ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

Convergence rates of high dimensional Smolyak quadrature

J. Zech and Ch. Schwab

Research Report No. 2017-27
June 2017

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 159940

Convergence rates of high dimensional Smolyak quadrature [*]

Jakob Zech and Christoph Schwab

June 19, 2017

Abstract

We analyze convergence rates of Smolyak integration for parametric maps $u : U \to X$ taking values in a Banach space X, defined on the parameter domain $U = [-1, 1]^{\mathbb{N}}$. For parametric maps which are sparse, as quantified by summability of their Taylor polynomial chaos coefficients, dimension-independent convergence rates superior to N-term approximation rates under the same sparsity are achievable.

We propose a concrete Smolyak algorithm to apriori identify integrand-adapted sets of active multiindices (and thereby unisolvent sparse grids of quadrature points) via upper bounds for the integrands' Taylor gpc coefficients. For so-called $(\boldsymbol{b}, \varepsilon)$-holomorphic integrands u with $\boldsymbol{b} \in \ell^p$, $p \in (0, 1)$, we prove the dimension independent convergence rate $2/p - 1$ in terms of number N of quadrature points. The proposed Smolyak algorithm is proved to yield (essentially) the same rate in terms of the total computational cost and memory consumption. Numerical experiments and a mathematical sparsity analysis accounting for cancellations in quadratures and in the combination formula demonstrate that the asymptotic rate $2/p - 1$ is achieved for a moderate number of quadrature points *provided the integrand's deviation around its mean is small.* By a refined analysis of model integrand classes we show that a generally large preasymptotic range otherwise precludes reaching the asymptotic rate $2/p - 1$ for practically relevant numbers of quadrature points.
Key words: generalized polynomial chaos, Smolyak Quadrature, sparsity, holomorphy

1 Introduction

The efficient numerical approximation of formally infinite-dimensional integrals

$$\int_U u(\boldsymbol{y}) \mathrm{d}\mu(\boldsymbol{y}), \tag{1.1}$$

of strongly μ-measurable, parametric maps $u : U \to X$ where $U = [-1, 1]^{\mathbb{N}}$, X is a Banach space and where μ denotes the product probability measure $\bigotimes_{j\in\mathbb{N}} \lambda/2$ with λ being the Lebesgue measure on $[-1, 1]$ is a key problem in computational uncertainty quantification ("UQ" for short). In computational UQ, the integrand function u in (1.1) is implicitly given as solution of a so-called *forward model*, typically an operator equation parametrized by a sequence $\boldsymbol{y} \in U$. The parameter sequences $\boldsymbol{y}$ can, for example, describe distributed uncertain constitutive relations or uncertain geometric shape. Equation (1.1) then describes an "ensemble average" (with respect to μ) of the parametric solution, over all admissible realizations of the uncertainty.

The high (in this case infinite) dimension of the integration domain U demands the integrand to possess appropriate sparsity properties in order to make a numerical computation feasible,

[*]Supported in part by SNSF Grant No. 159940

and overcome the so-called curse of dimensionality. For this reason, the integrand is typically assumed to be very smooth, e.g. to allow a bounded holomorphic extension into certain cylindrical subsets of $\mathbb{C}^{\mathbb{N}}$: here, as in [23], we consider parametric integrands which are holomorphic in cartesian products of discs with increasing radii. The rate at which those radii increase is a measure of the sparsity of the function, and as was observed in [34, 24, 23] and as we shall precise below, governs the (dimension-independent) rate of convergence of the quadrature.

Throughout, we shall work under an assumption of this type which is known as $(\boldsymbol{b}, \varepsilon)$-*holomorphy* and has similarly been employed in several papers [11, 12, 10]. It is made precise in Assumption 2.1. In the context of parametrized differential equations (PDEs), this kind of sparsity can be verified for a large variety of linear and nonlinear equations see for example [10, 28, 26, 13, 30], to which our new results may consequently be applied.

One possibility to numerically approximate the integral (1.1) is with a Monte Carlo method. Its advantage is that the convergence rate does not depend on the dimension of the integration domain. Its main disadvantage is the notoriously slow convergence rate of 1/2. For this reason, quasi Monte Carlo (QMC) methods exploiting the integrands' sparsity to attain higher order dimension-independent rates have been developed; we refer to [15, 17], to the surveys [18, 32] and to the references there. QMC quadrature is free from the curse of dimensionality, and additionally retains the Monte-Carlo feature of "embarrassingly parallel" integrand evaluation at the quadrature points. For high numbers of computationally intensive function evaluations (as arise in numerical PDE solution in the context of computational UQ) this becomes an important feature.

The present error analysis is based on so-called generalized polynomial chaos ("gpc" for short) expansions of the parametric integrand function. Expansions of gpc type have proved a valuable tool in regularity and sparsity analysis of countably-parametric functions taking values in a Banach space X; we refer to [40, 11, 12, 10] and to the survey [38] and the references there. The idea is to expand the integrand in a polynomial basis, and approximate the integral (1.1) with an interpolatory quadrature rule that is exact for the terms contributing most in the expansion. Such reasoning gives best N-term results, but in practice the optimal set of quadrature points is not known. The effectiveness of the method is due to the high smoothness of the integrand (it is holomorphic on certain sets), which is why polynomial approximations converge very fast. We refer to [41] and [20, 4] for a general description of sparse grid quadrature. For our proofs, as a basis we shall use the monomials, i.e. as in [40, 12, 10], we consider Taylor gpc expansions around $\mathbf{0} = (0, 0, \dots) \in U$. Unconditional convergence of such Taylor gpc expansion stipulates holomorphy of the integrand in polydiscs around $\mathbf{0}$. We choose the monomials for ease of presentation, but point out that a more general theory may be obtained by considering expansions in orthogonal bases such as the Legendre polynomials. Those merely require holomorphy on ellipses (cp. [10]), which results in weaker assumptions and will be worked out in [42]. The question remains on how to choose the quadrature points such that possibly few function evaluations result in a minimal error. In [21] an adaptive strategy has been proposed. The algorithm does not allow for parallel function evaluations in general however. Nonetheless, it delivers good results and has also been applied for parametrized PDEs in [37]. In the case of apriorily chosen quadrature points, the convergence for isotropic and anisotropic sparse grids was investigated in [2, 35, 34], and more recently in [24, 23]. The last two papers can be considered as the closest to ours. Numerical experiments in these works often revealed much better convergence rates, than what the theoretical findings suggested, see in particular [37, 24].

The aim of the present paper is to prove new convergence rates for an apriori choice of the sparse grid, which are stated in Thm. 3.3, Cor. 3.13. This will also shed some new light on the previously observed discrepancy between the observed convergence rates, and the proven ones. As a general idea, we use the known information on the function as stated in Assumption 2.1 and presumed throughout, to estimate the norm of the Taylor coefficients. Based on these estimates, a sparse grid is defined. The crucial observation, allowing us to improve earlier estimates, is then the following: The linear term $y \mapsto y$ has integral 0 over $[-1,1]$, and is integrated exactly by the one point Gauss rule (i.e. by an evaluation at $y = 0$ multiplied with the weight 1 corresponding to the measure $\lambda/2$). Consequently, we shall see that any polynomial in the multivariate Taylor expansion containing a linear term will always be integrated exactly by our quadrature operator. Subsequently, it is shown that the remaining Taylor coefficients in our setting have twice the decay rate of the one including all Taylor coefficients. This has a severe impact on the asymptotic convergence rate, and indeed our new results improve previously established convergence rates, given integrand sparsity, by more than a factor two. This last point will be discussed in more detail in the later sections, see in particular Rmk. 3.5 and Examples 4.1, 4.2.

The second contribution concerns an algorithm to efficient predict sequences of active gpc indices which are near optimal, and a bound on their complexity. Whereas many authors consider the number of quadrature points as a measure for the work, in fact, due to its structure based on differences of tensor product quadratures, the actual cost of the Smolyak algorithm does in general not behave linearly in the number of quadrature points. The mentioned convergence rates are therefore proven with respect to the total number of quadrature points in case of nested point sets such as Leja points, see Rmk. 3.1 and Thm. 3.3. In addition, we show that essentially the same rate can be obtained also for non-nested point sets, such as the Gauss points, see Cor. 3.13. Finally, this rate is also proven in terms of the total number of floating point operations, as is also stated in Cor. 3.13. The proven rates are asymptotic, and might not always be observable in the range of "small" numbers of quadrature points that are realizable in practice, as our numerical experiments and further analysis of particular model parametric integrand families in Section 4 reveal.

1.1 Outline

This paper is structured as follows: In Section 1.2 we set up notation and state a few assumptions used throughout. Section 2 deals with the decay of the Taylor coefficients, for functions exhibiting the sparsity properties of Assumption 2.1. The main result of the section is Thm. 2.11. In Section 3 we briefly recall the Smolyak algorithm, and then prove a first convergence result in Thm. 3.3. In Section 3.2 a finer investigation of the error in terms of the algorithm's complexity is conducted, and the results are summarized in Cor. 3.13. Finally, Section 4 is devoted to numerical experiments. We give more details on the implementation in Section 4.1. As already mentioned above, a large preasymptotic range is observed in certain situations. This is numerically investigated in Section 4.2, and we give heuristic arguments why it occurs. Finally, in Section 4.3 the convergence of our algorithm is tested for two exemplary real valued functions.

1.2 Notation and preliminaries

Throughout we let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbol C will stand for a generic, positive constant independent of any quantities determining the asymptotic behaviour of an estimate. It may change even within the same formula.

Multiindices are denoted by $\boldsymbol{\nu} = (\nu_j)_{j=1}^M \in \mathbb{N}_0^M$ where either $M \in \mathbb{N}$ or $M = \infty$. For the *order* of a multiindex we write $|\boldsymbol{\nu}| := \sum_{j=1}^M \nu_j$ and introduce the countable set

$$\mathcal{F} := \{\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} \,:\, |\boldsymbol{\nu}| < \infty\}. \tag{1.2}$$

The notation $\operatorname{supp} \boldsymbol{\nu}$ stands for the *support* of the multiindex, i.e. the set $\{j \in \{1, \dots, M\} : \nu_j \neq 0\}$, so that $\mathcal{F}$ consists of all finitely supported multiindices in $\mathbb{N}_0^{\mathbb{N}}$. A subset $\Lambda \subseteq \mathcal{F}$ is labelled *downward closed*, if $\boldsymbol{\nu} = (\nu_j)_{j\geq 1} \in \Lambda$ implies $\boldsymbol{\mu} = (\mu)_{j\geq 1} \in \Lambda$ for all $\boldsymbol{\mu} \leq \boldsymbol{\nu}$, by which we mean $\mu_j \leq \nu_j$, for all $j \geq 1$. For $p > 0$ we let $\ell^p(\mathcal{F})$ be the space of $\mathbb{R}$-valued sequences $\mathbf{t} = (t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$, satisfying $\|\mathbf{t}\|_{\ell^p} := (\sum_{\boldsymbol{\nu}\in\mathcal{F}} t_{\boldsymbol{\nu}}^p)^{1/p} < \infty$.

Let again $M \in \mathbb{N} \cup \{\infty\}$. As a topology on $\mathbb{C}^M$ we choose the product topology, and any subset such as $[-1,1]^M$ is equipped with the subspace topology. For a ball of radius $\varepsilon > 0$ in $\mathbb{C}$ we write $B_\varepsilon \subseteq \mathbb{C}$, and $\mathrm{clos}(B_\varepsilon) \subseteq \mathbb{C}$ for the closed ball. Furthermore $B_\varepsilon^M := \bigtimes_{j=1}^M B_\varepsilon \subseteq \mathbb{C}^M$. Moreover, the set $[-1,1]^M$ is considered as a probability space with the Borel sigma algebra and the (possibly countable) product measure $\mu := \bigotimes_{j=1}^M \lambda/2$, where λ denotes the Lebesgue measure on $[-1,1]$. For simplicity, the notation for μ does not reflect the dependence of μ on M. This will not be mentioned at every instance, and integrals or L^q-norms for $q \in [1,\infty]$ over $[-1,1]^M$ are always understood with respect to this measure. Elements of $\mathbb{C}^M$ are denoted by boldface characters such as $\boldsymbol{y} = (y_j)_{j=1}^M \in [-1,1]^M$. The standard multivariate notations $\boldsymbol{y}^{\boldsymbol{\nu}} := \prod_{j=1}^M y_j^{\nu_j}$ and $\boldsymbol{\nu}! = \prod_{j=1}^M \nu_j!$ will be employed.

Finally, for a Banach space X over $\mathbb{R}$ or $\mathbb{C}$ we introduce its complexification $X_\mathbb{C}$: By $X_\mathbb{C}$ we mean elements in the set $X_\mathbb{C} := X + \mathrm{i}X$, with i denoting the imaginary unit in $\mathbb{C}$. The vector space $X_\mathbb{C}$ is endowed with the norm $\|x_1 + \mathrm{i}x_2\|_{X_\mathbb{C}} := \sup_{0\le t\le 2\pi} \|x_1 \cos t - x_2 \sin t\|_X$. This norm extends the norm on X (cf. [33]). In case X is already a Banach space over $\mathbb{C}$, we have $X_\mathbb{C} = X$ with equivalent norms, which is why we do not distinguish between the two in this case.

2 Summability of Taylor coefficients

With $U := [-1,1]^\mathbb{N}$, consider $u : U \to X$, for some fixed Banach space X over $\mathbb{R}$ or $\mathbb{C}$. In the following we are concerned with the Taylor expansion

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} \tag{2.1}$$

of u and the summability properties of the Taylor coefficients $(\|u_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}}$.

2.1 $(\boldsymbol{b},\varepsilon)$ Holomorphy

For the formal gpc expansion (2.1) to be meaningful, p-summability of the sequence of (norms of) the Taylor coefficients $\{u_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu}\in\mathcal{F}} \subset X$ is required for some $0 < p \le 1$. A sufficient condition on the parametric map $U \ni \boldsymbol{y} \mapsto u(\boldsymbol{y}) \in X$ is the following assumption, which has similarly been stated in [11] and [10]. It will be employed in Sec. 2.3.

Assumption 2.1 ($(\boldsymbol{b},\varepsilon)$-Holomorphy)**.** *Assume given a sequence $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ of positive reals b_j such that $\boldsymbol{b} \in \ell^p(\mathbb{N})$ for some $p \in (0,1]$, and such that b_j is monotonically decreasing.*

We say that $\boldsymbol{\rho} \in [1,\infty)^\mathbb{N}$ is $(\boldsymbol{b},\varepsilon)$-admissible for some $\varepsilon > 0$ if

$$\sum_{j\in\mathbb{N}} b_j(\rho_j - 1) \le \varepsilon\ . \tag{2.2}$$

With

$$O_{\boldsymbol{b}} := \bigcup_{\{\boldsymbol{\rho}\,:\,\boldsymbol{\rho}\ is\ (\boldsymbol{b},\varepsilon)\text{-}admissible\}} \mathrm{clos}(B_{\boldsymbol{\rho}}) \subseteq \mathbb{C}^N, \tag{2.3}$$

the function $u : O_{\boldsymbol{b}} \to X_\mathbb{C}$ is continuous. Moreover u is holomorphic on an open superset of $O_{\boldsymbol{b}}$ as a function of each y_j. Additionally, there exists a constant $C_u < \infty$ such that $\sup_{\boldsymbol{y}\in O_{\boldsymbol{b}}} \|u\|_{X_\mathbb{C}} \le C_u$.

In case u satisfies Assumption 2.1, we will also say that u is $(\boldsymbol{b},\varepsilon)$-holomorphic. Note that by continuity in the above assumption we mean continuity with respect to the subspace topology on $O_{\boldsymbol{\rho}} \subseteq \mathbb{C}^\mathbb{N}$, where $\mathbb{C}^\mathbb{N}$ is equipped with the product topology. We now recall the well-known fact, that the Taylor expansion in (2.1) converges on finite dimensional polydiscs in $\mathbb{C}^M$, $M \in \mathbb{N}$. In the following, by an absolutely convergent series $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in Y^\mathcal{S}$, with Y some Banach space and $\mathcal{S}$ some countable set such as $\mathcal{F}$, we mean a sequence for which there exists a bijection $\pi : \mathbb{N} \to \mathcal{S}$ such that the sum $\sum_{j\in\mathbb{N}} \|t_{\pi(j)}\|_Y$ converges. This is sensible due to the countability of $\mathcal{S}$, and the fact that the existence of one such bijection guarantees the series to converge for any bijection $\pi : \mathbb{N} \to \mathcal{S}$. In the below proposition, for $\boldsymbol{\rho} = (\rho_j)_{j=1}^M$ we write $\mathrm{clos}(B_{\boldsymbol{\rho}})$ to denote the closed polydisc $\bigtimes_{j=1}^M \mathrm{clos}(B_{\rho_j}) \subseteq \mathbb{C}^M$, $M < \infty$.

Proposition 2.2. *Let $M \in \mathbb{N}$ and $\boldsymbol{\rho} = (\boldsymbol{\rho}_j)_{j=1}^M \in (1,\infty)^M$. Suppose that $u : O \to X$ is holomorphic on the open superset $O \supseteq \mathrm{clos}(B_{\boldsymbol{\rho}})$ and satisfies $\sup_{\boldsymbol{y}\in\mathrm{clos}(B_{\boldsymbol{\rho}})} \|u\|_{X_{\mathbb{C}}} \le C_u < \infty$. Then, for $\boldsymbol{y} \in [-1,1]^M$, u allows the expansion*

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathbb{N}_0^M} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} \qquad \textit{where} \qquad u_{\boldsymbol{\nu}} = \frac{1}{\boldsymbol{\nu}!} \frac{\partial u(\boldsymbol{y})}{\partial y_1^{\nu_1} \dots \partial y_M^{\nu_M}}\Big|_{\boldsymbol{y}=\boldsymbol{0}} \in X, \tag{2.4}$$

which is absolutely convergent in $L^\infty([-1,1]^M, X)$. It holds

$$\|u_{\boldsymbol{\nu}}\|_X \le C_u \boldsymbol{\rho}^{-\boldsymbol{\nu}}. \tag{2.5}$$

The bound (2.5) is a consequence of the Cauchy integral theorem [27, Thm. 2.1.2], see the proof of [11, Lemma 2.4]. The convergence of the series (2.4) is for example discussed in [27, Sec. 2.1].

In the next subsection, we proceed with proving and recalling summability results for real valued sequences. Those will then serve to verify ℓ^p-summability of certain subsequences of $(\|u_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}}$ in Section 2.3 under the holomorphy Assumption 2.1.

2.2 ℓ^p-summability of multiindex sequences

We give here a variant of Lemma 7.1 and Thm. 7.2 in [11]. Let in the following $\boldsymbol{\alpha} = (\alpha_j)_{j\in\mathbb{N}}$ be a sequence (not necessarily monotonic) of nonnegative numbers.

Definition 2.3. *For $k \in \mathbb{N}$, define $\mathcal{F}_k := \{\boldsymbol{\nu} \in \{0, k, k+1, \dots\}^{\mathbb{N}} : |\boldsymbol{\nu}| < \infty\}$.*

Lemma 2.4. *For $p \in (0,\infty)$ and $k \in \mathbb{N}$, the sequence $(\boldsymbol{\alpha}^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_k}$ belongs to $\ell^{p/k}(\mathcal{F}_k)$, iff $\|\boldsymbol{\alpha}\|_{\ell^p(\mathbb{N})} < \infty$ and $\|\boldsymbol{\alpha}\|_{\ell^\infty(\mathbb{N})} < 1$.*

Proof. Observe that $\|\boldsymbol{\alpha}\|_{\ell^\infty} < 1$ and $\|\boldsymbol{\alpha}\|_{\ell^p} < \infty$ are necessary, since $(\alpha_j^{lp/k})_{l\in\mathbb{N}}$, $j \in \mathbb{N}$, and $(\alpha_j^p)_{j\in\mathbb{N}}$ are all subsequences of $(\boldsymbol{\alpha}^{\boldsymbol{\nu}p/k})_{\boldsymbol{\nu}\in\mathcal{F}_k}$.

On the other hand, we have

$$\begin{aligned}\|\boldsymbol{\alpha}^{\boldsymbol{\nu}}\|_{\ell^{p/k}(\mathcal{F}_k)}^{p/k} &= \sum_{\boldsymbol{\nu}\in\mathcal{F}_k} \boldsymbol{\alpha}^{\boldsymbol{\nu}p/k} = \prod_{j\in\mathbb{N}} \left(1 + \sum_{\{l\,:\,l\ge k\}} \alpha_j^{lp/k}\right) = \prod_{j\in\mathbb{N}} \left(1 + \frac{\alpha_j^{pk/k}}{1-\alpha_j^{p/k}}\right) \\ &= \exp\left(\sum_{j\in\mathbb{N}} \log\left(1 + \frac{\alpha_j^p}{1-\alpha_j^{p/k}}\right)\right) \le \exp\left(\sum_{j\in\mathbb{N}} \frac{\alpha_j^p}{1-\alpha_j^{p/k}}\right) \le \exp\left(\frac{1}{1-\|\boldsymbol{\alpha}\|_{\ell^\infty}^{p/k}} \|\boldsymbol{\alpha}\|_{\ell^p}^p\right). \ \square\end{aligned}$$

Lemma 2.5. *Let $\rho > 1$ and $k \in \mathbb{N}$. Then there exist a constant C_k such that for all $\boldsymbol{0} \ne \boldsymbol{\nu} \in \mathcal{F}$ with $k\boldsymbol{\nu} = (k\nu_j)_{j\in\mathbb{N}} \in \mathcal{F}$*

$$(2\pi)^{(1-k)/2} \left(\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\right)^k \le \frac{|k\boldsymbol{\nu}|!}{(k\boldsymbol{\nu})!} \le \mathrm{e}(2\pi)^{-k/2} C_k^{|\operatorname{supp}\boldsymbol{\nu}|} \rho^{|\boldsymbol{\nu}|} \left(\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\right)^k. \tag{2.6}$$

Proof. We begin with the lower bound. Using Stirling's inequalities $\sqrt{2\pi}n^{n+1/2}\exp(-n) \le n! \le n^{n+1/2}\exp(-n+1)$ (see for example [36]) we get

$$\begin{aligned}
\frac{|k\boldsymbol{\nu}|!}{(k\boldsymbol{\nu})!} &\ge \frac{\sqrt{2\pi}(k|\boldsymbol{\nu}|)^{k|\boldsymbol{\nu}|+1/2}\exp(-k|\boldsymbol{\nu}|)}{\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}(k\nu_j)^{k\nu_j+1/2}\exp(-k\nu_j+1)} \\
&= \frac{\sqrt{2\pi}}{\exp(|\operatorname{supp}\boldsymbol{\nu}|)}\frac{k^{k|\boldsymbol{\nu}|+1/2}}{k^{k|\boldsymbol{\nu}|+|\operatorname{supp}\boldsymbol{\nu}|/2}}\frac{|\boldsymbol{\nu}|^{k|\boldsymbol{\nu}|+1/2}\exp(-k|\boldsymbol{\nu}|)}{\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\nu_j^{k\nu_j+1/2}\exp(-k\nu_j)} \\
&= \frac{\sqrt{2\pi}k^{(1-|\operatorname{supp}\boldsymbol{\nu}|)/2}}{\exp(|\operatorname{supp}\boldsymbol{\nu}|)}\frac{(2\pi)^{-k/2}|\boldsymbol{\nu}|^{(1-k)/2}(\sqrt{2\pi}|\boldsymbol{\nu}|^{|\boldsymbol{\nu}|+1/2}\exp(-|\boldsymbol{\nu}|))^k}{\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\exp(-k)\nu_j^{(1-k)/2}(\nu_j^{\nu_j+1/2}\exp(-\nu_j+1))^k} \\
&\ge (2\pi)^{(1-k)/2}\left(\frac{\exp(k)}{k^{1/2}\,\mathrm{e}}\right)^{|\operatorname{supp}\boldsymbol{\nu}|}\left(\frac{\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\nu_j}{|\boldsymbol{\nu}|}\right)^{(k-1)/2}\left(\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\right)^k. \qquad (2.7)
\end{aligned}$$

We claim that

$$f(\boldsymbol{\nu}) := \left(\frac{\exp(k-1)}{k^{1/2}}\right)^{|\operatorname{supp}\boldsymbol{\nu}|}\left(\frac{\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\nu_j}{|\boldsymbol{\nu}|}\right)^{(k-1)/2} \ge 1, \qquad (2.8)$$

for all $\mathbf{0} \ne \boldsymbol{\nu} \in \mathcal{F}$ which then gives the assertion. In order to see this we use induction on $n = |\boldsymbol{\nu}|$. The case $n = 1$ is trivial because $\exp(k-1)/k^{1/2} \ge 1$ for all $k \in \mathbb{N}$ and the expression involving the multiindex $\boldsymbol{\nu}$ in the right-hand side of(2.7) equals one if $|\operatorname{supp}\boldsymbol{\nu}| = 1$ (which holds in particular if $|\boldsymbol{\nu}| = 1$). For the induction step let $\mathbf{e}_i = (\delta_{ji})_{j\in\mathbb{N}}$ and presume the induction hypothesis $f(\boldsymbol{\nu}) \ge 1$ for arbitrary but fixed $\boldsymbol{\nu} \in \mathcal{F}$ with $|\boldsymbol{\nu}| = n$. First assume $i \in \operatorname{supp}\boldsymbol{\nu}$ so that $|\operatorname{supp}\boldsymbol{\nu}| = |\operatorname{supp}(\boldsymbol{\nu}+\mathbf{e}_i)|$. Then

$$f(\boldsymbol{\nu}+\mathbf{e}_i) \ge f(\boldsymbol{\nu}) \qquad \Leftrightarrow \qquad \frac{(\nu_i+1)\prod_{j\ne i}\nu_j}{|\boldsymbol{\nu}|+1} \ge \frac{\prod_j \nu_j}{|\boldsymbol{\nu}|} \qquad \Leftrightarrow \qquad \frac{\nu_i+1}{|\boldsymbol{\nu}|+1} \ge \frac{\nu_i}{|\boldsymbol{\nu}|}, \qquad (2.9)$$

which is true so that $f(\boldsymbol{\nu}+\mathbf{e}_i) \ge f(\boldsymbol{\nu}) \ge 1$. Next let $i \notin \operatorname{supp}\boldsymbol{\nu}$. Then $\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\nu_j = \prod_{j\in\operatorname{supp}(\boldsymbol{\nu}+\mathbf{e}_i)}(\boldsymbol{\nu}+\mathbf{e}_i)_j$ and with $n = |\boldsymbol{\nu}|$

$$\frac{f(\boldsymbol{\nu}+\mathbf{e}_i)}{f(\boldsymbol{\nu})} = \frac{\exp(k-1)}{k^{1/2}}\left(\frac{n}{n+1}\right)^{(k-1)/2} \ge \frac{\exp(k-1)}{k^{1/2}}\left(\frac{1}{2}\right)^{(k-1)/2} =: g(k). \qquad (2.10)$$

We have $g(1) = 1$. Moreover for $k \ge 1$

$$g'(k) = \frac{\exp(k-1)2^{(1-k)/2}\left(\left(1-\frac{\log(2)}{2}\right)\sqrt{k}-\frac{1}{2\sqrt{k}}\right)}{k} \ge 0, \qquad (2.11)$$

which shows $g(k) \ge g(1) \ge 1$ for all $k \in \mathbb{N}$ and therefore $f(\boldsymbol{\nu}+\mathbf{e}_i) \ge f(\boldsymbol{\nu}) \ge 1$ by (2.10). This concludes the proof of the claim (2.8) which further implies the lower bound in (2.6).

