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A single trace integral formulation of the second kind for acoustic scattering

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Abstract

We study the scattering of acoustic waves by an object composed of several adjacent parts with different material properties. For this problem we derive an integral formulation of the second kind. This formulation only involves one Dirichlet datum and one Neumann datum at each point of each interface of the object, so that our formulation can be considered to belong to the same family as the formulation of the first kind that was analyzed by von Petersdorff in [16] for scalar problems and by Buffa in [3] for Maxwell's equations.

The simulation of wave propagation in a medium with piecewise constant wave number is of practical interest in many applications related to acoustics and electromagnetics. To tackle this type of problem one possible approach consists of formulating the problem as an integral equation. As regards integral formulations though, most of the literature deals with geometries where at most two different media of propagation are adjacent to each other. However, in practice, there are many relevant geometrical configurations where three or more different media are adjacent to each other; what we call multiple subdomain scattering is the study of wave propagation in arrangements with this type of geometry.

Concerning multiple subdomain scattering, a first approach derived from Rumsey's principle, was analyzed by von Petersdorff in [16] for scalar problems. This analysis was extended to Maxwell's equations by Buffa in [3]. In this approach the transmission conditions are taken into account directly via the choice of well chosen variational spaces. Such a formulation turns out to be the generalization for multiple subdomain configurations of the classical first kind formulation well known for transmission problems where interfaces separate at most two different media. One interesting feature of this formulation is that, at each point of each interface, it involves only one Dirichlet datum and one Neumann datum. As a consequence, we call it single trace formulation of the first kind. However no efficient preconditionner has been proposed so far for this formulation in the case of multiple subdomain scattering.

In [8], Steinbach and co-workers developped another formulation of the first kind involving only one Dirichlet datum and two Neumann data at each interface. Several variants of this formulation were proposed later, see [11, 19] and references therein. It consists in a domain decomposition approach where part of the transmission conditions are imposed by means of Lagrange multipliers. Such a method can be readily preconditionned. However it requires the inversion of a Steklov-Poincare operator in each subdomain.

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More recently, in [7], Hiptmair and Jerez-Hanckes developped yet another integral formulation of the first kind that has also good properties in terms of preconditionning possibilities. This approach is different since transmission conditions are not imposed through Lagrange multipliers. The authors named this formulation multitrace formulation, as all unknows of the problem are doubled on each interface. This formulation does not require the inversion of any Steklov-Poincare operator, although preconditioning it requires the solution to integral equations local to each subdomain.

In the present report, we propose yet another single trace formulation, that generalizes second kind formulations i.e. the so-called Rokhlin-Müller scheme, already well established for the case of a transmission problems for a single homogeneous scatterer, see [10, 14, 17, 1]. An important advantage of such an approach is that second kind formulations are intrisically well conditionned provided that the identity operator can be discretized in a stable manner.

This paper is structured as follows. First of all we describe the problem that we consider. It consists in the scattering of a scalar wave in a medium with piecewise constant caracteristics. In Section 2 we introduce a functional setting that is well adapted to our problem. In Section 3 we recall some classical result about integral formulations. In Section 4 we provide a brief review of the formulation that was introduced by VonPetersdorff in [16]. In Section 5 we introduce an integral operator that will be the central ingredient in our formulation. We show in particular that this operator can be used to caracterize Cauchy data associated to our problem, at least in the case where contrasts between wave numbers are small enough. In Section 6 we derive an integral formulation of the form "identity+compact" for our problem. In Section 7 we examine this formulation in the case of only two subdomains and one interface, and we show that we recover the classical second kind formulation. In the last section we present the results of a toy numerical experiment in a 2-D situation.

1 Setting of the problem

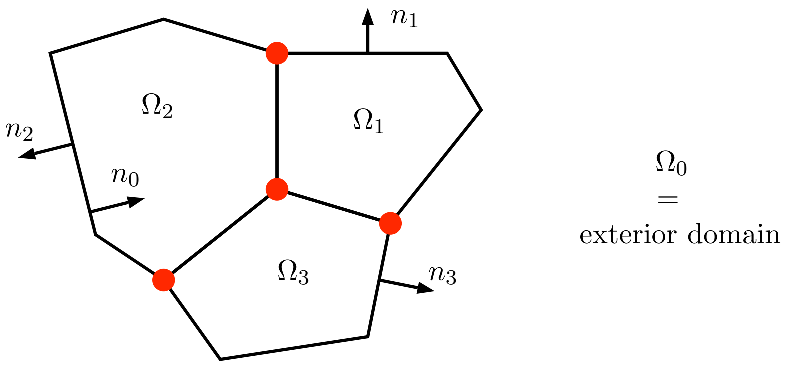
We consider a partition $\mathbb{R}^d = \cup_{i=0}^n \overline{\Omega}_i$ where $\cup_{i=1}^n \overline{\Omega}_i$ is bounded and each Ω_i is a connnected open Lipschitz subset. We also set $\Gamma = \cup_{i=1}^n \partial\Omega_i$. Note that there may exist points where three or more sub-domains would be contiguous, which is precisely the situation that we wish to tackle. For each i the vector n_i refers to the normal vector on $\partial\Omega_i$ directed toward the exterior of Ω_i.

Elementary functional spaces Let us recall the definition of some elementary functional spaces. For an open set $\omega \subset \mathbb{R}^d$, the space $\mathrm{L}^2(\omega)$ will refer to the set of measurable functions v such that $\|v\|^2_{\mathrm{L}^2(\omega)} = \int_\omega |v|^2 d\mathbf{x} < +\infty$, and we set

$$\mathrm{H}^1(\omega) = \left\{ v \in \mathrm{L}^2(\omega) \;\middle|\; \|v\|^2_{\mathrm{H}^1(\omega)} = \int_\omega |v|^2 + |\nabla v|^2 d\mathbf{x} < +\infty \right\}$$

$$\mathrm{H}(\mathrm{div},\omega) = \left\{ \mathbf{q} \in \mathrm{L}^2(\omega)^3 \;\middle|\; \|\mathbf{q}\|^2_{\mathrm{H}(\mathrm{div},\omega)} = \int_\omega |\mathbf{q}|^2 + |\mathrm{div}(\mathbf{q})|^2 d\mathbf{x} < +\infty \right\}$$

We also define the space $\mathrm{H}^1(\Delta,\omega) = \{\, v \in \mathrm{H}^1(\omega) \mid \nabla v \in \mathrm{H}(\mathrm{div},\omega) \,\}$ equipped with the norm $\|u\|^2_{\Delta,\omega} = \|u\|^2_{\mathrm{H}^1(\omega)} + \|\Delta u\|^2_{\mathrm{L}^2(\omega)}$. If $\mathrm{H}(\omega)$ is any one of the spaces $\mathrm{H}^1(\omega), \mathrm{H}(\mathrm{div},\omega)$ or $\mathrm{H}^1(\Delta,\omega)$, then we set $\mathrm{H}_{\mathrm{loc}}(\overline{\omega}) = \{\, v \text{ such that } \varphi v \in \mathrm{H}(\omega)\ \forall \varphi \in \mathscr{D}(\mathbb{R}^d) \,\}$ where $\mathscr{D}(\mathbb{R}^d)$ refers to the set of compactly supported C^∞ functions.



Trace spaces For an open set $\omega \subset \mathbb{R}^d$ with Lipschitz boundary, it is well known that the trace $v \mapsto v|_{\partial\omega}$ can be extended to a continuous map from $\mathrm{H}^1(\omega)$ into $\mathrm{L}^2(\partial\omega)$, and that the space $\mathrm{H}^{\frac{1}{2}}(\partial\omega) = \{\, v|_{\partial\omega} \mid v \in \mathrm{H}^1(\omega) \,\}$ equipped with the norm

$$\|v\|_{\mathrm{H}^{\frac{1}{2}}(\partial\omega)} = \inf \Big\{ \; \|u\|_{\mathrm{H}^1(\omega)} \;\text{ such that }\; u \in \mathrm{H}^1(\omega), u|_{\partial\omega} = v \; \Big\},$$

is a Banach space. Let us denote $\mathrm{H}^{-1/2}(\partial\omega)$ its topological dual, that we equip with the corresponding dual norm

$$\|q\|_{\mathrm{H}^{-\frac{1}{2}}(\partial\omega)} = \sup_{v \in \mathrm{H}^{\frac{1}{2}}(\partial\omega)} \frac{1}{\|v\|_{\mathrm{H}^{\frac{1}{2}}(\partial\omega)}} \Big| \int_{\partial\omega} v\, q\, d\sigma \Big| \,.$$

If n refers to the normal vector field over $\partial\omega$, it is also well known that $\mathbf{q} \mapsto n \cdot \mathbf{q}|_{\partial\omega}$ defines a continuous map from $\mathrm{H}(\mathrm{div}, \omega)$ onto $\mathrm{H}^{-1/2}(\partial\omega)$. Moreover the trace $n \cdot \nabla v|_{\partial\omega}$ is well defined whenever $v \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\omega})$.

Trace operators For $i = 0 \dots n$ and $v \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_i)$ define $\gamma^i_{\mathrm{D}}(v) = v|_{\partial\Omega_i}$ and $\gamma^i_{\mathrm{N}}(v) = n_i \cdot \nabla v|_{\partial\Omega_i}$ where the Dirichlet and Neumann traces are taken from the *interior* of Ω_i. The exterior Dirichlet and Neumann traces on $\partial\Omega_i$ will be denoted $\gamma^i_{\mathrm{D},c}$ and $\gamma^i_{\mathrm{N},c}$ (with normal vector directed toward the exterior of Ω_i). We also consider mean and jump combinations of these operators

$$\begin{aligned} &[\gamma^j_{\mathrm{D}}] = \gamma^j_{\mathrm{D}} - \gamma^j_{\mathrm{D},c} \qquad \text{and} \qquad [\gamma^j_{\mathrm{N}}] = \gamma^j_{\mathrm{N}} - \gamma^j_{\mathrm{N},c}, \\ &\{\gamma^j_{\mathrm{D}}\} = \frac{1}{2}(\gamma^j_{\mathrm{D}} + \gamma^j_{\mathrm{D},c}) \qquad \text{and} \qquad \{\gamma^j_{\mathrm{N}}\} = \frac{1}{2}(\gamma^j_{\mathrm{N}} + \gamma^j_{\mathrm{N},c}). \end{aligned} \tag{1}$$

The problem that we study Let $u_{\mathrm{inc}} \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ satisfy $\Delta u_{\mathrm{inc}} + \kappa_0^2\, u_{\mathrm{inc}} = 0$ in $\mathbb{R}^d$ for some $\kappa_0 \in \mathbb{R}$. This function will plays the role of incident field. In the present report we study the following problem:

$$\text{Find } \; u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d) \; \text{ such that} \tag{2}$$

$$\begin{cases} \Delta u + \kappa_i^2 u = 0 \quad \text{in } \Omega_i\,, \quad i = 0,\dots n \\ u - u_{\text{inc}} \;\text{ outgoing radiating in } \Omega_0 \end{cases} \tag{3}$$

where each $\kappa_i \in \mathbb{R}_+$ refers to the wave number inside Ω_i. In Equation (3), the outgoing radiation condition refers to the standard Sommerfeld radiation condition, see [13, 5]. It can be reformulated as

$$\lim_{r\to\infty} \int_{\partial \mathrm{B}_r} |\partial_r u - i\kappa_0 u|^2 d\sigma_r = 0 \quad \text{with} \quad r = |\mathbf{x}|\,. \tag{4}$$

where $\mathrm{B}_r = \{\mathbf{x} \in \mathbb{R}^d |\ |\mathbf{x}| < r\}$. Problem (2)-(3) is a transmission problem whose well posedness is a classical result, see [13, 5] for exemple. The transmission conditions are imposed through Equation (2). Indeed $u \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \mathbb{R}^d)$ implies that $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla u \in \mathrm{H}_{\mathrm{loc}}(\mathrm{div}, \mathbb{R}^d)$, which implies that u has no Dirichlet jump across any interface $\partial\Omega_k \cap \partial\Omega_j$, $j, k = 0\dots n$, and that ∇u has no normal jump across such interfaces.

