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Abstract

A variational principle for the shift of eigenvalues caused by a domain perturbation is derived for a class of operators which includes elliptic differential operators. This result allows the direct extension of asymptotic formulae from simple eigenvalues to repeated ones. Some interesting examples for the Laplacian are worked out explicitly for the following types of perturbation: excision of a small hole, local change of conductivity, small boundary deformation.

Keywords: eigenvalue perturbation, repeated eigenvalue, elliptic differential operator, asymptotic expansion, small inclusion, shape deformation.
Mathematics subject classification: 35P15, 35J20, 35J25, 35C20.

1. Introduction

Many asymptotic formulae for the shift of eigenvalues of differential operators caused by small domain perturbations have been obtained in the case of multiplicity one (see for instance [1] and references therein, [2, Chapter 9], [3], [4]). The generalization of such expressions to higher multiplicities often requires non-trivial calculations and is restricted to specific cases (see for example [5], [6], [7, Theorem 2.5.8]). Moreover, such an effort is usually considered unnecessary due to genericity results for simple eigenvalues (see [8], [9], [10]). Nonetheless, higher multiplicities appear in many natural situations; for example the Laplacian has non-simple spectrum whenever the domain presents some symmetries. In this paper we derive a tool to study the behavior and properties of repeated eigenvalues for general types of perturbations. Namely, the main result consists in a variational minimum principle (Theorem 2.5) which allows the direct extension of asymptotic formulae which are valid for simple eigenvalues to non-simple ones.

More in detail the topics of the paper are subdivided as follows. In Section 2 we introduce the abstract framework and the linear operators which will be considered; they are required to satisfy the key assumptions of genericity of simple eigenvalues and their stability. Then the main variational principle is derived by a technique which involves a double domain perturbation at asymptotically different speeds. In Section 3 we recall some results that show that second order elliptic operators satisfy the spectral stability and genericity assumptions. Finally in Section 4 we consider some domain perturbations of particular interest in applications: grounded inclusions, conductivity inclusions, and boundary deformations. We show how the general variational asymptotic formula applies in each of these cases and derive new interesting properties for the Dirichlet or Neumann Laplacian. We refer to Section 4.5 for a concise summary of the results obtained.

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The reader interested in grasping the intuition behind the general variational principle might first read the derivation of the asymptotic expansion for the perturbation caused by the excision of a small hole in the proof of Proposition 4.5, which does not depend on the concepts introduced in Section 2.

1.1. Notation used

In this paper we adopt the following conventions:

- we say that X is a domain if $X \subseteq \mathbb{R}^d$ is open, bounded, and connected;
- we denote as Ω an arbitrary domain with Lipschitz boundary;
- given a compact self-adjoint linear operator $L(\Omega)$, we will usually denote as λ an eigenvalue of $L(\Omega)$, as m its multiplicity, and as $u_1, \dots, u_m$ an arbitrary basis of the eigenspace of λ;
- given an arbitary operator with discrete spectrum, whenever we will count its eigenvalues, we will do so from lowest to highest and according to their multiplicity;
- given f a real valued function defined on a set S, we indicate as $\operatorname{argmin}_{x\in S} f(x)$ the set of x's which minimize f (or the single x when the minimum is unique).

Let us also briefly recall three types of convergence of domains. We say that Ω_ε converges to Ω as $\varepsilon \to 0$ in:

- Hausdorff distance, if $d_H(\Omega_\varepsilon, \Omega) \to 0$, where
$$d_H(A, B) := \max\{\sup_{a\in A}\inf_{b\in B} |a-b|,\ \sup_{b\in B}\inf_{a\in A} |a-b|\};$$
- measure, if $|\Omega_\varepsilon \triangle \Omega| \to 0$, where $\triangle$ indicates symmetric difference and $|\cdot|$ the Lebesgue measure;
- C^k-topology, if there is a family $(\phi_\varepsilon)_\varepsilon$ of C^k-diffeomorphisms such that
$$\Omega + \phi_\varepsilon(\Omega) = \Omega_\varepsilon \quad \text{and} \quad |\phi_\varepsilon|_{C^k} \xrightarrow{\varepsilon\to 0} 0. \tag{1}$$

We recall some relationships between these types of convergence which are relevant to us. C^k-convergence implies Hausdorff convergence, since if (1) holds then $d_H(\Omega_\varepsilon, \Omega) \le |\phi_\varepsilon(\Omega)|_{C^0} \xrightarrow{\varepsilon\to 0} 0$. If the sets considered have Lipschitz and connected boundaries, Hausdorff convergence implies measure convergence, since in this case $|\Omega_\varepsilon \triangle \Omega| \le d_H(\Omega_\varepsilon, \Omega) \min\{|\Omega_\varepsilon|, |\Omega|\}$. Although the converse result fails in general, if the converging sequence's boundaries have uniformly bounded Lipschitz constants, then by a local patching argument it can be easily shown that also measure convergence implies Hausdorff convergence.

Therefore, although we will state all our results referring to Hausdorff convergence, they can be adapted to other types of convergence whenever the domains under consideration have enough regularity.

2. Abstract variational formulation

Let $L(\Omega)$ be a compact self-adjoint linear operator on $\mathrm{L}^2(\Omega)$ with domain $H(\Omega)$. Recall that the eigenvalues and eigenfunctions of $L(\Omega)$ can be rewritten respectively as minima and minimizers of a quadratic functional $F(\Omega) : H(\Omega) \to \mathbb{R}$. More in detail, if we write $\lambda_1 \leq \lambda_2 \leq \dots$ for the eigenvalues of $L(\Omega)$ and $u_1, u_2, \dots$ for some corresponding orthonormal eigenfunctions, we have that for $F(\Omega)u := \langle L(\Omega)u, u\rangle_H$ it holds

$$u_n = \operatorname*{argmin}_{u \in S_n} F(\Omega)u, \qquad \lambda_n = F(\Omega)u_n, \tag{2}$$

where $S_n = \Big\{u \in H(\Omega) : u \perp \{u_1, \dots, u_{n-1}\}\,,\ |u|_{\mathrm{L}^2(\Omega)} = 1\Big\}$.

However, there is an ambiguity in the choice of eigenfunctions which is particularly relevant to us: the sequence of eigenfunctions is univocally determined only up to eigenspaces. That is if $\lambda_n = \dots = \lambda_{n+m}$, any choice of orthonormal eigenfunctions in the linear space spanned by $\{u_n, \dots, u_{n+m}\}$ is still a basis. Our minimum principle will select the right basis for the problem considered, whenever possible and up to a predetermined asymptotic error.

The notations and assumptions we adopt are the following.

Assumption 2.1. We suppose to have a family of domains $(\Omega_\varepsilon)_{0<\varepsilon\leq\varepsilon_0}$ such that $\Omega_\varepsilon \to \Omega$ in Hausdorff distance as $\varepsilon \to 0$.

We also assume that for any couple of Lipschitz domains $\Omega, \tilde{\Omega}$ such that $\Omega \subseteq \tilde{\Omega}$, there is a continuous restriction operator $\mathscr{R} : H(\tilde{\Omega}) \to H(\Omega)$, and for any $\delta > 0$ there is a continuous extension operator $\mathscr{E} : H(\Omega) \to H(\tilde{\Omega})$ such that $\mathscr{R} \circ \mathscr{E}$ is the identity and $|\mathscr{E}u| \leq |u| + \delta$. We will not use the notation $\mathscr{R}, \mathscr{E}$ explicitly; if $\tilde{u} \in H(\tilde{\Omega})$, we will denote still as $\tilde{u}$ the restriction $\mathscr{R}\tilde{u}$, while if $u \in H(\Omega)$ we will denote still as u any extension $\mathscr{E}u$.

Assumption 2.2. We suppose that the spectrum is continuous with respect to Hausdorff convergence. More precisely, if λ_ε is an eigenvalue of $L(\Omega_\varepsilon)$ and u_ε is an eigenfunction associated to λ_ε, then there exists λ an eigenvalue of $L(\Omega)$ and u an eigenfunction of λ such that $\lambda_\varepsilon \to \lambda$ and $u_\varepsilon \to u$ in norm in $H(E)$ as $\varepsilon \to 0$, for any Lipschitz domain E that contains Ω and Ω_ε for all ε.

