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Electromagnetic Wave Scattering by Random Surfaces: Shape Holomorphy

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For time-harmonic electromagnetic waves scattered by either perfectly conducting or dielectric bounded obstacles, we show that the fields depend holomorphically on the shape of the scatterer. In the presence of random geometrical perturbations, our results imply strong measurability of the fields, in weighted spaces in the exterior of the scatterer. These findings are key to prove dimension-independent convergence rates of sparse approximation techniques of polynomial chaos type for forward and inverse computational uncertainty quantification. Also, our shape-holomorphy results imply parsimonious approximate representations of the corresponding parametric solution families, which are produced, for example, by greedy strategies such as model order reduction or reduced basis approximations. Finally, the presently proved shape holomorphy results imply convergence of shape Taylor expansions far-field patterns for fixed amplitude domain perturbations in a vicinity of the nominal domain, thereby extending the widely used asymptotic linearizations employed in first-order, second moment domain uncertainty quantification.

Keywords: Electromagnetic Scattering; Shape Calculus; Uncertainty Quantification; Smolyak quadrature

AMS Subject Classification: 35A20, 35B30, 32D05, 35Q61

1. Introduction

The efficient quantification of engineering systems' responses subject to random input data has attracted considerable attention in recent years, leading to the development of *computational Uncertainty Quantification* (UQ). Loosely speaking, *aleatoric uncertainty* deals with the propagation of uncertain inputs–material parameters, domains of definition, source terms, etc.–throughout otherwise assumed known models into so-called *Quantities of Interest* (QoI). In computational electromagnetics (EM), this constitutes the core computational UQ problem as *epistemic uncertainty*, i.e. uncertainty in the mathematical model, is considered negligible. In the present paper, we consider the impact of *uncertainty in obstacle's shapes on diffracted time-harmonic EM waves*.

Techniques such as polynomial chaos, Model Order Reduction (MOR) or the stochastic finite-element method[40,3,22] have been used by engineers for UQ in EM scattering. Domain mapping approaches, alternatively, transfer uncertainty in the scatterers' geometry into parameter uncertainty[8]. Along these lines, one may stipulate small (random) deviations with respect to both wavelength and scatterers' shape from a nominal, reference shape, and perform a linearization of the scattered fields with respect to the obstacle's geometry. This naturally introduces the mathematical concept of *shape derivative*, as has been elaborated in several monographs in recent years; we mention only Refs. 38, 32 wherein problems in mechanics have been analyzed. For the EM problems considered here, by Hadamard's theorem, the resulting linearized equations for the first order shape sensitivities are homogeneous Maxwell equations posed on the nominal geometry, with inhomogeneous boundary data only.[35,25,32] In Ref. 27, we performed a sparse First-Order Second Moment (FOSM) domain perturbation analysis and derived tensorized boundary integral equations for direct computation of second order statistics (covariance) of the scattered, time-harmonic EM fields. This approach, while being computationally efficient, is allowed only for small amplitude domain perturbations and yields only second order statistics. Nonetheless, in some cases the scattered field is required in, as far as possible, explicit, parametric form with the parameters describing completely an admissible class of domain variations.

In the present note, rather than addressing the most general setting, we perform a *shape holomorphy* analysis for the time-harmonic, scattered EM fields arising from two types of bounded obstacles: **perfect conductor** (PC) and **dielectric** (DE) ones. Explicitly, we show that the electric and magnetic fields solving these problems depend holomorphically on the geometry of the corresponding scatterer's surface. Hence, for certain countably-parametric regular scattering surfaces, the scattered waves are, as functions of the coordinates y_j in the shape parametrization, a holomorphic map from the parameter space into certain Hilbert spaces. In particular, when interpreted as elements of weighted Sobolev spaces imposing Silver-Müller (outgoing) radiation conditions essentially (*cp.* Refs. 30, 1). Naturally, these holomorphy results are closely related to so-called *material derivatives* of problems (PC) and (DE). Our analysis therefore draws upon analytical tools from shape optimization, (*cf.* Ref. 38) and implies, as corollaries, the existence and local error bounds for *shape Taylor expansions*, which arise in sparse tensor discretization of FOSM approaches to computational UQ (*cf.* Ref. 24, 27 and references therein). However, unlike the FOSM analysis in Ref. 27, the presently developed holomorphy results apply also for large shape variations, albeit precluding topology changes.

The analytical results obtained here facilitate efficient computational treatment of shape uncertainty in computational EM scattering. Indeed, the presently obtained shape holomorphy also implies *sparsity results* on generalized polynomial chaos representations of the parametric EM fields corresponding to the uncertain scatterers' geometries. Adaptive sparse tensor interpolation and numerical quadrature schemes for countably-parametric functions were shown[16,13,12] to yield dimension-independent convergence rates for Hilbert space-valued functions of a sequence $\boldsymbol{y} = (y_j)_{j\geq 1}$ of parameters $y_j \in [-1, 1]$, *provided* that the parametric function has certain sparsity properties. For example, based on the shape-holomorphy results established here, *sparse adaptive collocation approximation algorithms* allow to compute dimensionally adaptive, sparse polynomial chaos surrogates of the scattered EM fields, as well as far-field expansions for the considered problem classes.

Our results also justify the use of *high-order Quasi-Monte-Carlo quadrature* (QMC) methods in order to efficiently compute numerically *ensemble averages* over all possible shapes, for example to calculate far-field statistics from known shape statistics. We refer to Refs. 21 and 20, and the references there for details. They also allow to employ recently developed, high-order QMC methods for *Bayesian shape inversion* (*cf.* Refs. 36, 37). Similar results as we shall prove in the following are obtained in 14 for the nonlinear stationary Navier-Stokes equation.

The outline of this paper is as follows: in Section 2, notation and preliminary results are introduced. Section 2.2 provides mathematical tools for the two Maxwell problems considered, with particular emphasis on the underlying functional spaces and a general strategy to prove existence and uniqueness. Section 2.5 contains key domain transformation results for rotational fields. Precise formulations of our model problems are given in Section 3, including their variational formulations pulled back onto the nominal domain. Main results concerning shape holomorphy are contained in Section 4. In Section 5, we establish $(\boldsymbol{b}, \epsilon)$-*holomorphy* for parametric *domain-to-solution maps* related to the Maxwell problems considered. Finally, conclusions and possible directions of future work are succinctly indicated in Section 6.

2. Preliminaries

To set notation and to prepare the ensuing development, we review some known results that will be required in the subsequent development. In particular, we specify functional spaces and trace operators appearing in variational formulations of the model problems considered. Furthermore, we comment on the existence and uniqueness of solutions for Maxwell problems in general, and introduce domain transformations as they will be used throughout.

2.1. *Notation*

Let $d \in \{1, 2, 3\}$. For a bounded Lipschitz domain $D \subseteq \mathbb{R}^d$, the set $C^m(D)$, $m \in \mathbb{N}_0$, denotes the space of m-times differentiable scalar functions on D, and similarly for the space of infinitely differentiable, scalar continuous functions, we write $C^\infty(D)$.

The notation $C_0^m(D)$ denotes the space of compactly supported C^m-functions in D. Let $L^p(D)$ denote the class of p-integrable functions over D. Throughout, for any real or complex Banach space X, X' denotes the dual space of X. Generally, boldface symbols for functional spaces represent vector-valued counterparts, e.g., $\boldsymbol{L}^2(D)$ is the space of vector-valued functions with d components in $L^2(D)$. Dual spaces are defined in standard fashion with duality products denoted by angular brackets $\langle\cdot,\cdot\rangle_D$. For the L^2-inner product on D we write $(\cdot,\cdot)_D$. If it will not cause any confusion, the subscript indicating the underlying domain is omitted. The inner product as well as any occurring dual products are always understood in the bilinear, rather than sesquilinear, sense (*cf.* Remark 2.1).

We shall use standard Sobolev spaces $W^{s,p}(D)$ of functions with $s \in \mathbb{R}$ weak derivatives in $L^p(D)$ for $p \geq 1$. In the case of $p = 2$, we use the standard notation $H^s(D)$, of complex-valued scalar functions with the customary convention $H^0(D) = L^2(D)$.

2.2. *Maxwell Equations*

Let now $D \subset \mathbb{R}^3$ be an open, bounded Lipschitz domain with simply connected boundary surface ∂D. We denote the unbounded exterior domain corresponding to D by $D^c := \mathbb{R}^3 \backslash \overline{D}$.

We consider the time-harmonic propagation of EM waves for a circular frequency $\omega > 0$. Material parameters ε and μ denote the dielectric permittivity and magnetic permeability, respectively, assumed to be positive and piecewise constant with further restrictions imposed later. We may also consider conductive media characterized by the conductivity parameter $\sigma \geq 0$. Denoting as usual by $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic fields, respectively, Maxwell equations without sources read[a]

$$\mathbf{curl}\,\mathbf{E} - \imath\omega\mu\mathbf{H} = \mathbf{0}\ , \quad \mathbf{curl}\,\mathbf{H} + (\imath\omega\varepsilon - \sigma)\mathbf{E} = \mathbf{0}. \tag{2.1}$$

Setting $\kappa^2 := \omega^2\mu\varepsilon + \imath\omega\mu\sigma$, (2.1) can be reduced to

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E} - \mu^{-1}\kappa^2\mathbf{E} = \mathbf{0}, \tag{2.2}$$

and, in case μ is constant, to

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} - \kappa^2\mathbf{E} = \mathbf{0}. \tag{2.3}$$

For $\sigma = 0$, the magnetic flux density is $\mathbf{H} = \frac{1}{\imath\omega\mu}\,\mathbf{curl}\,\mathbf{E}$, which is computed *a posteriori*. In D^c, we impose the Silver-Müller radiation condition:

$$\left|\mathbf{curl}\,\mathbf{E}(\mathbf{x}) \times \frac{\mathbf{x}}{r} - \imath\kappa\mathbf{E}(\mathbf{x})\right| = \mathcal{O}\left(\frac{1}{r^2}\right)\ , \quad \mathbf{x} \in \mathbb{R}^3,\ r \to \infty\ , \tag{2.4}$$

where $r := \|\mathbf{x}\|_2$ and $\|\cdot\|_2$ denotes the standard Euclidean norm.

[a]In the following, we denote scalars in simple typeface, vector fields with boldface. Quantities defined over volumes are written in capital letters and surface ones in lower case.

As mentioned in the Introduction, we will consider two particular time-harmonic EM wave scattering problems:

(PC) A *bounded perfect conductor* occupying D. This leads to an exterior Dirichlet problem for (2.2) with boundary ∂D;

(DE) A *dielectric interface*, with different material parameters. This leads to a transmission problem in $D \cup D^c$ where ∂D is the interface.

Detailed discussions for each model problem will be given in Section 3.

2.3. *Functional spaces*

We recall the following vectorial spaces to formulate Maxwell problems:

$$\boldsymbol{H}(\mathbf{curl}, D) := \{\mathbf{U} \in \boldsymbol{L}^2(D) \ : \ \mathbf{curl}\,\mathbf{U} \in \boldsymbol{L}^2(D)\} \,,$$
$$\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D) := \{\, \mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D) \mid \mathbf{curl}\,\mathbf{curl}\,\mathbf{U} \in \boldsymbol{L}^2(D)\,\} \,.$$

In the ensuing discussion of strong measurability of scattered fields for random shape perturbations as well as in the verification of shape holomorphy for exterior problems, we find it convenient to work in weighted Hilbert spaces over the exterior domain (*cf.* Remark 3.2 and Section 5.3 in Ref. 30):

$$\begin{aligned} \boldsymbol{H}_\kappa(\mathbf{curl}, D^c) := \big\{\mathbf{U} \,:\, &\mathbf{U} \text{ satisfies (2.4)},\ \mathbf{U}/r \in \boldsymbol{L}^2(D^c),\ \mathbf{curl}\,\mathbf{U}/r \in \boldsymbol{L}^2(D^c)\,, \\ &\tfrac{\mathbf{U}\cdot\mathbf{x}}{r} \in L^2(D^c),\ \tfrac{\mathbf{curl}\,\mathbf{U}\cdot\mathbf{x}}{r} \in L^2(D^c)\big\}\,, \end{aligned} \tag{2.5}$$
$$\boldsymbol{H}_\kappa(\mathbf{curl}\,\mathbf{curl}, D^c) := \big\{\mathbf{U} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c) \ : \ \mathbf{curl}\,\mathbf{U} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c)\big\} \,.$$

Observe that if $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D)$ (resp. $\mathbf{U} \in \boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$) solves the homogeneous Maxwell equations, then $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D)$ (resp. $\mathbf{U} \in \boldsymbol{H}_\kappa(\mathbf{curl}\,\mathbf{curl}, D^c)$). The dependence on κ in $\boldsymbol{H}_\kappa$ enters through the radiation condition (2.4).

Moreover, we denote by γ the standard trace operator mapping $\gamma : H^{s+1/2}(D) \to H^s(\partial D)$, $u \mapsto u|_{\partial D}$, $s \in (0,1)$, continuously. Similar considerations hold component-wise for vector spaces $\boldsymbol{H}^s(\partial D)$. For a Lipschitz surface $\Gamma = \partial D$, we will mainly be concerned with the trace spaces:

$$\boldsymbol{H}^{-1/2}_{\mathrm{div}}(\Gamma) := \{\mathbf{U} \in \boldsymbol{H}^{-1/2}(\Gamma) : \mathbf{U}\cdot\mathbf{n} = 0,\ \mathrm{div}_\Gamma\,\mathbf{U} \in H^{-1/2}(\Gamma)\} \,,$$

$$\boldsymbol{H}^{-1/2}_{\mathrm{curl}}(\Gamma) := \{\mathbf{U} \in \boldsymbol{H}^{-1/2}(\Gamma) : \mathbf{U}\cdot\mathbf{n} = 0,\ \mathrm{curl}_\Gamma\,\mathbf{U} \in H^{-1/2}(\Gamma)\},$$

endowed with corresponding graph norms. Here, the normal vector $\mathbf{n}$ on the scattering boundary points from D to D^c and div_Γ, curl_Γ denote surface divergence and scalar surface curl, respectively (*cp.* Refs. 30, Chap. 2.5, and Ref. 5 for definitions of these operators).

Definition 2.1. For $\mathbf{U} \in \boldsymbol{C}^\infty(\overline{D})$, we define the tangential Dirichlet and Neumann traces by

$$\gamma_{\mathrm{D}}\mathbf{U} := \mathbf{n} \times (\mathbf{U} \times \mathbf{n})|_{\partial D} \qquad \text{and} \qquad \gamma_{\mathrm{N}}\mathbf{U} := (\mathbf{n} \times \mathbf{curl}\,\mathbf{U})|_{\partial D} \ ,$$

respectively. The *flipped tangential trace* $\gamma_{\mathrm{D}}^{\times}$ is $\gamma_{\mathrm{D}}^{\times}\mathbf{U} := (\mathbf{n}\times\mathbf{U})|_{\partial D}$.

The trace operators $\gamma_{\mathrm{D}}^{\times}$ and γ_{N} can be extended to linear and continuous operators from $\boldsymbol{H}(\mathbf{curl}, D)$ and $\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl}, D)$, respectively, to $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$. Likewise, $\gamma_{\mathrm{D}} : \boldsymbol{H}(\mathbf{curl}, D) \to \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$ continuously [5]. Moreover, the traces γ_{D}, $\gamma_{\mathrm{D}}^{\times}$ and γ_N admit linear and continuous right inverses. For $\mathbf{U} \in \boldsymbol{H}_{\kappa}(D^c)$, $\mathbf{V} \in \boldsymbol{H}_{\kappa}(\mathbf{curl}, D^c)$, we define $\gamma_{\mathrm{D}}^{c}\mathbf{U}$ and $\gamma_{\mathrm{N}}^{c}\mathbf{V}$ in the same way and similar mapping properties hold.

