathrm{d}t \ , \quad \boldsymbol{a} \in \mathbb{R}^2 \ . \tag{4.16}$$

It satisfies $\mathrm{div}_\Gamma \mathsf{R}^{2D}_{\boldsymbol{a}}(u) = u$ for all $u \in C^\infty(\mathbb{R}^2)$. We point out that, along with (4.9), the formulas (4.14) and (4.15) are special versions of the general representation formula for discrete 1-forms, see [34, Formula (16)] and [4, Sect. 3.2]. Special facet tangential trace spaces with "zero boundary conditions" will also be needed:

$$\mathring{\mathcal{W}}^1_p(e) := \{u \in \mathcal{W}^1_p(e) : \int_e u \,\mathrm{d}l = 0\} , \quad e \in \mathcal{F}_1(T) , \tag{4.17}$$

$$\mathring{\mathcal{W}}^1_p(f) := \{\mathbf{u} \in \mathcal{W}^1_p(f) : \mathbf{u} \cdot \boldsymbol{n}_{e,f} \equiv 0 \;\forall e \in \mathcal{F}_1(T),\; e \subset \partial f\} , \quad f \in \mathcal{F}_2(T) , \tag{4.18}$$

$$\mathring{\mathcal{W}}^1_p(T) := \{\mathbf{u} \in \mathcal{W}^1_p(T) : \mathbf{u} \times \boldsymbol{n}_f \equiv 0 \;\forall f \in \mathcal{F}_2(T)\} . \tag{4.19}$$

Here $\boldsymbol{n}_f$ represents an exterior face unit normal of T, $\boldsymbol{n}_{e,f}$ the in-plane normal of a face w.r.t. an edge $e \subset \partial f$.

According to [54, Sect. 1.2], [34, Sect. 4], and [4, Sect. 4.3], the local degrees of freedom for $\mathcal{W}^1_p(T)$ are given by the first $p-2$ vectorial moments on the cells of $\mathcal{T}$, the first $p-1$ vectorial moments of the tangential components on the faces of $\mathcal{T}$ and the first p tangential moments along the edges of T, see (4.21) for concrete formulas. Then the set $\mathrm{dof}^1_p(T)$ of local degrees of freedom can be partitioned as [49, Ch. 3], [4, Sect. 4.5],

$$\mathrm{dof}^1_p(T) = \bigcup_{e \in \mathcal{F}_1(T)} \mathrm{ldf}^1_p(e) \;\cup \bigcup_{f \in \mathcal{F}_2(T)} \mathrm{ldf}^1_p(f) \;\cup\; \mathrm{ldf}^1_p(T) , \tag{4.20}$$

where the functionals in $\mathrm{ldf}^1_p(e)$, $\mathrm{ldf}^1_p(f)$, and $\mathrm{ldf}^1_p(T)$ are supported on an edge, face, and T, respectively, and read

$$\begin{aligned}
\kappa \in \mathrm{ldf}^1_p(e) \;&\Rightarrow\; \kappa(\mathbf{u}) = \textstyle\int_e q\boldsymbol{\xi} \cdot \boldsymbol{t}_e \,\mathrm{d}l && \text{for } e \in \mathcal{F}_1(T),\ \text{suitable } q \in \mathcal{P}_p(e) ,\\
\kappa \in \mathrm{ldf}^1_p(f) \;&\Rightarrow\; \kappa(\mathbf{u}) = \textstyle\int_f \mathbf{q} \cdot (\boldsymbol{\xi} \times \mathbf{n}) \,\mathrm{d}S && \text{for } f \in \mathcal{F}_2(T),\ \text{suitable } \mathbf{q} \in \boldsymbol{\mathcal{P}}_{p-1}(f) ,\\
\kappa \in \mathrm{ldf}^1_p(T) \;&\Rightarrow\; \kappa(\mathbf{u}) := \textstyle\int_T \mathbf{q} \cdot \boldsymbol{\xi} \,\mathrm{d}\boldsymbol{x} && \text{for certain } \mathbf{q} \in \boldsymbol{\mathcal{P}}_{p-2}(T) .
\end{aligned} \tag{4.21}$$

These functionals are unisolvent on $\mathcal{W}^1_p(T)$ and locally fix the tangential trace of $\mathbf{u} \in \mathcal{W}^1_p(T)$. There is a splitting of $\mathcal{W}^1_p(T)$ dual to (4.20): Defining

$$\mathcal{Y}^1_p(F) := \{\mathbf{v} \in \mathcal{W}^1(T) : \kappa(\mathbf{v}) = 0 \;\forall \kappa \in \mathrm{dof}^1_p(T) \setminus \mathrm{ldf}^1_p(F)\} \tag{4.22}$$

for $F \in \mathcal{F}_m(T)$, $m = 1, 2, 3$, we find the direct sum decomposition

$$\mathcal{W}^1_p(T) = \sum_{m=1}^{3} \sum_{F \in \mathcal{F}_m(T)} \mathcal{Y}^1_p(F) . \tag{4.23}$$

In addition, note that the tangential trace of $\mathbf{u} \in \mathcal{Y}_p^1(F)$ vanishes on all facets $\neq F$, whose dimension is smaller or equal the dimension of F. By the unisolvence of $\mathrm{dof}_p^1(T)$, there are bijective linear *extension operators*

$$\mathsf{E}^1_{e,p} : \mathcal{W}_p^1(e) \mapsto \mathcal{Y}_p^1(e)\ , \quad e \in \mathcal{F}_1(T)\ , \tag{4.24}$$

$$\mathsf{E}^1_{f,p} : \mathring{\mathcal{W}}_p^1(f) \mapsto \mathcal{Y}_p^1(f)\ , \quad f \in \mathcal{F}_2(T)\ . \tag{4.25}$$

Similar relationships hold for discrete 2-forms, for which we have the following alternative representation of the local space [34, Formula (16) for $l = 2$, $n = 3$]:

$$\mathcal{W}_p^2(T) = \boldsymbol{\mathcal{P}}_p(T) + \mathsf{D}_{\boldsymbol{a}}(\mathcal{P}_p(T))\ , \tag{4.26}$$

where the appropriate version of the Poincaré lifting reads

$$(\mathsf{D}_{\boldsymbol{a}} u)(\boldsymbol{x}) := \int_0^1 t^2 u(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a}))(\boldsymbol{x} - \boldsymbol{a})\,\mathrm{d}t\ , \quad \boldsymbol{a} \in T\ . \tag{4.27}$$

Like (4.9) this is a special incarnation of the general formula (16) in [34]. Again, dimensional arguments based on [54, Sect. 1.3] and [34, Thm. 6] confirm the representation (4.27). We remark that $\operatorname{div} \mathsf{D}_{\boldsymbol{a}} u = u$, see [30, Prop. 1.2].

The normal trace space of $\mathcal{W}_p^2(T)$ onto a face is

$$\mathcal{W}_p^2(f) := \mathcal{W}_p^2(T) \cdot \boldsymbol{n}_f = \mathcal{P}_p(f)\ , \quad f \in \mathcal{F}_2(T)\ , \tag{4.28}$$

and as relevant space "with zero trace" we are going to need

$$\mathring{\mathcal{W}}_p^2(f) := \{u \in \mathcal{W}_p^2(f) : \int_f u\,\mathrm{d}S = 0\}\ , \quad f \in \mathcal{F}_2(T)\ , \tag{4.29}$$

$$\mathring{\mathcal{W}}_p^2(T) := \{\mathbf{u} \in \mathcal{W}_p^2(T) :\ \mathbf{u} \cdot \boldsymbol{n}_{\partial T} = 0\}\ . \tag{4.30}$$

The connection between the local spaces $\mathcal{W}_p^1(T)$, $\mathcal{W}_p^2(T)$ and full polynomial spaces is established through a local exact sequence [34, Sect. 5]. To elucidate the relationship between differential operators and various traces onto faces and edges, we also include those in the statement of the following theorem. There $\boldsymbol{n}_f$ stands for an exterior face unit normal of T, $\boldsymbol{n}_{e,f}$ for the in-plane normal of a face w.r.t. an edge $e \subset \partial f$, and $\frac{d}{ds}$ is the differentiation w.r.t. arclength on an edge.

Theorem 4.4 (Local exact sequences). *For $f \in \mathcal{F}_2(T)$, $e \in \mathcal{F}_1(T)$, $e \subset \partial f$, all the sequences in*

$$\begin{array}{ccccccccccc}
\text{const} & \xrightarrow{\mathsf{Id}} & \mathcal{P}_{p+1}(T) & \xrightarrow{\mathbf{grad}} & \mathcal{W}_p^1(T) & \xrightarrow{\mathbf{curl}} & \mathcal{W}_p^2(T) & \xrightarrow{\operatorname{div}} & \mathcal{P}_p(T) & \xrightarrow{0} & \{0\} \\
 & & \Big\downarrow{\scriptstyle \cdot_{|f}} & & \Big\downarrow{\scriptstyle \cdot\times\boldsymbol{n}_{f\,|f}} & & \Big\downarrow{\scriptstyle \cdot\cdot\boldsymbol{n}_{f\,|f}} & & & & \\
\text{const} & \xrightarrow{\mathsf{Id}} & \mathcal{P}_{p+1}(f) & \xrightarrow{\mathbf{curl}_\Gamma} & \mathcal{W}_p^1(f) & \xrightarrow{\operatorname{div}_\Gamma} & \mathcal{P}_p(f) & \xrightarrow{0} & \{0\} & & \\
 & & \Big\downarrow{\scriptstyle \cdot_{|e}} & & \Big\downarrow{\scriptstyle \cdot\cdot\boldsymbol{n}_{e,f\,|e}} & & & & & & \\
\text{const} & \xrightarrow{\mathsf{Id}} & \mathcal{P}_{p+1}(e) & \xrightarrow{\frac{d}{ds}} & \mathcal{P}_p(e) & \xrightarrow{0} & \{0\} & & & &
\end{array}$$

are exact and the diagram commutes.

4.1.3 Projections, liftings, and extensions

Following the developments of [36, Sect. 3.5], projection based interpolation requires building blocks in the form of local *orthogonal* projections P^l_* and liftings L^l_*[2]. Some operators will depend on a regularity parameter $0 < \epsilon < \frac{1}{2}$, which is considered fixed below and will be specified in Sect. 4.1.5. To begin with, we define for every $e \in \mathcal{F}_1(T)$

$$\mathsf{P}^1_{e,p} : H^{-1+\epsilon}(e) \mapsto \frac{d}{ds}\mathring{\mathcal{P}}_{p+1}(e) = \mathring{\mathcal{W}}^1_p(e) \tag{4.31}$$

as the $H^{-1+\epsilon}(e)$-orthogonal projection. Here, $\mathring{\mathcal{P}}_p(F)$ denotes the space of degree p polynomials on a facet F that vanish on ∂F.

Similarly, for every face $f \in \mathcal{F}_2(T)$ introduce

$$\mathsf{P}^1_{f,p} : \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f) \mapsto \mathbf{curl}_\Gamma \mathring{\mathcal{P}}_{p+1}(f) = \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(f) :\ \mathrm{div}_\Gamma \mathbf{v} = 0\}\ , \tag{4.32}$$

$$\mathsf{P}^2_{f,p} : \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f) \mapsto \mathrm{div}_\Gamma \mathring{\mathcal{W}}^1_p(f) = \mathring{\mathcal{W}}^2_p(f)\ , \tag{4.33}$$

as the corresponding $\boldsymbol{H}^{-\frac{1}{2}+\epsilon}(f)$-orthogonal projections. Eventually, let

$$\mathsf{P}^1_{T,p} : \boldsymbol{L}^2(T) \mapsto \mathbf{grad}\, \mathring{\mathcal{P}}_{p+1}(T) = \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) :\ \mathbf{curl}\, \mathbf{v} = 0\}\ , \tag{4.34}$$

$$\mathsf{P}^2_{T,p} : \boldsymbol{L}^2(T) \mapsto \mathbf{curl}\, \mathring{\mathcal{W}}^1_p(T) = \{\mathbf{v} \in \mathring{\mathcal{W}}^2_p(T) :\ \mathrm{div}\, \mathbf{v} = 0\}\ , \tag{4.35}$$

$$\mathsf{P}^3_{T,p} : L^2(T) \mapsto \mathrm{div}\, \mathring{\mathcal{W}}^2_p(T) = \{v \in \mathcal{P}_p(T) :\ \int_T v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0\}\ , \tag{4.36}$$

stand for the respective $L^2(T)$-orthogonal projections. Local exact sequences have tacitly been used in these statements, see (4.46) below.