We will often say that λ_ε is perturbed from λ if $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$.

Assumption 2.3. We suppose that the spectrum of $L(\Omega)$ is "generically" simple. By this we mean that one can always find a domain arbitrarily near to Ω in Hausdorff distance whose eigenvalues are all simple.

Assumption 2.4. We suppose to already know an asymptotic expansion for simple eigenvalues. In particular we assume that if λ is a simple eigenvalue of $L(\Omega)$ with associated eigenfunction u, then

$$\lambda_\varepsilon - \lambda = f(\varepsilon, u) + r(\varepsilon),$$

where $r(\varepsilon) = o(f(\varepsilon, u))$ as $\varepsilon \to 0$, and f is a known function. We also require f to be continuous.

Under these assumptions we can prove the following result.

Theorem 2.5 (Variational minimum principle)**.** *Let λ be an eigenvalue of $L(\Omega)$ of multiplicity m, and let $\lambda_{\varepsilon,1} \leq \dots \leq \lambda_{\varepsilon,m}$ be all the eigenvalues of $L(\Omega_\varepsilon)$ perturbed from λ. Then for any $n \in \{1, \dots, m\}$ it holds*

$$\lambda_{\varepsilon,n} - \lambda = f(\varepsilon, v_n) + O(r(\varepsilon)),$$

where

$$v_n \in \operatorname*{argmin}_{v \in S_n} f(\varepsilon, v), \tag{3}$$

with

$$S_n = \Big\{ v \in H(\Omega) : v \perp \{v_1, \dots, v_{n-1}\}\,,\ |v|_{\mathrm{L}^2(\Omega)} = 1 \Big\}. \tag{4}$$

Proof. Step 1. For $\delta \in (0, \delta_0)$ let Ω_δ be a perturbation of Ω not farther than δ in Hausdorff distance from Ω, and such that λ splits into $\lambda_{\delta,1} < \cdots < \lambda_{\delta,m}$ simple eigenvalues with corresponding associated eigenfunctions $u_{\delta,1}, \dots, u_{\delta,m}$. Existence of such perturbations is guaranteed by Assumption 2.3. Let $\Omega_{\varepsilon,\delta}$ be an arbitrary set not farther than ε from Ω_δ in Hausdorff distance, and let $\lambda_{\varepsilon,\delta,1}, \dots, \lambda_{\varepsilon,\delta,m}$ indicate the eigenvalues of $L(\Omega_{\varepsilon,\delta})$ perturbed respectively from $\lambda_{\delta,1}, \dots, \lambda_{\delta,m}$. Then from Assumption 2.4

$$\lambda_{\varepsilon,\delta,n} - \lambda_{\delta,n} = f(\varepsilon, u_{\delta,n}) + r(\varepsilon).$$

Choosing δ appropriately small with respect to ε so that $\lambda_{\varepsilon,n} - \lambda_{\varepsilon,\delta,n} = O(r(\varepsilon))$ and $\lambda_{\delta,n} - \lambda = O(r(\varepsilon))$, we have that

$$\lambda_{\varepsilon,n} - \lambda = \lim_{\delta \to 0} (\lambda_{\varepsilon,n} - \lambda_{\varepsilon,\delta,n} + \lambda_{\varepsilon,\delta,n} - \lambda_{\delta,n} + \lambda_{\delta_n} - \lambda) = \lim_{\delta \to 0} (\lambda_{\varepsilon,\delta,n} - \lambda_{\delta,n}) + O(r(\varepsilon)).$$

By Assumption 2.2 as $\delta \to 0$ we have $u_{\delta,n} \to u_n$, where u_n is a certain eigenfunction of Ω with λ its associated eigenvalue. By the continuity of f we thus have

$$\lambda_{\varepsilon,n} - \lambda = f(\varepsilon, u_n) + r(\varepsilon). \tag{5}$$

Step 2. To retrieve the variational characterization of u_n, recall that we assumed $\lambda_{\varepsilon,1} \le \cdots \le \lambda_{\varepsilon,m}$. Therefore, for $\lambda_{\varepsilon,1}$ to be the smallest eigenvalue, u_1 must be a minimizer of the right hand side of (5) among all eigenfunctions of λ of norm 1, that is $u_1 \in V_1$. Considering now $\lambda_{\varepsilon,2}$, we have that u_2 must also be in the eigenspace of λ, but for λ to have multiplicity at least m, u_2 must be linearly independent with respect to u_1; thus we take $u_2 \in V_2$. With the same reasoning for $\lambda_{\varepsilon,3}, \dots, \lambda_{\varepsilon,m}$, we can show that (3) must hold for every n. □

Notice that we can recover some partial information also on the eigenfunctions' perturbation as in the following remark.

Remark 2.6. With the same notation of Theorem 2.5, let $u_{\varepsilon,1}, \dots u_{\varepsilon,m}$ be eigenfunctions associated to $\lambda_{\varepsilon,1}, \dots, \lambda_{\varepsilon,m}$. Since by Assumption 2.2 we have that $u_{\varepsilon,n} \to u_n$ for some eigenfunction u_n which is associated to λ, we will have that for ε small enough, an approximation up to an $O(r(\varepsilon))$ error of u_n will be given by (3). ■

2.1. Asymptotic formulae involving a bilinear function

It is useful to consider more carefully the case where the expression in the asymptotic formula admits a separation of variables as

$$f(\varepsilon, u) = E(\varepsilon) b(u, u), \tag{6}$$

with $E : X \to \mathbb{R}$ and $b : H \times H \to \mathbb{R}$ a symmetric bilinear form. This happens for many useful types of domain perturbations (see Section 4). The advantage of this case is that the minimizer of (3) can be easily computed as follows. Choosing $u_1, \dots, u_m$ an arbitrary orthonormal basis of the eigenspace of λ, condition (3) can be rewritten as

$$v_n \in E(\varepsilon) \operatorname{argmin} \Big\{ a \cdot Ba : a \in \mathbb{R}^m,\ |a| = 1,\ a_1 u_1 + \cdots + a_m u_m \perp \{v_1, \dots, v_{n-1}\} \Big\}, \tag{7}$$

where B is a symmetric matrix with elements

$$B_i^j := b(u_i, u_j). \tag{8}$$

Then by diagonalizing B we obtain the following result.

Corollary 2.7. *If f can be rewritten as in (6), a minimum and a minimizer of (3) are respectively the n-th eigenvalue of B and $a_{1,n}u_1 + \cdots + a_{m,n}u_m$, where $(a_{1,n}, \ldots, a_{m,n})$ is the normalized eigenfunction of B associated to its n-th eigenvalue.*

Notice that if the bilinear form b has the further decomposition

$$b(u_i, u_j) = l(u_i)l(u_j) \quad \forall i, j, \tag{9}$$

where $l : H \to \mathbb{R}$ is linear, an easy computation shows that the first $m-1$ eigenvalues of B are zero and the m-th one is $l(u_1)^2 + \cdots + l(u_m)^2$. Therefore we have the following result.

Corollary 2.8. *If the bilinear form b can be rewritten as in (9), then*

$$\lambda_{\varepsilon,n} - \lambda = \begin{cases} E(\varepsilon) \displaystyle\sum_{n=1}^{m} l(u_n)^2 + r(\varepsilon) & \text{if } n = m, \\ r(\varepsilon) & \text{if } n < m. \end{cases}$$

Moreover, by Remark 2.6, the $O(r(\varepsilon))$-best approximation of $u_{\varepsilon,m}$ in the eigenspace of λ is given by

$$\frac{\sum_{n=1}^{m} l(u_n)u_n}{\sqrt{\sum_{n=1}^{m} l(u_n)^2}}.$$

3. Stability and simplicity of the spectrum of elliptic operators

We recall hereafter some results which show that eigenvalues of linear elliptic operators are stable under Hausdorff convergence of the underlying domains, and that they are generically simple. In particular, from these results we will have that Assumptions 2.2-2.3 hold. For simplicity we restrict our attention to second order operators, but the same results for higher order operators can be found in the reference provided in the text.