With the trace operator $\gamma_{\mathrm{D}}^{\times}$, we define for a subset S of ∂D of positive surface measure:

$$\boldsymbol{H}_S(\mathbf{curl}, D) := \{\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D) \ : \ \gamma_{\mathrm{D}}^{\times}\mathbf{U} = \mathbf{0} \text{ on } S\} \, . \tag{2.6}$$

If $S = \partial D$, we set

$$\boldsymbol{H}_0(\mathbf{curl}, D) := \{\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D) \ : \ \gamma_{\mathrm{D}}^{\times}\mathbf{U} = \mathbf{0} \text{ on } \partial D\} \, . \tag{2.7}$$

By continuity of $\gamma_{\mathrm{D}}^{\times}$, $\boldsymbol{H}_S(\mathbf{curl}, D)$ is a closed subspace of $\boldsymbol{H}(\mathbf{curl}, D)$.

Finally, for $\mathbf{U}$, $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, D)$, where D is again bounded Lipschitz, there holds Green's formula [5] :

$$(\mathbf{U}, \mathbf{curl}\,\mathbf{V})_D - (\mathbf{curl}\,\mathbf{U}, \mathbf{V})_D = -\langle \gamma_{\mathrm{D}}^{\times}\mathbf{U}, \gamma_{\mathrm{D}}\mathbf{V}\rangle_{\partial D} = \langle \gamma_{\mathrm{D}}\mathbf{U}, \gamma_{\mathrm{D}}^{\times}\mathbf{V}\rangle_{\partial D} \, , \tag{2.8}$$

where $\langle\cdot,\cdot\rangle_{\partial D}$ denotes the $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)$ dual product since $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial D)' = \boldsymbol{H}_{\mathrm{curl}}^{-1/2}(\partial D)$ (*cf.* Thm. 2 in 6).

2.4. *Existence and uniqueness of solutions*

Existence proofs for time-harmonic scattering of EM waves are typically based on[29]: (i) a Helmholtz (Hodge) decomposition; (ii) the Fredholm alternative; and, (iii) a unique continuation result.

Even though not always explicitly stated in the respective references, this approach in general implies that the according operator is an isomorphism. Since we shall need this property in our analysis, we outline the general line of arguments for future reference.

Lemma 2.1. *Let X be a Banach space and $A : X \to X'$ a bounded linear operator. Furthermore, let $X = X_1 + X_2$ for some linear subspaces X_1, X_2 of X, and suppose that*

$$\langle Ax_2, y_1\rangle = 0 \qquad\qquad \forall\, x_2 \in X_2,\ \forall\, y_1 \in X_1. \tag{2.9}$$

Set the embedding $\iota_j : X' \hookrightarrow X_j'$ and define $A_j := \iota_j \circ A|_{X_j} : X_j \to X_j'$, $j = 1, 2$. Then,

(i) assuming injectivity of A_1, A is injective iff A_2 is injective;
(ii) if $A_j : X_j \to X_j'$ are isomorphisms with $\|A_j^{-1}\| \le C_j$ for $j = 1, 2$, then $A : X \to X'$ is an isomorphism with $\|A^{-1}\| \le C_1 + C_2 + C_1C_2\|A\|$.

Proof. In the following, $x = x_1 + x_2 \in X$, $y = y_1 + y_2 \in X$ with x_j, $y_j \in X_j$ for $j = 1, 2$.

(i) Let A be injective and $A_2 x_2 = 0$. Then $\langle Ax_2, y_2\rangle = \langle Ax_2, y_1 + y_2\rangle = 0$ for all $y_1 \in X_1$ and $y_2 \in X_2$, hence $x_2 = 0$ and therefore A_2 is injective. To show the converse, assume that A_1 and A_2 are injective. It then follows that $Ax = 0$ implies $\langle A(x_1 + x_2), y_1\rangle = \langle Ax_1, y_1\rangle = 0$ for all $y_1 \in X_1$, and thus $x_1 = 0$. Now, $\langle Ax, y_2\rangle = \langle Ax_2, y_2\rangle = 0$ for all $y_2 \in X_2$ gives $x_2 = 0$, and consequently A is injective.

(ii) The first item implies injectivity of A, so that it remains to show surjectivity of A and the bound on the inverse. For $f \in X'$ arbitrary, let x_1 such that $A_1 x_1 = f|_{X_1}$, and choose x_2 with $A_2 x_2 = f|_{X_2} - Ax_1|_{X_2}$. We observe that $Ax = f$, and consequently $A : X \to X'$ is bijective. Furthermore, $\|x_1\|_X \leq C_1 \|f\|_{X'}$ and $\|x_2\|_X \leq C_2(\|f\|_{X'} + \|A\|\|x_1\|_X)$ imply $\|A^{-1}\| \leq \|f\|_{X'}(C_1 + C_2 + C_1 C_2 \|A\|)$. □

Let us recall the Fredholm alternative as in Thm. 5.4.5, Ref. 30:

Proposition 2.1. *Let V, H be real or complex Hilbert spaces with injective compact embedding $V \hookrightarrow H$. Assume given $A : V \to V'$ continuous and satisfying a Gårding inequality: there exist constants $\alpha > 0$ and $c \geq 0$ and a linear isomorphism $\Theta : V \to V$ such that*

$$\forall\, u \in V : \quad \mathrm{Re}[\langle Au, \Theta u\rangle] \geq \alpha \|u\|_V^2 - c\|u\|_H^2 \,. \tag{2.10}$$

Then, $A : V \to V'$ satisfies the Fredholm dichotomy: either, A is an isomorphism, or A has a finite dimensional kernel $\mathcal{N} \subset V$. In this case, for $g \in V'$ such that $g(\mathcal{N}) = 0$, the problem of finding u with $Au = g$ admits a solution $u \in V$ which is unique up to $\mathcal{N}$.

The splitting in Lemma 2.1, is achieved through a Helmholtz or Hodge decomposition. It then remains to show that A_1 and A_2 are isomorphisms. This can be established by verifying Fredholmness and injectivity of A_j for $j = 1, 2$. According to Lemma 2.1 (i), it may alternatively be shown that A—rather than A_2—is one-to-one. In this step, unique continuation results[42,33,31,2] play an important role. Once this is obtained, Lemma 2.1 shows that A is an isomorphism.

Remark 2.1. As mentioned in Section 2.1, all inner products and duality pairings will be considered in the bilinear sense, i.e. without conjugation of the second argument as customary when dealing with complex-valued functions. This will also hold for all occurring bilinear forms, and similarly, we will consider linear instead of antilinear functionals. The reason for this convention is that we aim to show holomorphy of certain maps, and the operation of conjugation is not holomorphic. At the same time, the references we cite work with conjugation. These results are not affected by our convention, and we will not comment on this at every instance: let X be a complex Banach space, X' the continuous linear forms and X^* the continuous antilinear forms on X. Then, the map $\iota : X' \to X^*$ defined via $\langle \iota f, v\rangle := \langle f, \overline{v}\rangle$ is an isomorphism. Hence, $A : X \to X'$ is an isomorphism iff $\iota A : X \to X^*$ is.

2.5. *Domain Transformation*

We shall work with families of bounded domains $\{D_T\}_T$ parametrized by domain mappings T: there holds $D_T = T(\hat{D})$, where $\hat{D} \subset \mathbb{R}^3$ is a bounded Lipschitz domain henceforth referred to as *nominal domain.* We denote sets of admissible domain transformations by the symbol $\mathfrak{T}$, with a superscript *pc* and *de* indicating corresponding transformations for each problem class mentioned in Section 2.2. For all domain transformations $T \in \mathfrak{T}$, we shall assume the following:

Assumption 2.1. The set $\mathfrak{T} \subseteq W^{1,\infty}(\hat{D}, \mathbb{R}^3)$ is compact. Moreover, every map $T : \hat{D} \to D_T$ in $\mathfrak{T}$ is bijective and bi-Lipschitz, and $\hat{D}$, $D_T \subseteq \mathbb{R}^3$ are bounded Lipschitz domains.

In the ensuing analysis of shape holomorphy, a crucial role will be played by the transformation of the operator **curl** under domain transformations T (*cf.* [29]). Subsequently, for any T as in Assumption 2.1, $dT : D \to \mathbb{R}^{3\times 3}$ denotes the Jacobian of T.

Lemma 2.2. *Let $\hat{D}$, D_T, T be as in Assumption 2.1. The map*

$$\mathbf{U} \mapsto \hat{\mathbf{U}} := dT^\top(\mathbf{U} \circ T) \tag{2.11}$$

admits a bounded extension from $\boldsymbol{H}(\mathbf{curl}, D_T) \to \boldsymbol{H}(\mathbf{curl}, \hat{D})$, such that this extension is an isomorphism. The result remains true if we replace the spaces $\boldsymbol{H}(\mathbf{curl}, \cdot)$ by $\boldsymbol{H}_0(\mathbf{curl}, \cdot)$. Furthermore, there holds in $L^2(\hat{D})$

$$\mathbf{curl}\,\hat{\mathbf{U}} = (\det dT) dT^{-1}((\mathbf{curl}\,\mathbf{U}) \circ T)\ . \tag{2.12}$$

Proof. We begin with the case of $\boldsymbol{H}(\mathbf{curl}, D_T)$ and first assume $\mathbf{U} \in \boldsymbol{C}^\infty(\overline{D_T})$ and $T \in C^\infty(\overline{\hat{D}}, \mathbb{R}^3)$. Then, $dT^{\mathrm{co}}\,\mathbf{curl}\,\mathbf{U} \circ T = \mathbf{curl}(dT^\top \mathbf{U} \circ T)$, where dT^{co} denotes the cofactor matrix of dT. Now, assume again that T is only Lipschitz but still $\mathbf{U} \in \boldsymbol{C}^\infty(\overline{D_T})$. Then $\hat{\mathbf{U}} \in \boldsymbol{L}^\infty(\hat{D})$ is well defined. We claim that there exists a sequence $(T_n)_{n\in\mathbb{N}} \subseteq C^\infty(\overline{\hat{D}}, \mathbb{R}^3)$, such that $T_n \to T$ in $(H^1 \cap L^\infty)(\hat{D}, \mathbb{R}^3)$ as $n \to \infty$. First, since $\hat{D}$ is Lipschitz, we can extend T to a locally supported Lipschitz function $\tilde{T}$ from $\mathbb{R}^3 \to \mathbb{R}^3$ [39] . For a mollifier $\phi(\mathbf{x})$ with the properties that: $\phi \in C^\infty(\mathbb{R}^3)$, $\operatorname{supp}\phi \subseteq B_1$—the ball of radius one centered at the origin—, $\phi \geq 0$ and $\int_{\mathbb{R}^3} \phi = 1$, we set for $\epsilon > 0$, $\phi_\epsilon(\mathbf{x}) := \frac{1}{\epsilon^3}\phi(\frac{\mathbf{x}}{\epsilon})$ and $\tilde{T}_\epsilon := \tilde{T} * \phi_\epsilon$. Using $\tilde{T} \in (W^{1,\infty} \cap H^1)(\mathbb{R}^3, \mathbb{R}^3)$, it is not hard to see that $\tilde{T}_\epsilon \to \tilde{T}$ in $(H^1 \cap L^\infty)(\mathbb{R}^3, \mathbb{R}^3)$. Consequently, the same is true for the restriction of the $\tilde{T}_\epsilon$ to $\hat{D}$, which proves our claim with $T_n := (\tilde{T}_{1/n})|_{\hat{D}}$. Furthermore, again by Ref. 39, Sec. VI, 3.2, we may extend $\mathbf{U}$ to a compactly supported $\tilde{\mathbf{U}} \in \boldsymbol{C}^\infty(\mathbb{R}^3)$. Now, with $\tilde{\mathbf{U}}_n := \tilde{\mathbf{U}} \circ T_n$ and for $\hat{\mathbf{V}} \in \boldsymbol{C}_0^\infty(\hat{D})$ arbitrary, it holds

$$(\mathbf{curl}\,\hat{\mathbf{V}}, dT_n^\top \tilde{\mathbf{U}} \circ T_n)_{\hat{D}} = (\hat{\mathbf{V}}, \mathbf{curl}(dT_n^\top \tilde{\mathbf{U}} \circ T_n))_{\hat{D}} = (\hat{\mathbf{V}}, dT_n^{\mathrm{co}}\,\mathbf{curl}\,\tilde{\mathbf{U}} \circ T_n)_{\hat{D}} \,. \tag{2.13}$$

Since $\mathbf{curl}\,\tilde{\mathbf{U}} \in \boldsymbol{C}^1(\mathbb{R}^3)$ and $T_n \to T$ in $L^\infty(\hat{D}, \mathbb{R}^3)$, we observe that $\mathbf{curl}\,\tilde{\mathbf{U}} \circ T_n \to \mathbf{curl}\,\tilde{\mathbf{U}} \circ T = \mathbf{curl}\,\mathbf{U} \circ T$ in $\boldsymbol{L}^\infty(\hat{D})$ as $n \to \infty$. Next, dT_n^{co} consists of sums of elements of the type $\frac{\partial (T_n)_j}{\partial x_i} \frac{\partial (T_n)_l}{\partial x_m}$, for $i, j, l, m \in \{1, 2, 3\}$. Due to $T_n \to T$ in $H^1(\hat{D}, \mathbb{R}^3)$, we thus obtain $dT_n^{\mathrm{co}} \to dT^{\mathrm{co}}$ in $L^1(\hat{D}, \mathbb{R}^{3\times 3})$ as $n \to \infty$. Hence, the right-hand side in (2.13) converges to $(\hat{\mathbf{V}}, dT^{\mathrm{co}}\,\mathbf{curl}\,\mathbf{U} \circ T)_{\hat{D}}$, whereas, by similar arguments, the left-hand side converges to $(\mathbf{curl}\,\hat{\mathbf{V}}, dT^\top \mathbf{U} \circ T)_{\hat{D}} = (\mathbf{curl}\,\hat{\mathbf{V}}, \hat{\mathbf{U}})_{\hat{D}}$. Since $dT^{\mathrm{co}} = (\det dT)^{-1} dT^{-1}$— as T is bi-Lipschitz—we arrive at $\mathbf{curl}\,\hat{\mathbf{U}} = (\det dT) dT^{-1}(\mathbf{curl}\,\mathbf{U} \circ T)$ in the sense of distributions. The latter function is in $\boldsymbol{L}^\infty(\hat{D})$ by our assumptions on T, so that $\hat{\mathbf{U}} \in \boldsymbol{H}(\mathbf{curl}, \hat{D})$.

Next, we consider $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D_T)$. By the above,

$$\|\hat{\mathbf{U}}\|_{\boldsymbol{H}(\mathbf{curl}, \hat{D})} \leq C(T) \|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}, D_T)}, \quad \forall\ \mathbf{U} \in \boldsymbol{C}^\infty(\hat{D}).$$

Since the map $\mathbf{U} \mapsto \hat{\mathbf{U}}$ is linear, it allows a bounded extension from the dense subset $\boldsymbol{C}^\infty(\overline{D_T}) \subseteq \boldsymbol{H}(\mathbf{curl}, D_T)$ (*cf.* Thm. 3.26, Ref. 29) to the whole of $\boldsymbol{H}(\mathbf{curl}, D_T)$ as D_T is Lipschitz. By symmetry, also $\hat{\mathbf{U}} \mapsto d(T^{-1})^\top \hat{\mathbf{U}} \circ T^{-1}$ from $\boldsymbol{H}(\mathbf{curl}, \hat{D})$ to $\boldsymbol{H}(\mathbf{curl}, D_T)$ is well defined and bounded. Clearly, it is the inverse of the former map. We conclude that they are isomorphisms.