The lifting operators

$$\mathsf{L}^1_{e,p} : \mathring{\mathcal{W}}^1_p(e) \mapsto \mathring{\mathcal{P}}_{p+1}(e)\ , \quad e \in \mathcal{F}_1(T)\ , \tag{4.37}$$

$$\mathsf{L}^1_{f,p} : \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(f) :\ \mathrm{div}_\Gamma \mathbf{v} = 0\} \mapsto \mathring{\mathcal{P}}_{p+1}(f)\ , \quad f \in \mathcal{F}_2(T)\ , \tag{4.38}$$

$$\mathsf{L}^1_{T,p} : \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) :\ \mathbf{curl}\, \mathbf{v} = 0\} \mapsto \mathring{\mathcal{P}}_{p+1}(T)\ , \tag{4.39}$$

are uniquely defined by requiring

$$\frac{d}{ds}\mathsf{L}^1_{e,p} u = u \quad \forall u \in \mathring{\mathcal{W}}^1_p(e)\ , \tag{4.40}$$

$$\mathbf{curl}_\Gamma \mathsf{L}^1_{f,p} \mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \{\mathring{\mathcal{W}}^1_p(f) :\ \mathrm{div}_\Gamma \mathbf{v} = 0\}\ , \tag{4.41}$$

$$\mathbf{grad}\, \mathsf{L}^1_{T,p} \mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \{\mathbf{v} \in \mathring{\mathcal{W}}^1_p(T) :\ \mathbf{curl}\, \mathbf{v} = 0\}\ . \tag{4.42}$$

[2] The parameter l in the notations for the extension operators E^l_*, the projections P^l_*, and the liftings L^l_* refers to the degree of the discrete differential form they operate on. This is explained in more detail in [36, Sect. 3.5].

Another class of liftings provides right inverses for $\mathbf{curl}$ and div_Γ: Pick a face $f \in \mathcal{F}_2(T)$, and, without loss of generality, assume the vertex opposite to the edge $\widetilde{e}$ to coincide with 0. Then define

$$\mathsf{L}^2_{f,p} : \begin{cases} \mathrm{div}_\Gamma \overset{\circ}{\mathcal{W}}{}^1_p(f) & \mapsto \overset{\circ}{\mathcal{W}}{}^1_p(f) \\ u & \mapsto \mathsf{R}_0^{2D} u - \mathbf{curl}_\Gamma \mathsf{E}^0_{\widetilde{e},p} \mathsf{L}^1_{\widetilde{e},p}(\mathsf{R}_0^{2D} u \cdot \boldsymbol{n}_{\widetilde{e},f}) \, . \end{cases} \tag{4.43}$$

This is a valid definition, since, by virtue of definition (4.16), the normal components of $\mathsf{R}_0^{2D} u$ will vanish on $\partial f \setminus \widetilde{e}$. Moreover, $\mathrm{div}_\Gamma \mathsf{R}_0^{2D} u = u$ ensures that the normal component of $\mathsf{R}_0^{2D} u$ has zero average on $\widetilde{e}$. We infer

$$\Big(\mathbf{curl}_\Gamma \mathsf{E}^0_{\widetilde{e},p} \mathsf{L}^1_{\widetilde{e},p}\big((\mathsf{R}_0^{2D}\mathbf{u} \cdot \boldsymbol{n}_{\widetilde{e},f})_{|\widetilde{e}}\big) \cdot \boldsymbol{n}_{\widetilde{e},f}\Big)_{|\widetilde{e}} = \frac{d}{ds} \mathsf{L}^1_{\widetilde{e},p}\big((\mathsf{R}_0^{2D}\mathbf{u}) \cdot \boldsymbol{n}_{\widetilde{e},f}\big)_{|\widetilde{e}} = \mathsf{R}_0^{2D}\mathbf{u} \cdot \boldsymbol{n}_{\widetilde{e},f} \quad \text{on } \widetilde{e} \, ,$$

and see that the zero trace condition on ∂f is satisfied. The same idea underlies the definition of

$$\mathsf{L}^2_{T,p} : \begin{cases} \mathbf{curl}\, \overset{\circ}{\mathcal{W}}{}^1_p(T) & \mapsto \overset{\circ}{\mathcal{W}}{}^1_p(T) \\ \mathbf{u} & \mapsto \mathsf{R}_0 \mathbf{u} - \mathbf{grad}\, \mathsf{E}^0_{\widetilde{f},p} \mathsf{L}^1_{\widetilde{f},p}\big(((\mathsf{R}_0\mathbf{u}) \times \boldsymbol{n}_{\widetilde{f}})_{|\widetilde{f}}\big) \, , \end{cases} \tag{4.44}$$

where $\widetilde{f}$ is the face opposite to vertex 0, and the definition of

$$\mathsf{L}^3_{T,p} : \begin{cases} \mathrm{div}\, \overset{\circ}{\mathcal{W}}{}^2_p(T) & \mapsto \overset{\circ}{\mathcal{W}}{}^2_p(T) \\ u & \mapsto \mathsf{D}_0 u - \mathbf{curl}\, \mathsf{E}^1_{\widetilde{f},p} \mathsf{L}^2_{\widetilde{f},p}\big((\mathsf{D}_0 u \cdot \boldsymbol{n}_{\widetilde{f}})_{|\widetilde{f}}\big) \, . \end{cases} \tag{4.45}$$

The relationships between the various facet function spaces with vanishing traces can be summarized in the following exact sequences:

$$\begin{array}{l} \{0\} \xrightarrow{\mathsf{Id}} \overset{\circ}{\mathcal{P}}_{p+1}(T) \xrightarrow{\mathbf{grad}} \overset{\circ}{\mathcal{W}}{}^1_p(T) \xrightarrow{\mathbf{curl}} \overset{\circ}{\mathcal{W}}{}^2_p(T) \xrightarrow{\mathrm{div}} \overline{\mathcal{P}}_p(T) \xrightarrow{0} \{0\}, \\ \{0\} \xrightarrow{\mathsf{Id}} \overset{\circ}{\mathcal{P}}_{p+1}(f) \xrightarrow{\mathbf{curl}_\Gamma} \overset{\circ}{\mathcal{W}}{}^1_p(f) \xrightarrow{\mathrm{div}_\Gamma} \overline{\mathcal{P}}_p(f) \xrightarrow{0} \{0\}, \\ \{0\} \xrightarrow{\mathsf{Id}} \overset{\circ}{\mathcal{P}}_{p+1}(e) \xrightarrow{\frac{d}{ds}} \overline{\mathcal{P}}_p(e) \xrightarrow{0} \{0\} \, , \end{array} \tag{4.46}$$

where $\overline{\mathcal{P}}_p(F)$ designates degree p polynomial spaces on F with vanishing mean. These relationships and the lifting mappings $\mathsf{L}^l_{*,p}$ are studied in [36, Sect. 3.4].

Finally we need polynomial extension operators

$$\mathsf{E}^0_{e,p} : \overset{\circ}{\mathcal{P}}_{p+1}(e) \mapsto \mathcal{P}_{p+1}(T) \, , \tag{4.47}$$

$$\mathsf{E}^0_{f,p} : \overset{\circ}{\mathcal{P}}_{p+1}(f) \mapsto \mathcal{P}_{p+1}(T) \tag{4.48}$$

that satisfy

$$\mathsf{E}^0_{e,p} u_{|e'} = 0 \quad \forall e' \in \mathcal{F}_1(T) \setminus \{e\} \; , \tag{4.49}$$

$$\mathsf{E}^0_{f,p} u_{|f'} = 0 \quad \forall f' \in \mathcal{F}_2(T) \setminus \{f\} \; . \tag{4.50}$$

Such extension operators can be constructed relying on a representation of a polynomial on F, $F \in \mathcal{F}_m(T)$, $m = 1, 2$, as a homogeneous polynomial in the barycentric coordinates of F, see [36, Lemma 3.4] of [49, Sect. IV.3]. As an alternative, one may use the polynomial preserving extension operators proposed in [53, 21] and [1]. We stress that continuity properties of these extensions do not matter for our purpose.

4.1.4 Interpolation operators

Now we are in a position to define the projection based interpolation operators locally on a generic tetrahedron T with vertices $\boldsymbol{a}_i$, $i = 1, 2, 3, 4$.

First, we devise a suitable projection (depending on the regularity parameter $0 < \epsilon < \frac{1}{2}$, which is usually suppressed to keep notations manageable)

$$\Pi^0_{T,p} (= \Pi^0_{T,p}(\epsilon)) \; : \; C^\infty(\overline{T}) \mapsto \mathcal{P}_{p+1}(T) \tag{4.51}$$

for degree p Lagrangian $H^1(\Omega)$-conforming finite elements. For $u \in C^\infty(\overline{T})$ define (λ_i is the barycentric coordinate function belonging to vertex $\boldsymbol{a}_i$ of T)

$$u^{(0)} := u - \underbrace{\sum_{i=1}^{4} u(\boldsymbol{a}_i)\lambda_i}_{:=w^{(0)}} \; , \tag{4.52}$$

$$u^{(1)} := u^{(0)} - \underbrace{\sum_{e\in\mathcal{F}_1(T)} \mathsf{E}^0_{e,p}\mathsf{L}^1_{e,p}\mathsf{P}^1_{e,p}\frac{d}{ds}u^{(0)}{}_{|e}}_{:=w^{(1)}} \; , \tag{4.53}$$

$$u^{(2)} := u^{(1)} - \underbrace{\sum_{f\in\mathcal{F}_1(T)} \mathsf{E}^0_{f,p}\mathsf{L}^1_{f,p}\mathsf{P}^1_{f,p}\mathbf{curl}_\Gamma(u^{(1)}{}_{|f})}_{:=w^{(2)}} \; , \tag{4.54}$$

$$\Pi^0_{T,p}u := \mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\,\mathbf{grad}\,u^{(2)} + w^{(2)} + w^{(1)} + w^{(0)} \; . \tag{4.55}$$

Observe that $w^{(i)}{}_{|F} = 0$ for all $F \in \mathcal{F}_m(T)$, $0 \leq m < i \leq 3$. We point out that $w^{(0)}$ is the standard linear interpolant of u.

Lemma 4.5. *The linear mapping $\Pi^0_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $\mathcal{P}_{p+1}(T)$*

Proof. Assume $u \in \mathcal{P}_{p+1}(T)$, which will carry over to all intermediate functions. Since $u^{(0)}(\boldsymbol{a}_i) = 0$, $i = 1, \dots, 4$, we conclude from the projection property of $\mathsf{P}^1_{e,p}$ that $\mathsf{L}^1_e \mathsf{P}^1_e \frac{d}{ds} u^{(0)}{}_{|e} = u^{(0)}{}_{|e}$ for any edge $e \in \mathcal{F}_1(T)$. As a consequence

$$u^{(1)} = u^{(0)} - \sum_{e \in \mathcal{F}_1(T)} \mathsf{E}^0_{e,p} u^{(0)}{}_{|e} \quad \Rightarrow \quad u^{(1)}_{|e} = 0 \quad \forall e \in \mathcal{F}_1(T) \ . \tag{4.56}$$

We infer $\mathsf{L}^1_{f,p} \mathsf{P}^1_f \mathbf{curl}_\Gamma (u^{(1)}{}_{|f}) = u^{(1)}{}_{|f}$ on each face $f \in \mathcal{F}_2(T)$, which implies

$$u^{(2)} = u^{(1)} - \sum_{f \in \mathcal{F}_1(T)} \mathsf{E}^0_{f,p}(u^{(1)}{}_{|f}) \quad \Rightarrow \quad u^{(2)}{}_{|f} = 0 \quad \forall f \in \mathcal{F}_2(T) \ . \tag{4.57}$$

This means that $\mathsf{L}^1_{T,p} \mathsf{P}^1_{T,p} \, \mathbf{grad} \, u^{(2)} = u^{(2)}$ and a telescopic sum argument finishes the proof. $\square$

A similar stage by stage construction applies to edge elements and gives a projection

$$\Pi^1_{T,p} (= \Pi^1_{T,p}(\epsilon)) \ : \ \boldsymbol{C}^\infty(\overline{T}) \mapsto \mathcal{W}^1(T) \ : \tag{4.58}$$

for a directed edge $e := [\mathbf{a}_i, \mathbf{a}_j]$ we introduce the Whitney-1-form basis function

$$\mathbf{b}_e = \lambda_i \, \mathbf{grad} \, \lambda_j - \lambda_j \, \mathbf{grad} \, \lambda_i \ . \tag{4.59}$$

