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CH-8092 Zürich
Switzerland

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# Novel Multi-Trace Boundary Integral Equations for Transmission Boundary Value Problems

Xavier Claeys, Ralf Hiptmair, Carlos Jerez-Hanckes and Simon Pintarelli

**Abstract.** We consider scalar 2nd-order transmission problems in the exterior of a bounded domain $\Omega_Z \subset \mathbb{R}^d$. The coefficients are assumed to be piecewise constant with respect to a partition of $\mathbb{R}^d \setminus \overline{\Omega}_Z$ into subdomains. Dirichlet boundary conditions are imposed on $\partial\Omega_Z$.

We recast the transmission problems into two novel well-posed *multi-trace boundary integral equations*. Their unknowns are functions on the product of subdomain boundaries. Compared to conventional single-trace formulations they offer the big benefit of being amenable to operator preconditioning. We outline the analysis of the new formulations, give the details of operator preconditioning applied to them, and, for one type of a multi-trace formulation, report numerical tests confirming the efficacy of operator preconditioning.

**Keywords.** Multi-trace boundary integral equations; boundary element methods; 1st-kind integral equations; operator preconditioning; domain decomposition.

**AMS classification.** 74J20,65N38,65N55.

## 1 Introduction

This is the story of a marriage between boundary element methods (BEM) and domain decomposition (DD). In fact, viewed from the angle of boundary element methods, this relationship may be labelled a forced marriage, because, as the reader will certainly remember, boundary element methods can only cope with linear boundary value problems with constants coefficients. Piecewise constant coefficients are still within their scope, but in this case, the computational domain has to be decomposed into *subdomains*, on which the coefficients are constant. Subsequently, boundary integral equations have to be devised for the resulting transmission problems. They feature traces on the interfaces between subdomains as unknowns. Then, for the sake of discretization, these interfaces are triangulated and the degrees of freedom of the boundary element method will be located on the union of the interfaces, the so-called skeleton. All this very much resembles what is done in domain decomposition methods, even in the context of finite elements.

The most important representatives of these skeleton based BEM approaches rely on what we have dubbed the (direct) *classical single-trace boundary integral formula-*

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*tion* (STF). It had long been known for simple transmission problems comprising two subdomains [24], and was presented in full generality in [79] in the setting of strongly elliptic second-order scalar problems. More exotic, indirect variants have also been developed, for instance in [68, 45, 69]. Second-kind STFs are known, too, and covered in [17, 13, 31, 78, 81] for scalar transmission problems, and in [58, § 23] for electromagnetics. The classical STF is particularly popular in computational electromagnetics, where it is known as Poggio-Miller-Chew-Harrington-Wu-Tsai (PMCHWT) integral equations [67, 10, 80, 33]. Their numerical analysis for two subdomains was first accomplished in [9], and later extended in [7].

As unknowns the direct STF features a full set of Cauchy data, that is, pairs of Dirichlet and Neumann traces, on each interface between adjacent subdomains. In variational form, its associated bilinear form is built from local subdomain contributions, very much in the spirit of domain decomposition. This dispenses with global interactions in the assembly of the discrete boundary integral operators. The benefit of localization may be so big that it can be worthwhile to pursue "genuine domain decomposition" by introducing extra artificial interfaces inside regions with constant coefficients.

However, the classical STF renounces the spirit of DD in the choice of unknowns, which establish the coupling between the subdomains in strong form. As a consequence, the variational STF employs a function space that fails to be a simple product space of subdomain contributions. At first glance without a penalty, since the STF turns out to be unconditionally well posed, see [79] and [16, Sect. 3.2]. A brief review of the derivation and analysis of STF will be given in Section 3 of this article.

A drawback of the strong coupling imposed through the function space has surfaced recently: it compounds the difficulties of designing preconditioners. This matters, because modern boundary element applications are inconceivable without the use of local low-rank matrix compression implemented in techniques like fast multipole methods [30, 25], $\mathcal{H}$-matrix compression [32], or adaptive cross approximation [5, Ch. 3]. Compressed matrices allow only the use of iterative solvers, whose speed of convergence will deteriorate for ill-conditioned linear systems. Yet, standard low-order boundary element Galerkin discretization of the classical STF, which amounts to a *first-kind boundary integral equation*, will invariably produce ill-conditioned linear systems on fine triangulations. Thus, effective preconditioning becomes crucial.

Many preconditioning strategies have been suggested for discretized first kind boundary integral equations. Among them are geometric multilevel subspace correction methods (two-grid or multigrid) [71, 1, 75, 36, 43, 61, 52], as well as attempts to bring algebraic multigrid to bear on BEM [60, 48, 49]. However, no idea has revolutionized preconditioning for BEM as much as an approach known as *Calderón preconditioning*, invented by O. Steinbach and co-workers in [72, 57] and later extended by A. Buffa and S. Christiansen in [12, 8]. It has had and continues to have a massive impact, for instance in BEM based simulations in computational electromagnetism, witness the flurry of papers that has been devoted to its use [77, 4, 76, 2, 3, 21]. We point out that Calderón preconditioning fits the more general policy of *operator preconditioning* introduced in [38, 55], see also [16, Sect. 4].

Awareness of the gist of operator preconditioning as presented in [38] is key to understanding why it encounters problems for the STF. Thus, we briefly recall the main result of [38].

**Theorem 1.1** (Theorem 2.1 of[38]). *Let $X, Y$ be Hilbert spaces, and $X_h := \text{span}\{\varphi_i\}_{i=0}^N \subset X$ and $Y_h := \text{span}\{\phi_j\}_{j=0}^M \subset Y$ finite-dimensional subspaces with bases $\{\varphi_i\}_{i=0}^N$ and $\{\phi_j\}_{j=0}^M$. Further, let $\mathsf{a} \in L(X \times X, \mathbb{R})$ and $\mathsf{b} \in L(Y \times Y, \mathbb{R})$ be continuous bilinear forms (with norms $\|\mathsf{a}\|$ and $\|\mathsf{b}\|$, resp.), each satisfying discrete inf-sup conditions with constants $c_A, c_B > 0$ on $X_h$ and $Y_h$, respectively. If there is a continuous bilinear form $\mathsf{d} \in L(X \times Y, \mathbb{R})$ that also satisfies a discrete inf-sup condition on $X_h \times Y_h$ with constant $c_T > 0$, then the associated Galerkin matrices:*

$$\mathbf{A}_h := (\mathsf{a}(\varphi_i, \varphi_j))_{i,j=1}^N \,,\ \mathbf{B}_h := (\mathsf{b}(\phi_i, \phi_j))_{i,j=1}^M \,,\ \mathbf{D}_h := (\mathsf{d}(\varphi_i, \phi_j))_{i,j=1}^{N,M} \,,$$

*satisfy*

$$\kappa(\mathbf{D}_h^{-1}\mathbf{B}_h\mathbf{D}_h^{-T}\mathbf{A}_h) \leq \frac{\|\mathsf{a}\|\|\mathsf{b}\|\|\mathsf{d}\|^2}{c_A c_B c_T^2} \,, \tag{1.2}$$

*where $\kappa$ designates the spectral condition number.*

This theorem clearly reveals what it takes to build a viable "operator preconditioner" $\mathbf{M}_h := \mathbf{D}_h^{-1}\mathbf{B}_h\mathbf{D}_h^{-T}$ when given a stable Galerkin discretization of a variational problem involving the bilinear form $\mathsf{a}$. We have to find

- a space $Y$, equipped with a bilinear form $\mathsf{b}$,
- a suitable Galerkin trial space $Y_h \subset Y$ that yields a stable discretization of $\mathsf{b}$,
- and a "pairing bilinear form" $\mathsf{d}$ that induces a stable discrete duality pairing between $X_h$ and $Y_h$ (independent of the choice of discretization parameters),
- and that gives rise to a square Galerkin matrix $\mathbf{D}_h$, for which linear systems can be solved with little computational effort.

In the context of boundary element methods the last item usually means that the Galerkin matrix $\mathbf{D}_h$ is *sparse*. Thus, its formal inverse in the definition of the preconditioner $\mathbf{M}_h$ can be evaluated by means of direct Gaussian elimination. In general, sparsity of $\mathbf{D}_h$ can be achieved only if $\mathsf{d}$ is local, for instance, a simple $L^2$-inner product.

Unfortunately, for situations with more than two subdomains, so far no space $Y$ has been found that is in duality with the variational space $X$ of the STF with respect to a local pairing bilinear form $\mathsf{d}$. It goes without saying that suitable boundary element spaces $Y_h$ also remain elusive. This seems to be a fundamental obstacle to the application of Calderón preconditioning to the STF, as explained in [16, Sect. 4.5].

$L^2$-dual pairs of trace spaces are well known for boundaries of individual subdomains. Thus, if a stable variational boundary integral equation can be posed on their product space, local operator preconditioning becomes a straightforward option. This insight was the main motivation behind the development of *multi-trace formulations* (MTF). Truly in the spirit of domain decomposition, they rely on simple products of local trace spaces and impose coupling between different subdomains weakly. They owe their name to the use of two pairs of unknown Cauchy traces on every interface. It is this new class of boundary integral formulations that this article is devoted to.

Multi-trace formulations come in two different flavors; we distinguish between *global* and *local* MTFs. The former can be deduced from the STF through a vanishing gap limit and they will be treated in Section 4. The latter employ local transmission conditions and they are discussed in Section 5, with numerical results reported in Section 6. We emphasize that this article is largely meant to be a review. Occasionally, rigorous proofs and technical details are skipped. Those can be found in the original publications

- [18], as concerns the global MTF for acoustic scattering,
- [14], where the global MTF for electromagnetic scattering was introduced,
- [39], which proposed the local MTF,
- [40], where the local MTF is extended to more general transmission conditions.
- [15], where a global MTF for scattering problems with homogeneous Dirichlet boundary conditions is derived with focus on avoiding spurious resonances by means of combined field integral equations (CFIE).

Parallel to the developments in numerical analysis, local multi-trace BIE have recently been devised for large scale parallel simulations in computational electromagnetism by J.-F. Lee, Z. Peng, and collaborators [64, 65, 66]. This underscores their relevance for computational engineering.

In parts, this manuscript runs parallel to the survey article [16]. What is new is the treatment of essential boundary conditions, because, apart from [15], earlier work has always been concerned with (scattering) transmission problems posed on the entire space $\mathbb{R}^d$, $d = 2, 3$. Also new are the numerical investigations of the local MTF in 3D reported in Section 6.

Admittedly, multi-trace formulations are by no means the only attempt to harness ideas from domain decomposition for boundary element methods. Prominently, this was also pursued by O. Steinbach and co-workers with the Boundary Element Tearing and Interconnecting (BETI) method [62, 47, 51, 50, 53]. This is a specimen of the class of BEM based domain decomposition methods that involve approximate realizations of Dirichlet-to-Neumann maps by means of boundary elements [53, 73]. Other approaches marrying boundary elements and domain decomposition employ Lagrangian multipliers, *cf.* [46].

A word of warning; throughout we address domain decomposition in the volume. It must not be mixed up with domain decomposition on surfaces meant to break apart the linear systems arising from BEM. Only recently various kinds of such schemes have been proposed, see [11, 37, 35, 20, 63].

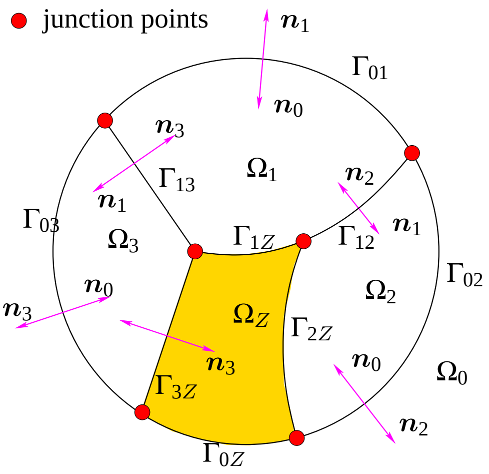
## List of notations

| | |
|---|---|
| $\Omega_j, \Omega_Z$ | Lipschitz domains "subdomains", see (2.1) |
| $\mathcal{L}_j$ | second-order diffusion operator associated with $\Omega_j$, see (2.3a), Page 6 |
| $\mathsf{T}_{D,i}, \mathsf{T}_{N,i}$ | Dirichlet and Neumann trace operators onto $\partial\Omega_i$ |
| $\langle\cdot,\cdot\rangle_{\partial\Omega}$ | $L^2$ duality pairing between $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ |
| $\mathbb{T}_i$ | Cauchy trace operator on $\partial\Omega_i$ |
| $\mathcal{T}(\partial\Omega_i)$ | Cauchy trace space on $\partial\Omega_i$, see Page 8 |
| $\langle\langle\cdot,\cdot\rangle\rangle_{\partial\Omega}$ | duality pairing in Cauchy trace space, see (3.3), Page 8 |
| $\mathbb{G}_i$ | potential associated with $\Omega_i$, see (3.4), Page 8 |
| $\mathbb{P}_i$ | Calderón projector onto $\mathbb{T}_i$, see (3.6), Page 8 |
| $\mathbb{A}_i$ | Calderón operator belonging to $\Omega_i$, see (3.7), Page 9 |
| $\boldsymbol{\mathcal{MT}}(\Sigma)$ | multi-trace space, see (3.8), Page 9 |
| $\mathbb{L}_i$ | localization operator $\boldsymbol{\mathcal{MT}}(\Sigma)\top\mathcal{T}(\partial\Omega_i)$, see (3.9), Page 9 |
| $\langle\langle\cdot,\cdot\rangle\rangle_{\Sigma}$ | self-duality pairing for $\boldsymbol{\mathcal{MT}}(\Sigma)$, see (3.10), Page 9 |
| $\boldsymbol{\mathcal{ST}}(\Sigma), \boldsymbol{\mathcal{ST}}_0(\Sigma)$ | single-trace spaces, see (3.11), Page 9, (3.13), Page 10 |
| $\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma), \widehat{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$ | clipped multi-trace spaces, see (4.3), (4.4), Page 14 |
| $\mathbb{C}_{i\to j}$ | remote coupling operators, see (4.7), Page 16 |
| $\mathcal{G}_i$ | triangular surface mesh on $\partial\Omega_i$ |
| $\mathcal{T}_h(\partial\Omega_i)$ | boundary element Cauchy trace space on $\partial\Omega_i$, see (4.9), Page 17 |
| $\mathbb{S}_{ij}$ | trace restriction operator, see (5.2), Page 19 |
| $\mathbb{X}_{i\to j}$ | trace transfer operator, see (5.2), Page 19 |
| $\widetilde{\mathcal{T}}_{\mathrm{pw}}(\partial\Omega_i)$ | piecewise Cauchy trace space, see (5.6), Page 20 |
| $\boldsymbol{\mathcal{MT}}_0(\Sigma), \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$ | multi-trace spaces with local regularity, see (5.10a), Page 21 |
| $\widetilde{\boldsymbol{\mathcal{MT}}}_0^\sharp(\Sigma)$ | multi-trace space with locally regular Neumann components, see (5.14), Page 22 |

# 2 Exterior Transmission Boundary Value Problems

As a model problem we study a scalar second-order elliptic transmission boundary value problem in the exterior $\mathbb{R}^d \setminus \overline{\Omega}_Z$ of a bounded Lipschitz domain $\Omega_Z \subset \mathbb{R}^d$, $d = 2, 3$. We restrict ourselves to spatially varying diffusion coefficients $\mu = \mu(\boldsymbol{x})$,

Figure 1: Geometric situation for the second-order scalar elliptic exterior transmission model problem. The $\boldsymbol{n}_j$'s stand for the *exterior* unit normal vector-fields on the subdomain boundaries $\partial\Omega_j$.




which are *piecewise constant* with respect to a partition of $\mathbb{R}^d \setminus \overline{\Omega}_Z$

$$\mathbb{R}^d \setminus \Omega_Z = \overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \ldots \cup \overline{\Omega}_N \;, \quad N \in \mathbb{N} \;, \tag{2.1}$$

into $N+1$ subdomains $\Omega_i$, of which only $\Omega_0$ is unbounded, see Figure 1 for an illustration. The two subdomains $\Omega_i$ and $\Omega_j$, $i \neq j$, are separated by the interface $\Gamma_{ij}$. The piece of boundary separating $\Omega_i$ and $\Omega_Z$ is denoted by $\Gamma_{iZ}$. The union of all boundaries forms the skeleton $\Sigma$: $\Sigma = \bigcup_i \partial\Omega_i$. If $\Omega_Z \neq \emptyset$ or $N > 1$, junction points will usually occur and then the skeleton may not be orientable, nor be a manifold.