2 Adapted functional spaces

In this paragraph we adopt a perspective of trace spaces that seems less standard but that appears much more convenient for the integral formulation that we consider in Section 4 and 6.

Multi trace space In order to reformulate Problem (3) as an integral equation posed over Γ, a natural functional setting consists in taking cartesian products of trace spaces, namely

$$\mathbb{H}(\Gamma) = \prod_{j=0}^{n} \Big[\mathrm{H}^{\frac{1}{2}}(\partial\Omega_j) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)\Big] \quad \text{equipped with the norm}$$

$$\|U\| = \Big(\sum_{j=0}^{n} \|u_j\|^2_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega_j)} + \|p_j\|^2_{\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)} \Big)^{\frac{1}{2}} \quad \text{when } U = (u_j, p_j)_{0\le j\le n}$$

It is a Hilbert space. Observe that this space can be identified to its own dual by means of the following duality pairing

$$\mathrm{B}(U, V) = \sum_{i=0}^{n} \int_{\partial\Omega_i} u_i\, \overline{q_i}\, d\sigma + \int_{\partial\Omega_i} p_i\, \overline{v_i}\, d\sigma$$
$$\text{where} \quad U = (u_j, p_j)_{0\le j\le n} \in \mathbb{H}(\Gamma) \quad \text{and} \quad V = (v_j, q_j)_{0\le j\le n} \in \mathbb{H}(\Gamma)$$

This bilinear form is non-degenerate: if $\mathrm{B}(U, V) = 0\ \forall V \in \mathbb{H}(\Gamma)$ then $U = 0$.

Single trace space Now we introduce spaces that seem more adapted to the treatment of transmission conditions. This setting is inspired by [2]. We set

$$\mathbb{X}^{+\frac{1}{2}}(\Gamma) = \Big\{ (v_i) \in \prod_{i=0}^{n} \mathrm{H}^{\frac{1}{2}}(\partial\Omega_i) \;\Big|\; \exists v \in \mathrm{H}^1(\mathbb{R}^d) \text{ with } v|_{\partial\Omega_j} = v_j,\ \forall j = 0\dots n \Big\}$$
$$\mathbb{X}^{-\frac{1}{2}}(\Gamma) = \Big\{ (q_i) \in \prod_{i=0}^{n} \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_i) \;\Big|\; \exists \mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d) \text{ with } n_j \cdot \mathbf{q}|_{\partial\Omega_j} = q_j,\ \forall j = 0\dots n \Big\}$$
$$\mathbb{X}(\Gamma) = \Big\{ (v_j, q_j)_{0\le j\le n} \in \mathbb{H}(\Gamma) \;\Big|\; (v_j) \in \mathbb{X}^{+\frac{1}{2}}(\Gamma) \text{ and } (q_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma) \Big\}\,.$$

To get an intuition of these spaces, observe that in the case where $\mathbb{R}^d = \overline{\Omega}_0 \cup \overline{\Omega}_1$ so that $\Gamma = \partial\Omega_0 = \partial\Omega_1$, there holds $\mathbb{X}(\Gamma) = \{\ (v, q, v, -q) \mid v \in \mathrm{H}^{1/2}(\Gamma),\ q \in \mathrm{H}^{-1/2}(\Gamma)\ \}$.

Proposition 2.1.
Let $(u_j) \in \Pi_{j=0}^{n}\mathrm{H}^{+\frac{1}{2}}(\partial\Omega_j)$ and $(p_j) \in \Pi_{j=0}^{n}\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)$. We have

$$\begin{aligned}
&\text{i)}\quad (u_j) \in \mathbb{X}^{+\frac{1}{2}}(\Gamma) \quad\Longleftrightarrow\quad \sum_{j=0}^{n}\int_{\partial\Omega_j} u_j q_j\, d\sigma = 0 \quad \forall (q_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma)\\
&\text{ii)}\quad (p_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma) \quad\Longleftrightarrow\quad \sum_{j=0}^{n}\int_{\partial\Omega_j} p_j v_j\, d\sigma = 0 \quad \forall (v_j) \in \mathbb{X}^{+\frac{1}{2}}(\Gamma)
\end{aligned}$$

Proof:
We only present the proof for i) since the proof for ii) is very similar. First assume that $(u_j) \in \mathbb{X}^{1/2}(\Gamma)$ and take $u \in \mathrm{H}^1(\mathbb{R}^d)$ such that $u|_{\partial\Omega_j} = u_j$, $j = 0 \dots n$. Since Γ is bounded, we may assume that $\mathrm{supp}(u)$ is bounded, using a cut-off function if necessary. Consider an arbitrary $(q_j) \in \mathbb{X}^{-1/2}(\Gamma)$ and take $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$ such that $n_j \cdot \mathbf{q}|_{\partial\Omega_j} = q_j$. Applying Green's Formula, and taking into account that $\mathrm{supp}(u)$ is bounded, we obtain

$$\sum_{j=0}^{n}\int_{\partial\Omega_j} u_j q_j d\sigma = \sum_{j=0}^{n}\int_{\Omega_j} u\,\mathrm{div}(\mathbf{q}) + \nabla u \cdot \mathbf{q}\, d\mathbf{x} = \int_{\mathbb{R}^d} u\,\mathrm{div}(\mathbf{q}) + \nabla u \cdot \mathbf{q}\, d\mathbf{x} = 0$$

Now let us consider an arbitrary $(u_j) \in \Pi_{j=0}^{n}\mathrm{H}^{+\frac{1}{2}}(\partial\Omega_j)$ and assume that it satisfies the condition in the right hand side of i). For any $j = 0 \dots n$ there exists $v_j \in \mathrm{H}^1(\Omega_j)$ such that $v_j|_{\partial\Omega_j} = u_j$. Define $u \in \mathrm{L}^2(\mathbb{R}^d)$ by $u|_{\Omega_j} = v_j$, and $\mathbf{p} \in \mathrm{L}^2(\mathbb{R}^d)^3$ by $\mathbf{p}|_{\Omega_j} = \nabla v_j$. For any $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$, setting $q_j = n_j \cdot \mathbf{q}|_{\partial\Omega_j}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} u\,\mathrm{div}(\mathbf{q})\, d\mathbf{x} = \sum_{j=0}^{n}\int_{\Omega_j} v_j\,\mathrm{div}(\mathbf{q})\, d\mathbf{x} &= \sum_{j=0}^{n}\int_{\partial\Omega_j} u_j\, q_j\, d\sigma - \sum_{j=0}^{n}\int_{\Omega_j} \nabla v_j \cdot \mathbf{q}\, d\mathbf{x}\\
&= -\int_{\mathbb{R}^d} \mathbf{p} \cdot \mathbf{q}\, d\mathbf{x}
\end{aligned}$$

as $(q_j) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma)$ by definition. Since the preceding identity holds for any $\mathbf{q} \in \mathrm{H}(\mathrm{div}, \mathbb{R}^d)$, this proves that $u \in \mathrm{H}^1(\mathbb{R}^d)$, so that $(u_j) \in \mathbb{X}^{1/2}(\Gamma)$. $\square$

Clearly $\mathbb{X}(\Gamma)$ is closed in $\mathbb{H}(\Gamma)$ for $\|\ \|$ since, according to the previous proposition, the constraints caracterizing $\mathbb{X}^{1/2}(\Gamma)$ (resp. $\mathbb{X}^{-1/2}(\Gamma)$) involve continuous functionals over $\mathrm{H}^{1/2}(\partial\Omega_j)$ (resp. $\mathrm{H}^{-1/2}(\partial\Omega_j)$), $j = 0 \dots n$. Moreover, one obvious consequence of the preceding proposition is that $\mathbb{X}(\Gamma)$ can be identified with its own polar set under the duality pairing $\mathrm{B}(\,,\,)$. More precisely: for any $U \in \mathbb{H}(\Gamma)$ we have

$$U \in \mathbb{X}(\Gamma) \quad\Longleftrightarrow\quad \mathrm{B}(U, V) = 0 \quad \forall V \in \mathbb{X}(\Gamma) \tag{5}$$

Our motivation for introducing the space $\mathbb{X}(\Gamma)$ is that the transmission conditions contained in Equation (2) can be restated as follows

$$u \text{ satisfies (2)} \quad\Longleftrightarrow\quad \left\{\begin{array}{l} u|_{\Omega_j} \in \mathrm{H}^1_{\mathrm{loc}}(\Delta, \overline{\Omega}_j),\ j = 0 \dots n,\\[1ex] (\gamma_{\mathrm{D}}^{j} u, \gamma_{\mathrm{N}}^{j} u)_{0 \le j \le n} \in \mathbb{X}(\Gamma)\ . \end{array}\right. \tag{6}$$

3 Classical results on potential operators

In this section we recall very classical results related to integral formulations for Helmholtz equation. Since what follows is already well known, we do not provide any proof for these results and refer the reader to the textbooks [13, 15, 18].

In the sequel $\mathrm{G}_\kappa(\mathbf{x})$ will refer to the Green Kernel of the operator $-\Delta-\kappa^2$ that satisfies the Sommerfeld radiation condition at infinity. For any subdomain Ω_i, consider

$$\begin{aligned}
&\mathrm{SL}^i_\kappa\{q\}(\mathbf{x}) = \int_{\partial\Omega_i} \mathrm{G}_\kappa(\mathbf{x}-\mathbf{y})q(\mathbf{y})d\sigma(\mathbf{y}) && \forall q\in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_i)\\
&\mathrm{DL}^i_\kappa\{v\}(\mathbf{x}) = -\int_{\partial\Omega_i} n_i(\mathbf{y})\cdot\nabla\mathrm{G}_\kappa(\mathbf{x}-\mathbf{y})v(\mathbf{y})d\sigma(\mathbf{y}) && \forall v\in \mathrm{H}^{+\frac{1}{2}}(\partial\Omega_i)
\end{aligned}\tag{7}$$

The potentials $\mathrm{SL}^i_{\kappa_i}\{p_i\}(\mathbf{x})$ and $\mathrm{DL}^i_{\kappa_i}\{u_i\}(\mathbf{x})$ are well defined functions over $\mathbb{R}^d\setminus\Gamma$. They induce continuous maps $\mathrm{SL}^i_{\kappa_i} : \mathrm{H}^{-1/2}(\partial\Omega_i)\to\Pi_{j=0}^n\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j)$ and $\mathrm{DL}^i_{\kappa_i} : \mathrm{H}^{+1/2}(\partial\Omega_i)\to\Pi_{j=0}^n\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j)$. Let us recall a crucial result about these potential operators, see for exemple [13, 15, 18]

Proposition 3.1.
Let $u\in\mathrm{H}^1_{\mathrm{loc}}(\overline{\Omega}_j)$ such that $\Delta u+\kappa_j^2 u = 0$ in Ω_j and u outgoing if $j=0$. We have the representation formula

$$\mathrm{DL}^j_{\kappa_j}\{\gamma^j_{\mathrm{D}}u\}(\mathbf{x}) + \mathrm{SL}^j_{\kappa_j}\{\gamma^j_{\mathrm{N}}u\}(\mathbf{x}) = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x}\in\Omega_j\\ 0 & \text{if } \mathbf{x}\in\mathbb{R}^d\setminus\overline{\Omega}_j\end{cases}$$

We shall say that $(v,q)\in\mathrm{H}^{1/2}(\partial\Omega_j)\times\mathrm{H}^{-1/2}(\partial\Omega_j)$ is a Cauchy datum of Ω_j whenever there exists $u\in\mathrm{H}^1_{\mathrm{loc}}(\Delta,\overline{\Omega}_j)$ such that $\Delta u+\kappa_j^2u=0$ in Ω_j, with u outgoing if $j=0$, and such that $\gamma^j_{\mathrm{D}}u = v$ and $\gamma^j_{\mathrm{N}}u=q$. We set

$$\mathcal{C}(\partial\Omega_j) = \{\text{ Cauchy data of } \Omega_j\ \}\quad\text{and}\quad \mathcal{C}(\Gamma) = \prod_{j=0}^{n}\mathcal{C}(\partial\Omega_j)$$

The operators $\mathrm{DL}^j_{\kappa_j},\mathrm{SL}^j_{\kappa_j}$ provide a convenient caracterization of Cauchy data of Ω_j. The following result is once again very classical, see [13, 15, 18].