These functions span $\mathcal{W}^1_0(T)$. Next, for $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ define

$$\mathbf{u}^{(0)} := \mathbf{u} - \underbrace{\Big(\sum_{e \in \mathcal{F}_1(T)} \int_e \mathbf{u} \cdot d\vec{s} \Big) \, \mathbf{b}_e}_{:= \mathbf{w}^{(0)}} \ , \tag{4.60}$$

$$\mathbf{u}^{(1)} := \mathbf{u}^{(0)} - \underbrace{\sum_{e \in \mathcal{F}_1(T)} \mathbf{grad} \, \mathsf{E}^0_{e,p} \mathsf{L}^1_{e,p} \mathsf{P}^1_{e,p} ((\mathbf{u}^{(0)} \cdot \boldsymbol{t}_e)_{|e})}_{:= \mathbf{w}^{(1)}} \ , \tag{4.61}$$

$$\mathbf{u}^{(2)} := \mathbf{u}^{(1)} - \underbrace{\sum_{f \in \mathcal{F}_2(T)} \mathsf{E}^1_{f,p} \mathsf{L}^2_{f,p} \mathsf{P}^2_{f,p} \mathrm{div}_\Gamma ((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f})}_{:= \mathbf{w}^{(2)}} \ , \tag{4.62}$$

$$\mathbf{u}^{(3)} := \mathbf{u}^{(2)} - \underbrace{\sum_{f \in \mathcal{F}_2(T)} \mathbf{grad} \, \mathsf{E}^0_{f,p} \mathsf{L}^1_{f,p} \mathsf{P}^1_{f,p} ((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f})}_{:= \mathbf{w}^{(3)}} \ , \tag{4.63}$$

$$\mathbf{u}^{(4)} := \mathbf{u}^{(3)} - \underbrace{\mathsf{L}^2_{T,p} \mathsf{P}^2_{T,p} \, \mathbf{curl} \, \mathbf{u}^{(3)}}_{:= \mathbf{w}^{(4)}} \ , \tag{4.64}$$

$$\Pi^1_{T,p} \mathbf{u} := \mathbf{grad} \, \mathsf{L}^1_{T,p} \mathsf{P}^1_{T,p} \mathbf{u}^{(4)} + \mathbf{w}^{(4)} + \mathbf{w}^{(3)} + \mathbf{w}^{(2)} + \mathbf{w}^{(1)} + \mathbf{w}^{(0)} \ . \tag{4.65}$$

The contribution $\mathbf{w}^{(0)}$ is the standard interpolant $\Pi^1_{T,0}$ of $\mathbf{u}$ onto the local space of Whitney-1-forms (lowest order edge elements). The extension operators were chosen in a way that guarantees that $\mathbf{w}^{(2)} \cdot \boldsymbol{t}_e = 0$ and $\mathbf{w}^{(3)} \cdot \boldsymbol{t}_e = 0$ for all $e \in \mathcal{F}_1(T)$.

Lemma 4.6. *The linear mapping* $\Pi^1_{T,p}$, $p \in \mathbb{N}_0$, *is a projection onto* $\mathcal{W}^1_p(T)$ *and satisfies the* commuting diagram property

$$\Pi^1_{T,p} \circ \mathbf{grad} = \mathbf{grad} \circ \Pi^0_{T,p} \quad \text{on } C^\infty(\overline{T}) . \tag{4.66}$$

Proof. The proof of the projection property runs parallel to that of Lemma 4.5. Assuming $\mathbf{u} \in \mathcal{W}^1_p(T)$, it is obvious that the same will hold for all $\mathbf{u}^{(i)}$ and $\mathbf{w}^{(i)}$ from (4.60)-(4.65). In order to confirm that all projections can be discarded, we have to check that their arguments satisfy conditions of zero trace on the facet boundaries and, in some cases, belong to the kernel of differential operators.

First, recalling the properties of the interpolation operator Π^1_0 for Whitney-1-forms, we find $(\mathbf{u}^{(0)} \cdot \boldsymbol{t}_e)_{|e} \in \mathring{\mathcal{W}}^1_p(e)$. This implies

$$\mathbf{grad}\, \mathsf{E}^0_{e,p} \mathsf{L}^1_{e,p} \mathsf{P}^1_{e,p}((\mathbf{u}^{(0)} \cdot \boldsymbol{t}_e)_{|e}) = (\mathbf{u}^{(0)} \cdot \boldsymbol{t}_e)_{|e} \quad \forall e \in \mathcal{F}_1(T) , \tag{4.67}$$

and

$$(\mathbf{u}^{(1)} \cdot \boldsymbol{t}_e)_{|e} \equiv 0 \quad \forall e \in \mathcal{F}_1(T) . \tag{4.68}$$

We see that $(\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f} \in \mathring{\mathcal{W}}^1_p(f)$ for any $f \in \mathcal{F}_2(T)$, so that

$$\mathsf{P}^2_{f,p} \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) \tag{4.69}$$

$$\Rightarrow \quad \mathrm{div}_\Gamma \mathsf{L}^2_{f,p} \mathsf{P}^2_{f,p} \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) \tag{4.70}$$

$$\Rightarrow \quad \mathrm{div}_\Gamma((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) = 0 \quad \forall f \in \mathcal{F}_2(T) , \quad (\mathbf{u}^{(2)} \cdot \boldsymbol{t}_e)_{|e} \equiv 0 \quad \forall e \in \mathcal{F}_1(T) \tag{4.71}$$

$$\Rightarrow \quad \mathsf{P}^1_{f,p}((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) = (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} \quad \forall f \in \mathcal{F}_2(T) \tag{4.72}$$

$$\Rightarrow \quad \mathbf{grad}\, \mathsf{E}^0_{f,p} \mathsf{L}^1_{f,p} \mathsf{P}^1_{f,p}((\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f}) \times \boldsymbol{n}_f = (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} \quad \forall f \in \mathcal{F}_2(T) \tag{4.73}$$

$$\Rightarrow \quad (\mathbf{u}^{(3)} \times \boldsymbol{n}_f)_{|f} = 0 \quad \forall f \in \mathcal{F}_2(T) \tag{4.74}$$

$$\Rightarrow \quad \mathsf{P}^2_{T,p}\, \mathbf{curl}\, \mathbf{u}^{(3)} = \mathbf{curl}\, \mathbf{u}^{(3)} \tag{4.75}$$

$$\Rightarrow \quad \mathbf{curl}\, \mathsf{L}^2_{T,p} \mathsf{P}^2_{T,p}\, \mathbf{curl}\, \mathbf{u}^{(3)} = \mathbf{curl}\, \mathbf{u}^{(3)} \tag{4.76}$$

$$\Rightarrow \quad \mathbf{curl}\, \mathbf{u}^{(4)} = 0 \quad \Rightarrow \quad \mathsf{P}^1_T \mathbf{u}^{(4)} = \mathbf{u}^{(4)} \tag{4.77}$$

$$\Rightarrow \quad \mathbf{grad}\, \mathsf{L}^1_T \mathsf{P}^1_T \mathbf{u}^{(4)} = \mathbf{u}^{(4)} , \tag{4.78}$$

which confirms the projector property.

Now assume $\mathbf{u} = \mathbf{grad}\, u$ for some $u \in C^\infty(\overline{T})$. The commuting diagram property will follow, if we manage to show $\mathbf{grad}\, u^{(0)} = \mathbf{u}^{(0)}$, $\mathbf{grad}\, u^{(1)} = \mathbf{u}^{(1)}$, $\mathbf{grad}\, u^{(2)} = \mathbf{u}^{(3)}$, etc., for the intermediate functions in (4.52)-(4.55) and (4.60)-(4.65), respectively.

By the commuting diagram property for the standard local interpolation operators onto the spaces of Whitney-0-forms (linear polynomials) and Whitney-1-forms, we conclude

$$\mathbf{grad}\, u^{(0)} = \mathbf{u}^{(0)} \quad \Rightarrow \quad \frac{d}{ds} u^{(0)}{}_{|e} = (\mathbf{u}^{(0)} \cdot \boldsymbol{t}_e)_{|e} \quad \forall e \in \mathcal{F}_1(T) \tag{4.79}$$

$$\Rightarrow \quad \mathbf{u}^{(1)} = \mathbf{grad}\, u^{(1)} \quad \Rightarrow \quad \mathrm{div}_\Gamma((\mathbf{u}^{(1)} \times \boldsymbol{n}_f)_{|f}) = 0 \quad \forall f \in \mathcal{F}_2(T) \tag{4.80}$$

$$\Rightarrow \quad \mathbf{u}^{(2)} = \mathbf{u}^{(1)} \tag{4.81}$$

$$\Rightarrow \quad (\mathbf{u}^{(2)} \times \boldsymbol{n}_f)_{|f} = \mathbf{curl}_\Gamma u^{(1)}{}_f \quad \forall f \in \mathcal{F}_2(T) \quad \Rightarrow \quad \mathbf{u}^{(3)} = \mathbf{grad}\, u^{(2)} \tag{4.82}$$

$$\Rightarrow \quad \mathbf{u}^{(4)} = \mathbf{u}^{(3)} \,. \tag{4.83}$$

Of course, analogous relationships for the functions $w^{(i)}$ and $\mathbf{w}^{(i)}$ hold, which yields $\Pi^1_{T,p}\mathbf{u} = \mathbf{grad}\, \Pi^0_{T,p} u$. □

Following [36, Sect. 3.5], a projection based interpolation onto $\mathcal{W}^2_p(T)$, the operator $\Pi^2_{T,p} (= \Pi^2_{T,p}(\epsilon)) : \boldsymbol{C}^\infty(\overline{T}) \mapsto \mathcal{W}^2_p(T)$, involves the stages

$$\mathbf{u}^{(0)} := \mathbf{u} - \underbrace{\Big(\sum_{f \in \mathcal{F}_2(T)} \int_f \mathbf{u} \cdot \boldsymbol{n}_f \, \mathrm{d}S \Big) \mathbf{b}_f}_{:= \mathbf{w}^{(0)}} \,, \tag{4.84}$$

$$\mathbf{u}^{(1)} := \mathbf{u}^{(0)} - \underbrace{\sum_{f \in \mathcal{F}_2(T)} \mathbf{curl}\, \mathsf{E}^1_{f,p} \mathsf{L}^2_{f,p} \mathsf{P}^2_{f,p} \big((\mathbf{u}^{(0)} \cdot \boldsymbol{n}_f)_{|f} \big)}_{:= \mathbf{w}^{(1)}} \tag{4.85}$$

$$\mathbf{u}^{(2)} := \mathbf{u}^{(1)} - \underbrace{\mathsf{L}^3_{T,p} \mathsf{P}^3_{T,p} \,\mathrm{div}\, \mathbf{u}^{(1)}}_{:= \mathbf{w}^{(2)}} \tag{4.86}$$

$$\Pi^2_{T,p}\mathbf{u} := \mathbf{curl}\, \mathsf{L}^2_{T,p} \mathsf{P}_{T,p} \mathbf{u}^{(2)} + \mathbf{w}^{(0)} + \mathbf{w}^{(1)} + \mathbf{w}^{(2)} \,. \tag{4.87}$$

Here, $\mathbf{b}_f$ refers to the local basis functions for Whitney-2-forms [36, Sect. 3.2]:

$$\mathbf{b}_f = \lambda_i \,\mathbf{grad}\, \lambda_j \times \mathbf{grad}\, \lambda_k + \lambda_j \,\mathbf{grad}\, \lambda_k \times \lambda_i + \lambda_k \,\mathbf{grad}\, \lambda_i \times \lambda_j \,. \tag{4.88}$$

Analogous to Lemma 4.6 one proves the following result.