Given diffusion coefficients $\mu_i > 0$, $i = 0, \ldots, N$, and Dirichlet data $g \in H^{\frac{1}{2}}(\partial\Omega_Z)$, our model transmission problem seeks $U \in H^1_{\text{loc}}(\mathbb{R}^d \setminus \overline{\Omega}_Z)$ that satisfies the Dirichlet boundary conditions

$$U = g \quad \text{on } \partial\Omega_Z \;, \tag{2.2}$$

that complies with suitable decay conditions at infinity (depending on the space dimension $d$, see [56, Ch. 8]), and whose restrictions $U_i := U|_{\Omega_i} \in H^1_{\text{loc}}(\Omega_i)$, $i = 0, \ldots, N$, fulfill

$$\mathcal{L}_i\, U_i := -\operatorname{div}(\mu_i\, \mathbf{grad}\, U_i) = 0 \quad \text{in } \Omega_i \;, \tag{2.3a}$$

$$U_i|_{\Gamma_{ij}} - U_j|_{\Gamma_{ij}} = 0 \;, \quad \mu_i \frac{\partial U_i}{\partial \boldsymbol{n}_i}\bigg|_{\Gamma_{ij}} + \mu_j \frac{\partial U_j}{\partial \boldsymbol{n}_j}\bigg|_{\Gamma_{ij}} = 0 \quad \text{on } \Gamma_{ij} \;. \tag{2.3b}$$

Equivalently, $U$ can be characterized as the solution of a variational problem posed on a Sobolev space with weighted $H^1(\mathbb{R}^d \setminus \overline{\Omega}_Z)$-norm, *cf.* [70, Sect. 2.9.2.4.] and [26, Chap. XI, Part B]. Existence and uniqueness of solutions of this variational problem can be established by standard techniques, see [70, Sect. 2.10.2.2] or [56, Ch. 8].

*Remark* 2.1 (More general boundary conditions on $\partial\Omega_Z$). We confine ourselves to Dirichlet boundary conditions on $\partial\Omega_Z$ just for the sake of simplicity. Neumann boundary conditions can be treated alike and BIE can even accomodate a mixture of both following the ideas of [70, Sect. 3.5.2]. △

*Remark* 2.2. As already remarked in the Introduction, multi-trace boundary integral equations can be derived for many more transmission problems beside the simple model problem (2.3). In fact, in [16] and [18] multi-trace boundary integral equations were first proposed and analyzed for acoustic scattering. Recall that the *acoustic transmission scattering problem* involves the local partial differential equations

$$-\operatorname{div}(\mu_i \,\mathbf{grad}\, U_i) - \kappa_i^2 U_i = 0 \quad \text{in } \mathbf{\Omega}_i \;, \tag{2.4}$$

with locally constant wave numbers $\kappa_i > 0$, and Sommerfeld radiation conditions at infinity [19, Ch. 2], [59, Ch. 2]. The transmission conditions (2.3b) and boundary conditions (2.2) apply unchanged. An extension to transmission problems for time-harmonic electromagnetic waves is pursued in [14]. Those read

$$\mathbf{U} \times \mathbf{n}_Z = \mathbf{g} \quad \text{on } \partial\Omega_Z \;, \tag{2.5}$$

$$\mathbf{curl}(\mu_i \,\mathbf{curl}\,\mathbf{U}_i) - \kappa_i^2 \mathbf{U}_i = 0 \quad \text{in } \mathbf{\Omega}_i \;, \quad i = 0, \ldots, N \;, \tag{2.6}$$

$$\left.\begin{aligned} \boldsymbol{n}_i \times (\mathbf{U}_i|_{\Gamma_{ij}} \times \boldsymbol{n}_i) - \boldsymbol{n}_j \times (\mathbf{U}_j|_{\Gamma_{ij}} \times \boldsymbol{n}_j) = 0 \;, \\ \mu_i \,\mathbf{curl}\,\mathbf{U}_i|_{\Gamma_{ij}} \times \boldsymbol{n}_i + \mu_j \,\mathbf{curl}\,\mathbf{U}_j|_{\Gamma_{ij}} \times \boldsymbol{n}_j = 0 \end{aligned}\right\} \quad \text{on } \Gamma_{ij} \;, \tag{2.7}$$

$$+ \quad \text{Silver-Müller radiation conditions at } \infty \text{ for } \mathbf{U} - \mathbf{U}_{\text{inc}}.$$

A detailed and comprehensive presentation of multi-trace BIE for the scattering transmission problems (2.4) and (2.5) is given in [16]. In all these works essential boundary conditions are not taken into account.

# 3 Single-trace Boundary Integral Equations (STF)

## 3.1 Calderón Projectors

Two trace operators are naturally associated with the second-order scalar differential operator $\mathcal{L}_i U_i := -\operatorname{div}(\mu_i \,\mathbf{grad}\, U_i)$. These are the Dirichlet trace $\mathsf{T}_{D,i}$, and Neumann trace $\mathsf{T}_{N,i}$, defined for smooth functions $V$ on $\overline{\mathbf{\Omega}}_i$ through

$$\mathsf{T}_{D,i}\, U := U|_{\partial\Omega_i} \quad , \quad \mathsf{T}_{N,i}\, U := \mu_i \,\mathbf{grad}\, U \cdot \boldsymbol{n}_i|_{\partial\Omega_i} \;. \tag{3.1}$$

They can be extended to continuous and surjective operators [70, Sect. 2.6 & 2.7] [1]

$$\mathsf{T}_{D,i} : H^1(\mathbf{\Omega}_i) \to H^{\frac{1}{2}}(\partial\mathbf{\Omega}_i) \quad , \quad \mathsf{T}_{N,i} : H(\Delta, \mathbf{\Omega}_i) \to H^{-\frac{1}{2}}(\partial\mathbf{\Omega}_i) \;. \tag{3.2}$$

[1] As usual, $H(\Delta, \Omega) := \{U \in H^1(\Omega) : \Delta U \in L^2(\Omega)\}$.

Their ranges are known as *trace spaces* and form the Gelfand triple $H^{\frac{1}{2}}(\partial\Omega_i) \subset L^2(\partial\Omega_i) \subset H^{-\frac{1}{2}}(\partial\Omega_i)$ with a duality $H^{\frac{1}{2}}(\partial\Omega_i)' \cong H^{-\frac{1}{2}}(\partial\Omega_i)$ effected by the $L^2(\partial\Omega_i)$ inner product. We write $\langle\cdot,\cdot\rangle_{\partial\Omega_i}$ for the associated duality pairing, which agrees with the $L^2(\partial\Omega_i)$-inner product for sufficiently regular functions. Trace spaces and operators may be combined into *Cauchy trace spaces* and *Cauchy trace operators*:

$$\mathcal{T}(\partial\Omega_i) := H^{\frac{1}{2}}(\partial\Omega_i) \times H^{-\frac{1}{2}}(\partial\Omega_i)\ , \quad \mathbb{T}_i : \begin{cases} H(\Delta,\Omega_i) & \to \quad \mathcal{T}(\partial\Omega_i) \\ U & \mapsto \quad (\mathsf{T}_{D,i}\, U, \mathsf{T}_{N,i}\, U)\ . \end{cases}$$

The ranges of the $\mathbb{T}_i$ are dense in $\mathcal{T}(\partial\Omega_i)$ [22, Lemma 3.5]. Obviously, the Cauchy trace spaces are in self-duality with respect to the skew-symmetric pairing [2]

$$\langle\langle \mathfrak{u}, \mathfrak{v} \rangle\rangle_{\partial\Omega_i} := \langle u, \varphi\rangle_{\partial\Omega_i} - \langle v, \nu\rangle_{\partial\Omega_i}\ , \quad \mathfrak{u} := \begin{pmatrix} u \\ \nu \end{pmatrix},\ \mathfrak{v} := \begin{pmatrix} v \\ \varphi \end{pmatrix} \in \mathcal{T}(\partial\Omega)\ . \tag{3.3}$$

Beside trace operators, potential *representation formulas* are the linchpin of the derivation of boundary integral equations. Here, the term “potential” is reserved for mappings that take functions on boundaries to functions in domains. These functions usually provide solutions of the homogeneous PDE. For our concrete operators $\mathcal{L}_i := -\operatorname{div}(\mu_i\, \mathbf{grad}\,\cdot)$ the key potential is formally given by

$$\mathbb{G}_i(\mathfrak{u})(\boldsymbol{x}) := \langle\langle \mathbb{T}_i\, \Phi(\boldsymbol{x} - \cdot), \mathfrak{u} \rangle\rangle_{\partial\Omega_i}\ , \quad \boldsymbol{x} \in \mathbb{R}^d \setminus \partial\Omega_i,\ \mathfrak{u} \in \mathcal{T}(\partial\Omega_i)\ , \tag{3.4}$$

where $\Phi$ is the fundamental solution for $\mathcal{L}_i$, see [70, Eq. (3.3)]. A comprehensive discussion of potentials can be found in [70, Sect. 3.1]. Then, *every* $U \in H_{\text{loc}}(\Delta,\Omega_i)$ that satisfies $\mathcal{L}_i\, U = 0$ and, for $i = 0$, the appropriate decay conditions at $\infty$, has the representation (in the sense of distributions)

$$U = \mathbb{G}_i(\mathbb{T}_i\, U)\ , \tag{3.5}$$

*cf.* [70, Sect. 3.11] and [56, Ch. 6]. Applying the Cauchy trace to the potential yields the *Calderón projector*, see [16, Sect. 2.3], [70, Sect. 3.6], and [44, Sect. 5.6]:

$$\mathbb{P}_i := \mathbb{T}_i\, \mathbb{G}_i : \mathcal{T}(\partial\Omega_i) \to \mathcal{T}(\partial\Omega_i)\ , \tag{3.6}$$

This operator turns out to be a projector: $\mathbb{P}_i^2 = \mathbb{P}_i$. The Calderón projectors owe their importance to the following fundamental theorem, [16, Thm. 2.6] and [70, Prop. 3.6.2(ii)].

**Theorem 3.1.** *$U_i \in H(\Delta,\Omega_i)$ solves $\mathcal{L}_i\, U_i = 0$ in $\Omega_i$ (and satisfies appropriate decay conditions at $\infty$ for $i = 0$), if and only if* $(\mathbb{P}_i - \mathsf{Id})\, \mathbb{T}_i\, U_i = 0$.

[2] Fraktur font is used to designate functions in the Cauchy trace space, whereas Roman typeface is reserved for Dirichlet traces, and Greek symbols for Neumann traces.

We remark that the Calderón projector $\mathbb{P}_i$ implicitly contains the customary four different boundary integral operators associated with 2nd-order scalar PDEs [70, Eq. (3.122)], because the *Calderón operator*

$$\mathbb{A}_i := \mathbb{P}_i - \tfrac{1}{2}\mathsf{Id} \quad \text{can be written as} \quad \mathbb{A}_i = \begin{pmatrix} -\mathsf{K}_i & \mathsf{V}_i \\ \mathsf{W}_i & \mathsf{K}'_i \end{pmatrix} , \tag{3.7}$$

where we have adopted the notations from [70, Sect. 3.1]: $\mathsf{K}_i$, $\mathsf{V}_i$, $\mathsf{W}_i$, and $\mathsf{K}'_i$ stand for the double layer, single layer, hypersingular, and adjoint double layer boundary integral operators on $\partial\Omega_i$, respectively.

## 3.2 Skeleton trace spaces

Boundary integral equations arising from the transmission problem (2.3) invariably involve unknown functions in trace spaces on the skeleton $\Sigma$. The largest and simplest such space is the (skeleton) *multi-trace space*:

$$\boldsymbol{\mathcal{MT}}(\Sigma) := \mathcal{T}(\partial\Omega_0) \times \mathcal{T}(\partial\Omega_1) \times \cdots \times \mathcal{T}(\partial\Omega_N) , \tag{3.8}$$

which comprises completely decoupled local traces. It owes its name to the fact that on each interface $\Gamma_{ij}$ a function $\underline{\mathfrak{u}} \in \boldsymbol{\mathcal{MT}}(\Sigma)$[3] comprises two pairs of Dirichlet and Neumann data, each stemming from a subdomain on either side. The simple component projections $\mathbb{L}_i : \boldsymbol{\mathcal{MT}}(\Sigma) \to \mathcal{T}(\partial\Omega_i)$ isolate the contribution of individual subdomains:

$$\mathbb{L}_i\, \underline{\mathfrak{u}} := \begin{pmatrix} u_i \\ \nu_i \end{pmatrix}, \text{for} \quad \underline{\mathfrak{u}} := \left( \begin{pmatrix} u_0 \\ \nu_0 \end{pmatrix}, \dots, \begin{pmatrix} u_N \\ \nu_N \end{pmatrix} \right) \in \boldsymbol{\mathcal{MT}}(\Sigma) . \tag{3.9}$$

Properties of its local components carry over to the multi-trace space, for instance, self-duality with respect to the symplectic $L^2$-type bilinear pairing, *cf.* (3.3),

$$\langle\!\langle \underline{\mathfrak{u}}, \underline{\mathfrak{v}} \rangle\!\rangle_\Sigma := \sum_{i=0}^{N} \langle\!\langle \mathbb{L}_i\, \underline{\mathfrak{u}}, \mathbb{L}_i\, \underline{\mathfrak{v}} \rangle\!\rangle_{\partial\Omega_i} , \quad \underline{\mathfrak{u}}, \underline{\mathfrak{v}} \in \boldsymbol{\mathcal{MT}}(\Sigma) . \tag{3.10}$$

**Corollary 3.2.** *The pairing* (3.10) *induces an isomorphism* $\boldsymbol{\mathcal{MT}}(\Sigma) \cong \boldsymbol{\mathcal{MT}}(\Sigma)'$.

Generically, multi-trace functions on different subdomains are utterly disconnected. If we impose the continuity stipulated by the transmission conditions (2.3b) we arrive at the so-called *single-trace spaces*

$$\boldsymbol{\mathcal{ST}}(\Sigma) := \left\{ \left( \begin{pmatrix} u_i \\ \nu_i \end{pmatrix} \right)_{i=0}^{N} \in \boldsymbol{\mathcal{MT}}(\Sigma) : \begin{array}{l} \exists V \in H^1(\mathbb{R}^d) : u_i = \mathsf{T}_{D,i}\, V , \\ \exists \mathbf{W} \in \boldsymbol{H}(\mathrm{div}, \mathbb{R}^d) : \nu_i = \mathsf{T}_{\boldsymbol{n},i}\, \mathbf{W} \end{array} \right\} . \tag{3.11}$$

[3] Functions in a multi-trace space will be tagged by an underline, e.g., $\underline{\mathfrak{u}}$, $\underline{\mathfrak{v}}$.

Here, $\mathsf{T}_{\boldsymbol{n},i}\,\mathbf{W}$ is the normal component trace $(\mathbf{W}\cdot\boldsymbol{n}_i)|_{\partial\Omega_i}$ on $\partial\Omega_i$ for a vector-field $\mathbf{W}$ in $\Omega_i$. For traces in $L^2(\partial\Omega_i)$, the formal definition (3.11) implies

$$\left(\begin{pmatrix}u_i\\ \nu_i\end{pmatrix}\right)_{i=0}^{N}\in\boldsymbol{\mathcal{ST}}(\Sigma)\quad\Rightarrow\quad\begin{matrix}u_i|_{\Gamma_{ij}}=u_j|_{\Gamma_{ij}}\ \text{and}\ \nu_i|_{\Gamma_{ij}}=-\,\nu_j|_{\Gamma_{ij}}\,,\\ 0\le i,j\le N\,,\end{matrix}\tag{3.12}$$

that is, the $H^{\frac{1}{2}}$-components are continuous across interfaces, and so are the $H^{-\frac{1}{2}}$-components, the latter up to a change of sign. The sign change reflects the opposite relative orientation of an interface with respect to the two adjacent subdomains, see Figure 1. The definition (3.11) may be modified slightly in order to impose zero Dirichlet boundary conditions on $\partial\Omega_Z$.

$$\boldsymbol{\mathcal{ST}}_0(\Sigma):=\left\{\begin{pmatrix}u_i\\ \nu_i\end{pmatrix}_{i=0}^{N}:\quad\begin{matrix}\exists U\in H^1_{\partial\Omega_Z}(\mathbb{R}^d\setminus\Omega_Z):\ u_i=\mathsf{T}_{D,i}\,U\,,\\ \exists\mathbf{V}\in\boldsymbol{H}(\mathrm{div},\mathbb{R}^d):\ \nu_i=\mathsf{T}_{\boldsymbol{n},i}\,\mathbf{V}\end{matrix}\right\}.\tag{3.13}$$

Here, we wrote $H^1_{\partial\Omega_Z}(\mathbb{R}^d\setminus\Omega_Z)$ for the space of functions in $H^1(\mathbb{R}^d\setminus\Omega_Z)$ that vanish on $\partial\Omega_Z$.