Proposition 3.2.
For any $j=0\dots n$ and any $(v,q)\in\mathrm{H}^{\frac{1}{2}}(\partial\Omega_j)\times\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_j)$, we have

$$(v,q)\in\mathcal{C}(\partial\Omega_j)\quad\Longleftrightarrow\quad\begin{cases}\gamma^j_{\mathrm{D}}\cdot\mathrm{DL}^j_{\kappa_j}\{v\}+\gamma^j_{\mathrm{D}}\cdot\mathrm{SL}^j_{\kappa_j}\{q\} = v\\[2mm] \gamma^j_{\mathrm{N}}\cdot\mathrm{DL}^j_{\kappa_j}\{v\}+\gamma^j_{\mathrm{N}}\cdot\mathrm{SL}^j_{\kappa_j}\{q\} = q\end{cases}$$

In the sequel, we will also need the jump relations that describe the behaviour of potentials across $\partial\Omega_j$. For any $(v,q)\in\mathrm{H}^{1/2}(\partial\Omega_j)\times\mathrm{H}^{-1/2}(\partial\Omega_j)$ we have (cf [13, 15, 18])

$$\begin{aligned}
&[\gamma^j_{\mathrm{D}}]\cdot\mathrm{DL}^j_{\kappa_j}\{v\} = v,\quad [\gamma^j_{\mathrm{D}}]\cdot\mathrm{SL}^j_{\kappa_j}\{q\} = 0,\\
&[\gamma^j_{\mathrm{N}}]\cdot\mathrm{DL}^j_{\kappa_j}\{v\} = 0,\quad [\gamma^j_{\mathrm{N}}]\cdot\mathrm{SL}^j_{\kappa_j}\{q\} = q\ .
\end{aligned}\tag{8}$$

We conclude this section by recalling that the set of Cauchy data can also be caracterized by means of a continuous operator $\mathrm{C}^j_{\kappa_j} : \mathrm{H}^{1/2}(\partial\Omega_j)\times\mathrm{H}^{-1/2}(\partial\Omega_j) \to \mathrm{H}^{1/2}(\partial\Omega_j)\times\mathrm{H}^{-1/2}(\partial\Omega_j)$ named as Calderon projector and defined by

$$\frac{\mathrm{Id}}{2} + \mathrm{C}^j_{\kappa_j} \;=\; \begin{bmatrix} \gamma^j_{\mathrm{D}}\cdot\mathrm{DL}^j_{\kappa_j} & \gamma^j_{\mathrm{D}}\cdot\mathrm{SL}^j_{\kappa_j} \\ \gamma^j_{\mathrm{N}}\cdot\mathrm{DL}^j_{\kappa_j} & \gamma^j_{\mathrm{N}}\cdot\mathrm{SL}^j_{\kappa_j} \end{bmatrix}$$

This operator is indeed a projector: it satisfies $(\frac{\mathrm{Id}}{2}+\mathrm{C}^j_{\kappa_j})^2 = \frac{\mathrm{Id}}{2}+\mathrm{C}^j_{\kappa_j}$. Besides $\mathrm{Im}(\frac{\mathrm{Id}}{2}+\mathrm{C}^j_{\kappa_j}) = \mathcal{C}(\partial\Omega_j)$. As a consequence, for any $U\in\mathbb{H}(\Gamma)$, we have

$$U\in\mathcal{C}(\Gamma) \iff \Big(\frac{\mathrm{Id}}{2}+\mathrm{C}_\kappa\Big)U = U \quad\text{where}\quad \mathrm{C}_\kappa = \begin{bmatrix} \mathrm{C}^0_{\kappa_0} & 0 & \cdots & 0 \\ 0 & \mathrm{C}^1_{\kappa_1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{C}^n_{\kappa_n} \end{bmatrix}$$

This also shows that $\mathcal{C}(\Gamma)$ is a closed subspace of $\mathbb{H}(\Gamma)$ since $\mathcal{C}(\Gamma) = \mathrm{Ker}(\frac{\mathrm{Id}}{2}-\mathrm{C}_\kappa)$. Finally recall also that we have $(v,q)\in\mathrm{Im}(\,\mathrm{Id}/2-\mathrm{C}^j_{\kappa_j})$ if and only if there exists $u\in\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d\setminus\Omega_j)$ such that $\Delta u+\kappa_j^2 u = 0$ in $\mathbb{R}^d\setminus\overline{\Omega}_j$, u outgoing if $j\neq 0$, and $\gamma^j_{\mathrm{D},c}(u) = v$ and $\gamma^j_{\mathrm{N},c}(u) = q$.

4 Classical single trace formulation of the first kind

In this section we give a brief review of the formulation that was first analyzed by Von-Petersdorff in [16]. Note however that we state this formulation relying on a different functional setting. First of all, set $U_{\mathrm{inc}} = (\gamma^0_{\mathrm{D}}u_{\mathrm{inc}}, \gamma^0_{\mathrm{N}}u_{\mathrm{inc}}, 0\ldots,0)^\top$ and observe that, according to (6), Problem (2)-(3) can be reformulated as

$$\begin{aligned} &\text{Find } U\in\mathbb{X}(\Gamma) \qquad \text{such that}\\ &\Big(\frac{\mathrm{Id}}{2}-\mathrm{C}_\kappa\Big)\big(\,U-U_{\mathrm{inc}}\,\big) = 0 \end{aligned} \tag{9}$$

Equation (9) is well posed, as Problem (2)-(3) is. Recalling that $\mathrm{B}(U,V) = 0$ whenever both U and V belong to $\in\mathbb{X}(\Gamma)$, the above equation implies

$$\begin{aligned} &\text{Find } U\in\mathbb{X}(\Gamma) \quad \text{such that}\\ &\mathrm{B}(\mathrm{C}_\kappa U, V) = -\mathrm{B}\big(\,(\,\mathrm{Id}/2-\mathrm{C}_\kappa\,)U_{\mathrm{inc}}, V\,\big) \quad \forall V\in\mathbb{X}(\Gamma)\;. \end{aligned} \tag{10}$$

At first sight though, it is not clear whether (10) implies (9). Actually both equations are equivalent, since Formulation (10) admits a unique solution as well.

Proposition 4.1.
For any $F\in\mathbb{H}(\Gamma)$, there exists a unique $U\in\mathbb{X}(\Gamma)$ such that

$$\mathrm{B}(\mathrm{C}_\kappa U, V) = \mathrm{B}(F,V) \quad \forall V\in\mathbb{X}(\Gamma)\;.$$

For the sake of completeness, the proof of Proposition 4.1 is put in appendix. We use the existence part of this proposition to prove what seems to be another valuable result that was not pointed out in [16].

Proposition 4.2.
We have $\mathbb{H}(\Gamma) = \mathbb{X}(\Gamma) \oplus \mathcal{C}(\Gamma)$.

Proof:
First of all, we have $\mathbb{X}(\Gamma) \cap \mathcal{C}(\Gamma) = \{0\}$. Indeed take any $U \in \mathbb{X}(\Gamma) \cap \mathcal{C}(\Gamma)$ with $U = (u_j, p_j)_{0\leq j\leq n}$ and define $u(\mathbf{x}) = \mathrm{DL}^{j}_{\kappa_j}\{u_j\}(\mathbf{x}) + \mathrm{SL}^{j}_{\kappa_j}\{p_j\}(\mathbf{x})$ for $\mathbf{x} \in \Omega_j$. Clearly we have $\Delta u + \kappa_j^2 u = 0$ in Ω_j. Besides, for any $j = 0 \dots n$, we have $\gamma^j_{\mathrm{D}}(u) = u_j$ and $\gamma^j_{\mathrm{N}}(u) = p_j$ since $(u_j, p_j) \in \mathcal{C}(\partial\Omega_j)$, hence $(\gamma^j_{\mathrm{D}} u, \gamma^j_{\mathrm{N}} u)_{0\leq j\leq n} \in \mathbb{X}(\Gamma)$ since $U \in \mathbb{X}(\Gamma)$. As a consequence u would satisfy Problem (2)-(3) with no incident field, so that $u = 0$, and actually $U = 0$.

Second, we prove that $\mathbb{X}(\Gamma)+\mathcal{C}(\Gamma) = \mathbb{H}(\Gamma)$. Take any $U \in \mathbb{H}(\Gamma)$. According to Proposition 4.1, there exists a unique $\widehat{U} \in \mathbb{X}(\Gamma)$ such that $\mathrm{B}(\,(\mathrm{Id}/2+\mathrm{C}_\kappa)\widehat{U}, V) = \mathrm{B}(U, V)$ for all $V \in \mathbb{X}(\Gamma)$. Setting $\widetilde{U} = U - (\mathrm{Id}/2 + \mathrm{C}_\kappa)\widehat{U}$, we clearly have $\mathrm{B}(\widetilde{U}, V) = 0, \forall V \in \mathbb{X}(\Gamma)$. As a consequence, according to (5) we have $\widetilde{U} \in \mathbb{X}(\Gamma)$. Since $U = \widetilde{U} + (\mathrm{Id}/2 + \mathrm{C}_\kappa)\widehat{U}$, and since $(\mathrm{Id}/2 + \mathrm{C}_\kappa)\widehat{U} \in \mathcal{C}(\Gamma)$, this proves that $U \in \mathbb{X}(\Gamma) + \mathcal{C}(\Gamma)$. $\square$

5 Multi potential operator

In this section we introduce a way to combine the classical results of Section 3 so as to obtain another formulation of our multi sub-domain problem. Let us define a continuous operator $\mathrm{A}_\kappa : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ by

$$\begin{aligned}
&\Phi(U)(\mathbf{x}) \;=\; \sum_{i=0}^{n} \mathrm{DL}^{i}_{\kappa_i}\{u_i\}(\mathbf{x}) + \mathrm{SL}^{i}_{\kappa_i}\{p_i\}(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^d \setminus \Gamma \\
&\text{and} \quad \mathrm{A}_\kappa = (\gamma^0_{\mathrm{D}} \cdot \Phi, \gamma^0_{\mathrm{N}} \cdot \Phi, \dots, \gamma^n_{\mathrm{D}} \cdot \Phi, \gamma^n_{\mathrm{N}} \cdot \Phi)^\top \\
&\text{for} \quad U = (u_0, p_0, \dots, u_n, p_n)^\top
\end{aligned} \tag{11}$$

Important here is to note that, in Definition (11), all potentials are considered as functions defined *everywhere* except on Γ.

5.1 The case where all wave numbers are equal

Before studying the operators defined above in the general case, we state two lemmas in the particular situation where $\kappa_j = \kappa_0$ for all $j = 1 \dots n$.

Lemma 5.1.
Assume that $\kappa_j = \kappa_0, \forall j = 1 \dots n$. *In this case,* $\Phi(U)(\mathbf{x}) = 0\ \forall \mathbf{x} \in \mathbb{R}^d$ *for any* $U \in \mathbb{X}(\Gamma)$.