Lemma 4.7. *The linear operator $\Pi^2_{T,p}$, $p \in \mathbb{N}_0$, is a projection onto $\mathcal{W}^2_p(T)$ and satisfies the* commuting diagram property

$$\Pi^2_{T,p} \circ \mathbf{curl} = \mathbf{curl} \circ \Pi^1_{T,p} \quad \text{on } \boldsymbol{C}^\infty(\overline{T}) \,. \tag{4.89}$$

The next lemma makes it possible to patch together the local projection based interpolation operator to obtain global interpolation operators

$$\Pi_p^l : \boldsymbol{C}^\infty(\overline{\Omega}) \mapsto \mathcal{W}_p^l(\mathcal{T}) \;, \quad l = 1, 2 \;. \tag{4.90}$$

Lemma 4.8. *For any* $F \in \mathcal{F}_m(T)$, $m = 0, 1, 2$, *and* $u \in C^\infty(\overline{T})$ *the restriction* $\Pi^0_{T,p} u_{|F}$ *depends only on* $u_{|F}$.

For any $F \in \mathcal{F}_m(T)$, $m = 1, 2$, *and* $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ *the tangential trace of* $\Pi^1_{T,p}\mathbf{u}$ *onto* F *depends only on the tangential trace of* $\mathbf{u}$ *on* F.

For any face $f \in \mathcal{F}_2(T)$ *and* $\mathbf{u} \in \boldsymbol{C}^\infty(\overline{T})$ *the normal trace of* $\Pi^2_{T,p}\mathbf{u}$ *onto* f *depends only on the normal component of* $\mathbf{u}$ *on* f.

Proof. The assertion is immediate from the construction, in particular, the properties of the extension operators used therein. □

It goes without saying that density arguments permit us to extend Π_p^l, $l = 0, 1, 2$, to Sobolev spaces, as long as they are continuous in the respective norms. (Repeated) application of trace theorems [33, Sect. 1.5] reveals that it is possible to obtain continuous projectors

$$\Pi_p^0 : H^{1+s}(\Omega) \mapsto \mathcal{W}_p^0(\mathcal{T}) \;, \tag{4.91}$$

$$\Pi_p^1 : \boldsymbol{H}^{\frac{1}{2}+s}(\Omega) \mapsto \boldsymbol{\mathcal{W}}_p^1(\mathcal{T}) \;, \tag{4.92}$$

$$\Pi_p^2 : \boldsymbol{H}^{s}(\Omega) \mapsto \boldsymbol{\mathcal{W}}_p^2(\mathcal{T}) \;, \tag{4.93}$$

for any $s > \frac{1}{2}$. In addition, by virtue of Lemma 4.8 and the resolution of Γ_D by $\mathcal{T}$, zero pointwise/tangential/normal trace on Γ_D of the argument function will be preserved by Π_p^l, $l = 0, 1, 2$, for instance,

$$\Pi_p^1 \left(\boldsymbol{H}^{\frac{1}{2}+s}(\Omega) \cap \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \right) = \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T}) \cap \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \;. \tag{4.94}$$

4.1.5 Local Interpolation error estimates

Closely following [20, Section 6] we first examine the interpolation error for $\Pi^0_{T,p}$. Please notice that $\Pi^0_{T,p}$ still depends on the fixed regularity parameter $0 < \epsilon < \frac{1}{2}$. The argument function of $\Pi^0_{T,p}$ is assumed to lie in $H^2(T)$. The continuous embedding $H^2(T) \hookrightarrow C^0(\overline{T})$ plus trace theorems for Sobolev spaces render all operators well defined in this case.

We start with an observation related to the local best approximation properties of the projection based interpolant.

Lemma 4.9. *For any $u \in H^2(T)$ holds*

$$\big(\mathbf{grad}(u - \Pi^0_{T,p}u), \mathbf{grad}\, v\big)_{L^2(T)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(T)\ , \tag{4.95}$$

$$\Big(\mathbf{curl}_\Gamma(u - \Pi^0_{T,p}u)_{|f}, \mathbf{curl}_\Gamma v\Big)_{H^{-\frac{1}{2}+\epsilon}(f)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(f),\ f \in \mathcal{F}_2(T)\ , \tag{4.96}$$

$$\left(\frac{d}{ds}(u - \Pi^0_{T,p}u)_{|e}, \frac{d}{ds}v\right)_{H^{-1+\epsilon}(e)} = 0 \quad \forall v \in \mathring{\mathcal{P}}_{p+1}(e),\ e \in \mathcal{F}_1(T)\ . \tag{4.97}$$

Proof. We use the notations of (4.52)-(4.55). Setting $w := w^{(0)} + w^{(1)} + w^{(2)}$, we find

$$\Pi^0_{T,p}u = \mathsf{L}^1_{T,p}\mathsf{P}^1_{T,p}\,\mathbf{grad}(u - w) + w\ , \tag{4.98}$$

which implies, because $\mathsf{L}^1_{T,p}$ is a right inverse of $\mathbf{grad}$,

$$\mathbf{grad}\,\Pi^0_{T,p}u = \mathsf{P}^1_{T,p}\,\mathbf{grad}\,u + (\mathsf{Id} - \mathsf{P}^1_{T,p})\,\mathbf{grad}\,w\ . \tag{4.99}$$

This means that $\mathbf{grad}\,u - \mathbf{grad}\,\Pi^0_{T,p}u$ belongs to the range of $\mathsf{Id} - \mathsf{P}^1_{T,p}$ and (4.95) follows from (4.34) and the properties of orthogonal projections. Similar manipulations establish (4.96):

$$\begin{aligned}
\mathbf{curl}_\Gamma\Pi^0_{T,p}u_{|f} &= \mathbf{curl}_\Gamma w_{|f} \\
&= \underbrace{\mathbf{curl}_\Gamma\mathsf{L}^1_{f,p}}_{=\mathsf{Id}}\,\mathsf{P}^1_{f,p}\mathbf{curl}_\Gamma u^{(1)} + \mathbf{curl}_\Gamma(w^{(0)} + w^{(1)})_{|f} \\
&= \mathsf{P}^1_{f,p}\mathbf{curl}_\Gamma u_{|f} + (\mathsf{Id} - \mathsf{P}^1_{f,p})\mathbf{curl}_\Gamma(w^{(0)} + w^{(1)}) \quad \forall f \in \mathcal{F}_2(T)\ .
\end{aligned}$$

The same arguments as above verify (4.97). □

From this we can conclude the result of [20, Section 6, Corollary 1]. To state it we now assume a dependence

$$0 < \epsilon = \epsilon(p) := \frac{1}{10\log(p+2)} < \frac{1}{4}\ , \quad p \in \mathbb{N}\ , \tag{4.100}$$

of the parameter ϵ in the definition of the local projection based interpolation operators. Below, all parameters ϵ are linked to p via (4.100). Please note that we retain the notation $\left(\Pi^l_{T,p}\right)_{p\in\mathbb{N}}$, $l = 0, 1, 2$, for these new families of operators.

Theorem 4.10 (Spectral interpolation error estimate for $\Pi^0_{T,p}$). *With a constant merely depending on the shape-regularity of T*

$$\big|(\mathsf{Id} - \Pi^0_{T,p})v\big|_{1,T} \lesssim (1 + \log^{3/2}(p+1))\frac{h_T}{p+1}|v|_{2,T} \quad \forall v \in H^2(T)\ . \tag{4.101}$$

Stable polynomial extensions are instrumental for the proof, which will be postponed until Page 47. First, we recall the results of [53, Thm. 1] and [1, Thm. 1]:

Theorem 4.11 (Stable polynomial extension for tetrahedra). *For a tetrahedron T there is linear operator $\mathsf{S}_T : H^{\frac{1}{2}}(\partial T) \mapsto H^1(T)$ such that*

$$\mathsf{S}_T u_{|\partial T} = u \quad \forall u \in H^{\frac{1}{2}}(\partial T) \,, \tag{4.102}$$

$$|\mathsf{S}_T u|_{1,T} \lesssim |u|_{\frac{1}{2},\partial T} \quad \forall u \in H^{\frac{1}{2}}(\partial T) \,, \tag{4.103}$$

$$\mathsf{S}_T w \in \mathcal{P}_{p+1}(T) \quad \forall w \in \mathcal{P}_{p+1}(T)_{|\partial T} \,. \tag{4.104}$$

Theorem 4.12 (Stable polynomial extension for triangles). *Given a triangle F, there is a linear mapping $\mathsf{S}_F : L^2(\partial F) \mapsto H^{\frac{1}{2}}(F)$ such that*

$$|\mathsf{S}_F u|_{\frac{1}{2},F} \lesssim \|u\|_{0,\partial F} \quad \forall u \in L^2(\partial F) \,, \tag{4.105}$$

$$|\mathsf{S}_F u|_{1,F} \lesssim |u|_{\frac{1}{2},\partial F} \quad \forall u \in H^{\frac{1}{2}}(\partial F) \,, \tag{4.106}$$

$$\mathsf{S}_F w \in \mathcal{P}_{p+1}(F) \quad \forall w \in \mathcal{P}_{p+1}(F)_{|\partial F} \,, \tag{4.107}$$

where the constants depend only on the shape regularity measure of T.

By interpolation in Sobolev scale from the last theorem we can conclude

$$|\mathsf{S}_F u|_{s,F} \lesssim |u|_{s-\frac{1}{2},\partial F} \quad \forall u \in H^{s-\frac{1}{2}}(\partial F),\ \frac{1}{2} \le s \le 1 \,. \tag{4.108}$$

We also need to deal with the awkward property of the $H^{\frac{1}{2}}(\partial T)$-norm that it cannot be split into face contributions. To that end we resort to a result from [50, Proof of Lemma 3.31], see also [20, Lemma 13].

Lemma 4.13 (Splitting of $H^{\frac{1}{2}}(\partial T)$-norm). *With a constants depending only on the shape regularity of the tetrahedron T holds*

$$|u|_{s,\partial T} \lesssim \frac{1}{s-\frac{1}{2}} \sum_{f \in \mathcal{F}_2(T)} |u|_{s,f} \quad \forall u \in H^{\frac{1}{2}+s}(\partial T),\ \frac{1}{2} < s \le 1 \,. \tag{4.109}$$

Another natural ingredient for the proof are polynomial best approximation estimates, see [59] or [53, Sect. 3].

Lemma 4.14. *Let $0 \le r \le 1$, $0 \le s \le 2$, and F be either a tetrahedron or a triangle. Then,*

$$\inf_{v_p \in \mathcal{P}_{p+1}(F)} |u - v_p|_{r,F} \lesssim \left(\frac{h_F}{p}\right)^{s+1-r} |u|_{s+1,F} \quad \forall u \in H^{s+1}(F) \,. \tag{4.110}$$

Define a semi-norm projection $\mathsf{Q}_{T,p} : H^1(T) \mapsto \mathcal{P}_{p+1}(T)$ on the tetrahedron T by

$$\begin{aligned}
&\int_T \mathbf{grad}(u - \mathsf{Q}_{T,p}u) \cdot \mathbf{grad}\, v_p \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v_p \in \mathcal{P}_{p+1}(T) \;, \\
&\int_T u - \mathsf{Q}_{T,p}u \,\mathrm{d}\boldsymbol{x} = 0 \;,
\end{aligned} \tag{4.111}$$

and, for $\frac{1}{2} \leq s \leq 1$, semi-norm projections $\mathsf{Q}_{f,p} : H^{s-\frac{1}{2}}(f) \mapsto \mathcal{P}_{p+1}(f)$, $f \in \mathcal{F}_2(T)$, by

$$\begin{aligned}
&\left(\mathbf{curl}_\Gamma(u - \mathsf{Q}_{f,p}u), \mathbf{curl}_\Gamma v_p\right)_{H^{s-\frac{1}{2}}(f)} = 0 \quad \forall v_p \in \mathcal{P}_{p+1}(T) \;, \\
&\int_f u - \mathsf{Q}_{f,p}u \,\mathrm{d}\boldsymbol{x} = 0 \;.
\end{aligned} \tag{4.112}$$

These definitions involve best approximation properties of $\mathsf{Q}_{T,p}u$ and $\mathsf{Q}_{f,p}u$. Thus, we learn from Lemma 4.14 that with constants independent of $0 < \epsilon < \frac{1}{2} < s \leq 1$

$$|u - \mathsf{Q}_{T,p}u|_{1,T} \lesssim \left(\frac{h_T}{p+1}\right)^s |u|_{1+s,T} \quad \forall u \in H^s(T) \;, \tag{4.113}$$

$$|u - \mathsf{Q}_{f,p}u|_{\frac{1}{2}+\epsilon,f} \lesssim \left(\frac{h_T}{p+1}\right)^{s-\epsilon} |u|_{\frac{1}{2}+s,T} \quad \forall u \in H^{\frac{1}{2}+s}(f) \;. \tag{4.114}$$

The latter estimate follows from the fact that $|\cdot|_{\frac{1}{2}+\epsilon,f}$ and $\|\mathbf{curl}_\Gamma \cdot\|_{-\frac{1}{2}+\epsilon,f}$ are equivalent semi-norms, uniformly in ϵ.