Let L be a second order elliptic differential operator defined as

$$L(\Omega)u := -\nabla \cdot (A\nabla u) + b \cdot \nabla u + c,$$

where $A : \mathbb{R}^{d\times d} \to \mathbb{R}$, $b : \mathbb{R}^d \to \mathbb{R}$, $c : \mathbb{R} \to \mathbb{R}$ are smooth. Let a be the associated coercive, continuous, and bounded bilinear form defined as

$$a_\Omega(u, v) := \int_\Omega A\nabla u \cdot \nabla v + (b \cdot \nabla u)v + cuv, \qquad \forall u, v \in \mathrm{H}^1(\Omega).$$

We say that λ is an eigenvalue of Ω with associated eigenfunction $u \in V(\Omega)$ if

$$a_\Omega(u, v) = \lambda \int_\Omega uv, \qquad \forall v \in V(\Omega),$$

where $V(\Omega) = \mathrm{H}^1_0(\Omega)$ in the case of homogeneous Dirichlet boundary conditions, or $V(\Omega) = \mathrm{H}^1(\Omega)/\mathbb{R}$ in the case of homogeneous Neumann boundary conditions. From standard results in spectral theory, we know that the eigenvalues of any Lipschitz domain have finite multiplicity and can be arranged in a non-decreasing sequence, where each eigenvalue is repeated as many times as its multiplicity; we also assume that the associated eigenfunctions are orthonormal in L^2. With these conventions we can state the following stability result.

Theorem 3.1. *For any $n \in \mathbb{N}$, Let λ be the n-th eigenvalue of $L(\Omega)$ and λ_ε be the n-th eigenvalue of $L(\Omega_\varepsilon)$ for a certain $\varepsilon \in X$. Then there exists a constant C, which depends only on n, on the Lipschitz constants of Ω, on the dimension d, and on the coercivity, continuity, boundedness constants of a, such that*

$$|\lambda_\varepsilon - \lambda| \leq C d_H(\Omega_\varepsilon, \Omega).$$

A proof of this result can be obtained as a consequence of the theory of transition operators and its applications (see [11] for a survey of the technique). In what follows we outline another approach which relies on stability results for boundary value problems.

Outline of the proof. We adapt the argument from [12, Section 4.4] to our case. Let E be an arbitary Lipschitz domain containing $\Omega_\varepsilon \cup \Omega$ and consider u_ε extended to the whole E. Let u be the orthogonal projection of u_ε from $V(E)$ onto $V(\Omega)$. Let $\overline{u}$ be the unique solution in $V(\Omega)$ of

$$a(\overline{u}, v) = \lambda \int uv, \qquad \forall v \in V(\Omega).$$

From [13, Inequality (3.2)] we have that

$$|\overline{u} - u|_{\mathrm{H}^1(E)} \leq C |\lambda u|_{\mathrm{L}^2(E)} |\lambda u|_{V(E)'} d_H(\Omega, \Omega_\varepsilon),$$

where C is a constant which depends only on the Lipschitz constant of Ω, and the constants involved in the continuity, boundedness and coercivity assumptions on a. Since $|u|_{V(E)'} \leq |u|_{\mathrm{L}^2(E)}$, and by Weyl's law there is a constant $\tilde{C}$ which depends only on the area of Ω and the dimension d such that $\lambda \leq \tilde{C} n^{d/2}$, then

$$|\overline{u} - u|_{\mathrm{H}^1(E)} \leq C \tilde{C} n^{2/d} |u|^2_{\mathrm{L}^2(E)} d_H(\Omega, \Omega_\varepsilon).$$

Then by [12, Equation (4.31) and Lemma 14], we obtain the estimate in the thesis. □

The following result concerning stability of eigenfunctions is a particular case of [14, Theorem 1.2 and subsequent discussion].

Theorem 3.2. *Let Ω_ε be a family of Lipschitz domains converging to Ω in Hausdorff distance as $\varepsilon \to 0$. Suppose that either one of the two following hypotheses holds:*

- *$\Omega_\varepsilon \subseteq \Omega$ for all ε,*
- *the Lipschitz constant of $\partial\Omega_\varepsilon$ are uniformly bounded.*

Let λ and $\tilde{\lambda}$ be the n-th eigenvalues respectively of Ω and $\tilde{\Omega}$ and let $u_{\varepsilon,1}, \ldots, u_{\varepsilon,m}$ be an orthonormal basis of the eigenspace of $\tilde{\lambda}$. Then there exists $u_1, \ldots, u_m$ an orthonormal basis of λ such that as $\varepsilon \to 0$ it holds

$$u_{\varepsilon,n} \to u_n \text{ in } \mathrm{H}^1(\Omega), \qquad \forall n \in \{1, \ldots, m\}.$$

We move on to the issue of genericity of simple eigenvalues.

Theorem 3.3. *There are smooth domains arbitrarily near to Ω in Hausdorff distance whose eigenvalues are all simple.*

This result has been proven in [8] for elliptic operators of arbitrary order on C^3-domains, and can be easily adapted to our case as follows. If $\partial\Omega$ is smooth, then we can find a smooth domain $\tilde{\Omega}$ whose eigenvalues are all simple and which is arbitrarily near to Ω in the C^3-topology. Since C^3-convergence implies Hausdorff convergence, we have the thesis. When $\partial\Omega$ is only Lipschitz, we can find a domain arbitrarily near to Ω in Hausdorff distance which has smooth boundary, reducing to the previous case.

Remark 3.4. In the case of the Laplacian, it has been proven in [15] that the perturbation can be chosen localized. This means that not only the perturbation can be chosen near in Hausdorff or C^k-distance, but also restricted to an arbitrarily small ball centered at an arbitrary point of Ω. ■

4. Applications to eigenvalues of the Laplacian

Definition 4.1. For this section we will use the following conventions:

- we say that λ is an eigenvalue of a domain X if it is an eigenvalue of the negative Laplacian on X, with either Dirichlet or Neumann conditions on ∂X;
- $(\Omega_\varepsilon)_\varepsilon$ denotes a family of Lipschitz domains perturbed from Ω. Although the nature of the perturbation may vary, we assume that $\Omega_\varepsilon \to \Omega$ in Hausdorff distance as $\varepsilon \to 0$.

Under these hypotheses Assumption 2.1 is fulfilled. Assumptions 2.2 and 2.3 have been shown to hold in the more general case considered in Section 3. Finally, in order to apply Theorem 2.5, we will verify Assumption 2.4 on a case by case basis.

4.1. Fundamental solutions and layer potentials

Before focusing on the particular domain perturbations considered, we recall some concepts which will be useful later.

Definition 4.2. If $\omega \in \mathbb{C}\setminus 0$, we define as fundamental solution for Helmholtz equation $\Delta u + \omega^2 u = 0$ the function

$$\Gamma_\omega(r) := \frac{\mathrm{i}}{4}\left(\frac{\omega r}{2\pi}\right)^{\frac{d}{2}-1} H^1_{\frac{d}{2}-1}(\omega r),$$

where H^1_n is the Hankel function of the first kind of order n (see [16, Chapter 9] for its definition). If $\omega = 0$, we define as fundamental solution for Laplace equation the function

$$\Gamma_0(r) := \begin{cases} -\dfrac{\log r}{2\pi} & \text{if } d = 2, \\ \dfrac{1}{|S^{d-1}| r^{d-2}} & \text{if } d \geq 3, \end{cases} \tag{10}$$

where S^{d-1} is the unit sphere of dimension $d-1$.

Recall that from the theory of Green's functions, for any $\omega \in \mathbb{C}$, we have that

$$(\Delta_x + \omega^2)\Gamma_\omega(|x-y|) = -\delta_y(x),$$

where δ_y is the Dirac delta distribution at y.

We introduce in the following definition two convolution operators, commonly called *layer potentials* in the theory of boundary value problems (for the latter we refer to [17]).