Proof:
Pick an arbitrary $\mathbf{x} \in \mathbb{R}^d \setminus \Gamma$, and consider a cut-off function $\chi : \mathbb{R}^d \to \mathbb{R}_+$ such that $\chi = 0$ over a neighborhood of $\mathbf{x}$, and $\chi = 1$ over a neighborhood of Γ. Let $v_{\mathbf{x}} \in C^\infty(\mathbb{R}^d)$ be defined by $v_{\mathbf{x}}(\mathbf{y}) = \chi(\mathbf{y}) G_{\kappa_0}(\mathbf{x} - \mathbf{y})$. Let $V = (\gamma^0_{\mathrm{D}}(\overline{v}_{\mathbf{x}}), \gamma^0_{\mathrm{N}}(\overline{v}_{\mathbf{x}}), \dots, \gamma^n_{\mathrm{D}}(\overline{v}_{\mathbf{x}}), \gamma^n_{\mathrm{N}}(\overline{v}_{\mathbf{x}}))^\top \in \mathbb{X}(\Gamma)$. Since $v_{\mathbf{x}}(\mathbf{y}) = G_{\kappa_0}(\mathbf{x} - \mathbf{y})$ for any $\mathbf{y}$ chosen sufficiently close to Γ, we have $\Phi(U)(\mathbf{x}) = \mathrm{B}(U, V) = 0$ $\forall U \in \mathbb{X}(\Gamma)$ $\square$

Lemma 5.2.
Assume that $\kappa_j = \kappa_0, \forall j = 1 \dots n$. *In this case, for any* $U \in \mathbb{H}(\Gamma)$, *we have* $\mathrm{A}_\kappa U = U$ *if and only if* $U \in \mathcal{C}(\Gamma)$.

Proof:

Assume first that $U \in \mathcal{C}(\Gamma)$. According to Proposition 3.1, we have $\gamma_{\mathrm{D}}^{j}(\mathrm{DL}_{\kappa_i}^{i}\{u_i\} + \mathrm{SL}_{\kappa_i}^{i}\{p_i\}) = 0$ and $\gamma_{\mathrm{N}}^{j}(\mathrm{DL}_{\kappa_i}^{i}\{u_i\} + \mathrm{SL}_{\kappa_i}^{i}\{p_i\}) = 0$ if $i \neq j$, and $\gamma_{\mathrm{D}}^{i}(\mathrm{DL}_{\kappa_i}^{i}\{u_i\} + \mathrm{SL}_{\kappa_i}^{i}\{p_i\}) = u_i$ and $\gamma_{\mathrm{N}}^{i}(\mathrm{DL}_{\kappa_i}^{i}\{u_i\} + \mathrm{SL}_{\kappa_i}^{i}\{p_i\}) = p_i$. Hence $\gamma_{\mathrm{D}}^{i} \cdot \Phi(U) = u_i$ and $\gamma_{\mathrm{N}}^{i} \cdot \Phi(U) = p_i$, for any $i = 0 \dots n$. Thus, according to (11), we have $\mathrm{A}_\kappa \cdot U = U$. Now assume that $U \in \mathbb{H}(\Gamma)$ such that $\mathrm{A}_\kappa U = U$. Since $\kappa_0 = \cdots = \kappa_n$, it is clear that $\mathrm{Im}(\mathrm{A}_\kappa) \subset \mathcal{C}(\Gamma)$, so that $U \in \mathcal{C}(\Gamma)$. $\square$

5.2 The case of arbitrary wave numbers

It would be interesting to establish the same result as Lemma 5.2, without the restrictive assumption that $\kappa_j = \kappa_0, \forall j = 1 \dots n$. Although we were not able to prove such a result in the general case, we prove a result slightly stronger than Lemma 5.2. To do so, we need a key proposition.

Proposition 5.1.
We have $\mathrm{B}(\mathrm{A}_\kappa U, V) = \mathrm{B}(U, V)$, $\forall U \in \mathbb{H}(\Gamma)$, $\forall V \in \mathbb{X}(\Gamma)$

Proof:

Choose $U \in \mathbb{H}(\Gamma)$ and $V \in \mathbb{X}(\Gamma)$ arbitrarily, denoting $U = (u_0, p_0, \dots, u_n, p_n)$ and $V = (v_0, q_0, \dots, v_n, q_n)$. Let us develop the expression of $\mathrm{B}(\mathrm{A}_\kappa U, V)$. According to (7), we have

$$\begin{aligned} \mathrm{B}(\mathrm{A}_\kappa U, V) = \quad & \sum_{j=0}^{n}\sum_{i=0}^{n} \Big(\int_{\partial\Omega_i} \overline{q}_i\, \gamma_{\mathrm{D}}^{i} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\} d\sigma + \int_{\partial\Omega_i} \overline{v}_i\, \gamma_{\mathrm{N}}^{i} \cdot \mathrm{SL}_{\kappa_j}^{j}\{p_j\} d\sigma \Big) \\ & + \sum_{j=0}^{n}\sum_{i=0}^{n} \Big(\int_{\partial\Omega_i} \overline{q}_i\, \gamma_{\mathrm{D}}^{i} \cdot \mathrm{SL}_{\kappa_j}^{j}\{p_j\} d\sigma + \int_{\partial\Omega_i} \overline{v}_i\, \gamma_{\mathrm{N}}^{i} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\} d\sigma \Big) \end{aligned} \tag{12}$$

Take a fixed j arbitrarily. Then according to jump relations (8), we have $(\gamma_{\mathrm{D}}^{0} \cdot \mathrm{SL}_{\kappa_j}^{j}\{p_j\}, \dots, \gamma_{\mathrm{D}}^{n} \cdot \mathrm{SL}_{\kappa_j}^{j}\{p_j\}) \in \mathbb{X}^{1/2}(\Gamma)$ and $(\gamma_{\mathrm{N}}^{0} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\}, \dots, \gamma_{\mathrm{N}}^{n} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\}) \in \mathbb{X}^{-1/2}(\Gamma)$. So according to the very definition of $\mathbb{X}^{\pm 1/2}(\Gamma)$, we have

$$\sum_{i=0}^{n} \Big(\int_{\partial\Omega_i} \overline{q}_i\, \gamma_{\mathrm{D}}^{i} \cdot \mathrm{SL}_{\kappa_j}^{j}\{p_j\} d\sigma - \int_{\partial\Omega_i} \overline{v}_i\, \gamma_{\mathrm{N}}^{i} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\} d\sigma \Big) = 0 \qquad \forall j = 0 \dots n\,.$$

Since this holds for any j, we see that all terms in the second line of (12) vanish. To study the remaining terms, take once again a fixed j chosen arbitrarily. Let $\gamma_{\mathrm{D},c}^{j}$ refer to the Dirichlet trace taken from the exterior of Ω_j. There exists a function $w \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\mathrm{DL}_{\kappa_j}^{j}(u_j) = w$ in $\mathbb{R}^d \setminus \overline{\Omega}_j$. Observe that

$$\gamma_{\mathrm{D}}^{i}(w) = \gamma_{\mathrm{D}}^{i} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\} \quad \text{ if } \; i \neq j \qquad \text{and} \qquad \gamma_{\mathrm{D}}^{j}\{w_j\} = \gamma_{\mathrm{D},c}^{j} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\}\,.$$

Observe that $(\gamma_{\mathrm{D}}^{0}(w), \dots, \gamma_{\mathrm{D}}^{n}(w)) \in \mathbb{X}^{\frac{1}{2}}(\Gamma)$ since $w \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$. Denote $[\gamma_{\mathrm{D}}^{j}] = \gamma_{\mathrm{D}}^{j} - \gamma_{\mathrm{D},c}^{j}$ and recall that $[\gamma_{\mathrm{D}}^{j}] \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\} = u_j$ according to jump relations (8). As a consequence we have

$$\begin{aligned} \sum_{i=0}^{n} \int_{\partial\Omega_i} \overline{q}_i\, \gamma_{\mathrm{D}}^{i} \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\} d\sigma \quad & = \int_{\partial\Omega_j} \overline{q}_j\, [\gamma_{\mathrm{D}}^{j}] \cdot \mathrm{DL}_{\kappa_j}^{j}\{u_j\} d\sigma + \underbrace{\sum_{i=0}^{n} \int_{\partial\Omega_i} \overline{q}_i\, \gamma_{\mathrm{D}}^{i}(w)\, d\sigma}_{=0} \\ & = \int_{\partial\Omega_j} \overline{q}_j\, u_j d\sigma \end{aligned}$$

Since j has been chosen arbitrarily, this holds for any j. As a consequence, summing over all $j = 0 \dots n$, we obtain

$$\sum_{j=0}^{n} \sum_{i=0}^{n} \int_{\partial\Omega_i} \overline{q}_i \, \gamma_{\mathrm{D}}^i \cdot \mathrm{DL}_{\kappa_j}^j(u_j) d\sigma = \sum_{j=0}^{n} \int_{\partial\Omega_j} \overline{q}_j \, u_j \, d\sigma \; .$$

A similar technique can be applied to treat the terms containing $\gamma_{\mathrm{N}}^i \cdot \mathrm{SL}_{\kappa_j}^j(p_j)$, the only difference is that we have to consider extensions that match Neumann data. □

Using the caracterization of $\mathbb{X}(\Gamma)$ given by (5), we see that the following stability statement is a direct consequence of the preceding proposition.

Corollary 5.1.
The operator A_κ *maps* $\mathbb{X}(\Gamma)$ *into itself.*

The operator A_κ actually coincides with a compact operator when restricted to $\mathbb{X}(\Gamma)$.

Proposition 5.2.
There exists a compact operator $\mathrm{K}_\kappa : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ *satisfying* $\mathrm{Im}(\mathrm{K}_\kappa) \subset \mathbb{X}(\Gamma)$ *and such that* $\mathrm{K}_\kappa U = \mathrm{A}_\kappa U$, $\forall U \in \mathbb{X}(\Gamma)$.

Proof:
Since $\mathbf{x} \mapsto G_{\kappa_i}(\mathbf{x}) - G_{\kappa_0}(\mathbf{x})$ is continuous over $\mathbb{R}^d$, using the arguments of Remark 3.1.3 and Lemma 3.9.8 in [18], we see that the following operators are compact

$$\gamma_{\mathrm{D}}^j \cdot (\mathrm{SL}_{\kappa_i}^i - \mathrm{SL}_{\kappa_0}^i), \quad \gamma_{\mathrm{D}}^j \cdot (\mathrm{DL}_{\kappa_i}^i - \mathrm{DL}_{\kappa_0}^i), \quad \gamma_{\mathrm{N}}^j \cdot (\mathrm{SL}_{\kappa_i}^i - \mathrm{SL}_{\kappa_0}^i) \quad \text{and} \quad \gamma_{\mathrm{N}}^j \cdot (\mathrm{DL}_{\kappa_i}^i - \mathrm{DL}_{\kappa_0}^i)$$

Denote by $\mathrm{A}_0 : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ the operator defined in the same manner as A_κ except that we replace G_{κ_i} by G_{κ_0} in each sub-domain, so that $\mathrm{K}_\kappa = \mathrm{A}_\kappa - \mathrm{A}_0$ is compact according to the remarks above.