We also need error estimates for the $L^2(e)$-orthogonal projections,

$$\mathsf{Q}^*_{e,p} : L^2(e) \mapsto \overset{\circ}{\mathcal{P}}_{p+1}(e) \;, \quad e \in \mathcal{F}_1(T) \;. \tag{4.115}$$

Lemma 4.15 (see [20, Lemma 18]). *With a constant independent of* p, $0 \leq \epsilon \leq \frac{1}{2}$, *and* $2\epsilon \leq r \leq 1 + \epsilon$

$$\left|u - \mathsf{Q}^*_{e,p}u\right|_{\epsilon,e} \lesssim \left(\frac{h_e}{p+1}\right)^{r-2\epsilon} |u|_{r,e} \quad \forall u \in H^r(e) \cap H^1_0(e) \;.$$

Proof. By scaling arguments we may assume $h_e = 1$. Write $\mathsf{I}_{e,p} : H^1_0(e) \mapsto \overset{\circ}{\mathcal{P}}_{p+1}$ for the interpolation operator

$$(\mathsf{I}_{e,p}u)(\xi) = u(0) + \int_0^\xi \big(\mathsf{Q}_{e,p}\frac{du}{d\xi}\big)(\tau)\,\mathrm{d}\tau \;, \quad 0 \leq \xi \leq |e| \;,$$

where ξ is the arclength parameter for the edge e and $\mathsf{Q}_{e,p} : L^2(\Omega) \mapsto \mathcal{P}_p(e)$ is the $L^2(e)$-orthogonal projection. From [59, Sect. 3.3.1, Thm. 3.17] we learn that

$$|u - \mathsf{I}_{e,p}u|_{1,e} \lesssim (p+1)^{-1} |u|_{2,e} \quad \forall u \in H^2(e) \;, \tag{4.116}$$

$$\|u - \mathsf{I}_{e,p}u\|_{0,e} \lesssim (p+1)^{-m} |u|_{m,e} \quad \forall u \in H^m(e) \;, \quad m = 1, 2 \;. \tag{4.117}$$

As $\mathsf{I}_{e,p}u \in \overset{\circ}{\mathcal{P}}_{p+1}(e)$ for $u \in H_0^1(e)$, this permits us to conclude

$$\left\|u - \mathsf{Q}^*_{e,p}u\right\|_{0,e} \leq \|u - \mathsf{I}_{e,p}u\|_{0,e} \lesssim (p+1)^{-1}\,\|u\|_{1,e}\ , \tag{4.118}$$

which yields, by interpolation between $H^1(e)$ and $L^2(e)$,

$$\left\|u - \mathsf{Q}^*_{e,p}u\right\|_{0,e} \lesssim (p+1)^{-q}\|u\|_{q,e}\ , \quad 0 \leq q \leq 1\ , \tag{4.119}$$

where the constant is independent of q. On the other hand, using the inverse inequality [7, Lemma 1]

$$\|u\|_{1,e} \lesssim (p+1)^2\,\|u\|_{0,e} \quad \forall u \in \mathcal{P}_{p+1}(e) \tag{4.120}$$

and (4.116), (4.117) we find the estimate

$$\begin{aligned}\left|u - \mathsf{Q}^*_{e,p}u\right|_{1,e} &\leq |u - \mathsf{I}_{e,p}u|_{1,e} + \left|\mathsf{Q}^*_{e,p}u - \mathsf{I}_{e,p}u\right|_{1,e} \\ &\leq |u - \mathsf{I}_{e,p}u|_{1,e} + (p+1)^2\left\|\mathsf{Q}^*_{e,p}u - \mathsf{I}_{e,p}u\right\|_{0,e} \\ &\lesssim |u - \mathsf{I}_{e,p}u|_{1,e} + (p+1)^2\,\|u - \mathsf{I}_{e,p}u\|_{0,e} \lesssim \|u\|_{2,e}\ .\end{aligned} \tag{4.121}$$

Interpolation between (4.119) with $q = \frac{r-2\epsilon}{1-\epsilon}$ and (4.121) finishes the proof. □

Proof of Thm. 4.10, cf. *[20, Sect. 6].* Orthogonality (4.95) of Lemma 4.9 combined with the definition of $\mathsf{Q}_{T,p}$ involves

$$\int_T \mathbf{grad}((\Pi^0_{T,p} - \mathsf{Q}_{T,p})u)\cdot\mathbf{grad}\,v_p\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall v_p \in \overset{\circ}{\mathcal{P}}_{p+1}(T)\ . \tag{4.122}$$

Hence, $(\Pi^0_{T,p} - \mathsf{Q}_{T,p})u$ turns out to be the $|\cdot|_{1,T}$-minimal degree $p+1$ polynomial extension of $(\Pi^0_{T,p} - \mathsf{Q}_{T,p})u_{|\partial T}$, which, thanks to Thm. 4.11, implies

$$\begin{aligned}\left|(\Pi^0_{T,p} - \mathsf{Q}_{T,p})u\right|_{1,T} &\leq \left|\mathsf{S}_T((\Pi^0_{T,p}u - \mathsf{Q}_{T,p}u)_{|\partial T})\right|_{1,T} \\ &\lesssim \left|(\Pi^0_{T,p}u - \mathsf{Q}_{T,p}u)_{|\partial T}\right|_{\frac{1}{2},\partial T}\ .\end{aligned} \tag{4.123}$$

Thus, by the continuity of the trace operator $H^1(T) \mapsto H^{\frac{1}{2}}(\partial T)$,

$$\begin{aligned}&\left|u - \Pi^0_{T,p}u\right|_{1,T} \lesssim \\ &\qquad |u - \mathsf{Q}_{T,p}u|_{1,T} + \left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{\frac{1}{2},\partial T} + \left|(u - \mathsf{Q}_{T,p}u)_{|\partial T}\right|_{\frac{1}{2},\partial T} \\ &\qquad\qquad \lesssim \left(|u - \mathsf{Q}_{T,p}u|_{1,T} + \left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{\frac{1}{2},\partial T}\right)\ .\end{aligned} \tag{4.124}$$

To estimate $\left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{\frac{1}{2},\partial T}$ we appeal to Lemma 4.13 and get

$$\begin{aligned}\left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{\frac{1}{2},\partial T} &\leq \left|(u - \Pi^0_{T,p}u)_{|\partial T}\right|_{\frac{1}{2}+\epsilon,\partial T} \\ &\lesssim \frac{1}{\epsilon} \sum_{f\in\mathcal{F}_2(T)} \left|(u - \Pi^0_{T,p}u)_{|f}\right|_{\frac{1}{2}+\epsilon,f} .\end{aligned} \tag{4.125}$$

Next, we use (4.96) from Lemma 4.9 together with (4.112), which confirms that $(\Pi^0_{T,p}u)_{|f} - \mathsf{Q}_{f,p}u$ is the minimum $|\cdot|_{\frac{1}{2}+\epsilon,f}$-seminorm polynomial extension of $(\Pi^0_{T,p}u)_{|\partial f} - \mathsf{Q}_{f,p}(u)_{|\partial f}$. Hence, based on arguments parallel to the derivation of (4.124), this time using Thm. 4.12, we can bound

$$\left|(u - \Pi^0_{T,p}u)_{|f}\right|_{\frac{1}{2}+\epsilon,f} \lesssim \left|u_{|f} - \mathsf{Q}_{f,p}u\right|_{\frac{1}{2}+\epsilon,f} + \left|(\Pi^0_{T,p}u - \mathsf{Q}_{f,p}u)_{|\partial f}\right|_{\epsilon,\partial f} , \tag{4.126}$$

where the (ϵ-independent !) continuity constant of the trace mapping S_f enters the constant. Also recall the continuity of the trace mapping $H^{\frac{1}{2}+\epsilon}(f) \mapsto H^{\epsilon}(\partial f)$ [50, Proof of Lemma 3.35]: with a constant independent of ϵ,

$$\left\|u_{|\partial f}\right\|_{\epsilon,\partial f} \lesssim \frac{1}{\sqrt{\epsilon}} \|u\|_{\frac{1}{2}+\epsilon,f} \quad \forall u \in H^{\frac{1}{2}+\epsilon}(f) . \tag{4.127}$$

Use this to continue the estimate (4.126)

$$\left|(u - \Pi^0_{T,p}u)_{|f}\right|_{\frac{1}{2}+\epsilon,f} \lesssim \frac{1}{\sqrt{\epsilon}} \left|u_{|f} - \mathsf{Q}_{f,p}u\right|_{\frac{1}{2}+\epsilon,f} + \left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{\epsilon,\partial f} . \tag{4.128}$$

As $\epsilon < \frac{1}{2}$, we can localize the norm $\left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{\epsilon,\partial f}$ to the edges of f, similarly to Lemma 4.13:

$$\left|(u - \Pi^0_{T,p}u)_{|\partial f}\right|_{\epsilon,\partial f} \lesssim \frac{1}{\frac{1}{2} - \epsilon} \sum_{e\in\mathcal{F}_1(T), e\subset\partial f} \left|(u - \Pi^0_{T,p}u)_{|e}\right|_{\epsilon,e} . \tag{4.129}$$

Recall the ϵ-uniform equivalence of the norms $|\cdot|_{\epsilon,e}$ and $\left\|\frac{d}{ds}\cdot\right\|_{-1+\epsilon,e}$. Hence, owing to (4.97), we have from Lemma 4.15 with $r = 1$:

$$\begin{aligned}\left|(u - \Pi^0_{T,p}u)_{|e}\right|_{\epsilon,e} &\lesssim \inf_{v_p \in \mathring{\mathcal{P}}_{p+1}} \left|(u - \Pi^0_{T,0}u)_{|e} - v_p\right|_{\epsilon,e} \\ &\lesssim \left|(u - \Pi^0_{T,0}u)_{|e} - \mathsf{Q}^*_{e,p}((u - \Pi^0_{T,0}u)_{|e})\right|_{\epsilon,e} \\ &\lesssim \left(\frac{h_T}{p+1}\right)^{1-2\epsilon} \left|(u - \Pi^0_{T,0}u)_{|e}\right|_{s,e} .\end{aligned} \tag{4.130}$$

Moreover, $H^2(T)$ is continuously embedded into $C^0(\overline{T})$. Consequently, applying trace theorems twice and appealing to the equivalence of all norms on the finite dimensional space $\mathcal{P}_1(T)$,

$$\left|(u - \Pi^0_{T,0}u)_{|e}\right|_{s,e} \leq \left|u_{|e}\right|_{s,e} + \left|(\Pi^0_{T,0}u)_{|e}\right|_{s,e} \lesssim |u|_{1+s,T} \ , \tag{4.131}$$

where the constant may depend on s. Combining the estimates (4.124), (4.125), (4.128), and (4.129), (4.130) with (4.131), we find

$$\begin{aligned} \left|u - \Pi^0_{T,p}u\right|_{1,T} \lesssim |u - \mathsf{Q}_{T,p}u|_{1,T} + \frac{1}{\epsilon^{3/2}} \sum_{f\in\mathcal{F}_2(T)} \left|u_{|f} - \mathsf{Q}_{f,p}(u_{|f})\right|_{\frac{1}{2}+\epsilon,f} + \\ \left(\frac{h_T}{p+1}\right)^{s-2\epsilon} \frac{1}{\epsilon(\frac{1}{2}-\epsilon)} \sum_{e\in\mathcal{F}_1(T)} |u|_{2,T} \ . \end{aligned} \tag{4.132}$$

Finally, we plug in the projection error estimates (4.113), (4.114), and arrive at

$$\begin{aligned} \left|u - \Pi^0_{T,p}(\epsilon)u\right|_{1,T} \lesssim \frac{h_T}{p+1}|u|_{2,T} + \left(\frac{h_T}{p+1}\right)^{1-+\epsilon} \frac{1}{\epsilon^{3/2}} \sum_{f\in\mathcal{F}_2(T)} |u|_{3/2,f} + \\ \left(\frac{h_T}{p+1}\right)^{1-2\epsilon} \frac{1}{\epsilon(\frac{1}{2}-\epsilon)} \sum_{e\in\mathcal{F}_1(T)} |u|_{1,e} \ . \end{aligned} \tag{4.133}$$

with constants also indepedent of ϵ. The choice (4.100) of ϵ together with an application of trace theorems then finishes the proof. □

The next lemma plays the role of [9, Lemma 9] and makes it possible to adapt the approach of [9, Sect. 4.4] to 3D edge elements.