Obviously, the restriction of single traces to $\partial\Omega_Z$ is well defined by taking the point trace $\mathsf{T}_{D,Z}\,V$ and normal component trace $\mathsf{T}_{\boldsymbol{n},Z}\,\mathbf{W}$ onto $\partial\Omega_Z$, respectively, of their extensions $V$ and $\mathbf{W}$ according to (3.11). For this restriction operation we introduce the localization onto $\partial\Omega_Z$

$$\mathbb{L}_Z=\begin{pmatrix}\mathsf{L}_{D,Z}\\ \mathsf{L}_{N,Z}\end{pmatrix}:\boldsymbol{\mathcal{ST}}(\Sigma)\to\mathcal{T}(\partial\Omega_Z):=H^{\frac{1}{2}}(\partial\Omega_Z)\times H^{-\frac{1}{2}}(\partial\Omega_Z)\,.\tag{3.14}$$

Thanks to (3.12) the pairing of single trace functions leads to massive cancellations, because for sufficiently smooth multi-trace functions we can rewrite (3.10) as a sum of contributions of interfaces:

$$\langle\!\langle\underline{\mathfrak{u}},\underline{\mathfrak{v}}\rangle\!\rangle_\Sigma=\sum_{\substack{i,j=0\\ i<j}}^{N}\int_{\Gamma_{ij}}u_i\psi_i-v_i\nu_i+u_j\psi_j-v_j\nu_j\,\mathrm{d}S+\sum_{i=0}^{N}\int_{\Gamma_{iZ}}u_i\psi_i-v_i\nu_i\,\mathrm{d}S\,,\tag{3.15}$$

where $\underline{\mathfrak{u}}=\left(\binom{u_0}{\nu_0},\dots,\binom{u_N}{\nu_N}\right)$, $\underline{\mathfrak{v}}=\left(\binom{v_0}{\psi_0},\dots,\binom{v_N}{\psi_N}\right)$. Remember that $\Gamma_{iZ}$ are the interfaces separating subdomains and $\Omega_Z$. The identity (3.15) holds since each interior interface is visited twice in the evaluation of the pairing. Now, combine (3.15) with the insight from (3.12) and conclude for $\underline{\mathfrak{u}},\underline{\mathfrak{v}}\in\boldsymbol{\mathcal{ST}}(\Sigma)$

$$\langle\!\langle\underline{\mathfrak{u}},\underline{\mathfrak{v}}\rangle\!\rangle_\Sigma=\sum_{i,j}\int_{\Gamma_{ij}}\underbrace{(u_i-u_j)}_{=0}\psi_i-\underbrace{(v_i-v_j)}_{=0}\nu_i\,\mathrm{d}S+\sum_{i=0}^{N}\int_{\Gamma_{iZ}}u_i\psi_i-v_i\nu_i\,\mathrm{d}S\,.$$

We immediately get

$$\langle\!\langle\underline{\mathfrak{u}},\underline{\mathfrak{v}}\rangle\!\rangle_\Sigma=-\langle\!\langle\mathbb{L}_Z\,\underline{\mathfrak{u}},\mathbb{L}_Z\,\underline{\mathfrak{v}}\rangle\!\rangle_{\partial\Omega_Z},\quad\forall\underline{\mathfrak{u}},\underline{\mathfrak{v}}\in\boldsymbol{\mathcal{ST}}(\Sigma)\,,\tag{3.16}$$

and note that the minus sign is due to the opposite orientation of normals on $\partial\Omega_Z$.

*Remark* 3.1. A fundamental result is the characterization of single-trace spaces as complete Lagrangian subspaces w.r.t. the pairing $\langle\langle\cdot,\cdot\rangle\rangle_\Sigma$ of multi-trace spaces as given in [18, Prop. 2.1] and [16, Thm. 2.23]. It can be adapted to homogeneous Dirichlet boundary on $\partial\Omega_Z$:

$$\boldsymbol{\mathcal{ST}}_0(\Sigma) = \{\underline{\mathfrak{u}} \in \boldsymbol{\mathcal{MT}}(\Sigma) : \langle\langle \underline{\mathfrak{u}}, \underline{\mathfrak{v}} \rangle\rangle_\Sigma = 0 \ \forall \underline{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}_0(\Sigma)\} \ .$$

The intuition behind this characterization is clear from (3.15): Varying $\underline{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}_0(\Sigma)$ enforces (3.12) and vanishing Dirichlet components on the boundaries $\Gamma_{iZ}$. △

### 3.3 First-kind boundary integral equations

Assume that $U \in H^1_{\text{loc}}(\mathbb{R}^d)$ solves the transmission problem (2.3). Then Theorem 3.1 permits us to conclude

$$(\mathbb{P}_j - \mathsf{Id})\, \mathbb{T}_j\, U = 0 \quad \text{in}\, \mathcal{T}(\partial\Omega_j)\ , \quad j = 0, \ldots, N\ . \tag{3.17}$$

Using (3.7), this implies that for all $\underline{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}_0(\Sigma)$

$$\sum_{j=0}^{N} \left\langle\left\langle (\mathbb{A}_j - \tfrac{1}{2}\mathsf{Id})\, \mathbb{T}_j\, U, \mathbb{L}_j\, \underline{\mathfrak{v}} \right\rangle\right\rangle_{\partial\Omega_j} = 0\ . \tag{3.18}$$

Next, recall (3.16), that the Dirichlet components of $\underline{\mathfrak{v}}$ vanish on $\partial\Omega_Z$, $\mathsf{L}_{D,Z}\underline{\mathfrak{v}} = 0$, and that $U$ satisfies the Dirichlet boundary conditions (2.2), that is, $\mathsf{T}_{D,Z}\, U = g$ on $\partial\Omega_Z$. Thus, we find ($\mathbb{T}_Z\, U$ is the Cauchy trace of $U$ on $\partial\Omega_Z$)

$$\sum_{j=0}^{N} \left\langle\left\langle -\tfrac{1}{2}\, \mathbb{T}_j\, U, \mathbb{L}_j\, \underline{\mathfrak{v}} \right\rangle\right\rangle_{\partial\Omega_j} = -\tfrac{1}{2}\langle\langle \mathbb{T}_Z\, U, \mathbb{L}_Z\, \underline{\mathfrak{v}} \rangle\rangle_{\partial\Omega_Z} = -\tfrac{1}{2}\langle g, \mathsf{L}_{N,Z}\underline{\mathfrak{v}} \rangle_{\partial\Omega_Z},$$

and (3.18) can be converted into the equation:

$$\sum_{j=0}^{N} \langle\langle \mathbb{A}_j\, \mathbb{T}_j\, U, \mathbb{L}_j\, \underline{\mathfrak{v}} \rangle\rangle_{\partial\Omega_j} = -\tfrac{1}{2}\langle g, \mathsf{L}_{N,Z}\underline{\mathfrak{v}} \rangle_{\partial\Omega_Z}\ . \tag{3.19}$$

Eventually, we have derived a variational equation satisfied by $\mathbb{T}_\Sigma\, U := (\mathbb{T}_i\, U)_{i=0}^N \in \boldsymbol{\mathcal{ST}}(\Sigma)$. In order to balance trial and test spaces, we employ the customary offset function technique. We rely on a function $G \in H^1_{\text{loc}}(\mathbb{R}^d)$, whose point trace on $\partial\Omega_Z$ agrees with the Dirichlet data $g$: $\mathsf{T}_{D,Z}\, G = g$. Thus, we can define the skeleton extension of the Dirichlet boundary values, $\underline{\mathfrak{g}} := \binom{G|_\Sigma}{0} \in \boldsymbol{\mathcal{ST}}(\Sigma)$. Introducing the skeleton Cauchy trace of $U$ minus this extension as unknown $\underline{\mathfrak{u}}$, that is, $\underline{\mathfrak{u}} = \mathbb{T}_\Sigma\, U - \underline{\mathfrak{g}}$ yields the classical single-trace boundary integral equation (STF): seek $\underline{\mathfrak{u}} \in \boldsymbol{\mathcal{ST}}_0(\Sigma)$ such that

$$\boxed{\sum_{j=0}^{N} \langle\langle \mathbb{A}_j\, \mathbb{L}_j\, \underline{\mathfrak{u}}, \mathbb{L}_j\, \underline{\mathfrak{v}} \rangle\rangle_{\partial\Omega_j} = -\tfrac{1}{2}\langle g, \mathsf{L}_{N,Z}\underline{\mathfrak{v}} \rangle_{\partial\Omega_Z} - \sum_{j=0}^{N} \left\langle\left\langle \mathbb{A}_j\, \mathbb{L}_j\, \underline{\mathfrak{g}}, \mathbb{L}_j\, \underline{\mathfrak{v}} \right\rangle\right\rangle_{\partial\Omega_j},} \tag{3.20}$$

for all $\underline{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}_0(\Sigma)$. This is a first kind boundary integral equation in weak form. We write $\mathsf{m}_{\text{STF}}$ for its associated bilinear form $\mathsf{m}_{\text{STF}} : \boldsymbol{\mathcal{ST}}_0(\Sigma) \times \boldsymbol{\mathcal{ST}}_0(\Sigma) \to \mathbb{R}$.

### 3.4 Existence and Uniqueness of Solutions

The bilinear form $\mathsf{m}_{\text{STF}}$ from (3.20) is a sum of local contributions associated with the integral operators $\mathbb{A}_j$, $j = 0, \ldots, N$. This permits us to appeal to well established results asserting the ellipticity of single layer and hypersingular boundary integral operators. The proofs can be found in [56, Cor. 3.13 & Thm. 8.18], [70, Thm. 3.5.3], and [74, Sect. 6.6.1].

**Theorem 3.3.** *Let $D \subset \mathbb{R}^d$ be a Lipschitz domain, $\partial D$ connected and bounded, and write $\mathsf{V}$ and $\mathsf{W}$ for the single layer boundary integral operator and hypersingular boundary integral operator on $\partial D$, respectively, associated with $-\Delta$. If $d = 2$, assume that $\partial D$ has a diameter smaller than $1$. Then there are constants $c_V > 0$ and $c_W > 0$ such that*

$$\langle \mathsf{V}\varphi, \varphi \rangle_{\partial D} \geq c_V \, \|\varphi\|^2_{H^{-\frac{1}{2}}(\partial D)} , \qquad \forall \varphi \in H^{-\frac{1}{2}}(\partial D) , \tag{3.21}$$

$$\langle \mathsf{W}v, v \rangle_{\partial D} \geq c_W \, \|v\|^2_{H^{\frac{1}{2}}(\partial D)} , \qquad \forall v \in H^{\frac{1}{2}}(\partial D)/\mathbb{R} . \tag{3.22}$$

*If $D$ is not bounded, then* (3.22) *holds on the entire space $H^{\frac{1}{2}}(\partial D)$.*

To apply this theorem recall the symmetry of the double layer boundary integral operators for $-\Delta$: $\langle \mathsf{K}v, \varphi \rangle_{\partial D} = \langle v, \mathsf{K}'\varphi \rangle_{\partial D}$ for all $v \in H^{\frac{1}{2}}(\partial D)$ and $\varphi \in H^{-\frac{1}{2}}(\partial D)$. Thus, we obtain $D = \boldsymbol{\Omega}_j$

$$\begin{aligned} \left\langle\!\!\left\langle \mathbb{A}_j \begin{pmatrix} v \\ \varphi \end{pmatrix}, \begin{pmatrix} -v \\ \varphi \end{pmatrix} \right\rangle\!\!\right\rangle_{\partial\Omega_j} &= \left\langle\!\!\left\langle \begin{pmatrix} -\mathsf{K}_j v + \mathsf{V}_j \varphi \\ \mathsf{W}_j v + \mathsf{K}'_j \varphi \end{pmatrix}, \begin{pmatrix} -v \\ \varphi \end{pmatrix} \right\rangle\!\!\right\rangle_{\partial\Omega_j} \\ &= \langle \mathsf{V}_j \varphi, \varphi \rangle_{\partial\Omega_j} + \langle \mathsf{W}_j v, v \rangle_{\partial\Omega_j} , \end{aligned} \tag{3.23}$$

and from Theorem 3.3 we immediately conclude that there are constants $c_j > 0$, $j = 1, \ldots, N$, such that

$$\langle\!\langle \mathbb{A}_j \, \mathfrak{v}, \Xi_j \mathfrak{v} \rangle\!\rangle_{\partial\Omega_j} \geq c_j \inf_{\alpha \in \mathbb{R}} \left\| \mathfrak{v} - \tbinom{\alpha}{0} \right\|^2_{\mathcal{T}(\partial\Omega_j)} , \quad \forall \mathfrak{v} \in \mathcal{T}(\partial\Omega_j) , \tag{3.24}$$

where the simple isometric isomorphism $\Xi_j : \mathcal{T}(\partial\Omega_j) \to \mathcal{T}(\partial\Omega_j)$ is defined as $\Xi_j \binom{v}{\varphi} := \binom{-v}{\varphi}$. Again the quotient norm is redundant for $j = 0$:

$$\exists c_0 > 0 : \quad \langle\!\langle \mathbb{A}_0 \, \mathfrak{v}, \Xi_0 \mathfrak{v} \rangle\!\rangle_{\partial\Omega_0} \geq c_0 \, \|\mathfrak{v}\|^2_{\mathcal{T}(\partial\Omega_0)} , \quad \forall \mathfrak{v} \in \mathcal{T}(\partial\Omega_0) . \tag{3.25}$$

In the interest of concise notation we merge the local sign-change isomorphisms $\Xi_j$ into the operator $\Xi$ on $\boldsymbol{\mathcal{MT}}(\Sigma)$ that amounts to component-wise application of the $\Xi_j$'s.

**Theorem 3.4** ($\Xi$-ellipticity of STF bilinear form). *The STF bilinear form* $\mathsf{m}_{\mathrm{STF}}$ *satisfies the "$\Xi$-ellipticity" estimate*

$$\mathsf{m}_{\mathrm{STF}}(\underline{\mathfrak{v}}, \Xi\underline{\mathfrak{v}}) \geq c\,\|\mathfrak{v}\|^2_{\boldsymbol{\mathcal{MT}}(\Sigma)} \quad \forall \mathfrak{v} \in \boldsymbol{\mathcal{ST}}_0(\Sigma)\ ,$$

*for some constant* $c > 0$.

*Proof.* Invoking the definition of $\mathsf{m}_{\mathrm{STF}}$ from (3.20) together with (3.24) and (3.25) confirms that $\mathsf{m}_{\mathrm{STF}}$ is positive semidefinite with an at most finite-dimensional kernel

$$\mathsf{m}_{\mathrm{STF}}(\underline{\mathfrak{v}}, \Xi\underline{\mathfrak{v}}) \geq c_0\,\|\mathfrak{v}\|^2_{\mathcal{T}(\partial\Omega_0)} + \sum_{j=1}^{N} c_j \inf_{\alpha\in\mathbb{R}} \left\|\mathfrak{v} - \binom{\alpha}{0}\right\|^2_{\mathcal{T}(\partial\Omega_j)}\ .$$

It remains to show that the kernel can only be trivial. Let us consider the variational problem with vanishing right hand side: $\mathsf{m}_{\mathrm{STF}}(\mathfrak{u}, \mathfrak{v}) = 0$ for all $\mathfrak{v} \in \boldsymbol{\mathcal{ST}}_0(\Sigma)$. Choosing $\underline{\mathfrak{v}} = \underline{\mathfrak{u}}$ we infer from (3.24) that $\mathbb{L}_j\,\mathfrak{u} = \binom{c_j}{0}$ for some $c_j \in \mathbb{R}$, that is, the Neumann components of $\mathfrak{u}$ vanish and its Dirichlet components are constant on the subdomain boundaries. In fact, by (3.25), on $\partial\Omega_0$ those have to vanish, too. Hence, they have to be zero on all subdomains abutting $\partial\Omega_0$, and a simple induction arguments confirms $\mathfrak{u} = 0$. □

**Corollary 3.5.** *The variational problem* (3.20) *has a unique solution for any* $g \in H^{\frac{1}{2}}(\partial\Omega_Z)$.