For the upper bound, we use again Stirling's inequalities to obtain

$$\begin{aligned}
\frac{|k\boldsymbol{\nu}|!}{(k\boldsymbol{\nu})!} &\le \frac{(k|\boldsymbol{\nu}|)^{k|\boldsymbol{\nu}|+1/2}\exp(-k|\boldsymbol{\nu}|+1)}{\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\sqrt{2\pi}(k\nu_j)^{k\nu_j+1/2}\exp(-k\nu_j)} \\
&= \frac{\mathrm{e}(2\pi)^{-k/2}|\boldsymbol{\nu}|^{(1-k)/2}(\sqrt{2\pi}|\boldsymbol{\nu}|^{|\boldsymbol{\nu}|+1/2}\exp(-|\boldsymbol{\nu}|))^k}{\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\sqrt{2\pi}\exp(-k)\nu_j^{(1-k)/2}(\nu_j^{\nu_j+1/2}\exp(-\nu_j+1))^k} \\
&\le \frac{\mathrm{e}(2\pi)^{-k/2}|\boldsymbol{\nu}|^{(1-k)/2}}{(\sqrt{2\pi}\exp(-k))^{|\operatorname{supp}\boldsymbol{\nu}|}\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\nu_j^{(1-k)/2}}\left(\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\right)^k \\
&\le \mathrm{e}(2\pi)^{-k/2}\left(\frac{\exp(k)}{\sqrt{2\pi}}\right)^{|\operatorname{supp}\boldsymbol{\nu}|}\prod_{j\in\operatorname{supp}\boldsymbol{\nu}}\nu_j^{(k-1)/2}\left(\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\right)^k. \qquad (2.12)
\end{aligned}$$

Since $\rho > 1$, there exists a constant $\tilde{C}$ such that $n^{(k-1)/2} \leq \tilde{C}\rho^n$ for all $n \in \mathbb{N}$. Thus $\prod_{j\in\operatorname{supp}\boldsymbol{\nu}} \nu_j^{(k-1)/2} \leq \tilde{C}^{|\operatorname{supp}\boldsymbol{\nu}|}\rho^{|\boldsymbol{\nu}|}$. The upper bound in (2.6) then follows via (2.12) with $C_k := \tilde{C}\exp(k)(2\pi)^{-1/2}$. □

Lemma 2.6. *Let $p \in (0,1]$ and $k \in \mathbb{N}$. Then the sequence $(\boldsymbol{\alpha}^{\boldsymbol{\nu}}|\boldsymbol{\nu}|!/\boldsymbol{\nu}!)_{\boldsymbol{\nu}\in\mathcal{F}_k}$ belongs to $\ell^{p/k}(\mathcal{F}_k)$ iff $\|\boldsymbol{\alpha}\|_{\ell^p} < \infty$ and $\|\boldsymbol{\alpha}\|_{\ell^1} < 1$.*

Proof. Without loss of generality, we assume throughout $\alpha_j > 0$ for all $j \in \mathbb{N}$. The proof of [11, Thm. 7.2] covers the case $k = 1$, $p = 1$. Then

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\boldsymbol{\alpha}^{\boldsymbol{\nu}} = \sum_{l\geq 0}\left(\sum_{j\in\mathbb{N}}\alpha_j\right)^l = \frac{1}{1-\|\boldsymbol{\alpha}\|_{\ell^1(\mathbb{N})}} < \infty, \tag{2.13}$$

which, due to $\mathcal{F}_1 = \mathcal{F}$, gives $(\boldsymbol{\alpha}^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_1} \in \ell^1(\mathcal{F}_1)$ iff $\|\boldsymbol{\alpha}\|_{\ell^1} < 1$.

The stated necessity of $\|\boldsymbol{\alpha}\|_{\ell^p} < \infty$ (for general p and k) is clear, since $(\alpha_j^p)_{j\in\mathbb{N}}$ is a subsequence of $((\boldsymbol{\alpha}^{\boldsymbol{\nu}}|\boldsymbol{\nu}|!/\boldsymbol{\nu}!)^{p/k})_{\boldsymbol{\nu}\in\mathcal{F}_k}$. In order to verify necessity of $\|\boldsymbol{\alpha}\|_{\ell^1} < 1$, it is sufficient to do so for $p = 1$. So let now $k > 1$, $p = 1$. With Lemma 2.5 it then holds

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}_k}\left(\boldsymbol{\alpha}^{\boldsymbol{\nu}}\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\right)^{1/k} \geq \sum_{\boldsymbol{\nu}\in\mathcal{F}}\left(\boldsymbol{\alpha}^{k\boldsymbol{\nu}}\frac{|k\boldsymbol{\nu}|!}{k\boldsymbol{\nu}!}\right)^{1/k} \geq C\sum_{\boldsymbol{\nu}\in\mathcal{F}}\left(\boldsymbol{\alpha}^{k\boldsymbol{\nu}}\left(\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\right)^k\right)^{1/k} = C\sum_{\boldsymbol{\nu}\in\mathcal{F}}\boldsymbol{\alpha}^{\boldsymbol{\nu}}\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}. \tag{2.14}$$

According to (2.13), the last sum is finite iff $\|\boldsymbol{\alpha}\|_{\ell^1} < 1$.

Since $k = 1$, $p = 1$ has been treated in (2.13), it remains to prove that our assumptions are sufficient for $k > 1$, $p \in (0,1]$ and $k = 1$, $p \in (0,1)$. We begin again with the case $k > 1$, $p = 1$, and claim that for every $\boldsymbol{\nu} \in \mathcal{F}_k$, there exists $\boldsymbol{\mu} \in \{0, k, 2k, \dots\}^{\mathbb{N}}$ such that

$$\boldsymbol{\alpha}^{\boldsymbol{\nu}}\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \leq k^{k|\operatorname{supp}\boldsymbol{\nu}|}\boldsymbol{\alpha}^{\boldsymbol{\mu}}\frac{|\boldsymbol{\mu}|!}{\boldsymbol{\mu}!} \qquad \text{and} \qquad |\nu_j - \mu_j| < k \quad \forall\, j \in \mathbb{N}. \tag{2.15}$$

For $\boldsymbol{\nu} \in \mathcal{F}_k$ fixed, we construct $\boldsymbol{\mu}$ as follows: let j such that $\nu_j \notin \{0, k, 2k, \dots\}$, which by definition of $\mathcal{F}_k$ implies in particular $\nu_j > k$. Assume first that

$$\alpha_j^{-1}\frac{\nu_j}{|\boldsymbol{\nu}|} \geq 1. \tag{2.16}$$

Then for $r \in \{1, \dots, k-1\}$

$$\alpha_j^{-1}\frac{\nu_j - r}{|\boldsymbol{\nu}| - r} = \alpha_j^{-1}\frac{\nu_j}{|\boldsymbol{\nu}|}\frac{|\boldsymbol{\nu}|}{|\boldsymbol{\nu}| - r}\frac{\nu_j - r}{\nu_j} \geq \frac{\nu_j - r}{\nu_j} \geq \frac{1}{k}, \tag{2.17}$$

because $\nu_j > k$ and $r < k$. With $\tilde{\mu}_i := \nu_i$ for $i \neq j$ and $\tilde{\mu}_j := \max\{nk : n \in \mathbb{N},\ nk \leq \nu_j\}$, we have $|\nu_j - \tilde{\mu}_j| < k$ and

$$\boldsymbol{\alpha}^{\boldsymbol{\nu}}\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \leq \boldsymbol{\alpha}^{\boldsymbol{\nu}}\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}k^{\nu_j-\tilde{\mu}_j}\prod_{r=1}^{\nu_j-\tilde{\mu}_j}\alpha_j^{-1}\frac{\nu_j - r}{|\boldsymbol{\nu}| - r} = k^{\nu_j-\tilde{\mu}_j}\boldsymbol{\alpha}^{\tilde{\boldsymbol{\mu}}}\frac{|\tilde{\boldsymbol{\mu}}|!}{\tilde{\boldsymbol{\mu}}!} \leq k^k\boldsymbol{\alpha}^{\tilde{\boldsymbol{\mu}}}\frac{|\tilde{\boldsymbol{\mu}}|!}{\tilde{\boldsymbol{\mu}}!}. \tag{2.18}$$

On the other hand, if (2.16) does not hold, then $\alpha_j|\boldsymbol{\nu}|/\nu_j > 1$ and therefore for $r \in \{1, \dots, k-1\}$

$$\alpha_j\frac{|\boldsymbol{\nu}| + r}{\nu_j + r} \geq \alpha_j\frac{|\boldsymbol{\nu}|}{\nu_j}\frac{|\boldsymbol{\nu}| + r}{|\boldsymbol{\nu}|}\frac{\nu_j}{\nu_j + r} \geq \frac{\nu_j}{\nu_j + r} \geq \frac{1}{k}. \tag{2.19}$$

With $\tilde{\mu}_i := \nu_i$ for $i \neq j$ and $\tilde{\mu}_j := \min\{nk : n \in \mathbb{N},\ nk \geq \nu_j\}$, we then have $|\tilde{\mu}_j - \nu_j| < k$ and similar as before

$$\boldsymbol{\alpha}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \leq \boldsymbol{\alpha}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} k^{\tilde{\mu}_j - \nu_j} \prod_{r=1}^{\tilde{\mu}_j - \nu_j} \alpha_j \frac{|\boldsymbol{\nu}| + r}{\nu_j + r} = k^{\tilde{\mu}_j - \nu_j} \boldsymbol{\alpha}^{\tilde{\boldsymbol{\mu}}} \frac{|\tilde{\boldsymbol{\mu}}|!}{\tilde{\boldsymbol{\mu}}!} \leq k^k \boldsymbol{\alpha}^{\tilde{\boldsymbol{\mu}}} \frac{|\tilde{\boldsymbol{\mu}}|!}{\tilde{\boldsymbol{\mu}}!}. \tag{2.20}$$

Repeating this procedure for all j with $\nu_j \notin \{0, k, 2k, \dots\}$, we find $\boldsymbol{\mu}$ satisfying (2.15). Next, note that for $\boldsymbol{\mu} \in \mathcal{F}_k$ with $\mu_j \in \{0, k, 2k, \dots\}$, $j \in \mathbb{N}$,

$$|\{\boldsymbol{\nu} \in \mathcal{F}_k : |\nu_j - \mu_j| < k,\ \forall\, j \in \mathbb{N}\}| \leq (2k-1)^{|\operatorname{supp} \boldsymbol{\mu}|}. \tag{2.21}$$

With $\boldsymbol{\mu}(\boldsymbol{\nu})$ denoting the above constructed multiindex satisfying (2.15), we then get with (2.21)

$$\sum_{\boldsymbol{\nu} \in \mathcal{F}_k} \left(\boldsymbol{\alpha}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \right)^{1/k} \leq \sum_{\boldsymbol{\nu} \in \mathcal{F}_k} k^{|\operatorname{supp} \boldsymbol{\nu}|} \left(\boldsymbol{\alpha}^{\boldsymbol{\mu}(\boldsymbol{\nu})} \frac{|\boldsymbol{\mu}(\boldsymbol{\nu})|!}{\boldsymbol{\mu}(\boldsymbol{\nu})!} \right)^{1/k} \leq \sum_{\boldsymbol{\nu} \in \mathcal{F}} (2k-1)^{|\operatorname{supp} \boldsymbol{\nu}|} k^{|\operatorname{supp} \boldsymbol{\nu}|} \left(\boldsymbol{\alpha}^{k\boldsymbol{\nu}} \frac{|k\boldsymbol{\nu}|!}{(k\boldsymbol{\nu})!} \right)^{1/k}. \tag{2.22}$$

Now let $\rho > 1$ so small that $\|\rho^{1/k} \boldsymbol{\alpha}\|_{\ell^1} < 1$, which is possible because $\|\boldsymbol{\alpha}\|_{\ell^1} < 1$ by assumption. Then, employing Lemma 2.5, we further bound the right-hand side of (2.22) by

$$C \sum_{\boldsymbol{\nu} \in \mathcal{F}} (k(2k-1))^{|\operatorname{supp} \boldsymbol{\nu}|} C_k^{|\operatorname{supp} \boldsymbol{\nu}|} (\rho^{1/k} \boldsymbol{\alpha})^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \leq C \sum_{\boldsymbol{\nu} \in \mathcal{F}} \tilde{C}_k^{|\operatorname{supp} \boldsymbol{\nu}|} (\rho^{1/k} \boldsymbol{\alpha})^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}, \tag{2.23}$$

with $\tilde{C}_k := k(2k-1)C_k$. Now let $J \in \mathbb{N}$ be so large that with $\tilde{\alpha}_j := \rho^{1/k} \alpha_j$ if $j \leq J$ and $\tilde{\alpha}_j := \tilde{C}_k \rho^{1/k} \alpha_j$ if $j > J$, it holds $\|\tilde{\boldsymbol{\alpha}}\|_{\ell^1} < 1$. With this choice, by (2.22), (2.23) we arrive at

$$\sum_{\boldsymbol{\nu} \in \mathcal{F}_k} \left(\boldsymbol{\alpha}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \right)^{1/k} \leq C \tilde{C}_k^{J/k} \sum_{\boldsymbol{\nu} \in \mathcal{F}} \tilde{\boldsymbol{\alpha}}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} < \infty, \tag{2.24}$$

which is finite by (2.13) and because $\|\tilde{\boldsymbol{\alpha}}\|_{\ell^1} < 1$. This concludes the proof for $k > 1$, $p = 1$.

Finally let $k \geq 1$ and $p \in (0,1)$. As shown in the proof of [11, Thm. 7.2], with $p' := p/(1-p)$ one can construct sequences $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$ such that

$$\|\boldsymbol{\gamma}\|_{\ell^1(\mathbb{N})} < 1, \qquad \|\boldsymbol{\delta}\|_{\ell^\infty(\mathbb{N})} < 1, \qquad \|\boldsymbol{\delta}\|_{\ell^{p'}(\mathbb{N})} < \infty \qquad \text{and} \qquad \alpha_j \leq \delta_j \gamma_j \quad \forall\, j \in \mathbb{N} \tag{2.25}$$

(essentially $\gamma_j \sim \alpha_j^p$ and $\delta_j \sim \alpha_j^{1-p}$). We get

$$\sum_{\boldsymbol{\nu} \in \mathcal{F}_k} \left(\boldsymbol{\alpha}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \right)^{p/k} \leq \sum_{\boldsymbol{\nu} \in \mathcal{F}_k} \left(\boldsymbol{\gamma}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \right)^{p/k} \boldsymbol{\delta}^{p/k} \leq \left(\sum_{\boldsymbol{\nu} \in \mathcal{F}_k} \left(\boldsymbol{\gamma}^{\boldsymbol{\nu}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} \right)^{1/k} \right)^p \left(\sum_{\boldsymbol{\nu} \in \mathcal{F}_k} \boldsymbol{\delta}^{\boldsymbol{\nu} \frac{p}{k(1-p)}} \right)^{1-p}. \tag{2.26}$$

Using (2.25), the first sum is finite by the statement for $p = 1$ shown in (2.24), and the second one since $(\boldsymbol{\delta}^{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathcal{F}_k} \in \ell^{p'/k}(\mathcal{F}_k)$ according to Lemma 2.4. This proves $(\boldsymbol{\alpha}^{\boldsymbol{\nu}} |\boldsymbol{\nu}|!/\boldsymbol{\nu}!)_{\boldsymbol{\nu} \in \mathcal{F}_k} \in \ell^{p/k}(\mathcal{F}_k)$. □

2.3 ℓ^p-summability of Taylor coefficients

We now show that under Assumption 2.1, the Taylor coefficients of u as in (2.1) are in $\ell^{p/k}(\mathcal{F}_k)$. This improved summability is the essential property in order to verify improved, dimension-independent algebraic convergence rates for suitably adapted Smolyak quadratures, see Sec. 3. N term approximation rate bounds for Taylor and other gpc expansions have previously been established by several authors, we only mention [11, 12, 10] and the references therein. Our new contribution here is twofold: first, instead of $\mathcal{F}$ we consider the smaller set $\mathcal{F}_k$ and in particular

$\mathcal{F}_2$, which as we shall see in Section 3, is better suited for analyzing Smolyak-style quadrature algorithms. Our second contribution concerns a computable estimator bounding the norm of the Taylor coefficients. We show that, without loss of convergence order, it can be chosen constant on certain subsets of $\mathcal{F}$. This is to be contrasted with greedy computational schemes based on computational solution of knapsack problems as, for example, in [4, 3]. Our new, *a-priori construction* allows to localize the multiindex set for the Smolyak quadrature in near linear complexity (work and memory), as explained in Section 3.2.

Before stating our theorem, we introduce the set Z and show two simple lemmata. The meaning of Z is essentially, that our subsequently presented algorithm will only operate on indices in (a set like) $Z^{\mathbb{N}} \cap \mathcal{F}$, allowing to reduce the overall complexity.

Assumption 2.7. *The set $Z = \{\zeta_j \,:\, j \in \mathbb{N}_0\} \subseteq \mathbb{N}_0$ consists of the strictly monotonically growing, nonnegative sequence $(\zeta_j)_{j\in\mathbb{N}_0}$, where $\zeta_0 = 0$ and $\zeta_1 = 1$. There exists a constant $1 \leq C_Z < \infty$ such that*

$$\forall j \in \mathbb{N}: \qquad \zeta_{j+1} - \zeta_j \leq C_Z(\zeta_j - \zeta_{j-1}). \tag{2.27}$$

This assumption basically states that ζ_j grows at most exponentially:

$$\zeta_{j+1} \leq C_Z\zeta_j + (\zeta_j - C_Z\zeta_{j-1}) \leq (1 + C_Z)\zeta_j. \tag{2.28}$$

With $Z \subseteq \mathbb{N}_0$ as in the assumption and, for a real number $x \geq 0$, define

$$\lfloor x \rfloor_Z := \max\{a \in Z \,:\, a \leq x\} \qquad \text{and} \qquad \lceil x \rceil_Z := \min\{a \in Z \,:\, a \geq x\}. \tag{2.29}$$

Application to sequences of these operators is understood componentwise. For future reference, we note that by (2.28) for any $n \in \mathbb{N}$ with $\lfloor n \rfloor_Z = \zeta_j$

$$\frac{n}{1 + C_Z} \leq \frac{\zeta_{j+1} - 1}{1 + C_Z} < \frac{\zeta_{j+1}}{1 + C_Z} \leq \zeta_j = \lfloor n \rfloor_Z. \tag{2.30}$$

Lemma 2.8. *Let Assumption 2.7 be satisfied. Let $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ be a nonnegative monotonically decreasing sequence, i.e. $t_{\boldsymbol{\nu}} \geq t_{\boldsymbol{\mu}}$ whenever $\boldsymbol{\nu} \leq \boldsymbol{\mu}$ componentwise. Then*

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} t_{\lfloor \boldsymbol{\nu} \rfloor_Z} \leq \sum_{\boldsymbol{\nu}\in\mathcal{F}} C_Z^{|\operatorname{supp} \boldsymbol{\nu}|} t_{\boldsymbol{\nu}}. \tag{2.31}$$

Proof. Let $A := \{\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{F} \,:\, \nu_j \in Z \;\forall j\}$, and for each $\boldsymbol{\mu} \in A$ let $\boldsymbol{\mu}^+ \in A$ such that $\mu_j^+ = \lceil \mu_j + 1 \rceil_Z$ if $\mu_j > 0$ and $\mu_j^+ = \mu_j = 0$ otherwise. Similarly $\boldsymbol{\mu}^- \in A$ is such that $\mu_j^- = \lfloor \mu_j - 1 \rfloor_Z$ if $\mu_j > 0$ and $\mu_j^- = \mu_j = 0$ otherwise. Then for $\boldsymbol{\mu} \in A$

$$[\boldsymbol{\mu}, \boldsymbol{\mu}^+) := \{\boldsymbol{\nu} \in \mathcal{F} \,:\, \mu_j \leq \nu_j < \mu_j^+ \text{ if } \mu_j > 0,\ \nu_j = 0 \text{ otherwise}\}, \tag{2.32}$$

and

$$(\boldsymbol{\mu}^-, \boldsymbol{\mu}] := \{\boldsymbol{\nu} \in \mathcal{F} \,:\, \mu_j^- < \nu_j \leq \mu_j \text{ if } \mu_j > 0,\ \nu_j = 0 \text{ otherwise}\}. \tag{2.33}$$

Note that these sets are nonempty, and any $\boldsymbol{\nu} \in (\boldsymbol{\mu}^-, \boldsymbol{\mu}] \cup [\boldsymbol{\mu}, \boldsymbol{\mu}^+)$ satisfies $\operatorname{supp} \boldsymbol{\nu} = \operatorname{supp} \boldsymbol{\mu}$. For every $\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{F}$ there exists a $\boldsymbol{\mu} \in A$ with $\boldsymbol{\nu} \in [\boldsymbol{\mu}, \boldsymbol{\mu}^+)$, namely $\boldsymbol{\mu} = \lfloor \boldsymbol{\nu} \rfloor_Z$. Furthermore if $\boldsymbol{\mu}$, $\boldsymbol{\eta} \in A$ with $\boldsymbol{\mu} \neq \boldsymbol{\eta}$, then for at least one j it holds wlog $0 \leq \mu_j < \eta_j$ and thus $\mu_j^+ \leq \eta_j$. Distinguishing between the cases $j \in \operatorname{supp} \boldsymbol{\mu}$, $j \notin \operatorname{supp} \boldsymbol{\mu}$ we easily conclude $[\boldsymbol{\mu}, \boldsymbol{\mu}^+) \cap [\boldsymbol{\eta}, \boldsymbol{\eta}^+) = \emptyset$. Consequently we have the partition

$$\mathcal{F} \backslash \{\mathbf{0}\} = \dot{\bigcup}_{\boldsymbol{\mu}\in A} [\boldsymbol{\mu}, \boldsymbol{\mu}^+), \tag{2.34}$$

where $\dot{\cup}$ denotes the union of disjoint sets. Before estimating the sum in (2.31), we also point out that by a similar argument as above $(\boldsymbol{\mu}^-, \boldsymbol{\mu}] \cap (\boldsymbol{\eta}^-, \boldsymbol{\eta}] = \emptyset$ if $\boldsymbol{\mu}$, $\boldsymbol{\eta} \in A$ with $\boldsymbol{\mu} \neq \boldsymbol{\eta}$.

Let $\boldsymbol{\mu} \in A$ and $j \in \operatorname{supp} \boldsymbol{\mu}$, i.e. $0 < \mu_j \in Z$. Then $\mu_j^- < \mu_j < \mu_j^+$, and due to Assumption 2.7, $\mu_j^+ - \mu_j \le C_Z(\mu_j - \mu_j^-)$. Therefore, for $\boldsymbol{\mu} \in A$

$$
\begin{aligned}
|[\boldsymbol{\mu}, \boldsymbol{\mu}^+)| &= \prod_{j \in \operatorname{supp} \boldsymbol{\mu}} (\mu_j^+ - \mu_j) \le \prod_{j \in \operatorname{supp} \boldsymbol{\mu}} C_Z(\mu_j - \mu_j^-) \\
&= C_Z^{|\operatorname{supp} \boldsymbol{\mu}|} \prod_{j \in \operatorname{supp} \boldsymbol{\mu}} (\mu_j - \mu_j^-) = C_Z^{|\operatorname{supp} \boldsymbol{\mu}|} |(\boldsymbol{\mu}^-, \boldsymbol{\mu}]|.
\end{aligned} \tag{2.35}
$$

Moreover, if $\tilde{\boldsymbol{\nu}} \in (\boldsymbol{\mu}^-, \boldsymbol{\mu}]$ and $\boldsymbol{\nu} \in [\boldsymbol{\mu}, \boldsymbol{\mu}^+)$, then $\tilde{\boldsymbol{\nu}} \le \boldsymbol{\mu} \le \lfloor \boldsymbol{\nu} \rfloor_Z$ and by monotonicity $t_{\tilde{\boldsymbol{\nu}}} \ge t_{\boldsymbol{\mu}} \ge t_{\lfloor \boldsymbol{\nu} \rfloor_Z}$. Hence with (2.34)

$$
\sum_{\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{F}} t_{\lfloor \boldsymbol{\nu} \rfloor_Z} = \sum_{\boldsymbol{\mu} \in A} \sum_{\boldsymbol{\nu} \in [\boldsymbol{\mu}, \boldsymbol{\mu}^+)} t_{\lfloor \boldsymbol{\nu} \rfloor_Z} \le \sum_{\boldsymbol{\mu} \in A} C_Z^{|\operatorname{supp} \boldsymbol{\mu}|} \sum_{\boldsymbol{\nu} \in (\boldsymbol{\mu}^-, \boldsymbol{\mu}]} t_{\boldsymbol{\mu}} \le \sum_{\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{F}} C_Z^{|\operatorname{supp} \boldsymbol{\nu}|} t_{\boldsymbol{\nu}}, \tag{2.36}
$$

where for the last inequality we have used $\mathbf{0} \notin (\boldsymbol{\mu}^-, \boldsymbol{\mu}]$ and $(\boldsymbol{\mu}^-, \boldsymbol{\mu}] \cap (\boldsymbol{\nu}^-, \boldsymbol{\nu}] = \emptyset$ whenever $\boldsymbol{\mu} \neq \boldsymbol{\nu}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu} \in A$, as pointed out above. This concludes the proof. □

In the previous Lemma we required $\zeta_1 = 1$ as presumed in Assumption 2.7, to avoid an infinite number of multiindices being rounded to $\mathbf{0}$: if $\zeta_1 > 1$, then $\lfloor \boldsymbol{\nu} \rfloor_Z = \mathbf{0}$ for all $\boldsymbol{\nu} \in \{0,1\}^{\mathbb{N}} \cap \mathcal{F}$.

Lemma 2.9. *Let $p \in (0,\infty)$ and $(t_j)_{j\in\mathbb{N}}$ nonnegative and monotonically decreasing. Then, for all $N \in \mathbb{N}$,*

$$
t_N \le \left(\sum_{j=1}^N t_j^p\right)^{\frac{1}{p}} N^{-\frac{1}{p}}. \tag{2.37}
$$

Proof. This follows from Hölder's inequality: assume $p \in (0,1)$ and define $\tilde{p} := \frac{1}{p} \in [1,\infty)$ and its Hölder conjugate $\tilde{q} = \frac{1}{1-p} \in [1,\infty)$. Since t_j is monotonically decreasing,

$$
t_N^{p^2} \le \frac{1}{N} \sum_{j=1}^N t_j^{p^2} \le \frac{1}{N} \left(\sum_{j=1}^N (t_j^{p^2})^{\tilde{p}} \right)^{\frac{1}{\tilde{p}}} \left(\sum_{j=1}^N 1^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} = \left(\sum_{j=1}^N t_j^p \right)^p N^{-p} . \tag{2.38}
$$

For $p \ge 1$, $t_N \le N^{-1} \sum_{j=1}^N t_j \le N^{-1} (\sum_{j=1}^N t_j^p)^{1/p} N^{1-1/p}$. □

Definition 2.10. *The space $\ell^p_{\mathrm{m}}(\mathcal{F})$ consists of all $\ell^p(\mathcal{F})$ sequences $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$, for which the monotone majorant $t^{\mathrm{m}}_{\boldsymbol{\nu}} := \sup_{\boldsymbol{\mu}\ge\boldsymbol{\nu}} t_{\boldsymbol{\mu}}$ is also in ℓ^p.*

We are now in position to formulate the following theorem, which is an extension of some results given in [11], [10], see in particular [11, Thm. 1.3], [10, Thm. 2.2]. The first item states that the Taylor coefficients corresponding to the multiindices in $\mathcal{F}_k$ are in $\ell^{p/k}$. The second item, which gives further information on the estimator bounding the norm of the Taylor coefficients, will become relevant when analyzing the complexity of our algorithm in Sec. 3.2.

Theorem 2.11. *Let $k \in \mathbb{N}$, $0 \le \tau < \infty$ and Assumption 2.7 be satisfied. Let $U := [-1,1]^{\mathbb{N}}$ and let $u : U \to X$ be $(\boldsymbol{b}, \varepsilon)$-holomorphic, i.e. u satisfies Assumption 2.1 with $\boldsymbol{b} \in \ell^p$ for some $p \in (0,1)$ and $q < p$. Then*

(i) there exists C independent of u, as well as a sequence $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p_{\mathrm{m}}(\mathcal{F})$ such that with C_u as in Assumption 2.1, and $u_{\boldsymbol{\nu}}$ as in (2.4)

$$
\|u_{\boldsymbol{\nu}}\|_X \prod_{j\in\mathbb{N}} (\nu_j + 1)^\tau \le C C_u t_{\boldsymbol{\nu}} \qquad \forall\ \boldsymbol{\nu} \in \mathcal{F}, \tag{2.39}
$$

and (2.1) *holds in the sense of absolute convergence in $L^\infty(U, X)$. Moreover, there exists a monotonically decreasing majorant $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ of the extension by zero of $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_k}$ to $\mathcal{F}$ satisfying $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^{p/k}(\mathcal{F})$. This sequence depends on $\boldsymbol{b}$, ε, τ but is independent of u, C_u.*

(ii) With $x \mapsto g(x)$ and $x \mapsto h(x)$ defined for $x \le m_{\mathbf{0}}$ by

$$g(x) := \min\{d \in \mathbb{N}_0 \,:\, \sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,\max_j \nu_j \ge d\}} m_{\boldsymbol{\nu}} \le x\}, \tag{2.40a}$$

$$h(x) := \min\{d \in \mathbb{N}_0 \,:\, \sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,|\operatorname{supp}\boldsymbol{\nu}| \ge d\}} m_{\boldsymbol{\nu}} \le x\}, \tag{2.40b}$$

it holds

$$g(x) = O(-\log(x)), \quad h(x) = o(-\log(x)) \text{ as } x \to 0. \tag{2.41}$$

Additionally, with $Z_k := \{0,1\} \cup \{a \in Z \,:\, a > k\}$

$$m_{\boldsymbol{\nu}} = m_{\boldsymbol{\mu}} \quad \text{if and only if} \quad \lfloor \boldsymbol{\nu} \rfloor_{Z_k} = \lfloor \boldsymbol{\mu} \rfloor_{Z_k} \tag{2.42}$$

and rearranging the sequence $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ to a monotonically decreasing sequence $(m_j)_{j\in\mathbb{N}}$ it holds $Cm_j \ge j^{-k/q}$.