Finally, consider the transformation of functions in $\boldsymbol{H}_0(\mathbf{curl}, D_T)$. In this case, we argue analogously as above except that we start with $\mathbf{U} \in \boldsymbol{C}_0^\infty(D_T)$ instead of $\mathbf{U} \in \boldsymbol{C}^\infty(\overline{D_T})$, and use density of $\boldsymbol{C}_0^\infty(D_T)$ in $\boldsymbol{H}_0(\mathbf{curl}, D_T)$ (*cp.* (3.42) and Thm. 3.33 in Ref. 29). We merely need to check whether $\hat{\mathbf{U}} \in \boldsymbol{H}_0(\mathbf{curl}, \hat{D})$ holds true. For $\mathbf{x} \in \partial D_T$ with normal vector $\mathbf{n}(\mathbf{x})$, the corresponding normal vector $\hat{\mathbf{n}}(T^{-1}(\mathbf{x}))$ at $\hat{\mathbf{x}} := T^{-1}(\mathbf{x}) \in \partial\hat{D}$ is given by

$$\hat{\mathbf{n}}(T^{-1}(\mathbf{x})) = \frac{dT^\top \mathbf{n}(\mathbf{x})}{\|dT^\top \mathbf{n}(\mathbf{x})\|_2}$$

almost everywhere on $\partial\hat{D}$ (see, e.g., Eq. (2.1.94) in Ref. 4). Then,

$$\hat{\mathbf{n}}(\hat{\mathbf{x}}) \times \hat{\mathbf{U}}(\hat{\mathbf{x}}) = \frac{1}{\|dT^\top \mathbf{n}(\mathbf{x})\|_2^2} dT^\top \mathbf{n}(\mathbf{x}) \times dT^\top \mathbf{U}(\mathbf{x}) \,.$$

The identity $\mathbf{M}\boldsymbol{a} \times \mathbf{M}\boldsymbol{m} = \mathbf{M}(\boldsymbol{a} \times \boldsymbol{m})$ gives $\hat{\mathbf{n}}(\hat{\mathbf{x}}) \times \hat{\mathbf{U}}(\hat{\mathbf{x}}) = \left\|dT^\top \mathbf{n}(\mathbf{x})\right\|_2^{-1} (dT^\top)^{\mathrm{co}}(\mathbf{n}(\mathbf{x}) \times \mathbf{U}(\mathbf{x}))$. Therefore, if $\gamma_{\mathrm{D}}^\times \mathbf{U}$ vanishes, so does $\gamma_{\mathrm{D}}^\times \hat{\mathbf{U}}$. □

Remark 2.2. Suppose that D is bounded Lipschitz with $\partial D = \Gamma_1 \cup \Gamma_2$ where Γ_1, Γ_2 are closed and $\Gamma_1 \cap \Gamma_2 = \emptyset$. Then, we can find Lipschitz domains D_1, D_2 such that $D_1 \cup D_2 = D$ and $\overline{D_j} \cap \Gamma_i = \emptyset$ if $i \neq j$ as well as $\overline{D_i} \cap \Gamma_i = \Gamma_i$ for all i, $j \in \{1, 2\}$. Furthermore, let $\psi \in C_0^\infty(\mathbb{R}^3)$ such that $\psi \equiv 1$ in a neighbourhood of Γ_1 and $\operatorname{supp}\psi \subseteq D_1$. For a function $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D)$ with $\gamma_{\mathrm{D}}^\times \mathbf{U}|_{\Gamma_1} = 0$, using density of $\boldsymbol{C}_0^\infty(D_1)$ in $\boldsymbol{H}_0(\mathbf{curl}, D_1)$ respectively density of $\boldsymbol{C}^\infty(D)$ in $\boldsymbol{H}_0(\mathbf{curl}, D)$, it is then easy to see, that we may find functions $\mathbf{U}_n^1 \in \boldsymbol{C}_0^\infty(D_1)$ converging to $\mathbf{U}\psi|_{D_1} \in \boldsymbol{H}_0(\mathbf{curl}, D_1)$ and $\mathbf{U}_n^2 \in \boldsymbol{C}^\infty(D)$ with $\mathbf{U}_n^2|_{\Gamma_1} \equiv 0$ converging to $(1-\psi)\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}, D)$ as $n \to \infty$. Overall, we obtain density of $\{\varphi \in \boldsymbol{C}^\infty(D) : \varphi|_{\Gamma_1} \equiv 0\}$ in $\{\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D) : \gamma_{\mathrm{D}}^\times \mathbf{U}|_{\Gamma_1} \equiv 0\}$. Consequently, with an analogous proof, the statement of Lemma 2.2 also remains true if $\gamma_{\mathrm{D}}^\times \mathbf{U} = 0$ does not hold on ∂D, but merely on some part Γ_1 of ∂D, closed and separated from $\partial D \backslash \Gamma_1$ in the above sense.

Lemma 2.3. *Let Assumption 2.1 be satisfied and let* ε, $\mu \in L^\infty(D_T, \mathbb{C}^{3\times 3})$ *be given. For* $\mathbf{U}$, $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, D_T)$ *there holds*

$$\int_{D_T} \varepsilon\, \mathbf{curl}\, \mathbf{U} \cdot \mathbf{curl}\, \mathbf{V} = \int_{\hat{D}} \frac{1}{\det dT} (\varepsilon \circ T) dT\, \mathbf{curl}\, \hat{\mathbf{U}} \cdot dT\, \mathbf{curl}\, \hat{\mathbf{V}} \tag{2.14}$$

$$\int_{D_T} \mu \mathbf{U} \cdot \mathbf{V} = \int_{\hat{D}} \det dT (\mu \circ T) dT^{-\top} \hat{\mathbf{U}} \cdot dT^{-\top} \hat{\mathbf{V}}\,. \tag{2.15}$$

Proof. According to (2.11) and (2.12), $\mathbf{U} \circ T = dT^{-\top} \hat{\mathbf{U}}$ and $\mathbf{curl}\, \mathbf{U} \circ T = (\det dT)^{-1} dT\, \mathbf{curl}\, \hat{\mathbf{U}}$ for all $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl}, D_T)$. Since all integrands are in $\boldsymbol{L}^1(D_T)$ or $\boldsymbol{L}^1(\hat{D})$, (2.14), (2.15) are an immediate consequence of the transformation formulae (2.11), (2.12). □

3. Maxwell Model Problems

We introduce two model problems to be considered in what follows. At this stage, we formulate models for a generic bounded scatterer geometry denoted by $\tilde{D}$; it is associated with the computational domain D, wherein the corresponding variational formulation will be stated and our analysis performed. The relation between $\tilde{D}$ and D depends on the model under consideration and will be made clear subsequently though attention should be given to this. Furthermore, D will stand either for the nominal domain $\hat{D}$ or for an instance D_T of a parametric family of domains diffeomorphic to $\hat{D}$ as in Section 2.5. Later on, we will denote by $S \subseteq \partial\tilde{D}$ the part of the volume boundary subject to shape uncertainty.

The occurring bilinear forms for problems (PC) and (DE) will be denoted by a^{pc}, a^{de}, respectively. Furthermore, we write A^{pc} and A^{de} for the operators induced by those bilinear forms accordingly.

3.1. *Scattering by a perfect conductor*

The scattering of time-harmonic, EM waves by a perfect electrical conductor is a classical problem in computational EM; we refer, for example, to Sec. 5.4 in Ref. 30, and Chap. 10 in Ref. 29.

We assume the bounded Lipschitz domain $\tilde{D}$ to be filled by either a PC or a purely dielectric exterior domain $\tilde{D}^c = \mathbb{R}^3 \backslash \overline{\tilde{D}}$ with material constants:

$$\mu, \varepsilon > 0, \qquad \sigma = 0 \qquad \text{and} \qquad \kappa := \omega\sqrt{\mu\varepsilon}\,. \tag{3.1}$$

Here, the uncertain scatterer geometry is given by the surface $S = \partial\tilde{D}$. We assume the incident field $\mathbf{E}^{\rm inc} \in \boldsymbol{H}_{\rm loc}(\mathbf{curl}, \tilde{D}^c)$ as given and such that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^{\rm inc} - \kappa^2\mathbf{E}^{\rm inc} = \mathbf{0}$. If $\mathbf{E}^{\rm scat}$ denotes the field scattered by the PC, the total electric field in $\tilde{D}^c$ can be written as $\mathbf{E} = \mathbf{E}^{\rm inc} + \mathbf{E}^{\rm scat}$, satisfying $\gamma_{\rm D}\mathbf{E} = \mathbf{0}$ over $S = \partial\tilde{D}$. This implies

$$\gamma_{\rm D}\mathbf{E}^{\rm scat} = -\gamma_{\rm D}\mathbf{E}^{\rm inc} =: \boldsymbol{m}^{\rm pc} \quad \text{on } S. \tag{3.2}$$

The scattering of a time-harmonic incident wave by a perfect conductor can be stated as follows: given $\boldsymbol{m}^{\rm pc} \in \boldsymbol{H}_{\rm div}^{-1/2}(S)$, we seek a scattered field $\mathbf{E}^{\rm scat}$ satisfying (2.2) in $\tilde{D}^c$, as well as (2.4) and (3.2) such that

$$\mathbf{E}^{\rm scat} \in \boldsymbol{H}_\kappa(\mathbf{curl}, \tilde{D}^c)\ . \tag{3.3}$$

3.1.1. *Reduction to a bounded domain*

As is customary (*cf.* Ref. 29, Sec. 10.2, and Ref. 30, Sec. 5.4.2), we reduce the exterior problem (PC) to a bounded domain using a Calderón operator: to this end, fix $R > 0$ such that the closure of $\tilde{D}$ is contained in the ball $B_{R/2}$ of radius $R/2$ around the origin. According to Sec. 9.4.1 in Ref. 29 or Sec. 5.3.2 in Ref. 30, there exists a bounded linear *Dirichlet-to-Neumann* (DtN) operator $\Lambda^{\rm pc} : \boldsymbol{H}_{\rm div}^{-1/2}(\partial B_R) \to \boldsymbol{H}_{\rm div}^{-1/2}(\partial B_R)$ on ∂B_R corresponding to radiating solutions of (2.2), (2.4) in $B_R^c := \mathbb{R}^3 \setminus \overline{B}_R$.

Remark 3.1. In fact, the Calderón operator and general discussion in Secs. 9-10 in Ref. 29 only consider the case $\varepsilon = \mu = 1$. However, this is no real restriction: let $\mathbf{E}$, $\mathbf{H}$ solve (2.1) in $\tilde{D}^c$ (again $\sigma = 0$), together with (2.4). Set $\tilde{\mathbf{H}} := \sqrt{\frac{\mu}{\varepsilon}}\mathbf{H}$. Then $\mathbf{E}$, $\tilde{\mathbf{H}}$ satisfy

$$\mathbf{curl}\,\mathbf{E} - \imath\omega\sqrt{\varepsilon\mu}\,\tilde{\mathbf{H}} = \mathbf{0}\ , \quad \mathbf{curl}\,\tilde{\mathbf{H}} + \imath\omega\sqrt{\varepsilon\mu}\,\mathbf{E} = \mathbf{0} \quad \text{in} \quad B_R^c, \tag{3.4}$$

and (2.4) remains true for $\mathbf{E}$ with $\kappa = \omega\sqrt{\varepsilon\mu}$ corresponding to the occurring factor in (3.4). Thus, problem (3.4)—covered by the analysis in Ref. 29 with wavenumber $\kappa = \omega\sqrt{\varepsilon\mu}$—leads to the same partial differential equation for the electrical field $\mathbf{E}$ when eliminating the magnetic field $\tilde{\mathbf{H}}$ from the equation. Applying the results from Sec. 9.4.1 in Ref. 29 to the solution of (3.4), the Calderón operator $\Lambda^{\rm pc}$ then satisfies $\Lambda^{\rm pc}\gamma_{\rm D}^\times\mathbf{E} = \gamma_{\rm D}^\times\tilde{\mathbf{H}} = \frac{1}{\imath\omega\sqrt{\varepsilon\mu}}\gamma_{\rm N}\mathbf{E}$.

Defining $D := \tilde{D}^c \cap B_R$, the Maxwell equation (2.2) in $\tilde{D}^c$ together with the radiation condition (2.4) for $\mathbf{E}^{\rm scat}$, is equivalent to

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E} - \omega^2\varepsilon\mathbf{E} = 0 \qquad \text{in } D, \tag{3.5a}$$

$$\gamma_{\rm D}^\times\mathbf{E} = 0 \qquad \text{on } S, \tag{3.5b}$$

$$\gamma_{\rm N}(\mathbf{E} - \mathbf{E}^{\rm inc}) = \imath\omega\sqrt{\varepsilon\mu}\Lambda^{\rm pc}\gamma_{\rm D}^\times(\mathbf{E} - \mathbf{E}^{\rm inc}) \qquad \text{on } \partial B_R, \tag{3.5c}$$

for the total electric field $\mathbf{E} = \mathbf{E}^{\rm inc} + \mathbf{E}^{\rm scat}$, with $\Lambda^{\rm pc}$ as in Remark 3.1, and as before $S = \partial\tilde{D}$. We have applied the Calderón operator $\Lambda^{\rm pc}$ to $\mathbf{E}^{\rm scat} = \mathbf{E} - \mathbf{E}^{\rm inc}$, since per assumption only $\mathbf{E}^{\mathrm{scat}}$—not necessarily $\mathbf{E}^{\mathrm{inc}}$—satisfies the radiation condition (2.4).