*Proof.* The continuity of the bilinear form $\mathsf{m}_{\mathrm{STF}}$ from (3.20) and of the right hand side linear form on $\boldsymbol{\mathcal{ST}}_0(\Sigma)$ is obvious. Moreover, the ellipticity result of Theorem 3.4 implies an inf-sup condition for $\mathsf{m}_{\mathrm{STF}}$. □

### 3.5 Obstruction to operator preconditioning

According to Theorem 3.4 the single trace variational boundary integral equation (3.20) is stable. Moreover, its boundary element discretization is straightforward, once a triangulation of the skeleton is given.

However, (3.20) is posed on the single-trace space $\boldsymbol{\mathcal{ST}}_0(\Sigma)$ and there is no known trace space that is dual to $\boldsymbol{\mathcal{ST}}_0(\Sigma)$ with respect to an $L^2$-type duality pairing. Of course, uniformly stable discrete $L^2$-dualities are even more elusive for conventional boundary element subspaces of $\boldsymbol{\mathcal{ST}}_0(\Sigma)$. Thus, as remarked in the Introduction, operator preconditioning for (3.20) is *not possible*. This is further elaborated in [16, Sect. 4.5].

# 4 Global Multi-trace Boundary Integral Equations

## 4.1 Preface

In this and the next section we present multi-trace boundary integral equations for the transmission problem (2.3). They have in common that they give rise to variational problems posed on products of simple trace spaces on subdomain boundaries.

To understand the benefit of a product space framework in terms of operator preconditioning, recall Theorem 1.1 and its straightforward generalization to a setting where the bilinear form $\mathsf{a}$ is defined on a product of Hilbert spaces[4] $X := X_1 \times \dots X_n$, $n \in \mathbb{N}$. We also need Hilbert spaces $Y_1, \dots, Y_n$ equipped with continuous bilinear forms $\mathsf{b}_j \in L(Y_j \times Y_j, \mathbb{R})$. Then, given finite-dimensional subspaces $X_{j,h} \subset X_j$, $Y_{j,h} \subset Y_j$, assuming inf-sup conditions for $\mathsf{a}$ on $X_h := X_{1,h} \times \cdots \times X_{n,h}$ and for the $\mathsf{b}_j$'s on $Y_{j,h}$, and with continuous and stable discrete pairings $\mathsf{d}_j : X_{j,h} \times Y_{j,h} \to \mathbb{R}$ at our disposal, we find that the matrix

$$\mathbf{M}_h^{\times} := \sum_{j=1}^{n} \mathbf{D}_j^{-1} \mathbf{B}_j \mathbf{D}_j^{-T} \tag{4.1}$$

provides operator preconditioning for the Galerkin matrix $\mathbf{A}_h$ of $\mathsf{a}$ on $X_h$. Here, the matrices $\mathbf{B}_j$ and $\mathbf{D}_j$ are the Galerkin matrices spawned by $\mathsf{b}_j$ on $Y_{j,h} \times Y_{j,h}$ and $\mathsf{d}_j$ on $X_{j,h} \times Y_{j,h}$.

## 4.2 Heuristic gap construction

Consider the special situation that none of the subdomains $\Omega_1, \dots, \Omega_N$ and $\Omega_Z$ touch as in Figure 2 (right). In this case the skeleton can be partitioned according to

$$\Sigma = \partial\Omega_0 = \partial\Omega_1 \cup \cdots \cup \partial\Omega_N \cup \partial\Omega_Z \; . \tag{4.2}$$

We arrive at this arrangement of "separated subdomains" when introducing a small gap between all bounded subdomains, which is illustrated in Figure 2.

Now consider the single-trace first kind BIE presented in Section 3 in the case of separated subdomains. Obviously, this special setting permits us to identify the single-trace spaces with products of Cauchy trace spaces on subdomain boundaries. The resulting space which can be viewed as a truncated multi-trace space with omitted (except for the part on $\partial\Omega_Z$) $\partial\Omega_0$-contribution [18, Sect. 7]:

$$\boldsymbol{\mathcal{ST}}(\Sigma) \cong \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) := \mathcal{T}(\partial\Omega_1) \times \cdots \times \mathcal{T}(\partial\Omega_N) \times \mathcal{T}(\partial\Omega_Z) \; , \tag{4.3}$$

$$\boldsymbol{\mathcal{ST}}_0(\Sigma) \cong \widehat{\boldsymbol{\mathcal{MT}}}_0(\Sigma) := \mathcal{T}(\partial\Omega_1) \times \cdots \times \mathcal{T}(\partial\Omega_N) \times H^{-\frac{1}{2}}(\partial\Omega_Z) \; . \tag{4.4}$$

[4] The notations introduced in the context of Theorem 1.1 are used tacitly.

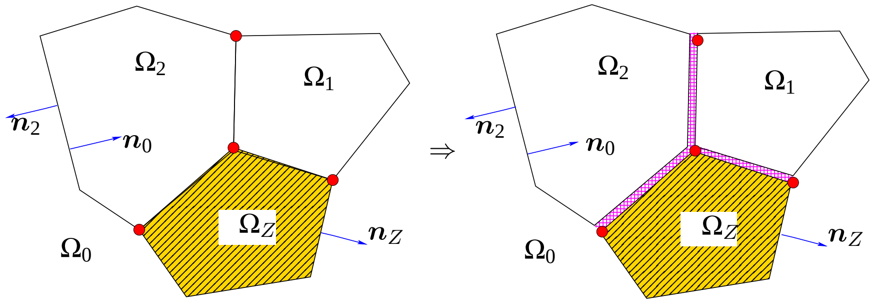



Figure 2: Illustration of the gap idea

The related isomorphisms are $\mathbb{I} : \widehat{\mathcal{MT}}(\Sigma) \to \mathcal{ST}(\Sigma)$ and $\mathbb{I}_0 : \widehat{\mathcal{MT}}_0(\Sigma) \to \mathcal{ST}(\Sigma)$. Specifically, with $\mathfrak{v}_j = \binom{v_j}{\psi_j} \in \mathcal{T}(\partial\Omega_j)$, $j = 1, \dots, N, Z$, we have

$$\mathbb{I}(\mathfrak{v}_1, \dots, \mathfrak{v}_N, \mathfrak{v}_Z) := \left( \begin{pmatrix} v_1 \vee \cdots \vee v_N \vee v_Z \\ -(\psi_1 \vee \cdots \vee \psi_N \vee \psi_Z) \end{pmatrix}, \mathfrak{v}_1, \dots, \mathfrak{v}_N \right) , \tag{4.5}$$

where $\vee$ designates the joining of functions on subdomain boundaries to form a function on $\partial\Omega_0$. This is made possible by the partitioning (4.2). The minus-sign reflects the opposite orientations of the normals $\boldsymbol{n}_0$ and $\boldsymbol{n}_j$.

We observe that for separated subdomains the STF gives rise to a variational problem on a genuine product Hilbert space of subdomain trace spaces. As explained above in Section 4.1, this facilitates the construction of operator preconditioners of the form (4.1). Algorithmic details are postponed to Section 4.4.

Next, we take a closer look at the STF variational formulation (3.20) in the situation of separated subdomains, with the aim of recasting it into a problem posed on $\widehat{\mathcal{MT}}_0(\Sigma)$. Since,

$$\mathbb{L}_j \, \mathbb{I}(\mathfrak{v}_1, \dots, \mathfrak{v}_N, \mathfrak{v}_Z) = \mathfrak{v}_j \ , \quad j = 1, \dots, N \text{ or } j = Z \ , \tag{4.6}$$

it is only the term in (3.20) contributed by $\partial\Omega_0$ that needs closer scrutiny. We write $\underline{\widehat{\mathfrak{u}}} = (\mathfrak{u}_1, \dots, \mathfrak{u}_N, \mathfrak{u}_Z) \in \widehat{\mathcal{MT}}(\Sigma)$, $\underline{\widehat{\mathfrak{v}}} = (\mathfrak{v}_1, \dots, \mathfrak{v}_N, \mathfrak{v}_Z) \in \widehat{\mathcal{MT}}(\Sigma)$, appeal to the lengthy manipulations from [16, Eq. (5.8)] or [18, Sect. 8], and end up with

$$\left\langle\!\left\langle \mathbb{A}_0 \, \mathbb{L}_0 \, \mathbb{I} \, \underline{\widehat{\mathfrak{u}}}, \mathbb{L}_0 \, \mathbb{I} \, \underline{\widehat{\mathfrak{v}}} \right\rangle\!\right\rangle_{\partial\Omega_0} = \sum_{j=1,Z}^{N} \left( \left\langle\!\left\langle \mathbb{A}_j^{\mu_0} \, \mathfrak{u}_j, \mathfrak{v}_j \right\rangle\!\right\rangle_{\partial\Omega_j} + \sum_{\substack{i=1,Z \\ i \neq j}}^{N} \left\langle\!\left\langle \mathbb{T}_j^{\mu_0} \, \mathbb{G}_i^{\mu_0}(\mathfrak{u}_i), \mathfrak{v}_j \right\rangle\!\right\rangle_{\partial\Omega_j} \right) .$$

Here, $Z$ in the range of summation indices means that $\partial\Omega_Z$ is also covered by the sum. Moreover a superscript $\mu_0$ indicates that the Calderón operators $\mathbb{A}_j^{\mu_0}$, potentials $\mathbb{G}_j^{\mu_0}$, or Cauchy trace operators $\mathbb{T}_j^{\mu_0}$ are defined using the diffusion coefficient $\mu_0$, but still live on $\partial\Omega_j$, $j = 1, \dots, N, Z$.

For the sake of brevity, we introduce the "remote coupling operators" for $i \neq j$

$$\mathbb{C}_{i\to j} : \mathcal{T}(\partial\Omega_i) \to \mathcal{T}(\partial\Omega_j) \quad , \quad \mathbb{C}_{i\to j}(\mathfrak{v}_i) := \mathbb{T}_j^{\mu_0}\big(\mathbb{G}_i^{\mu_0}(\mathfrak{v}_i)\big) \; . \tag{4.7}$$

Observe that these operators take the trace of a potential at another boundary. We also note that for separated subdomains the offset function $\underline{\mathfrak{g}}$ from (3.20) may be supported on $\partial\Omega_Z$ alone, that is, we can choose $\underline{\mathfrak{g}} = \mathbb{I}\left(0, \ldots, 0, \binom{g}{0}\right)$. In particular, this means that $\mathbb{L}_j\, \underline{\mathfrak{g}} = 0$ for all $j = 1, \ldots, N$. Then, using the identity stated above, we can rewrite (3.20) as the following variational problem posed on $\widehat{\mathcal{MT}}_0(\Sigma)$: seek $\widehat{\underline{\mathfrak{u}}} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_N, \nu_Z) \in \widehat{\mathcal{MT}}_0(\Sigma)$ such that

$$\begin{aligned}
&\langle \mathsf{V}_Z \nu_Z, \psi_Z\rangle_{\partial\Omega} + \sum_{j=1}^{N} \left\langle\!\left\langle (\mathbb{A}_j^{\mu_0} + \mathbb{A}_j)\mathfrak{u}_j, \mathfrak{v}_j \right\rangle\!\right\rangle_{\partial\Omega_j} \\
&\qquad + \sum_{i=1}^{N} \left\langle\!\left\langle \mathbb{C}_{i\to Z}\, \mathfrak{u}_i, \tbinom{0}{\psi_Z} \right\rangle\!\right\rangle_{\partial\Omega_Z} + \sum_{j=1}^{N} \sum_{\substack{i=1\\ i\neq j}}^{N} \left\langle\!\left\langle \mathbb{C}_{i\to j}\, \mathfrak{u}_i, \mathfrak{v}_j \right\rangle\!\right\rangle_{\partial\Omega_j} \\
&\qquad\qquad + \sum_{j=1}^{N} \left\langle\!\left\langle \mathbb{C}_{Z\to j} \tbinom{0}{\nu_Z}, \mathfrak{v}_j \right\rangle\!\right\rangle_{\partial\Omega_j} \\
&\qquad\qquad\qquad = -\tfrac{1}{2}\left\langle (\mathsf{Id} - 2\mathsf{K}_Z^{\mu_0}) g, \psi_Z \right\rangle_{\partial\Omega_Z} - \sum_{j=1}^{N} \left\langle\!\left\langle \mathbb{C}_{Z\to j} \tbinom{g}{0}, \mathfrak{v}_j \right\rangle\!\right\rangle_{\partial\Omega_j} ,
\end{aligned} \tag{4.8}$$

for all $\widehat{\underline{\mathfrak{v}}} = (\mathfrak{v}_1, \ldots, \mathfrak{v}_N, \psi_Z) \in \widehat{\mathcal{MT}}_0(\Sigma)$.

So far, this bulky variational BIE was derived for separated subdomains, depicted on the right in Figure 2. The key insight gleaned in [18, Sect. 5 "Gap idea"] and also discussed in [16, Sect. 5.2 "Gap idea"] is *that* (4.8) *remains well defined even without a gap* between the $\Omega_j$ and $\Omega_Z$, $j = 1, \ldots, N$, as shown on the left in Figure 2. Indeed, reading the trace $\mathbb{T}_j^{\mu_0}$ in (4.7) as an *exterior* Cauchy trace onto $\partial\Omega_j$ with respect to $\Omega_j$, the remote coupling operators remain well-defined continuous mappings. Hence, (4.8) defines the variational form of the *global multi-trace boundary integral equation formulation* for the transmission problem (2.3) even in the general setting outlined in Section 2.

*Remark* 4.1. As hinted at by their name, in (4.8) the "remote coupling operators" $\mathbb{C}_{i\to j}$ from (4.7) establish a variational coupling between subdomains, even it they do not have a common interface. These long-range interaction made us choose the attribute "global" for this kind of MTF. △

## 4.3 Global MTF: Existence and uniqueness of solutions

In light of the "gap idea" it is not surprising that the variational problem (4.8) inherits quite a few of the remarkable properties of the original STF (3.20). The first result mirrors the "$\Xi$-ellipticity" of $\mathsf{m}_{\mathrm{STF}}$ asserted in Theorem 3.4.

**Theorem 4.1.** *The global multi-trace bilinear form defined by the left hand side of* (4.8) *is* continuous *and* $\Xi$-elliptic *on* $\widehat{\mathcal{MT}}_0(\Sigma)$.

*Proof.* Continuity of the remote coupling operators $\mathbb{C}_{i\to j}$ is immediate from the mapping properties of potentials and traces. This implies continuity of the bilinear form.

As regards $\Xi$-ellipticity on $\widehat{\mathcal{MT}}_0(\Sigma)$, we merely sketch the idea of the proof: First note that the first line of (4.8) enjoys $\widehat{\mathcal{MT}}_0(\Sigma)$-ellipticity on a subspace, defined by imposing vanishing averages of the Dirichlet components on $\partial\Omega_z, \ldots, \partial\Omega_N, \partial\Omega_Z$. The remaining constant Dirichlet components are taken care of by the remote coupling terms in lines two and three of (4.8), see the proof of [18, Prop. 10.3] for details. □

The next result confirms that the unique solution of (4.8) yields the Cauchy traces of the unique solution of the transmission problem (2.3).

**Theorem 4.2.** *If* $\underline{\widehat{\mathfrak{u}}} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_N, \nu_Z) \in \widehat{\mathcal{MT}}_0(\Sigma)$ *solves* (4.8)*, then* $\mathfrak{u}_j = \mathbb{T}_j U - \binom{\mathsf{T}_{D,j} G}{0}$ *and* $\nu_Z = \mathsf{T}_{N,Z} U$*, where* $U$ *is the unique solution of* (2.3).