Proof. We proceed in four steps. In the first two steps $t_{\boldsymbol{\nu}}$ as stated in the Theorem is constructed, and the absolute convergence of (2.1) is verified. In the third step we introduce $m_{\boldsymbol{\nu}}$, and finally in the fourth step the properties in item (ii) are shown.

1st Step: We introduce the estimator $t_{\boldsymbol{\nu}}$ in the spirit of [12, Sec. 3] and [10], and prove some summability properties. The construction is shortly recalled since we need some additional properties not mentioned in the cited references, and also to make the proof self-contained. The idea is to appropriately choose $\boldsymbol{\rho}$ in (2.5) using Assumption 2.1. With $B \ge 1$ fixed for the moment (and chosen subsequently) we now construct $t_{\boldsymbol{\nu}}$ depending on B. To this end we first observe that it is possible to find constants $\kappa_0 > 1$, $\delta > 0$, $C_\tau > 1$ and $J \in \mathbb{N}$ with the properties

$$(1+n)^\tau \le C_\tau \kappa_0^n \;\forall\, n \in \mathbb{N}, \quad (\kappa_0^2 - 1)\sum_{j=1}^J b_j + \kappa_1 \sum_{j>J} b_j < \varepsilon - \delta, \quad \text{and} \quad \sum_{j>J} b_j < \frac{\delta B^{-k/p}}{C_\tau \kappa_0 \,\mathrm{e}}, \tag{2.43}$$

where $\kappa_1 := \mathrm{e} C_\tau \kappa_0$ and $\mathrm{e} = \exp(1)$. They are obtained employing $\|\boldsymbol{b}\|_{\ell^1} < \infty$ to choose $\kappa_0 > 1$ with $(\kappa_0^2 - 1)\sum_{j\in\mathbb{N}} b_j < \varepsilon - 2\delta$ for some $\delta > 0$, then C_τ such that $(1+n)^\tau \le C_\tau \kappa_0^n$ and afterwards $J \in \mathbb{N}$ large enough such that $\kappa_1 \sum_{j>J} b_j < \delta$ and the last condition in (2.43) hold.

In the following for $\boldsymbol{\nu} \in \mathcal{F}$, $\boldsymbol{\nu}_E$ denotes the multiindex which coincides with $\boldsymbol{\nu}$ in the first J components and is zero otherwise. Furthermore $\boldsymbol{\nu}_F := \boldsymbol{\nu} - \boldsymbol{\nu}_E$ and $\mathcal{F}_G := \{\boldsymbol{\nu}_G \,:\, \boldsymbol{\nu} \in \mathcal{F}\}$, $G \in \{E, F\}$. Set

$$\rho_{\boldsymbol{\nu};j} := \begin{cases} \kappa_0^2 & \text{if } j \le J, \\ \max\left\{\kappa_1, \frac{\delta \nu_j}{|\boldsymbol{\nu}_F| b_j}\right\} & \text{if } j > J. \end{cases} \tag{2.44}$$

Then with (2.43)

$$\sum_{j\in\mathbb{N}} (\rho_{\boldsymbol{\nu};j} - 1) b_j \le (\kappa_0^2 - 1)\sum_{j=1}^J b_j + \sum_{j>J} \rho_{\boldsymbol{\nu};j} b_j \le (\kappa_0^2 - 1)\sum_{j\in\mathbb{N}} b_j + \kappa_1 \sum_{j>J} b_j + \delta \sum_{j>J} \frac{\nu_j}{|\boldsymbol{\nu}_F|} \le \varepsilon, \tag{2.45}$$

i.e. $\boldsymbol{\rho}_{\boldsymbol{\nu}}$ is $(\boldsymbol{b}, \varepsilon)$-admissible in the sense of Assumption 2.1. Therefore, with C_u and C_τ as in (2.5) and with (2.43),

$$\begin{aligned}
\frac{1}{C_u}\|u_{\boldsymbol{\nu}}\|_X \prod_{j\in\mathbb{N}} (\nu_j + 1)^\tau &\le \left(C_\tau^{|\operatorname{supp}\boldsymbol{\nu}|} \prod_{j\in\operatorname{supp}\boldsymbol{\nu}} \kappa_0^{\nu_j} \right) \prod_{j\in\mathbb{N}} \rho_{\boldsymbol{\nu};j}^{-\nu_j} \\
&\le C_\tau^{|\operatorname{supp}\boldsymbol{\nu}|} \kappa_0^{|\boldsymbol{\nu}|} \prod_{j=1}^J \kappa_0^{-2\nu_j} \prod_{j>J} \max\left\{\kappa_1, \frac{\delta \nu_j}{|\boldsymbol{\nu}_F| b_j}\right\}^{-\nu_j} \\
&\le C_\tau^J \prod_{j=1}^J \kappa_0^{-\nu_j} \prod_{j>J} \max\left\{\frac{\kappa_1}{C_\tau \kappa_0}, \frac{\delta \nu_j}{C_\tau \kappa_0 |\boldsymbol{\nu}_F| b_j}\right\}^{-\nu_j} =: t_{\boldsymbol{\nu}}\,.
\end{aligned} \tag{2.46}$$

We point out that $\kappa_1/(C_\tau\kappa_0) = \mathrm{e}$ by definition of κ_1.

We now prove that $t_{\boldsymbol{\nu}}$ is monotonically decreasing in $\boldsymbol{\nu}$. For $j \le J$ and with $\mathbf{e}_j := (\delta_{ji})_{i\in\mathbb{N}}$, since $\kappa_0 > 1$ we have $t_{\boldsymbol{\nu}+\mathbf{e}_j} \le \kappa_0^{-1} t_{\boldsymbol{\nu}} < t_{\boldsymbol{\nu}}$. Next, fix $j > J$. Note that

$$\max\left\{\mathrm{e}, \frac{\delta\nu_j}{C_\tau\kappa_0|\boldsymbol{\nu}_F|b_j}\right\} = \max\left\{\mathrm{e}, \frac{\delta\nu_j}{C_\tau\kappa_0 b_j(\nu_j + \sum_{\{i>J\,:\,i\neq j\}}\nu_i)}\right\} \tag{2.47}$$

is monotonically growing as a function of ν_j, and the maximum is always larger or equal to e. Therefore

$$\frac{t_{\boldsymbol{\nu}+\mathbf{e}_j}}{t_{\boldsymbol{\nu}}} \le \mathrm{e}^{-1} \prod_{\{i>J\,:\,i\neq j\}} \frac{\max\left\{\mathrm{e}, \frac{\delta\nu_i}{C_\tau\kappa_0|\boldsymbol{\nu}_F|b_i}\right\}^{\nu_i}}{\max\left\{\mathrm{e}, \frac{\delta\nu_i}{C_\tau\kappa_0(|\boldsymbol{\nu}_F|+1)b_i}\right\}^{\nu_i}} \le \mathrm{e}^{-1}\left(1 + \frac{1}{|\boldsymbol{\nu}_F|}\right)^{|\boldsymbol{\nu}_F|} \le 1. \tag{2.48}$$

Next we show $(B^{|\operatorname{supp}\boldsymbol{\nu}|} t_{\boldsymbol{\nu}}^{p/l})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^1(\mathcal{F}_l)$ where $1 \le l \le k$ arbitrary. With Stirling's inequalities $n^n \le \exp(n)n!$ and thus

$$\frac{|\boldsymbol{\nu}_F|^{|\boldsymbol{\nu}_F|}}{\boldsymbol{\nu}_F^{\boldsymbol{\nu}_F}} \le \exp(|\boldsymbol{\nu}_F|)\frac{|\boldsymbol{\nu}_F|!}{\boldsymbol{\nu}_F!}. \tag{2.49}$$

Employing (2.46), $d_j := (B^{l/p}C_\tau\kappa_0\mathrm{e}b_j/\delta)^{-1}$ and $\tilde{d}_j := d_{j+J}$, $j \in \mathbb{N}$, we get

$$\begin{aligned}
\sum_{\boldsymbol{\nu}\in\mathcal{F}_l} B^{|\operatorname{supp}\boldsymbol{\nu}|} t_{\boldsymbol{\nu}}^{p/l} &= \sum_{\boldsymbol{\nu}\in\mathcal{F}_l} B^{|\operatorname{supp}\boldsymbol{\nu}_E|} B^{|\operatorname{supp}\boldsymbol{\nu}_F|} t_{\boldsymbol{\nu}}^{p/l} \le \sum_{\boldsymbol{\nu}\in\mathcal{F}_l} B^J B^{|\boldsymbol{\nu}_F|} t_{\boldsymbol{\nu}}^{p/l} \\
&\le (C_\tau^{p/l}B)^J \sum_{\boldsymbol{\mu}\in\mathcal{F}_E\cap\mathcal{F}_l} \kappa_0^{-|\boldsymbol{\mu}|p/l} \sum_{\boldsymbol{\nu}\in\mathcal{F}_F\cap\mathcal{F}_l} \left(\frac{|\boldsymbol{\nu}_F|^{|\boldsymbol{\nu}_F|}}{\boldsymbol{\nu}_F^{\boldsymbol{\nu}_F}} \prod_{\{j>J\,:\,j\in\operatorname{supp}\boldsymbol{\nu}\}} \left(\frac{B^{l/p}C_\tau\kappa_0 b_j}{\delta}\right)^{\nu_j}\right)^{p/l} \\
&\le (C_\tau^{p/l}B)^J \sum_{\boldsymbol{\mu}\in\mathcal{F}_E\cap\mathcal{F}_l} \kappa_0^{-|\boldsymbol{\mu}|p/l} \sum_{\boldsymbol{\nu}\in\mathcal{F}_F\cap\mathcal{F}_l} \left(\frac{|\boldsymbol{\nu}_F|!}{\boldsymbol{\nu}_F!}\boldsymbol{d}^{-\boldsymbol{\nu}_F}\right)^{p/l} \\
&\le (C_\tau B)^{Jp/l} \sum_{\boldsymbol{\mu}\in\mathcal{F}_E} \kappa_0^{-|\boldsymbol{\mu}|p/l} \sum_{\boldsymbol{\nu}\in\mathcal{F}_l} \left(\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\tilde{\boldsymbol{d}}^{-\boldsymbol{\nu}}\right)^{p/l}.
\end{aligned} \tag{2.50}$$

Both sums on the right-hand side are finite according to Lemmata 2.4 and 2.6, and because $\|(\tilde{d}_j^{-1})_{j\in\mathbb{N}}\|_{\ell^p(\mathbb{N})} \le C\|\boldsymbol{b}\|_{\ell^p(\mathbb{N})} < \infty$ by Assumption 2.1 as well as

$$\|(\tilde{d}_j^{-1})_{j\in\mathbb{N}}\|_{\ell^1} = \frac{B^{l/p}C_\tau\kappa_0\mathrm{e}}{\delta}\sum_{j>J} b_j \le \frac{B^{k/p}C_\tau\kappa_0\mathrm{e}}{\delta}\sum_{j>J} b_j < 1 \tag{2.51}$$

by (2.43). In the particular case $l = 1$ it holds $\mathcal{F}_1 = \mathcal{F}$, and letting $B := 1$ we have proven $t_{\boldsymbol{\nu}} \in \ell^p(\mathcal{F})$.

Finally, we point out that there is no loss of generality in assuming

a) $b_j > j^{-1/q}$ in the above construction: in case this does not hold, define $\tilde{b}_j := \max(b_j, j^{-1/q}) \ge b_j$ and note that this new sequence is also in $\ell^p(\mathbb{N})$ and satisfies for any sequence $\boldsymbol{\rho} \in (1,\infty)^{\mathbb{N}}$ that $\sum_{j\in\mathbb{N}}(\rho_j - 1)\tilde{b}_j \ge \sum_{j\in\mathbb{N}}(\rho_j - 1)b_j$. Hence, any $\boldsymbol{\rho}$ which is $(\tilde{\boldsymbol{b}}, \varepsilon)$-admissible in the sense of Assumption 2.1 is also $(\boldsymbol{b}, \varepsilon)$-admissible. This implies that Assumption 2.1 holds in particular for the new sequence $\tilde{\boldsymbol{b}}$.

b) $t_{\boldsymbol{\nu}} \neq t_{\boldsymbol{\mu}}$ for all $\boldsymbol{\nu} \neq \boldsymbol{\mu}$: in case this is wrong, let $\pi : \mathbb{N} \to \mathcal{F}$ be a bijection such that $t_{\pi(j)}$ decays monotonically and $\{t_{\pi(1)}, \dots, t_{\pi(n)}\}$ is downward closed for any $n \in \mathbb{N}$. This is possible because $t_{\boldsymbol{\nu}}$ decays monotonically. Now define a strictly monotonically decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers with $(\varepsilon_n)_{n\in\mathbb{N}} \le t_{\pi(n)}$ and introduce $\tilde{t}_{\pi(n)} := t_{\pi(n)} + \varepsilon_n$ (note that a) implies $t_{\boldsymbol{\nu}} \neq 0$ for all $\boldsymbol{\nu} \in \mathcal{F}$). Then $\tilde{t}_{\boldsymbol{\nu}}$ is bounded by two times the right-hand side of (2.46), so that the above estimates remain true up to a factor, but $t_{\boldsymbol{\nu}} \neq t_{\boldsymbol{\mu}}$ whenever $\boldsymbol{\nu} \neq \boldsymbol{\mu}$.

2nd Step: We prove absolute convergence of (2.1) to u in $L^\infty(U, X)$. In the previous step, we have shown that $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p(\mathcal{F})$, and consequently $(\|u_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^1(\mathcal{F})$. Hence (2.1) is absolutely convergent to some function $\tilde{u} \in C^0(U, X)$. Fix $\boldsymbol{y} \in U$. By Assumption 2.1, u depends continuously on $\boldsymbol{y}$ w.r.t. the product topology on U. Thus we can find $N \in \mathbb{N}$ so large that $\|u(y_1, \dots, y_N, 0, \dots) - u(\boldsymbol{y})\|_X \le \varepsilon/2$. Furthermore, according to Prop. 2.2, there must exist $\Lambda \subseteq \mathbb{N}_0^N$ downward closed with $\sum_{\boldsymbol{\nu}\in\Lambda^c} \|u_{\boldsymbol{\nu}}\|_X \le \varepsilon/2$. This shows $\|u(\boldsymbol{y}) - \sum_{\boldsymbol{\nu}\in\tilde{\Lambda}} \boldsymbol{y}^{\boldsymbol{\nu}} u_{\boldsymbol{\nu}}\|_X \le \varepsilon$ for any superset $\tilde{\Lambda} \supseteq \Lambda$. Thus $\tilde{u}(\boldsymbol{y}) = u(\boldsymbol{y})$, i.e. the Taylor series converges to u.

3rd Step: We construct $m_{\boldsymbol{\nu}} \in \ell^{p/k}(\mathcal{F})$. To this end, define $\tilde{Z} := Z \cup \{k\}$ and

$$(\boldsymbol{\nu}_k)_j := \begin{cases} \nu_j & \text{if } \nu_j \notin \{1, \dots, k-1\} \\ k & \text{otherwise} \end{cases} \qquad \text{and} \qquad m_{\boldsymbol{\nu}} := t_{\lfloor \boldsymbol{\nu}_k \rfloor_{\tilde{Z}}} \tag{2.52}$$

for all $\boldsymbol{\nu} \in \mathcal{F}$. For $\boldsymbol{\mu} \le \boldsymbol{\nu} \in \mathcal{F}$ arbitrary, we have $\boldsymbol{\mu}_k \le \boldsymbol{\nu}_k$ and thus $\lfloor \boldsymbol{\mu}_k \rfloor_{\tilde{Z}} \le \lfloor \boldsymbol{\nu}_k \rfloor_{\tilde{Z}}$, which by monotonicity of $t_{\boldsymbol{\nu}}$ implies monotonicity of $m_{\boldsymbol{\nu}}$. Additionally, if $\boldsymbol{\nu} \in \mathcal{F}_k$, then $\boldsymbol{\nu}_k = \boldsymbol{\nu}$ and hence $\lfloor \boldsymbol{\nu}_k \rfloor_{\tilde{Z}} = \lfloor \boldsymbol{\nu} \rfloor_{\tilde{Z}}$ so that $m_{\boldsymbol{\nu}} = t_{\lfloor \boldsymbol{\nu} \rfloor_{\tilde{Z}}} \ge t_{\boldsymbol{\nu}}$. Therefore $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ constitutes a monotonically decreasing majorant of the extension by zero of $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_k}$ to $\mathcal{F}$. We wish to show $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^{p/k}(\mathcal{F})$. Set $\tilde{t}_{\boldsymbol{\nu}} := t_{\boldsymbol{\nu}_k}$. Distinguishing between the cases $x = 0$, $x \in \{1, \dots, k-1\}$ and $x > k$, with the subscript k having the same meaning as in (2.52) we get $(\lfloor x \rfloor_{\tilde{Z}})_k = \lfloor x_k \rfloor_{\tilde{Z}}$ for all $x \in \mathbb{N}_0$ (here we need $k \in \tilde{Z}$, which is why we introduced this set). Thus $\tilde{t}_{\lfloor \boldsymbol{\nu} \rfloor_{\tilde{Z}}} = t_{(\lfloor \boldsymbol{\nu} \rfloor_{\tilde{Z}})_k} = t_{\lfloor \boldsymbol{\nu}_k \rfloor_{\tilde{Z}}}$ for all $\boldsymbol{\nu} \in \mathcal{F}$. Moreover, for $\boldsymbol{\mu} \in \mathcal{F}_k$ we observe

$$|\{\boldsymbol{\nu} \in \mathcal{F} : \boldsymbol{\nu}_k = \boldsymbol{\mu}\}| = k^{|\{j\in\mathbb{N} : \mu_j = k\}|} \le k^{|\operatorname{supp} \boldsymbol{\mu}|}. \tag{2.53}$$

Note that $\tilde{Z} = Z \cup \{k\}$ also satisfies Assumption 2.7, but possibly with a different constant in (2.27) which we denote also by C_Z. Then, with Lemma 2.8

$$\begin{aligned} \sum_{\boldsymbol{\nu}\in\mathcal{F}} m_{\boldsymbol{\nu}}^{p/k} &= \sum_{\boldsymbol{\nu}\in\mathcal{F}} t_{\lfloor \boldsymbol{\nu}_k \rfloor_{\tilde{Z}}}^{p/k} = \sum_{\boldsymbol{\nu}\in\mathcal{F}} \tilde{t}_{\lfloor \boldsymbol{\nu} \rfloor_{\tilde{Z}}}^{p/k} \le \sum_{\boldsymbol{\nu}\in\mathcal{F}} C_Z^{|\operatorname{supp} \boldsymbol{\nu}|} \tilde{t}_{\boldsymbol{\nu}}^{p/k} \\ &= \sum_{\boldsymbol{\nu}\in\mathcal{F}} C_Z^{|\operatorname{supp} \boldsymbol{\nu}_k|} t_{\boldsymbol{\nu}_k}^{p/k} \le \sum_{\boldsymbol{\nu}\in\mathcal{F}_k} (kC_Z)^{|\operatorname{supp} \boldsymbol{\nu}|} t_{\boldsymbol{\nu}}^{p/k}, \end{aligned} \tag{2.54}$$

which is finite according to the computation in (2.50) if we let $B = kC_Z \ge 1$ and $l = k$ in the first step. It is clear from the construction that the sequence $m_{\boldsymbol{\nu}}$ is independent of u and C_u. This concludes the proof of (i).

4th Step: We prove (ii). Equation (2.42) holds because the $t_{\boldsymbol{\nu}}$ are distinct by assumption b) in the first step, because Z_k is chosen such that $\lfloor \boldsymbol{\nu}_k \rfloor_{\tilde{Z}} = \lfloor \boldsymbol{\mu}_k \rfloor_{\tilde{Z}}$ is equivalent to $\lfloor \boldsymbol{\nu} \rfloor_{Z_k} = \lfloor \boldsymbol{\mu} \rfloor_{Z_k}$ as is readily verified, and because of the definition of $m_{\boldsymbol{\nu}}$ in (2.52). The last claim in (ii), stating that the nth largest element of the sequence $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ is bounded from below by $Cn^{-k/q}$, holds because with $k\mathbf{e}_n$ denoting the index which has a k at position n and zeros otherwise, and our assumption a) stating that $b_n \ge Cn^{-1/q}$ from the first step, we get for all $n > J$ so large that $\kappa_1 < b_n^{-1}$ (cp. (2.46))

$$m_{k\mathbf{e}_n} = t_{\lfloor k\mathbf{e}_n \rfloor_Z} \ge t_{k\mathbf{e}_n} = C_\tau^{J+1} \kappa_0 b_n^k \ge Cn^{-k/q}. \tag{2.55}$$

For the first property in (2.41) we use $\boldsymbol{b} \in \ell^p$, so that by Lemma 2.9 we have $b_j \le C_{\boldsymbol{b}} j^{-1/p}$

for some $C_{\boldsymbol{b}} < \infty$ and for $d > J$ with (2.46)

$$H(d) := \sup_{|\operatorname{supp}\boldsymbol{\nu}|\geq d} m_{\boldsymbol{\nu}} \leq \sup_{|\operatorname{supp}\boldsymbol{\nu}|\geq d} t_{\boldsymbol{\nu}} = \sup_{\substack{\boldsymbol{\nu}\in\{0,1\}^{\mathbb{N}}\\ |\boldsymbol{\nu}|=d}} t_{\boldsymbol{\nu}} \leq C_\tau^J \kappa_0^{-J} \prod_{j=J+1}^{d} C_\tau\kappa_0/\delta(d-J)C_{\boldsymbol{b}} j^{-1/p}$$
$$\leq C(C_{\boldsymbol{b}}C_\tau\kappa_0/\delta)^{d-J} d^d \prod_{j=J+1}^{d} j^{-1/p} \leq C\exp\left(\tilde{C}(d-J) + d\log(d) - \frac{1}{p}\int_{J+1}^{d}\log(x)\mathrm{d}x\right), \quad (2.56)$$

with $\tilde{C} := \log(C_{\boldsymbol{b}}C_\tau\kappa_0/\delta)$ and where we have employed $\sum_{j=J+1}^{d}\log(j) \geq \int_{J+1}^{d}\log(x)\mathrm{d}x$. Here we have also used that $t_{\boldsymbol{\nu}}$ decays monotonically, so that the argmax after the first inequality must satisfy $\boldsymbol{\nu}\in\{0,1\}^{\mathbb{N}}$ and $|\boldsymbol{\nu}| = d$. The term in the exponential function is bounded by

$$\tilde{C}(d-J) + d\log(d) - \frac{1}{p}\left(d(\log(d)-1) - (J+1)(\log(J+1)-1)\right)$$
$$\leq C + \left(\tilde{C} + \frac{1}{p}\right)d + \left(1 - \frac{1}{p}\right)d\log(d) \leq C - C_0 d\log(d), \quad (2.57)$$

for all $d \geq d_0$ and some $d_0 \in \mathbb{N}$ large enough as well as $C > 0$ depending on J and $C_0 > 0$ small enough (such C_0 exists because $1 - 1/p < 0$ due to the additional assumption $p < 1$). Hence $H(d) = \sup_{|\operatorname{supp}\boldsymbol{\nu}|=d} m_{\boldsymbol{\nu}} \leq C_1 d^{-C_0 d} =: \tilde{H}(d)$, for some fixed C_0, $C_1 > 0$. Now let $\tilde{F}$ be the inverse function of the strictly monotonic mapping $d \mapsto \tilde{H}(d)$, $d \geq 1$. Then, writing $\tilde{F}(x) = -\log(x)r(x)$ for some function r, it holds

$$x = C_1(-\log(x)r(x))^{-C_0\log(x)r(x)} = C_1 x^{\log(-C_0\log(x)r(x))r(x)}. \quad (2.58)$$

In order for the exponent to be bounded as $x \to 0$ we must have $r(x) \to 0$, implying $\tilde{F}(x) = o(-\log(x))$ as $x \to 0$. Now let $x_d > 0$ for $d \in \mathbb{N}$ such that $\tilde{F}(x_d) = d$. Since $\tilde{H}$ is strictly monotonic, the same holds for $\tilde{F}$, and $x_d \to 0$ as $d \to \infty$ with $x_{d+1} < x_d$ for all $d \in \mathbb{N}$. Using again this strict monotonicity and $\tilde{H}(d) \geq H(d)$ we have (cp. (2.40b))

$$\tilde{F}(x_d) = d \leq d \quad \Rightarrow \quad x_d = \tilde{H}(\tilde{F}(x_d)) \geq \tilde{H}(d) \quad \Rightarrow \quad x_d \geq H(d) \quad \Leftrightarrow \quad h(x_d) \leq d. \quad (2.59)$$

Thus $\tilde{F}(x_d) = d \geq h(x_d)$ for all x_d, $d \in \mathbb{N}$. Since h is also strictly monotonically decreasing, for any $x \in (x_{d+1}, x_d)$ it must hold $h(x) \leq h(x_{d+1}) \leq d+1 = \tilde{F}(x_d) + 1 \leq \tilde{F}(x) + 1$. Hence for all $0 < x \leq x_1$ we have $h(x) \leq \tilde{F}(x) + 1$. With $\tilde{F}(x) = o(-\log(x))$ we conclude $h(x) = o(-\log(x))$ as $x \to 0$.

Now consider g in (2.40a). As noted above, the set $\tilde{Z} = Z \cup \{k\}$ also satisfies Assumption 2.7, and wlog we assume the constant C_Z in (2.27) to be the same. Using monotonicity of $m_{\boldsymbol{\nu}}$, $t_{\boldsymbol{\nu}}$ in $\boldsymbol{\nu}$, we deduce from (2.46) for $\lfloor d\rfloor_{\tilde{Z}} > J$

$$\sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,\max_j \nu_j \geq d\}} m_{\boldsymbol{\nu}} = \sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,\max_j \nu_j\geq d,\ |\operatorname{supp}\boldsymbol{\nu}|=1\}} m_{\boldsymbol{\nu}} = \sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,\max_j \nu_j\geq d,\ |\operatorname{supp}\boldsymbol{\nu}|=1\}} t_{\lfloor\boldsymbol{\nu}_k\rfloor_{\tilde{Z}}}$$
$$\leq \sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,\max_j \nu_j\geq \lfloor d\rfloor_{\tilde{Z}},\ |\operatorname{supp}\boldsymbol{\nu}|=1\}} t_{\boldsymbol{\nu}} \leq C\kappa_0^{-J}\exp(-(\lfloor d\rfloor_{\tilde{Z}} - J)) \leq C\exp(-d), \quad (2.60)$$

where we have used in the first inequality that $\nu_j \geq d$ implies $\lfloor(\boldsymbol{\nu}_k)_j\rfloor_{\tilde{Z}} \geq \lfloor\nu_j\rfloor_{\tilde{Z}} \geq \lfloor d\rfloor_{\tilde{Z}}$, in the second inequality that $\kappa_1/(C_\tau\kappa_0) = \mathrm{e}$ in (2.46) by definition of κ_1, and finally in the third inequality that $\lfloor d\rfloor_{\tilde{Z}} \geq d/(1+C_Z)$ (cp. (2.30)). Similar as before we obtain $g(x) = O(-\log(x))$ as $x \to 0$. □

3 Smolyak Quadrature

For $n \in \mathbb{N}_0$, let $(\chi_{n;j})_{j=0}^{n} \subset [-1,1]$ be a sequence of pairwise distinct points in $[-1,1]$. Define $\chi_{n;\boldsymbol{\nu}} := (\chi_{n;\nu_1}, \chi_{n;\nu_2}, \dots) \in [-1,1]^{\mathbb{N}}$ for $\boldsymbol{\nu} \in \mathcal{F}$. Throughout we assume that there exists

$0 < \tau < \infty$ such that the Lebesgue constant $L((\chi_{n;j})_{j=0}^n)$ of $(\chi_{n;j})_{j=0}^n$ satisfies

$$L((\chi_{n;j})_{j=0}^n) \leq (n+1)^\tau \qquad \forall\, n \in \mathbb{N}. \tag{3.1}$$

The univariate quadrature $Q_n : C^0([-1,1], X) \to \mathbb{R}$ is the interpolatory quadrature rule with quadrature points $(\chi_{n;0}, \dots, \chi_{n;n})$ (i.e. $Q_n P = \int_U P(\boldsymbol{y}) \mathrm{d}\mu(\boldsymbol{y})$ for all polynomials P of degree n). Furthermore with $Q_{-1} := 0$, we introduce the $\boldsymbol{\nu}$-increment operators

$$\Delta_{\boldsymbol{\nu}} := \bigotimes_{j\in\mathbb{N}} (Q_{\nu_j} - Q_{\nu_j - 1}). \tag{3.2}$$

For a downward closed index set $\Lambda \subseteq \mathcal{F}$ of finite cardinality, the corresponding Smolyak quadrature is defined by

$$Q_\Lambda := \sum_{\boldsymbol{\nu}\in\Lambda} \Delta_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\nu}\in\Lambda} \Bigg(\sum_{\{\mathbf{e}\in\{0,1\}^{\mathbb{N}}\,:\,\boldsymbol{\nu}+\mathbf{e}\in\Lambda\}} (-1)^{|\mathbf{e}|} \Bigg) Q_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\nu}\in\Lambda} c_{\Lambda;\boldsymbol{\nu}} Q_{\boldsymbol{\nu}}, \tag{3.3}$$

where $c_{\Lambda;\boldsymbol{\nu}} := \sum_{\{\mathbf{e}\in\{0,1\}^{\mathbb{N}}\,:\,\boldsymbol{\nu}+\mathbf{e}\in\Lambda\}} (-1)^{|\mathbf{e}|}$. The latter representation of Q_Λ in (3.3) is easily verified by induction over $d = |\operatorname{supp} \boldsymbol{\nu}|$, and is closely related to the so-called "combination technique". It is often preferred in implementations, since it allows to avoid unnecessary computation of $Q_{\boldsymbol{\nu}}$, whenever the combination coefficient $c_{\Lambda;\boldsymbol{\nu}}$ vanishes.