Observe also that, for $p_i \in \mathrm{H}^{-1/2}(\partial\Omega_i)$, $i = 0 \dots n$, we have $[\gamma_{\mathrm{D}}^i] \cdot \mathrm{SL}_{\kappa_i}^i\{p_i\} = 0 = [\gamma_{\mathrm{D}}^i] \cdot \mathrm{SL}_{\kappa_0}^i\{p_i\}$ and $[\gamma_{\mathrm{N}}^i] \cdot \mathrm{SL}_{\kappa_i}^i\{p_i\} = p_i = [\gamma_{\mathrm{N}}^i] \cdot \mathrm{SL}_{\kappa_0}^i\{p_i\}$. If $v = (\mathrm{SL}_{\kappa_i}^i - \mathrm{SL}_{\kappa_0}^i)\{p_i\}$, the preceding remark shows that $(\gamma_{\mathrm{D}}^0(v), \gamma_{\mathrm{N}}^0(v), \dots, \gamma_{\mathrm{D}}^n(v), \gamma_{\mathrm{N}}^n(v)) \in \mathbb{X}(\Gamma)$. The same remark can be formulated, replacing the operators $\mathrm{SL}_{\kappa_i}^i - \mathrm{SL}_{\kappa_0}^i$ by the operators $\mathrm{DL}_{\kappa_i}^i - \mathrm{DL}_{\kappa_0}^i$. Since K_κ is defined by composing $\gamma_{\mathrm{D}}^j, \gamma_{\mathrm{N}}^j, j = 0 \dots n$ with operators of the form $\mathrm{SL}_{\kappa_i}^i - \mathrm{SL}_{\kappa_0}^i$ and $\mathrm{DL}_{\kappa_i}^i - \mathrm{DL}_{\kappa_0}^i$, $i = 0 \dots n$, the preceding remarks show that $\mathrm{Im}(\mathrm{K}_\kappa) \subset \mathbb{X}(\Gamma)$. To finish the proof, it remains to show that $\mathrm{A}_0 U = 0$ if $U \in \mathbb{X}(\Gamma)$. This is a straightforward consequence of Lemma 5.1. □

Proposition 5.2 shows that the operator $\mathrm{Id} - \mathrm{A}_\kappa$ is of Fredholm type whenever it is restricted to $\mathbb{X}(\Gamma)$. Besides Lemma 5.2 shows that $(\mathrm{Id} - \mathrm{A}_\kappa)|_{\mathbb{X}(\Gamma)}$ is one-to-one hence isomorphic whenever $\kappa_j = \kappa_0$, $\forall j = 0 \dots n$. This can be generalized at least in the case of small contrasts.

Theorem 5.1.
For any $\kappa_0 \in \mathbb{R}_+$ *there exists* $\delta > 0$ *such that, if* $\max_{j=1\dots n} |\kappa_j - \kappa_0| < \delta$, *then the operator* $\mathrm{Id} - \mathrm{A}_\kappa$ *isomorphically maps* $\mathbb{X}(\Gamma)$ *onto itself.*

Proof:

We simply use a continuity argument. First of all consider $\mathcal{L}(\mathbb{X}(\Gamma), \mathbb{X}(\Gamma))$ the space of continuous linear operators over $\mathbb{X}(\Gamma)$. Equip this space with the following norm

$$\|\mathrm{L}\| = \sup_{V \in \mathbb{X}(\Gamma)} \frac{\|\mathrm{L}V\|}{\|V\|} \quad \text{when} \quad \mathrm{L} : \mathbb{X}(\Gamma) \to \mathbb{X}(\Gamma) \text{ is continuous.} \tag{13}$$

The set of continuous isomorphisms of $\mathbb{X}(\Gamma)$ is open in the space of continuous linear operators over $\mathbb{X}(\Gamma)$ for the norm above. According to Proposition 8.2 in appendix, the operator A_κ continuously depends on the wave numbers $\kappa_j, j = 0 \dots n$ with respect to this norm. Since $\mathrm{Id} - \mathrm{A}_\kappa$ is an isomorphism of $\mathbb{X}(\Gamma)$ when $\kappa_j = \kappa_0, j = 0 \dots n$, this concludes the proof. $\square$

Corollary 5.2.
For any $\kappa_0 \in \mathbb{R}_+$ there exists $\delta > 0$ such that, if $\max_{j=1\dots n}|\kappa_j - \kappa_0| < \delta$, then the operator $\mathrm{Id} - \mathrm{K}_\kappa$ isomorphically maps $\mathbb{H}(\Gamma)$ onto itself.

Proof:

If $\max_{j=1\dots n}|\kappa_j - \kappa_0| < \delta$ where δ is chosen as in the statement of Theorem 5.1, then $\mathrm{Id} - \mathrm{K}_\kappa$ is one-to-one over $\mathbb{H}(\Gamma)$. Indeed take $U \in \mathbb{H}(\Gamma)$ such that $U = \mathrm{K}_\kappa U$. Since $\mathrm{Im}(\mathrm{K}_\kappa) \subset \mathbb{X}(\Gamma)$ according to Proposition 5.1, we have $U \in \mathbb{X}(\Gamma)$ so $U - \mathrm{K}_\kappa U = U - \mathrm{A}_\kappa U = 0$ which implies $U = 0$. Since $\mathrm{Id} - \mathrm{K}_\kappa$ is of Fredholm type, the proof is finished. $\square$

6 Variational formulation

In Theorem 5.1 and Corollary 5.2 we assumed that the wave numbers are sufficiently close to each other. We shall refer to this hypothesis as the "small contrast assumption".

The operator $\mathrm{Id} - \mathrm{A}_\kappa$ leads to an integral formulation of Problem (2). Set $U_{\mathrm{inc}} = (\gamma_{\mathrm{D}}^0 u_{\mathrm{inc}}, \gamma_{\mathrm{N}}^0 u_{\mathrm{inc}}, 0 \dots, 0)$. The following lemma is a simple rewriting of (2)-(3) by means of Theorem 5.2.

Proposition 6.1.
Suppose that the small contrast assumption of Theorem 5.1 is satisfied. Then $u \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^d)$ is solution to Problem (2)-(3) if and only if $U = (\,\gamma_{\mathrm{D}}^0(u), \gamma_{\mathrm{N}}^0(u),\, \dots, \gamma_{\mathrm{D}}^n(u), \gamma_{\mathrm{N}}^n(u)\,)^\top$ satisfies

$$U \in \mathbb{X}(\Gamma) \quad \textit{and} \quad (\mathrm{Id} - \mathrm{A}_\kappa)U = (\mathrm{Id} - \mathrm{A}_\kappa)U_{\mathrm{inc}} \tag{14}$$

Equation (14) is particularly interesting because, in the present case, it takes the form "identity + compact" according to Proposition 5.2. To obtain a variational formulation, we need to select a space of test functions. Proposition 5.1 shows that taking $\mathbb{X}(\Gamma)$ as the set of test functions is pointless.

A relevant space of test functions may consist in any closed space $\mathbb{Y}$ of $\mathbb{H}(\Gamma)$ such that $\mathbb{X}(\Gamma) \oplus \mathbb{Y} = \mathbb{H}(\Gamma)$. For small contrasts, $\mathrm{Im}(\mathrm{Id} - \mathrm{A}_\kappa) = \mathbb{X}(\Gamma)$, and Equation (14) is then equivalent to the following variational formulation

$$\begin{aligned} &\text{Find } \; U \in \mathbb{X}(\Gamma) \quad \text{such that} \\ &\mathrm{B}\big(\,(\mathrm{Id} - \mathrm{A}_\kappa)U, V\,\big) = \mathrm{B}\big(\,(\mathrm{Id} - \mathrm{A}_\kappa)U_{\mathrm{inc}}, V\,\big) \quad \forall V \in \mathbb{Y}\,. \end{aligned} \tag{15}$$

It is not clear to us wether there exists a distinguished choice of $\mathbb{Y}$. One may take $\mathbb{Y} = \mathcal{C}(\Gamma)$, but such a choice does not seem to be really interesting from a computational point of view.

7 The case of two sub-domains

In this section we apply the preceding analysis to the particular case where $n = 1$. This is an already well known transmission problem where the diffracting object is homogeneous i.e. only contains one sub-domain. We show that our analysis leads to Rokhlin-Müller scheme i.e. the classical second kind integral equation, see [10, 14, 17, 1]. Note that this analysis is interesting also because it will be possible to write every calculation explicitely.

First of all, in this case $\Gamma = \partial\Omega_0 = \partial\Omega_1$ and $(v_0, v_1) \in \mathbb{X}^{+\frac{1}{2}}(\Gamma)$ if and only if $v_0 = v_1$. Similarly $(q_0, q_1) \in \mathbb{X}^{-\frac{1}{2}}(\Gamma)$ if and only if $q_1 = -q_0$. As a consequence, in this particular case

$$\mathbb{X}(\Gamma) = \left\{ \begin{bmatrix} \mathrm{Id} \\ Q \end{bmatrix} \cdot V \;\middle|\; V \in \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma) \right\}$$

$$\text{with} \qquad \mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In the particular case of two sub-domains, there is a choice for $\mathbb{Y}(\Gamma)$ that appears rather natural, namely

$$\mathbb{Y}(\Gamma) = \left\{ \begin{bmatrix} \mathrm{Id} \\ -Q \end{bmatrix} \cdot V \;\middle|\; V \in \mathrm{H}^{\frac{1}{2}}(\Gamma) \times \mathrm{H}^{-\frac{1}{2}}(\Gamma) \right\}$$

Here, it is straightforward to verify that $\mathbb{H}(\Gamma) = \mathbb{X}(\Gamma) \oplus \mathbb{Y}(\Gamma)$. The expression of the duality pairing in the matrix form used above is the following

$$\mathrm{B}\Big(\begin{bmatrix} U_0 \\ U_1 \end{bmatrix}, \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} \Big) = \begin{bmatrix} U_0^\top & U_1^\top \end{bmatrix} \cdot \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \cdot \begin{bmatrix} V_0 \\ V_1 \end{bmatrix} \qquad \text{with} \quad P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Finally, it is possible to give a very explicit expression for the operator A, using the Calderon projectors that have been defined in Section 3. Indeed we have

$$\mathrm{A}_\kappa = \begin{bmatrix} \mathrm{Id}/2 + \mathrm{C}^0_{\kappa_0} & Q(-\mathrm{Id}/2 + \mathrm{C}^1_{\kappa_1}) \\ Q(-\mathrm{Id}/2 + \mathrm{C}^0_{\kappa_0}) & \mathrm{Id}/2 + \mathrm{C}^1_{\kappa_1} \end{bmatrix}.$$

Then we obtain an explicit expression for $\mathrm{B}(U, V)$ when $U \in \mathbb{X}(\Gamma)$, $U^\top = [U_0^\top \; U_1^\top]$ and $V \in \mathbb{Y}(\Gamma)$, $V^\top = [V_0^\top \; V_1^\top]$, using the fact that $Q^2 = \mathrm{Id}$ and $QP = -PQ$, we have

$$\mathrm{B}(U, V) \;=\; V_0^\top \cdot [\mathrm{Id} \;\; -Q] \cdot \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \cdot \begin{bmatrix} \mathrm{Id} \\ Q \end{bmatrix} \cdot U_0 \;=\; 2\, V_0^\top \cdot P \cdot U_0$$

The same calculus can be achieved with U replaced by $U_{\mathrm{inc}} = (\gamma^0_{\mathrm{D}}(u_{\mathrm{inc}}), \gamma^0_{\mathrm{N}}(u_{\mathrm{inc}}), 0, 0)^\top$, which yields $\mathrm{B}(U_{\mathrm{inc}}, V) = V_0^\top \cdot P \cdot U^0_{\mathrm{inc}}$ where $U^0_{\mathrm{inc}} = (\gamma^0_{\mathrm{D}}(u_{\mathrm{inc}}), \gamma^0_{\mathrm{N}}(u_{\mathrm{inc}})\,)^\top$. A similar calculus yields

$$\mathrm{B}(\mathrm{A}_\kappa\, U, V) \;=\; V_0^\top \cdot [\mathrm{Id} \;\; -Q] \cdot \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} \cdot \begin{bmatrix} \mathrm{Id}/2 + \mathrm{C}^0_{\kappa_0} & Q(-\mathrm{Id}/2 + \mathrm{C}^1_{\kappa_1}) \\ Q(-\mathrm{Id}/2 + \mathrm{C}^0_{\kappa_0}) & \mathrm{Id}/2 + \mathrm{C}^1_{\kappa_1} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{Id} \\ Q \end{bmatrix} \cdot U_0$$

hence

$$\mathrm{B}(\mathrm{A}_\kappa\, U, V) \;=\; 2\, V_0^\top \cdot P \cdot \Big(\mathrm{C}^0_{\kappa_0} + Q\mathrm{C}^1_{\kappa_1} Q \Big) \cdot U_0$$