Lemma 4.16. *If* $\mathbf{u} \in \boldsymbol{H}^1(T) \cap \boldsymbol{H}(\mathbf{curl}, T)$ *possesses a polynomial* $\mathbf{curl}$ *in the sense that* $\mathbf{curl}\,\mathbf{u} \in \boldsymbol{\mathcal{P}}_p(T)$*, then*

$$\left\|(\mathsf{Id} - \Pi^1_p)\mathbf{u}\right\|_{0,\Omega} \lesssim (1 + \log^{3/2}(p+1))\frac{h_T}{p} |\mathbf{u}|_{1,T} \ . \tag{4.134}$$

Proof. Pick any $\mathbf{u}$ complying with the assumptions of the lemma and split

$$\mathbf{u} = (\mathbf{u} - \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}) + \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u} \ . \tag{4.135}$$

Note that the properties of the smoothed Poincaré lifting R_T stated in Thm. 4.3 imply

(i) $\mathbf{curl}(\mathbf{u} - \mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}) = 0$ on T, as a consequence of (4.8), and
(ii) $\mathsf{R}_T\,\mathbf{curl}\,\mathbf{u} \in \boldsymbol{H}^1(T)$ and the bound

$$\|\mathsf{R}_T\,\mathbf{curl}\,\mathbf{u}\|_{1,T} \lesssim \|\mathbf{curl}\,\mathbf{u}\|_{0,\Omega} \ , \tag{4.136}$$

where here and below no constant may depend on $\mathbf{u}$ or p.

Hence, as $\mathbf{u} \in \boldsymbol{H}^1(T)$, there exists $v \in H^2(T)$ such that

$$\mathbf{u} = \mathbf{grad}\, v + \mathsf{R}_T\, \mathbf{curl}\, \mathbf{u}\ . \tag{4.137}$$

The continuity of R_T reveals that

$$|v|_{2,T} \leq \|\mathbf{u}\|_{1,T} + |\mathsf{R}_T\, \mathbf{curl}\, \mathbf{u}|_{1,T} \lesssim \|\mathbf{u}\|_{1,T} + \|\mathbf{curl}\, \mathbf{u}\|_{0,T}\ . \tag{4.138}$$

By the assumptions of the lemma and (4.11) we know that

$$\mathsf{R}_T\, \mathbf{curl}\, \mathbf{u} \in \mathcal{W}^1_p(T)\ . \tag{4.139}$$

By the commuting diagram property from Lemma 4.6 and the projector property of $\Pi^1_{T,p}$ the task is reduced to an interpolation estimate for $\Pi^0_{T,p}$:

$$(\mathsf{Id} - \Pi^1_{T,p})\mathbf{u} \overset{(4.137)}{=} \mathbf{grad}(\mathsf{Id} - \Pi^0_{T,p})v + \underbrace{(\mathsf{Id} - \Pi^1_{T,p})\mathsf{R}_T\, \mathbf{curl}\, \mathbf{u}}_{=0}\ . \tag{4.140}$$

As a consequence, invoking Theorem 4.10,

$$\begin{aligned} \left\|(\mathsf{Id} - \Pi^1_{T,p})\mathbf{u}\right\|_{0,T} &\overset{(4.140)}{=} \left|(\mathsf{Id} - \Pi^0_{T,p})v\right|_{1,T} \lesssim (1 + \log^{2/3}(p+1))\frac{h_T}{p}\, |v|_{2,T} \\ &\overset{(4.138)}{\lesssim} (1 + \log^{3/2}(p+1))\frac{h_T}{p}\left(\|\mathbf{u}\|_{1,T} + \|\mathbf{curl}\, \mathbf{u}\|_{0,T}\right)\ , \end{aligned} \tag{4.141}$$

which gives the assertion of the lemma. □

Remark 4.17. In principle, the very construction of projection based interpolation operators well fits spaces of discrete differential forms with variable polynomial degree ("hp-spaces) as long as the so-called minimum rule for the degrees, see [49, Rem. IV.3.2] or [22], is fulfilled. Unfortunately, it is not clear how to adapt the splitting (4.135) to the hp setting and our proof of the key Theorem 4.16 cannot be extended.

4.2 Boundary-Aware p-Stable Quasi-Interpolation for Lagrangian Finite Elements

In this section we sketch the construction of a local quasi-interpolation operator into $\mathcal{W}^0_{\Gamma_D}(\mathcal{T})$ following the policy of smoothing projections by local regularization that as developed in [14], [26], [49, Ch. VII] and [47]. The latter fundamental work is our main source and [47, Cor. 3.7] already asserts the existence of suitable quasi-interpolation operator in the case $\Gamma_D = \partial\Omega$. We extend this to zero boundary conditions on parts of $\partial\Omega$, borrowing a distortion technique from [49, Sect. VII.2]. We point out that [51, Thm. 3.3] provides excatly the kind of quasi-interpolation we need, unfortunately only in two dimensions. The extension to 3D looks formidably technical.

According to [14, Sect. 4.1] the flow induced by the vector field $\widetilde{\boldsymbol{n}}$ introduced in Section 2.1 can be used to define a “reflection at the boundary Γ”, a map $\mathsf{R}_\Gamma : \Omega_\Gamma \to \Omega_\Gamma$ satisfying

(R_1) $\mathsf{R}_\Gamma(\Omega \cap \Omega_\Gamma) = (\mathbb{R}^3 \setminus \overline{\Omega}) \cap \Omega_\Gamma$,

(R_2) $\mathsf{R}_\Gamma(\boldsymbol{x}) = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \Gamma$,

(R_3) R_Γ is bi-Lipschitz with Lipschitz constants depending only on Γ.

We introduce the p-scaled mesh width function $\varepsilon_h \in L^\infty(\Omega)$, $\varepsilon_h(\boldsymbol{x}) = {}^{h_T}/_{p+1}$ on $T \in \mathcal{T}$: $\varepsilon_h := {}^{h}/_{p+1}$. We can extend it to a function $\varepsilon_h \in L^\infty(\Omega)$ on the expanded domain $\widetilde{\Omega} := \Omega \cup \Omega_\Gamma$ by reflection:

$$\varepsilon_h(\boldsymbol{x}) := \varepsilon_h(\mathsf{R}_\Gamma^{-1}(\boldsymbol{x})) \quad \text{for almost all } \boldsymbol{x} \in \Omega_\Gamma \setminus \Omega \,.$$

From [47, Lemma 3.1] or [49, Lemma VII.8.2] we learn that convolution of ε_h with a simple mollifier yields a smoothed extended mesh width function with bounded derivatives.

Lemma 4.18 (Smooth extended mesh width funnction). *The exists a smooth function $\varepsilon \in C^\infty(\widetilde{\Omega})$ such that*

(E_1) $\varepsilon \eqsim \varepsilon_h$ almost everywhere in $\widetilde{\Omega}$,

(E_2) $|D^{\mathbf{r}}\varepsilon| \lesssim |\varepsilon|^{1-|\mathbf{r}|}$ for all $\mathbf{r} \in \mathbb{N}_0^3$ pointwise in $\widetilde{\Omega}$,

Thus, ε qualifies as an *admissible lenth scale function* in the parlance of [47, Def. 2.1]. In particular, ε is uniformly positive and Lipschitz continuous; we write $L_\varepsilon > 0$ for its Lipschitz constant that depends on Ω and $\rho(\mathcal{T})$ alone.

To handle zero boundary conditions on Γ_D, we take the cue from [49, Sect. VII.2] and consider a blow-up map for the bulge domain Υ_D introduced in Section 2.1, Theorem 2.2.

Lemma 4.19 (Shrinkage mapping for bulge domain [49, Thm .VII.2.1]). *We can find constants $\delta_D > 0$ and $L_D > 0$ depending only on $\widetilde{\Omega}$ and Υ_D such that for* any *function $\xi : \widetilde{\Omega} \to \mathbb{R}^+$ with*

- $|\xi(\boldsymbol{x}) - \xi(\boldsymbol{y})| \le \delta_D \|\boldsymbol{x} - \boldsymbol{y}\|$, *for all $\boldsymbol{x}, \boldsymbol{y} \in \widetilde{\Omega}$,*
- $|\xi(\boldsymbol{x})| \le \delta_D$ *for all $\boldsymbol{x} \in \widetilde{\Omega}$,*

there exists a bi-Lipschitz *mapping $\mathsf{T}_\xi : \widetilde{\Omega} \to \widetilde{\Omega}$ with* [3]

(T_1) $\|\mathsf{T}_\xi(\boldsymbol{x}) - \mathsf{T}_\xi(\boldsymbol{y})\| \le L_D(1+\delta_D)\|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \widetilde{\Omega}$,

(T_2) $\left\|\mathsf{T}_\xi^{-1}(\boldsymbol{x}) - \mathsf{T}_\xi^{-1}(\boldsymbol{y})\right\| \le L_D(1+\delta_D)\|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \widetilde{\Omega}$,

[3] The symbol $B_r(\boldsymbol{z})$ designates the open ball around $\boldsymbol{z} \in \mathbb{R}^3$ with radius $r > 0$.

(T₃) $\|\mathsf{T}_\xi(\boldsymbol{x}) - \boldsymbol{x}\| \leq L_D \xi(\boldsymbol{x})\,, \quad \boldsymbol{x} \in \widetilde{\Omega}$,

(T₄) $\mathsf{T}_\xi(\boldsymbol{x}) = \boldsymbol{x}$ *for all* $\boldsymbol{x} \in \widetilde{\Omega}$ *with* $\mathrm{dist}(\boldsymbol{x}, \partial\Upsilon_D) \geq L_D\xi(\boldsymbol{x})$,

(T₅) *for all* $\boldsymbol{x} \in \overline{\Upsilon_D}$ *holds* $\quad \mathsf{T}_\xi(B_{\xi(\boldsymbol{x})/L_D}(\boldsymbol{x}) \cap \widetilde{\Omega}) \subset \Upsilon_D$,

(T₆) $\det D\mathsf{T}_\xi(\boldsymbol{x}) \approx 1$ *for all* $\boldsymbol{x} \in \widetilde{\Omega}$.

Casually speaking, by (T_5) T_ξ is a mapping that pulls a neighborhood of Υ_D into Υ_D. The property (T_3) ensures that the amount of local distortion effected by T_ξ can be controlled by ξ. The next result is borrowed from [47, Lemma 5.1 and 5.7] and paves the way for localization arguments.

Lemma 4.20 (Finite cover). *We can find "small constants"*

$$\alpha, \beta > 0\,, \quad \alpha < \beta\,, \quad \beta < \min\Big\{1, \frac{\mathrm{dist}(\Omega^e, \partial\widetilde{\Omega})}{\|\varepsilon\|_{\infty,\widetilde{\Omega}}}, \frac{1}{2L_\varepsilon}\Big\}, \tag{4.142}$$

and a finite set of points $\mathcal{Z} \subset \widetilde{\Omega}$ *such that*

(C₁) $\widetilde{\Omega} \subset \bigcup\{B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z}),\, \boldsymbol{z} \in \mathcal{Z}\}$ *(covering property)*

(C₂) $\mathrm{card}\{\boldsymbol{z} \in \mathcal{Z} :\, \boldsymbol{x} \in B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\} \lesssim 1$ *for all* $\boldsymbol{x} \in \widetilde{\Omega}$ *(uniform finite overlap).*

From now we fix α, β according to Theorem 4.20. From the covering and finite overlap property we conclude for any $m \in \mathbb{N}_0$

$$\sum_{\boldsymbol{z}\in\mathcal{Z}} \|v\|^2_{m,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \approx \sum_{\boldsymbol{z}\in\mathcal{Z}} \|v\|^2_{m,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \approx \|v\|^2_{m,\widetilde{\Omega}} \quad \forall v \in H^m(\widetilde{\Omega})\,. \tag{4.143}$$

In addition, by the triangle inequality the bound on β ensures that for any $\boldsymbol{z} \in \Omega^e$

$$B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \subset \widetilde{\Omega} \quad \text{and} \quad \tfrac{1}{2}\varepsilon(\boldsymbol{z}) \leq \varepsilon(\boldsymbol{x}) \leq \tfrac{3}{2}\varepsilon(\boldsymbol{z}) \quad \forall \boldsymbol{x} \in B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\,. \tag{4.144}$$

Next, set $\tau := \frac{1}{2}(\alpha + \beta)$ and choose a small number $\delta > 0$ satisfying the following inequalities

(δ_1) $2L_D^2\delta \leq \beta - \tau$, with L_D from Theorem 4.19,

(δ_2) $\delta L_\varepsilon \leq 1$ for the Lipschitz constant L_ε of ε,

(δ_3) $\delta L_D L_\varepsilon < \delta_D$, and $\delta L_D \|\varepsilon\|_{\infty,\widetilde{\Omega}} \leq \delta_D$,

(δ_4) $2\delta + \alpha < \tau$ and $2\delta + \tau < \beta$.

