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# EXPONENTIALLY LOCALISED INTERFACE EIGENMODES IN FINITE CHAINS OF RESONATORS

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ABSTRACT. This paper studies wave localisation in chains of finitely many resonators. There is an extensive theory predicting the existence of localised modes induced by defects in infinitely periodic systems. This work extends these principles to finite-sized systems. We consider finite systems of subwavelength resonators arranged in dimers that have a geometric defect in the structure. This is a classical wave analogue of the Su-Schrieffer-Heeger model. We prove the existence of a spectral gap for defectless finite dimer structures and then show the existence of an eigenvalue in the gap of the defect structure. We find a direct relationship between an eigenvalue being within the spectral gap and the localisation of its associated eigenmode, which we show is exponentially localised. To the best of our knowledge, our method, based on Chebyshev polynomials, is the first to characterise quantitatively the localised interface modes in systems of finitely many resonators.

**Keywords.** Finite Hermitian systems, subwavelength resonators, capacitance matrix, topological protection, Chebyshev polynomials, wave localisation

**AMS Subject classifications.** 34L40, 34L20, 34L25,

## 1. Introduction

Wave localisation at subwavelength scales has many important applications in nanophotonics and nanophononics [9, 18, 20]. Here, *subwavelength* means that the incident wavelengths are much larger than the size of the building blocks of the structure. When these relatively small building blocks are locally resonant (which, in the case studied here, will be due to large material contrasts) this allows for waves to be localised at subwavelength scales, thereby beating traditional diffraction limits [4, 5, 7]. This principle has unlocked a wealth of novel nanotechnologies.

In this paper, we consider wave localisation at the interface between two systems of *finitely many* subwavelength resonators. These resonators are arranged in pairs or dimers, such that the model we consider shares many of the features of the Su-Schrieffer-Heeger (SSH) model in quantum mechanics [23]. We prove the existence of exponentially localised interface eigenmodes in this finite structure. These interface modes have been subject to numerous studies in the setting of infinite structures. A particular focus has been put on studying the topological properties of infinite periodic structures and then introducing carefully designed interfaces so as to create so-called *topologically protected* eigenmodes [8]. These modes have significant implications for applications since they are expected to be robust with respect to imperfections in the design. These concepts have been widely studied in a variety of settings, most notably in quantum mechanics for the Schrödinger operator [15, 16] and more recently, for related continuous models of classical wave systems in [12, 19, 22, 24, 25]. In finite-sized systems, these eigenmodes have been observed both experimentally and numerically; see, for instance, [2, 8, 10, 28] and references therein.

The present work considers the far less-explored but more realistic physical setting of interface modes in finite dimer structures. As far as we know, it is the first work to deal with

the existence of interface modes in finite structures. It provides a one-to-one correspondence between the position of the eigenfrequency in the spectrum of the corresponding infinite periodic structure (*i.e.*, in the asymptotic spectral bulk, its boundary or in the asymptotic spectral gap; see Definition 4.1) and the behaviour (localised versus delocalised) of the corresponding eigenmode. Our results hold for any finite system of dimer structure with defect satisfying a mild condition on the size of the resonators. Furthermore, we show that the eigenfrequencies lying in the gap converge exponentially as the size of the structure goes to infinity and provide an explicit and simple formula for the limit. We also show that the Hermitian nature of the system together with the position of the interface eigenfrequency in the gap yields a very strong stability property when the geometry of the system is perturbed.

Our approach is based on the capacitance matrix, which in general is a powerful tool for characterising the subwavelength eigenfrequencies of a system of high-contrast resonators [2, 6, 17]. In essence, the capacitance formulation provides a discrete approximation to the continuous spectral problem of the differential model, valid in the high-contrast asymptotic limit. This approximation is based solely on first principles and provides a natural starting point for both theoretical analysis and numerical simulation of wave localisation in the subwavelength regime [3, 4, 7]. In the case of our one-dimensional, finite system of dimer resonators with a geometric defect, the capacitance matrix is a perturbed tridiagonal block 2-Toeplitz matrix; see (2.6). Based on some properties of the Chebyshev polynomials, we prove existence and uniqueness of an eigenfrequency in the gap for the finite dimer structures with a geometric defect and show exponential localisation of the corresponding eigenmode. Our proof does not rely on any perturbation argument neither on any a priori assumption on the band gaps of the two dimer systems in the structure.