To compute the right hand side, first observe that $(\mathrm{Id}/2+\mathrm{C}^0_{\kappa_0})U^0_{\mathrm{inc}} = 0$ since U^0_{inc} is a Cauchy datum for the interior problem i.e. $U^0_{\mathrm{inc}} \in \mathcal{C}(\partial\Omega_1)$. This remark yields $\mathrm{B}(\mathrm{A}_\kappa\, U_{\mathrm{inc}}, V) = -V_0^\top P U^0_{\mathrm{inc}}$. According to the expressions that we derived above, Formulation (15) explicitely writes as follows

$$\begin{aligned}
&\text{Find } U_0 \in \mathrm{H}^{\frac{1}{2}}(\Gamma)\times\mathrm{H}^{-\frac{1}{2}}(\Gamma) \text{ such that}\\
&V_0^\top\cdot P\cdot\Big(\mathrm{Id}-\mathrm{C}^0_{\kappa_0}-Q\mathrm{C}^1_{\kappa_1}Q\Big)\cdot U_0 = V_0^\top P U^0_{\mathrm{inc}}\;,\quad \forall V_0\in\mathrm{H}^{\frac{1}{2}}(\Gamma)\times\mathrm{H}^{-\frac{1}{2}}(\Gamma)\;.
\end{aligned}\tag{16}$$

Note that $\mathrm{C}^0_{\kappa_0} + Q\mathrm{C}^1_{\kappa_1}Q$ is a compact operator. Indeed, $\mathrm{C}^1_{\kappa_1} - \mathrm{C}^1_{\kappa_0}$ is a compact operator from $\mathrm{H}^{1/2}(\Gamma)\times\mathrm{H}^{-1/2}(\Gamma)$ into $\mathrm{H}^{1/2}(\Gamma)\times\mathrm{H}^{-1/2}(\Gamma)$. In this case, it is easy to check that $Q\mathrm{C}^0_{\kappa_0} = -\mathrm{C}^1_{\kappa_0}Q$. This means that $\mathrm{C}^0_{\kappa_0} + Q\mathrm{C}^1_{\kappa_1}Q = Q(\mathrm{C}^1_{\kappa_1} - \mathrm{C}^1_{\kappa_0})Q$.

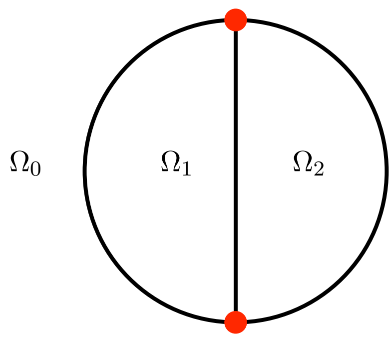
A careful inspection should convince the reader that Equation (16) takes a very standard form such as in chapter 3 of [4], see also [10, 14, 17, 1]. The only reason why it does not take strictly the same form lies in our choice of orientation for normal vectors: we have $n_1 = -n_0$ whereas most articles take the opposite convention.

The conclusion of this section is that formulation (15) is a "multiple sub-domain generalization" of the already well established second kind integral formulation of transmission problems.

8 Numerical experiments

In this section we present numerical results obtained by discretizing Formulation (15) by means of a Galerkin method that we describe now. Note however that we have not been able to prove any theoretical result concerning the consistency of this numerical scheme.

In this numerical experiment, we considered a problem of diffraction of a plane wave by a dielectric object that has the shape of a disk. This disk is artificially split in two subdomains, as is represented in the picture below.



$\Omega_1\cup\Omega_2 = \overline{\mathrm{D}}(0,1)$,
$\kappa_0 = 1$,
$\kappa_1 = \kappa_2 = \kappa_{\mathrm{d}}$

We take $u_{\mathrm{inc}}(x,y) = e^{-i\kappa_0 x}$ as incident field. For this geometry and this incident field, the solution is obtained using the method of separation of variables and the Jacobi-Anger Formula $u_{\mathrm{inc}}(r,\theta) = \sum_{p\in\mathbb{Z}}(-i)^{|p|}\mathrm{J}_{|p|}(\kappa_0 r)e^{ip\theta}$, see Identity (5.12.2) in [12]. This leads to the following expression for the exact solution to our problem,

$$
\text{for } r>1 \quad \begin{cases} u(r,\theta) = \displaystyle\sum_{p=-\infty}^{+\infty} \big[\, \alpha_p \, \mathrm{H}^{(1)}_{|p|}(\kappa_0 r) + (-1)^{|p|} \mathrm{J}_{|p|}(\kappa_0 r) \,\big]\, e^{ip\theta} \\[2ex] \text{with} \quad \alpha_p = (-1)^{|p|+1} \, \dfrac{\kappa_0 \mathrm{J}'_{|p|}(\kappa_0)\mathrm{J}_{|p|}(\kappa_\mathrm{d}) - \kappa_\mathrm{d}\mathrm{J}_{|p|}(\kappa_0)\mathrm{J}'_{|p|}(\kappa_\mathrm{d})}{\kappa_0 \mathrm{H}^{(1)\prime}_{|p|}(\kappa_0)\mathrm{J}_{|p|}(\kappa_\mathrm{d}) - \kappa_\mathrm{d}\mathrm{H}^{(1)}_{|p|}(\kappa_0)\mathrm{J}'_{|p|}(\kappa_\mathrm{d})} \end{cases}
$$

$$
\text{for } r<1 \quad \begin{cases} u(r,\theta) = \displaystyle\sum_{p=-\infty}^{+\infty} \beta_p \, \mathrm{J}_{|p|}(\kappa_\mathrm{d} r)\, e^{ip\theta} \\[2ex] \text{with} \quad \beta_p = (-1)^{|p|+1} \, \dfrac{\kappa_0 \mathrm{J}'_{|p|}(\kappa_0)\mathrm{H}^{(1)}_{|p|}(\kappa_0) - \kappa_\mathrm{d}\mathrm{J}_{|p|}(\kappa_0)\mathrm{H}^{(1)\prime}_{|p|}(\kappa_0)}{\kappa_0 \mathrm{H}^{(1)\prime}_{|p|}(\kappa_0)\mathrm{J}_{|p|}(\kappa_\mathrm{d}) - \kappa_\mathrm{d}\mathrm{H}^{(1)}_{|p|}(\kappa_0)\mathrm{J}'_{|p|}(\kappa_\mathrm{d})} \end{cases} \tag{17}
$$

For discretization, we considered a paneling $\Gamma = \cup_{q=1}^{\mathrm{Q}} \Gamma^q_\mathrm{h}$ where each Γ^q_h is a segment. Then we considered the space $\mathbb{V}_h(\Gamma) = \Pi_{j=0}^n \mathbb{V}(\partial\Omega_j)$ where

$$
\mathbb{V}(\partial\Omega_j) = \{ v_h \in C^0(\partial\Omega_j) \mid v_h|_{\Gamma^q_\mathrm{h}} \in \mathbb{P}_1 \;\text{ for }\; \Gamma^q_\mathrm{h} \subset \partial\Omega_j,\; q = 1\dots \mathrm{Q} \}
$$

We take $\mathbb{X}^+_h = \mathbb{V}_h(\Gamma) \cap \mathbb{X}^{1/2}(\Gamma)$ as an approximation space for $\mathbb{X}^{1/2}(\Gamma)$. For the approximation of $\mathbb{X}^{-1/2}(\Gamma)$, we base our construction on a discrete counterpart of Proposition 2.1. We approximate $\mathbb{X}^{-1/2}(\Gamma)$ by a space $\mathbb{X}^+_h$ defined by

$$
\mathbb{X}^-_h = \Big\{ \, (q_j)_{j=0\dots n} \in \mathbb{V}_h(\Gamma) \;\Big|\; \sum_{j=0}^{n} ⨍_{\partial\Omega_j} q_j \, v_j \, d\sigma = 0 \quad \forall (v_j)_{j=0\dots n} \in \mathbb{X}^+_h \, \Big\}
$$

where the symbol $⨍$ means that the quadrature has been achieved using the trapezoidal rule (which provides mass lumping). The discrete counterpart that we considered for $\mathbb{X}(\Gamma)$ and $\mathbb{Y}$ are given by

$$
\begin{aligned}
\mathbb{X}_h(\Gamma) \;\; &= \{ \, (v_j, q_j)_{j=0\dots n} \mid (v_j) \in \mathbb{X}^+_h, \; (q_j) \in \mathbb{X}^-_h \, \} \\
\mathbb{Y}_h \;\; &= \{ \, (v_j, q_j)_{j=0\dots n} \mid (v_j) \in \mathbb{X}^-_h, \; (q_j) \in \mathbb{X}^+_h \, \} \, .
\end{aligned}
$$

Thanks to this particular choice for the space $\mathbb{Y}_h$, the matrix associated to the term $\mathrm{B}(U_h, V_h)$ is symmetric. For the assembly of the matrix associated to the integral operator A, we used the MATLAB toolbox ie2m developped by A.Bendali. We used uniform meshes so that there exists a constant $C > 0$ such that

$$
\text{Number of degrees of freedom} = C/\mathrm{h}
$$

where h is the caracteristic size of the length of the segments in the mesh i.e. there exists $\alpha > 0$ such that $\alpha\,\mathrm{h} \leq \mathrm{length}(\Gamma^q_h) \leq \mathrm{h}/\alpha$ for any segment Γ^q_h of the mesh.

In figures **Fig.1** and **Fig.2** below we represent the relative errors $u_0 - u_{0,h}$ and $p_0 - p_{0,h}$, where on the one hand u_0, p_0 refer to the Dirichlet and Neumann component on $\partial\Omega_0$ of the exact solution, and on the other hand $u_{0,h}, p_{0,h}$ refer to the Dirichlet and Neumann component on

$\partial\Omega_0$ of the approximate solution. We represented these errors both for Formulation (15) and Formulation (10).

These pictures clearly show that relative errors are in $O(\mathrm{h})$. Although we use piecewise linear shape functions, we do not use curved boundary elements. As a consequence, this rate of convergence seems optimal because of the error induced by the approximation of Γ by means of simple segments.

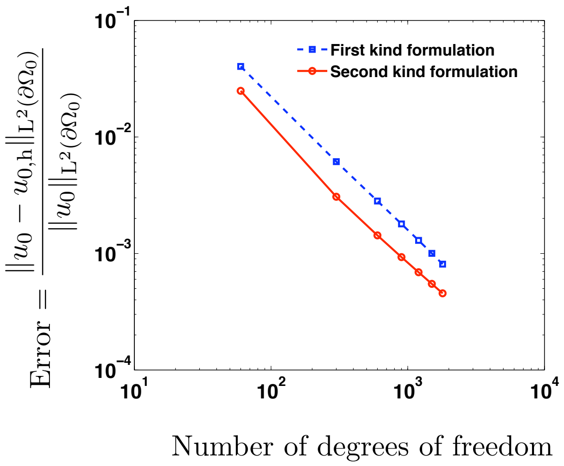


Fig.1 Consistency results for the Dirichlet traces

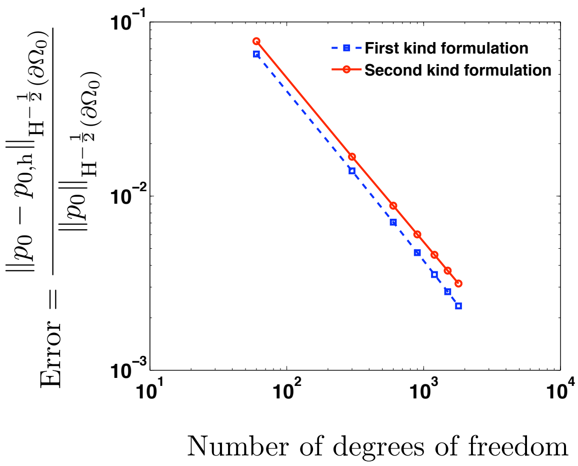


Fig.2 Consistency results for the Neumann traces

We also see that the second kind formulation (15) is as precise as the equation of the first kind studied in [16]. In figure **Fig.3** we represent the L^2 norm of the far field pattern for the first and second kind formulation.