Remark 3.1. *Assume that for some sequence $(\chi_j)_{j\in\mathbb{N}_0}$ with $\chi_0 = 0$ we have $\chi_{n;j} = \chi_j$ for all $n \geq j$. Such sequences fulfilling* (3.1) *are known. One example are the so called Leja points, see [6, 5, 9]. In this case, the evaluation of* (3.3) *requires the value of u at all points $\chi_{\boldsymbol{\nu}} := (\chi_{\nu_j})_{j\in\mathbb{N}} \in U$ for $\boldsymbol{\nu} \in \Lambda$. Thus the number of quadrature points employed by Q_Λ equals the number of multiindices in Λ. For the more general case described above, this is not true.*

Hereafter the main results of this paper are established. First, the dimension-independent convergence rate of $2/p - 1$ for the Smolyak quadrature in terms of number of multiindices (or quadrature points under Rmk. 3.1) is given for $(\boldsymbol{b}, \varepsilon)$-holomorphic functions with $\boldsymbol{b} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$. Then, we prove that the actual cost of the algorithm is near linear in the number of quadrature points, thus yielding (up to an epsilon) the same convergence rate with respect to the overall complexity. Additionally, we shall see that almost the same convergence rate, in terms of number of quadrature points, can be retained for the general case described above (i.e. without nested quadrature points as in Rmk. 3.1).

3.1 Convergence for $(\boldsymbol{b}, \varepsilon)$-holomorphic functions

The Smolyak operator Q_Λ in (3.3) satisfies the following elementary properties.

Lemma 3.2. *Let $\Lambda \subseteq \mathcal{F}$ be downward closed and finite. Then, the Smolyak operator Q_Λ in* (3.3) *satisfies the following properties:*

(i) $Q_\Lambda P = \int_{[-1,1]^{\mathbb{N}}} P(\boldsymbol{y}) \mathrm{d}\mu(\boldsymbol{y})$ *for all* $P \in \operatorname{span}\{\boldsymbol{y}^{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \Lambda\}$.

(ii) If additionally $\chi_{0;0} = 0$, then $Q_\Lambda P = \int_U P(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} = 0$ for all $P \in \operatorname{span}\{\boldsymbol{y}^{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathcal{F}\backslash\mathcal{F}_2\}$.

(iii) If additionally (3.1) *holds, then there exists a constant $C > 0$ independent of Λ such that*

$$|Q_\Lambda P| \leq C |\{\boldsymbol{\eta} : \boldsymbol{\eta} \in \Lambda,\ \boldsymbol{\eta} \leq \boldsymbol{\nu}\}|^{\tau+1} \|P\|_{L^\infty(U)} \qquad \forall P \in \operatorname{span}\{\boldsymbol{y}^{\boldsymbol{\mu}} : \boldsymbol{\mu} \leq \boldsymbol{\nu}\}.$$

Proof. Items (i), (iii) were shown in [8] for Smolyak interpolation operators, but since we consider interpolatory quadrature rules, they hold verbatim for the quadrature operators.

For (ii) consider the one dimensional quadrature operator $Q_n : C_0([-1,1]) \to \mathbb{R}$, employing $n+1$ quadrature points in $[-1,1]$. The monomial $y \mapsto y$ satisfies $Q_n y = \int_{-1}^1 y \mathrm{d}y = 0$ for all

$n \in \mathbb{N}_0 \cup \{-1\}$: this is clear for $n \geq 1$, since every interpolatory quadrature rule with at least two points integrates polynomials of degree 1 exactly. It is true for $n = 0$, because by assumption the first quadrature point fulfills $\chi_{0;0} = 0$, so that Q_0 is the one point Gauss (midpoint) rule. By definition, $Q_{-1}y = 0$. For $\boldsymbol{\nu} \in \mathcal{F}$ and $\boldsymbol{\mu} \in \mathcal{F} \backslash \mathcal{F}_2$ arbitrary there exists j with $\mu_j = 1$ and thus

$$Q_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\mu}} = \bigotimes_{j\in\mathbb{N}} (Q_{\nu_j} - Q_{\nu_j-1}) \boldsymbol{y}^{\boldsymbol{\mu}} = \prod_{j\in\mathbb{N}} (Q_{\nu_j} - Q_{\nu_j-1}) y_j^{\mu_j} = 0 = \int_U \boldsymbol{y}^{\boldsymbol{\mu}} \mathrm{d}\mu(\boldsymbol{y}), \tag{3.4}$$

which by (3.3) gives $Q_\Lambda \boldsymbol{y}^{\boldsymbol{\mu}} = 0 = \int_U \boldsymbol{y}^{\boldsymbol{\nu}} \mathrm{d}\mu(\boldsymbol{y})$ for all $\boldsymbol{\mu} \in \mathcal{F} \backslash \mathcal{F}_2$. ☐

We next formulate our main convergence result, which assumes $\chi_{n;j}$ and Q_Λ as in the beginning of the section, in particular $\chi_{0;0} = 0$. We also assume polynomial growth of the univariate Lebesgue constants, (3.1), holds for some $\tau > 0$. It gives a convergence rate in terms of number of multiindices. However, note that with nested sequences $\chi_{n;j}$ as in Rmk. 3.1, the convergence rate is also in terms of number of quadrature points.

Theorem 3.3. *Let X be a Banach space, $U = [-1,1]^{\mathbb{N}}$ and let $u : U \to X$ satisfy Assumption 2.1 with $\boldsymbol{b} \in \ell^p$, for some $p \in (0,1)$. Then there exists a constant C and for every $N \in \mathbb{N}$ there exists a downward closed set $\Lambda_N \subseteq \mathcal{F}$ with $|\Lambda_N| \leq N$ such that*

$$\left\| \int_U u(\boldsymbol{y}) \mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_N} u \right\|_X \leq C N^{-(\frac{2}{p}-1)}. \tag{3.5}$$

Furthermore under the assumptions of Thm. 2.11, the set Λ_N can be chosen as follows

$$\Lambda_N = \{\boldsymbol{\nu} \in \mathcal{F} \,:\, m_{\boldsymbol{\nu}} \geq \varepsilon_N\} \tag{3.6}$$

for some threshold $\varepsilon_N > 0$ and a sequence $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ as in item (ii) of Thm. 2.11 with $k = 2$.

Proof. We will employ Thm. 2.11 with $k = 2$, $\tilde{\tau} = \tau + 1$ (here τ comes from assumption (3.1) on the univariate Lebesgue constant), and in case no set Z as in Assumption 2.7 is given, we let $Z := \mathbb{N}_0$. Then, by Thm. 2.11, the weighted Taylor coefficient $\prod_{j\in\mathbb{N}} (\nu_j+1)^{\tau+1} \|u_{\boldsymbol{\nu}}\|_X$ is bounded by $t_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} \in \mathcal{F}$ and by $m_{\boldsymbol{\nu}}$ whenever $\boldsymbol{\nu} \in \mathcal{F}_2$. For $\varepsilon > 0$ we define $\Lambda_{\mathrm{m};\varepsilon} := \{\boldsymbol{\nu} \in \mathcal{F} \,:\, m_{\boldsymbol{\nu}} \geq \varepsilon\}$, which is a finite downward closed index set, thanks to the monotonicity of $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^{p/2}(\mathcal{F})$. Furthermore $u(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}}$ converges absolutely in $L^\infty(U, X)$.

Fix $\varepsilon > 0$. As $Q_{\Lambda_{\mathrm{m};\varepsilon}} : C^0(U) \to X$ is a bounded linear operator, by the absolute convergence of $u(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}}$ and Lemma 3.2 (i)

$$Q_{\Lambda_{\mathrm{m};\varepsilon}} u = Q_{\Lambda_{\mathrm{m};\varepsilon}} \sum_{\boldsymbol{\nu}\in\mathcal{F}} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} = \sum_{\boldsymbol{\nu}\in\mathcal{F}} u_{\boldsymbol{\nu}} Q_{\Lambda_{\mathrm{m};\varepsilon}} \boldsymbol{y}^{\boldsymbol{\nu}} = \int_U \sum_{\boldsymbol{\nu}\in\Lambda_{\mathrm{m};\varepsilon}} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} \mathrm{d}\mu(\boldsymbol{y}) + \sum_{\boldsymbol{\nu}\in(\Lambda_{\mathrm{m};\varepsilon})^c} u_{\boldsymbol{\nu}} Q_{\Lambda_{\mathrm{m};\varepsilon}} \boldsymbol{y}^{\boldsymbol{\nu}}, \tag{3.7}$$

where the latter sum is absolutely convergent in X. Here, $(\Lambda_{\mathrm{m};\varepsilon})^c = \mathcal{F} \backslash \Lambda_{\mathrm{m};\varepsilon}$. With Lemma 3.2 (ii) we arrive at

$$Q_{\Lambda_{\mathrm{m};\varepsilon}} u = \int_U \sum_{\boldsymbol{\nu}\in\Lambda_{\mathrm{m};\varepsilon}} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} \mathrm{d}\mu(\boldsymbol{y}) + \sum_{\boldsymbol{\nu}\in\mathcal{F}_2\cap(\Lambda_{\mathrm{m};\varepsilon})^c} u_{\boldsymbol{\nu}} Q_{\Lambda_{\mathrm{m};\varepsilon}} \boldsymbol{y}^{\boldsymbol{\nu}}. \tag{3.8}$$

Note that item (ii) of Lemma 3.2 also implies $\int_U u(\boldsymbol{y}) \mathrm{d}\mu(\boldsymbol{y}) = \int_U \sum_{\boldsymbol{\nu}\in\mathcal{F}_2} u_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} \mathrm{d}\mu(\boldsymbol{y})$. Using

Lemma 3.2 (iii) and (3.8) we get

$$\begin{aligned}\left\|\int_U u(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_{\mathrm{m};\varepsilon}}u\right\|_X &\le \left\|\int_U \sum_{\boldsymbol{\nu}\in\mathcal{F}_2\cap(\Lambda_{\mathrm{m};\varepsilon})^c} u_{\boldsymbol{\nu}}\boldsymbol{y}^{\boldsymbol{\nu}}\mathrm{d}\mu(\boldsymbol{y})\right\|_X + \sum_{\boldsymbol{\nu}\in\mathcal{F}_2\cap(\Lambda_{\mathrm{m};\varepsilon})^c} \|u_{\boldsymbol{\nu}}\|_X |Q_{\Lambda_{\mathrm{m};\varepsilon}}\boldsymbol{y}^{\boldsymbol{\nu}}| \\ &\le \sum_{\boldsymbol{\nu}\in\mathcal{F}_2\cap(\Lambda_{\mathrm{m};\varepsilon})^c} \|u_{\boldsymbol{\nu}}\|_X \|\boldsymbol{y}^{\boldsymbol{\nu}}\|_{L^\infty(U)}(1+C|\{\boldsymbol{\mu}\in\mathcal{F}\,:\,\boldsymbol{\mu}\le\boldsymbol{\nu}\}|^{\tau+1}) \\ &\le C\sum_{\boldsymbol{\nu}\in\mathcal{F}_2\cap(\Lambda_{\mathrm{m};\varepsilon})^c} \|u_{\boldsymbol{\nu}}\|_X \prod_{j\in\mathbb{N}}(\nu_j+1)^{\tau+1} \le C\sum_{\boldsymbol{\nu}\in(\Lambda_{\mathrm{m};\varepsilon})^c} m_{\boldsymbol{\nu}} = C\sum_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,m_{\boldsymbol{\nu}}<\varepsilon\}} m_{\boldsymbol{\nu}}, \end{aligned} \tag{3.9}$$

where the final equality holds by definition of $\Lambda_{\mathrm{m};\varepsilon}$. Exploiting $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^{p/2}(\mathcal{F})$, a result known as the Stechkin Lemma then allows to bound the last sum by $C|\Lambda_{\mathrm{m};\varepsilon}|^{1-2/p}$. One way to see this, is to note that rearranging the sequence $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ as a monotonically decreasing sequence $(m_j)_{j\in\mathbb{N}}$, Lemma 2.9 gives $m_j \le Cj^{-2/p}$ and thus $\sum_{j>N} m_j \le \int_N^\infty x^{-2/p} \le CN^{1-2/p}$.

So far we have shown that (3.5) holds for $N = |\Lambda_{\mathrm{m};\varepsilon}|$ with the index set $\Lambda_{\mathrm{m};\varepsilon}$, and it remains to define Λ_N for arbitrary $N\in\mathbb{N}$. Since Z satisfies Assumption 2.7, so does Z_k in Thm. 2.11, and wlog we assume that the constant C_Z is the same as for Z. Now fix $N\in\mathbb{N}$. By Thm. 2.11, $m_{\boldsymbol{\nu}} \ne m_{\boldsymbol{\mu}}$ whenever $\lfloor\boldsymbol{\nu}\rfloor_{Z_k} \ne \lfloor\boldsymbol{\mu}\rfloor_{Z_k}$. Thus $m_{\mathbf{0}} > m_{\boldsymbol{\nu}}$ for all $\boldsymbol{\nu}\ne\mathbf{0}$, and so $|\Lambda_{\mathrm{m};m_{\mathbf{0}}}| = |\{\mathbf{0}\}| = 1$. Therefore we can find $0<\varepsilon<\tilde\varepsilon$ such that $|\Lambda_{\mathrm{m};\tilde\varepsilon}| \le N \le |\Lambda_{\mathrm{m};\varepsilon}|$ with $\varepsilon>0$ minimal, $\tilde\varepsilon>0$ maximal. Set $\Lambda_N := \Lambda_{\mathrm{m};\tilde\varepsilon}$. If $|\Lambda_{\mathrm{m};\tilde\varepsilon}| = N$ we are done. Otherwise $|\Lambda_{\mathrm{m};\varepsilon}| > |\Lambda_{\mathrm{m};\tilde\varepsilon}|$. Since $m_{\boldsymbol{\nu}} = m_{\lfloor\boldsymbol{\nu}\rfloor_{Z_k}}$ for all $\boldsymbol{\nu}\in\mathcal{F}$, there exists $\boldsymbol{\nu}\in\mathcal{F}\cap(Z^k)^{\mathbb{N}}$ with

$$\Lambda_{\mathrm{m};\varepsilon}\backslash\Lambda_{\mathrm{m};\tilde\varepsilon} = \{\boldsymbol{\mu}\in\mathcal{F}\,:\,\boldsymbol{\nu} = \lfloor\boldsymbol{\mu}\rfloor_{Z_k}\} =: A. \tag{3.10}$$

We have $|A| = \prod_{j\in\mathrm{supp}_{\boldsymbol{\nu}}} \lceil\nu_j+1\rceil_{Z_k} - \nu_j$. Wlog assume $\nu_1\ge1$ and set $\tilde\nu_1 := \lfloor\nu_1-1\rfloor_{Z_k}\ge0$ and $\tilde{\boldsymbol{\nu}} := (\tilde\nu_1,\nu_2,\nu_3,\dots)$. Then $\tilde{\boldsymbol{\nu}}\le\boldsymbol{\nu}$, $\lfloor\tilde{\boldsymbol{\nu}}\rfloor_{Z_k} = \tilde{\boldsymbol{\nu}} \ne \boldsymbol{\nu} = \lfloor\boldsymbol{\nu}\rfloor_{Z_k}$ and thus $m_{\tilde{\boldsymbol{\nu}}} > m_{\boldsymbol{\nu}}$. Since $\Lambda_{\mathrm{m};\varepsilon}$ is downward closed and $\tilde{\boldsymbol{\nu}}\le\boldsymbol{\nu}\in\Lambda_{\mathrm{m};\varepsilon}$, $B := \{\boldsymbol{\mu}\in\mathcal{F}\,:\,\tilde{\boldsymbol{\nu}} = \lfloor\boldsymbol{\mu}\rfloor_{Z_k}\}$ must be a subset of Λ_ε because $m_{\tilde{\boldsymbol{\nu}}} = m_{\boldsymbol{\mu}}$ for all $\lfloor\boldsymbol{\mu}\rfloor_{Z_k} = m_{\boldsymbol{\nu}}$. Since $A\cap B=\emptyset$, by (3.10) also $B\subseteq\Lambda_{\mathrm{m};\tilde\varepsilon}$. With Assumption 2.7 we get

$$|B| = (\nu_1 - \lfloor\nu_1-1\rfloor_{Z_k})\prod_{j\in\mathrm{supp}_{\boldsymbol{\nu}},\ j\ge2}(\lceil\nu_j+1\rceil_{Z_k}-\nu_j) \ge \frac{1}{C_Z}\prod_{j\in\mathrm{supp}_{\boldsymbol{\nu}}}(\lceil\nu_j+1\rceil_{Z_k}-\nu_j) = C_Z^{-1}|A| \tag{3.11}$$

and thus $|\Lambda_{\mathrm{m};\varepsilon}| \le C_Z|\Lambda_{\mathrm{m};\tilde\varepsilon}|$. Consequently $|\Lambda_{\mathrm{m};\tilde\varepsilon}| \ge C_Z^{-1}N$ and ultimately with $\Lambda_N = \Lambda_{\mathrm{m};\tilde\varepsilon}$ the left-hand side of (3.5) is bounded by

$$\left\|\int_U u(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_{\mathrm{m};\tilde\varepsilon}}u\right\|_X \le C|\Lambda_{\mathrm{m};\tilde\varepsilon}|^{-(\frac{2}{p}-1)} \le CC_Z^{\frac{2}{p}-1}N^{-(\frac{2}{p}-1)}. \qquad \square$$

Remark 3.4. *The requirement $\chi_{0;0}=0$ is always satisfied for symmetric probability measures, in particular for the uniform measure μ, for centered Gaussian measures and, generally, for marginal probability measures on the parameters y_j which are symmetric and centered.*

Remark 3.5. *In the papers [24], [23], rather than Assumption 2.1, a requirement of the following type is presumed:*

$$u \text{ is holomorphic and uniformly bounded on some polydisc } B_{\boldsymbol{\rho}} := \bigtimes_{j\in\mathbb{N}} B_{\rho_j} \subseteq \mathbb{C}^{\mathbb{N}}, \text{ with } \rho_j>1 \text{ for all } j\in\mathbb{N} \text{ and } (\rho_j^{-1})_{j\in\mathbb{N}}\in\ell^p,\ p\in(0,1). \tag{3.12}$$

In these references, under assumptions similar to (3.12) *dimension-independent convergence rates* $(1/p-1)/2$ *and* $(1/p-1)$*, respectively, are established (cp. [24, Assumption 4.2, Thm. 5.5], [23, Cor. 5.9] for the precise assumptions and statements).*

Let Assumption 2.1 be satisfied. Let $\kappa > 1$ so small and $J \in \mathbb{N}$ so large that $(\kappa-1)\sum_{j\in\mathbb{N}} b_j + \sum_{j>J} b_j^p < \varepsilon$. This is possible because $\|\boldsymbol{b}\|_{\ell^1}$, $\|\boldsymbol{b}\|_{\ell^p} < \infty$. Set $\rho_j := \kappa$ for $j \le J$ and $\rho_j := \max\{\kappa, b_j^{p-1}\}$. Then $\sum_{j\in\mathbb{N}} b_j(\rho_j - 1) \le \sum_{j\in\mathbb{N}}(\kappa-1)b_j + \sum_{j>J} b_j^p \le \varepsilon$. Thus Assumption 2.1 implies (3.12) with this $\boldsymbol{\rho}$. Note that $\boldsymbol{\rho} \in \ell^{p/(1-p)}$ and $p/(1-p) > p$. On the other hand, (3.12) implies Assumption 2.1, with $b_j := (\rho_j - 1)^{-1}\varepsilon$ and $(b_j)_{j\in\mathbb{N}} \in \ell^p$: if $\tilde{\boldsymbol{\rho}}$ is arbitrary with $\sum_j b_j(\tilde{\rho}_j - 1) < \varepsilon$, then $b_j(\tilde{\rho}_j - 1) < \varepsilon$, and thus $\varepsilon(\tilde{\rho}_j - 1)/(\rho_j - 1) < \varepsilon$ implying $\tilde{\rho}_j < \rho_j$ for each $j \in \mathbb{N}$. Since u allows a bounded holomorphic extension to $B_{\boldsymbol{\rho}}$ by (3.12), it also allows a bounded holomorphic extension to $B_{\tilde{\boldsymbol{\rho}}} \subseteq B_{\boldsymbol{\rho}}$. Hence Assumption 2.1 is more general than (3.12).

In summary, Theorem 3.3 improves the dimension-independent convergence rates $(1/p-1)/2$, $(1/p-1)$ for the anisotropic Smolyak quadrature proved in [24], [23] by at least a factor 4 and 2, respectively, under slightly weaker assumptions, as we explain in Examples 4.1, 4.2 ahead.

3.2 Complexity

Let $\Lambda \subseteq \mathcal{F}$ be a finite downward closed index set. To bound the cost of evaluating the Smolyak quadrature $Q_\Lambda u$, we shall work with the representation (3.3), and assume each integrand evaluation to be of cost $O(1)$. For $\Lambda \subseteq \mathcal{F}$ with $|\Lambda| < \infty$ let us introduce its effective dimension and maximal degree by

$$d(\Lambda) := \max_{\boldsymbol{\nu}\in\Lambda} |\operatorname{supp} \boldsymbol{\nu}| \qquad \text{and} \qquad m(\Lambda) := \max_{\boldsymbol{\nu}\in\Lambda} \max_{j\in\mathbb{N}} \nu_j. \tag{3.13}$$

We begin our cost analysis with the computation of $(c_{\Lambda;\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\Lambda}$.

Lemma 3.6. *For any finite, downward closed set $\Lambda \subset \mathcal{F}$, the coefficients $(c_{\Lambda;\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\Lambda}$ in (3.3) can be computed with cost bounded by $C2^{d(\Lambda)}|\Lambda|$, for some C independent of Λ and for $d(\Lambda)$ as in (3.13).*

Proof. This is achieved by looping over all $\boldsymbol{\nu} \in \Lambda$, and updating the coefficient of all (at most $2^{d(\Lambda)}$) neighbours in Λ of the type $\boldsymbol{\nu} - \mathbf{e}$ for some $\mathbf{e} \in \{0,1\}^{\mathbb{N}}$. □

Next, we consider the evaluation of $Q_{\boldsymbol{\nu}} u$, which corresponds to the tensor quadrature $\bigotimes_{j\in\operatorname{supp}\boldsymbol{\nu}} Q_{\nu_j}$. The one dimensional quadrature rule Q_n with $n+1$ points in $[-1,1]$, can be evaluated with complexity $O(n)$, assuming that the quadrature weights have been precomputed with complexity $O(n^3)$, e.g. by solving a linear system. The cost of computing those weights sums up to $\sum_{j=1}^{m(\Lambda)} j^3 = O(m(\Lambda)^4)$, and once this is done each $Q_{\boldsymbol{\nu}}$ in (3.3) adds $O(\prod_j(\nu_j+1))$ to the total complexity (this equals the number of quadrature points employed by $Q_{\boldsymbol{\nu}}$ and here we use our assumption that each evaluation of u is of cost $O(1)$). Finally, we need to evaluate the sum $\sum_{\{\boldsymbol{\nu}\in\Lambda\,:\,c_{\Lambda;\boldsymbol{\nu}}\neq 0\}} c_{\Lambda;\boldsymbol{\nu}}(Q_{\boldsymbol{\nu}} u)$, which adds another $O(|\{\boldsymbol{\nu}\in\Lambda : c_{\Lambda;\boldsymbol{\nu}} \neq 0\}|)$ floating point operations. In summary, we introduce

$$\operatorname{cost}(Q_\Lambda) := \underbrace{m(\Lambda)^4}_{\text{precomp. of weights}} + \underbrace{2^{d(\Lambda)}|\Lambda|}_{\text{comp. of } (c_{\Lambda;\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\Lambda}} + \sum_{\{\boldsymbol{\nu}\in\Lambda\,:\,c_{\Lambda;\boldsymbol{\nu}}\neq 0\}} \underbrace{\prod_{j\in\mathbb{N}}(\nu_j+1)}_{\text{evaluation of } Q_{\boldsymbol{\nu}}}, \tag{3.14}$$

as a measure for the cost of evaluating the Smolyak quadrature $Q_\Lambda u$, where the cost of the final summation was absorbed in the third term in (3.14).

In Section 2.3 we required the sequence $(\zeta_j)_{j\in\mathbb{N}}$ in Assumption 2.7 to grow at most exponentially. In this section, where the meaning of Z will become more clear, in order to profit from our previous estimates, we shall need that the sequence grows at least exponentially. For this reason we make the following assumption.

Assumption 3.7. *The set $Z = \{\zeta_j : j \in \mathbb{N}_0\} \subseteq \mathbb{N}_0$ consists of the strictly monotonically growing, nonnegative sequence $(\zeta_j)_{j\in\mathbb{N}_0}$, where $\zeta_0 = 0$. There exists a constant C_Z with $\sum_{j=1}^m (\zeta_j + 1) \le C_Z \zeta_m$ for all $m \in \mathbb{N}$.*

In order to exploit the vanishing of certain $c_{\Lambda;\boldsymbol{\nu}}$ in (3.3), we consider index sets of the following type: Under Assumption 3.7, suppose that Λ has the property (cp. (2.29))

$$\boldsymbol{\nu} \in \Lambda \Longrightarrow \lceil \boldsymbol{\nu} \rceil_Z \in \Lambda. \tag{3.15}$$

For $\Lambda \subseteq \mathcal{F}$, set

$$\Lambda^Z := \{\boldsymbol{\nu} \in \Lambda \,:\, \nu_j \in Z \;\forall\, j \in \mathbb{N}\}. \tag{3.16}$$

Lemma 3.8. *Let Λ be downward closed with the property* (3.15)*. Then for all $\boldsymbol{\nu} \in \Lambda \backslash \Lambda^Z$*

$$c_{\Lambda;\boldsymbol{\nu}} := \sum_{\{\mathbf{e}\in\{0,1\}^{\mathbb{N}}\,:\,\boldsymbol{\nu}+\mathbf{e}\in\Lambda\}} (-1)^{|\mathbf{e}|} = 0. \tag{3.17}$$

Proof. Let $\mathbf{0} \neq \boldsymbol{\nu} \in \Lambda\backslash\Lambda^Z$. Since $\boldsymbol{\nu} \notin \Lambda^Z$, there exists $j \in \mathbb{N}$ with $\nu_j \notin Z$. Set $A := \{\mathbf{e} = (e_i)_{i\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \,:\, \boldsymbol{\nu} + \mathbf{e} \in \Lambda,\; e_j = 0\}$, and let $\mathbf{e} \in A$ arbitrary. Then, with $\mathbf{e}_j = (\delta_{ij})_{i\in\mathbb{N}}$ we get $\boldsymbol{\nu} + \mathbf{e} + \mathbf{e}_j \leq \lceil \boldsymbol{\nu} + \mathbf{e} \rceil_Z \in \mathcal{F}$, so that by the downward closedness of Λ it holds $\boldsymbol{\nu} + \mathbf{e} + \mathbf{e}_j \in \Lambda$. Thus

$$\sum_{\{\mathbf{e}\in\{0,1\}^{\mathbb{N}}\,:\,\boldsymbol{\nu}+\mathbf{e}\in\Lambda\}} (-1)^{|\mathbf{e}|} = \sum_{\mathbf{e}\in A} (-1)^{|\mathbf{e}|} - \sum_{\mathbf{e}\in A} (-1)^{|\mathbf{e}|} = 0. \qquad \square$$

Note that in fact all we required for Z in the above Lemma is $Z \subseteq \mathbb{N}_0$. The next Lemma provides an upper bound for the third term in (3.14), in the general case.