*Proof.* We refer to [18, Sect. 9], in particular the proof of [18, Thm. 9.1], which carries over to (4.8) with only slight modifications. □

## 4.4 Global MTF: Operator preconditioning in 3D

We restrict ourselves to $d = 3$ and the simplest boundary element Galerkin discretization of (4.8). To define trial and test spaces we *independently* equip each boundary $\partial\Omega_j$, $j = 1, \ldots, N$, and $\partial\Omega_Z$ with a conforming triangular surface mesh $\mathcal{G}_j$ [70, Sect. 4.1.2]. On these meshes we define the spaces $S^{-1}(\mathcal{G}_j) \subset H^{-\frac{1}{2}}(\partial\Omega_j)$ of (discontinuous) $\mathcal{G}_j$-piecewise constant functions [70, Ex. 4.1.16], and $S^0(\mathcal{G}_j) \subset H^{\frac{1}{2}}(\partial\Omega_j)$ of $\mathcal{G}_j$-piecewise linear and *continuous* functions [70, Ex. 4.1.37]. Their product provides discrete local Cauchy trace spaces

$$\mathcal{T}_h(\partial\Omega_j) := S^0(\mathcal{G}_j) \times S^{-1}(\mathcal{G}_j) \subset \mathcal{T}(\partial\Omega_j)\ , \quad j = 1, \ldots, N\ , \tag{4.9}$$

which are the building blocks for the discrete counterpart of $\widehat{\mathcal{MT}}_0(\Sigma)$ (as defined in (4.4)):

$$\widehat{\mathcal{MT}}_{0,h}(\{\mathcal{G}_j\}) := \mathcal{T}_h(\partial\Omega_1) \times \cdots \times \mathcal{T}_h(\partial\Omega_N) \times S^{-1}(\mathcal{G}_Z) \subset \widehat{\mathcal{MT}}_0(\Sigma)\ . \tag{4.10}$$

This completely defines the Galerkin BEM for (4.8). If standard localized nodal basis functions are used for $S^{-1}(\mathcal{G}_j)$ and $S^0(\mathcal{G}_j)$, then for shape-regular and quasi-uniform families of surface meshes we will encounter a growth of the spectral condition numbers of the resulting linear systems of equations like $O(h^{-2})$ as the meshwidth $h \to 0$ [70, Sect. 4.5].

Operator preconditioning will be carried out in the product space framework outlined in Section 4.1. Retaining the notations from there we have $n = N+1$, $X_j := \mathcal{T}(\partial\Omega_j)$, $j = 1, \dots, N$, $X_n := H^{-\frac{1}{2}}(\partial\Omega_Z)$, and the bilinear form $\mathsf{a}$ is that of (4.8), for which Theorem 4.1 guarantees uniform stability. From (4.4) we see that the role of $X_{j,h}$ is played by $\mathcal{T}_h(\partial\Omega_j)$, $j = 1, \dots, N$, and that of $X_{n,h}$ by $S^{-1}(\mathcal{G}_Z)$.

As suggested by $L^2$-duality, we pick

$$Y_j := X_j \ , \quad j = 1, \dots, N \quad \text{and} \quad Y_n := H^{\frac{1}{2}}(\partial\Omega_Z) \ , \tag{4.11}$$

because we want to employ the local $L^2$-type pairings on $\partial\Omega_j$ and $\partial\Omega_Z$, respectively, as pairing bilinear forms $\mathsf{d}_j$, that is,

$$\mathsf{d}_j := \langle\!\langle \cdot, \cdot \rangle\!\rangle_{\partial\Omega_j} \ , \quad j = 1, \dots, N \ , \quad \mathsf{d}_n := \langle \cdot, \cdot \rangle_{\partial\Omega_Z} \ . \tag{4.12}$$

Next, the Calderón operators from (3.7) supply the bilinear forms $\mathsf{b}_j$, $j = 1, \dots, N$:

$$\mathsf{b}_j(\mathfrak{u}_j, \mathfrak{v}_j) := \langle\!\langle \mathbb{A}_j \, \mathfrak{u}_j, \mathfrak{v}_j \rangle\!\rangle_{\partial\Omega_j} \ , \quad \mathfrak{u}_j, \mathfrak{v}_j \in \mathcal{T}(\partial\Omega_j) \ , \tag{4.13}$$

whereas the hypersingular boundary integral operator on $\partial\Omega_Z$ gives us $\mathsf{b}_n$: $\mathsf{b}_n(u_Z, v_Z) := \langle \mathsf{W}_Z u_Z, v_Z \rangle_{\partial\Omega_Z}$, $u_Z, v_Z \in H^{\frac{1}{2}}(\partial\Omega_Z)$.

The choice of the boundary element spaces $Y_{j,h} \subset Y_j$ poses a challenge, because just using lowest order boundary elements on the same meshes $\mathcal{G}_j$ fails to deliver uniformly stable discretizations of the pairing bilinear forms $\mathsf{d}_j$ even on sequences of shape-regular and quasi-uniform meshes. Instead we resort to the breakthrough idea from [72] and [8] and use boundary element spaces on *dual meshes* $\widehat{\mathcal{G}}_j$ defined via barycentric subdivisions of $\mathcal{G}_j$ as displayed in Figure 3 [73, Sect. 2.2].

The spaces $S^{-1}(\widehat{\mathcal{G}}_j) \subset H^{-\frac{1}{2}}(\partial\Omega_j)$ comprise functions that are piecewise constant on dual cells [8, Sect. 2, Fig. 3]. Further, $S^0(\widehat{\mathcal{G}}_j)$ is spanned by continuous functions that are piecewise linear on the barycentric refinement of $\mathcal{G}_j$, and whose values at nodes of $\mathcal{G}_j$ and midpoints of edges of $\mathcal{G}_j$ are determined by the average of their values in adjacent barycenters [8, Sect. 2, Fig. 1]. Then the theoretical developments of [73, Sect. 2] confirm that the pairs of spaces $S^{-1}(\mathcal{G}_j) \times S^0(\widehat{\mathcal{G}}_j)$ and $S^{-1}(\widehat{\mathcal{G}}_j) \times S^0(\mathcal{G}_j)$ provide stable Galerkin discretizations of the duality pairing $\langle \cdot, \cdot \rangle_{\partial\Omega_j} : H^{-\frac{1}{2}}(\partial\Omega_j) \times H^{\frac{1}{2}}(\partial\Omega_j) \to \mathbb{R}$. This hinges on certain assumptions on the geometry of $\mathcal{G}_j$, which are satisfied for shape-regular and quasi-uniform families and even for a wide range of locally refined meshes. Appealing to this theory, the next theorem is a consequence of Theorem 1.1.

**Theorem 4.3.** *Operator preconditioning of the Galerkin boundary element discretization of* (4.8) *as outlined in Section 4.1 leads to uniformly bounded spectral condition numbers of the preconditioned linear systems in the case of shape-regular and quasi-uniform families of triangular surface meshes.*

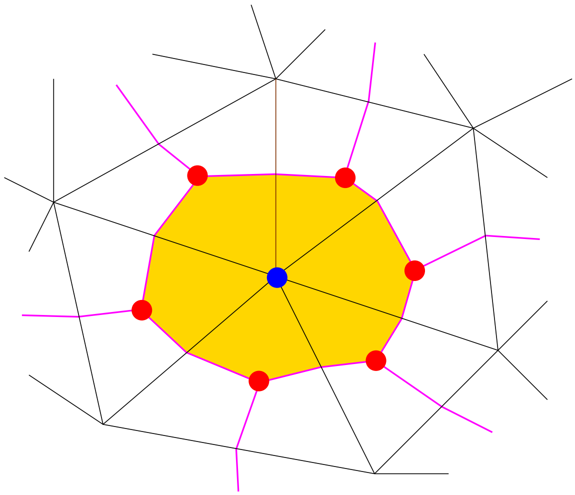


Figure 3: Barycentric dual mesh for a triangular primal mesh (black lines): the shaded region marks a dual cell associated with a primal node (blue disk), the magenta lines represent dual edges, the red disks dual nodes.

# 5 Local Multi-trace Boundary Integral Equations

## 5.1 Localized transmission conditions

In the global multi-trace formulation introduced in Section 4 the transmission conditions are implicitly contained in the variational formulation. Conversely, the local multi-trace approach takes into account the transmission conditions in their local form (2.3b), which can be expressed as

$$\begin{pmatrix} \mathsf{T}_{D,i} \\ \mathsf{T}_{N,i} \end{pmatrix} U_i = \begin{pmatrix} \mathsf{Id} & 0 \\ 0 & -\mathsf{Id} \end{pmatrix} \begin{pmatrix} \mathsf{T}_{D,j} \\ \mathsf{T}_{N,j} \end{pmatrix} U_j \quad \text{on } \Gamma_{ij} \ , \tag{5.1}$$

for which we embrace the compact notation:

$$\mathbb{S}_{ij}\,\mathbb{T}_i\,U_i = \mathbb{X}_{j\to i}\,\mathbb{T}_j\,U_j \ . \tag{5.2}$$

Here $\mathbb{S}_{ji} : \mathcal{T}(\partial\Omega_j) \to \mathcal{T}(\Gamma_{ij})$ restricts traces on $\partial\Omega_j$ to the interface $\Gamma_{ij} \subset \partial\Omega_j$. We used the natural notation $\mathcal{T}(\Gamma_{ij}) := H^{\frac{1}{2}}(\Gamma_{ij}) \times H^{-\frac{1}{2}}(\Gamma_{ij})$. The action of the operator $\mathbb{X}_{j\to i} : \mathcal{T}(\partial\Omega_j) \to \mathcal{T}(\Gamma_{ij})$ is immediate from (5.1): $\mathbb{X}_{j\to i}\binom{v}{\psi} := \mathbb{S}_{ji}\binom{v}{-\psi}$; in addition to restricting traces, $\mathbb{X}_{j\to i}$ flips the sign of the Neumann component, thus adjusting it to the orientation of the other subdomain boundary $\partial\Omega_i$.

Since we aim for boundary integral equations in weak form, we need to cast (5.2) into a variational equation. Formally, this can be accomplished by pairing with test functions in the dual space of $\mathcal{T}(\Gamma_{ij})$ (with respect to the pivot space $L^2(\Gamma_{ij})$). However, be aware that, in contrast to $\mathcal{T}(\partial\Omega_j)$, the $L^2(\Gamma_{ij})$ inner product does **not** induce a self-duality of $\mathcal{T}(\Gamma_{ij})$. Rather, the dual space is [56, Ch. 3], [70, Sect. 2.4.2],

$$\mathcal{T}(\Gamma_{ij})' = (H^{\frac{1}{2}}(\Gamma_{ij}))' \times (H^{-\frac{1}{2}}(\Gamma_{ij}))' \cong \widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij}) \times \widetilde{H}^{\frac{1}{2}}(\Gamma_{ij}) \; . \tag{5.3}$$

Here $\widetilde{H}^{\frac{1}{2}}(\Gamma_{ij})$ and $\widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})$ designate distributions on $\Gamma_{ij}$, whose extensions by zero to $\partial\Omega_i$ belong to $H^{\frac{1}{2}}(\partial\Omega_i)$ and $H^{-\frac{1}{2}}(\partial\Omega_i)$, respectively. Thus, $\widetilde{H}^{\frac{1}{2}}(\Gamma_{ij})$ and $\widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})$ can be identified with subspaces of $H^{\frac{1}{2}}(\Gamma_{ij})$ and $H^{-\frac{1}{2}}(\Gamma_{ij})$, respectively, which are actually *dense*. Yet, the norms of $\widetilde{H}^{\frac{1}{2}}(\Gamma_{ij})$ and $\widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})$ are *strictly stronger* than those of $H^{\frac{1}{2}}(\Gamma_{ij})$ and $H^{-\frac{1}{2}}(\Gamma_{ij})$.

We conclude that an equivalent weak form of (5.2) is

$$\langle\!\langle \mathbb{S}_{ij}\,\mathbb{T}_i\,U_i - \mathbb{X}_{j\to i}\,\mathbb{T}_j\,U_j, \mathfrak{v}_{ij} \rangle\!\rangle_{\Gamma_{ij}} = 0, \quad \forall\, \mathfrak{v}_{ij} \in \widetilde{\mathcal{T}}(\Gamma_{ij}) \; , \tag{5.4}$$

where $\widetilde{\mathcal{T}}(\Gamma_{ij}) := \widetilde{H}^{\frac{1}{2}}(\Gamma_{ij}) \times \widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})$. Here, the *continuous* pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\Gamma_{ij}} : \mathcal{T}(\Gamma_{ij}) \times \widetilde{\mathcal{T}}(\Gamma_{ij}) \to \mathbb{R}$ of Cauchy traces on $\Gamma_{ij}$ is defined in analogy to (3.3) based on $L^2(\Gamma_{ij})$-inner products.

## 5.2 Local MTF: Variational formulation

We now combine Theorem 3.1 in the form of (3.17) and (5.4) into a set of variational equations that *every* solution $U \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ of (2.3) will satisfy:

$$(3.17) \quad \Rightarrow \quad \langle\!\langle (\mathsf{Id} - \mathbb{P}_j)\,\mathbb{T}_j\,U, \mathfrak{v}_j \rangle\!\rangle_{\partial\Omega_j} = 0, \quad \forall\, \mathfrak{v}_j \in \mathcal{T}(\partial\Omega_j) \; , \tag{5.5a}$$

$$(5.4) \quad \Rightarrow \quad \Big\langle\!\!\Big\langle \mathbb{S}_{ji}\,\mathbb{T}_j\,U - \mathbb{X}_{i\to j}\,\mathbb{T}_i\,U, \mathbb{S}_{ji}\,\widetilde{\mathfrak{v}}_j \Big\rangle\!\!\Big\rangle_{\Gamma_{ij}} = 0, \quad \forall\, \widetilde{\mathfrak{v}}_j \in \widetilde{\mathcal{T}}_{\mathrm{pw}}(\partial\Omega_j) \; . \tag{5.5b}$$

Note that the duality (5.3), which underlies (5.4), enforces the use of the following special test space in the second equation

$$\widetilde{\mathcal{T}}_{\mathrm{pw}}(\partial\Omega_j) := \{ \widetilde{\mathfrak{v}} \in \mathcal{T}(\partial\Omega_j) : \; \mathbb{S}_{ji}\,\widetilde{\mathfrak{v}} \in \widetilde{\mathcal{T}}(\Gamma_{ij}), \; i \in \mathcal{N}_j \} \subset \mathcal{T}(\partial\Omega_j) \; . \tag{5.6}$$

Here and below, we write $\mathcal{N}_j \subset \{0,\ldots,N\}$ for the set of indices of subdomains that have an interface in common with $\partial\Omega_j$. Obviously, the restriction operators map continuously $\mathbb{S}_{ji} : \widetilde{\mathcal{T}}_{\mathrm{pw}}(\partial\Omega_j) \to \widetilde{\mathcal{T}}(\Gamma_{ij})$.

By adding (5.5a) and several instances of (5.5b), from (5.5) we conclude, that a solution $U \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ of (2.3) also fulfills, *cf.* [40, Sect. 3.2]

$$\Big\langle\!\!\Big\langle (\mathsf{Id} - \mathbb{P}_j)\,\mathbb{T}_j\,U, \widetilde{\mathfrak{v}}_j \Big\rangle\!\!\Big\rangle_{\partial\Omega_j} + \sum_{i\in\mathcal{N}_j} \sigma_{ij} \Big\langle\!\!\Big\langle \mathbb{S}_{ji}\,\mathbb{T}_j\,U - \mathbb{X}_{i\to j}\,\mathbb{T}_i\,U, \mathbb{S}_{ji}\,\widetilde{\mathfrak{v}}_j \Big\rangle\!\!\Big\rangle_{\Gamma_{ij}} = 0 \; , \tag{5.7}$$

for all $\widetilde{\mathfrak{v}}_j \in \widetilde{\mathcal{T}}_{\mathrm{pw}}(\partial\Omega_j)$, $j = 0,\ldots,N$. The $\sigma_{ij} \in \mathbb{R}$ are arbitrary non-zero combination coefficients.

Again, we employ an offset function technique to deal with the non-homogeneous Dirichlet boundary conditions (2.2). As in Section 3.3 we introduce a function $G \in H^1_{\text{loc}}(\mathbb{R}^d)$, whose Dirichlet trace on $\partial\Omega_Z$ agrees with the given data $g \in H^{\frac{1}{2}}(\partial\Omega_Z)$: $\mathsf{T}_{D,Z}\, G = g$. Then $\bar{U} := U - G \in \{V \in H^1_{\text{loc}}(\mathbb{R}^d) :\ V|_{\partial\Omega_Z} = 0\}$ solves

$$\begin{aligned}
&\left\langle\!\left\langle (\mathsf{Id} - \mathbb{P}_j)\, \mathbb{T}_j\, \bar{U}, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\partial\Omega_j} + \sum_{i\in\mathcal{N}_j} \sigma_{ij} \left\langle\!\left\langle \mathbb{S}_{ji}\, \mathbb{T}_j\, \bar{U} - \mathbb{X}_{i\to j}\, \mathbb{T}_i\, \bar{U}, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}} \\
&= \underbrace{-\left\langle\!\left\langle (\mathsf{Id} - \mathbb{P}_j)\, \mathbb{T}_j\, G, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\partial\Omega_j} - \sum_{i\in\mathcal{N}_j} \sigma_{ij} \left\langle\!\left\langle \mathbb{S}_{ji}\, \mathbb{T}_j\, G - \mathbb{X}_{i\to j}\, \mathbb{T}_i\, G, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}}}_{=:\Psi_j(\widetilde{\mathfrak{v}}_j)} \ , \qquad (5.8)
\end{aligned}$$

for all $\widetilde{\mathfrak{v}}_j \in \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_j)$, $j = 0, \ldots, N$.