Now, recall Theorem 4.19 and define a concrete distortion map T_ε by setting $\mathsf{T}_\varepsilon := \mathsf{T}_\xi$ with the particular control function $\xi(\boldsymbol{x}) := L_D\delta\varepsilon(\boldsymbol{x})$, $\boldsymbol{x} \in \widetilde{\Omega}$. Owing to ($\delta_3$), this choice of $\xi : \widetilde{\Omega} \to \mathbb{R}^+$ satisfies the assumptions of Theorem 4.19. Thanks to Theorem 4.19, (T_5) we infer

$$\mathsf{T}_\varepsilon(B_{\delta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})) \subset \Upsilon_D \quad \forall \boldsymbol{z} \in \Gamma_D\,. \tag{4.145}$$

As a consequence of (4.144), (δ_1), and Theorem 4.19, (T_3) we note

$$\forall \boldsymbol{z} \in \widetilde{\Omega} : \quad \begin{aligned} &\mathsf{T}_\varepsilon(B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \cap \widetilde{\Omega}) \subset B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \;, \\ &\mathsf{T}_\varepsilon(B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \cap \widetilde{\Omega}) \subset B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z}) \;. \end{aligned} \tag{4.146}$$

We now study the pullback of functions under the distortion $\mathsf{T}_\varepsilon : \widetilde{\Omega} \to \widetilde{\Omega}$,

$$(\mathsf{T}_\varepsilon^* v)(\boldsymbol{x}) := v(\mathsf{T}_\varepsilon(\boldsymbol{x})) \quad \boldsymbol{x} \in \widetilde{\Omega} \quad \text{for} \quad v : \widetilde{\Omega} \to \mathbb{R} \;. \tag{4.147}$$

Lemma 4.21 (Estimates for pullback). *With constants depending only on Ω and the Lipschitz constant L_ε of ε the following estimates hold true:*

(PB_1) $\|\mathsf{T}_\varepsilon^* v\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}} \eqsim \|v\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}}$ *for all $\boldsymbol{z} \in \widetilde{\Omega}$, $v \in L^2(\widetilde{\Omega})$,*

(PB_2) $|\mathsf{T}_\varepsilon^* v|_{1,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}} \lesssim |v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})\cap\widetilde{\Omega}}$ *for all $\boldsymbol{z} \in \widetilde{\Omega}$, $v \in H^1(\widetilde{\Omega})$,*

(PB_3) $\|(\mathsf{Id} - \mathsf{T}_\varepsilon^*)v\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})\,|v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}$ *for all $\boldsymbol{z} \in \Omega^e$ and $v \in H^1(\widetilde{\Omega})$.*

Proof. The assertions (PB_1) and (PB_2) follow from (4.146), $\|D\mathsf{T}_\varepsilon\|_{\infty,\widetilde{\Omega}}, \left\|D\mathsf{T}_\varepsilon^{-1}\right\|_{\infty,\widetilde{\Omega}} \lesssim 1$, Theorem 4.19, ($T_6$), the chain rule and the transformation formula for integrals.

To show (PB_3) we resort to convolution with a mollifier $\rho \in C^\infty(\mathbb{R}^3)$ that satisfies $\rho \geq 0$, $\text{supp}(\rho) \subset B_1(0)$, and $\int_{\mathbb{R}^3} \rho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$. Writing $\rho_\nu(\boldsymbol{x}) := \nu^{-3}\rho(\boldsymbol{x}/\nu)$, $\nu > 0$, we define for some function $\xi : \widetilde{\Omega} \to \mathbb{R}^+$

$$(\mathsf{M}_\xi v)(\boldsymbol{x}) := \int_{\mathbb{R}^3} v(\boldsymbol{x} - \boldsymbol{y})\rho_{\xi(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{y} \;, \quad v \in L^1(\mathbb{R}^3) \;. \tag{4.148}$$

Since $\|\rho_\nu\|_{0,\mathbb{R}^3}^2 = \nu^{-3}\,\|\rho\|_{0,\mathbb{R}^3}^2$, the Cauchy-Schwarz inequality yields

$$|(\mathsf{M}_\xi v)(\boldsymbol{x})| \leq \left\|\rho_{\xi(\boldsymbol{x})}\right\|_{0,\mathbb{R}^3} \|v\|_{0,B_{\xi(\boldsymbol{x})}(\boldsymbol{x})} \lesssim \xi(\boldsymbol{x})^{-3/2}\,\|v\|_{0,B_{\xi(\boldsymbol{x})}(\boldsymbol{x})} \;. \tag{4.149}$$

From now we set $\xi(\boldsymbol{x}) := L_D\delta\varepsilon(\boldsymbol{x})$ and, by (4.144), (4.149) and (δ_1), conclude for every $\boldsymbol{z} \in \Omega^e$

$$\|\mathsf{M}_\xi(\boldsymbol{x})\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}^2 \leq (\tau\varepsilon(\boldsymbol{z}))^3 \max_{\boldsymbol{z}\in B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} L_D\delta\varepsilon(\boldsymbol{x})^{-3} \lesssim \|v\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}^2 \;. \tag{4.150}$$

The properties of ρ ensure that M_ξ preserves constants, so that we obtain by a scaling argument and the Bramble-Hilbert lemma [47, Lemma 4.3]:

$$\begin{aligned} \|v - \mathsf{M}_\xi v\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} &= \inf_{c\in\mathbb{R}} \|(v - c) - \mathsf{M}_\xi(v - c)\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \\ &\lesssim \inf_{c\in\mathbb{R}} \|v - c\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \beta\varepsilon(\boldsymbol{z})\,|v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \;, \end{aligned} \tag{4.151}$$

for any $v \in H^1(\widetilde{\Omega})$. Fixing $v \in H^1(\widetilde{\Omega})$ and $\boldsymbol{z} \in \Omega^e$ we continue with the triangle inequality

$$\|v - \mathsf{T}_\varepsilon^* v\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \leq \|v - \mathsf{M}_\xi v\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} + \|(\mathsf{Id} - \mathsf{T}_\varepsilon^*)\mathsf{M}_\xi v\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} + \|\mathsf{T}_\varepsilon^*(\mathsf{M}_\xi - \mathsf{Id})v\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \ . \quad (4.152)$$

By means of (4.151) and Theorem 4.21, (PB_1) the first and last term can be estimated by $\lesssim \varepsilon(\boldsymbol{z})\,|v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}$. Concerning the middle term we appeal to the mean value theorem applied to $w := \mathsf{M}_\xi v$ and, by Theorem 4.19, (T_3), (4.146), get for $\boldsymbol{x} \in B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})$

$$|w(\boldsymbol{x}) - w(\mathsf{T}_\varepsilon(\boldsymbol{x}))| \leq \|\mathbf{grad}\, w\|_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})}\ \|\boldsymbol{x} - \mathsf{T}_\varepsilon(\boldsymbol{x})\| \lesssim \|\mathbf{grad}\,\mathsf{M}_\xi v\|_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \ .$$

Since $\mathbf{grad}$ commutes with convolution, the maximum norm of $\mathbf{grad}\,\mathsf{M}_\xi v$ can be estimated as in (4.149) above:

$$\|\mathbf{grad}\,\mathsf{M}_\xi v\|_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{-3/2}\,\|\mathbf{grad}\,v\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \ .$$

Ultimately this yields

$$\|(\mathsf{Id} - \mathsf{T}_\varepsilon^*)\mathsf{M}_\xi v\|^2_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^3\,\|\mathbf{grad}\,\mathsf{M}_\xi v\|^2_{\infty,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \|\mathbf{grad}\,v\|^2_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \ ,$$

and the assertion (PB_3) when plugged into (4.152). □

Following [47, Sect. 5.2], we now outline the key idea of regularization by mollification. We employ a mollifier of oder $k_{\max} \in \mathbb{N}_0 := 6$, that is, a function $\rho \in C^\infty(\mathbb{R}^3)$ with $\operatorname{supp}(\rho) \subset B_1(0)$, and [47, Equ. (4.1)],

$$\int_{\mathbb{R}^3} \boldsymbol{y}^{\mathbf{r}}\rho(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \begin{cases} 1 & \text{if } \mathbf{r} = 0 \ , \\ 0 & \text{else,} \end{cases} \quad (4.153)$$

for every multi-index $\mathbf{r} \in \mathbb{N}_0^3$ with $|\mathbf{r}| \leq k_{\max}$. This property leads to the preservation of polynomials of degree up to $k_{\max}$ under convolution with ρ_ν. Analogously to (4.148) we define the mollification

$$(\mathsf{E}v)(\boldsymbol{x}) := \int_{\mathbb{R}^3} v(\boldsymbol{y})\rho_{\delta\varepsilon(\boldsymbol{x})}(\boldsymbol{x} - \boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \ , \quad \boldsymbol{x} \in \Omega^e \ , \quad v \in L^1(\widetilde{\Omega}) \ . \quad (4.154)$$

From [47, Lemma 5.3] we learn that for every $\boldsymbol{z} \in \Omega^e$ and integers $0 \leq m \leq \ell$, with $\ell, m \leq k_{\max} + 1$,

$$|\mathsf{E}v|_{\ell,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{m-\ell}\,|v|_{m,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \quad \forall v \in H^\ell(\widetilde{\Omega}) \ , \quad (4.155)$$

$$|(\mathsf{Id} - \mathsf{E})v|_{m,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{\ell-m}\,|v|_{\ell,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \quad \forall v \in H^\ell(\widetilde{\Omega}) \ . \quad (4.156)$$

The composition of mollificaton and distortion pullback yields the regularizing operator

$$\mathsf{J} := \mathsf{E} \circ \mathsf{T}_\varepsilon^* : L^1(\widetilde{\Omega}) \to C^\infty(\Omega^e) \ . \tag{4.157}$$

In light of (4.145) it is immediate that

$$\boxed{v_{|\Upsilon_D} = 0 \quad \Rightarrow \quad \mathsf{J}v_{|\Gamma_D} = 0} \ . \tag{4.158}$$

Using (4.155) for $m = \ell = 0$, (4.156) for $m = 0, \ell = 1$, and Theorem 4.21, (PB$_3$), for any $\boldsymbol{z} \in \Omega^e$ we find the bound

$$\begin{aligned} \|(\mathsf{Id} - \mathsf{J})v\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} &\leq \|(\mathsf{Id} - \mathsf{E})v\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} + \|\mathsf{E}(\mathsf{Id} - \mathsf{T}_\varepsilon^*)v\|_{0,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \\ &\lesssim \varepsilon(\boldsymbol{z})\, |v|_{1,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} + \|(\mathsf{Id} - \mathsf{T}_\varepsilon^*)v\|_{0,B_{\tau\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})\, |v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \ . \end{aligned} \tag{4.159}$$

By means of (4.155) for $m = 0, 1$ we get for any $1 \leq \ell \leq k_{max} + 1$

$$|\mathsf{J}v|_{\ell,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{-\ell}\, \|\mathsf{T}_\varepsilon^* v\|_{0,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \ , \quad \forall v \in H^1(\widetilde{\Omega}) \ , \tag{4.160}$$

$$|\mathsf{J}v|_{\ell,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim \varepsilon(\boldsymbol{z})^{1-\ell}\, |\mathsf{T}_\varepsilon^* v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \ , \quad \forall v \in H^1(\widetilde{\Omega}) \ . \tag{4.161}$$

Further, (4.155) for $m = \ell = 1$ and Theorem 4.21, (PB$_3$) lead to

$$|\mathsf{J}v|_{1,B_{\alpha\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \lesssim |v|_{1,B_{\beta\varepsilon(\boldsymbol{z})}(\boldsymbol{z})} \quad \forall v \in H^1(\widetilde{\Omega}) \ . \tag{4.162}$$

The final step is inspired by [47, Sect. 3.1]. To build the desired quasi-interpolation operator we apply the perfectly local projection-based interpolation operators I_p : $H^6(\Omega) \to \mathcal{W}_p^0(\mathcal{T})$ from [52, Cor. 7.4] to the regularized function:

$$\boxed{\mathsf{Q}_p := \mathsf{I}_p \circ \mathsf{J} : L^1(\widetilde{\Omega}) \to \mathcal{W}_{p,\Gamma_D}^0(\mathcal{T})} \ . \tag{4.163}$$

We recall properties of I_p from [52, Sect. 7]. Firstly, it enjoys locality in the sense that

- $(\mathsf{I}_p v)(\boldsymbol{a}) = v(\boldsymbol{a})$ for every vertex $\boldsymbol{a}$ of the mesh $\mathcal{T}$,
- $(\mathsf{I}_p v)_{|e}$ is uniquely determined by $v_{|e}$ for every edge e,
- $(\mathsf{I}_p v)_{|F}$ depends only on $v_{|F}$ for every face F,
- and $(\mathsf{I}_p v)_{|T}$ exclusively relies on $v_{|T}$ for all tetrahedra $T \in \mathcal{T}$.