Definition 4.3. Given $\phi \in \mathrm{L}^2(\partial\Omega)$, we define:

$$\text{(Single layer potential)} \qquad \mathcal{S}_\Omega^\omega[\phi](x) := \int_{\partial\Omega} \Gamma_\omega(x,y)\phi(y)\,\mathrm{d}\sigma(y) \qquad \text{for } x \in \mathbb{R}^2,$$

$$\text{(Neumann-Poincaré operator)} \qquad \mathcal{K}_\Omega^\omega[\phi](x) := \int_{\partial\Omega} \frac{\partial\Gamma_\omega(x,y)}{\partial\nu(y)}\phi(y)\,\mathrm{d}\sigma(y) \qquad \text{for } x \in \partial\Omega.$$

4.2. Perturbation by a grounded inclusion

Let B be a Lipschitz domain in $\mathbb{R}^d$ with connected boundary, with volume $|B| = |\Omega|$, and centered at the origin in the sense that $\int_{\partial B} y_1\,\mathrm{d}\sigma(y_1,y_2) = \int_{\partial B} y_2\,\mathrm{d}\sigma(y_1,y_2) = 0$. Fix a point $z \in \Omega$ and consider a scaling coefficient $\varepsilon > 0$. Let $D := z + \varepsilon B$ and $\Omega_\varepsilon := \Omega \setminus D$.

Suppose then that the domain Ω is perturbed into Ω_ε by requiring homogeneous Dirichlet conditions to hold on ∂D. Let λ be an eigenvalue of Ω with associated eigenfunction u. Then λ_ε is an eigenvalue of Ω_ε perturbed from λ with associated eigenfunction u_ε, if $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to 0$ and

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 \text{ or } \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta u_\varepsilon + \lambda_\varepsilon u_\varepsilon = 0 & \text{in } \Omega \setminus D, \\ u_\varepsilon = 0 & \text{on } \partial D, \\ u_\varepsilon = 0 \text{ or } \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

In [1, Chapter 3] the leading order term for the perturbation of a simple eigenvalue is obtained in the case of dimension 2 and 3. However these computations can be repeated exactly in the same way for $d \geq 4$, and the resulting asymptotic formula can be restated as follows.

Lemma 4.4. *Given λ a simple eigenvalue of Ω with associated eigenfunction u, and λ_ε the eigenvalue of Ω_ε perturbed from λ, then*

$$\lambda_\varepsilon - \lambda = \frac{u(z)^2}{\Gamma_0(\varepsilon)} + o\left(1/\Gamma_0(\varepsilon)\right). \tag{11}$$

We seek to apply the variational principle from Theorem 2.5 to our case. Although the expression in (11) is not continuous under L^2 convergence of u, we can easily rewrite it so that Assumption 2.4 holds. In fact, taking B_δ a ball centered at z of radius $\delta = \delta(\varepsilon)$ small enough, by the continuity of Laplacian eigenfunctions we have that $u(z) = \int_{B_\delta} u + o(1)$ as $\varepsilon \to 0$. Thus we also have that

$$\lambda_\varepsilon - \lambda = \frac{\int_{\partial B_\delta} u^2}{\Gamma_0(\varepsilon)} + o\left(1/\Gamma_0(\varepsilon)\right), \tag{12}$$

an expression which now is continuous under L^2 convergence. Then by Corollary 2.8 we obtain the following result.

Proposition 4.5. *Let λ be an eigenvalue of multiplicity m of the negative Laplacian on Ω and $u_1, \dots, u_m$ some associated eigenfunctions orthonormal in $\mathrm{L}^2(\Omega)$. Let*

$$U(z) := (u_1(z), \dots, u_m(z)).$$

Then the largest perturbed eigenvalue behaves like

$$\lambda_{\varepsilon,m} - \lambda = \frac{|U(z)|^2}{\Gamma_0(\varepsilon)} + o\left(1/\Gamma_0(\varepsilon)\right), \tag{13}$$

while all the other eigenvalues behave like

$$\lambda_{\varepsilon,n} - \lambda = o(1/\Gamma_0(\varepsilon)), \qquad \textit{for } n < m,$$

where Γ_0 is the fundamental solution of the Laplacian as defined in (10).

Remark 4.6. We collect some interesting consequences of Proposition 4.5.

1. For ε small enough, the largest perturbed eigenvalue $\lambda_{\varepsilon,m}$ will always be simple as long as at least one of the eigenfunctions $u_1, \dots, u_m$ is not zero in z.

2. If $u_{\varepsilon,m}$ is the eigenfunction associated to $\lambda_{\varepsilon,m}$, then by Corollary 2.8 the $o(1/\Gamma_0(\varepsilon))$-best approximation to $u_{\varepsilon,m}$ in the eigenspace of λ will be

$$\frac{u_1(z)u_1 + \cdots + u_m(z)u_m}{\sqrt{u_1(z)^2 + \cdots + u_m(z)^2}}.$$

3. It can be shown that in two dimensions the higher order terms in formula (11) can be further computed as

$$\lambda_\varepsilon - \lambda = \frac{u(z)^2}{\log(\varepsilon) + R(z)} + O\left(\varepsilon^2\right), \tag{14}$$

   where R is a function of z which does not depend on u (see [18] for its computation and expression). Therefore from Corollary 2.8 we have the stronger approximation

$$\lambda_{\varepsilon,n} - \lambda = \begin{cases} \dfrac{|U(z)|^2}{\log(\varepsilon) + R(z)} + O\left(\varepsilon^2\right) & \text{if } n = m, \\ O\left(\varepsilon^2\right) & \text{if } n < m. \end{cases} \quad \blacksquare$$

Hereafter we give an additional proof of Proposition 4.5 without relying on the general framework introduced in Section 2, but mimicking the proof of Theorem 2.5. It serves the dual purpose of giving a self-contained proof and motivating the choice of Assumptions 2.1-2.4. We also exploit some specific features of the case considered.

Proof. Step 1. For any $\varepsilon, \delta > 0$ consider a localized deformation of Ω into Ω_δ such that

- λ splits into simple eigenvalues $\lambda_{\delta,1} < \cdots < \lambda_{\delta,m}$ with $u_{\delta,1}, \dots, u_{\delta,m}$ associated eigenfunctions;
- $\Omega_\delta \triangle \Omega$ is contained in a ball of radius δ;
- the ε and δ-perturbations are well separated, that is the closures of $\Omega_\delta \triangle \Omega$ and $\Omega_\varepsilon \triangle \Omega$ are disjoint.

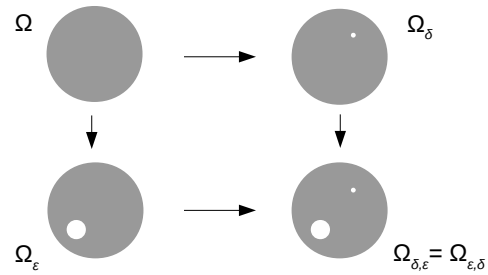

Figure 1: An illustration of the successive perturbations used in the proof of Proposition 4.5 in the case of a disk.

(Existence of such perturbations is guaranteed by Remark 3.4). Consider now the further perturbation of Ω_δ into $\Omega_{\varepsilon,\delta}$ obtained by inserting in Ω_δ the inclusion D, and call $\lambda_{\varepsilon,\delta,1} < \cdots < \lambda_{\varepsilon,\delta,m}$ the eigenvalues perturbed respectively from $\lambda_{\delta,1}, \ldots, \lambda_{\delta,m}$. Notice that from the continuity of the spectrum (Theorem 3.1), $\lambda_{\varepsilon,\delta,1}, \ldots, \lambda_{\varepsilon,\delta,m}$ are still simple for ε small enough. From (11) we have that

$$\lambda_{\varepsilon,\delta,n} - \lambda_{\delta,n} = \frac{u_{\delta,n}(z)^2}{\Gamma_0(\varepsilon)} + o\left(1/\Gamma_0(\varepsilon)\right). \tag{15}$$

From Theorem 3.1 we know that $u_{\delta,n}$ converges in H^1-norm as $\delta \to 0$ to u_n, a certain element in the eigenspace of λ. Unfortunately we can not pass directly to the limit in the right hand side of (15), since from the H^1-convergence it can not be deduced that $u_{\delta,n}$ converges to u_n at the point z. However we can exploit the continuity of $u_{\delta,n}$ and u_n at z to rewrite

$$u_{\delta,n}(z)^2 + O(\varepsilon^2) = \int_{B_r} u_{\delta,n}^2 \xrightarrow[\delta\to 0]{} \int_{B_r} u_n^2 = u_n(z)^2 + O(\varepsilon^2) \tag{16}$$

where B_r is a small enough ball centered at z.