In order to arrive at a variational boundary integral equation for the unknown Cauchy traces of $\bar{U}$, in the next key step we introduce all local subdomain traces $\mathfrak{u}_j := \binom{u_j}{\mu_j} := \mathbb{T}_j\, \bar{U} \in \mathcal{T}(\partial\Omega_j)$, $j = 0, \ldots, N$, as unknowns. By construction, they satisfy $u_j|_{\Gamma_{jZ}} = 0$. In addition we use $\mathbb{A}_j = \mathbb{P}_j - \frac{1}{2}\mathsf{Id}$ from (3.7) and obtain a *first kind BIE*: seek $\underline{\mathfrak{u}} = (\mathfrak{u}_0, \ldots, \mathfrak{u}_N) \in \boldsymbol{\mathcal{MT}}_0(\Sigma)$ such that

$$\left\langle\!\left\langle (\mathbb{A}_j - \tfrac{1}{2}\mathsf{Id})\mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\partial\Omega_j} + \sum_{i\in\mathcal{N}_j} \sigma_{ij} \left\langle\!\left\langle \mathbb{S}_{ji}\, \mathfrak{u}_j - \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}} = \Psi_j(\widetilde{\mathfrak{v}}_j) \ , \qquad (5.9)$$

for all $\underline{\mathfrak{v}} := (\widetilde{\mathfrak{v}}_0, \ldots, \widetilde{\mathfrak{v}}_N) \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$, $j = 0, \ldots, N$. This variational problem is posed on multi-trace spaces that respect homogeneous Dirichlet boundary conditions on $\partial\Omega_Z$:

$$\boldsymbol{\mathcal{MT}}_0(\Sigma) := \left\{ \left( \begin{pmatrix} v_j \\ \nu_j \end{pmatrix} \right)_{j=0}^{N} \in \boldsymbol{\mathcal{MT}}(\Sigma) :\ v_j|_{\Gamma_{jZ}} = 0 \right\} , \qquad (5.10a)$$

$$\widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma) := \left( \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_0) \times \cdots \times \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_N) \right) \cap \boldsymbol{\mathcal{MT}}_0(\Sigma) \ . \qquad (5.10b)$$

Again, we point out that the trial functions must allow extension by zero from each interface $\Gamma_{ij}$ to the associated subdomain boundaries $\partial\Omega_i$ and $\partial\Omega_j$.

There is a "magic" choice for the parameters $\sigma_{ij}$, because, by density arguments,

$$\sum_{i\in\mathcal{N}_j} \left\langle\!\left\langle \mathbb{S}_{ji}\, \mathfrak{u}_j, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}} = \left\langle\!\left\langle \mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\partial\Omega} \quad \forall \mathfrak{u}_j \in \mathcal{T}(\partial\Omega_j),\ \widetilde{\mathfrak{v}}_j \in \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_j) \ .$$

Hence, if we set $\sigma_{ij} = \frac{1}{2}$ in (5.9), we can benefit from cancellation and convert (5.9) into: seek $\underline{\mathfrak{u}} = (\mathfrak{u}_0, \ldots, \mathfrak{u}_N) \in \boldsymbol{\mathcal{MT}}_0(\Sigma)$ such that

$$\boxed{\left\langle\!\left\langle \mathbb{A}_j\, \mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\partial\Omega_j} - \frac{1}{2} \sum_{i\in\mathcal{N}_j} \left\langle\!\left\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}} = \Psi_j(\widetilde{\mathfrak{v}}_j)} \ , \qquad (5.11)$$

for all $\underline{\mathfrak{v}} := (\widetilde{\mathfrak{v}}_0, \dots, \widetilde{\mathfrak{v}}_N) \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$. This is the "classical" MTF for the transmission problem (2.3) as proposed in [39] and discussed in [16, Sect. 6] and [40, Sect. 3.3]. It is a variational problem posed on $\boldsymbol{\mathcal{MT}}_0(\Sigma) \times \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$, whose underlying bilinear form we are going to denote by $\mathsf{m}_{\mathrm{LMF}}$:

$$\mathsf{m}_{\mathrm{LMF}}(\underline{\mathfrak{u}}, \underline{\mathfrak{v}}) := \sum_{j=0}^{N} \left\langle\!\left\langle \mathbb{A}_j\, \mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\partial\Omega_j} - \sum_{j=0}^{N} \sum_{i\in\mathcal{N}_j} \left\langle\!\left\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}} , \tag{5.12}$$

$\underline{\mathfrak{u}} = (\mathfrak{u}_0, \dots, \mathfrak{u}_N) \in \boldsymbol{\mathcal{MT}}_0(\Sigma)$, $\underline{\mathfrak{v}} = (\widetilde{\mathfrak{v}}_0, \dots, \widetilde{\mathfrak{v}}_N) \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$.

In light of (3.3) and (3.7) and the sign flip effected by $\mathbb{X}_{i\to j}$, the compact notation (5.11) can be unravelled into an explicit variational problem for Dirichlet and Neumann traces: seek $\underline{\mathfrak{u}} = \left(\binom{u_0}{\nu_0}, \dots, \binom{u_N}{\nu_N}\right) \in \boldsymbol{\mathcal{MT}}_0(\Sigma)$ such that

$$\begin{aligned} \left\langle \mathsf{V}_j \nu_j, \widetilde{\psi}_j \right\rangle_{\partial\Omega_j} &- \left\langle \mathsf{K}_j u_j, \widetilde{\psi}_j \right\rangle_{\partial\Omega_j} - \tfrac{1}{2} \sum_{i\in\mathcal{N}_j} \left\langle u_i|_{\Gamma_{ij}}, \widetilde{\psi}_j\Big|_{\Gamma_{ij}} \right\rangle_{\Gamma_{ij}} = \dots , \\ -\left\langle \mathsf{K}'_j \nu_j, \widetilde{v}_j \right\rangle_{\partial\Omega_j} &- \left\langle \mathsf{W}_j u_j, \widetilde{v}_j \right\rangle_{\partial\Omega_j} - \tfrac{1}{2} \sum_{i\in\mathcal{N}_j} \left\langle \nu_i|_{\Gamma_{ij}}, \widetilde{v}_j|_{\Gamma_{ij}} \right\rangle_{\Gamma_{ij}} = \dots , \end{aligned} \tag{5.13}$$

for all $\binom{\widetilde{v}_j}{\widetilde{\psi}_j} \in \mathcal{T}_{\mathrm{pw}}(\partial\Omega_j)$, $j = 0, \dots, N$, which satisfy $\widetilde{v}_j|_{\Gamma_{jZ}} = 0$. Please refer to (3.7), p. 9, for the definition of the boundary integral operators. The right hand side functionals $\Psi_j$ have been suppressed for the sake of brevity.

*Remark* 5.1. As explained in [39, Sect. 3.2.4], the MTF bilinear form $\mathsf{m}_{\mathrm{LMF}}$ from (5.12) will remain well defined, when *both arguments* belong to

$$\widetilde{\boldsymbol{\mathcal{MT}}}^{\sharp}_0(\Sigma) := \left\{ \left( \begin{pmatrix} v_j \\ \nu_j \end{pmatrix} \right)_{j=0}^{N} \in \boldsymbol{\mathcal{MT}}(\Sigma) : \begin{array}{l} v_j|_{\Gamma_{jZ}} = 0 , \\ \nu_j \in \widetilde{H}^{-\frac{1}{2}}_{\mathrm{pw}}(\partial\Omega_j) \end{array} \right\} , \tag{5.14}$$

where $\widetilde{H}^{-\frac{1}{2}}_{\mathrm{pw}}(\partial\Omega_j)$ comprises functions in $H^{-\frac{1}{2}}(\partial\Omega_j)$, whose restrictions to any interface $\Gamma_{ij}$ belong to $\widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})$: $\mathsf{m}_{\mathrm{LMF}} \in L(\widetilde{\boldsymbol{\mathcal{MT}}}^{\sharp}_0(\Sigma) \times \widetilde{\boldsymbol{\mathcal{MT}}}^{\sharp}_0(\Sigma), \mathbb{R})$. This paves the way to an alternative formulation of the local MTF with equal trial and test spaces [39, Sect. 3.2]. △

*Remark* 5.2. From [39, Sect. 3.2.5] recall that the variational formulation (3.20) of the STF can formally be obtained by restricting (5.11) to test and trial functions in $\boldsymbol{\mathcal{ST}}_0(\Sigma)$. △

### 5.3 Local MTF: Existence and uniqueness of solutions

The choice of the particular broken trace spaces was made to ensure the continuity of the MTF bilinear form.

**Lemma 5.1.** *The MTF bilinear form* $\mathsf{m}_{\text{LMF}} : \boldsymbol{\mathcal{MT}}_0(\Sigma) \times \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma) \to \mathbb{R}$ *from* (5.12) *is continuous.*

*Proof.* Thanks to the continuity properties of the boundary integral operators the first term in (5.12) is even continuous on $\boldsymbol{\mathcal{MT}}(\Sigma) \times \boldsymbol{\mathcal{MT}}(\Sigma)$.

The continuity of $\mathbb{S}_{ji} : \mathcal{T}(\partial\Omega_j) \to \mathcal{T}(\Gamma_{ij})$ and $\mathbb{S}_{ji} : \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_j) \to \widetilde{\mathcal{T}}(\Gamma_{ij})$ is clear. Besides, the $L^2(\Gamma_{ij})$-inner product can be extended to continuous pairings on $H^{\frac{1}{2}}(\Gamma_{ij}) \times \widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})$ and $\widetilde{H}^{\frac{1}{2}}(\Gamma_{ij}) \times H^{-\frac{1}{2}}(\Gamma_{ij})$. This ensures continuity of the second part of (5.12). $\square$

The following lemma reveals a particular "block skew-symmetric" structure of (5.11). It makes use of the isometric local "sign change isomorphisms" $\Xi_j : \mathcal{T}(\partial\Omega_j) \to \mathcal{T}(\partial\Omega_j)$, $\Xi_j \binom{v}{\varphi} := \binom{-v}{\varphi}$, introduced in Section 3.4, p. 12.

**Lemma 5.2** ([39, Sect 2.2.3, Lemma 1]). *For all* $\mathfrak{u}_i \in \mathcal{T}(\partial\Omega_i)$ *and* $\widetilde{\mathfrak{v}} \in \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_j)$ *holds*

$$\left\langle\!\left\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ij}\, \Xi_j \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}} = -\left\langle\!\left\langle \mathbb{X}_{j\to i}\, \widetilde{\mathfrak{v}}_j, \mathbb{S}_{ji}\, \Xi_i \mathfrak{u}_i \right\rangle\!\right\rangle_{\Gamma_{ij}} .$$

*Proof.* The identity follows from straightforward computations using the definitions of the operators and the pairings. We write $\mathfrak{u}_i = \binom{u}{\nu}$, $\widetilde{\mathfrak{v}}_j = \binom{\widetilde{v}}{\widetilde{\psi}}$ and find

$$\begin{aligned}
&\left\langle\!\left\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ij}\, \Xi_i \widetilde{\mathfrak{v}}_j \right\rangle\!\right\rangle_{\Gamma_{ij}} \\
&\quad = \left\langle\!\!\left\langle \mathbb{S}_{ij} \begin{pmatrix} u \\ -\nu \end{pmatrix}, \mathbb{S}_{ji} \begin{pmatrix} -\widetilde{v} \\ \widetilde{\psi} \end{pmatrix} \right\rangle\!\!\right\rangle_{\Gamma_{ij}} \\
&\quad\quad = \left\langle u, \widetilde{\psi} \right\rangle_{\Gamma_{ij}} - \langle \widetilde{v}, \nu \rangle_{\Gamma_{ij}} ,
\end{aligned}
\qquad
\begin{aligned}
&\left\langle\!\left\langle \mathbb{X}_{j\to i}\, \widetilde{\mathfrak{v}}_j, \mathbb{S}_{ji}\, \Xi_i \mathfrak{u}_i \right\rangle\!\right\rangle_{\Gamma_{ij}} \\
&\quad = \left\langle\!\!\left\langle \mathbb{S}_{ji} \begin{pmatrix} \widetilde{v} \\ -\widetilde{\psi} \end{pmatrix}, \mathbb{S}_{ij} \begin{pmatrix} -u \\ \nu \end{pmatrix} \right\rangle\!\!\right\rangle_{\Gamma_{ij}} \\
&\quad\quad = \langle \widetilde{v}, \nu \rangle_{\Gamma_{ij}} - \left\langle u, \widetilde{\psi} \right\rangle_{\Gamma_{ij}} .
\end{aligned}$$

$\square$

Very much in analogy to Theorem 4.1 for the global MTF, the relationship from Lemma 5.2 guarantees ellipticity of the bilinear form of (5.11) up to a simple local change of sign.

**Theorem 5.3** ("Ξ-ellipticity" of bilinear form for MTF). *There is a constant* $c > 0$ *such that*

$$\mathsf{m}_{\text{LMF}}(\underline{\mathfrak{v}}, \Xi\underline{\mathfrak{v}}) \geq c\, \|\underline{\mathfrak{v}}\|^2_{\boldsymbol{\mathcal{MT}}(\Sigma)} \quad \forall \underline{\mathfrak{v}} \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma) .$$

*Proof.* Throughout the proof we write $\underline{\mathfrak{v}} = (\widetilde{\mathfrak{v}}_0, \ldots, \widetilde{\mathfrak{v}}_N) = \left(\binom{\widetilde{v}_0}{\widetilde{\psi}_0}, \ldots, \binom{\widetilde{v}_N}{\widetilde{\psi}_N}\right) \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$.

❶ As a consequence of Lemma 5.2 all the “off-diagonal” coupling terms cancel and we end up with, *cf.* (3.23),

$$\mathsf{m}_{\mathrm{LMF}}(\underline{\mathfrak{v}},\Xi\underline{\mathfrak{v}}) = \sum_{j=0}^{N} \langle\!\langle \mathbb{A}_j\,\mathfrak{v},\Xi_j\mathfrak{v}\rangle\!\rangle_{\partial\Omega} = \sum_{j=0}^{N} \left\langle \mathsf{V}_j\widetilde{\psi}_j,\widetilde{\psi}_j\right\rangle_{\partial\Omega_j} + \langle \mathsf{W}_j\widetilde{v}_j,\widetilde{v}_j\rangle_{\partial\Omega_j}\ . \quad (5.15)$$

Then (3.24) immediately shows that $\mathsf{m}_{\mathrm{LMF}}(\underline{\mathfrak{v}},\Xi\underline{\mathfrak{v}}) \geq 0$ for all $\underline{\mathfrak{v}} \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$, because $\mathcal{T}_{\mathrm{pw}}(\partial\Omega_j) \subset \mathcal{T}(\partial\Omega_j)$. In addition, $\underline{\mathfrak{v}} \mapsto \mathsf{m}_{\mathrm{LMF}}(\underline{\mathfrak{v}},\Xi\underline{\mathfrak{v}})$ inherits $\boldsymbol{\mathcal{MT}}(\Sigma)$-coercivity (up to finite-dimensional perturbations) from the Calderón operators, recall (3.24) and (3.25).

❷ Next, single out a $\underline{\mathfrak{v}} \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$ such that $\mathsf{m}_{\mathrm{LMF}}(\underline{\mathfrak{v}},\Xi\underline{\mathfrak{v}}) = 0$. Then (5.15) and (3.24) imply that $\widetilde{\psi}_j = 0$ for all $j = 0,\dots,N$, and that $\widetilde{v}_j$ is *constant* on $\partial\Omega_j$, $j = 0,\dots,N$. So, $\underline{\mathfrak{v}}$ belongs to a space of finite dimension and it remains to show the injectivity of $\mathsf{m}_{\mathrm{LMF}}$ on this space.

To begin with, we conclude from (3.25) that $\widetilde{v}_0 = 0$. Moreover, if the subdomain $\Omega_j$ has a common interface with $\Omega_Z$, then the definition (5.10b) makes $\widetilde{v}_j$ vanish on $\partial\Omega_j \cap \partial\Omega_Z$ and, since it is constant on $\partial\Omega_j$, it has to vanish on the entire boundary $\partial\Omega_j$.