Lemma 3.9. *Let Z satisfy Assumption 3.7 for some constant C_Z. Let $\Lambda \subseteq \mathcal{F}$ be downward closed and $|\Lambda| < \infty$. Then with $d(\Lambda)$, Λ^Z as in* (3.13), (3.16)

$$\sum_{\boldsymbol{\nu}\in\Lambda} \prod_{j\in\mathbb{N}} (\nu_j + 1) \leq |\Lambda|^2 \qquad \textit{and} \qquad \sum_{\boldsymbol{\nu}\in\Lambda^Z} \prod_{j\in\mathbb{N}} (\nu_j + 1) \leq C_Z^{d(\Lambda)} |\Lambda|. \tag{3.18}$$

Proof. We start with the first inequality. Due to the downward closedness of Λ it holds

$$\sum_{\boldsymbol{\nu}\in\Lambda} \prod_{j\in\operatorname{supp}\boldsymbol{\nu}} (\nu_j + 1) = \sum_{\boldsymbol{\nu}\in\Lambda} |\{\boldsymbol{\mu} \in \Lambda \,:\, \boldsymbol{\mu} \leq \boldsymbol{\nu}\}| \leq \sum_{\boldsymbol{\nu}\in\Lambda} |\Lambda| = |\Lambda|^2. \tag{3.19}$$

To prove the second inequality, define for $n \in \mathbb{N}$ and for any $\Gamma \subseteq \mathcal{F}$

$$\Gamma_n := \{\boldsymbol{\nu} \in \Gamma \,:\, |\operatorname{supp}\boldsymbol{\nu}| = n\}, \qquad \Gamma_n^Z := \{\boldsymbol{\nu} \in \Gamma \,:\, |\operatorname{supp}\boldsymbol{\nu}| = n,\; \nu_j \in Z \;\forall j \in \mathbb{N}\}. \tag{3.20}$$

The goal is to show that there exists a constant $C_Z > 0$ such that for every $n \in \mathbb{N}$

$$\sum_{\boldsymbol{\nu}\in\Lambda_n^Z} \prod_{j\in\mathbb{N}} (\nu_j + 1) \leq C_Z^n |\Lambda_n| \;. \tag{3.21}$$

The case $n = 0$ with $\Lambda_0 = \{\mathbf{0}\}$ is trivial. Next, we let $n = 1$. Using downward closedness of Λ (and therefore of Λ_1), with $\mathbf{e}_j := (\delta_{ji})_{i\in\mathbb{N}}$ it holds that $i\mathbf{e}_j \in \Lambda_1$ implies $k\mathbf{e}_j \in \Lambda_1$ for all $k \in \{1, \dots, i\}$. Therefore

$$|\Lambda_1| = \sum_{\{j\in\mathbb{N}\,:\,\mathbf{e}_j\in\Lambda\}} \max\{i \,:\, i\mathbf{e}_j \in \Lambda\} \tag{3.22}$$

and with Assumption 3.7

$$\sum_{\boldsymbol{\nu}\in\Lambda_1^Z} \prod_{j\in\mathbb{N}} (\nu_j + 1) = \sum_{j\in\mathbb{N}} \sum_{\{0<i\in Z\,:\,i\mathbf{e}_j\in\Lambda_1\}} (i+1) \leq \sum_{j\in\mathbb{N}} C_Z \max\{i \,:\, i\mathbf{e}_j \in \Lambda\} = C_Z |\Lambda_1|. \tag{3.23}$$

We proceed by induction over n, and assume (3.21) to be satisfied for $n-1$. Additionally, for the moment we restrict ourselves to the case $\operatorname{supp}\boldsymbol{\nu} = \{1,\dots,n\}$ for all $\boldsymbol{\nu} \in \Lambda_n$. For all $i \in \mathbb{N}$ and $\Gamma \subseteq \mathcal{F}$ arbitrary set

$$\Gamma(i) := \{(\nu_2,\nu_3,\dots) \in \mathcal{F} \,:\, (i,\nu_2,\nu_3,\dots) \in \Gamma\}. \tag{3.24}$$

Furthermore $\pi_1(\Gamma) := \bigcup_{j\in\mathbb{N}} \Gamma(j)$. We have for $0 < i \in Z$

$$\begin{aligned}\Lambda_n^Z(i) &= \{(\nu_2,\nu_3,\dots) \,:\, (\nu_1,\nu_2,\nu_3,\dots) \in \Lambda,\ \nu_1 = i,\ \nu_j \in Z\ \forall j,\ |\operatorname{supp}\boldsymbol{\nu}| = n\}\\ &= (\Lambda_n(i))_{n-1}^Z \end{aligned} \tag{3.25}$$

in the sense of (3.20), and one readily verifies that $\Lambda_n(i)$ is downward closed. It holds $\Lambda_n^Z = \bigcup_{\{0<i\,:\,i\in S_n^Z\}} \Lambda_n^Z(i)$, since $\nu_1 > 0$ if $\boldsymbol{\nu} \in \Lambda_n$ by our assumption that $\operatorname{supp}\boldsymbol{\nu} = \{1,\dots,n\}$. Hence, using the induction hypothesis (3.21) for $n-1$ on $(\Lambda_n(i))_{n-1}^Z$ we obtain with $(\Lambda_n(i))_{n-1} = \Lambda_n(i)$ and (3.25)

$$\begin{aligned}\sum_{\boldsymbol{\nu}\in\Lambda_n^Z}\prod_{j\in\mathbb{N}}(\nu_j+1) &= \sum_{0<i\in Z}(i+1)\sum_{\boldsymbol{\mu}\in\Lambda_n^Z(i)}\prod_{j\in\mathbb{N}}(\mu_j+1) = \sum_{0<i\in Z}(i+1)\sum_{\boldsymbol{\mu}\in(\Lambda_n(i))_{n-1}^Z}\prod_{j\in\mathbb{N}}(\mu_j+1)\\ &\le \sum_{0<i\in Z}(i+1)C_Z^{n-1}|(\Lambda_n(i))_{n-1}| = C_Z^{n-1}\sum_{0<i\in Z}(i+1)\sum_{\{\boldsymbol{\mu}\in\mathcal{F}\,:\,(i,\boldsymbol{\mu})\in\Lambda_n\}}1\\ &= C_Z^{n-1}\sum_{\boldsymbol{\mu}\in\pi_1(\Lambda_n)}\ \sum_{\{0<i\in Z\,:\,(i,\boldsymbol{\mu})\in\Lambda_n\}}(i+1)\\ &\le C_Z^{n-1}\sum_{\boldsymbol{\mu}\in\pi_1(\Lambda_n)} C_Z \max\{i\in\mathbb{N}\,:\,(i,\boldsymbol{\mu})\in\Lambda_n\}\\ &= C_Z^n \sum_{\boldsymbol{\mu}\in\pi_1(\Lambda_n)}\ \sum_{\{i\in\mathbb{N}\,:\,(i,\boldsymbol{\mu})\in\Lambda_n\}} 1 = C_Z^n|\Lambda_n|,\end{aligned} \tag{3.26}$$

where we have employed Assumption 3.7.

For the general case where $\operatorname{supp}\boldsymbol{\nu} = \{1,\dots,n\}$ is not fulfilled for all $\boldsymbol{\nu} \in \Lambda_n$ we distinguish between all possible sets $\{a_1,\dots,a_n\} \subseteq \mathbb{N}$ of cardinality n, and use the above argument for every subset of Λ_n containing all $\boldsymbol{\nu} \in \mathcal{F}$ satisfying $\operatorname{supp}\boldsymbol{\nu} = \{a_1,\dots,a_n\}$. Since those subsets of Λ_n have empty intersection, and the claim holds for each of them, we have shown (3.21) for all $n \in \mathbb{N}_0$, which is a stronger statement than the second inequality in (3.18). □

Remark 3.10. *Estimates* (3.18) *are sharp in the following sense: Let* $\Lambda = \{\boldsymbol{\nu} \in \mathcal{F} : \operatorname{supp}\boldsymbol{\nu} \subseteq \{1,\dots,d\},\ \nu_j \le N\ \forall j\}$ *and set* $Z := \{0\} \cup \{2^j : j \in \mathbb{N}_0\}$. *Then, with* $N = 2^m$, *we have* $|\Lambda| = (N+1)^d$ *and*

$$\sum_{\boldsymbol{\nu}\in\Lambda}\prod_{j\in\mathbb{N}}(\nu_j+1) = \prod_{j=1}^d\sum_{i=1}^{N+1} i = \left(\frac{(N+1)(N+2)}{2}\right)^d \ge 2^{-d}((N+1)^d)^2 = 2^{-d}|\Lambda|^2, \tag{3.27}$$

as well as (for $m \ge 1$*)*

$$\begin{aligned}\sum_{\boldsymbol{\nu}\in\Lambda^Z}\prod_{j\in\mathbb{N}}(\nu_j+1) &= \prod_{j=1}^d\left(1+\sum_{i=0}^m(2^i+1)\right) \ge \prod_{j=1}^d(1+2^{m+1}-1+m+1)\\ &\ge (2(2^m+1))^d \ge 2^d(N+1)^d = 2^d|\Lambda|.\end{aligned} \tag{3.28}$$

Letting $N \to \infty$ *in* (3.27) *and* $d \to \infty$ *in* (3.28), *a better asymptotic behaviour than quadratic in* $|\Lambda|$ *in the first case, and linear in* $|\Lambda|$ *with a constant depending exponentially on* $d(\Lambda)$ *in the second case therefore cannot be expected in general.*

However, these estimates may not accurately measure the actual cost of evaluating (3.3)*, since they do not take into account the fact that some (further) coefficients in* (3.3) *might vanish. Indeed, for the above example* Q_Λ *is the tensor product quadrature* $Q_{\boldsymbol{\nu}}$ *with* $\nu_j = N$ *if* $j \leq d$ *and* $\nu_j = 0$ *otherwise. The cost of its evaluation is proportional to* $|\Lambda| = (N+1)^d$ *if* d *and or* $N \to \infty$.

As an example, we shall now consider the situation of Thm. 3.3. For ease of exposition, the requirements in the following lemma are shortly recalled, but note that the presumptions on $m_{\boldsymbol{\nu}}$ and Λ_N are as stated in Thms. 2.11, 3.3, and thus satisfied in the settings of these theorems.

Lemma 3.11. *Let* Z *as in Assumption 2.7,* $r > 0$ *and* $C_r > 0$*. Let* $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^1(\mathcal{F})$ *monotonically decreasing such that*

(i) $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ *satisfies* (2.41)-(2.42)*,*

(ii) $m_{\boldsymbol{\nu}} = m_{\boldsymbol{\mu}}$ *iff* $\lfloor \boldsymbol{\nu} \rfloor_Z = \lfloor \boldsymbol{\nu} \rfloor_Z$*,*

(iii) the j*th largest element of* $\{m_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathcal{F}\}$ *is bounded from below by* $C_r j^{-r}$*,*

(iv) for all $N \in \mathbb{N}$*,* $\Lambda_N := \{\boldsymbol{\nu} \in \mathcal{F} : m_{\boldsymbol{\nu}} \geq \varepsilon_N\}$*, for some monotonically decreasing sequence* $\varepsilon_N \to 0$*.*

Then, for the quantities defined in (3.13) *it holds*

$$d(\Lambda_N) = o(\log(|\Lambda_N|)) \qquad \textit{and} \qquad m(\Lambda_N) = O(\log(|\Lambda_N|)) \quad \textit{as } |\Lambda_N| \to \infty. \tag{3.29}$$

Additionally assume that Z *fulfills Assumption 3.7. Then for every* $\varepsilon > 0$ *there exists* C *such that with* (3.14)*, and for all* $N \in \mathbb{N}$ *holds*

$$\operatorname{cost}(Q_{\Lambda_N}) \leq C|\Lambda_N|^{1+\varepsilon}\,. \tag{3.30}$$

Proof. 1st Step: We show the first equality in (3.29). Let

$$H(d) := \sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,|\operatorname{supp}\boldsymbol{\nu}|\geq d\}} m_{\boldsymbol{\nu}}. \tag{3.31}$$

Then $H(d+1) < H(d)$ for all $d \in \mathbb{N}$: Assume $H(d+1) = H(d)$ for some $d \in \mathbb{N}$. Since $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^1(\mathcal{F})$, the sequence tends to zero and there must exist $\boldsymbol{\nu}_{d+1} \in \mathcal{F}$ and $\boldsymbol{\nu}_d \in \mathcal{F}$ such that the supremum in (3.31) is obtained at these multiindices. By (ii) we have $\lfloor \boldsymbol{\nu}_{d+1} \rfloor_Z = \lfloor \boldsymbol{\nu}_d \rfloor_Z$. Since $1 \in Z$ it must hold $\operatorname{supp}\boldsymbol{\nu}_d = \operatorname{supp}\boldsymbol{\nu}_{\lfloor d \rfloor_Z} = \operatorname{supp}\boldsymbol{\nu}_{\lfloor d \rfloor_Z} = \operatorname{supp}\boldsymbol{\nu}_{d+1}$ and thus $|\operatorname{supp}\boldsymbol{\nu}_d| \geq d+1$. Wlog assume $(\boldsymbol{\nu}_d)_1 \neq 0$. Set $\boldsymbol{\mu} := (0, (\boldsymbol{\nu}_d)_2, (\boldsymbol{\nu}_d)_3, \dots)$. Then $\boldsymbol{\mu} \leq \boldsymbol{\nu}_d$ and $\lfloor \boldsymbol{\mu} \rfloor_Z \neq \lfloor \boldsymbol{\nu}_d \rfloor_Z$, which again by Thm. 2.11 (ii) gives $m_{\boldsymbol{\mu}} > m_{\boldsymbol{\nu}_d}$. This contradicts the definition of $\boldsymbol{\nu}_d$ because $|\operatorname{supp}\boldsymbol{\mu}| \geq |\operatorname{supp}\boldsymbol{\nu}_d| - 1 \geq d$.

Now, with $d(\Lambda_N)$ as in (3.13), the $|\Lambda_N|$th largest element of $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ must be less or equal to $H(d(\Lambda_N))$, which is a consequence of the definition of Λ_N in (iv). Therefore, $H(d(\Lambda_N)) \geq C|\Lambda_N|^{-r}$ by definition of $r > 0$. Let h be defined as in (2.40b). We have

$$h(H(d(\Lambda_N))) = \min\{d \in \mathbb{N} : \sup_{|\operatorname{supp}\boldsymbol{\nu}|\geq d} m_{\boldsymbol{\nu}} \leq \sup_{|\operatorname{supp}\boldsymbol{\nu}|\geq d(\Lambda_N)} m_{\boldsymbol{\nu}}\} \leq d(\Lambda_N). \tag{3.32}$$

Since $H(d)$ decreases strictly in $d \in \mathbb{N}$ as observed above, we get for any $\tilde{d} \in \mathbb{N}_0$, $\tilde{d} < d(\Lambda_N)$,

$$\sup_{|\operatorname{supp}\boldsymbol{\nu}|\geq \tilde{d}} m_{\boldsymbol{\nu}} = H(\tilde{d}) > H(d(\Lambda_N)) = \sup_{|\operatorname{supp}\boldsymbol{\nu}|\geq d(\Lambda_N)} m_{\boldsymbol{\nu}}, \tag{3.33}$$

and conclude with (3.32) that it must hold $h(H(d(\Lambda_N))) = d(\Lambda_N)$. The fact that h is monotonically decreasing then implies $d(\Lambda_N) = h(H(d(\Lambda_N))) \leq h(C|\Lambda_N|^{-r})$. By (2.41), we obtain $h(C|\Lambda_N|^{-r}) = o(\log(|\Lambda_N|))$ and therefore $d(\Lambda_N) = o(\log(|\Lambda_N|))$ as $|\Lambda_N| \to \infty$.

2nd Step: We show the second equality in (3.29). We proceed similarly as in the first step. With

$$G(m) := \sup_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,\max_j \nu_j\ge m\}} m_{\boldsymbol{\nu}}, \tag{3.34}$$

by the same argument as before it holds $G(m(\Lambda_N)) \ge C|\Lambda_N|^{-r}$ and G is monotonically decreasing. Moreover for $m_0 \in \mathbb{N}$, $m_0 > 1$ we have $G(\lfloor m_0/(1+C_Z)\rfloor_Z) > G(m_0)$ (here C_Z is as in Assumption 2.7): assume the contrary, i.e. $G(\lfloor m_0/(1+C_Z)\rfloor_Z) = G(m_0)$. Then, with $G(\lfloor m_0/(1+C_Z)\rfloor_Z) = m_{\boldsymbol{\nu}_0}$ and $G(m_0) = m_{\boldsymbol{\nu}_1}$ we have $m_{\boldsymbol{\nu}_0} = m_{\boldsymbol{\nu}_1}$. Note that by (2.30)

$$\left\lfloor \frac{m_0}{1+C_Z} \right\rfloor_Z \le \frac{m_0}{1+C_Z} < \lfloor m_0 \rfloor_Z. \tag{3.35}$$

Set $\boldsymbol{\mu} := (\lfloor \nu_{1;j}/(1+C_Z)\rfloor_Z)_{j\in\mathbb{N}}$. Then $\boldsymbol{\mu} \le \boldsymbol{\nu}_1$ and by (3.35) $\lfloor\boldsymbol{\mu}\rfloor_Z \ne \lfloor\boldsymbol{\nu}_1\rfloor_Z$ which gives $m_{\boldsymbol{\nu}_0} = m_{\boldsymbol{\nu}_1} < m_{\boldsymbol{\mu}}$ (cp. (2.42)) and $\max_j \mu_j \ge \lfloor \max_j \nu_{1;j}/(1+C_Z)\rfloor_Z \ge \lfloor m_0/(1+C_Z)\rfloor_Z$, thus contradicting the definition of $m_{\boldsymbol{\nu}_0}$. The definition of the monotonically decreasing function g in (2.40a) and $G(\lfloor m(\Lambda_N)/(1+C_Z)\rfloor_Z) > G(m(\Lambda_N))$ now allow to conclude

$$g(G(m(\Lambda_N))) := \min\{d \in \mathbb{N}_0 \,:\, G(d) \le G(m(\Lambda_N))\} \ge \left\lfloor \frac{m(\Lambda_N)}{1+C_Z} \right\rfloor_Z, \tag{3.36}$$

if $m(\Lambda_N) > 1$. Employing the property of g in (2.41) we arrive at

$$\frac{m(\Lambda_N)}{(1+C_Z)^2} \le \left\lfloor \frac{m(\Lambda_N)}{1+C_Z} \right\rfloor_Z \le g(G(m(\Lambda_N))) \le g(|\Lambda_N|^{-r}) = O(\log(|\Lambda_N|)) \text{ as } |\Lambda_N| \to \infty, \tag{3.37}$$

where the first inequality is again due to (2.30).

3rd Step: Additionally presuming Assumption 3.7 for Z we verify (3.30). Fix $N \in \mathbb{N}$. First we note that with $Z := \{a-1 \,:\, 0 < a \in Z\} \cup \{0\}$ (cp. Assumption 2.7), we have

$$\boldsymbol{\nu} \in \Lambda_N \Longrightarrow \lceil \boldsymbol{\nu} \rceil_Z \in \Lambda_N. \tag{3.38}$$

In order to see this, let $\boldsymbol{\nu} \in \Lambda_N$ and thus $m_{\boldsymbol{\nu}} \ge \varepsilon_N$. Now, by definition of Z, $\lfloor\lceil\boldsymbol{\nu}\rceil_Z\rfloor_Z = \lfloor\boldsymbol{\nu}\rfloor_Z$ and hence (ii) gives $m_{\boldsymbol{\nu}} = m_{\lfloor\boldsymbol{\nu}\rfloor_Z} = m_{\lceil\boldsymbol{\nu}\rceil_Z} \ge \varepsilon_N$ implying $\lceil\boldsymbol{\nu}\rceil_Z \in \Lambda_N$. Note that since Z satisfies Assumption 3.7, so does Z (possibly with a different constant, which we also denote by C_Z). We may then invoke Lemma 3.9 to get for any $\varepsilon > 0$

$$\sum_{\boldsymbol{\nu}\in\Lambda_N^Z} \prod_{j\in\mathbb{N}} (\nu_j + 1) \le C C_Z^{d(\Lambda_N)} |\Lambda_N| = O(|\Lambda_N|^{1+\varepsilon}) \qquad \text{as } |\Lambda_N| \to \infty, \tag{3.39}$$

due to the first equality in (3.29). According to (3.38) and Lemma 3.8 it holds $c_{\Lambda;\boldsymbol{\nu}} = 0$ for all $\boldsymbol{\nu} \notin \Lambda_N^Z$. Thus (3.39) gives an estimate on the last term in (3.14). The above shown asymptotics in (3.29), now allow to conclude with (3.14) that (3.30) is satisfied. □

Remark 3.12. *The computation of the tensor operator $Q_{\boldsymbol{\nu}} u = \bigotimes_{j\in\mathbb{N}} Q_{\nu_j} u$, requires the values of u at $\prod_{j\in\mathbb{N}}(\nu_j+1)$ points in $U = [-1,1]^{\mathbb{N}}$. Consequently, by* (3.3) *the total number of necessary function evaluations needed to compute $Q_\Lambda u$ is bounded by $\sum_{\{\boldsymbol{\nu}\in\Lambda_N\,:\,c_{\Lambda_N;\boldsymbol{\nu}}\ne 0\}} \prod_{j\in\mathbb{N}}(\nu_j+1)$. Under the presumptions of Lemma 3.11 this quantity grows as $O(N^{1+\varepsilon})$, for any $\varepsilon > 0$. Therefore (in particular for non-nested quadrature point sets) we obtain Thm. 3.3 also in the number of quadrature points, with almost the same rate, see Cor. 3.13 below.*

The previous Lemma states, that the asymptotic complexity of evaluating Q_{Λ_N} is near linear in the number of floating point operations, which are measured by (3.14). Similarly, the storage of the index set Λ_N together with all its forward and backward neighbours is bounded by $Cd(\Lambda_N)|\Lambda_N|$, and consequently for any $\varepsilon > 0$ by (3.30). Summarizing Thm. 3.3, Lemma 3.11

and Rmk. 3.12, we obtain the below corollary (with the set Z in Thm. 3.3 and Lemma 3.11 for example being $\{0\} \cup \{2^j : j \in \mathbb{N}\}$, which satisfies Assumptions 2.7, 3.7). It gives (up to an $\varepsilon > 0$) the convergence rate in Thm. 3.3 also for non-nested sets (cp. Rmk. 3.1) in the number of quadrature points, and in addition w.r.t. to the evaluation cost of the Smolyak algorithm.

Corollary 3.13. *Let the assumptions of Thm. 3.3 be satisfied. Then, for any $\varepsilon > 0$ there exists $C > 0$ such that for all $N \in \mathbb{N}$ there exists a downward closed set $\Lambda_N \subseteq \mathcal{F}$ such that $\mathrm{cost}(Q_{\Lambda_N}) \le CN$ (see (3.14) for the definition of $\mathrm{cost}(\cdot)$), the number of quadrature points employed by Q_{Λ_N} is bounded as $\sum_{\boldsymbol{\nu}\in\Lambda_N}\prod_{j\in\mathbb{N}}(\nu_j+1) \le N$, and*

$$\left\| \int_U u(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_N} u \right\|_X \le C N^{-(\frac{2}{p}-1)+\varepsilon}. \tag{3.40}$$

4 Numerical experiments

This section reports on the numerical testing, which we have performed for the presented algorithm. To begin with, more details on the construction of the index sets will be given in Sec. 4.1. We shall see, that there is a large preasymptotic range, which is addressed in Sec. 4.2. Afterwards, in Sec. 4.3 we consider the integration of two real valued test functions.

Throughout, the quadrature points $(\chi_{n;0}, \dots, \chi_{n;n})$, $n \in \mathbb{N}_0$, described at the beginning of Sec. 3, are chosen as a section of a Leja sequence as e.g. discussed in [6, 5, 9]. More precisely, for a Leja sequence $(\chi_0, \chi_1, \dots)$ of distinct points in $[-1, 1]$, we set $\chi_{n;j} := \chi_j$ for all $n \in \mathbb{N}_0$ and all $0 \le j \le n$. As pointed out in Remark 3.1, this implies that the number of quadrature points used by the quadrature operator Q_Λ defined in (3.3) equals the number of multiindices in the employed multiindex set $\Lambda \subseteq \mathcal{F}$, for all of the following experiments.

Before continuing, we take a look at two exemplary integrands, and discuss the proven convergence rate of the Smolyak quadrature in both cases, comparing them with the results in the recent papers [24, 23].

Example 4.1. *For a monotonically decreasing sequence $\boldsymbol{b}$ of positive numbers, consider*

$$u_1(\boldsymbol{y}) := \prod_{j\in\mathbb{N}} (1 + b_j y_j)^{-1}. \tag{4.1}$$

Assume $\|\boldsymbol{b}\|_{L^\infty} < 1$ and $\|\boldsymbol{b}\|_{\ell^p} < \infty$, for some $p \in (0, 1)$. We now discuss the proven convergence rate, in terms of number of quadrature points based on Thm. 3.3, as well as based on the results in [24, 23].

(i) Fix $0 < \varepsilon < 1 - \|\boldsymbol{b}\|_{\ell^\infty}$ and let $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}} \in (1,\infty)^{\mathbb{N}}$ be an arbitrary sequence such that $\sum_{j\in\mathbb{N}} b_j(\rho_j - 1) < \varepsilon$, i.e. $\boldsymbol{\rho}$ is $(\boldsymbol{b}, \varepsilon)$-admissible as in (2.2). Let $\boldsymbol{z} \in B_{\boldsymbol{\rho}} = \times_{j\in\mathbb{N}} B_{\rho_j} \subseteq \mathbb{C}^{\mathbb{N}}$, where $B_{\rho_j} \subseteq \mathbb{C}$, denotes the disc with radius ρ_j and center 0. For $\delta < 1$ we can find a constant C_δ such that for $0 \le x \le \delta$ it holds $\log(1/(1-x)) \le C_\delta x$. Since $b_j\rho_j = b_j(\rho_j - 1) + b_j \le \varepsilon + \|\boldsymbol{b}\|_{\ell^\infty} =: \delta < 1$, by definition of ε, we obtain

$$|u_1(\boldsymbol{z})| = \left| \prod_{j\in\mathbb{N}} (1 + b_j z_j)^{-1} \right| \le \prod_{j\in\mathbb{N}} (1 - b_j\rho_j)^{-1} \le \exp\left(C_\delta \sum_{j\in\mathbb{N}} b_j\rho_j \right). \tag{4.2}$$

The last term is finite because $\sum_{j\in\mathbb{N}} b_j\rho_j = \sum_{j\in\mathbb{N}} b_j(\rho_j - 1) + \sum_{j\in\mathbb{N}} b_j \le \varepsilon + \|\boldsymbol{b}\|_{\ell^1}$. Therefore u allows a well-defined uniformly bounded holomorphic extension onto $B_{\boldsymbol{\rho}}$. Continuity of $B_{\boldsymbol{\rho}} \ni \boldsymbol{z} \mapsto u_1(\boldsymbol{z})$ is easily checked, and holomorphy of the function in each z_j is obvious. Assumption 2.1 can now readily be verified. By Thm. 3.3, the convergence rate of the Smolyak quadrature is then at least $2/p - 1$.