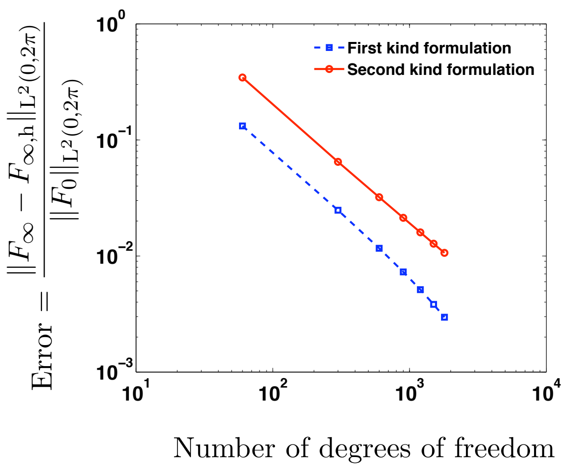


Fig.3 Consistency results for the far field pattern

In **Fig.3** the far field patterns $F_\infty(\theta)$ and $F_{\infty,\mathrm{h}}(\theta)$ are defined in the sense of Corollary 3.7 of chapter 3 in [4]. The main davantage of Formulation (15) is the condition number of the boundary element matrices that it induces. It remains bounded independently of the step of the mesh.

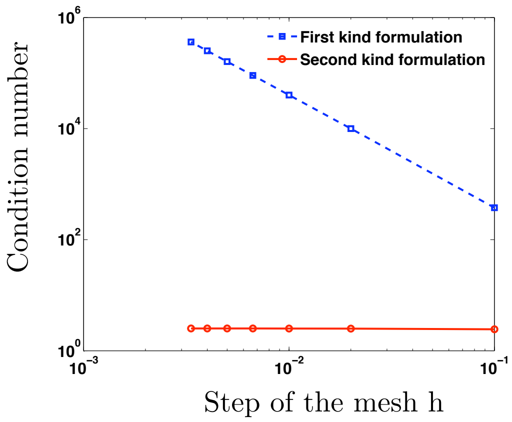


Fig.4 Stable conditioning

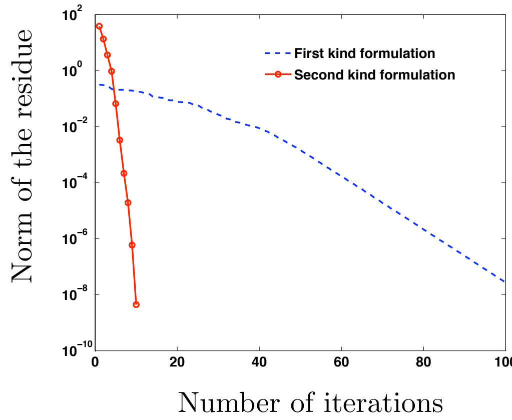


Fig.5 Convergence of GMRES with 1800 degrees of freedom

This is put into evidence in figure **Fig.4** and **Fig.5** above. In particular, we observe that the condition number of the matrix associated to (15) by means of a Galerkin discretization remains low, in accordance with §14.1 in [9], while the condition number of the matrix for the first kind formulation diverges like $O(h^{-1})$ which is standard, see §4.5 in [18]. Note however that we did not try to apply any preconditioner to the first kind formulation (10).

Appendix

Proposition 8.1.
For any $F \in \mathbb{H}(\Gamma)$, there exists a unique $U \in \mathbb{X}(\Gamma)$ such that

$$\mathrm{B}(\mathrm{C}_\kappa U, V) = \mathrm{B}(F, V) \quad \forall V \in \mathbb{X}(\Gamma)\,.$$

Proof:
According to (ii) §2.1 in [16], there exists a compact operator $\mathrm{T} : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ and a constant $\alpha > 0$ such that $\Re e\{\mathrm{B}(\,(\mathrm{C}_\kappa + \mathrm{T}\,)V, V)\} \geq \alpha \|V\|, \forall V \in \mathbb{H}(\Gamma)$ (note that our choice for the duality pairing differs from the one in [16]). As a consequence, according to Fredholm alternative, in order to prove the result, we only need to show that the only $U \in \mathbb{X}(\Gamma)$ satisfying $\mathrm{B}(\mathrm{C}_\kappa U, V) = 0$, $\forall V \in \mathbb{X}(\Gamma)$ is $U = 0$.

Take any $U = (U_0, \dots, U_n)^\top \in \mathbb{X}(\Gamma)$ satisfying $\mathrm{B}(\mathrm{C}_\kappa U, V) = 0$, $\forall V \in \mathbb{X}(\Gamma)$. Define $\psi_j(\mathbf{x}) = \mathrm{G}^j_{\kappa_j}\{U_j\}(\mathbf{x})$. First, let us prove that $\psi_j = 0$ in Ω_j for all $j = 0 \dots n$. Define $\varphi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\varphi|_{\Omega_j} = \psi_j$, and set $W_{\rm int} = (\mathrm{Id}/2 + \mathrm{C}_\kappa)U$. We have $W_{\rm int} = (\gamma^0(\varphi), \dots, \gamma^n(\varphi))$ and since $\mathrm{B}(W_{\rm int}, V) = \mathrm{B}((\mathrm{Id}/2 + \mathrm{C}_\kappa)U, V) = \mathrm{B}(\mathrm{C}_\kappa U, V) = 0\,, \forall V \in \mathbb{X}(\Gamma)$, we deduce that $W_{\rm int} \in \mathbb{X}(\Gamma)$ hence $\varphi \in \mathrm{H}^1_{\rm loc}(\Delta, \mathbb{R}^d)$. To sum up

$$\begin{aligned}
&\varphi \in \mathrm{H}^1_{\rm loc}(\Delta, \mathbb{R}^d) \quad \text{such that}\\
&\Delta\varphi + \kappa_j^2\varphi = 0 \quad \text{in } \Omega_j,\ j = 0 \dots n\\
&\varphi \text{ is outgoing radiating in } \Omega_0.
\end{aligned}$$

As a consequence φ is solution to an homogeneous transmission problem (that is classically well posed). Hence $\varphi = 0$ i.e. $\psi_j = 0$ in Ω_j for all $j = 0 \dots n$.

Now let us show that $\psi_j = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}_j$ for all $j = 0 \dots n$. Set $W_{\rm ext} = -(\mathrm{Id}/2 - \mathrm{C}_\kappa)U$. We have $\mathrm{B}(W_{\rm ext}, V) = -\mathrm{B}((\mathrm{Id}/2 - \mathrm{C}_\kappa)U, V) = \mathrm{B}(\mathrm{C}_\kappa U, V) = 0\,, \forall V \in \mathbb{X}(\Gamma)$ so that $W_{\rm ext} \in \mathbb{X}(\Gamma)$ according to (5). Clearly

$$\begin{aligned}
&\Delta\psi_j + \kappa_j^2\psi_j = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{\Omega}_j\\
&\text{and } \psi_j \text{ is outgoing (for the wave number } \kappa_j) \text{ for } j \neq 0.
\end{aligned}$$

Since $W_{\rm ext} \in \mathrm{Im}(\mathrm{Id}/2 - \mathrm{C}_\kappa)$, we have $W_{\rm ext} = (\gamma_c^0(\psi_0), \dots, \gamma_c^n(\psi_n)) \in \mathbb{X}(\Gamma)$. Take $r > 0$ large enough to ensure that $(\mathbb{R}^d \setminus \Omega_0) \subset \mathrm{B}_r = \{\, \mathbf{x} \in \mathbb{R}^d \mid |\mathbf{x}| < r \,\}$. Applying Green formulas in each $\mathrm{B}_r \setminus \overline{\Omega}_j$ we obtain

$$\begin{aligned}
\int_{\partial \mathrm{B}_r} \psi_j\, \partial_r \overline{\psi}_j d\sigma &= \int_{\mathrm{B}_r \setminus \overline{\Omega}_j} |\nabla\psi_j|^2 - \kappa_j^2\, |\psi_j|^2\, d\mathbf{x} + \int_{\partial\Omega_j} \gamma^j_{\mathrm{D},c}(\psi_j)\gamma^j_{\mathrm{N},c}(\overline{\psi}_j) d\sigma \quad \forall j \neq 0\\
0 &= \int_{\mathrm{B}_r \setminus \overline{\Omega}_0} |\nabla\psi_0|^2 - \kappa_0^2\, |\psi_0|^2\, d\mathbf{x} + \int_{\partial\Omega_0} \gamma^0_{\mathrm{D},c}(\psi_0)\gamma^0_{\mathrm{N},c}(\overline{\psi}_0) d\sigma.
\end{aligned}$$

In the equations above ∂_r refers to the radial derivative. Take the imaginary part of the identity above, and sum over $j = 0 \dots n$, taking into account that $(\gamma^j_{\mathrm{D},c}(\psi_j))_{0 \leq j \leq n} \in \mathbb{X}^{1/2}(\Gamma)$ and $(\gamma^j_{\mathrm{N},c}(\psi_j))_{0 \leq j \leq n} \in \mathbb{X}^{-1/2}(\Gamma)$ (since $W_{\rm ext} \in \mathbb{X}(\Gamma)$). This yields

$$\sum_{j=1}^n \mathrm{Im}\Big\{ \int_{\partial \mathrm{B}_r} \psi_j\, \partial_n \overline{\psi}_j d\sigma \Big\} = \mathrm{Im}\Big\{ \sum_{j=0}^n \int_{\partial\Omega_j} \gamma_{\mathrm{D},c}(\psi_j)\gamma_{\mathrm{N},c}(\overline{\psi}_j)\, d\sigma \Big\} = 0.$$

In the last equality above we used Proposition 2.1. Note that, by construction, ψ_j is outgoing radiating with respect to the wave number κ_j. Combining this condition at infinity with the identity above for $j = 1 \dots n$ yields

$$\begin{aligned}
&\sum_{j=1}^n \int_{\partial \mathrm{B}_r} |\partial_r\psi_j|^2 + \kappa_j^2 |\psi_j|^2 d\sigma\\
&\qquad = \sum_{j=1}^n \int_{\partial \mathrm{B}_r} |\partial_r\psi_j - i\kappa_j\psi_j|^2 d\sigma - \sum_{j=1}^n \mathrm{Im}\Big\{ \int_{\partial \mathrm{B}_r} \psi_j\, \partial_r \overline{\psi}_j d\sigma \Big\} \underset{r\to\infty}{\longrightarrow} 0.
\end{aligned}$$

This shows in particular that $\lim_{r\to\infty} \int_{\partial \mathrm{B}_r} |\psi_j|^2 d\sigma = 0$ for all $j = 1 \dots n$. As a consequence, we can apply Rellich Lemma, see Lemma 2.11 in [5], which implies that $\psi_j = 0$ in $\mathbb{R}^d \setminus \overline{\Omega}_j, j =$

$1\dots n$. There only remains to deal with ψ_0. According to the transmission conditions satisfied by ψ_0 we have $\gamma^0_{\mathrm{D},c}(\psi_0) = 0$ and $\gamma^0_{\mathrm{N},c}(\psi_0) = 0$. Hence $-\psi_0(\mathbf{x}) = \mathrm{G}^0_{\kappa_0}\{\gamma^0_c(\psi_0)\}(\mathbf{x}) = 0$ in $\mathbb{R}^d\setminus\overline{\Omega}_0$.