3.1.2. *Variational formulation*

Again with $D := \tilde{D}^c \cap B_R$, there holds $\partial D = S \cup \partial B_R$. Multiplying (3.5a) with $\mathbf{V} \in \boldsymbol{H}_S(\mathbf{curl}, D)$ (*cf.* (2.6)) and integrating by parts as in (2.8), there holds

$$\int_D \mu^{-1}\,\mathbf{curl}\,\mathbf{E}\cdot\mathbf{curl}\,\mathbf{V} - \omega^2 \int_D \varepsilon \mathbf{E}\cdot\mathbf{V}$$
$$= -\int_S \gamma_{\mathrm{D}}^{\times}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E})\cdot\gamma_{\mathrm{D}}\mathbf{V} - \int_{\partial B_R} \gamma_{\mathrm{D}}^{\times}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E})\cdot\gamma_{\mathrm{D}}\mathbf{V},$$

where the normal vectors used to define the trace operators on S and ∂B_R are always considered pointing outwards of D. Using (3.5c) we derive

$$\gamma_{\mathrm{D}}^{\times}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}) = \mu^{-1}\left[\imath\omega\sqrt{\varepsilon\mu}\Lambda^{\mathrm{pc}}\gamma_{\mathrm{D}}^{\times}(\mathbf{E}-\mathbf{E}^{\mathrm{inc}}) + \gamma_{\mathrm{D}}^{\times}(\mathbf{curl}\,\mathbf{E}^{\mathrm{inc}})\right] \quad \text{on } \partial B_R.$$

Since $\gamma_{\mathrm{D}}\mathbf{V}|_S = 0$, and thus $\gamma_{\mathrm{D}}^{\times}\mathbf{V}|_S = 0$, we obtain the following *variational formulation of problem (PC)* (3.5) (see, e.g., Eq. (10.2) in Ref. 29): find $\mathbf{E} \in \boldsymbol{H}_S(\mathbf{curl}, D)$ such that

$$\mathsf{a}^{\mathrm{pc}}(\mathbf{E},\mathbf{V}) = \mathsf{f}^{\mathrm{pc}}(\mathbf{V}) \qquad \forall\, \mathbf{V} \in \boldsymbol{H}_S(\mathbf{curl}, D), \tag{3.6}$$

where the bilinear and linear forms are defined by

$$\mathsf{a}^{\mathrm{pc}}(\mathbf{E},\mathbf{V}) := (\mu^{-1}\,\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{V})_D - \omega^2(\varepsilon\mathbf{E},\mathbf{V})_D + \imath\omega\left\langle \sqrt{\tfrac{\varepsilon}{\mu}}\Lambda^{\mathrm{pc}}\gamma_{\mathrm{D}}^{\times}\mathbf{E}, \gamma_{\mathrm{D}}\mathbf{V}\right\rangle_{\partial B_R}, \tag{3.7}$$

$$\mathsf{f}^{\mathrm{pc}}(\mathbf{V}) := -\left\langle \mu^{-1}\gamma_{\mathrm{N}}\mathbf{E}^{\mathrm{inc}}, \gamma_{\mathrm{D}}\mathbf{V}\right\rangle_{\partial B_R} + \imath\omega\left\langle \sqrt{\tfrac{\varepsilon}{\mu}}\Lambda^{\mathrm{pc}}\gamma_{\mathrm{D}}^{\times}\mathbf{E}^{\mathrm{inc}}, \gamma_{\mathrm{D}}\mathbf{V}\right\rangle_{\partial B_R}, \tag{3.8}$$

with $\langle\cdot,\cdot\rangle_{\partial B_R}$ denoting, in both definitions, the dual pairing in $\boldsymbol{H}_{\mathrm{div}}^{-1/2}(\partial B_R)$ without conjugating the second argument.

3.1.3. *Well-posedness*

Problem (3.6) is well posed, in the sense that the operator $A^{\mathrm{pc}} : \boldsymbol{H}_S(\mathbf{curl}, D) \to \boldsymbol{H}_S(\mathbf{curl}, D)'$ induced by a^{pc} is an isomorphism. According to Lemma 10.3 in Ref. 29, it holds $\boldsymbol{H}_S(\mathbf{curl}, D) = X_1 \oplus X_2$ with the subspaces:

$$X_1 := \{\nabla V \,:\, V \in H^1(D),\ V|_S = 0\}, \tag{3.9}$$

$$X_2 := \left\{\mathbf{E} \in \boldsymbol{H}_S(\mathbf{curl}, D) \,:\, -\omega^2(\varepsilon\mathbf{E},\mathbf{V}) + \imath\omega\left\langle \sqrt{\tfrac{\varepsilon}{\mu}}\Lambda^{\mathrm{pc}}\gamma_{\mathrm{D}}^{\times}\mathbf{E}, \gamma_{\mathrm{D}}\mathbf{V}\right\rangle = 0,\ \forall\, \mathbf{V} \in X_1\right\}, \tag{3.10}$$

of $\boldsymbol{H}_S(\mathbf{curl}, D)$. It is easily verified that $\mathsf{a}^{\mathrm{pc}}(\mathbf{E},\mathbf{V}) = 0$ for all $\mathbf{E} \in X_2$, $\mathbf{V} \in X_1$. With $A := A^{\mathrm{pc}}$ and the notation from Sec. 2.4, Theorem 10.2 in Ref. 29 states that $A_1 : X_1 \to X_1'$ is an isomorphism, and by Thm. 10.6 in Ref. 29, $A_2 : X_2 \to X_2'$ is an isomorphism. Hence, $A^{\mathrm{pc}} = A$ is an isomorphism by Lemma 2.1.

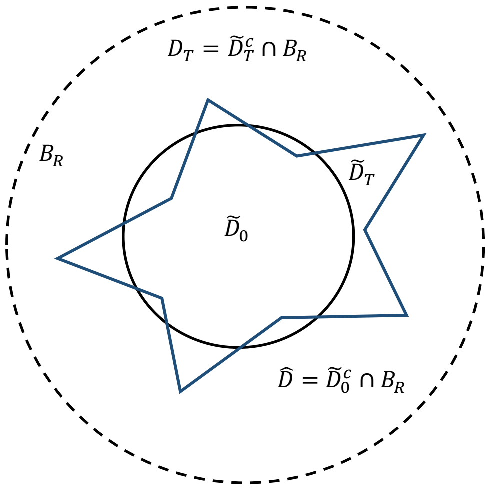

Fig. 1. Nominal domain $\hat{D}$ and physical domain D_T for problem (PC): bi-Lipschitzian transformation T allows for corners and edges in D_T with smooth nominal $\hat{D}$.

Remark 3.2. The variational formulation (3.6) is posed on the bounded domain $\tilde{D}^c \cap B_R$, using the DtN operator Λ^{pc}. The unique solution $\mathbf{E}^{\mathrm{scat}}$ admits a unique extension to B_R^c belonging to the weighted space $\boldsymbol{H}_\kappa(\mathbf{curl}, B_R^c)$ (*cf.* Thm. 2.6.5 and Lemma 2.6.5 in Ref. 30), and which satisfies in $\tilde{D}^c$ the Silver-Müller radiation condition (2.4). This avoids the use of non-Hilbertian spaces of locally integrable functions on $\tilde{D}^c$. The separability and reflexivity of the Hilbert spaces $\boldsymbol{H}_\kappa(\mathbf{curl}, \tilde{D}^c)$ introduced in Section 2.1 facilitates the use of Bochner integrals in the mathematical description of random input data and in the formulation of polynomial chaos approximations from Refs. 13 and 16.

3.1.4. *Transformed problem*

We work under the following assumption:

Assumption 3.1. Assumption 2.1 is satisfied for some $\mathfrak{T} = \mathfrak{T}^{\mathrm{pc}} \subseteq W^{1,\infty}(\hat{D}, \mathbb{R}^3)$. Furthermore, there exists $R > 0$ such that $B_{R/2}^c \cap B_R \subseteq \hat{D} \subseteq B_R$ and $T|_{B_R \cap B_{R/2}^c} = \mathsf{I}$ for all $T \in \mathfrak{T}^{\mathrm{pc}}$.

For $T \in \mathfrak{T}^{\mathrm{pc}}$, we recall $D_T = T(\hat{D})$. This signifies that the scatterer occupies the domain $\tilde{D}_T := D_T^c \cap B_R$, whereas D_T constitutes the computational domain surrounding the scatterer, restricted to the artificial ball B_R as shown in Figure 1. As an immediate consequence of the unique solvability of problem (3.6) in $D := D_T$, we obtain well-posedness of the transformed problem on the nominal domain $\hat{D}$, for every $T \in \mathfrak{T}^{\mathrm{pc}}$.

Lemma 3.1. *For every $T \in \mathfrak{T}^{\mathrm{pc}}$, problem (PC)* (3.6) *with data ε, $\mu \in L^\infty(D_T)$,* *such that* ε, μ *are constant on* B_r^c *for some* $r < R$ *with* R *as in Assumption 3.1, admits a unique solution* $\mathbf{E}_T \in \boldsymbol{H}_S(\mathbf{curl}, D_T)$ *iff there exists a unique* $\hat{\mathbf{E}}_T \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ *such that*

$$\hat{\mathsf{a}}_T^{\mathrm{pc}}(\hat{\mathbf{E}}_T, \hat{\mathbf{V}}) = \hat{\mathsf{f}}^{\mathrm{pc}}(\hat{\mathbf{V}}) \ \text{for all} \ \hat{\mathbf{V}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D}), \tag{3.11}$$

where $\mu_T := \mu \circ T$ *and* $\varepsilon_T := \varepsilon \circ T$,

$$\begin{aligned}\hat{\mathsf{a}}_T^{\mathrm{pc}}(\hat{\mathbf{E}}, \hat{\mathbf{V}}) :=& \left(\frac{1}{\det dT}\mu_T^{-1} dT\,\mathbf{curl}\,\hat{\mathbf{E}}, dT\,\mathbf{curl}\,\hat{\mathbf{V}}\right)_{\hat{D}} \\ &- \omega^2\left((\det dT)\varepsilon_T dT^{-\top}\hat{\mathbf{E}}, dT^{-\top}\hat{\mathbf{V}}\right)_{\hat{D}} + \imath\omega\left\langle\sqrt{\tfrac{\varepsilon_T}{\mu_T}}\Lambda^{\mathrm{pc}}\gamma_{\mathrm{D}}^{\times}\hat{\mathbf{E}}, \gamma_{\mathrm{D}}\hat{\mathbf{V}}\right\rangle_{\partial B_R}, \\ \hat{\mathsf{f}}_T^{\mathrm{pc}}(\hat{\mathbf{V}}) :=& \left\langle -\mu_T^{-1}\gamma_{\mathrm{D}}^{\times}\,\mathbf{curl}\,\mathbf{E}^{\mathrm{inc}} + \imath\omega\sqrt{\tfrac{\varepsilon_T}{\mu_T}}\Lambda^{\mathrm{pc}}\gamma_{\mathrm{D}}\mathbf{E}^{\mathrm{inc}}, \gamma_{\mathrm{D}}\hat{\mathbf{V}}\right\rangle_{\partial B_R}.\end{aligned}$$

In this case, $\hat{\mathbf{E}}_T = \hat{\mathbf{E}}_T$, *with* $\hat{\mathbf{E}}_T$ *related to* $\mathbf{E}_T$ *as in* (2.12). *Furthermore, if* A_T^{pc} *denotes the operator associated to the bilinear form in* (3.6) *on* D_T, *then* $\hat{A}_T^{\mathrm{pc}}$ *is an isomorphism iff* A_T^{pc} *is.*

Proof. Let $T \in \mathfrak{T}^{\mathrm{pc}}$ and let $\mathbf{E}$, $\mathbf{V} \in \boldsymbol{H}_S(\mathbf{curl}, D_T)$ be fixed. Then, the $\boldsymbol{L}^2$-inner products:

$$(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{V})_{D_T} - \omega^2(\varepsilon\mathbf{E}, \mathbf{V})_{D_T} \tag{3.12}$$

are well defined and may be transformed into integrals over $\hat{D}$ as follows: by (2.12) in Lemma 2.2, $\det(dT)(\mathbf{curl}\,\mathbf{E}) \circ T = dT^{-\top}\,\mathbf{curl}\,\hat{\mathbf{E}} \in \boldsymbol{L}^2(\hat{D})$, and $\hat{\mathbf{E}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ is well defined. Based on (2.14) and (2.15), (3.12) is equal to

$$\left(\frac{1}{\det dT}\mu_T^{-1} dT\,\mathbf{curl}\,\hat{\mathbf{E}}, dT\,\mathbf{curl}\,\hat{\mathbf{V}}\right)_{\hat{D}} - \omega^2\left((\det dT)\varepsilon_T dT^{-\top}\hat{\mathbf{E}}, dT^{-\top}\hat{\mathbf{V}}\right)_{\hat{D}}.$$

Due to Assumption 3.1, the dual products over ∂B_R are not affected by the transformation, and are well defined since ε, μ are constant on B_R^c. Therefore, $\hat{\mathsf{a}}_T^{\mathrm{pc}}(\hat{\mathbf{E}}_T, \hat{\mathbf{V}}) = \hat{\mathsf{f}}_T^{\mathrm{pc}}(\hat{\mathbf{V}})$. Note that $\partial D = S \cup \partial B_R$, with $S \cap \partial B_r = \emptyset$. By Lemma 2.2 and Remark 2.2, $\boldsymbol{H}_S(\mathbf{curl}, D_T) \ni \mathbf{V} \mapsto \hat{\mathbf{V}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ is an isomorphism, and since $\mathbf{V} \in \boldsymbol{H}_S(\mathbf{curl}, D_T)$ is arbitrary, we obtain $\hat{\mathsf{a}}_T^{\mathrm{pc}}(\hat{\mathbf{E}}_T, \hat{\mathbf{W}}) = \hat{\mathsf{f}}_T^{\mathrm{pc}}(\hat{\mathbf{W}})$ for all $\hat{\mathbf{W}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$. By the same arguments, one shows the converse direction, which concludes the first part of the proof.

The last statement is an immediate consequence of Lemma 2.2, Remark 2.2 and $\mathsf{a}_T^{\mathrm{pc}}(\mathbf{U}, \mathbf{V}) = \hat{\mathsf{a}}_T^{\mathrm{pc}}(\hat{\mathbf{U}}, \hat{\mathbf{V}})$ for all $\mathbf{U}$, $\mathbf{V} \in \boldsymbol{H}_S(\mathbf{curl}, D_T)$. □

We refer to (3.11) as problem $\widehat{(\mathrm{PC})}$ in the following.

3.2. *Scattering by a dielectric obstacle*

For this case, we follow primarily Ref. 25 wherein shape and material derivatives for the scattering problem of time-harmonic, EM waves were considered.

We assume given a dielectric medium occupying a bounded obstacle $\tilde{D}$, with electric conductivity $\sigma_1 \geq 0$, electric permittivity $\varepsilon_1 > 0$ and magnetic permeability

$\mu_1 > 0$. Its complement, $\tilde{D}^c := \mathbb{R}^3 \backslash \overline{\tilde{D}}$, is assumed to be occupied by an isotropic, homogeneous medium with constant parameters ε_0, $\mu_0 > 0$ and $\sigma_0 = 0$. In summary,

$$\varepsilon = \begin{cases} \varepsilon_1 > 0 & \text{in } \tilde{D}, \\ \varepsilon_0 > 0 & \text{in } \tilde{D}^c, \end{cases} \qquad \mu = \begin{cases} \mu_1 > 0 & \text{in } \tilde{D}, \\ \mu_0 > 0 & \text{in } \tilde{D}^c, \end{cases} \qquad \text{and} \quad \sigma = \begin{cases} \sigma_1 \geq 0 & \text{in } \tilde{D}, \\ 0 & \text{in } \tilde{D}^c, \end{cases} \tag{3.13}$$

with corresponding wavenumbers denoted by $\kappa_i^2 := \omega^2 \mu_i \varepsilon_i + \imath \omega \mu_i \sigma_i$, $i = 0, 1$.