(ii) *Consider now* (3.12), *i.e. the assumption given in Rmk. 3.5, and similarly presumed in [24, 23]. We wish to find a fixed polydisc* $B_{\boldsymbol{\rho}}$ *onto which* u *allows a uniformly bounded holomorphic extension. In view of Rmk. 3.5, the sequence* $\boldsymbol{\rho}$ *should be chosen such that* $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^{\tilde{p}}$ *for some possibly small* $\tilde{p} > 0$.
For $0 \leq x \leq 1$ *we have* $1/(1-x) \geq 1+x$ *and furthermore* $\log(1+x) \geq x/2$, *which gives* $-\log(1-x) \geq x/2$. *Thus for* $\boldsymbol{z} := (-\rho_j/2)_{j\in\mathbb{N}} \in B_{\boldsymbol{\rho}}$

$$|u_1(\boldsymbol{z})| = \prod_{j\in\mathbb{N}} (1 - b_j\rho_j/2)^{-1} = \exp\left(-\sum_{j\in\mathbb{N}} \log(1 - b_j\rho_j/2)\right) \geq \exp\left(\frac{1}{4}\sum_{j\in\mathbb{N}} b_j\rho_j\right). \quad (4.3)$$

Hence $\boldsymbol{\rho}$ *must satisfy* $\sum_{j\in\mathbb{N}} \rho_j b_j < \infty$. *This implies* $\rho_j^{-1} = b_j/c_j$ *for some sequence* $(c_j)_{j\in\mathbb{N}} \in \ell^1$. *Suppose that* $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^{\tilde{p}}$ *for some* $0 < \tilde{p} < 1$. *Then with* $\hat{p} := \tilde{p}/(1+\tilde{p}) < 1$

$$\sum_{j\in\mathbb{N}} b_j^{\hat{p}} = \sum_{j\in\mathbb{N}} \left(\frac{b_j}{c_j}\right)^{\hat{p}} c_j^{\hat{p}} \leq \left(\sum_{j\in\mathbb{N}} \left(\frac{b_j}{c_j}\right)^{\frac{\hat{p}}{1-\hat{p}}}\right)^{1-\hat{p}} \left(\sum_{j\in\mathbb{N}} c_j\right)^{\hat{p}} = \left(\sum_{j\in\mathbb{N}} \left(\frac{b_j}{c_j}\right)^{\tilde{p}}\right)^{1-\hat{p}} \left(\sum_{j\in\mathbb{N}} c_j\right)^{\hat{p}} \quad (4.4)$$

and we obtain $\boldsymbol{b} \in \ell^{\hat{p}}$. *Assuming that* $p > 0$ *was an optimal choice, in the sense that* $\boldsymbol{b} \in \ell^p$ *but* $\boldsymbol{b} \notin \ell^q$ *with* $q < p$, *we obtain* $\hat{p} = \tilde{p}/(1+\tilde{p}) \geq p$, *and therefore* $\tilde{p} \geq p/(1-p)$. *Hence* $(\rho_j^{-1})_{j\in\mathbb{N}}$, *can at best be in* $\ell^{p/(1-p)}$. *One possible choice achieving this is* $\rho_j := \max\{\kappa, b_j^{p-1}\}$, *with* $\kappa > 1$ *fulfilling* $\kappa\|\boldsymbol{b}\|_{\ell^\infty} < 1$. *One checks that* u *then allows a uniformly bounded extension onto* $B_{\boldsymbol{\rho}}$ *and it holds* $(\rho_j^{-1}) \in \ell^{\tilde{p}}$ *with* $\tilde{p} := p/(1-p)$. *The statements in [24, Assumption 4.2, Thm. 5.5], [23, Cor. 5.9] then essentially give the convergence rates* $(\tilde{p}^{-1} - 1)/2$, $\tilde{p}^{-1} - 1$, *i.e.* $1/(2p) - 1$, $1/p - 2$ *respectively.*

Example 4.2. *For a monotonically decreasing sequence* $\boldsymbol{b}$ *of positive numbers, consider*

$$u_2(\boldsymbol{y}) := \left(1 + \sum_{j\in\mathbb{N}} b_j y_j\right)^{-1}. \quad (4.5)$$

Assume $\|\boldsymbol{b}\|_{\ell^1} < 1$, $\|\boldsymbol{b}\|_{\ell^p} < \infty$ *for some* $p \in (0,1)$. *Let* $B_{\boldsymbol{\rho}}$ *be a polydisc onto which* u *allows a uniformly bounded holomorphic extension with* $|u_2(\boldsymbol{z})| \leq a$, $1 \leq a < \infty$ *for all* $\boldsymbol{z} \in B_{\boldsymbol{\rho}}$. *This is equivalent to*

$$1 - \sum_{j\in\mathbb{N}} b_j\rho_j \geq a^{-1} \qquad \text{or} \qquad \sum_{j\in\mathbb{N}} b_j\rho_j \leq 1 - a^{-1}. \quad (4.6)$$

Similar as in Example 4.1, we find that Assumption 2.1 is satisfied for some ε, *whereas* (3.12) *in Rmk. 3.5 is satisfied for some* $\boldsymbol{\rho}$ *with* $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^{p/(1-p)}$. *Hence Thm. 3.3 states that the Smolyak quadrature (with nested points as in Rmk. 3.1) converges with rate* $2/p - 1$ *in terms of number of quadrature points. Again, the corresponding results in [24, Assumption 4.2, Thm. 5.5], [23, Cor. 5.9] merely give the convergence rates* $1/(2p) - 1$, $1/p - 2$ *respectively.*

Remark 4.3. *Differentiating* u_1, u_2 *in* (4.1), (4.5) *for some* $\boldsymbol{\nu} \in \mathcal{F}$ *we find*

$$\frac{1}{\boldsymbol{\nu}!}\frac{\partial}{\partial y_1^{\nu_1} y_2^{\nu_2}\ldots} u_1(\boldsymbol{y})\bigg|_{\boldsymbol{y}=\boldsymbol{0}} = (-1)^{|\boldsymbol{\nu}|}\boldsymbol{b}^{\boldsymbol{\nu}} \qquad \text{and} \qquad \frac{1}{\boldsymbol{\nu}!}\frac{\partial}{\partial y_1^{\nu_1} y_2^{\nu_2}\ldots} u_2(\boldsymbol{y})\bigg|_{\boldsymbol{y}=\boldsymbol{0}} = (-1)^{|\boldsymbol{\nu}|}\frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}\boldsymbol{b}^{\boldsymbol{\nu}}. \quad (4.7)$$

Thus the modulus of the Taylor coefficients at $\mathbf{0}$ *agree with the sequences in Lemmata 2.4, 2.6.*

4.1 Estimators and construction of index sets

Mimicking the estimates in the proof of Thm. 2.11, appropriate definitions for $t_{\boldsymbol{\nu}}$, $m_{\boldsymbol{\nu}}$ as in the formulation of Thm. 2.11 are now given presuming Assumption 2.1.

4.1.1 Estimators

An estimator of the Taylor coefficients in a general setting, and for the particular case of Example 4.1 is obtained as follows:

(hol) Holomorphic extension to a union of polydiscs: Let Assumption 2.1 be satisfied for some $\boldsymbol{b} \in \ell^p$. With (2.46) in mind, we set $\boldsymbol{\rho_\nu} = (\rho_{\boldsymbol{\nu};j})_{j\in\mathbb{N}}$ with

$$\rho_{\boldsymbol{\nu};j} := \begin{cases} \kappa & \text{if } j \le J, \\ \max\{\kappa, \delta b_j^{-1}\} & \text{otherwise,} \end{cases} \tag{4.8}$$

for some fixed κ, $J \in \mathbb{N}$. In view of the proof of Thm. 2.11, this choice is justified for J large enough and $\kappa > 1$ small enough. We then define

$$t_{\boldsymbol{\nu}}^{(\mathrm{hol})} = t_{\boldsymbol{\nu}}^{(\mathrm{hol})}(\kappa, J, \delta) := \boldsymbol{\rho_\nu}^{-\boldsymbol{\nu}} = \prod_{j=1}^{J} \kappa^{-\nu_j} \prod_{j>J} \max\left\{\kappa, \delta \frac{\nu_j}{b_j \sum_{j>J} \nu_j}\right\}^{-\nu_j}. \tag{4.9}$$

(u1) Taylor coefficients of u_1: With $\boldsymbol{b} \in \ell^p$ fixed, we set

$$t_{\boldsymbol{\nu}}^{(\mathrm{u1})} = \boldsymbol{b}^{\boldsymbol{\nu}}. \tag{4.10}$$

As pointed out in Rmk. 4.3, this estimator coincides with the Taylor coefficients of u_1 in (4.1). More generally, in view of (2.5) the choice (4.10) is justified as an upper bound of the norm of the Taylor coefficients for functions allowing a uniformly bounded holomorphic extension onto the polydisc $B_{(b_j^{-1})_{j\in\mathbb{N}}} \subseteq \mathbb{C}^{\mathbb{N}}$.

As suggested in the previous sections, for performing quadrature on high dimensional integrands, we introduce the estimators $(m_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ for $t_{\boldsymbol{\nu}}$ as in (4.9), (4.10) via

$$(\boldsymbol{\nu}_2)_j := \begin{cases} \nu_j & \text{if } \nu_j \ne 1 \\ 2 & \text{if } \nu_j = 1 \end{cases} \qquad \text{and} \qquad m_{\boldsymbol{\nu}} := t_{\boldsymbol{\nu}_2}. \tag{4.11}$$

This estimator is targeted at quadrature algorithms, in the sense that it takes into account Lemma 3.2 (ii). Finally, with $Z := \{0,1\} \cup \{2^j : j \ge 2\}$ we set

$$m_{\boldsymbol{\nu}}^{(\cdot);2} := t_{\lfloor \boldsymbol{\nu}_2 \rfloor_Z}^{(\cdot)}. \tag{4.12}$$

Note that Z satisfies Assumptions 2.7, 3.7.

In our experiments we concentrate on the situation $b_j := \theta j^{-r}$ for some $\theta > 0$ and $r > 1$. The constants $J = 0$ and $\kappa = 2.8 > \mathrm{e}$ for (hol) in (4.9) are fixed throughout all what follows. The choice of κ for (hol) is motivated by (2.47), and ensures that $(t_{\boldsymbol{\nu}}^{(\mathrm{hol})})_{\boldsymbol{\nu}\in\mathcal{F}}$ is monotonically decreasing. In summary, we work with the estimators

$$t_{\boldsymbol{\nu}}^{(\mathrm{u1})} := \prod_{j\in\mathbb{N}} \left(\frac{j^r}{\theta}\right)^{-\nu_j}, \qquad t_{\boldsymbol{\nu}}^{(\mathrm{hol})} := \prod_{j\in\mathbb{N}} \max\left\{\kappa, \frac{j^r \nu_j}{\theta |\boldsymbol{\nu}|}\right\}^{-\nu_j} \tag{4.13}$$

and their variants defined in (4.11), (4.12). Whereas exact values of κ and J depending on θ and r could be computed as in the proof of Thm. 2.11 (cp. (2.43)), we opt here for this simplified version of the estimators.

4.1.2 Index sets

The above estimators will be referred to as $\mathbf{t}^{(\mathrm{hol})}$, $\mathbf{m}^{(\mathrm{hol})}$, $\mathbf{m}^{(\mathrm{hol});2}$ etc. With $\mathbf{e} = (e_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ being one of them, and for a prescribed threshold $\varepsilon > 0$ we build

$$\Lambda = \Lambda((e_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}) := \{\boldsymbol{\nu} \in \mathcal{F} \,:\, e_{\boldsymbol{\nu}} \geq \varepsilon\} \tag{4.14a}$$

and also write

$$\Lambda_N = \Lambda \quad \text{ where } \quad N := |\Lambda|. \tag{4.14b}$$

The subscript N therefore indicates the number of indices in Λ_N which, by definition of Q_{Λ_N}, equals the number of quadrature points employed by the Smolyak quadrature operator Q_{Λ_N} since we use nested points as explained in Rmk. 3.1 and at the beginning of Sec. 4.

4.1.3 Decay of the estimators

Figure 1 depicts the decay of the estimators $\mathbf{m}$ in (4.11), (4.13),for different values of r and θ. In all cases the largest ones have been computed and sorted according to decreasing size to some sequence $(x_j)_{j\in\mathbb{N}}$ which is plotted against j. Let us first consider $\mathbf{t}^{(\mathrm{u1})}$. It holds

$$t_{\boldsymbol{\nu}}^{(\mathrm{u1})} = \prod_{j\in\mathbb{N}} (\theta j^{-r})^{\boldsymbol{\nu}_j}. \tag{4.15}$$

For $\theta < 1$ and $\varepsilon > 0$ arbitrary, this sequence is in $\ell^{\tilde{p}}(\mathcal{F}_2)$ for $\tilde{p} := 1/(2r) + \varepsilon$ according to Lemma 2.4. Furthermore with the same argument as in (2.54) for $k = 2$ and $J \in \mathbb{N}$

$$\begin{aligned}\sum_{\boldsymbol{\nu}\in\mathcal{F}} (m_{\boldsymbol{\nu}}^{(\mathrm{u1})})^{\tilde{p}} = \sum_{\boldsymbol{\nu}\in\mathcal{F}} (t_{\boldsymbol{\nu}_2}^{(\mathrm{u1})})^{\tilde{p}} &\leq \sum_{\boldsymbol{\nu}\in\mathcal{F}_2} 2^{|\operatorname{supp}\boldsymbol{\nu}|} \left(\prod_{j\in\mathbb{N}} \left(\theta j^{-r}\right)^{\nu_j}\right)^{\tilde{p}} \\ &\leq 2^J \sum_{\boldsymbol{\nu}\in\mathcal{F}_2} \left(\prod_{j=1}^{J} \left(\theta j^{-r}\right)^{\nu_j} \prod_{j>J} \left(\theta 2^{\frac{1}{\tilde{p}}} j^{-r}\right)^{\nu_j}\right)^{\tilde{p}}. \end{aligned} \tag{4.16}$$

For J large enough $\sup_{j>J} \theta 2^{1/\tilde{p}} j^{-r} < 1$, and again Lemma 2.4 with $k = 2$ then gives that the right-hand side in (4.16) is finite with $\tilde{p} = 1/(2r) + \varepsilon$. Therefore Lemma 2.9 suggests the decay rate of $\mathbf{m}^{(\mathrm{u1})}$ in Figure 1 to be $2r - \varepsilon$ for $\varepsilon > 0$ arbitrary. Similar arguments apply to $\mathbf{m}^{(\mathrm{hol})}$: in this case we can proceed analogously to (2.50), invoke Lemma 2.6 and obtain $(m_{\boldsymbol{\nu}}^{(\mathrm{hol})}) \in \ell^{\tilde{p}}$ for any $\tilde{p} > 1/(2r)$, which by Lemma 2.9 again gives the asymptotic decay rate $2r - \varepsilon$ for any $\varepsilon > 0$. These rates are in general not obtained in Figure 1, as there appears to be a large preasymptotic range for larger θ. Decreasing θ improves the situation in the plotted range of j. For very small θ, the rates come close to the predicted ones.

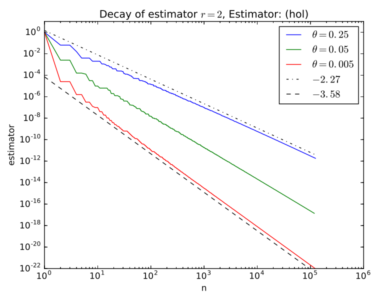
4.2 Preasymptotic behaviour

We analyze the convergence rates observed in Figure 1. *In the range of active indices shown in Figure 1* and *for large values of the scaling parameter $\theta \in (0, 1)$ close to* 1, the observed convergence rates appear to contradict the predicted asymptotic rates as noted in Sec. 4.1.3. To understand this, we consider the Taylor coefficients of the integrand in Example 4.1, and investigate in more detail the decay properties of

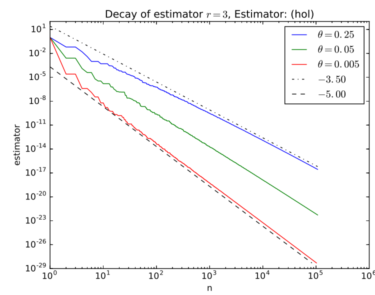
$$((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \qquad \text{where} \qquad \boldsymbol{\rho} = (j)_{j\in\mathbb{N}},\ \theta \in (0, 1),\ r > 1. \tag{4.17}$$

We partition $\mathcal{F}_k$, $k \in \{1, 2\}$ according to sets of multiindices of total order $m \in \mathbb{N}$ (cp. Def. 2.3), and define subsets of *m-homogeneous multiindices*

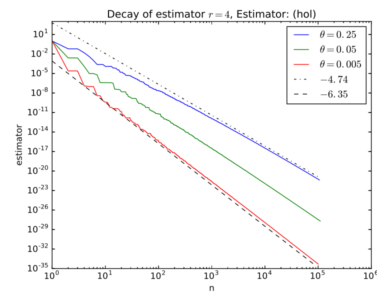
$$\mathcal{F}^m := \{\boldsymbol{\nu} \in \mathcal{F} \,:\, |\boldsymbol{\nu}| = m\} \qquad \text{and} \qquad \mathcal{F}_2^m := \{\boldsymbol{\nu} \in \mathcal{F}_2 \,:\, |\boldsymbol{\nu}| = m\}. \tag{4.18}$$



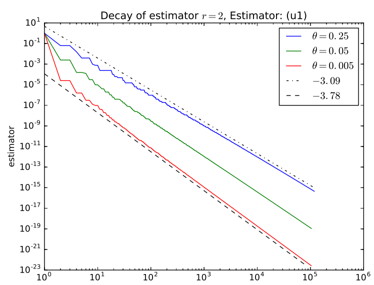
(a) (hol), $r = 2$



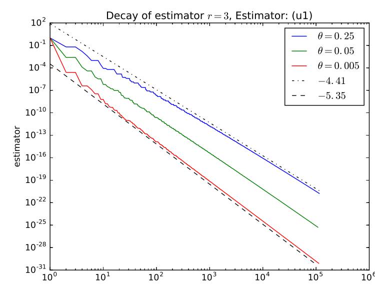
(b) (hol), $r = 3$



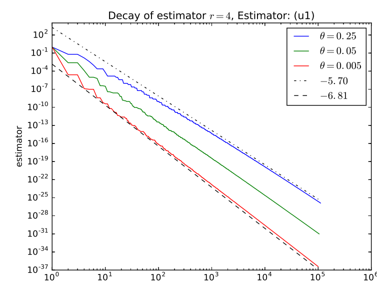
(c) (hol), $r = 4$



(d) (u1), $r = 2$



(e) (u1), $r = 3$



(f) (u1), $r = 4$

Figure 1: Decay of the estimators $(m_{\boldsymbol{\nu}}^{(\mathrm{hol})})_{\boldsymbol{\nu}\in\mathcal{F}}$, $(m_{\boldsymbol{\nu}}^{(\mathrm{u1})})_{\boldsymbol{\nu}\in\mathcal{F}}$ as in (4.13), (4.11) for different values of r and θ where $b_j := \theta j^{-r}$, $j \in \mathbb{N}$. Here $(x_j)_{j\in\mathbb{N}}$ denotes a decreasing rearrangement of the estimator sequences, and we plot x_j against j on the x-axis. Asymptotically, the algebraic decay rate is $2r-\varepsilon$ for any $\varepsilon > 0$ in all cases.

4.2.1 Decay with respect to $\mathcal{F}^m$

For $m \in \mathbb{N}$ and $\boldsymbol{\rho}$, r as in (4.17), consider

$$(x_{m;j})_{j\in\mathbb{N}}, \text{ a decreasing rearrangement of } (\boldsymbol{\rho}^{-\boldsymbol{\nu}r})_{\boldsymbol{\nu}\in\mathcal{F}^m}\ . \tag{4.19}$$

For any $\varepsilon > 0$ there then exists C (depending on ε and m) such that $x_{m;j} \le Cj^{-r+\varepsilon}$ for all $j \in \mathbb{N}$. To see this, fix for the moment $\varepsilon > 0$ and $0 < \theta < 1$. By Lemma 2.4, $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^{1/r+\varepsilon}$. Since $(\theta^m x_{m;j})_{j\in\mathbb{N}}$ is a subsequence, according to Lemma 2.9 our claim is true.

In the following, log denotes the natural logarithm.

Lemma 4.4. *Let r, $\boldsymbol{\rho}$ as in* (4.17)*, $m \in \mathbb{N}$ and for $R \ge 0$ set*

$$A_m(R) := \sum_{\{\boldsymbol{\nu}\in\mathcal{F}^m\,:\,\boldsymbol{\rho}^{-r\boldsymbol{\nu}}\ge R^{-r}\}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!}. \tag{4.20}$$

Then $A_m(R) = 0$ if $R < 1$ and with $c_0 := 1 - \log(2) \in (0,1)$ for all $R \ge 1$

$$c_0^m R \sum_{i=0}^{m-1} \frac{(c_0^{-1}\log(R))^i}{i!} \le A_m(R) \le R \sum_{i=0}^{m-1} \frac{(2\log(R))^i}{i!}. \tag{4.21}$$

Proof. For $R \in [0,1)$ the sum is over the empty set, so let $R \ge 1$ in the following. Then

$$A_m(R) = \sum_{\{\boldsymbol{\nu}\in\mathcal{F}\,:\,|\boldsymbol{\nu}|=m,\ \boldsymbol{\rho}^{-r\boldsymbol{\nu}}\ge R^{-r}\}} \frac{|\boldsymbol{\nu}|!}{\boldsymbol{\nu}!} = \left|\left\{(i_1,\dots,i_m)^{\mathbb{N}}\ :\ \prod_{j=1}^m i_j^{-r} \ge R^{-r}\right\}\right|, \tag{4.22}$$

since for every $\boldsymbol{\nu} \in \mathcal{F}$ with $|\boldsymbol{\nu}| = m$, there exist exactly $|\boldsymbol{\nu}|!/\boldsymbol{\nu}!$ elements $(i_1,\dots,i_m)$ of $\mathbb{N}^m$ such that $|\{j \in \{1,\dots,m\}\ :\ i_j = l\}| = \nu_l$ for all $l \in \mathbb{N}$. With $N := \lfloor R \rfloor \in \mathbb{N}$ we have

$$A_{m+1}(R) = \sum_{j=1}^N \sum_{\{(i_1,\dots,i_m)\,:\,j^{-r}\prod_{l=1}^m i_l^{-r}\ge R^{-r}\}} 1 = \sum_{j=1}^N \sum_{\{(i_1,\dots,i_m)\,:\,\prod_{l=1}^m i_l^{-r}\ge (R/j)^{-r}\}} 1 = \sum_{j=1}^N A_m(R/j). \tag{4.23}$$

To prove the upper bound in (4.21), we proceed by induction over m. For $m = 1$ it holds $i_1^{-r} \ge R^{-r}$ iff $i_1 \le R$, so that $A_1(R) = \lfloor R \rfloor$ and the estimate is satisfied. Next, employing (4.23) and the induction hypothesis

$$A_{m+1} \le \sum_{j=1}^N \frac{R}{j} \sum_{i=0}^{m-1} \frac{\log(R/j)^i}{i!} = R \sum_{j=1}^N \sum_{i=0}^{m-1} \frac{1}{i!} \frac{\log(R/j)^i}{j}. \tag{4.24}$$

Now $f : x \mapsto \sum_{i=0}^m 2^i \log(R/x)^i/(x \cdot i!)$ is monotonically decreasing because every summand is. Therefore we can estimate $\sum_{j=1}^N f(j) \le f(1) + \int_1^N f(x)\mathrm{d}x$, giving

$$\begin{aligned}\sum_{j=1}^N \sum_{i=0}^{m-1} \frac{2^i}{i!} \frac{\log(R/j)^i}{j} &\le f(1) + \int_1^N \sum_{i=0}^{m-1} \frac{2^i}{i!} \frac{\log(R/x)^i}{x}\mathrm{d}x = f(1) + \sum_{i=0}^{m-1} \frac{2^i}{i!} \int_0^{\log(N)} (\log(R) - y)^i \mathrm{d}y \\ &\le f(1) + \sum_{i=0}^{m-1} \frac{2^i}{i!} \int_0^{\log(R)} (\log(R) - y)^i \mathrm{d}y = \sum_{i=0}^{m-1} \frac{2^i \log(R)^i}{i!} + \sum_{i=0}^{m-1} \frac{2^i}{i!} \frac{\log(R)^{i+1}}{i+1} \le \sum_{i=0}^{m} \frac{2^i \log(R)^i}{i!},\end{aligned} \tag{4.25}$$

which concludes the proof of the upper bound.

For the lower bound, the case $m = 1$ follows by $Rc_0 \le \lfloor R \rfloor = A_1(R)$ where $c_0 = (1-\log(2)) < 1/2$. With (4.23) due to the induction hypothesis

$$A_{m+1}(R) = \sum_{j=1}^{N} A_m(R/j) \ge R \sum_{j=1}^{N} \frac{R}{j} \sum_{i=0}^{m-1} \frac{c_0^{m-i} \log(R/j)^i}{i!}. \tag{4.26}$$

Note that for $\lfloor R \rfloor = N \ge 1$

$$\sum_{j=1}^{N} \frac{1}{j} \ge 1 + \int_2^{N+1} \frac{1}{x} \mathrm{d}x \ge 1 - \int_1^2 \frac{1}{x} \mathrm{d}x + \int_1^R \frac{1}{x} \mathrm{d}x = c_0 + \log(R). \tag{4.27}$$

Hence, using that $f(x) := \log(R/x)^i/x$ is monotonically decreasing for $x \ge 1$ so that $\sum_{j=1}^{N} f(j) \ge \int_1^{N+1} f(x)\mathrm{d}x \ge \int_1^R f(x)\mathrm{d}x$ similar as in (4.25),

$$\begin{aligned}\sum_{j=1}^{N} \sum_{i=0}^{m-1} \frac{1}{i!} \frac{c_0^{m-i} \log(R/j)^i}{j} &\ge \sum_{j=1}^{N} \frac{c_0^{m-1}}{j} + \sum_{j=1}^{N} \sum_{i=1}^{m-1} \frac{c_0^{m-i}}{i!} \frac{\log(R/j)^i}{j} \\ &\ge c_0^{m+1} + c_0^m \log(R) + \sum_{i=1}^{m-1} \int_1^R \frac{c_0^{m-i}}{i!} \frac{\log(R/x)^i}{x} \mathrm{d}x = \sum_{i=0}^{m} \frac{c_0^{m+1-i} \log(R)^i}{i!},\end{aligned} \tag{4.28}$$

which proves the lower bound in (4.21). □

With Lemma 4.4 and $c_0 := 1 - \log(2) \in (0, 1)$, we observe for $R \ge 1$

$$f_m(R) := \frac{c_0^m}{m!} R \sum_{i=0}^{m-1} \frac{(c_0^{-1} \log(R))^i}{i!} \le |\{\boldsymbol{\nu} \in \mathcal{F}^m \ : \ \boldsymbol{\rho}^{-r\boldsymbol{\nu}} \ge R^{-r}\}| \le R \sum_{i=0}^{m-1} \frac{(2\log(R))^i}{i!} =: g_m(R), \tag{4.29}$$

which immediately gives:

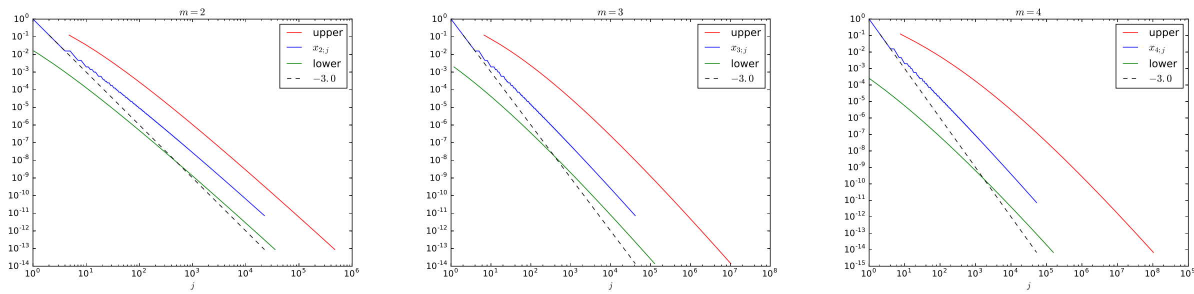
Lemma 4.5. *Let $j \in \mathbb{N}$ and $R_j \ge 1$, $S_j \ge 1$ such that $f_m(R_j) = j$ and $g_m(S_j) = j$. Then with $x_{m;j}$ as in* (4.19) *for all $j \in \mathbb{N}$*

$$R_j^{-r} \le x_{m;j} \le S_j^{-r}. \tag{4.30}$$

To estimate the local algebraic decay of the upper bound for $m = 2$ in Lemma 4.5 at position $j = g_2(R) = R(1 + 2\log(R))$, with $x = \log(R)$ we need to consider the derivative of the parametrized curve $(x+\log(1+2x), -rx)$. At $(x+\log(1+2x), -rx)$ it equals $-r/(1+2/(1+2x))$, so that our upper bound at $j = R(1 + 2\log(R))$ has the local algebraic decay rate (in the above sense)

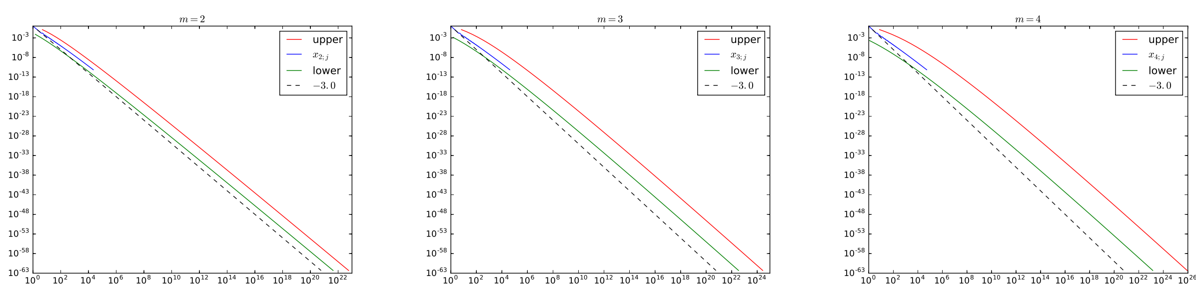
$$\frac{r}{1 + \frac{2}{1+2\log(R)}}. \tag{4.31}$$

A similar deliberation for the lower bound in (4.30) gives the same rate at position $j = f_2(r) = R(c_0 + \log(R))c_0/2$. This explains, why a rate close to r is only observed for rather large j. Furthermore, due to the additional (higher order) logarithmic terms in (4.29), in a given, fixed range of j, the rate of decay becomes worse as m grows. Indeed, Figure 2 shows the sequences $(x_{m,j})_{j\in\mathbb{N}}$ for $m = 2, 3, 4$, together with the bounds from Lemma 4.5. For the plotted range of small j, the behaviour is far from j^{-r}. Figure 3 shows that the rate will eventually approach r. Also note that $g_m(R) \le R^3$ for all $m \in \mathbb{N}$, which suggests that the worst we can expect for large m is a preasymptotic rate of $r/3$, although we do not claim that our estimate is optimal, so this might be too pessimistic.