Step 2. Consider now the domain Ω_ε instead. Perturb Ω_ε into $\Omega_{\delta,\varepsilon}$ by inserting an inclusion exactly as from Ω to Ω_δ; this is possible since we supposed that the ε and δ-perturbations are well separated (see for instance Figure 1). Call $\lambda_{\delta,\varepsilon,1} \leq \cdots \leq \lambda_{\delta,\varepsilon,m}$ the eigenvalues of $\Omega_{\delta,\varepsilon}$ perturbed from λ. We have that $\Omega_{\delta,\varepsilon} = \Omega_{\varepsilon,\delta}$ and $\lambda_{\varepsilon,\delta,n} = \lambda_{\delta,\varepsilon,n}$. Therefore we can rewrite

$$\lambda_{\varepsilon,n} = \lambda - (\lambda_{\delta,\varepsilon,n} - \lambda_{\varepsilon,n}) + (\lambda_{\delta,\varepsilon,n} - \lambda_{\delta,n}) + (\lambda_{\delta,n} - \lambda).$$

Thanks to the continuity of the spectrum, δ can be chosen small enough with respect to ε so that $\lambda_{\delta,\varepsilon,n} - \lambda_{\varepsilon,n} = o(1/\Gamma_0(\varepsilon))$ and $\lambda_{\delta,n} - \lambda = o(1/\Gamma_0(\varepsilon))$. Therefore, taking into consideration (15) and (16), we have that

$$\lambda_{\varepsilon,n} = \lambda + \frac{u_n(z)^2}{\Gamma_0(\varepsilon)} + o\left(1/\Gamma_0(\varepsilon)\right). \tag{17}$$

In the next steps we provide a variational characterization of u_n.

Step 3. Notice that the only "free variable" in the right hand side of (17) is the choice of u_n in the eigenspace of λ. From the fact that we already ordered the eigenvalues as $\lambda_{\varepsilon,1} \leq \cdots \leq \lambda_{\varepsilon,m}$ and the fact that $u_{\delta,k}$ cannot converge to u_n if $n \neq k$, it follows that $u_n \in \operatorname{argmin}_{v\in S_n} v(z)^2$, where

$$S_n = \Big\{ v \in \mathrm{H}^1(\Omega) : v \perp \{u_1, \ldots, u_{n-1}\},\ |v|_{\mathrm{L}^2(\Omega)} = 1 \Big\}.$$

Step 4. Finding a minimizer of $v(z)^2$ among all $v \in S_n$ amounts to finding unit vectors $a_1, \ldots, a_n \in \mathbb{R}^m$ such that $a_n \perp \{a_1, \ldots, a_{n-1}\}$ and $(a_n \cdot U(z))^2$ is minimum. Since $\mathbb{R}^m$ has

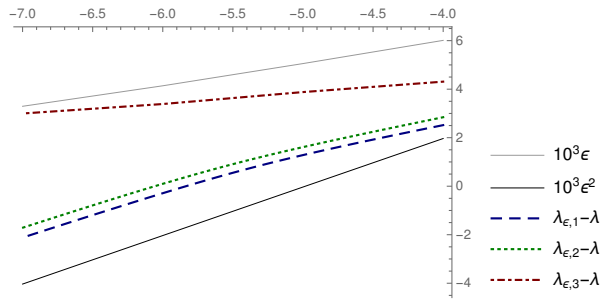

Figure 2: A $\log_2$-$\log_2$ plot of the behavior of an eigenvalue bifurcation from $50\pi^2$ as the size coefficient ε of the inclusion decreases. The original domain is the unit square $(0,1)^2$ and the inclusion is a disk of radius ε centered at $(1/4, 1/4)$.

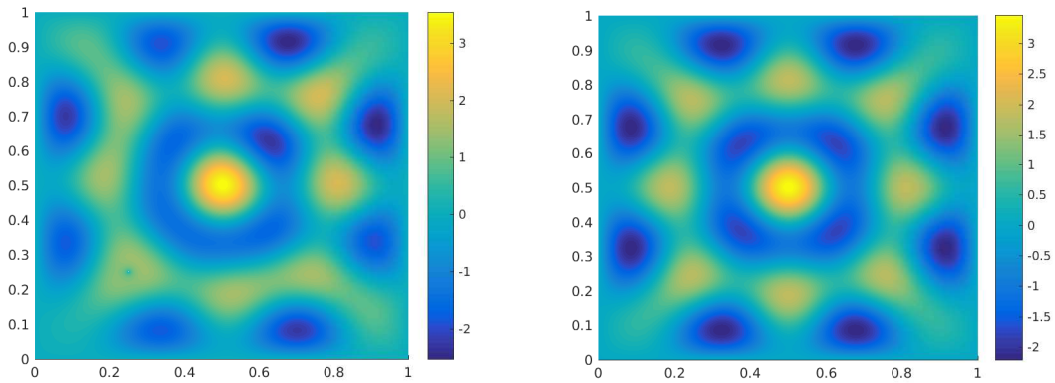

Figure 3: Eigenfunction associated to the largest perturbed eigenvalue from $50\pi^2$ caused by a grounded disk inclusion of radius 10^{-3} centered at $(1/4, 1/4)$; numerical computation (left) and the $O(\varepsilon^2)$-best approximation with an element of the eigenspace of $50\pi^2$ (right).

dimension m, we can find $m-1$ unit vectors $a_1, \ldots, a_{m-1}$ orthogonal to $U(z)$, while the last one must be $a_m = U(z)/|U(z)|$. Rewriting $u_n(z)$ as $a_n \cdot U(z)$ in (17) we obtain the formula in the thesis. □

Example 4.7. Let Ω be the unit square $(0,1)^2$ and consider the Dirichlet eigenvalue $\lambda = 50\pi^2$ with associated orthonormal eigenfunctions u_1, u_2, u_3 defined as

$$\begin{cases} u_1(x,y) = 2\sin(\pi x)\sin(7\pi y), \\ u_2(x,y) = 2\sin(7\pi x)\sin(\pi y), \\ u_3(x,y) = 2\sin(5\pi x)\sin(5\pi y). \end{cases}$$

Since for any point z in Ω there is at least one eigenfunction which is non-zero at z, the insertion at z of a small grounded inclusion will cause an eigenvalue bifurcation of λ; in particular one perturbed eigenvalue will shift from λ as $1/\log(\varepsilon)$ while the others will shift like $O(\varepsilon^2)$. The outcome of a numerical experiment for a disk inclusions centered at $(1/4, 1/4)$, is illustrated in Figure 2.