For a given incident field $\mathbf{E}^{\text{inc}} \in \boldsymbol{H}_{\text{loc}}(\mathbf{curl}, \tilde{D}^c)$ such that $\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^{\text{inc}} - \kappa_0^2 \mathbf{E}^{\text{inc}} = \mathbf{0}$, we seek again time-harmonic fields $\mathbf{E}^i$, for $i \in \{0, 1\}$, with common circular frequency ω satisfying (2.2) such that

$$\mathbf{E}^1 \in \boldsymbol{H}(\mathbf{curl}, \tilde{D}) \,, \qquad \mathbf{E}^0 \in \boldsymbol{H}_{\kappa_0}(\mathbf{curl}, \tilde{D}^c) \,, \tag{3.14}$$

$$\mathbf{curl}\, \mu_1^{-1}\, \mathbf{curl}\, \mathbf{E}^1 - \mu_1^{-1} \kappa_1^2 \mathbf{E}^1 = \mathbf{0} \text{ in } \tilde{D} \,, \qquad \mathbf{curl}\, \mu_0^{-1}\, \mathbf{curl}\, \mathbf{E}^0 - \mu_0^{-1} \kappa_0^2 \mathbf{E}^0 = \mathbf{0} \text{ in } \tilde{D}^c \,. \tag{3.15}$$

Observe that $\mathbf{E}^1$ in $\tilde{D}$ is the total field while $\mathbf{E}^0$ is the exterior scattered field, so that the total exterior field is $\mathbf{E}^{\text{inc}} + \mathbf{E}^0$ in $\tilde{D}^c$. The problem (3.14)–(3.15) is completed with *transmission conditions*:

$$[\gamma_{\text{D}} \mathbf{E}]_{\pm} = 0 \,, \quad [\gamma_{\text{D}} \boldsymbol{H}]_{\pm} = 0 \qquad \text{on} \quad S := \partial \tilde{D}, \tag{3.16}$$

and by the radiation condition (2.4). Here, $[\cdot]_{\pm}$ denotes subtraction of corresponding traces taken from $\tilde{D}$ and $\tilde{D}^c$ at S, respectively. Equation (3.16) states, that the tangential components of the electric field and magnetic flux densities should be continuous across S. The normal components of $\mathbf{E}$ and $\boldsymbol{H}$ satisfy

$$[\gamma\,(\varepsilon \mathbf{n} \cdot \mathbf{E})]_{\pm} = 0 \,, \quad [\gamma\,(\mu \mathbf{n} \cdot \boldsymbol{H})]_{\pm} = 0 \qquad \text{on} \quad S \,.$$

3.2.1. *Variational formulation*

The variational formulation of (3.14), (3.15) and (3.16) is standard (*cf.* Chap. 10 in Ref. 29, Sec. 2 in Ref. 25, and Chap. 5 in Ref. 30). Fix $R > 0$ such that $\tilde{D} \subseteq B_{R/2}$. We denote by $\Lambda^{\text{de}} : \boldsymbol{H}_{\text{div}}^{-1/2}(\partial B_R) \to \boldsymbol{H}_{\text{div}}^{-1/2}(\partial B_R)$ the dielectric Calderón operator, which coincides with Λ^{pc} introduced in Sec. 3.1.

Remark 3.3. Remark 3.1 also applies for the dielectric problem, with $\tilde{\mathbf{H}} := \sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{H}$. That is, if $(\mathbf{E}, \mathbf{H})$ is a solution pair of the dielectric Maxwell problem (2.1) for data (3.13), then $\mathbf{E}$, $\tilde{\mathbf{H}}$ solve the same problem for $\omega_r := \omega\sqrt{\varepsilon_0 \mu_0}$,

$$\sigma_r(x) := \begin{cases} \sigma_1 & x \in \tilde{D}, \\ 0 & x \in \tilde{D}^c, \end{cases} \quad \mu_r(x) := \begin{cases} \frac{\mu_1}{\mu_0} & x \in \tilde{D}, \\ 1 & x \in \tilde{D}^c, \end{cases} \quad \text{and} \quad \varepsilon_r(x) := \begin{cases} \frac{\varepsilon_1 + \imath \sigma_1/\omega}{\varepsilon_0} & x \in \tilde{D}, \\ 1 & x \in \tilde{D}^c. \end{cases} \tag{3.17}$$

The fact that $\varepsilon_r = \mu_r = 1$ and $\sigma_r = 0$ in $\tilde{D}^c$, allows the use of the Calderón operator from [29] on ∂B_R, and we obtain as in Remark 3.1 that $\Lambda^{\text{de}} \gamma_{\text{D}} \mathbf{E} = \gamma_{\text{D}} \tilde{\mathbf{H}} = \frac{1}{\imath\omega\sqrt{\varepsilon_0\mu_0}} \gamma_{\text{N}} \mathbf{E}$.

Denoting by $\mathbf{E}$ the electric field in $(\tilde{D} \cup \tilde{D}^c) \cap B_R$, we introduce on $\boldsymbol{H}(\mathbf{curl}, B_R) \times \boldsymbol{H}(\mathbf{curl}, B_R)$ the bilinear form:

$$\mathsf{a}^{\mathrm{de}}(\mathbf{E}, \mathbf{V}) = \left(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{V}\right)_{B_R} - \omega^2 (\varepsilon \mathbf{E}, \mathbf{V})_{B_R} + \imath\omega \left\langle \sqrt{\tfrac{\varepsilon_0}{\mu_0}} \Lambda^{\mathrm{de}} \gamma_{\mathrm{D}}^{\times} \mathbf{E}, \gamma_{\mathrm{D}} \mathbf{V} \right\rangle_{\partial B_R}, \tag{3.18}$$

and the functional:

$$\mathsf{f}^{\mathrm{de}}(\mathbf{V}) := \left\langle -\mu_0^{-1} \gamma_{\mathrm{N}} \mathbf{E}^{\mathrm{inc}} + \imath\omega \sqrt{\tfrac{\varepsilon_0}{\mu_0}} \Lambda^{\mathrm{de}} \gamma_{\mathrm{D}}^{\times} \mathbf{E}^{\mathrm{inc}}, \gamma_{\mathrm{D}} \mathbf{V} \right\rangle_{\partial B_R}. \tag{3.19}$$

Again, we point out that $\mathsf{a}^{\mathrm{de}}(\cdot, \cdot)$ is linear and not sesquilinear in both arguments, and that $\mathsf{f}^{\mathrm{de}}(\cdot)$ is also linear and not antilinear in its argument. The *variational formulation of problem (DE)* reads: find $\mathbf{E} \in \boldsymbol{H}(\mathbf{curl}, B_R)$ such that

$$\mathsf{a}^{\mathrm{de}}(\mathbf{E}, \mathbf{V}) = \mathsf{f}^{\mathrm{de}}(\mathbf{V}), \quad \forall\, \mathbf{V} \in \boldsymbol{H}(\mathbf{curl}, B_R). \tag{3.20}$$

3.2.2. *Well-posedness*

Existence and uniqueness of solutions for the exterior problem (3.20) is established, for example, in the Refs. 7 and 30. As in Section 3.1.3, and cited works therein, we observe that the operator $A^{\mathrm{de}} : \boldsymbol{H}(\mathbf{curl}, B_R) \to \boldsymbol{H}(\mathbf{curl}, B_R)'$, induced by a^{de}, is an isomorphism.

3.2.3. *Transformed problem*

For problem (DE), the computational domain B_R will be independent of the transformation, which in this case serves the purpose of determining the interface.

Assumption 3.2. Assumption 2.1 is satisfied for some $\mathfrak{T} = \mathfrak{T}^{\mathrm{de}} \subseteq W^{1,\infty}(\hat{D}, \mathbb{R}^3)$ with $\hat{D} := B_R$ and $R > 0$ fixed. For every $T \in \mathfrak{T}^{\mathrm{de}}$, $D_T = B_R$ and $T|_{B_R \cap B_{R/2}^c} = \mathsf{I}$.

For $\tilde{D} \subseteq B_{R/2}$ fixed, we set $\tilde{D}_T := T(\tilde{D})$, which will represent the transformed problem geometry with interface $\partial \tilde{D}_T$. It represents one of two different homogenous isotropic materials, the second one occupying its complement. This will be irrelevant in the next lemma, which is formulated for more general data than just piecewise constants in $\tilde{D}_T$, $\tilde{D}_T^c$, but we shall return to this viewpoint when we prove shape holomorphy. By the same reasoning as in Section 3.1.4, we obtain the well-posedness of the transformed problem on the nominal domain. We point out that in the following result $D_T = \hat{D} = B_R$.

Lemma 3.2. *For every $T \in \mathfrak{T}^{\mathrm{de}}$, problem (DE)* (3.20)*, with data ε, $\mu \in L^\infty(B_R)$ admits a unique solution $\mathbf{E}_T \in \boldsymbol{H}(\mathbf{curl}, D_T)$ iff there exists a unique $\hat{\mathbf{E}}_T \in \boldsymbol{H}(\mathbf{curl}, \hat{D})$ such that*

$$\hat{\mathsf{a}}_T^{\mathrm{de}}(\hat{\mathbf{E}}_T, \hat{\mathbf{V}}) = \hat{\mathsf{f}}_T^{\mathrm{de}}(\hat{\mathbf{V}}) \ \text{ for all } \ \hat{\mathbf{V}} \in \boldsymbol{H}(\mathbf{curl}, \hat{D}), \tag{3.21}$$

where the bilinear form $\hat{\mathsf{a}}_T^{\mathrm{de}}$ and the linear functional $\hat{\mathsf{f}}_T^{\mathrm{de}}$ are as in (3.18) *and* (3.19)*, respectively, with material parameters:*

$$\varepsilon_T = \varepsilon \circ T \in L^\infty(\hat{D}), \qquad \mu_T = \mu \circ T \in L^\infty(\hat{D}), \tag{3.22}$$

and domain of integration $\hat{D}$. In this case, $\hat{\mathbf{E}}_T = \hat{\mathbf{E}}_T$ and $\hat{\mathbf{E}}_T$ is related to $\mathbf{E}_T$ by (2.11)*. Furthermore, if $A_T^{\rm de}$ denotes the operator associated with the bilinear form* (3.20)*, then $\hat{A}_T^{\rm de}$ is an isomorphism iff $A_T^{\rm de}$ is.*

We refer to (3.21) as problem $\widehat{\text{(DE)}}$ in the following.

4. Shape Holomorphy

We are now ready to prove the main results of the present paper: the shape holomorphy of the scattering problems (PC) and (DE). Specifically, we show that the pulled back solutions of the Maxwell model problems in Sections 3.1 and 3.2 depend holomorphically on domain transformations T. They represent the scatterer as the image of a *nominal domain*, denoted throughout as $\hat{D}$, under T. We emphasize that, in fact, shape holomorphy will not require T to be smooth. Indeed, we shall admit bi-Lipschitz transformations in our analysis so that, in particular, $D_T = T(\hat{D})$ can be a polyhedron even in the case that $\hat{D}$ has a smooth boundary $\partial\hat{D}$.

Our general strategy to verify shape holomorphy is as follows: we first extend the variational formulations and domain mappings of the Maxwell model problems (*cf.* Sections 3.1 and 3.2) holomorphically to complex-valued transformations. We then verify, by previously known existence and uniqueness results as well as a perturbation argument, that the corresponding bilinear forms from Sections 3.1 and 3.2 remain continuous and boundedly invertible for such complex extensions of the transformations T. Thus, the model problems remain well-posed in the nominal domain within this framework.

Finally, the proof of shape holomorphy is accomplished. Complex differentiability of the solutions on the domain mappings T will be obtained by a difference quotient argument reminiscent to the differentials which appear in real-variable shape calculus. This amounts, in fact, to the calculation of what is referred to in the shape optimization literature as *material or domain derivatives* in a neighbourhood of T, and to the investigation of its isomorphism properties.

The outline of the rest of this Section is as follows. In Section 4.1, we extend the domain mappings T as well as several expressions appearing in the transformation formulas (2.14), (2.15) holomorphically to the complex domain. Lemma 4.1 will be instrumental for the verification of shape holomorphy of problems (PC) and (DE), in Sections 4.2 and 4.3, respectively. We start the analysis with auxiliary results on the holomorphy of expressions emerging from the transformed bilinear and linear forms of pullback formulations.

4.1. *Holomorphic extensions*

We denote by $D \subset \mathbb{R}^3$ a bounded Lipschitz domain. In the holomorphic extensions of problems $\widehat{\text{(PC)}}$ and $\widehat{\text{(DE)}}$, the following results will be used for $D = D_T, \hat{D}$, respectively.

Lemma 4.1. *The following two maps are holomorphic:*

$$F_1 : W^{1,\infty}(D, \mathbb{C}^3) \to L^\infty(D, \mathbb{C}^{3\times 3}) : T \mapsto dT \;, \tag{4.1}$$

$$F_2 : W^{1,\infty}(D, \mathbb{C}^3) \to L^\infty(D, \mathbb{C}) : T \mapsto \det dT \;, \tag{4.2}$$

with Fréchet derivatives along the direction $H \in W^{1,\infty}(D, \mathbb{C}^3)$ *at* T*:* $dF_1(T)(H) = dH$ *and* $dF_2(T)(H) = \mathrm{tr}(dT^{\mathrm{adj}} dH)$*, where* dT^{adj} *denotes the adjugate of* dT*, i.e. the transpose of the cofactor matrix. If additionally,* $T^{-1} \in W^{1,\infty}(T(D), D)$*, then also*

$$F_3 : W^{1,\infty}(D, \mathbb{C}^3) \to L^\infty(D, \mathbb{C}^{3\times 3}) : T \mapsto dT^{-1}, \tag{4.3}$$

is holomorphic at T *with differential* $dF_3(T)(H) = -dT^{-1} dH dT^{-1}$.

Proof. The map F_1 is linear and bounded, so the statement is trivial. For F_2, the Fréchet derivative of the determinant map $\mathbb{C}^{3\times 3} \ni A \mapsto \det A \in \mathbb{C}$ is given by Jacobi's formula, and reads for each $B \in \mathbb{C}^{3\times 3}$, $d\det(A)(B) = \mathrm{tr}(A^{\mathrm{adj}} B)$. Since $\det : \mathbb{C}^{3\times 3} \to \mathbb{C}$ is holomorphic, and in particular in $C^2(\mathbb{C}^{3\times 3}, \mathbb{C})$, for all A, $B \in \mathbb{C}^{3\times 3}$ we get $\det(A+B) = \det A + \mathrm{tr}(A^{\mathrm{adj}} B) + d^2(\det A)(\zeta B)(\zeta B)$ for some $\zeta \in [0,1]$. For $\|H\|_{W^{1,\infty}(D,\mathbb{C}^3)} \leq 1$, we thus obtain

$$\begin{aligned} &\| \det dT - \det d(T+H) - \mathrm{tr}(dT^{\mathrm{adj}} dH) \|_{L^\infty(D)} \\ &\quad \leq \sup_{\{A \in \mathbb{C}^{3\times 3} : \|A\| \leq \|dT(\mathbf{x})\|_{L^\infty(D)} + 1\}} \|d^2 \det A\|_{L^\infty(D)} \|H\|^2_{W^{1,\infty}(D)} = \mathcal{O}\left(\|H\|^2_{W^{1,\infty}(D,\mathbb{C}^3)} \right) \end{aligned}$$

as $\|H\|_{W^{1,\infty}(D,\mathbb{C}^3)} \to 0$, which shows holomorphy of F_2. The derivative of $\mathbb{C}^{3\times 3} \ni A \mapsto A^{-1} \in \mathbb{C}^{3\times 3}$ at A along the direction of B is $-A^{-1} B A^{-1}$, as can be shown by a Neumann series. Similarly as above, we obtain the statement about F_3. □

We now verify shape holomorphy for problems $\widehat{(\mathrm{PC})}$ and $\widehat{(\mathrm{DE})}$.

4.2. *Perfect conductor (PC)*

We continue to work under Assumption 3.1. Also, we assume that all materials are homogeneous and isotropic in $D_T \cup B_R^c$, i.e. outside of the perfect conductor, for all $T \in \mathfrak{T}^{\mathrm{pc}}$, with ε, μ, σ fixed as in (3.1).