(a) $m = 2$ (b) $m = 3$ (c) $m = 4$

Figure 2: The decay of the sequences $(x_{m;j})_{j\in\mathbb{N}}$ in (4.19), for $r = 3$ and for gpc coefficients of total orders $m = 2, 3, 4$. Additionally, the lower and upper bounds of $x_{2;m;j}$ in (4.30) are depicted. Asymptotically, for any $\varepsilon > 0$ the curves decay like $j^{-3+\varepsilon}$. In the plotted range of j a worse, preasymptotic rate is observed.



(a) $m = 2$ (b) $m = 3$ (c) $m = 4$

Figure 3: Same as Figure 2 but with the bounds plotted for a larger range.

4.2.2 Decay w.r.t. $\mathcal{F}_2^m$

So far, we have considered the sequences over the index set $\mathcal{F}$. For the convergence rate analysis of Smolyak quadrature, we are mainly interested in sequences with index set $\mathcal{F}_2$. For fixed $m \in \mathbb{N}$, $r > 1$, and with $\boldsymbol{\rho}$ as in (4.17) define (cp. (4.18), (4.19))

$$(x_{2;m;j})_{j\in\mathbb{N}}, \text{ a decreasing rearrangement of } (\boldsymbol{\rho}^{-\boldsymbol{\nu} r})_{\boldsymbol{\nu}\in\mathcal{F}_2^m}. \tag{4.32}$$

We claim that for any $\varepsilon > 0$ exists $C > 0$ (depending on ε) such that

$$\forall j \in \mathbb{N}: \quad x_{2;m;j} \le C j^{-2r+\varepsilon} . \tag{4.33}$$

According to Lemma 2.4, for $0 < \theta < 1$ fixed, it holds $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2} \in \ell^{2r-\varepsilon}(\mathcal{F}_2)$. Thus also $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2^m} \in \ell^{2r-\varepsilon}(\mathcal{F}_2^m)$ and consequently $(x_{2;m;j})_{j\in\mathbb{N}}$, being a rearrangement of $(\theta^{-m}(\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2^m}$, must be in $\ell^{2r-\varepsilon}(\mathbb{N})$. Lemma 2.9 then shows (4.33).

We discuss the decay of $(x_{2;m;j})_{j\in\mathbb{N}}$ for different m:

- $m = 1$: Observe that $\emptyset = \mathcal{F}_2^1 = \{\boldsymbol{\nu} \in \mathcal{F}_2 : |\boldsymbol{\nu}| = 1\}$, so this case is trivial.
- $m = 2$: With $\mathbf{e}_j = (\delta_{ij})_{i\in\mathbb{N}}$ we have $\mathcal{F}_2^2 = \{2\mathbf{e}_j : j \in \mathbb{N}\}$ and $\{\boldsymbol{\rho}^{-r\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathcal{F}_2^2\} = \{j^{-2r} : j \in \mathbb{N}\}$ so that $x_{2;2;j} = j^{-2r}$, and the decay predicted by (4.33) is apparent for small j.
- $m = 3$: It holds $\mathcal{F}_2^3 = \{3\mathbf{e}_j : j \in \mathbb{N}\}$ and thus $\{\boldsymbol{\rho}^{-r\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathcal{F}_2^3\} = \{j^{-3r} : j \in \mathbb{N}\}$. Hence $m = 3$ can be considered as a special case, since $x_{2;3;j} = j^{-3r}$ and the decay is even faster than j^{-2r}.
- $m = 4$: We have

  $$\left|\{\boldsymbol{\nu} \in \mathcal{F}_2^4 : \boldsymbol{\rho}^{-r\boldsymbol{\nu}} \ge R^{-r}\}\right| = \left|\{\boldsymbol{\nu} \in \mathcal{F}^2 : \boldsymbol{\rho}^{-r2\boldsymbol{\nu}} \ge R^{-r}\}\right| = \left|\{\boldsymbol{\nu} \in \mathcal{F}^2 : \boldsymbol{\rho}^{-r\boldsymbol{\nu}} \ge R^{-r/2}\}\right| \tag{4.34}$$

  and thus with (4.29)

  $$f_2(R^{1/2}) \le \left|\{\boldsymbol{\nu} \in \mathcal{F}_2^4 : \boldsymbol{\rho}^{-r\boldsymbol{\nu}} \ge R^{-r}\}\right| \le g_2(R^{1/2}). \tag{4.35}$$

  Considering the parametrized curves $(f_2(R^{1/2}), R^{-r})$, $(g_2(R^{1/2}), R^{-r})$ for $R \ge 1$, a computation similar to the one before (4.31) implies that the decay of $(x_{2;4;j})_{j\in\mathbb{N}}$ in the preasymptotic range is worse than what (4.33) suggests, due to the logarithmic factors occurring in f_2, g_2.
- $m > 4$: Similar arguments as in the case $m = 4$ apply, and we expect the decay rate to further diminish as m grows. The precise rate depends on the number of possibilities to write m as a sum of integers in $\mathbb{N}\backslash\{1\}$: for example $\{x_{2;5;j} : j \in \mathbb{N}\} = \{k^{-2}l^{-3} : k \ne l \in \mathbb{N}\}$ decreases faster than $\{x_{2;4;j} : j \in \mathbb{N}\} = \{k^{-2}l^{-2} : k < l \in \mathbb{N}\}$, as Fig. 4 (a) ahead shows.

Implications for $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2}$ are as follows. We write

$$(\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}} = \theta^m \boldsymbol{\rho}^{-r\boldsymbol{\nu}} \qquad \forall \boldsymbol{\nu} \in \mathcal{F}_2^m, \tag{4.36}$$

and note that *all terms belonging to $\mathcal{F}_2^m$ are scaled by the common factor θ^m: the smaller θ, the fewer multiindices of high total order m (which, in the preasymptotic range, decay slower than expected as we have noticed) will be among the N largest ones.* Denote now by $(x_{2;j})_{j\in\mathbb{N}}$ a decreasing rearrangement of the sequence in (4.36). By Lemma 2.9 and due to the fact that for any $p > 1/r$ it holds $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2} \in \ell^p(\mathcal{F}_2)$ by Lemma 2.4, for any $\varepsilon > 0$ there exists a constant $C > 0$ (depending on ε and on m) s.t.

$$x_{2;j} \le C j^{-2r+\varepsilon}. \tag{4.37}$$

If $0 < \theta < 1$ is small then, due to the factor θ^m in (4.36), only few multiindices of order $m \geq 4$ occur among the largest, and essentially $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2^2\cup\mathcal{F}_2^3}$ governs the decay of x_j for small j, thus yielding the expected rate $2r - \varepsilon$. On the other hand, as θ draws closer to 1, more higher order multiindices contribute to the best j terms, resulting in a longer preasymptotic range with slower decay.

To numerically verify these heuristic considerations, we find computable lower and upper bounds of $x_{2;j}$. First note that with f_m as in (4.29), for $R \geq 1$ there holds

$$f_m(R) \leq |\{\boldsymbol{\nu} \in \mathcal{F}^m \,:\, \boldsymbol{\rho}^{-r2\boldsymbol{\nu}} \geq R^{-2r}\}| \leq |\{\boldsymbol{\nu} \in \mathcal{F}_2^{2m} \,:\, \boldsymbol{\rho}^{-r\boldsymbol{\nu}} \geq R^{-2r}\}|. \tag{4.38}$$

We extend f_m via $f_m(R) := 0$ for all $R \in [0, 1)$, and (4.38) then remains true also for $R < 1$. Therefore

$$\begin{aligned} F(R) &:= 1 + \sum_{m\in\mathbb{N}} f_m(\theta^{2m/2r}R) \leq |\{\mathbf{0}\}| + \sum_{m\in\mathbb{N}} |\{\boldsymbol{\nu} \in \mathcal{F}_2^{2m} \,:\, \boldsymbol{\rho}^{-r\boldsymbol{\nu}} \geq (\theta^{2m/2r}R)^{-2r}\}| \\ &= |\{\mathbf{0}\}| + \sum_{m\in\mathbb{N}} |\{\boldsymbol{\nu} \in \mathcal{F}_2^{2m} \,:\, (\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}} \geq R^{-2r}\}| \leq |\{\boldsymbol{\nu} \in \mathcal{F}_2 \,:\, (\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}} \geq R^{-2r}\}|. \end{aligned} \tag{4.39}$$

A lower bound can now be derived as in Lemma 4.5. A (rough) upper bound can be derived for any $p > 1/r$ via Lemmata 2.4, 2.9 and the estimate (see the proof of Lemma 2.4)

$$\|(\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}}\|_{\ell^{p/2}(\mathcal{F}_2)} \leq \exp\left(\frac{2}{p}\frac{\theta^p}{1-\theta^{p/2}}\sum_{j\in\mathbb{N}} j^{-rp}\right) \leq \exp\left(\frac{2}{p}\frac{\theta^p}{1-\theta^{p/2}}(1 + (rp-1)^{-1})\right) < \infty, \tag{4.40}$$

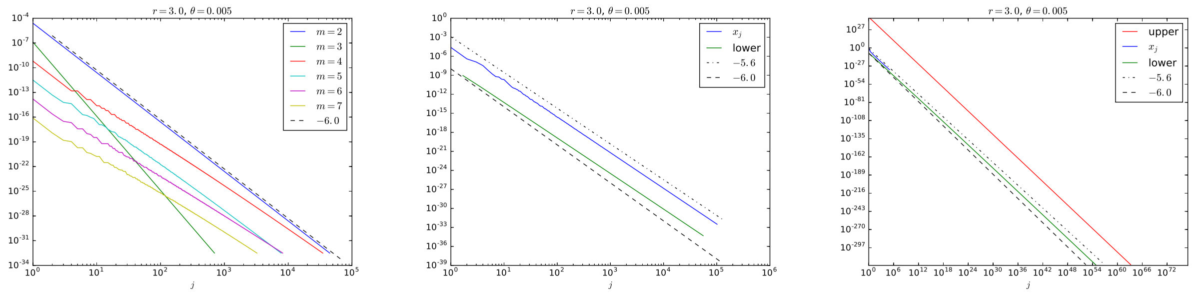
for any $p > 1/r$. We then obtain:

Lemma 4.6. *Let $j \in \mathbb{N}$ and $R_j \geq 1$ such that $F(R_j) = j$ and let $p > 1/r$. Then with $x_{2;j}$ denoting a decreasing rearrangement of the sequence in* (4.36)*, for all $j \in \mathbb{N}$*

$$R_j^{-2r} \leq x_{2;j} \leq \exp\left(\frac{2}{p}\frac{\theta^p}{1-\theta^{p/2}}(1 + (rp-1)^{-1})\right) j^{-2/p}. \tag{4.41}$$

Figure 4 depicts the decay of $(x_{2;m;j})_{j\in\mathbb{N}}$, $(x_{2;j})_{j\in\mathbb{N}}$ as well as the lower and upper bound in Lemma 4.6 for $r = 3$, $\theta = 0.005$. Due to θ being relatively small, the curves for $(x_{2;m;j})_{j\in\mathbb{N}}$ in subfigure (a) are far apart (cf. (4.36)). The measured decay rate of $(x_j)_{j\in\mathbb{N}}$ is 5.6, which is fairly close to $2r = 6$, cp. (4.37). Figure 5 shows the same but with $\theta = 0.25$. In this case the measured rate of $(x_j)_{j\in\mathbb{N}}$ in the observed range of j is merely 4.7. Subfigures (c) depict the upper and lower bounds in Lemma 4.6 also for (unrealistically) large values of j. The lower bound seems to capture the actual error convergence whereas the upper bound appears to be overly conservative. This is due to the fact, that the estimate of the $\ell^{p/2}$ norm in (4.40) tends to ∞ as $p \to 1/r$. The plots suggest, that the asymptotic rates are realized only for rather large numbers of quadrature points.

Finally, we have here considered the sequence $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2}$, which is not quite the same as $(m_{\boldsymbol{\nu}}^{(\mathrm{u1})})_{\boldsymbol{\nu}\in\mathcal{F}}$ shown in Figure (1). Indeed, the first is a subsequence of the latter. More precisely, the entries are the same, but $(m_{\boldsymbol{\nu}}^{(\mathrm{u1})})_{\boldsymbol{\nu}\in\mathcal{F}}$ contains some of them a multiple number of times (cp. (4.11), (4.13)). This has a further diminishing effect on the decay rates as a comparison between Figures 1 and 5 reveals. However, the asymptotic decay rate is the same for both sequences (viz $2r - \varepsilon$ for $\varepsilon > 0$ arbitrary). Moreover, the estimator $(m_{\boldsymbol{\nu}}^{(\mathrm{hol})})_{\boldsymbol{\nu}\in\mathcal{F}}$ in (4.13) also depicted in Figure 1, shows this preasymptotic effect even stronger. This does not come as a surprise, as for any $\boldsymbol{\nu} \in \mathcal{F}_2$, it even holds that if $\theta > 0$ is small enough, then $m_{\boldsymbol{\nu}}^{(\mathrm{hol})} \geq C(|\boldsymbol{\nu}|!/\boldsymbol{\nu}!)(\theta\boldsymbol{\rho}^{-r})^{-\boldsymbol{\nu}}$, as we have shown in the proof of Thm. 2.11.

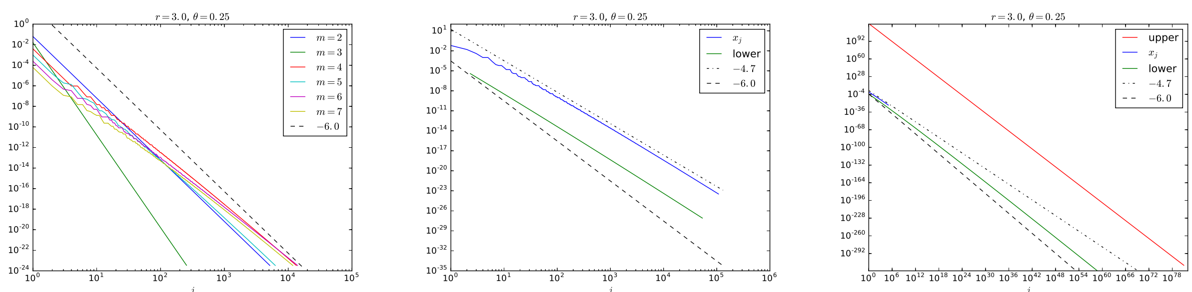


(a) $(x_{2;m;j})_{j\in\mathbb{N}}$ for different m

(b) $(x_{2;j})_{j\in\mathbb{N}}$ and lower bound

(c) $(x_{2;j})_{j\in\mathbb{N}}$ and both bounds for larger range of j

Figure 4: In this plot $(x_{2;m;j})_{j\in\mathbb{N}}$ is as in (4.32) for $\theta := 0.005$, $r := 3$ and $\boldsymbol{\rho} = (j)_{j\in\mathbb{N}}$. The sequence $(x_{2;j})_{j\in\mathbb{N}}$ denotes a decreasing rearrangement of $((\theta\boldsymbol{\rho}^{-r})^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}_2}$. For $\varepsilon > 0$ arbitrary, both $(x_{2;m;j})_{j\in\mathbb{N}}$ and $(x_{2;j})_{j\in\mathbb{N}}$ decay with algebraic rate $2r - \varepsilon$ (for any $m \in \mathbb{N}$, where in the case $m = 3$ we even have the rate $3r$ as is explained in the text). Subfigures (b), (c) show the sequence $(x_{2;j})_{j\in\mathbb{N}}$ together with the lower and upper bounds from Lemma 4.6. The summability exponent $p \in (1/3, 1]$ in (4.41) was chosen as $1/(3\cdot 0.97)$, implying that the upper bound in subfigure (c) has an algebraic convergence rate of $2/p = 5.82$.



(a) $(x_{2;m;j})_{j\in\mathbb{N}}$ for different m

(b) $(x_{2;j})_{j\in\mathbb{N}}$ and lower bound

(c) $(x_{2;j})_{j\in\mathbb{N}}$ and both bounds for larger range of j

Figure 5: Like Figure 4 but with $\theta := 0.25$ instead. Due to θ being closer to 1, the curves in subfigure (a) are shifted to the top as compared with Figure 4 (a) where $\theta = 0.005$. This results in a worse preasymptotic decay of $(x_{2;j})_{j\in\mathbb{N}}$, as depicted in subfigure (b). Subfigure (c) shows the same but with the bounds plotted for larger j. The summability exponent $p = 1/(3\cdot 0.9) > 1/3$ in (4.41), which means that the upper bound in (c) has an algebraic decay rate of $2/p = 5.4$.

4.3 Real valued model parametric integrand functions

We now test the convergence of the Smolyak quadrature for the functions u_1, u_2 in Examples 4.1, 4.2. For u_2 we also refer to [24] where computations for almost the same integrand were done with the method suggested in their paper.

4.3.1 Model integrand u_1

Let

$$u_1(\boldsymbol{y}) = \prod_{j\in\mathbb{N}} \frac{1}{1 + y_j\theta j^{-r}} \tag{4.42}$$

as in (4.1) with $b_j := \theta j^{-r}$, $0 < \theta < 1$, $r > 1$. As explained in Example 4.1, for this integrand Assumption 2.1 is satisfied. Since $\boldsymbol{b} \in \ell^p$ for any $p < 1/r$, Thm. 3.3 states that for suitable index sets Λ_N the Smolyak quadrature will reach the asymptotic convergence rate $2r - \varepsilon$, in terms of the number N of quadrature points. Figure 6 shows the absolute error $|\int_U u(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_N} u|$ for different values of r and θ, and with Λ_N in (4.14) based on the estimators $\mathbf{m}^{(\mathrm{hol})}$, $\mathbf{m}^{(\mathrm{u1})}$ given in (4.13), (4.11). The reference value can in this case be computed directly as $\int_U u(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) = \prod_{j\in\mathbb{N}} \log((1 + b_j)/(1 - b_j))/(2b_j)$.

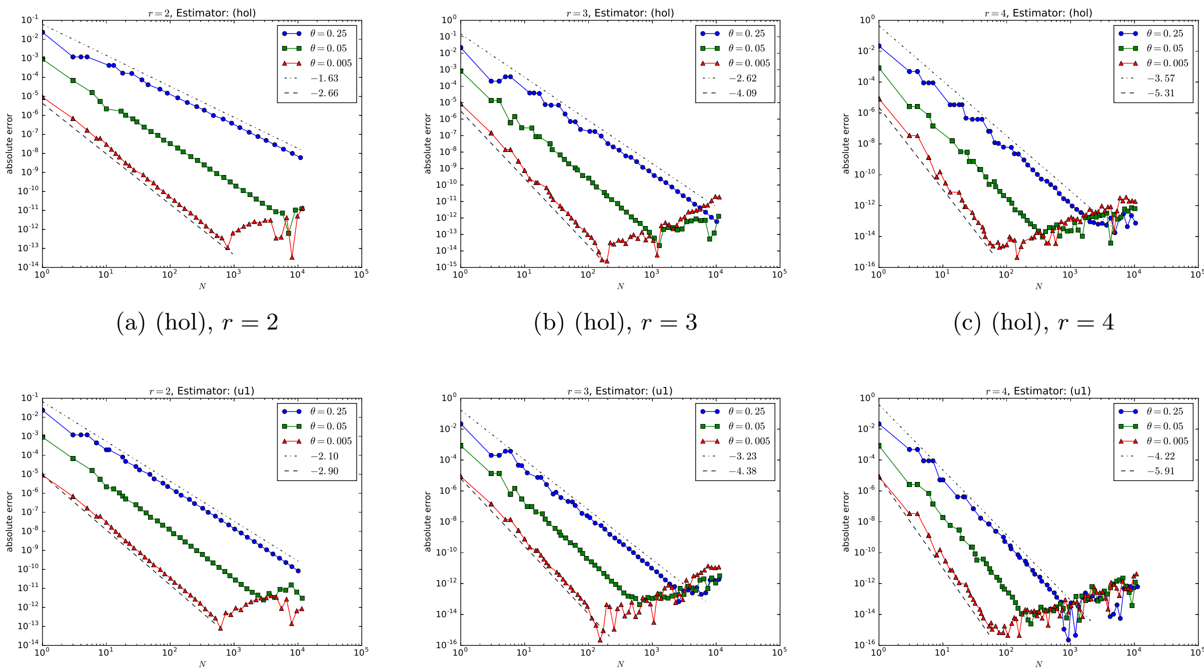
The estimator $\mathbf{m}^{(\mathrm{u1})}$ delivers the better rate, which is not surprising, as it is based on exact values of the modulus of the Taylor coefficients, see Rmk. 4.3. The estimator $\mathbf{m}^{(\mathrm{hol})}$ performs similarly but slightly worse. Evidently the constant θ has a significant influence on the error decay in both cases. In section 4.2.2, we noted that in the practical range of j, the Taylor coefficients of u_1 w.r.t. $\mathcal{F}_2$ do not converge as fast as expected (i.e. at least like $O(n^{-2r+\varepsilon})$), also see Figure 1 (d), (e), (f) and (4.11). Assuming that the error corresponds to the sum over the modulus of the Taylor coefficients with indices not in the index set, an error convergence rate better than the decay rate of the Taylor coefficients minus one, can in general not be expected. This is roughly in accordance with what is observed in in Figure 1 (d), (e), (f) and Figure 7 (d), (e), (f). Decreasing θ results in the error bounds approaching the asymptotic behaviour for smaller values of j. The plots confirm that considerably faster convergence than the previously proved $O(N^{1-r})$ is in principle attainable.

4.3.2 Model integrand u_2

Let

$$u_2(\boldsymbol{y}) = \frac{1}{1 + \theta \sum_{j\in\mathbb{N}} y_j j^{-r}} \tag{4.43}$$

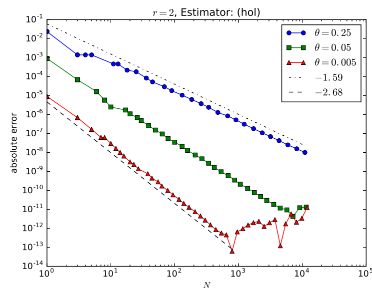
as in (4.5) with $b_j := \theta j^{-r}$, $r > 1$ and $\theta > 0$ small enough such that $\theta \sum_{j\in\mathbb{N}} j^{-r} < 1$. We have already noted in Example 4.2, that Assumption 2.1 is satisfied for any $p > 1/r$, so that by Thm. 3.3 we expect the convergence rate $2r - \varepsilon$ with $\varepsilon > 0$ arbitrary. Figure 7 shows the convergence of the absolute error $|\int_U u(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_N} u|$ for different values of r and θ, and with Λ_N in (4.14) based on the estimators $\mathbf{m}^{(\mathrm{hol})}$, $\mathbf{m}^{(\mathrm{u1})}$. Strictly speaking, the usage of $\mathbf{m}^{(\mathrm{u1})}$ in (4.13) is not in accordance with our theory, since the modulus of the Taylor coefficients of u_2 are $((|\boldsymbol{\nu}|!/\boldsymbol{\nu}!)\boldsymbol{b}^{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ (see Rmk. 4.3), and thus in general not bounded by $m_{\boldsymbol{\nu}}^{(\mathrm{u1})} = \boldsymbol{b}^{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} \in \mathcal{F}_2$. However, the performance of $\mathbf{m}^{(\mathrm{u1})}$ is slightly better also for this integrand. The reference value for $\int_U u(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y})$ has been computed with a higher order quasi Monte Carlo rule (a so-called high-order, Interlaced Polynomial Lattice rule adapted to the model integrand, with suitable digit interlacing parameter, see [19] and the references there) utilizing $2^{20} = 1048576$ quadrature points applied to the function u restricted to the first 1024 dimensions. This explains the saturation of the curves after a certain time, since our approximation at some point exceeds the precision of the reference value.



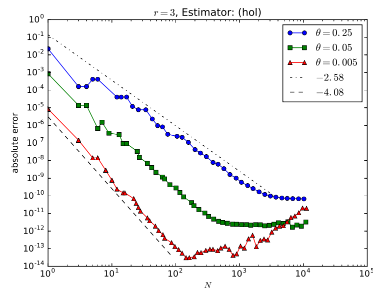
(a) (hol), $r = 2$ (b) (hol), $r = 3$ (c) (hol), $r = 4$

(d) (u1), $r = 2$ (e) (u1), $r = 3$ (f) (u1), $r = 4$

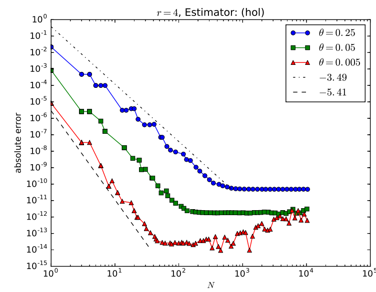
Figure 6: Quadrature error $|\int_U u_1(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_N} u_1|$ for u_1 in (4.42), and Λ_N in (4.14) built with the estimators $\mathbf{m}^{(\mathrm{hol})}$, $\mathbf{m}^{(\mathrm{u1})}$ in (4.13), (4.11), for different values of r and θ. The plot shows the absolute error in terms of the number of quadrature points $N = |\Lambda_N|$. The proven asymptotic convergence rate is $2r - 1 - \varepsilon$ with $\varepsilon > 0$ arbitrary in all cases.