Now, assume that $\mathsf{m}_{\mathrm{LMF}}(\underline{\mathfrak{v}},\underline{\mathfrak{w}}) = 0$ for all $\underline{\mathfrak{w}} \in \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$. In particular, we can choose $\underline{\mathfrak{w}}$ such that its Neumann component is equal to 1 on the boundary of a single subdomain $\Omega_k$ and zero everywhere else. If $\Omega_k$ is adjacent to a subdomain $\Omega_l$, where $\widetilde{\mathfrak{v}}_l = 0$ is already known, the coupling terms enforce that $\widetilde{v}_k|_{\Gamma_{lk}} = 0$ and $\widetilde{\mathfrak{v}}_k$ has to be zero, too. Thus, we can work our way through all subdomains, because $\mathbb{R}^d \setminus \Omega_Z$ is supposed to be connected. This finally establishes $\underline{\mathfrak{v}} = 0$. □

Unfortunately, this theorem does not settle the issue of existence and uniqueness of solutions of (5.11), because we encounter a mismatch of spaces as observed in [39, Sect. 3.2.8]: $\mathsf{m}_{\mathrm{LMF}}$ is $\boldsymbol{\mathcal{MT}}_0(\Sigma)$-elliptic, but continuous only on $\boldsymbol{\mathcal{MT}}_0(\Sigma) \times \widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$, so that we cannot instantly conclude an inf-sup condition from Ξ-ellipticity. We have to rely on a more sophisticated result known as “Lion’s projection lemma” [54, Ch. III, Thm. 1.1], see also [23, Sect. 2] and [28, Sect. 2].

**Lemma 5.4** ([39, Lemma 9]). *Let $H$ be a Hilbert space and $V$ be a subspace of $H$ (not necessarily closed in $H$). Moreover, let $\mathsf{b} : H \times V \to \mathbb{R}$ be a bilinear form satisfying the following properties:*

(i) *For every $\varphi \in V$, the linear form $u \mapsto \mathsf{b}(u,\varphi)$ is continuous on $H$.*

(ii) *There exists $c > 0$ such that*

$$|\mathsf{b}(\varphi,\varphi)| \geq c\,\|\varphi\|_H^2\,, \quad \forall\,\varphi \in V\ . \quad (5.16)$$

*Then for each continuous linear form $l \in H'$, there exists $u_0 \in H$ such that*

$$\mathsf{b}(u_0,\varphi) \;=\; \langle l,\varphi\rangle_H \quad \forall\,\varphi \in V \quad \textit{and} \quad \|u_0\|_H \leq \frac{1}{c}\,\|l\|_{H'}\,. \quad (5.17)$$

Clearly, Theorem 5.3 suggests that we apply this lemma with $\mathsf{b} := \mathsf{m}_{\text{LMF}}$, $H := \mathcal{MT}_0(\Sigma)$, and $V := \widetilde{\mathcal{MT}}_0(\Sigma)$. We immediately conclude existence of solutions of the MTF variational problem (5.11).

Yet, as pointed out in [54, Ch. III, Rem. 31], despite (5.16), Lemma 5.4 does not ensure uniqueness of solutions. To obtain it, we resort to considerations that directly exploit the boundary integral equations.

**Theorem 5.5.** *Solutions of* (5.11) *are unique.*

*Proof.* As in the proof of [39, Thm. 9], we show that (5.11) with $\Psi_j = 0$ involves $\underline{\mathfrak{u}} = 0$. Hence, let $\underline{\mathfrak{u}} \in \mathcal{MT}_0(\Sigma)$ satisfy

$$\mathsf{m}_{\text{LMF}}(\underline{\mathfrak{u}}, \underline{\mathfrak{v}}) = 0 \quad \forall \mathfrak{v} \in \widetilde{\mathcal{MT}}_0(\Sigma) \ . \tag{5.18}$$

❶ We set $U_j := \mathbb{G}_j(\mathfrak{u}_j)$, $j = 0, \dots, N$ and from (3.6) we get

$$\mathbb{T}_j\, U_j := (\mathbb{A}_j + \tfrac{1}{2}\mathsf{Id})\mathfrak{u}_j \ . \tag{5.19}$$

On the other hand, from (5.18) and (5.11) we infer

$$\left\langle\!\!\left\langle (\mathbb{A}_j + \tfrac{1}{2}\mathsf{Id})\mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\!\right\rangle_{\partial\Omega_j} = \tfrac{1}{2}\left\langle\!\!\left\langle \mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\!\right\rangle_{\partial\Omega_j} + \tfrac{1}{2}\sum_{i\in\mathcal{N}_j} \left\langle\!\!\left\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\!\right\rangle_{\Gamma_{ij}} , \tag{5.20}$$

for all $\widetilde{\mathfrak{v}} \in \mathcal{T}_{\text{pw}}(\partial\Omega_j)$.

Now, we single out an interface $\Gamma_{ij}$, choose arbitrary functions $\widetilde{v} \in \widetilde{H}^{\frac{1}{2}}(\Gamma_{ij})$ and $\widetilde{\psi} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})$, and obtain the components of the test functions $\widetilde{\mathfrak{v}}_j \in \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_j)$ and $\widetilde{\mathfrak{v}}_i \in \widetilde{\mathcal{T}}_{\text{pw}}(\partial\Omega_i)$ by extending $\widetilde{v}$ and $\widetilde{\psi}$ by zero onto $\partial\Omega_j$ and $\partial\Omega_i$, respectively. For $\widetilde{\mathfrak{v}}_i$ we also change the sign of the Neumann component. For these special test functions (5.20) together with (5.19) yields

$$\begin{aligned}
\langle\!\langle \mathbb{T}_j\, U_j, \mathbb{S}_{ji}\, \mathfrak{v}_j \rangle\!\rangle_{\partial\Omega} &= \tfrac{1}{2}\left\langle\!\!\left\langle \mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\!\right\rangle_{\partial\Omega_j} + \tfrac{1}{2}\left\langle\!\!\left\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\!\right\rangle_{\Gamma_{ij}} , \\
\langle\!\langle \mathbb{X}_{i\to j}\, \mathbb{T}_i\, U_i, \mathbb{S}_{ji}\, \mathfrak{v}_j \rangle\!\rangle_{\partial\Omega} &= -\langle\!\langle \mathbb{T}_i\, U_i, \mathbb{S}_{ij}\, \mathfrak{v}_i \rangle\!\rangle_{\partial\Omega} \\
&= -\tfrac{1}{2}\left\langle\!\!\left\langle \mathfrak{u}_i, \widetilde{\mathfrak{v}}_i \right\rangle\!\!\right\rangle_{\partial\Omega_i} - \tfrac{1}{2}\left\langle\!\!\left\langle \mathbb{X}_{j\to i}\, \mathfrak{u}_j, \mathbb{S}_{ij}\, \widetilde{\mathfrak{v}}_i \right\rangle\!\!\right\rangle_{\Gamma_{ij}} .
\end{aligned}$$

Owing to the local support of the test functions, a closer inspection reveals that

$$\left\langle\!\!\left\langle \mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \right\rangle\!\!\right\rangle_{\partial\Omega_j} = -\left\langle\!\!\left\langle \mathbb{X}_{j\to i}\, \mathfrak{u}_j, \mathbb{S}_{ij}\, \widetilde{\mathfrak{v}}_i \right\rangle\!\!\right\rangle_{\Gamma_{ij}} \quad \text{and} \quad \left\langle\!\!\left\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \right\rangle\!\!\right\rangle_{\Gamma_{ij}} = -\left\langle\!\!\left\langle \mathfrak{u}_i, \widetilde{\mathfrak{v}}_i \right\rangle\!\!\right\rangle_{\partial\Omega_i} ,$$

which means that

$$\langle\!\langle \mathbb{T}_j\, U_j, \mathbb{S}_{ji}\, \mathfrak{v}_j \rangle\!\rangle_{\partial\Omega} = \langle\!\langle \mathbb{X}_{i\to j}\, \mathbb{T}_i\, U_i, \mathbb{S}_{ji}\, \mathfrak{v}_j \rangle\!\rangle_{\partial\Omega} \ . \tag{5.21}$$

As a consequence, we find that

$$\mathbb{T}_j\, U_j = \mathbb{X}_{i\to j}\, \mathbb{T}_i\, U_i \quad \text{on } \Gamma_{ij} \ . \tag{5.22}$$

In words, the function $U \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ that is obtained by patching together the $U_j$'s, satisfies the transmission conditions (2.3b). Moreover, the representation formula guarantees $\mathcal{L}_j U_j = 0$. Finally, using (5.18) and (5.20) with test functions supported on $\Gamma_{jZ} \subset \partial\Omega_Z$, confirms that $U|_{\partial\Omega_Z} = 0$. Summing up, $U$ solves (2.3) with zero Dirichlet data on $\partial\Omega_Z$. Uniqueness of solutions of (2.3) tells us that $U = 0$.

❷ Having established $U_j = 0$, combining (5.19) and (5.20) yields

$$\langle\!\langle \mathfrak{u}_j, \widetilde{\mathfrak{v}}_j \rangle\!\rangle_{\partial\Omega_j} = -\sum_{i \in \mathcal{N}_j} \langle\!\langle \mathbb{X}_{i\to j}\, \mathfrak{u}_i, \mathbb{S}_{ji}\, \widetilde{\mathfrak{v}}_j \rangle\!\rangle_{\Gamma_{ij}} \quad \forall \widetilde{\mathfrak{v}} \in \mathcal{T}_{\mathrm{pw}}(\partial\Omega_j)\ , \tag{5.23}$$

which means $\mathfrak{u}_j = -\,\mathbb{X}_{i\to j}\,\mathfrak{u}_i$ on $\Gamma_{ij}$. In words, the Cauchy data $\mathfrak{u}_j$ satisfy *sign-flipped transmission conditions* across interfaces $\Gamma_{ij}$.

❸ Taking the Cauchy trace $\mathbb{T}_j$ of $U_j = \mathbb{G}_j(\mathfrak{u}_j) = 0$ reveals that $\mathbb{P}_j\,\mathfrak{u}_j = 0$. Hence, using the fact that interior and exterior Calderón projectors add up to zero [39, Sect. 2.3.3] and Theorem 3.1 on the complement domain $\Omega_j^c := \mathbb{R}^d \setminus \Omega_j$, we find that $\mathfrak{u}_j = \mathbb{T}_j^c\, V_j$ for a function $V_j \in H_{\mathrm{loc}}(\Delta, \Omega_j^c)$ that satisfies $\mathcal{L}_j V_j = 0$ in $\Omega_j^c$ and appropriate decay conditions at $\infty$ for $j \neq 0$. Here, $\mathbb{T}_j^c$ is the Cauchy trace operator on $\Omega_j^c$.

We adapt an idea from the proof of [39, Thm. 9]: For a "sign vector"

$$\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_N) \in \{-1, +1\}^N,$$

we define the *multi-valued function*

$$V^{\boldsymbol{\sigma}} := \sigma_j V_j \quad \text{on } \Omega_j^c \quad , \quad V^{\boldsymbol{\sigma}} := V_0 \quad \text{on } \Omega_0^c\ . \tag{5.24}$$

Case (i): Assume that there is a sign vector $\boldsymbol{\sigma}$ such that $\sigma_i = -\sigma_j$ for every interface $\Gamma_{ij}$; we are dealing with a bipartite connected graph of complement subdomains (whose edges correspond to non-empty interfaces $\Gamma_{ij}$). Then, by virtue of (5.23), $V^{\boldsymbol{\sigma}}$ is a *multi-valued* solution of a transmission problem of the type (2.3) on $\bigcup_{j=0}^{N} \Omega_j^c$, which features the right decay conditions at $\infty$ and has zero Dirichlet boundary conditions on $\partial\Omega_Z$. Hence, $V^{\boldsymbol{\sigma}} = 0$, which implies $V_j = 0$, and, immediately, $\mathfrak{u}_j = 0$.

Case (ii): Assume that the graph of complement subdomains is not bipartite. Regard two complement domains $\Omega_j^c$ and $\Omega_i^c$ as "linked", if they share an interface and if $\sigma_i = -\sigma_j$. We denote by $V_\sharp^{\boldsymbol{\sigma}}$ the restriction of $V^{\boldsymbol{\sigma}}$ to the union of complement domains, for which there is a chain of links to $\Omega_0^c$. Again thanks to (5.23), This multi-valued function satisfies transmission conditions (2.3b) between linked complement domains.

Next, we appeal to the *strong unique continuation principle* for solutions of transmission problems of type (2.3) [34, Sect. 3.4.1]. It confirms that any two functions $V_\sharp^{\boldsymbol{\sigma}}$ agree on complement domains, on which they are both defined, because they are the same on $\Omega_0^c$. Now, if the graph is not bipartite, for some $k \in \{1, \ldots, N\}$ we can find two sign vectors $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$ such that $\sigma_k = -\sigma_k'$ and such that for both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ there is a chain of links from $\Omega_k^c$ to $\Omega_0^c$. We conclude that both $V^{\boldsymbol{\sigma}}|_{\Omega_k^c} = V^{\boldsymbol{\sigma}'}\Big|_{\Omega_k^c}$ (from unique continuation) and $V^{\boldsymbol{\sigma}}|_{\Omega_k^c} = -\left. V^{\boldsymbol{\sigma}'}\right|_{\Omega_k^c}$ (by definition), which implies $V_k = 0$. By unique continuation we then conclude that $V_j = 0$ for all $j = 0, \ldots, N$, and also $\mathfrak{u}_j = 0$. □

*Remark* 5.3. In Remark 5.1 we pointed out that the variational formulation (5.11) of the classical MTF may be lifted to trace spaces $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$, for which additional regularity is imposed on the Neumann components only.

Now assume Dirichlet data $g \in H^1(\partial\Omega_Z)$. Then elliptic regularity results [56, Ch. 4] ensure that all local Neumann traces $\mathsf{T}_{N,j}\,U$ and $\mathsf{T}_{N,Z}\,U$ of the solution $U$ of the transmission problem (2.3) belong to $L^2(\partial\Omega_j)$, $j = 1, \ldots, N, Z$.

As a consequence the unique solution of (5.11) is contained in the space $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$ from (5.14), Page 22. Thus, the solution of (5.11) will be preserved when we switch to the trial space $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$. The Neumann components of Cauchy traces in that space possess more regularity compared to those of $\boldsymbol{\mathcal{MT}}_0(\Sigma)$. This enables us to use test functions with less regular Dirichlet components than stipulated by $\widetilde{\boldsymbol{\mathcal{MT}}}_0(\Sigma)$. More precisely, testing with functions merely belonging to $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$ becomes possible, which yields a variant of (5.11) with the same trial and test space, *cf.* [39, Sect. 3.2.1]. △

## 5.4 Local MTF: Operator preconditioning in 3D

The developments are largely parallel to that of Section 4.4 and we reuse the notations introduced there mostly without further mention. In the context of Galerkin boundary element discretization it is advisable to adopt the perspective of Remarks 5.1 and 5.3, and lift the MTF variational problem (5.11) into the space $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$. Then, since piecewise polynomial approximation invariably provides functions in $\widetilde{H}_{\mathrm{pw}}^{-\frac{1}{2}}(\partial\Omega_j)$, we deal with a $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$-conforming boundary element Galerkin approximation.

In order to take into account the Dirichlet boundary conditions, we rely on the boundary element spaces ($j = 0, \ldots, N$)

$$S_Z^0(\mathcal{G}_j) := \{v_h \in S^0(\mathcal{G}_j) : \; v_h|_{\partial\Omega_Z} = 0\} \;, \tag{5.25}$$

$$\mathcal{T}_{h,Z}(\partial\Omega_j) := S_Z^0(\mathcal{G}_j) \times S^{-1}(\mathcal{G}_j) \subset H^{\frac{1}{2}}(\partial\Omega_j) \times \widetilde{H}_{\mathrm{pw}}^{-\frac{1}{2}}(\partial\Omega_j) \;. \tag{5.26}$$

to build the trial and test space for (5.11):

$$\boldsymbol{\mathcal{MT}}_{0,h}(\{\mathcal{G}_j\}) := \mathcal{T}_{h,Z}(\partial\Omega_0) \times \cdots \times \mathcal{T}_{h,Z}(\partial\Omega_N) \subset \widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma) \;. \tag{5.27}$$

Existence, uniqueness, and convergence of Galerkin solutions in the absence of the impenetrable object $\Omega_Z$ have been established in [39, Section 4].