We extend the set $\mathfrak{T}^{\mathrm{pc}}$ of admissible domain transformations T to the complex domain, and consider the following *set of admissible, complex-valued domain transformations*: for $\mathfrak{T}^{\mathrm{pc}}$ as in Assumption 3.1 and for $\delta > 0$, it is given by

$$\mathfrak{T}^{\mathrm{pc}}_\delta := \{T \in W^{1,\infty}(\hat{D}, \mathbb{C}^3) \;:\; \exists \tilde{T} \in \mathfrak{T}^{\mathrm{pc}} \;:\; \|T - \tilde{T}\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)} \leq \delta\} \;. \tag{4.4}$$

4.2.1. *Shape Holomorphy*

In order to prove holomorphy of the *domain-to-solution* map

$$W^{1,\infty}(\hat{D}, \mathbb{C}^3) \ni T \mapsto \hat{\mathbf{E}}_T \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D}), \tag{4.5}$$

we verify Fréchet differentiability of this map locally, around a certain T. This has already been done for the dielectric problem in Ref. 25, Thm. 3.3, however only in the real valued setting at $T = \mathsf{I} \in C^1(\hat{D}, \mathbb{R}^3)$. Since in the complex-valued case the formal computations are analogous, we only sketch the technical arguments. We remark at this point that the condition $T|_{B_R \cap B^c_{R/2}} = \mathsf{I}$ in Assumption 3.1 implies in particular that the complex differential of the domain-to-solution map in the ensuing Theorem 4.1 will not depend on the far-field integral over ∂B_R.

Theorem 4.1. *Under Assumption 3.1, there exists $\delta > 0$ such that the domain-to-solution map of problem* $\widehat{(\mathrm{PC})}$

$$\mathcal{S}^{\mathrm{pc}} : \mathfrak{T}^{\mathrm{pc}} \to \boldsymbol{H}_S(\mathbf{curl}, \hat{D}) : T \mapsto \hat{\mathbf{E}}_T$$

admits a bounded holomorphic extension onto $\mathfrak{T}^{\mathrm{pc}}_\delta$, also denoted by $\mathcal{S}^{\mathrm{pc}}$. Furthermore, the Fréchet derivative $d\mathcal{S}^{\mathrm{pc}}(T)(H)$ of $\mathcal{S}^{\mathrm{pc}}$ in the space $W^{1,\infty}(\hat{D}, \mathbb{C}^3)$ at $T \in \mathfrak{T}^{\mathrm{pc}}$ in the direction $H \in W^{1,\infty}(\hat{D}, \mathbb{C}^3)$ is given by the unique solution $\hat{\mathbf{W}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ of the problem:

$$\hat{\mathsf{a}}^{\mathrm{pc}}_T(\hat{\mathbf{W}}, \hat{\mathbf{V}}) = \hat{\mathsf{k}}^{\mathrm{pc}}_{T,H}(\hat{\mathbf{E}}_T; \hat{\mathbf{V}}) \quad \text{for all } \hat{\mathbf{V}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D}) \,. \tag{4.6}$$

Here, for $T \in \mathfrak{T}^{\mathrm{pc}}$, $H \in W^{1,\infty}(\hat{D}, \mathbb{C}^3)$, and for all $\hat{\mathbf{V}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$, we have defined

$$\begin{aligned}
&\hat{\mathsf{k}}^{\mathrm{pc}}_{T;H}(\hat{\mathbf{E}}_T, \hat{\mathbf{V}}) := \\
&\left(\left[\frac{\mathrm{tr}(dT^{\mathrm{adj}} dH)}{(\det dT)^2} dT^\top \mu^{-1} dT - \frac{1}{\det dT} (dH^\top \mu^{-1} dT + dT^\top \mu^{-1} dH) \right] \mathbf{curl}\, \hat{\mathbf{E}}_T, \mathbf{curl}\, \hat{\mathbf{V}} \right)_{\hat{D}} \\
&+ \omega^2 \left(\left[\mathrm{tr}(dT^{\mathrm{adj}} dH) dT^{-1} \varepsilon dT^{-\top} - \det dT \big(dT^{-1} dH dT^{-1} \varepsilon dT^{-\top} + dT^{-1} \varepsilon dT^{-\top} dH^\top dT^{-\top} \big) \right] \hat{\mathbf{E}}_T, \hat{\mathbf{V}} \right)_{\hat{D}} .
\end{aligned}$$

Proof. We now consider $\hat{\mathbf{W}} : W^{1,\infty}(\hat{D}, \mathbb{C}^3) \to \boldsymbol{H}_S(\mathbf{curl}, \hat{D}) : H \mapsto \hat{\mathbf{W}}(H)$, the mapping taking H to the solution of (4.6). Since $\hat{\mathsf{f}}^{\mathrm{pc}}_T$ is independent of T (3.8), by (3.6) we have for $H \in W^{1,\infty}(\hat{D}, \mathbb{C}^3)$ and for every $\hat{\mathbf{V}} \in \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$:

$$\begin{aligned}
&\hat{\mathsf{a}}^{\mathrm{pc}}_T(\hat{\mathbf{E}}_{T+H} - \hat{\mathbf{E}}_T - \hat{\mathbf{W}}(H), \hat{\mathbf{V}}) = \hat{\mathsf{a}}^{\mathrm{pc}}_T(\hat{\mathbf{E}}_{T+H}) - \hat{\mathsf{a}}^{\mathrm{pc}}_{T+H}(\hat{\mathbf{E}}_{T+H}, \hat{\mathbf{V}}) - \hat{\mathsf{a}}^{\mathrm{pc}}_T(\hat{\mathbf{W}}(H), \hat{\mathbf{V}}) \\
&= - \left(\left[\frac{1}{\det d(T+H)} d(T+H)^\top \mu^{-1} d(T+H) - \frac{1}{\det dT} dT^\top \mu^{-1} dT \right] \mathbf{curl}\, \hat{\mathbf{E}}_{T+H}, \mathbf{curl}\, \hat{\mathbf{V}} \right)_{\hat{D}} \\
&\quad + \omega^2 \left(\left[\det d(T+H) d(T+H)^{-1} \varepsilon d(T+H)^{-\top} - \det dT dT^{-1} \varepsilon dT^{-\top} \right] \hat{\mathbf{E}}_{T+H}, \hat{\mathbf{V}} \right) - \hat{\mathsf{k}}^{\mathrm{pc}}_{T;H}(\hat{\mathbf{V}}) \,.
\end{aligned} \tag{4.7}$$

By Lemma 4.1

$$\begin{aligned}
&\frac{1}{\det d(T+H)} d(T+H)^\top \mu^{-1} d(T+H) \\
&= \left(\frac{1}{\det dT} - \frac{\operatorname{tr}(dT^{\mathrm{adj}} dH)}{(\det dT)^2} + R_1(H) \right) (dT + dH)^\top \mu^{-1} (dT + dH) \\
&= \frac{1}{\det dT} dT^\top \mu^{-1} dT \\
&\quad + \left(- \frac{\operatorname{tr}(dT^{\mathrm{adj}} dH)}{(\det dT)^2} dT^\top \mu^{-1} dT + \frac{1}{\det dT} (dH^\top \mu^{-1} dT + dT^\top \mu^{-1} dH) \right) + R_2(H),
\end{aligned} \tag{4.8}$$

where $\|R_1(H)\|_{L^\infty(\hat{D},\mathbb{C})} = o\left(\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)}\right)$ and $\|R_2(H)\|_{L^\infty(\hat{D},\mathbb{C}^{3\times 3})} = o\left(\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)}\right)$ as $\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)} \to 0$. Similarly,

$$\begin{aligned}
&\det \left(d(T+H) d(T+H)^{-1} \varepsilon d(T+H)^{-\top} \right) \\
&\qquad = \det dT dT^{-1} \varepsilon dT^{-\top} + \Big(\operatorname{tr}(dT^{\mathrm{adj}} dH) dT^{-1} \varepsilon dT^{-\top} \\
&\qquad\quad - \det dT \left(dT^{-1} dH dT^{-1} \varepsilon dT^{-\top} + dT^{-1} \varepsilon dT^{-\top} dH^\top dT^{-\top} \right) \Big) + R_3(H),
\end{aligned} \tag{4.9}$$

with $\|R_3(H)\|_{L^\infty(\hat{D},\mathbb{C}^{3\times 3})} = o\left(\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)}\right)$ as $\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)} \to 0$.

Consider now the set $\mathcal{L}_{\mathrm{iso}}(\boldsymbol{H}_S(\mathbf{curl}, \hat{D}), \boldsymbol{H}_S(\mathbf{curl}, \hat{D})')$ of boundedly invertible linear operators. This set is open and the map associating its elements with their inverses is continuous. According to Section 3.1.3, A_T^{pc} is as a mapping from $\boldsymbol{H}_S(\mathbf{curl}, D_T)$ to $\boldsymbol{H}_S(\mathbf{curl}, D_T)'$, such that it holds

$$\mathcal{A}_T^{\mathrm{pc}} \in \mathcal{L}_{\mathrm{iso}} \left(\boldsymbol{H}_S(\mathbf{curl}, D_T), \boldsymbol{H}_S(\mathbf{curl}, D_T)' \right).$$

By Lemma 3.1, we have

$$\hat{A}_T^{\mathrm{pc}} \in \mathcal{L}_{\mathrm{iso}} \left(\boldsymbol{H}_S(\mathbf{curl}, \hat{D}), \boldsymbol{H}_S(\mathbf{curl}, \hat{D})' \right).$$

The set of boundedly invertible linear operators $\mathcal{L}_{\mathrm{iso}}$ being open in the set of bounded linear operators $\mathcal{L}$, the continuous dependence of $\hat{A}_T^{\mathrm{pc}} \in \mathcal{L}_{\mathrm{iso}}$ on $T \in W^{1,\infty}(\hat{D}, \mathbb{C}^3)$ implies $\hat{A}_{T+H}^{\mathrm{pc}} \in \mathcal{L}_{\mathrm{iso}}$ provided that $\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)}$ is sufficiently small. Thus, $\hat{\mathbf{E}}_{T+H}$ is well defined, and furthermore, $\hat{\mathbf{E}}_{T+H} \to \hat{\mathbf{E}}_T$ in $\boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ as $\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)} \to 0$. The definition of $\hat{\mathbf{W}}(H)$, along with (4.7), (4.8) and (4.9) show that

$$\hat{\mathsf{a}}_T^{\mathrm{pc}}(\hat{\mathbf{E}}_{T+H} - \hat{\mathbf{E}}_T - \hat{\mathbf{W}}(H), \hat{\mathbf{V}}) = \hat{\mathsf{k}}_{T;H}^{\mathrm{pc}}(\hat{\mathbf{E}}_{T+H} - \hat{\mathbf{E}}_T, \hat{\mathbf{V}}) + R_4(H) \tag{4.10}$$

where $|R_4(H)| = o\left(\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)}\right)$. Note that

$$|\hat{\mathsf{k}}_T^{\mathrm{pc}}(\hat{\mathbf{E}}, \hat{\mathbf{V}})| \le C(T) \|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)} \|\hat{\mathbf{E}}\|_{\boldsymbol{H}_S(\mathbf{curl},\hat{D})} \|\hat{\mathbf{V}}\|_{\boldsymbol{H}_S(\mathbf{curl},\hat{D})}.$$

Therefore,

$$\|\hat{\mathsf{a}}_T^{\mathrm{pc}}(\hat{\mathbf{E}}_{T+H}-\hat{\mathbf{E}}_T-\hat{\mathbf{W}}(H),\cdot)\|_{\boldsymbol{H}_S(\mathbf{curl},\hat{D})'} = o\left(\|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)}\right) \quad \text{as} \quad \|H\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)} \to 0 \,. \tag{4.11}$$

Furthermore, the fact that $\hat{A}_T^{\mathrm{pc}}$ is an isomorphism entails the existence of $C = C(T) > 0$ such that

$$\|\hat{A}_T^{\mathrm{pc}}\mathbf{E}\|_{\boldsymbol{H}_S(\mathbf{curl},\hat{D})'} \geq C\|\mathbf{E}\|_{\boldsymbol{H}_S(\mathbf{curl},\hat{D})} \quad \forall\, \mathbf{E} \in \boldsymbol{H}_S(\mathbf{curl},\hat{D}) \,. \tag{4.12}$$

Together with (4.11), this proves $\hat{\mathbf{W}} = d\mathcal{S}^{\mathrm{pc}}(T)$ at every $T \in \mathfrak{T}^{\mathrm{pc}}$. We remark that the differential is *uniformly stable* with respect to T: there holds $C_-(\mathfrak{T}_\delta^{\mathrm{pc}}) := \inf_{T\in\mathfrak{T}_\delta^{\mathrm{pc}}} C(T) > 0$ as a consequence of the continuous dependence of $\hat{\mathsf{a}}_T^{\mathrm{pc}}$ on T and of the compactness of $\mathfrak{T}_\delta^{\mathrm{pc}} \subset W^{1,\infty}(\hat{D},\mathbb{C}^3)$ by Assumption 2.1. Finally, uniform boundedness of $\|\hat{\mathbf{E}}_T\|_{\boldsymbol{H}_S(\mathbf{curl},\hat{D})}$ for $T \in \mathfrak{T}_\delta$ can be ensured by choosing $\delta > 0$ small enough, since $\mathfrak{T}$ is compact and $\mathfrak{T}_\delta \ni T \mapsto \hat{\mathbf{E}}_T \in \boldsymbol{H}_S(\mathbf{curl},\hat{D})$ is continuous. □

In the preceding proof, the difference quotient argument to verify that the complex differential with respect to the transformation is regular, does not involve the DtN term. Based on Remark 3.2, this result implies in particular shape holomorphy of the exterior problem in the weighted spaces $\boldsymbol{H}_\kappa(\mathbf{curl}, D^c)$. We refer also to Remark 5.1 ahead.

4.3. *Dielectric scatterer (DE)*

Here, we consider sets $\mathfrak{T}^{\mathrm{de}} \subset W^{1,\infty}(B_R;\mathbb{R}^3)$ of holomorphic, bi-Lipschitz transformations as in Assumption 3.2. We again set $D_T := T(\hat{D})$. In order to give data determining the material, we assume $D \subseteq B_{R/2}$ to be fixed throughout what follows, with $R > 0$ as in Assumption 3.2 and set $D_T := T(\hat{D})$. We fix ε, μ, σ as in (3.13) for the domain D and define

$$\check{\sigma}_T := \sigma \circ T^{-1}, \qquad \check{\mu}_T := \mu \circ T^{-1}, \qquad \check{\varepsilon}_T := \varepsilon \circ T^{-1} \,, \tag{4.13}$$

which is piecewise constant data with respect to the transformed problem geometry D_T. Then, under Assumption 3.2, $D_T \subset B_{R/2}$, and in particular, the dielectric interface $\partial D_T \subset B_{R/2}$, for $T \in \mathfrak{T}^{\mathrm{de}}$. As before, the condition $T|_{B_R\cap B^c_{R/2}} = \mathsf{I}$ in Assumption 3.2 implies that the complex domain differential in Theorem 4.2 ahead will not involve the far-field integral over ∂B_R appearing in the bilinear form $\mathsf{a}^{\mathrm{de}}(\cdot,\cdot)$ in (3.20).