By Remark 2.6 we also know that the best $O(\varepsilon^2)$ approximation in the eigenspace of λ of the eigenfunction corresponding to the largest perturbed eigenvalue will be given by the sum of the eigenfunctions associated to λ; a comparison between a numerical computation of the eigenfunction associated to the largest perturbed eigenvalue and its $O(\varepsilon^2)$ approximation is given in Figure 3. ■

4.3. Perturbation by a conductivity inclusion

In this section we consider a perturbation of Ω caused by the insertion of a small inclusion with conductivity coefficient different from the background. This causes the eigenvalue λ to split into m (possibly distinct) eigenvalues $\lambda_{\varepsilon,1} \leq \cdots \leq \lambda_{\varepsilon,m}$ such that the following eigenvalue system holds:

$$\begin{cases} (\Delta + \lambda_{\varepsilon,n})u_{\varepsilon,n} = 0 & \text{in } \Omega \setminus \overline{D}, \\ (k\Delta + \lambda_{\varepsilon,n})u_{\varepsilon,n} = 0 & \text{in } D, \\ \displaystyle\lim_{\substack{x \to \partial D \\ x \in \Omega \setminus D}} \frac{\partial u_{\varepsilon,n}}{\partial \nu}(x) = k \lim_{\substack{x \to \partial D \\ x \in D}} \frac{\partial u_{\varepsilon,n}}{\partial \nu}(x), & \\ u_{\varepsilon,n} = 0 \text{ or } \partial_\nu u_{\varepsilon,n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $B, z, \varepsilon, D, \Omega_\varepsilon$ are defined as in Section 4.2, $n = 1, \ldots, m$ and $u_{\varepsilon,n}$ are eigenfunctions associated to $\lambda_{\varepsilon,n}$. It has been shown in [19] that if λ is a simple eigenvalue with associated eigenfunction u, and λ_ε is a perturbation of λ, then

$$\lambda_\varepsilon - \lambda = \varepsilon^d \langle \nabla u(z), \nabla u(z) \rangle_M + O(\varepsilon^{d+1}),$$

where $\langle x, y \rangle_M := x \cdot M(k, B) y$ for any $x, y \in \mathbb{R}^d$, and $M(k, B)$ is a $d \times d$ matrix known as polarization tensor, which can be defined as

$$(M(k, B))_i^j = \int_{\partial B} \left(\frac{k+1}{2(k-1)} I - (\mathcal{K}_B^0)^* \right)^{-1} (\nu_j) y_i \, \mathrm{d}\sigma(y).$$

Therefore in this case Corollary 2.7 specifies to the following result.

Proposition 4.8. *Let λ be an eigenvalue of multiplicity m of $L(\Omega)$ and $u_1, \ldots, u_m$ some associated eigenfunctions orthonormal in $\mathrm{L}^2(\Omega)$. Let $\lambda_{\varepsilon,1} \leq \cdots \leq \lambda_{\varepsilon,m}$ be the eigenvalues perturbed from λ. Then for every $n \in \{1, \ldots, m\}$, the $O(\varepsilon^{d+1})$ approximation of $\lambda_{\varepsilon,n} - \lambda$ is given by the n-th eigenvalue of the matrix with element*

$$\varepsilon^d \langle \nabla u_i(z), \nabla u_j(z) \rangle_M$$

in position (i, j).

Remark 4.9. We consider some extremal cases of Proposition 4.8 of particular interest.

1. If $\langle \nabla u_i(z), \nabla u_j(z) \rangle_M = 0$ for all $i \neq j$, then it is enough to reorder $u_1, \ldots, u_m$ according to the magnitude of $\langle \nabla u_1(z), \nabla u_1(z) \rangle_M, \ldots, \langle \nabla u_m(z), \nabla u_m(z) \rangle_M$, to obtain that for any n it holds
$$\lambda_{\varepsilon,n} - \lambda = \varepsilon^d \langle \nabla u_n(z), \nabla u_n(z) \rangle_M + O(\varepsilon^{d+1}).$$
We also remark that if the multiplicity m is larger than the dimension d, only d vectors can be linearly independent, and thus we will have that $\lambda_{\varepsilon,n} - \lambda = O(\varepsilon^{d+1})$ for $n \leq m - d$.

2. If $\nabla u_i(z), \nabla u_j(z)$ are parallel with respect to $\langle \cdot, \cdot \rangle_M$, then by Corollary 2.8 we will have
$$\lambda_{\varepsilon,n} - \lambda = \begin{cases} \varepsilon^d \sum_{n=1}^m \langle \nabla u_n(z), \nabla u_n(z) \rangle_M + O\left(\varepsilon^{d+1}\right) & \text{if } n = m, \\ O\left(\varepsilon^{d+1}\right) & \text{if } n < m. \end{cases} \quad \blacksquare$$

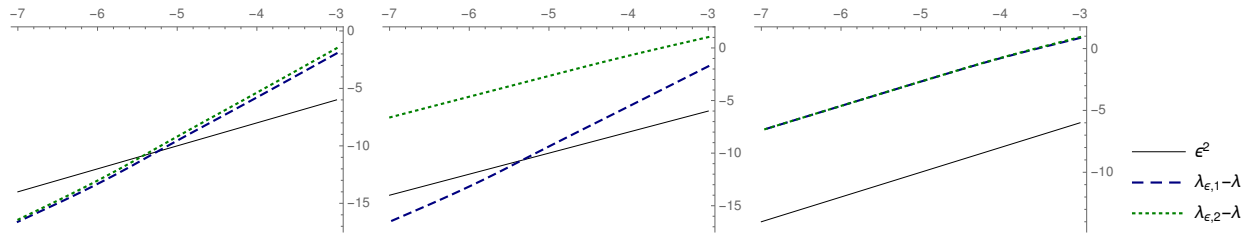

Figure 4: A $\log_2$-$\log_2$ plot of the behavior of an eigenvalue bifurcation as the size coefficient ε of the conductivity inclusion decreases. The original domain is the unit square $(0,1)^2$, the inclusion a disk of conductivity $k = 2$, centered respectively at $(1/2, 1/2)$ (left graph), at $(1/4, 1/2)$ (center graph), at $(1/4, 1/4)$ (right graph).

Example 4.10. Let Ω be the unit square $(0,1)^2$ and consider the Neumann eigenvalue $\lambda = 4\pi^2$ with associated eigenfunctions u_1, u_2 defined as

$$\begin{cases} u_1(x,y) = \sqrt{2}\cos(2\pi y), \\ u_2(x,y) = \sqrt{2}\cos(2\pi x). \end{cases}$$

Let B be the disk of radius $1/\pi^2$ centered at 0. Recall that in this case we have $M(k,B) = 2(k-1)I/(\pi^2(k+1))$. Although the first term in the asymptotic formula for $\lambda_{\varepsilon,n} - \lambda$ can be easily computed in this case, here we focus our attention only on the asymptotic order. We can easily determine, reasoning as in Remark 4.9, whether the first term in the asymptotic expansion of $\lambda_{e,n} - \lambda$ is zero; such behavior will depend on the choice of z. For example:

- for $z = (1/2, 1/2)$ we have that both eigenfunctions u_1, u_2 have zero gradient at 0, and thus both eigenvalues shift from λ as $O(\varepsilon^3)$;
- for $z = (1/4, 1/2)$ one of the eigenfunctions has zero gradient while the other has a non-zero entry, thus one eigenvalue shift behaves like ε^2 the other like $O(\varepsilon^3)$;
- for $z = (1/4, 1/4)$ the gradients of the two eigenfunctions are orthogonal and non-zero, thus both eigenvalues shift from λ as ε^2.

Numerical results (obtained with the finite element method) which confirm the previous calculations are presented in Figure 4. A comparison between a numerical computation of the eigenfunctions and their $O(\varepsilon^2)$ approximation for the case of a disk inclusion centered at $z = (1/4, 1/2)$ is given in Figure 5. ■

Example 4.11. Let Ω be the unit square $(0,1)^2$ and consider the Neumann eigenvalue $\lambda = 100\pi^2$ with associated eigenfunctions u_1, u_2, u_3, u_4 defined as

$$\begin{cases} u_1(x,y) = \sqrt{2}\cos(10\pi y), \\ u_2(x,y) = \sqrt{2}\cos(10\pi x), \\ u_3(x,y) = 2\cos(6\pi x)\cos(8\pi y), \\ u_4(x,y) = 2\cos(8\pi x)\cos(6\pi y). \end{cases}$$

Let B be the disk of radius $1/\pi^2$ centered at 0. Different choices for z showcase the asymptotically different splittings which can and cannot occur. For example:

- for $z = (1/2, 1/2)$, for any n it holds $\nabla u_n(z) = 0$, and therefore $\lambda_{\varepsilon,n} - \lambda = O(\varepsilon^3)$;