From (5.27) we identify the spaces $\mathcal{T}_{h,Z}(\partial\Omega_j)$ as the $X_j$'s in the product space setting for operator preconditioning (see the preface to Section 4 on Page 14). As in the case of the global MTF discussed in Section 4.4, the bilinear forms $\mathsf{b}_j$ are chosen according to (4.13), and the $Y_j$ are boundary element spaces on dual meshes $\widehat{\mathcal{G}}_j$, $j = 0, \dots, N$, arising from barycentric refinement:

$$Y_j := S^0(\widehat{\mathcal{G}}_j) \times S_0^{-1}(\widehat{\mathcal{G}}_j) \subset H^{\frac{1}{2}}(\partial\Omega_j) \times H^{-\frac{1}{2}}(\partial\Omega_j) \ . \tag{5.28}$$

Note that the $\widehat{\mathcal{G}}_j$-piecewise constant functions in the spaces $S_0^{-1}(\mathcal{G}_j)$ have to vanish on dual cells associated with nodes of $\mathcal{G}_j$ that are located on $\partial\Omega_Z$. This ensures equal dimensions of $S_Z^0(\mathcal{G}_j)$ and $S_0^{-1}(\mathcal{G}_j)$ and follows the policy of [8, Sect. 4.2]. The local pairing bilinear forms $\mathsf{d}_j$ are again chosen as duality pairings $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\partial\Omega_j}$.

The analysis of operator preconditioning is slightly more difficult than in Section 4.4, because the local MTF bilinear form $\mathsf{m}_{\mathrm{LMF}}$ enjoys ellipticity in $\boldsymbol{\mathcal{MT}}_0(\Sigma)$, recall Theorem 5.3, but is continuous only on $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$ as defined in (5.14). The norm of the latter space,

$$\|\underline{\mathfrak{v}}\|^2_{\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)} := \sum_{j=0}^{N} \Big( \|v_j\|^2_{H^{\frac{1}{2}}(\partial\Omega_j)} + \sum_{i\in\mathcal{N}_j} \Big\| \nu_j|_{\Gamma_{ij}} \Big\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})} \Big) \ , \tag{5.29}$$

is stronger than that of the former. This mismatch compounded the theoretical difficulties encountered in Section 5.3 and thwarts the straightforward application of Theorem 1.1.

To cope with this situation, we have to make another assumption concerning the meshes: we assume that each interface $\Gamma_{ij}$ is resolved by cells of the meshes $\mathcal{G}_j$ and $\mathcal{G}_i$. The same should apply to $\mathcal{G}_j$ and the boundary parts $\Gamma_{jZ}$.

**Theorem 5.6.** *In the setting detailed above let us consider families of surface meshes generated by* regular refinement *of coarse initial meshes. Then the spectral condition numbers of the preconditioned linear systems grow moderately like* $O(L^{\frac{3}{2}}) = O(|\log h|^{\frac{3}{2}})$ *in the level* $L$ *of refinement as* $L \to \infty$ *(*$h \to 0$*).*

*Proof.* We observe that the inf-sup constants and norms entering the bound in the estimate (1.2) of Theorem 1.1 refer to the discrete setting. In it all norms are equivalent and we can resort to the $\boldsymbol{\mathcal{MT}}(\Sigma)$-norm throughout.

On boundary element spaces we have to use *inverse inequalities* to relate the norms of $\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)$ and $\boldsymbol{\mathcal{MT}}(\Sigma)$. In particular, we appeal to the estimate of [41, Thm. 2.2]

$$\|\nu_h\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_{ij})} \leq CL^{\frac{3}{2}} \|\nu_h\|_{H^{-\frac{1}{2}}(\Gamma_{ij})} \quad \forall \nu_h \in S^{-1}(\mathcal{G}_j) \ , \tag{5.30}$$

when $\mathcal{G}_j$ is on level $L$ of the refinement hierarchy ($C > 0$ is constant that depends only on the geometry of $\Gamma_{ij}$ and the coarsest mesh). This implies, with another constant $C > 0$ independent of the level $L$ of refinement

$$\|\underline{\mathfrak{v}}_h\|_{\widetilde{\boldsymbol{\mathcal{MT}}}{}_0^\sharp(\Sigma)} \leq CL^{\frac{3}{2}} \|\underline{\mathfrak{v}}_h\|_{\boldsymbol{\mathcal{MT}}(\Sigma)} \quad \forall \underline{\mathfrak{v}}_h \in \boldsymbol{\mathcal{MT}}_{0,h}(\{\mathcal{G}_j\}) \ . \tag{5.31}$$

As a consequence, the norm of the bilinear form $\mathsf{a}$ on $\mathcal{MT}_{0,h}(\{\mathcal{G}_j\})$, when measured in the weaker $\mathcal{MT}(\Sigma)$-norm, increases mildly like $O(L^{\frac{3}{2}})$, when we keep on refining the meshes and send $L \to \infty$. Since all other norms and inf-sup constants do not depend on $L$, see [8] for results on the pairing bilinear forms $\mathsf{d}_j$, this accounts for the assertion of the theorem. □

# 6 Numerical Studies

In this section, we report numerical tests of the performance of operator preconditioning for the *local MTF* implementing the algorithms described in Section 5.4. All computations were carried out with the C++ boundary element template library BETL [42][5]. The matrices arising from Galerkin discretization were subject to local low-rank compression using the AHMED library[6] [5]. However, the compression parameters were chosen so that its impact can safely be ignored.

Due to its saddle point structure, the operator preconditioner based on (4.13) fails to be positive definite. Thus, the conjugate gradient method (CG) is not an option and the preconditioned generalized minimal residual method (GMRES) without restart is used as an iterative solver. The iterations are stopped as soon as a relative decrease of the Euclidean norm of the residual vector by a factor of $10^{-7}$ was achieved. Initial guess is zero throughout.

Quasi-uniform and shape-regular sequences (finite, of course) of triangular surface meshes with flat triangles are used for all experiments. They were produced by the mesh generator Gmsh[7] [29] and all of them are compatible with the interfaces.

In all experiments the behavior of the GMRES iterative solver without preconditioner and with operator preconditioning on meshes of different resolution is recorded. Total iteration counts versus global mesh widths are tabulated.

## 6.1 Experiment I: Two half-spheres

The first experiment uses $\Omega_Z = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| < 1,\, x_3 > 0\}$, $\Omega_1 = \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| < 1,\, x_3 < 0\}$, and Dirichlet data $g(\boldsymbol{x}) := \|\boldsymbol{x}\|^{-1}\big|_{\partial\Omega_Z}$. The geometric situation is displayed in Figure 4. Measured data on GMRES convergence are given in Figure 5 and Table 2.

[5] http://www.sam.math.ethz.ch/betl/
[6] http://bebendorf.ins.uni-bonn.de/AHMED.html
[7] http://geuz.org/gmsh/

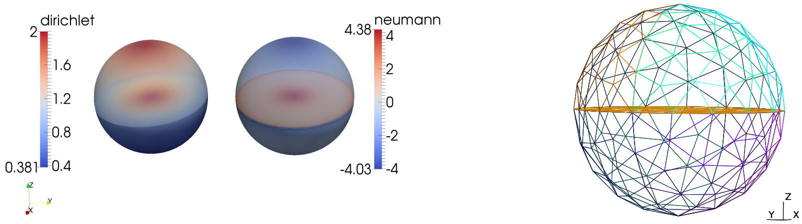
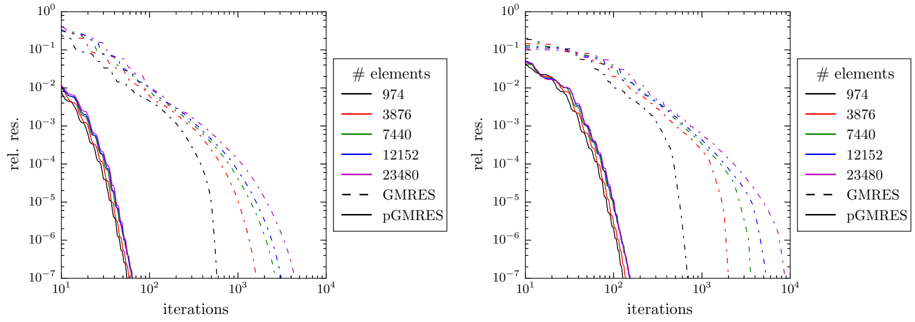



Figure 4: **Experiment I** (Two half-spheres): Dirichlet and Neumann traces of the solution (left) for $\mu_0 = 10, \mu_1 = 1$, 23480 elements, and coarse mesh with 296 flat triangles (right)




(a) $\mu_0 = 1.5$, $\mu_1 = 1$ (b) $\mu_0 = 10$, $\mu_1 = 1$

Figure 5: **Experiment I**: (preconditioned) GMRES iteration history; decay of relative Euclidean norm of the residual. Different diffusion coefficients were used on different subdomains.

| | | $\mu_0 = 1.5,\ \mu_1 = 1$ | | $\mu_0 = 10,\ \mu_1 = 1$ | |
|---|---|---|---|---|---|
| $N$ | mesh width | GMRES | pGMRES | GMRES | pGMRES |
| 974 | $1{,}25 \cdot 10^{-1}$ | 581 | 56 | 688 | 128 |
| 3876 | $6{,}18 \cdot 10^{-2}$ | 1617 | 59 | 1999 | 139 |
| 7440 | $4{,}50 \cdot 10^{-2}$ | 2668 | 65 | 3574 | 152 |
| 12152 | $3{,}52 \cdot 10^{-2}$ | 3098 | 64 | 5366 | 153 |
| 23480 | $2{,}52 \cdot 10^{-2}$ | 4426 | 64 | 8728 | 155 |

Table 2: **Experiment I**: Iteration counts for (preconditioned) GMRES for different choices of diffusion coefficients, $N$ denotes no. of elements

## 6.2 Experiment II: Glued boxes

The geometry of the second experiment is $\Omega_Z = \{\boldsymbol{x} \in \mathbb{R}^3 : -\frac{1}{2} \leq x_1, x_3 \leq \frac{1}{2}, -1 \leq x_2 \leq 0\}$, $\Omega_1 = \{\boldsymbol{x} \in \mathbb{R}^3 : -\frac{1}{2} \leq x_1, x_3 \leq \frac{1}{2},\ 0 \leq x_2 \leq 1\}$, with Dirichlet data $g(\boldsymbol{x}) := \left.\|\boldsymbol{x}\|^{-1}\right|_{\partial\Omega_Z}$, see Figure 6. GMRES behavior is documented in Figure 7 and Table 3.

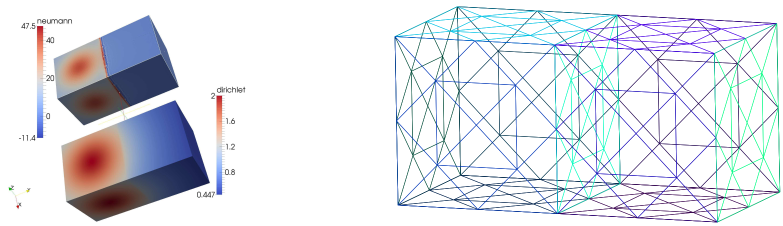
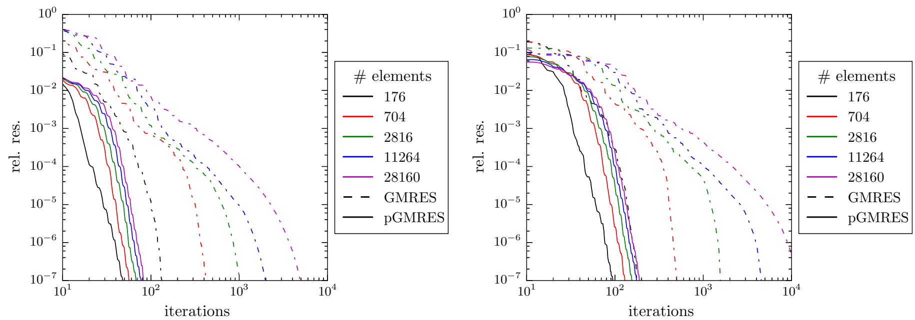



Figure 6: **Experiment II**: Left: Dirichlet (front) and Neumann traces (rear) of the solution. Right: coarsest mesh with 176 triangles




(a) $\mu_0 = 1.5$, $\mu_1 = 1$ (b) $\mu_0 = 10$, $\mu_1 = 1$

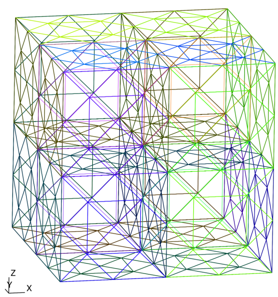
Figure 7: **Experiment II**: (preconditioned) GMRES iteration history. Diffusion coefficients attain different values on different subdomains.

| | | $\mu_0 = 1.5,\ \mu_1 = 1$ | | $\mu_0 = 10,\ \mu_1 = 1$ | |
|---|---|---|---|---|---|
| $N$ | mesh width | GMRES | pGMRES | GMRES | pGMRES |
| 176 | $2{,}64 \cdot 10^{-1}$ | 133 | 49 | 179 | 95 |
| 704 | $1{,}32 \cdot 10^{-1}$ | 422 | 57 | 501 | 130 |
| 2816 | $6{,}59 \cdot 10^{-2}$ | 1017 | 68 | 1564 | 156 |
| 11264 | $3{,}30 \cdot 10^{-2}$ | 1982 | 78 | 4552 | 176 |
| 28160 | $2{,}14 \cdot 10^{-2}$ | 4891 | 83 | 10961 | 189 |

Table 3: **Experiment II**: Iteration counts for (preconditioned) GMRES, $N$: # elements

### 6.3 Experiment III: Cube split into eight smaller cubes

The subdomains $\Omega_i$, $i = 1, \ldots, 8$, are the equal cubes of edge length $\frac{1}{2}$ created by splitting the unit cube. $\partial\Omega_z$ is centered at the origin, Dirichlet data are $g(\boldsymbol{x}) := \|\boldsymbol{x}\|^{-1}\big|_{\partial\Omega_Z}$, and the same diffusion coefficient $\mu_i = 1$ was used on all subdomains, see Figure 8. Information about the convergence of GMRES is provided in Figure 8 and Table 4



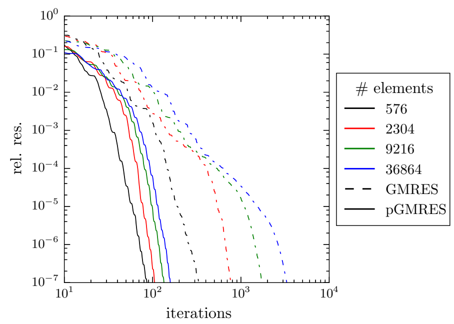




Figure 8: **Experiment III**: Coarsest mesh (left) and behavior of relative residual during (preconditioned) GMRES iterations (right).

| | | | | |
|---|---|---|---|---|
| $N$ | 576 | 2304 | 9216 | 36864 |
| mesh width | $1{,}32 \cdot 10^{-1}$ | $6{,}59 \cdot 10^{-2}$ | $3{,}30 \cdot 10^{-2}$ | $1{,}65 \cdot 10^{-2}$ |
| GMRES | 339 | 765 | 1728 | 3304 |
| pGMRES | 87 | 106 | 135 | 159 |

Table 4: **Experiment III**: Iteration counts for (preconditioned) GMRES vs. Number of triangles

**Summary of observations.** Obviously, in all numerical tests operator preconditioning substantially accelerates the convergence of GMRES. A moderate dependence of the number of iterations on the mesh width seems to persist, at least in Experiments II and III. We point out that matching the empiric data with the theoretical predictions of Theorem 5.6 is problematic, because (i) pre-asymptotic behavior may prevail in numerical experiments, (ii) logarithmic factors are hard to tell from measured data, and (ii) convergence rates of GMRES do not seem to be governed by the spectral condition number, but by the numerical range of the non-symmetric preconditioned system matrix [6, 27].

The data collected hint at a strong dependence of the iteration counts on the relative variation of the diffusion coefficients between the subdomains; Operator preconditioning as described in Section 5.4 does not seem to be robust with respect to the size of jumps of the coefficients.

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## Author information

Xavier Claeys, LJLL, UPMC Paris, France.
E-mail: `claeys@ann.jussieu.fr`

Ralf Hiptmair, SAM, ETH Zurich, Switzerland.
E-mail: `hiptmair@sam.math.ethz.ch`

Carlos Jerez-Hanckes, Escuela de Ingeniería, Pontificia Universidad Católica de Chile, Santiago, Chile.
E-mail: `cjerez@ing.puc.cl`

Simon Pintarelli, SAM, ETH Zurich, Switzerland.
E-mail: `pintarelli@sam.math.ethz.ch`

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