We set

$$\mathfrak{T}_\delta^{\mathrm{de}} := \{T \in W^{1,\infty}(\hat{D},\mathbb{C}^3) \,:\, \exists\, \tilde{T} \in \mathfrak{T}^{\mathrm{de}} \,:\, \|T-\tilde{T}\|_{W^{1,\infty}(\hat{D},\mathbb{C}^3)} \leq \delta\} \,. \tag{4.14}$$

4.3.1. *Shape Holomorphy*

In the following, let $\mathbf{E}_T$ be the solution of (3.20), for data (4.13), and let $\hat{\mathbf{E}}_T$ be related to $\mathbf{E}_T$ as in (2.11). Similar arguments as in Section 4.2.1 lead to:

Theorem 4.2. *Under Assumption 3.2, there exists $\delta > 0$ such that the domain-to-solution map of problem $\widehat{(\mathrm{DE})}$ and data (4.13), i.e. the map*

$$\mathcal{S}^{\mathrm{de}} : \mathfrak{T}^{\mathrm{de}} \to \boldsymbol{H}(\mathbf{curl}, B_R) : T \mapsto \hat{\mathbf{E}}_T,$$

admits a uniformly bounded holomorphic extension onto $\mathfrak{T}^{\mathrm{pc}}_\delta$, also denoted by $\mathcal{S}^{\mathrm{de}}$. Furthermore, the Fréchet derivative $d\mathcal{S}^{\mathrm{de}}(T)(H)$ of $\mathcal{S}^{\mathrm{de}}$ in the space $W^{1,\infty}(\hat{D}, \mathbb{C}^3)$ at $T \in \mathfrak{T}^{\mathrm{pc}}$ along the direction $H \in W^{1,\infty}(\hat{D}, \mathbb{C}^3)$ is given by the unique solution $\hat{\mathbf{W}} \in \boldsymbol{H}(\mathbf{curl}, \hat{D})$ of the following problem:

$$\hat{\mathsf{a}}^{\mathrm{de}}_T(\hat{\mathbf{W}}, \hat{\mathbf{V}}) = \hat{\mathsf{k}}^{\mathrm{de}}_{T;H}(\hat{\mathbf{E}}_T; \hat{\mathbf{V}}) \quad \text{for all } \hat{\mathbf{V}} \in \boldsymbol{H}(\mathbf{curl}, B_R)\,, \tag{4.15}$$

where

$$\hat{\mathsf{k}}^{\mathrm{de}}_{T;H}(\hat{\mathbf{E}}_T, \hat{\mathbf{V}}) :=$$
$$\left(\left[\frac{\mathrm{tr}(dT^{\mathrm{adj}} dH)}{(\det dT)^2} dT^\top \mu^{-1} dT - \frac{1}{\det dT}(dH^\top \mu^{-1} dT + dT^\top \mu^{-1} dH)\right] \mathbf{curl}\, \hat{\mathbf{E}}_T, \mathbf{curl}\, \hat{\mathbf{V}}\right)_{\hat{D}}$$
$$+ \omega^2 \left(\left[\mathrm{tr}(dT^{\mathrm{adj}} dH) dT^{-1} \varepsilon dT^{-\top} - \det dT \left(dT^{-1} dH dT^{-1} \varepsilon dT^{-\top} + dT^{-1} \varepsilon dT^{-\top} dH^\top dT^{-\top}\right)\right] \hat{\mathbf{E}}_T, \hat{\mathbf{V}}\right)_{\hat{D}}.$$

5. Parametric Holomorphy

In the previous section, we proved an abstract holomorphy result of *domain-to-solution* maps, $\mathcal{S}^{\mathrm{pc}}$ and $\mathcal{S}^{\mathrm{de}}$, where the admissible domain parameterizations ranged in a subset of $W^{1,\infty}(\hat{D}; \mathbb{R}^3)$. In computational applications, in particular from UQ, but also in Reduced Basis and MOR, error bounds are required for approximations of integrals over all shapes or of interpolants and other "surrogates" in the space of all shapes. To this end, *parametric representations of transformations T* are chosen from a suitable set of representations. We mention as examples B-splines, NURBS, Fourier and wavelet representations.[19,34,15,18] We also refer to Sec. 5 in Ref. 14 where the ensuing discussion is carried out for the Stokes and Navier-Stokes equation.

In general, transformations can be parametrized, for example, in an *affine-parametric* manner, i.e. in terms of a sequence $\boldsymbol{y} = (y_j)_{j\geq 1} \subset \mathbb{U} := [-1,1]^{\mathbb{N}}$ of the form:

$$T_{\boldsymbol{y}} = \mathsf{I} + \sum_{j\geq 1} y_j T_j\,, \tag{5.1}$$

where the sequence $(T_j)_{j\geq 1}$ is assumed to be an unconditional basis of a subspace $S(\hat{D})$ of $W^{1,\infty}(\hat{D}; \mathbb{R}^3)$, endowed with the norm $\|\cdot\|_{W^{1,\infty}(\hat{D})}$. For the sake of simplicity and clarity, we shall discuss the case of affine-parametric families of domain mappings (5.1) in more detail, but we point out that considerably more general parametrizations are possible with our approach, with precise requirements specified in Theorem 5.1 below.

Let us now assume

$$\exists\, \tau < 1 : \qquad \left\| \sum_{j\geq 1} |dT_j| \right\|_{L^\infty(\hat{D};\mathbb{R}^3)} \leq \tau\,, \tag{5.2}$$

and sparsity:

$$\exists p \in (0,1): \qquad \boldsymbol{b} \in \ell^p(\mathbb{N}), \text{ where } b_j := \|T_j\|_{W^{1,\infty}(\hat{D})}. \tag{5.3}$$

For parametric transformations $T_{\boldsymbol{y}}$, we denote the corresponding parametric Maxwell solutions by $\mathbf{E}_{\boldsymbol{y}} = \mathbf{E}_{T_{\boldsymbol{y}}}$ for the two problems considered. For $\mathbf{E}_{\boldsymbol{y}}$, the shape holomorphy of solutions will imply, via composition of holomorphic functions, *parametric holomorphy*. We next recall the definition from Ref. 16[b]. Throughout what follows, the parameter set $\mathbb{U}$ will be equipped with the product topology.

Definition 5.1. For a positive sequence $\boldsymbol{b} = (b_j)_{j\geq 1} \in \ell^1(\mathbb{N})$, a parametric mapping $\mathbb{U} \to X : \boldsymbol{y} \mapsto u_{\boldsymbol{y}}$ satisfies the $(\boldsymbol{b}, \epsilon)$-*holomorphy assumption* in the complex Banach space X if and only if for some $\epsilon \in (0,1)$ there exists a constant $C_\epsilon < \infty$ such that for any sequence $\boldsymbol{\rho} := (\rho_j)_{j\geq 1}$ of semi-axis sums $\rho_j > 1$ that is $(\boldsymbol{b}, \epsilon)$-*admissible*, i.e.

$$\sum_{j=1}^{\infty} (\rho_j - 1) b_j \leq \epsilon, \tag{5.4}$$

the parametric map $\boldsymbol{y} \mapsto u_{\boldsymbol{y}} \in X$ admits a continuous extension to complex parameters $\boldsymbol{z} \mapsto u_{\boldsymbol{z}}$ that is a holomorphic mapping with respect to each variable z_j in a cylindrical set of the form $\mathcal{O}_{\boldsymbol{\rho}} := \bigotimes_{j\geq 1} \mathcal{O}_{\rho_j}$. Herein, $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open set containing $\mathcal{E}_{\rho_j}$, the Bernstein ellipse in $\mathbb{C}$ with foci ± 1 and semiaxis sum $\rho_j > 1$, and the modulus $\|u_{\boldsymbol{z}}\|_X$ of this extension is bounded on the cylinder $\mathcal{E}_{\boldsymbol{\rho}} := \mathcal{E}_{\rho_1} \times \mathcal{E}_{\rho_2} \times \ldots \subset \mathbb{C}^{\mathbb{N}}$ according to

$$\sup_{\boldsymbol{z} \in \mathcal{E}_{\boldsymbol{\rho}}} \|u_{\boldsymbol{z}}\|_X \leq C_\epsilon. \tag{5.5}$$

The importance of the above definition lies in the fact that $(\boldsymbol{b}, \epsilon)$-holomorphic maps with $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}} \in \ell^p$ for some $p \in (0,1)$, are shown to allow sparse polynomial approximations on $\mathbb{U}$, with best n-term convergence rates $1/p - 1$ and $1/p - 1/2$ corresponding to the norms $\|\cdot\|_{L^\infty(\mathbb{U})}$, $\|\cdot\|_{L^2(\mathbb{U})}$ respectively. Furthermore, these convergence rates can be achieved, among others, by Smolyak type interpolation algorithms[16] (also see Cor. 5.1 ahead).

Assumptions (5.2) and (5.3) imply $(\boldsymbol{b}, \epsilon)$-holomorphy of the parametric family $\mathbb{U} \ni \boldsymbol{y} \mapsto T_{\boldsymbol{y}} \in W^{1,\infty}(\hat{D})$.

Proposition 5.1. *Under assumption* (5.2) *and* (5.3)*, the affine-parametric transformations* (5.1) *are continuous* $\mathbb{U} \ni \boldsymbol{y} \mapsto T_{\boldsymbol{y}} \in W^{1,\infty}(\hat{D})$*, bijective and* $(\boldsymbol{b}, \epsilon)$*-holomorphic, with* $T_{\boldsymbol{y}}^{-1} \in W^{1,\infty}(T_{\boldsymbol{y}}(\hat{D}))$ *for every* $\boldsymbol{y} \in \mathbb{U}$.

Proof. The continuity of $\mathbb{U} \ni \boldsymbol{y} \mapsto T_{\boldsymbol{y}} \in W^{1,\infty}(\hat{D})$ follows from (5.3): the inclusion $\ell^p(\mathbb{N}) \subset \ell^1(\mathbb{N})$ implies uniform unconditional convergence of the series (5.1), in the

[b] In fact, we use a slight adjustment of Def. 2.1 in Ref. 16: we state it here only for the case of X being a complex Banach space; and, more importantly, we additionally assume the extensions to be *continuous*. The latter is necessary, in order for certain results in Ref. 16 on the convergence of collocation approximations to be valid.

norm of $W^{1,\infty}$, with respect to $\boldsymbol{y} \in \mathbb{U}$. It also implies uniform–with respect to $\boldsymbol{z} \in \mathcal{O}_{\boldsymbol{\rho}}$–unconditional convergence of the series (5.1), with $z_j \in \mathbb{C}$ in place of y_j, in the norm of $S(\hat{D}) \subset W^{1,\infty}(\hat{D};\mathbb{C}^3)$.

The smallness condition (5.2) implies moreover that, for every $\mathbf{x}, \mathbf{x}' \in \hat{D}$ with $\operatorname{conv}(\mathbf{x}, \mathbf{x}') \subset \hat{D}$ where $\operatorname{conv}(\mathbf{x}, \mathbf{x}')$ denotes the line segment connecting $\mathbf{x}$ and $\mathbf{x}'$, and for every $\boldsymbol{y} \in \mathbb{U}$, it holds

$$(1-\tau)\|\mathbf{x}-\mathbf{x}'\|_2 \leq \|T_{\boldsymbol{y}}(\mathbf{x}) - T_{\boldsymbol{y}}(\mathbf{x}')\|_2 \leq (1+\tau)\|\mathbf{x}-\mathbf{x}'\|_2 \,. \tag{5.6}$$

This implies bijectivity of $T_{\boldsymbol{y}}$, and $\det(dT_{\boldsymbol{y}})(\mathbf{x}) \in [1-\tau, 1+\tau]$ for $\mathbf{x} \in \hat{D}$ and for every $\boldsymbol{y}$, e.g., Lemma 7.5 in Ref. 41. Furthermore, as a consequence of (5.6), we have $T_{\boldsymbol{y}}^{-1} \in W^{1,\infty}(T(\hat{D}), \hat{D})$, which concludes the proof. □

To state the next Lemma, we introduce the set

$$\mathcal{O}_\epsilon := \bigcup_{\{\boldsymbol{\rho}\,:\,\boldsymbol{\rho} \text{ is } (\boldsymbol{b},\epsilon)\text{-admissible}\}} \mathcal{O}_{\boldsymbol{\rho}} \subseteq \mathbb{C}^{\mathbb{N}}, \tag{5.7}$$

where $\mathbb{C}^{\mathbb{N}}$ is equipped with the product topology, and $\mathcal{O}_\epsilon \subseteq \mathbb{C}^{\mathbb{N}}$ with the subspace topology. Then, with X denoting a Banach space, there holds

Lemma 5.1. *Let $\mathbb{U} \ni \boldsymbol{y} \mapsto u_{\boldsymbol{y}} \in X$ be $(\boldsymbol{b},\epsilon)$-holomorphic in the sense of Definition 5.1. Then $\mathbb{U} \ni \boldsymbol{y} \mapsto u_{\boldsymbol{y}} \in X$ admits a unique extension onto $\mathcal{O}_\epsilon$, with the property that the map is continuous and holomorphic in each y_j. Furthermore*

$$\limsup_{\tilde{\epsilon}\to 0} \sup_{\boldsymbol{z}\in\mathcal{O}_{\tilde{\epsilon}}} \inf_{\boldsymbol{y}\in\mathbb{U}} \|u_{\boldsymbol{y}} - u_{\boldsymbol{z}}\|_X = 0. \tag{5.8}$$

Proof. We first show the existence of the extension. Let $(z_1, z_2, \dots) = \boldsymbol{z} \in \mathcal{O}_{\boldsymbol{\rho}_1} \cap \mathcal{O}_{\boldsymbol{\rho}_2}$, with $\boldsymbol{\rho}_j$ being $(\boldsymbol{b},\epsilon)$-admissible, $j \in \{1,2\}$. Denote the respective extensions onto $\mathcal{O}_{\boldsymbol{\rho}_j}$ by f_j, and set $\boldsymbol{z}_k := (z_1, \dots, z_k, 0, 0, \dots)$. Then, $f_j(\boldsymbol{z}_k) \to f_j(\boldsymbol{z})$ for $k \to \infty$ since the maps are continuous. Next, f_j is separately and therefore jointly holomorphic as a function of the first k arguments by Hartogs' theorem (*cf.* Thm. 2.2.8 in Ref. 26). Furthermore $f_1|_{[-1,1]^k} = f_2|_{[-1,1]^k}$, and thus f_1, f_2 coincide with the unique analytic extension of u in its first k arguments and we obtain $f_1(\boldsymbol{z}_k) = f_2(\boldsymbol{z}_k)$ for all $k \in \mathbb{N}$. Thus $f_1(\boldsymbol{z}) = f_2(\boldsymbol{z})$, and the extension to $\mathcal{O}_\epsilon$ is well-defined.

Fix $\varepsilon > 0$. In order to show (5.8), we note that the continuity of the extension of $u_{\boldsymbol{z}}$ on $\mathcal{O}_\epsilon$ implies that for any neighbourhood N_X of $\{u_{\boldsymbol{y}} : \boldsymbol{y} \in \mathbb{U}\} \subseteq X$, there exists a neighbourhood $N_{\mathbb{U}} \subseteq \mathcal{O}_\epsilon$ of the compact set $\mathbb{U}$ such that $\{u_{\boldsymbol{z}} : \boldsymbol{z} \in N_{\mathbb{U}}\} \subseteq N_X$. Setting $N_X := \{v \in X : \exists \boldsymbol{y} \in \mathbb{U} \text{ s.t. } \|v - u_{\boldsymbol{y}}\|_X \leq \delta\}$ for $\delta > 0$ arbitrary, it is sufficient to prove that $O_{\tilde{\epsilon}} \subseteq N_{\mathbb{U}}$ for all $\tilde{\epsilon} \leq \epsilon_0$, with $\epsilon_0 \leq \epsilon$ small enough.