Obviously, if $v_{|F} = 0$, then $(\mathsf{I}_p v)_{|F} = 0$. As $\mathcal{T}$ was supposed to resolve Γ_D, applying I_p to a smooth function vanishing on Γ_D will result in an interpolant with the same property. This accounts for the range of Q_p stated in (4.163).

The locality of I_p comes at the price of poor stability. In [52, Cor. 7.4] the authors showed p-uniform local continuity of

$$\mathsf{I}_{p\,|T} : H^6(T) \to \mathcal{P}_p(\mathbb{R}^3) \ , \tag{4.164}$$

and an estimate of the form

$$ {}^{h_T}\!/_{p}\,|(\mathsf{Id}-\mathsf{I}_p)v|_{1,T} + \|(\mathsf{Id}-\mathsf{I}_p)v\|_{0,T} \lesssim ({}^{h_T}\!/_{p})^6\,\|v\|_{6,T} \quad \forall v \in H^6(T)\ , \tag{4.165}$$

where the constants depends merely on the shape regularity measure of the tetrahedron $T \in \mathcal{T}$. Since, $\mathsf{J}v \in C^\infty(\Omega^e)$, the tight smoothness requirements of I_p can be accommodated. This is the main rationale behind using the regularizer J.

H^1-Stability of Q_p is straightforward from (4.165), (4.161), and the finite overlap property from Theorem 4.20. To begin with, we get

$$\begin{aligned} \|\mathsf{Q}_p u\|_{0,T} &= \|\mathsf{I}_p(\mathsf{J}u)\|_{0,T} \lesssim \|\mathsf{J}u\|_{0,T} + ({}^{h_Y}\!/_{p})^6\,\|\mathsf{J}u\|_{6,T} \lesssim \|u\|_{0,U_T}\ , \\ |\mathsf{Q}_p u|_{1,T} &= |\mathsf{I}_p(\mathsf{J}u)|_{1,T} \lesssim |\mathsf{J}u|_{1,T} + ({}^{h_Y}\!/_{p})^5\,\|\mathsf{J}u\|_{6,T} \lesssim |u|_{1,U_T}\ , \end{aligned} \tag{4.166}$$

where $U_T := \bigcup\{B_{\beta\varepsilon(\boldsymbol{x})}(\boldsymbol{x}),\ \boldsymbol{x} \in T\}$ is a local neighborhood of T. Local approximation estimates can be deduced from (4.159), (4.161), and (4.165):

$$\begin{aligned} \|(\mathsf{Id}-\mathsf{Q}_p)u\|_{0,T} &\le \|(\mathsf{Id}-\mathsf{J})u\|_{0,T} + \|(\mathsf{Id}-\mathsf{I}_p)\mathsf{J}u\|_{0,T} \\ &\lesssim {}^{h_T}\!/_{p}\,|u|_{1,U_T} + ({}^{h_T}\!/_{p})^6\,\|\mathsf{J}u\|_{6,T} \lesssim {}^{h_T}\!/_{p}\,|u|_{1,U_T}\ . \end{aligned} \tag{4.167}$$

Squaring and adding both (4.166) and (4.167) establishes global stability and approximation properties of our quasi-interpolation Q_p.

Theorem 4.22 (Quasi-Interpolation operator). *The operators* $\mathsf{Q}_p : L^1(\widetilde{\Omega}) \to \mathcal{W}_p^0(\mathcal{T}) \subset H^1_{\Gamma_D}(\Omega)$ *satisfy*

(Q₁) $\|\mathsf{Q}_p u\|_{0,\Omega} \lesssim \|u\|_{0,\widetilde{\Omega}}$ *for all* $u \in L^2(\widetilde{\Omega})$,

(Q₂) $|\mathsf{Q}_p u|_{1,\Omega} \lesssim |u|_{1,\widetilde{\Omega}}$ *for all* $u \in H^1(\widetilde{\Omega})$,

(Q₃) $\left\|\varepsilon^{-1}(\mathsf{Id}-\mathsf{Q}_p)u\right\|_{0,\Omega} \lesssim |u|_{1,\widetilde{\Omega}}$ *for all* $u \in H^1(\widetilde{\Omega})$,

with constants depending only on Ω, Γ_D, *and the shape regularity measure* $\rho(\mathcal{T})$.

4.3 Proof of Theorem 1.3

With local commuting projectors Π_p^1 from Section 4.1 and stable quasi-interpolation operator Q_p from Section 4.2 at our disposal, the construction and analysis of p-uniformly stable discrete regular decompositions of $\boldsymbol{\mathcal{W}}^1_{\Gamma_D}(\mathcal{T})$ runs rather parallel to the lowest-order case presented in Section 3.2.

We fix $\mathbf{v}_p \in \boldsymbol{\mathcal{W}}^1_{p,\Gamma_D}(\mathcal{T}) \subset \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$ and consider its regular decomposition supplied by Theorem 2.1:

$$\mathbf{v}_p = \mathbf{z} + \mathbf{grad}\,\varphi \quad \mathbf{z} \in \boldsymbol{H}^1(\mathbb{R}^3),\ \mathbf{z}_{|\Upsilon_D} \equiv 0\ , \quad \varphi \in H^1_{\Gamma_D}(\Omega)\ , \tag{4.168}$$

with norm bounds

$$\|\mathbf{z}\|_{0,\mathbb{R}^3} \lesssim \|\mathbf{v}_p\|_{0,\Omega} \ , \quad |\mathbf{z}|_{1,\mathbb{R}^3} \lesssim \|\mathbf{v}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \ , \quad \|\varphi\|_{1,\Omega} \lesssim \|\mathbf{v}_p\|_{0,\Omega} \ . \tag{4.169}$$

None of the constants depends on $\mathbf{v}_p$. Since $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_p$, that is, $\mathbf{z}$ has a piecewise polynomial $\mathbf{curl}$, Theorem 4.16 ensures that $\Pi_p^1\mathbf{z}$ is well-defined. In addition, for every $T \in \mathcal{T}$ we have $\mathbf{grad}\,\varphi_{|T} = \mathbf{v}_{p\,|T} - \mathbf{z}_{|T} \in \boldsymbol{H}^1(T)$, which implies $\varphi_{|T} \in H^2(T)$. Hence, φ possesses enough local regularity to render also $\Pi_p^0\varphi$ well-defined. This permits us to rely on the commuting diagram property of Theorem 4.6 when letting Π_p^1 act on $\mathbf{v}_p$:

$$\mathbf{v}_p = \Pi_p^1\mathbf{v}_p = \Pi_p^1\mathbf{z} + \mathbf{grad}\,\Pi_p^0\varphi \ .$$

In order to obtain a contribution in $\boldsymbol{H}^1_{\Gamma_D}(\Omega)$, we insert a boundary-aware quasi-interpolant to generate the regular part $\mathbf{z}_p$ of the decomposition ((III)):

$$\mathbf{v}_p = \Pi_p^1 \underbrace{\mathsf{Q}_p\mathbf{z}}_{=:\mathbf{z}_p} + \Pi_p^1(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z} + \mathbf{grad}\,\underbrace{\Pi_p^0\varphi}_{=:\varphi_p} \ , \qquad \begin{array}{l} \mathbf{z}_p \in \boldsymbol{\mathcal{V}}^0_{p,\Gamma_D}(\mathcal{T}) \ , \\ \varphi_p \in \mathcal{W}^0_{\Gamma_D}(\mathcal{T}) \ . \end{array} \tag{4.170}$$

Writing $\widetilde{\mathbf{v}}_p := \Pi_p^1(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z} \in \boldsymbol{\mathcal{W}}^1_{\Gamma_D}(\mathcal{T})$, we have split $\mathbf{v}_p \in \boldsymbol{\mathcal{W}}^1_{\Gamma_D}(\mathcal{T})$ as

$$\mathbf{v}_p = \Pi_p^1\mathbf{z}_p + \widetilde{\mathbf{v}}_p + \mathbf{grad}\,\varphi_p \ . \tag{(III)}$$

Next, we investigate the stability of this splitting, bounding norms of its terms by norms of $\mathbf{v}_p$.

❶ Estimating norms of $\mathbf{z}_p = \mathsf{Q}_p\mathbf{z}$ based on Theorem 4.22 is straightforward

$$\text{Theorem 4.22, (Q}_1) \quad \Rightarrow \quad \|\mathbf{z}_p\|_{0,\Omega} \lesssim \|\mathbf{z}\|_{0,\Omega} \lesssim \|\mathbf{v}_p\|_{0,\Omega} \ , \tag{4.171}$$

$$\text{Theorem 4.22, (Q}_2) \quad \Rightarrow \quad |\mathbf{z}_p|_{1,\Omega} \lesssim |\mathbf{z}|_{1,\Omega} \lesssim \|\mathbf{v}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \ . \tag{4.172}$$

❷ Interpolation error estimates from Theorem 4.16 for Π_p^1 give bounds for $\widetilde{\mathbf{v}}_p$, local ones first: for any tetrahedron $T \in \mathcal{T}$

$$\begin{aligned} \|\widetilde{\mathbf{v}}_p\|_{0,T} &\lesssim \left\|(\mathsf{Id} - \Pi_p^1)(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}\right\|_{0,T} + \|(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}\|_{0,T} \\ &\lesssim (1 + \log(p+1))^{3/2} \frac{h_T}{p+1} \, |(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}|_{1,T} + \frac{h_T}{p+1} \, |\mathbf{z}|_{1,T} \\ &\lesssim (1 + \log(p+1))^{3/2} \frac{h_T}{p+1} \, |(\mathsf{Id} - \mathsf{Q}_p)\mathbf{z}|_{1,T} \ , \end{aligned} \tag{4.173}$$

which implies after squaring and summing that

$$\left(\sum_{T \in \mathcal{T}} \left\| \frac{p+1}{h_T} \widetilde{\mathbf{v}}_p \right\|_{0,T}^2 \right)^{1/2} \lesssim (1 + \log(p+1))^{3/2} \, \|\mathbf{v}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \ . \tag{4.174}$$

❸ Norm estimates for φ_p rely on those for $\mathbf{z}_p$ and the local interpolation error estimate of Theorem 4.16:

$$\begin{aligned} |\varphi_p|_{1,T} &\leq \|\mathbf{v}_p\|_{0,T} + \left\|\Pi_p^1 \mathbf{z}\right\|_{0,T} \\ &\lesssim \|\mathbf{v}_p\|_{0,T} + \|\mathbf{z}\|_{0,T} + (1+\log(p+1))^{3/2} \frac{h_T}{p} |\mathbf{z}|_{1,T} \ . \end{aligned} \tag{4.175}$$

As a consequence of (4.169) we emd up with

$$|\varphi_p|_{1,\Omega} \lesssim \|\mathbf{v}_p\|_{0,\Omega} + \max_{T\in\mathcal{T}} \left\{ (1+\log(p+1))^{3/2} \frac{h_T}{p+1} \right\} \|\mathbf{curl}\,\mathbf{v}_p\|_{0,\Omega} \ . \tag{4.176}$$

Thus we are done, because Theorem 1.3 merely collects the estimates (4.171), (4.172), (4.174), and (4.176).

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