The paper is organised as follows. In Section 2, we introduce the capacitance matrix approximation of a finite, dimer system with a geometric defect in order to approximate its eigenfrequencies and associated eigenmodes. In Section 3, we first write the characteristic polynomial of the capacitance matrix associated to a defectless, finite dimer system in terms of Chebyshev polynomials. Then, we characterise the structure of the eigenvectors of the capacitance matrix associated with the perturbed dimer structure. Section 4 is devoted to the derivation of a direct relationship between an eigenvalue being within the spectral gap and the localisation of its corresponding eigenvector. In Section 5, we analyse the limiting behavior of the system as its size goes to infinity and show that the eigenvalue of the capacitance matrix lying in the asymptotic spectral gap converges exponentially fast to a value in the gap. In Section 6, we analyse the robustness of the interface localisation with respect to imperfections in the structure design. The paper ends with some concluding remarks in Section 7. In Appendix A, we recall some basic definitions and results on pseudo-spectra of normal matrices. Appendix B is devoted to the discussion of the topological origin of the robustness of the interface eigenmodes.

## 2. One-dimensional subwavelength resonator systems

We consider a one-dimensional chain of  $N$  disjoint identical high-contrast resonators  $D_i := (x_i^L, x_i^R)$ , where  $(x_i^L, x_i^R)_{1 \leq i \leq N} \subset \mathbb{R}$  are the  $2N$  boundaries satisfying  $x_i^L < x_i^R < x_{i+1}^L$  for any  $1 \leq i \leq N - 1$ . We fix the coordinates such that  $x_1^L = 0$ . We also denote by  $\ell_i = x_i^R - x_i^L$  the length of the each of the resonators, and by  $s_i = x_{i+1}^L - x_i^R$  the spacing between the  $i$ -th and  $(i + 1)$ -th inclusions. The system is illustrated in Figure 1.

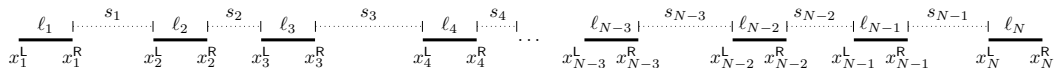


FIGURE 1. A chain of  $N$  resonators, with lengths  $(\ell_i)_{1 \leq i \leq N}$  and spacings  $(s_i)_{1 \leq i \leq N-1}$  (which will be chosen to alternate between two distinct values, as depicted).

We use

$$D := \bigcup_{i=1}^N (x_i^L, x_i^R)$$

to denote the set of subwavelength resonators. In this paper, we only consider systems of identically sized resonators, that is

$$\ell_i = \ell \in \mathbb{R}_{>0} \text{ for all } 1 \leq i \leq N.$$

This will simplify the formulas in our subsequent analysis and is sufficient to observe the physical phenomena we are interested in.

In this work, we consider the one-dimensional wave equation propagating in a heterogeneous medium with space-dependent material parameters:

$$\frac{\omega^2}{\kappa(x)} u(x) + \frac{d}{dx} \left( \frac{1}{\rho(x)} \frac{d}{dx} u(x) \right) = 0, \quad x \in \mathbb{R}. \quad (2.1)$$

The material parameters  $\kappa(x)$  and  $\rho(x)$  are piecewise constant in the interior and exterior of the resonators

$$\kappa(x) = \begin{cases} \kappa_b, & x \in D, \\ \kappa, & x \in \mathbb{R} \setminus D, \end{cases} \quad \text{and} \quad \rho(x) = \begin{cases} \rho_b, & x \in D, \\ \rho, & x \in \mathbb{R} \setminus D, \end{cases}$$

where the constants  $\rho_b, \rho, \kappa, \kappa_b \in \mathbb{R}_{>0}$ . The wave speeds inside the set  $D$  of resonators and inside the background medium  $\mathbb{R} \setminus D$ , are denoted respectively by  $v_b$  and  $v$ , the wave numbers respectively by  $k_b$  and  $k$ , and the contrast between the densities of the resonators and the background medium by  $\delta$ :

$$v_b := \sqrt{\frac{\kappa_b}{\rho_b}}, \quad v := \sqrt{\frac{\kappa}{\rho}}, \quad k_b := \frac{\omega}{v_b}, \quad k := \frac{\omega}{v}, \quad \delta := \frac{\rho_b}{\rho}. \quad (2.2)$$

For these step-wise defined material parameters, the wave problem determined by (2.1) reduces to the following system of coupled one-dimensional Helmholtz equations:

$$\begin{cases} \frac{d^2}{dx^2} u(x) + \frac{\omega^2}{v^2} u(x) = 0, & x \in \mathbb{R} \setminus D, \\ \frac{d^2}{dx^2} u(x) + \frac{\omega^2}{v_b^2} u(x) = 0, & x \in D, \\ u|_{\mathbb{R}}(x_i^{L,R}) - u|_{\mathbb{L}}(x_i^{L,R}) = 0, & 1 \leq i \leq N, \\ \frac{du}{dx} \Big|_{\mathbb{R}}(x_i^L) - \delta \frac{du}{dx} \Big|_{\mathbb{L}}(x_i^L) = 0, & 1 \leq i \leq N, \\ \delta \frac{du}{dx} \Big|_{\mathbb{R}}(x_i^R) - \frac{du}{dx} \Big|_{\mathbb{L}}(x_i^R) = 0, & 1 \leq i \leq N, \end{cases} \quad (2.3)$$