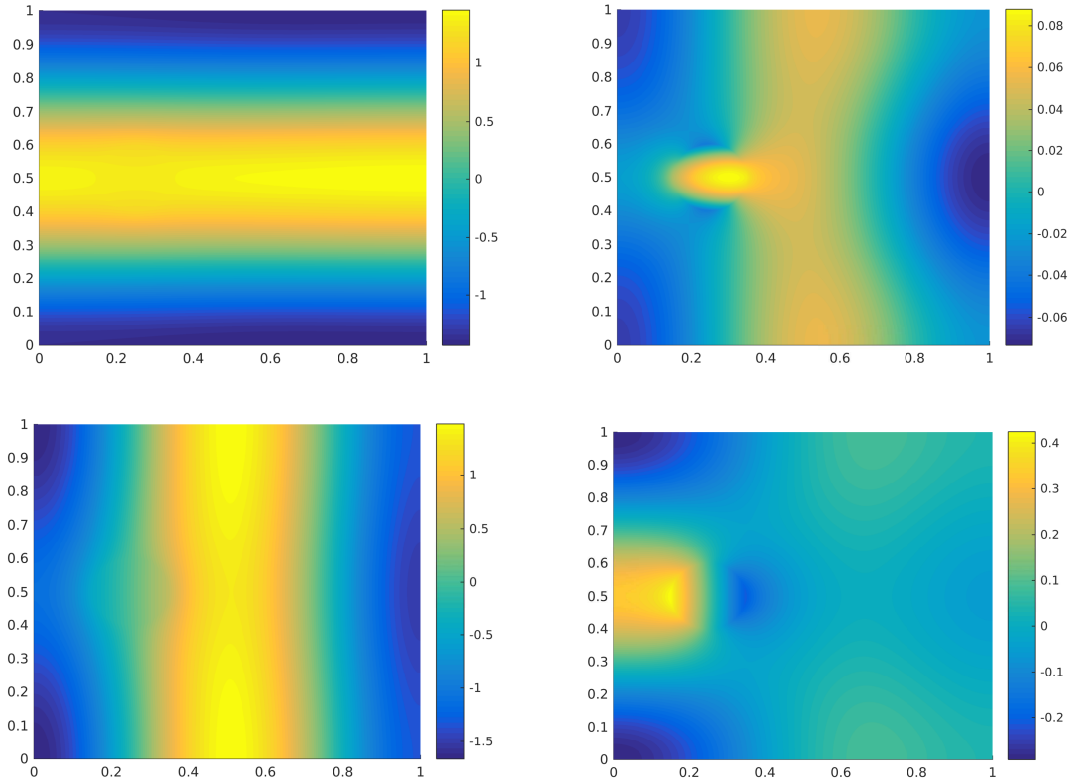

Figure 5: Eigenfunctions associated to the eigenvalues perturbed from $4\pi^2$. The perturbation consists in the excision of a disk of radius 0.1 centered at $(1/4, 1/2)$. Numerical computations of the eigenfunctions themselves are plotted in the left column, their difference with the $O(\varepsilon^2)$-best approximations in the two-dimensional eigenspace of $4\pi^2$ are plotted in the right column.

- for $z = (1/3, 1/2)$, $\nabla u_1(z) = \nabla u_3(z) = 0$ while $\nabla u_2(z), \nabla u_4(z)$ are all parallel and non-zero, thus from Point 2 of Remark 4.9 we have $\lambda_{\varepsilon,n} - \lambda = O(\varepsilon^3)$ for $n = 1, 2, 3$ while $\lambda_{\varepsilon,4} - \lambda = \Theta(\varepsilon^2)$;
- for $z = (1/7, 1/7)$ computations of the gradient of the eigenfunctions at z show that $\lambda_{\varepsilon,n} - \lambda = O(\varepsilon^3)$ for $n = 1, 2$ and $\lambda_{\varepsilon,n} - \lambda = \Theta(\varepsilon^2)$ for $n = 3, 4$;
- by Point 1 of Remark 4.9 there is no $z \in \Omega$ such that $\lambda_{\varepsilon,n} - \lambda = \Theta(\varepsilon^2)$ for more than two different indices n. ■

4.4. Perturbation by boundary deformation

In this section we consider Ω_ε obtained from Ω by a normal boundary deformation. For simplicity suppose that Ω is globally the epigraph of a Lipschitz function φ, that is $\Omega = \{x \in \mathbb{R}^d : \varphi(x) \leq 0\}$. Given $w \in C^\infty(\partial\Omega)$ we also suppose that the boundary perturbation is such that $\Omega_\varepsilon = \{x \in \mathbb{R}^d : \varphi(x) + \varepsilon w(x)\nu(x) \leq 0\}$. Recall that if λ is a simple eigenvalue of Ω with associated eigenfunction u, and $\lambda_\varepsilon \to \lambda$, then Hadamard's formula reads

$$\frac{\partial \lambda_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} = \int_{\partial\Omega} \Big(|\nabla u|^2 - \lambda u^2 - 2(\partial_\nu u)^2\Big) w, \tag{18}$$

for Dirichlet or Neumann conditions on $\partial\Omega$ (for its simple proof see [15, Lemma 5], or the more extensive discussions at [20], [21]). Therefore, if λ has multiplicity m, an application of Theorem 2.5 provides us with the variational formula

$$\lambda_{\varepsilon,n} - \lambda = \varepsilon \operatorname*{argmin}_{v \in S_n} \int_{\partial\Omega} \Big(|\nabla v|^2 - \lambda v^2 - 2(\partial_\nu v)^2\Big) w + O(\varepsilon^2). \tag{19}$$

Example 4.12. Let $\Omega = (0,1)^2$ and consider the Dirichlet eigenvalue $\lambda = 10\pi^2$ with associated orthonormal eigenfunctions $u_1(x,y) = 2\sin(\pi x)\sin(3\pi y)$, $u_2(x,y) = 2\sin(3\pi x)\sin(\pi y)$. Suppose the boundary of Ω is perturbed on the upper side of the square Ω with $w(x,y) = \sin(\pi x)$, that is $\partial\Omega \cap \{y = 1\}$ is deformed into $\{(x,y) : 0 \leq x \leq 1, y = 1 + \sin(\pi x)\}$. Notice that in the case of homogeneous Dirichlet boundary conditions, the integrand in (18) becomes just $-(\partial_\nu u)^2 w$. Then a direct computation shows that

$$\begin{aligned} -\int_{\partial\Omega} (a_1\partial_\nu u_1 + a_2\partial_\nu u_2)^2 w &= -\int_0^1 (6a_1\pi\sin(\pi x) + 2a_2\pi\sin(3\pi x))^2 \sin(\pi x) \\ &= \frac{16\pi}{35}(-105a_1^2 + 14a_1a_2 - 9a_2^2), \end{aligned}$$

thus by (19) we can explicitly calculate

$$\begin{aligned} u_{\varepsilon,1} &= c_1u_1 + c_2u_2 + O(\varepsilon^2), \qquad & \lambda_{\varepsilon,1} - \lambda &= C_1\varepsilon + O(\varepsilon^2), \\ u_{\varepsilon,2} &= c_2u_1 - c_1u_2 + O(\varepsilon^2), \qquad & \lambda_{\varepsilon,2} - \lambda &= C_2\varepsilon + O(\varepsilon^2), \end{aligned}$$

where $c_1 \simeq 0.9974, c_2 \simeq -0.07235, C_1 \simeq -151.5, C_2 \simeq -12.20$. A comparison of these results with a finite element method simulation is presented in Figure 6. A comparison between a numerical computation of the eigenfunctions and the minimizers of the expression in (19) is given in Figure 7. ■

Notice that in general minimizing the expression in (19) is a computationally expensive task. However, we can still obtain some cheaper, qualitative information if we approximate to a more treatable form the domain perturbation considered. We showcase such an heuristic in the following example, where a local perturbation is "singularized" to obtain an asymptotic formula easier to analyze.

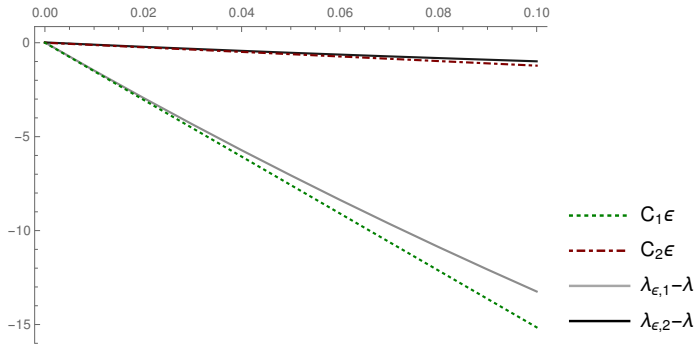

Figure 6: Behavior of an eigenvalue bifurcation as the scaling parameter ε of the boundary deformation decreases. The original domain is the unit square $(0,1)^2$ and the boundary deformation is given by $\varepsilon \sin(\pi x)\nu(x,1)$ on the upper side $\{(x,y) : x \in (0,1), y = 1\}$.