Since $U = (U_0, U_1, \dots, U_n)$ with $U_j = \gamma^j(\psi_j) - \gamma^j_c(\psi_j) = 0$ for all $j = 0\dots n$, we have $U = 0$. $\square$

Proposition 8.2.
The operator $\mathrm{A}_\kappa : \mathbb{H}(\Gamma) \to \mathbb{H}(\Gamma)$ defined by (11) has continuous dependency, in the sense of the operator norm (13), with respect to each wave number $\kappa_j, j = 0\dots n$.

Proof:
First of all recall that, for any open ball $\mathrm{B}_r \subset \mathbb{R}^d$ of radius $r > 0$ centered at 0 such that $(\mathbb{R}^d \setminus \Omega_0) \subset \mathrm{B}_r$, the trace operators $\gamma^i_{\mathrm{D}} : \mathrm{H}^1(\Delta, \mathrm{B}_r \cap \Omega_i) \to \mathrm{H}^{\frac{1}{2}}(\partial\Omega_i)$ and $\gamma^i_{\mathrm{N}} : \mathrm{H}^1(\Delta, \mathrm{B}_r \cap \Omega_i) \to \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_i)$ are continuous. As a consequence, according to the definition of A_κ, it suffices to show that $\mathrm{DL}^j_{\kappa_j}$ (resp. $\mathrm{SL}^j_{\kappa_j}$) continuously depends on κ_j as a continuous operator from $\mathrm{H}^{1/2}(\partial\Omega_j)$ (resp. $\mathrm{H}^{-1/2}(\partial\Omega_j)$) to $\mathrm{H}^1(\Delta, \mathrm{B}_r \cap \Omega_j)$ or $\mathrm{H}^1(\Delta, \mathrm{B}_r \setminus \overline{\Omega}_j)$, for any open ball $\mathrm{B}_r \subset \mathbb{R}^d$ of radius $r > 0$ centered at 0 such that $(\mathbb{R}^d \setminus \Omega_0) \subset \mathrm{B}_r$.

We will show this only for $j = 0$ and for a fixed B_r containing $\mathbb{R}^d\setminus\overline{\Omega}_0$, since for $j = 1, \dots n$ the proof is the same. Set $\psi_\kappa\{u, p\} = \mathrm{DL}^0_\kappa\{u\} + \mathrm{SL}^0_\kappa\{p\}$ for $(u, p) \in \mathrm{H}^{\frac{1}{2}}(\partial\Omega_0) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_0)$. Set also $\xi_\kappa\{u, p\} = \psi_\kappa\{u, p\} - \psi_{\kappa_0}\{u, p\}$. For each $\kappa \in \mathbb{C}$ define $\varphi_\kappa : \mathrm{H}^{\frac{1}{2}}(\partial\Omega_0) \times \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_0) \to \mathrm{H}^{\frac{1}{2}}(\partial\mathrm{B}_r) \times \mathrm{H}^{-\frac{1}{2}}(\partial\mathrm{B}_r)$ by

$$\varphi_\kappa\{u, p\} = -\partial_r\xi_\kappa\{u, p\}|_{\partial\mathrm{B}_r} - i\,\xi_\kappa\{u, p\}|_{\partial\mathrm{B}_r}\,.$$

Note that $\varphi_{\kappa_0} = 0$. Moreover, since $\partial\mathrm{B}_r \cap \partial\Omega_0 = \emptyset$, the integral kernel involved in the definition of φ_κ is of class C^∞, so the family φ_κ admits continuous dependency with respect to κ. To be more precise, there exists $\alpha : \mathbb{C} \to \mathbb{R}_+$ with $\lim_{\kappa\to\kappa_0} \alpha(\kappa) = 0$ such that

$$\|\varphi_\kappa\{u, p\}\|_{\mathrm{H}^{\frac{1}{2}}(\partial\mathrm{B}_r)} \leq\ \alpha(\kappa)\,\left(\|u\|_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega_0)} + \|p\|_{\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_0)}\right)$$

for any $(u, p) \in \mathrm{H}^{\frac{1}{2}}(\partial\mathrm{B}_r) \times \mathrm{H}^{-\frac{1}{2}}(\partial\mathrm{B}_r)$. Note also that $[\gamma^0_{\mathrm{D}}] \cdot \xi_\kappa\{u, p\} = 0$ and $[\gamma^0_{\mathrm{N}}] \cdot \xi_\kappa\{u, p\} = 0$ where $[\gamma^0_{\mathrm{D}}]$ and $[\gamma^0_{\mathrm{N}}]$ were defined in (1). Besides $\xi_\kappa\{u, p\}$ satisfies

$$\begin{aligned} &-\Delta\xi_\kappa\{u, p\} - \kappa^2\xi_\kappa\{u, p\} = (\kappa^2 - \kappa_0^2)\psi_{\kappa_0}\{u, p\} && \text{in } \mathrm{B}_r \\ &\partial_r\xi_\kappa\{u, p\}|_{\partial\mathrm{B}_r} + i\xi_\kappa\{u, p\}|_{\partial\mathrm{B}_r} = \varphi_\kappa\{u, p\} && \text{on } \partial\mathrm{B}_r \end{aligned}$$

The equations above can be put in a variational form: $\xi_\kappa\{u, p\} \in \mathrm{H}^1(\mathrm{B}_r)$ satisfies

$$\begin{aligned} &a_\kappa(\xi_\kappa\{u, p\}, \eta) = (\kappa^2 - \kappa_0^2)\int_{\mathrm{B}_r} \psi_{\kappa_0}\{u, p\}\,\overline{\eta}\,d\mathbf{x} + \int_{\partial\mathrm{B}_r} \varphi_\kappa\{u, p\}\,\overline{\eta}\,d\sigma \quad \forall\eta \in \mathrm{H}^1(\mathrm{B}_r) \\ &a_\kappa(\xi, \eta) = \int_{\mathrm{B}_r} \nabla\xi\cdot\nabla\overline{\eta}\,d\mathbf{x} - \kappa^2\int_{\mathrm{B}_r} \xi\overline{\eta}\,d\mathbf{x} + i\int_{\partial\mathrm{B}_r} \xi\,\overline{\eta}\,d\sigma \quad \forall\xi, \eta \in \mathrm{H}^1(\mathrm{B}_r)\,. \end{aligned}$$

The form $a_\kappa(\ ,\ )$ is sesquilinear and continuous. Consider a scalar product of $\mathrm{H}^1(\mathrm{B}_r)$ that we denote $\langle\ \rangle_{1,\mathrm{B}_r}$. According to Riesz representation theorem, there exists a unique continuous operator $\mathcal{R}_\kappa : \mathrm{H}^1(\mathrm{B}_r) \to \mathrm{H}^1(\mathrm{B}_r)$ and a unique $\mathcal{F}_\kappa\{V\} \in \mathrm{H}^1(\mathrm{B}_r)$ such that

$$\begin{aligned} &\langle\mathcal{R}_\kappa\xi, \eta\rangle_{1,\mathrm{B}_r} = a_\kappa(\xi, \eta) \\ &\langle\mathcal{F}_\kappa\{u, p\}, \eta\rangle_{1,\mathrm{B}_r} = \int_{\mathrm{B}_r} (\kappa^2 - \kappa_0^2)\psi_{\kappa_0}\{u, p\}\,\overline{\eta}\,d\mathbf{x} + \int_{\partial\mathrm{B}_r} \varphi_\kappa\{u, p\}\,\overline{\eta}\,d\sigma \quad \forall\xi, \eta \in \mathrm{H}^1(\mathrm{B}_r) \end{aligned}$$

As a family valued in $\mathcal{L}(\mathrm{H}^1(\mathrm{B}_r), \mathrm{H}^1(\mathrm{B}_r))$ (the space of continuous linear maps from $\mathrm{H}^1(\mathrm{B}_r)$ to $\mathrm{H}^1(\mathrm{B}_r)$), the function $\kappa \mapsto \mathcal{R}_\kappa$ has analytical dependency. For $\kappa \in i\mathbb{R} \setminus \{0\}$, the operator $\mathcal{R}_\kappa$ is invertible since $\Re e\{a_\kappa(\xi,\xi)\} \geq \min(1,|\kappa|)\, \|\xi\|_{\mathrm{H}^1(\mathrm{B}_r)}$ in this case. Besides, the operator $\mathcal{R}_\kappa$ is of Fredholm type for any κ as the only term depending on κ in $a_\kappa(\ ,\ )$ is a compact term. As a consequence we can apply analytic Fredholm theory to the family $\mathcal{R}_\kappa$, see for exemple Corollary 8.4 of chapter XI in [6], and conclude that $\mathcal{R}_\kappa^{-1}$ is well defined and analytic except at countably many isolated poles.

Let us verify that $\mathcal{R}_\kappa$ is invertible at any $\kappa \in \mathbb{R}_+$. Since $\mathcal{R}_\kappa$ is of Fredholm type, we only need to check that $\mathcal{R}_\kappa$ is one-to-one for $\kappa \in \mathbb{R}_+$. Assume that $\mathcal{R}_\kappa\xi = 0$ for some $\xi \in \mathrm{H}^1(\mathrm{B}_r)$. In this case we have $\mathrm{Im}\{a_\kappa(\xi,\xi)\} = \int_{\partial\mathrm{B}_r} |\xi|^2 d\sigma = 0$, so that $\xi|_{\partial\mathrm{B}_r} = 0$. From the variational formulation $a_\kappa(\xi,\eta) = 0$ for all $\eta \in \mathrm{H}^1(\mathrm{B}_r)$ we also obtain that $\Delta\xi + \kappa^2\xi = 0$ in B_r and $\partial_r\xi|_{\partial\mathrm{B}_r} + i\xi|_{\partial\mathrm{B}_r} = 0$. As a consequence we have both $\partial_r\xi|_{\partial\mathrm{B}_r} = 0$ and $u|_{\partial\mathrm{B}_r} = 0$. Since $\Delta\xi + \kappa^2\xi = 0$ in B_r, this is a consequence of Proposition 3.1 (considering B_r instead of Ω_j) that $u = 0$.

Since φ_κ continuously depends on κ, so does $\mathcal{F}_\kappa\{u,p\}$. Besides we also have $\mathcal{F}_{\kappa_0}\{u,p\} = 0$. Hence $\psi_\kappa\{u,p\} = \mathcal{R}_\kappa^{-1} \cdot \mathcal{F}_\kappa\{u,p\}$ admits continuous dependency with respect to κ, i.e. there exists $\beta : \mathbb{C} \to \mathbb{R}_+$ such that $\lim_{\kappa\to\kappa_0} \beta(\kappa) = 0$ and

$$\begin{aligned}
&\|\psi_\kappa\{u,p\}\|_{\mathrm{H}^1(\mathrm{B}_r)} \leq \beta(\kappa) \left(\|u\|_{\mathrm{H}^{\frac{1}{2}}(\partial\Omega_0)} + \|p\|_{\mathrm{H}^{-\frac{1}{2}}(\partial\Omega_0)} \right) \\
&\forall u \in \mathrm{H}^{\frac{1}{2}}(\partial\Omega_0)\ ,\ \ \forall p \in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega_0)
\end{aligned}$$

There only remain to check that continuity holds also for $\|\Delta\psi_\kappa\{u,p\}\|_{\mathrm{L}^2(\mathrm{B}_r)}$. This is a consequence of the equation $-\Delta\xi_\kappa\{u,p\} = \kappa^2\xi_\kappa\{u,p\} + (\kappa^2 - \kappa_0^2)\psi_{\kappa_0}\{u,p\}$ in B_r. □

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