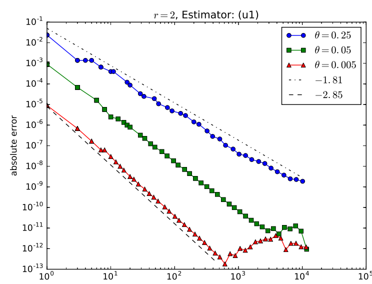
(a) (hol), $r = 2$



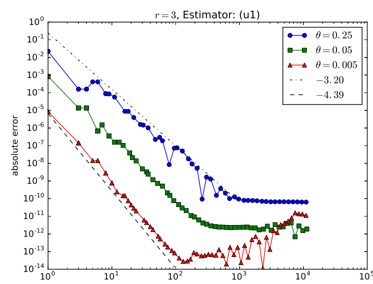
(b) (hol), $r = 3$



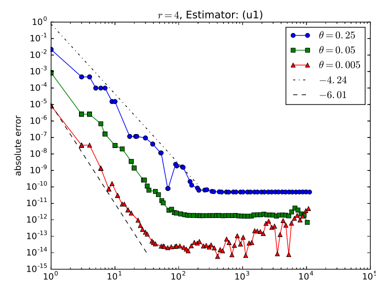
(c) (hol), $r = 4$



(d) (u1), $r = 2$



(e) (u1), $r = 3$



(f) (u1), $r = 4$

Figure 7: Quadrature error $|\int_U u_2(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_N} u_2|$ for u_2 in (4.43), and Λ_N in (4.14) built with the estimators $\mathbf{m}^{(\mathrm{hol})}$, $\mathbf{m}^{(\mathrm{u1})}$ in (4.13), (4.11), for different values of r and θ. The plot shows the absolute error in terms of the number of quadrature points $N = |\Lambda_N|$. The reference value for $\int_U u_2(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y})$ has been computed with a higher order quasi Monte Carlo rule utilizing 2^{20} lattice points applied to the function u_2 restricted to the first 1024 dimensions. The proven asymptotic convergence rate is $2r - 1 - \varepsilon$ with $\varepsilon > 0$ arbitrary in all cases.

4.3.3 Comparison with an adaptive method

We consider the model parametric integrand u_2 defined in (4.5), with $b_j := \theta j^{-r}$ for $r = 2$ and $\theta > 0$. In the following, our method is compared with a variant of the dimension adaptive algorithm described in [21] which we outline briefly for completeness. For some finite, downward closed set of multiindices $\{\mathbf{0}\} \neq \Lambda \subseteq \mathcal{F}$, following [7] we introduce the reduced set of neighbours

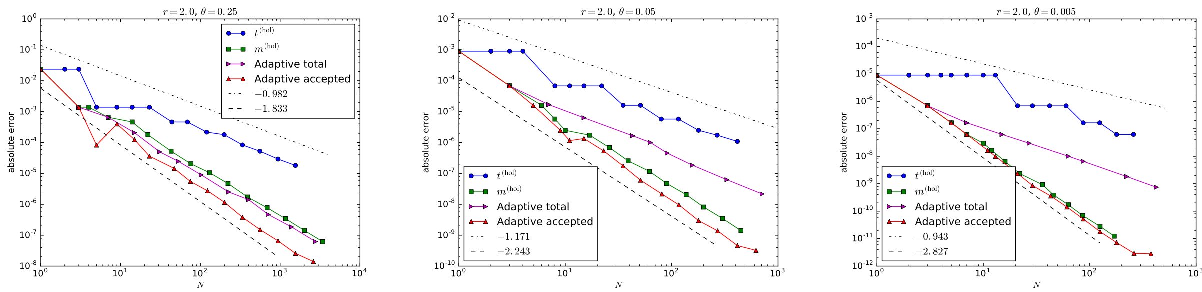
$$\mathcal{N}(\Lambda) := \{\boldsymbol{\nu} \in \mathcal{F} : \boldsymbol{\nu} \notin \Lambda,\ \boldsymbol{\nu} - \mathbf{e}_j \in \Lambda\ \forall j \in \operatorname{supp} \boldsymbol{\nu}, \\ \nu_j = 0\ \forall j > \max_{\boldsymbol{\mu} \in \Lambda} \max\{i \in \operatorname{supp} \boldsymbol{\mu} : \mu_i \neq 0\} + 1\}, \tag{4.44}$$

with the special case $\mathcal{N}(\{\mathbf{0}\}) := \{(1, 0, 0, \dots)\}$. Algorithm 1 shows the used adaptive method. Note that here, in contrast to our method described in Sec. 4.1, increasing a multiindex by one in some dimension adds two more quadrature points in this dimension. For this reason we use the notation $D_{\boldsymbol{\nu}}$ rather than $\Delta_{\boldsymbol{\nu}}$ as in (3.2), to avoid confusion. Also recall, that $Q_{-1} := 0$ and for $n \in \mathbb{N}_0$, Q_n stands for the one dimensional interpolatory quadrature employing the $n+1$ points $(\chi_j)_{j=0}^n$ in $[-1, 1]$. They are the first $n + 1$ points of a Leja sequence as in [6, 5, 9].

Algorithm 1 AdaptiveSmolyak(integrand $u : [-1, 1]^{\mathbb{N}} \to \mathbb{R}$, number of multiindices $M \in \mathbb{N}$)

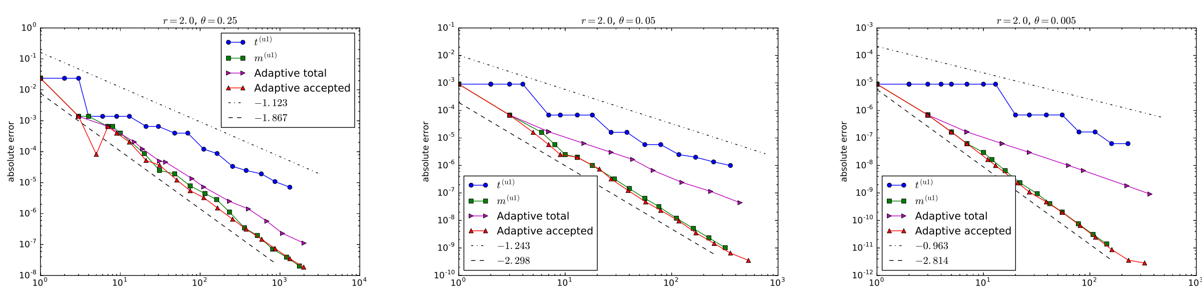
$\Lambda_{\mathrm{ad}} := \{\mathbf{0}\}$
$\Lambda_{\mathrm{tot}} := \Lambda_{\mathrm{ad}} \cup \mathcal{N}(\Lambda_{\mathrm{ad}})$
for all $\boldsymbol{\nu} \in \Lambda_{\mathrm{tot}}$ compute $D_{\boldsymbol{\nu}} := \bigotimes_{j \in \mathbb{N}} (Q_{2\nu_j + 1} - Q_{2(\nu_j - 1) + 1}) u$
while $|\Lambda_{\mathrm{ad}}| < M$ **do**
  $\boldsymbol{\mu} := \operatorname{argmax}\{|D_{\boldsymbol{\nu}}| : \boldsymbol{\nu} \in \Lambda_{\mathrm{tot}} \backslash \Lambda_{\mathrm{ad}}\}$
  $\Lambda_{\mathrm{ad}} := \Lambda_{\mathrm{ad}} \cup \{\boldsymbol{\mu}\}$
  $\Lambda_{\mathrm{new}} := \mathcal{N}(\Lambda_{\mathrm{ad}}) \backslash \Lambda_{\mathrm{tot}}$
  for all $\boldsymbol{\nu} \in \Lambda_{\mathrm{new}}$ compute $D_{\boldsymbol{\nu}} := \bigotimes_{j \in \mathbb{N}} (Q_{2\nu_j + 1} - Q_{2(\nu_j - 1) + 1}) u$
  $\Lambda_{\mathrm{tot}} := \Lambda_{\mathrm{ad}} \cup \Lambda_{\mathrm{new}}$
end while
$Q_{\Lambda_{\mathrm{ad}}} u := \sum_{\boldsymbol{\nu} \in \Lambda_{\mathrm{ad}}} D_{\boldsymbol{\nu}} u$
$Q_{\Lambda_{\mathrm{tot}}} u := \sum_{\boldsymbol{\nu} \in \Lambda_{\mathrm{tot}}} D_{\boldsymbol{\nu}} u$

Figure 8 depicts a comparison of the convergence for our apriori chosen index sets Λ_N in (4.14) based on the estimator $\mathbf{m}^{(\mathrm{hol})}$ with the results from Alg. 1, as well as with the estimator $\mathbf{t}^{(\mathrm{hol})}$, which does *not* incorporate the fact that polynomials of degree 1 need not be considered for the quadrature error. The plots show the error vs. number of quadrature points. In case of the adaptive algorithm, we plot the curve for the set of accepted indices Λ_{ad} and for the set of total indices Λ_{tot}, as computed by Alg. 1. First, we point out that exploiting Lemma 3.2 (ii) accomplishes a considerable improvement as the comparison between $\mathbf{m}^{(\mathrm{hol})}$ and $\mathbf{t}^{(\mathrm{hol})}$ reveals. Next, note that in order to find the set Λ_{ad}, Alg. 1 also requires to evaluate the integrand at quadrature points belonging to the total set Λ_{tot}. Thus, the curve for the accepted multiindices Λ_{ad} should be considered as a benchmark, whereas the curve for the total set of indices Λ_{tot} can be seen as a practically obtainable computation in terms of error vs. number of quadrature points (i.e. number of function evaluations). We observe, that our apriori chosen quadrature points are almost as good, as the ones obtained by the adaptive method and denoted by Λ_{ad} above. For small θ there is hardly any difference. In case of Λ_{tot}, our method even outperforms the adaptive algorithm when θ becomes small. Figure 9 shows the same comparison, but based on $\mathbf{m}^{(\mathrm{u1})}$. In this case the performance of the apriori chosen index sets and the adaptive ones is practically identical. Also, in case one integrand function evaluation is costly, determining the index set Λ_N in a precomputation step allows to compute all function evaluations in parallel, which is in general not possible for the adaptive algorithm in [21].



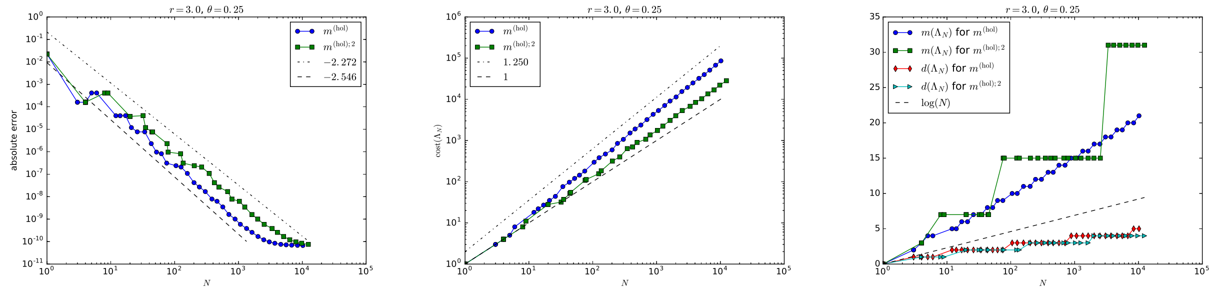
(a) $\theta = 0.25$ (b) $\theta = 0.05$ (c) $\theta = 0.005$

Figure 8: Comparison of the absolute quadrature error for u_2 in (4.43), with $r = 3$, different values of θ and different methods: either we use the apriori built sets Λ_N in (4.14) with the estimators $\mathbf{t}^{(\mathrm{hol})}$, $\mathbf{m}^{(\mathrm{hol})}$ (cp. (4.11), (4.13)) respectively, in which case the error is $|\int_U u_2(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_N} u_2|$. Or, we use the adaptive algorithm Alg. 1, in which case the error is $|\int_U u_2(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_{\mathrm{ad}}} u_2|$ for the accepted and $|\int_U u_2(\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) - Q_{\Lambda_{\mathrm{tot}}} u_2|$ for the total set. The x-axis shows the number N of employed quadrature points in each case. Our apriori determined index sets for this example, are almost as good as the adaptively constructed ones Λ_{ad} (whose construction forces sequential integrand evaluation and requires memory growing superlinearly with $\#(\Lambda_{\mathrm{ad}})$). Taking into account Lemma 3.2, item (ii) improves the method significantly, as the curves for $\mathbf{t}^{(\mathrm{hol})}$ and $\mathbf{m}^{(\mathrm{hol})}$ show.



(a) $\theta = 0.25$ (b) $\theta = 0.05$ (c) $\theta = 0.005$

Figure 9: Same test as shown in Figure 8, but employing $\mathbf{m}^{(\mathrm{u1})}$ instead of $\mathbf{m}^{(\mathrm{hol})}$.



(a) Absolute quadrature error

(b) Comparison of number of quadrature points $|\Lambda_N| = N$ versus $\mathrm{cost}(\Lambda_N)$

(c) The quantities defined in (3.13)

Figure 10: Effect of Z in Assumption 2.7, 3.7 for $r = 3$, $\theta = 0.25$: In this figure we denote $\mathrm{cost}(\Lambda_N) := \sum_{\{\boldsymbol{\nu}\in\Lambda_N\,:\,c_{\Lambda_N;\boldsymbol{\nu}}\neq 0\}} \prod_{j\in\mathbb{N}}(\nu_j+1)$, which is the third term in (3.14). In all three plots, we compare the respective quantities with Λ_N in (4.14) either constructed using the estimator $\mathbf{m}^{(\mathrm{hol})}$ or $\mathbf{m}^{(\mathrm{hol});2}$ (cp. (4.11), (4.12), (4.13)) as indicated in the legend.

4.3.4 Error vs. work

Let u_2 as in (4.43) and $r = 3$, $\theta = 0.25$. We now investigate the influence of using the estimator $\mathbf{m}^{(\mathrm{hol});2}$ as in (4.12) instead of $\mathbf{m}^{(\mathrm{hol})}$ in (4.11), where $Z = \{0\} \cup \{2^j \,:\, j \in \mathbb{N}_0\}$. Recall, that according to Lemma 3.11 and Corollary 3.13, using $\mathbf{m}^{(\mathrm{hol});2}$ means that the error will converge with the proven rate $2r-1-\varepsilon$ with respect to the overall complexity of the Smolyak algorithm, by which we mean (3.14). In the first case of $\mathbf{m}^{(\mathrm{hol})}$, this rate is only obtained with respect to the number of quadrature points. The first plot in Figure 10 depicts the error convergence w.r.t. the number of quadrature points in both cases. The loss of flexibility in choosing the set Λ_N based on Z, i.e. employing $\mathbf{m}^{(\mathrm{hol};2)}$ rather than $\mathbf{m}^{(\mathrm{hol})}$, leads to a (slightly) inferior performance, in the considered example.

In the second plot, we compare the third part of the cost in (3.14) for the sets generated with both estimators. It is observed, that with (cp. (3.14))

$$\mathrm{cost}(\Lambda) := \sum_{\{\boldsymbol{\nu}\in\Lambda\,:\,c_{\Lambda;\boldsymbol{\nu}}\}\neq 0} \prod_{j\in\mathbb{N}} (\nu_j + 1), \tag{4.45}$$

$\mathrm{cost}(\Lambda_N(\mathbf{m}^{(\mathrm{hol});2}))$ grows nearly linear, whereas the growth of $\mathrm{cost}(\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})$ is more noticeably superlinear in $N = |\Lambda_N|$. Overall the difference and improvement in terms of cost vs. error (if any) is marginal, since the number of quadrature points vs. cost behaviour for $\mathbf{m}^{(\mathrm{hol})}$ is already close to linear, and a more accentuated difference possibly might, similarly as in Sec. 4.2, again only be observed for (very) large numbers of quadrature points.

Finally, the third plot shows the quantities in (3.13). The behaviour in the second and third plot seems in accordance with Lemma 3.11, which states in particular that (with (4.45))

$$\mathrm{cost}(\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})) = O(|\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})|^{1+\varepsilon}), \tag{4.46}$$

$$d(\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})) = o(\log(|\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})|)), \tag{4.47}$$

$$m(\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})) = O(\log(|\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})|)), \tag{4.48}$$

as $|\Lambda_N(\mathbf{m}^{(\mathrm{hol});2})| \to \infty$, for any $\varepsilon > 0$. The last two asymptotics are also true for $\mathbf{m}^{(\mathrm{hol})}$.

5 Conclusions and Generalizations

We have analyzed convergence rates of Smolyak quadratures for classes of smooth, Banach space valued, parametric functions with a suitable sparsity as precised in Assumption 2.1. We proved that exploiting certain cancellation properties implied by the combination coefficients and the symmetry of the marginal probability measures allow for the dimension independent convergence rate $2/p - 1$ for p-summable sequences of (norms of) Taylor gpc coefficients of the parametric integrand functions. This is superior to previously known rates established, for example, in [25, 23], of N-term gpc approximation of the integrand obtained in [12], or for Higher Order Quasi-Monte Carlo integration in [16], under analogous sparsity assumptions on the parametric integrands. We also provided an a-priori construction algorithm of integrand-adapted sparse grids whose complexity (work and memory) scales near linearly with respect to the number quadrature points. Numerical experiments verify our findings and show that the dimension-independent convergence rates are achieved with a moderate number of quadrature points *provided the parametric integrand functions have small deviation from their 'nominal', average, values.* We explain, by a refined analysis of the error bounds for a model class of integrand functions, that the asymptotic range where the (dimension-independent) convergence rate $O(N^{-(2/p-1)})$ is visible could appear only for a prohibitively large number of quadrature points. Convergence rates which are superior to N-term approximation bounds for the parametric integrands have been reported in numerical experiments for example in [37]. Concrete a-priori estimates on gpc coefficients that may be exploited to apriori determine suitable index sets by e.g. greedy searches or by knapsack solvers were also given in these references. The presently proposed variants of the Smolyak algorithm, in particular exploiting multiindices containing a 1, appear to be new. As we prove and verify in numerical experiments, this results in an algorithm that performs comparably to the currently best (heuristic) adaptive algorithms, from [20, 21] as shown in in Figure 8.

The complexity of the Smolyak quadrature was investigated under p-summability of sequences of (X-norms of) Taylor coefficients, as implied under the $(\boldsymbol{b}, \varepsilon)$-holomorphy Assumption 2.1 of the parametric integrand function. This is known to hold for broad classes of holomorphic-parametric operator equations as shown in [10], and also for the corresponding Bayesian inverse problems [39, 37]. We emphasize that our key findings, notably the observation that all linear terms are integrated exactly by any Smolyak quadrature, remain valid for other measures μ, presuming that the one point rule in the Smolyak construction integrates linear polynomials exactly. In particular, similar improvements as shown in this paper can also be expected under different summability results. For example, for linear, affine-parametric diffusion problems with localized coefficient functions $\psi_j(x)$, a weighted summability condition as presumed in the below corollary was verified in [1, Theorem 1.2]. This gives a bound analogous to [1, Cor. 2.1]:

Corollary 5.1. *Let $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}} \in \ell^q(\mathbb{N})$ with $q > 0$ and $\rho_j > 1$ for all $j \in \mathbb{N}$, and assume $(\|u_{\boldsymbol{\nu}}\|_X \boldsymbol{\rho}^{\boldsymbol{\nu}}) \in \ell^2(\mathcal{F})$. Then $(\|u_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}_2} \in \ell^p(\mathcal{F}_2)$ with $p = 2q/(4+q)$.*

Proof. Hölder's inequality implies

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}_2} \|u_{\boldsymbol{\nu}}\|_X^p \leq \left(\sum_{\boldsymbol{\nu}\in\mathcal{F}_2} (\|u_{\boldsymbol{\nu}}\|_X \boldsymbol{\rho}^{\boldsymbol{\nu}})^2\right)^{\frac{p}{2}} \left(\sum_{\boldsymbol{\nu}\in\mathcal{F}_2} \boldsymbol{\rho}^{-\frac{2p}{2-p}}\right)^{\frac{2-p}{p}}. \tag{5.1}$$

According to Lemma 2.4, the last sum is finite iff $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^{\tilde{q}}(\mathbb{N})$ with $\tilde{q} = 4p/(2-p)$. Solving this equation for $\tilde{q}$ gives $p = 2\tilde{q}/(4+\tilde{q})$. □

With such a result, we expect following the arguments in Thm. 2.11 and Thm. 3.3 the dimension-independent convergence rate $1/p-1 = (4-q)/(2q)$ provided $q \in (0,4)$ (i.e. $p \in (0,1)$).

Another particular case in point are Gaussian measures μ. Here, for certain PDEs bounds on Hermite Chaos coefficients can be obtained by real-variable bootstrapping on the parametric PDE (see [22, 31, 29]), so that similar conclusions for the corresponding Smolyak algorithms could be expected.

In many practical settings the evaluation of the integrand is presumed to be far more costly than performing the quadrature itself. For integrands exhibiting low sparsity, using a large number of quadrature points becomes inevitable. The near linear scaling of the cost in terms of the number of quadrature points makes the algorithm feasible also for such problems.

In this paper we assumed the integrand to allow exact evaluation at each quadrature point with cost $O(1)$. In general, for UQ problems the integrand is given as the solution to some PDE, which needs to be approximated by a numerical scheme. This will be addressed in [14], where we perform a fully discrete error analysis taking into account the cost of approximating the function values at the quadrature points.

References

[1] M. Bachmayr, A. Cohen, and G. Migliorati. Sparse polynomial approximation of parametric elliptic PDEs. Part I: Affine coefficients. *ESAIM Math. Model. Numer. Anal.*, 51(1):321–339, 2017.

[2] V. Barthelmann, E. Novak, and K. Ritter. High dimensional polynomial interpolation on sparse grids. *Adv. Comput. Math.*, 12(4):273–288, 2000. Multivariate polynomial interpolation.

[3] J. Beck, R. Tempone, F. Nobile, and L. Tamellini. On the optimal polynomial approximation of stochastic PDEs by Galerkin and collocation methods. *Math. Models Methods Appl. Sci.*, 22(9):1250023, 33, 2012.

[4] H.-J. Bungartz and M. Griebel. Sparse grids. *Acta Numer.*, 13:147–269, 2004.

[5] J.-P. Calvi and P. V. Manh. Lagrange interpolation at real projections of Leja sequences for the unit disk. *Proc. Amer. Math. Soc.*, 140(12):4271–4284, 2012.

[6] J.-P. Calvi and M. Phung Van. On the Lebesgue constant of Leja sequences for the unit disk and its applications to multivariate interpolation. *J. Approx. Theory*, 163(5):608–622, 2011.

[7] A. Chkifa, A. Cohen, R. DeVore, and C. Schwab. Adaptive algorithms for sparse polynomial approximation of parametric and stochastic elliptic pdes. *M2AN Math. Mod. and Num. Anal.*, 47(1):253–280, 2013.

[8] A. Chkifa, A. Cohen, and C. Schwab. High-dimensional adaptive sparse polynomial interpolation and applications to parametric pdes. *Journ. Found. Comp. Math.*, 14(4):601–633, 2013.

[9] M. A. Chkifa. On the Lebesgue constant of Leja sequences for the complex unit disk and of their real projection. *J. Approx. Theory*, 166:176–200, 2013.

[10] A. Cohen, A. Chkifa, and C. Schwab. Breaking the curse of dimensionality in sparse polynomial approximation of parametric pdes. *Journ. Math. Pures et Appliquees*, 103(2):400–428, 2015.

[11] A. Cohen, R. DeVore, and C. Schwab. Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs. *Found. Comput. Math.*, 10(6):615–646, 2010.

[12] A. Cohen, R. Devore, and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's. *Anal. Appl. (Singap.)*, 9(1):11–47, 2011.

[13] A. Cohen, C. Schwab, and J. Zech. Shape holomorphy of the stationary Navier-Stokes equations. Technical Report 2016-45, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2016.

[14] D. Dũng, C. Schwab, and J. Zech. 2017. in preparation.

[15] J. Dick. Walsh spaces containing smooth functions and quasi-Monte Carlo rules of arbitrary high order. *SIAM J. Numer. Anal.*, 46(3):1519–1553, 2008.

[16] J. Dick, F. Y. Kuo, Q. T. L. Gia, D. Nuyens, and C. Schwab. Higher order QMC Petrov-Galerkin discretization for affine parametric operator equations with random field inputs. *SIAM J. Numerical Analysis*, 52(6):2676–2702, 2014.

[17] J. Dick, F. Y. Kuo, Q. T. Le Gia, D. Nuyens, and C. Schwab. Higher order QMC Petrov-Galerkin discretization for affine parametric operator equations with random field inputs. *SIAM J. Numer. Anal.*, 52(6):2676–2702, 2014.

[18] J. Dick, F. Y. Kuo, and I. H. Sloan. High-dimensional integration: the quasi-Monte Carlo way. *Acta Numer.*, 22:133–288, 2013.

[19] R. N. Gantner and C. Schwab. Computational higher order quasi-monte carlo integration. In *Monte Carlo and Quasi-Monte Carlo Methods: MCQMC, Leuven, Belgium, April 2014*, volume 163, pages 271–288, 2016.

[20] T. Gerstner and M. Griebel. Numerical integration using sparse grids. *Numer. Algorithms*, 18(3-4):209–232, 1998.

[21] T. Gerstner and M. Griebel. Dimension-adaptive tensor-product quadrature. *Computing*, 71(1):65–87, 2003.

[22] I. G. Graham, F. Y. Kuo, J. A. Nichols, R. Scheichl, C. Schwab, and I. H. Sloan. Quasi-monte carlo finite element methods for elliptic pdes with lognormal random coefficients. *Numerische Mathematik*, 131(2):329–368, 2015.

[23] M. Griebel and J. Oettershagen. On tensor product approximation of analytic functions. *J. Approx. Theory*, 207:348–379, 2016.

[24] A.-L. Haji-Ali, H. Harbrecht, M. Peters, and M. Siebenmorgen. Novel results for the anisotropic sparse grid quadrature. Technical report, 2015.

[25] A.-L. Haji-Ali, F. Nobile, L. Tamellini, and R. Tempone. Multi-Index Stochastic Collocation for random PDEs. *Comput. Methods Appl. Mech. Engrg.*, 306:95–122, 2016.

[26] H. Harbrecht, M. Peters, and M. Siebenmorgen. Analysis of the domain mapping method for elliptic diffusion problems on random domains. *Numer. Math.*, 134(4):823–856, 2016.

[27] M. Hervé. *Analyticity in infinite-dimensional spaces*, volume 10 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1989.

[28] R. Hiptmair, L. Scarabosio, C. Schillings, and C. Schwab. Large deformation shape uncertainty quantification in acoustic scattering. Technical Report 2015-31, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2015.

[29] V. H. Hoang and C. Schwab. N-term Wiener chaos approximation rate for elliptic PDEs with lognormal Gaussian random inputs. *Math. Models Methods Appl. Sci.*, 24(4):797–826, 2014.

[30] C. Jerez-Hanckes, C. Schwab, and J. Zech. Electromagnetic wave scattering by random surfaces: Shape holomorphy. Technical Report 2016-49, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2016.

[31] F. Kuo, R. Scheichl, C. Schwab, I. Sloan, and E. Ullmann. Multilevel quasi-monte carlo methods for lognormal diffusion problems. Technical Report 2015-22, Seminar for Applied Mathematics, ETH Zürich, 2016.

[32] F. Y. Kuo and D. Nuyens. Application of quasi-Monte Carlo methods to elliptic PDEs with random diffusion coefficients: a survey of analysis and implementation. *Found. Comput. Math.*, 16(6):1631–1696, 2016.

[33] G. A. Muñoz, Y. Sarantopoulos, and A. Tonge. Complexifications of real Banach spaces, polynomials and multilinear maps. *Studia Math.*, 134(1):1–33, 1999.

[34] F. Nobile, R. Tempone, and C. G. Webster. An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, 46(5):2411–2442, 2008.

[35] F. Nobile, R. Tempone, and C. G. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, 46(5):2309–2345, 2008.

[36] H. Robbins. A remark on Stirling's formula. *Amer. Math. Monthly*, 62:26–29, 1955.

[37] C. Schillings and C. Schwab. Sparse, adaptive Smolyak quadratures for Bayesian inverse problems. *Inverse Problems*, 29(6):065011, 28, 2013.

[38] C. Schwab and C. Gittelson. Sparse tensor discretizations of high-dimensional parametric and stochastic pdes. *Acta Numerica*, 20:291–467, 2011.

[39] C. Schwab and A. M. Stuart. Sparse deterministic approximation of Bayesian inverse problems. *Inverse Problems*, 28(4):045003, 32, 2012.

[40] C. Schwab and R. A. Todor. Convergence rates of sparse chaos approximations of elliptic problems with stochastic coefficients. *IMA Journal of Numerical Analysis*, 44:232–261, 2007.

[41] S. Smolyak. Quadrature and interpolation formulas for tensor products of certain classes of functions. *Soviet Mathematics, Doklady*, 4:240–243, 1963.

[42] J. Zech, 2018. Dissertation, ETH Zürich, in preparation.