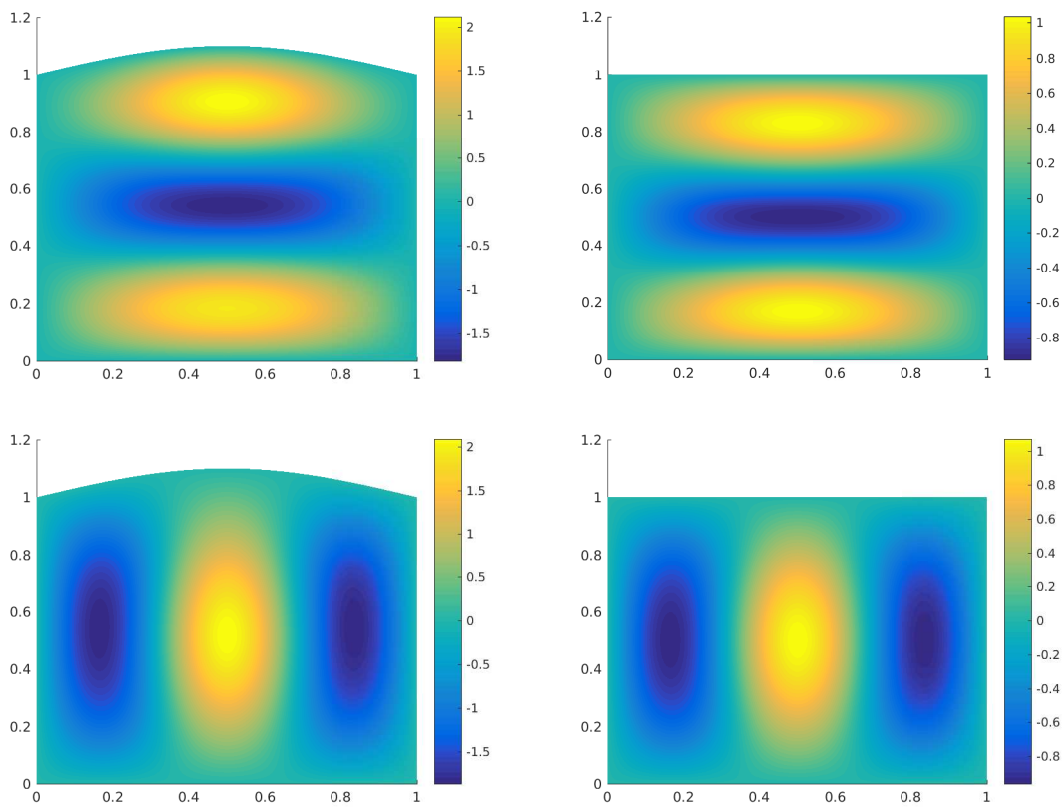

Figure 7: Eigenfunctions associated to the eigenvalues perturbed from $10\pi^2$. The perturbation consists in a boundary deformation $\varepsilon \sin(\pi x)\nu(x,1)$ of the upper side $\{(x,y) : x \in (0,1), y = 1\}$ for $\varepsilon = 1/10$. Numerical computations of the eigenfunctions are plotted in the left column, their $O(\varepsilon^2)$-best approximations in the two-dimensional eigenspace of $4\pi^2$ in the right column.

Example 4.13. Suppose a small dent is present on the surface $\partial\Omega$ at the point z, shaped as a cone with circular base of radius δ and height ε. Let at first λ be a simple Dirichlet eigenvalue with associated eigenfunction u. If we approximate

$$\int_{\partial\Omega} (\partial_\nu u)^2 w \simeq \delta^2 \pi (\partial_\nu u(z))^2,$$

then for frequencies low enough we will have

$$\lambda_\varepsilon - \lambda \simeq -\varepsilon \delta^2 \pi (\partial_\nu u(z))^2. \tag{20}$$

The right hand side in (20) is bilinear in u therefore, if we adopt such an approximation for $\lambda_\varepsilon - \lambda$, by Corollary 2.8 we have that for any non-simple eigenvalue the largest perturbed eigenvalue will shift like $O(\varepsilon)$ while all the smaller ones will shift like $O(\varepsilon^2)$. ■

4.5. Concluding remarks

We summarize hereafter the main results obtained for each of the perturbations considered. For this purpose we recall that Ω indicates an arbitrary Lipschitz domain in $\mathbb{R}^d$, λ an eigenvalue of the negative (Dirichlet or Neumann) Laplacian on Ω, $u_1, \dots, u_m$ an arbitrary orthonormal basis in $\mathrm{L}^2(\Omega)$ of the eigenspace of λ, and $\lambda_{\varepsilon,1} \le \dots \le \lambda_{\varepsilon,m}$ the eigenvalues perturbed from λ.

- When a hole D of volume ε^d and centered at z is cut out of Ω and homogeneous Dirichlet boundary conditions are imposed on ∂D we have

$$\begin{aligned} \lambda_{\varepsilon,m} - \lambda &= C\varepsilon^{d-1} \sum_{i=1}^{m} u_i(z)^2 + O(\varepsilon^d), \\ \lambda_{\varepsilon,n} - \lambda &= O(\varepsilon^d) \qquad \text{for } n < m, \end{aligned}$$

  in the case where $d \ge 3$ (see (14) for the case $d = 2$), where C is a constant which depends only on the dimension d. Therefore we have that the largest eigenvalue splits at a higher asymptotic order than all the others eigenvalues, as long as one among the quantities $u_1(z), \dots, u_m(z)$ is non-zero.

- In the case of a conductivity inclusion we do not have such an explicit formula, but still we can easily recover a first order approximation by computing the eigenvalues of a matrix. More precisely, if we suppose to change the conductivity coefficient from 1 to k only in D, a small disk of radius ε centered at a point z, then for any $n \in \{1, \dots, m\}$,

$$\lambda_{\varepsilon,n} - \lambda = 2\frac{k-1}{k+1}\varepsilon^d \mu_n + O(\varepsilon^{d+1}),$$

  where μ_n is the n-th eigenvalue of the matrix with element $\nabla u_i(z) \cdot \nabla u_j(z)$ in position (i, j).

- In the case of a normal boundary deformation of Ω with shape $w \in C^\infty(\partial\Omega)$, that is the perturbed domain boundary is given locally by $\partial\Omega + \varepsilon w \nu$, to find $\lambda_{\varepsilon,n} - \lambda$ for any $n \in \{1, \dots, m\}$, one has to find the minimizer v_n of

$$J(v) := \int_{\partial\Omega} \Big(|\nabla v|^2 - \lambda v^2 - 2(\partial_\nu v)^2 \Big) w$$

  among all v's of unit L^2-norm in the eigenspace of λ and perpendicular to $v_1, \dots, v_{n-1}$; then

$$\lambda_{\varepsilon,n} - \lambda = \varepsilon J(v_n).$$

We notice that all the deformations considered were such that $\lambda_\varepsilon - \lambda$ was in bijection with ε, for ε small enough. Therefore, if one could know a priori that a domain perturbation is of a certain type and that it depends only on the size parameter ε, one could reconstruct an approximation of ε from the knowledge of a single eigenvalue's shift.

As a final remark, let us point out that similar formulae can be derived for many other types of domain perturbations or other differential operators. For example, with the same approach of Section 4.4, it is immediate to generalize the asymptotic expansion of eigenvalues in the case of shape deformation of conductivity inclusions (see [22, Theorem 2.1]); or, with the same approach of Section 4.3, to generalize the asymptotic formulae for eigenvalues of the Lamé operator in the context of linear elasticity (see [23, Theorems 3.5 and 4.1] and [24, Theorem 2.1]).

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