Assume at first that there exists j with $b_j = 0$. Then, setting $\rho_j := \infty$, $u_{\boldsymbol{y}}$ as a function of y_j admits per assumption a bounded holomorphic extension onto $\mathbb{C}$. Thus, $u_{\boldsymbol{y}}$ is constant in and therefore independent of y_j. Hence, we may assume, without loss of generality, $b_j > 0$ for all $j \in \mathbb{N}$. Since $N_{\mathbb{U}}$ is a neighbourhood of $\mathbb{U}$ with respect to the product topology, we can find open sets $N_j \subseteq \mathbb{C}$ containing $[-1,1]$

s.t. $N_j = \mathbb{C}$ for all $j \geq J_0$ and $\bigotimes_j N_j \subseteq N_{\mathbb{U}}$. Since $b_j > 0$ for all j, and any $\boldsymbol{z} \in \mathcal{O}_{\tilde{\epsilon}}$ is contained in some $\mathcal{O}_{\boldsymbol{\rho}}$ with $\sum_j(\rho_j - 1)b_j \leq \tilde{\epsilon}$, we have $\rho_j - 1 \leq \tilde{\epsilon}/b_j$ and therefore (*cf.* Lemma 4.4 in Ref. 16) $\mathrm{dist}(z_j, [-1, 1]) \leq \rho_j - 1 \leq \tilde{\epsilon}/b_j$ for all $j = 1, \ldots, J_0$. Thus $\mathcal{O}_{\boldsymbol{\rho}} \subseteq \bigotimes_j N_j$ and consequently $\mathcal{O}_{\tilde{\epsilon}} \subseteq \bigotimes_j N_j \subseteq N_{\mathbb{U}}$ for $\tilde{\epsilon}$ sufficiently small. □

Next, we show that the property of the transformation family being $(\boldsymbol{b}, \epsilon)$-holomorphic is inherited by the parametric solution maps for problems (PC) and (DE). In Proposition 5.1, we have seen that $T_{\boldsymbol{y}}$ as in (5.1), satisfying (5.2) and (5.3) is *an example* of a $(\boldsymbol{b}, \epsilon)$-holomorphic parametrization, however.

Theorem 5.1. *Let $\boldsymbol{y} \mapsto T_{\boldsymbol{y}}$ be $(\boldsymbol{b}, \epsilon)$-holomorphic such that, with $\mathfrak{T} := \{T_{\boldsymbol{y}} : \boldsymbol{y} \in \mathbb{U}\}$, Assumptions 3.1 for problem (PC) and 3.2 in the case of (DE) are satisfied for some $\hat{D}$. Then, there exist $\epsilon = \epsilon(\hat{D}, \mathfrak{T}) > 0$ such that for problems (PC) and (DE), the parametric domain-to-solution maps $\boldsymbol{y} \mapsto \hat{\mathbf{E}}_{\boldsymbol{y}} = \hat{\mathbf{E}}_{T_{\boldsymbol{y}}}$ are $(\boldsymbol{b}, \epsilon)$-holomorphic.*

More precisely, for problem (PC) of a perfect conductor in Section 3.1 occupying a bounded domain $D^c_{T_{\boldsymbol{y}}} \cap B_r$ with respect to the Hilbert space $X = \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ and for problem (DE) of scattering at a dielectric interface in Section 3.2 with respect to the Hilbert space $X = \boldsymbol{H}(\mathbf{curl}, B_R)$ in (3.20).

Proof. Consider problem (PC) and note that compactness of $\mathfrak{T}^{\mathrm{pc}} = \mathfrak{T}$ already follows from $(\boldsymbol{b}, \epsilon)$-holomorphy of $\boldsymbol{y} \mapsto T_{\boldsymbol{y}}$, since it is then the image of the compact set $\mathbb{U}$ under a continuous map. Under Assumption 3.1, we showed in Theorem 4.1 the existence of a holomorphic extension of the domain-to-solution map $\mathcal{S}^{\mathrm{pc}} : \mathfrak{T}^{\mathrm{pc}} \to \boldsymbol{H}_S(\mathbf{curl}, \hat{D}) : T \mapsto \hat{\mathbf{E}}_T$ to the set $\mathfrak{T}^{\mathrm{pc}}_\delta$ of admissible maps defined in (4.4), for some $\delta > 0$.

For $\epsilon > 0$ sufficiently small, by Lemma 5.1 it holds that $T_{\boldsymbol{z}} \in \mathfrak{T}^{\mathrm{pc}}_\delta$ for $\boldsymbol{z} \in \mathcal{O}_\epsilon$ as defined in (5.7) and for the affine-parametric transformations (5.1). Thus, the composition $\mathcal{O}_\epsilon \ni \boldsymbol{z} \mapsto \hat{\mathbf{E}}_{\boldsymbol{z}} := \hat{\mathbf{E}}_{T_{\boldsymbol{z}}}$ is well defined. It is also $(\boldsymbol{b}, \epsilon)$-holomorphic: $\mathcal{O}_\epsilon \ni \boldsymbol{z} \mapsto \hat{\mathbf{E}}_{\boldsymbol{z}}$ is continuous, as composition of continuous maps. Uniform boundedness of $\mathfrak{T}^{\mathrm{pc}}_\delta \ni T \mapsto \|\hat{\mathbf{E}}_T\|_X$ by Theorem 4.1, implies $C_\epsilon := \sup_{\boldsymbol{z} \in \mathcal{O}_{\boldsymbol{\rho}}} \|\hat{\mathbf{E}}_{\boldsymbol{z}}\|_X < \infty$. Finally, $\mathcal{O}_\epsilon \ni \boldsymbol{z} \mapsto \hat{\mathbf{E}}_{\boldsymbol{z}}$ is also separately holomorphic with respect to each $z_j \in \mathcal{O}_{\rho_j} \subset \mathbb{C}$, as composition of holomorphic maps. The arguments for problem (DE) are along the same lines. □

Remark 5.1. For the variational formulation (3.6), we used a Calderón operator to be able to work on bounded domains. As already mentioned in Remark 3.2, problem (PC) on $D := B_R \cap \tilde{D}^c$ (i.e. with the scatterer $\tilde{D}$) can equivalently be stated in the weighted space $\boldsymbol{H}_\kappa(\mathbf{curl}, \tilde{D}^c)$ on the unbounded domain $\tilde{D}^c = D \cup B_R^c$. Let $T : \hat{D} \to D$ as in Assumption 3.1. Apart from problems (PC) and $\widehat{(\mathrm{PC})}$ given in (3.6), (3.11) respectively, we now additionally introduce problems $(\mathrm{PC})_\kappa$ and $\widehat{(\mathrm{PC})_\kappa}$, the former being the equivalent formulation of (PC) on $D \cup B_R^c$, and the latter its transformation to $\hat{D} \cup B_R^c$. Here we use that, thanks to Assumption 3.1, the transformation $T : \hat{D} \to D$ can naturally be extended from $\hat{D} \cup B_R^c$ to $D \cup B_R^c$, by defining it as the identity on B_R^c.

The mapping of a solution of (PC) in $\boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ to the corresponding solution of $\widehat{\text{(PC)}}$ in $\boldsymbol{H}_\kappa(\mathbf{curl}, \hat{D} \cup B_R^c)$ is bounded and linear (*cp.* Thm. 5.3.3 in Ref. 30). As a corollary, Theorem 5.1 therefore also yields shape holomorphy for problem $(\text{PC})_\kappa$, i.e. the existence of a holomorphic extension of the solution map to $\widehat{(\text{PC})_\kappa}$ formulated in the weighted space $\boldsymbol{H}_\kappa(\mathbf{curl}, \hat{D} \cup B_R^c)$ over the unbounded domain $\hat{D} \cup B_R^c$. An analogous statement is true for problem (DE).

As already indicated, the significance of our results lies in that they entail dimension independent convergence rates for polynomial approximations of the solution manifold, respectively for approximations of the expectation via quadrature. In order to state these results, we introduce some notation: Let $\boldsymbol{z} = (z_j)_{j \in \mathbb{N}_0}$ be a sequence of distinct points in $[-1, 1]$. For $\boldsymbol{\nu} = (\nu_j)_{j\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}}$ set $z_{\boldsymbol{\nu}} := (z_{\nu_j})_{j\in\mathbb{N}} \in \mathbb{U}$. We call a set $\Lambda \subseteq \{\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} : \sum_j \nu_j < \infty\}$ downward closed, if $\boldsymbol{\nu} \in \Lambda$ entails $\boldsymbol{\mu} \in \Lambda$ whenever $\mu_j \le \nu_j$ for all $j \in \mathbb{N}$. The interpolation operator mapping $\mathbf{E} \in C^0(\mathbb{U}, X)$ to the (unique) multivariate polynomial in $\text{span}\{\prod_{j\in\mathbb{N}} y_j^{\nu_j} t_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \Lambda, \ t_{\boldsymbol{\nu}} \in X\}$ coinciding with $\mathbf{E}$ at $\boldsymbol{z_\nu}$ for each $\boldsymbol{\nu} \in \Lambda$, is denoted by I_Λ.

Corollary 5.1. *Let the assumptions of Thm. 5.1 be satisfied and let X be either as in Thm. 5.1 or as in Rmk. 5.1. Assume that there exists $\theta < \infty$ such that the Lebesgue constant of $(z_1, \dots, z_n)$ is bounded by $(1+n)^\theta$ for all $n \in \mathbb{N}$. Then there exists $C < \infty$ and a sequence of nested downward closed sets $(\Lambda_N)_{N\in\mathbb{N}}$ such that, for each $N \in \mathbb{N}$, it holds $|\Lambda_N| \le N$ and*

$$\|\hat{\mathbf{E}}_{\boldsymbol{y}} - I_{\Lambda_N} \hat{\mathbf{E}}_{\boldsymbol{y}}\|_{L^\infty(\mathbb{U},X)} \le C N^{-r}, \qquad r = \frac{1}{p} - 1. \tag{5.9}$$

It is immediate that the interpolatory quadrature rules such as, for example, Smolyak type quadratures, originating from the interpolation operators I_{Λ_N} can also yield the convergence rate $r = 1/p - 1$ *provided* that the index sets Λ_N are known (see, e.g., Ref. 36), when approximating the expectation of $\hat{\mathbf{E}}_{\boldsymbol{y}}$. An alternative option are recently developed, higher-order quasi-Monte Carlo methods, which have been proven[21] to achieve the dimension-independent rate $r = 1/p$. For further details on this result and the above corollary we refer to Refs. 16 and 21.

Finally, we remark that the above deterministic statements can be employed in frameworks built upon *random* perturbations of the domain. In particular, the continuous dependence of $\hat{\mathbf{E}}_{\boldsymbol{y}}$ on $\boldsymbol{y} \in \mathbb{U}$, yields:

Corollary 5.2. *Let the assumption of Thm. 5.1 be satisfied. Let $(\Omega, \mathcal{A}, P)$ be a probability space and assume that $(X_j)_{j\in\mathbb{N}}$ is a sequence of independent uniformly distributed random variables on Ω mapping to $[-1, 1]$ equipped with the Borel σ-algebra. For $\omega \in \Omega$ denote by $\hat{\mathbf{E}}^{\text{pc}}(\omega)$ the solution to $\widehat{\text{(PC)}}$ (resp. $\widehat{(\text{PC})_\kappa}$) corresponding to the transformation $T_{\boldsymbol{y}(\omega)}$ where $\boldsymbol{y}(\omega) := (X_j(\omega))_{j\in\mathbb{N}}$. Then, for any $q \in (0, \infty]$ it holds $\hat{\mathbf{E}}^{\text{pc}} \in L^q(\Omega, X)$, where $X := \boldsymbol{H}_S(\mathbf{curl}, \hat{D})$ (resp. $X := \boldsymbol{H}_\kappa(\mathbf{curl}, \hat{D} \cup B_R^c)$) and the map is strongly measurable. An analogous statement is true for problem (DE).*

Proof. We have already established that the map $\mathbb{U} \to X : \boldsymbol{y} \mapsto \hat{\mathbf{E}}^{\mathrm{pc}}_{\boldsymbol{y}}$ is continuous on the compact set $\mathbb{U}$. Therefore $\|\hat{\mathbf{E}}^{\mathrm{pc}}(\omega)\|_X$ is uniformly bounded on Ω. It remains to verify strong measurability: again, by employing the compactness of $\mathbb{U}$, it is clear that the continuous map $\mathbb{U} \to X : \boldsymbol{y} \mapsto \hat{\mathbf{E}}^{\mathrm{pc}}_{\boldsymbol{y}}$ may be approximated uniformly by simple functions –with respect to the Borel σ-Algebra on $\mathbb{U}$. By measurability of $\Omega \to \mathbb{U} : \omega \mapsto (X_j(\omega))_{j\in\mathbb{N}}$, this implies the same for $\Omega \to X : \omega \mapsto \hat{\mathbf{E}}^{\mathrm{pc}}_{\boldsymbol{y}(\omega)} = \hat{\mathbf{E}}^{\mathrm{pc}}(\omega)$. □

6. Concluding Remarks

In the present derivations, we assumed *piecewise homogeneous material*, i.e. constitutive parameters are constant in each subdomain. This assumption allows to infer well-posedness for the physical problem by means of a Fredholm alternative and a unique continuation principle which is classical for homogeneous materials (*cf.* Ref. 29). Well-posedness of the transformed problem in the nominal domain is an immediate consequence in the case of sufficiently smooth transformations and domains. The presented analysis remains valid also for inhomogeneous materials in Lipschitz domains, defined as in (4.13) on the physical domains. The required smoothness of the inhomogeneous constitutive parameters describing the heterogeneous materials is dictated by the available unique continuation results; we refer to [28,33] and references therein.

For several time-harmonic, EM scattering problems that admit a shape derivative, we proved, for affine-parametric shape parametrization with a parameter sequence $\boldsymbol{y} = (y_j)_{j\geq 1}$, that the corresponding parametric solution families $\boldsymbol{y} \to \hat{\mathbf{E}}_{\boldsymbol{y}}$ on the nominal domain $\hat{D}$ admit a $(\boldsymbol{b}, \epsilon)$-holomorphic extension to complex parameter domains. The possibly infinite sequence $\boldsymbol{b}$ here belongs to $\ell^p(\mathbb{N})$ for some $0 < p < 1$ which summability implies, in turn, sparsity of gpc expansions of the parametric solution families. Specifically, the p-summability of the shape uncertainty parametrization implies dimension-independent polynomial chaos approximation rates of the parametric scattered fields, and a dimension-independent convergence rate $r = 1/p - 1$ of adaptive interpolation to compute parametric surrogate maps, in case $\boldsymbol{b} \in \ell^p$, $p \in (0, 1)$. Moreover, higher-order quasi Monte-Carlo quadratures for iterated high-dimensional integrals arising in Bayesian shape inversion converge at rate $r = 1/p$. We refer to Refs. 10, 9, 23, 20 and references there for further details on Bayesian shape inversion for holomorphic, parametric forward maps.

The presently established shape holomorphy implies, in particular, the convergence of so-called shape Taylor expansions which occur, for example, in shape Taylor expansion approaches for domain uncertainty analysis, for domain deformations in a neighborhood of the nominal domain. Such expansions are usually only justified *asymptotically*, i.e. as the size of the domain perturbation tends to zero (see, e.g., Refs. 11, 17). The shape holomorphy results in Thms. 4.1, 4.2 and 5.1 ensure, however, the convergence of higher-order truncations of shape Taylor expansions at perturbations of fixed size–but still sufficiently small–about the nominal domain. The holomorphic dependence of the scattered fields in the weighted spaces introduced in (2.5) in Section 2.3 will then ensure the convergence of the corresponding far-field patterns for the class of exterior EM scattering problems considered.

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