where for a one-dimensional function  $w$  we denote by

$$w|_{\mathbb{L}}(x) := \lim_{\substack{s \rightarrow 0 \\ s > 0}} w(x-s) \quad \text{and} \quad w|_{\mathbb{R}}(x) := \lim_{\substack{s \rightarrow 0 \\ s > 0}} w(x+s)$$

if the limits exist.

We are interested in the high-contrast regime characterised by the contrast parameter  $\delta$  being small. In this case, there exist resonant modes of the system at subwavelength frequencies, for which the size of the resonators is smaller than the wavelength in the background medium.

**DEFINITION 2.1.** A resonance  $\omega(\delta) \in \mathbb{C}$  for which there exists a non trivial solution to (2.3) is called *subwavelength* in the high-contrast regime if

$$\omega(\delta) \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0.$$





### 3. Perturbed tridiagonal 2-Toeplitz matrices

In this section we will briefly recall results established in [3] about eigenvalues and eigenvectors of tridiagonal 2-Toeplitz matrices with perturbations on the corners.

Denote by

$$A_{2k+1}^{(a,b)}(\alpha, \beta_1, \beta_2) := \begin{pmatrix} \alpha + a & \beta_1 & 0 & 0 & \dots & 0 & 0 \\ \beta_1 & \alpha & \beta_2 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \alpha & \beta_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \alpha & \beta_2 \\ 0 & 0 & 0 & 0 & \dots & \beta_2 & \alpha + b \end{pmatrix} \in \mathbb{R}^{(2k+1) \times (2k+1)} \quad (3.1)$$

and

$$A_{2k}^{(a,b)}(\alpha, \beta_1, \beta_2) := \begin{pmatrix} \alpha + a & \beta_1 & 0 & 0 & \dots & 0 & 0 \\ \beta_1 & \alpha & \beta_2 & 0 & \dots & 0 & 0 \\ 0 & \beta_2 & \alpha & \beta_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \alpha & \beta_1 \\ 0 & 0 & 0 & 0 & \dots & \beta_1 & \alpha + b \end{pmatrix} \in \mathbb{R}^{2k \times 2k} \quad (3.2)$$

the prototypical tridiagonal 2-Toeplitz matrices with perturbations on the corners.

#### 3.1. Eigenvalues

We define the polynomials

$$P_k^*(x) := (\beta_1 \beta_2)^k U_k \left( \frac{(x - \alpha)^2 - \beta_1^2 - \beta_2^2}{2\beta_1 \beta_2} \right), \quad (3.3)$$

where  $U_k$  is the Chebyshev polynomial of the second kind, as well as the function

$$y(z) := \frac{z^2 - \beta_1^2 - \beta_2^2}{2\beta_1 \beta_2}. \quad (3.4)$$

The following proposition holds.

**PROPOSITION 3.1 (Eigenvalues).** *The characteristic polynomials of  $A_{2k+1}^{(a,b)}$  and  $A_{2k}^{(a,b)}$  are respectively*

$$\chi_{A_{2k+1}^{(a,b)}}(x) = (x - \alpha - a - b) P_k^*(x) + (ab(x - \alpha) - a\beta_1^2 - b\beta_2^2) P_{k-1}^*(x) \quad (3.5)$$

and

$$\chi_{A_{2k}^{(a,b)}}(x) = P_k^*(x) + ((a + b)(\alpha - x) + ab + \beta_2^2) P_{k-1}^*(x) + ab\beta_1^2 P_{k-2}^*(x). \quad (3.6)$$

#### 3.2. Eigenvectors

We start by defining two families of polynomials  $\widehat{p}_{k+1}^{(\xi_p, \xi_q)}(x)$  and  $\widehat{q}_{k+1}^{(\xi_p, \xi_q)}(x)$  as solutions to the recursion relations

$$\widehat{p}_0^{(\xi_p, \xi_q)}(\mu) = \xi_p, \quad \widehat{p}_1^{(\xi_p, \xi_q)}(\mu) = 2\mu\xi_p + \frac{\xi_p - \xi_q}{\beta}, \quad (3.7)$$

$$\widehat{p}_{k+1}^{(\xi_p, \xi_q)}(\mu) = 2\mu\widehat{p}_k^{(\xi_p, \xi_q)}(\mu) - \widehat{p}_{k-1}^{(\xi_p, \xi_q)}(\mu),$$

and

$$\widehat{q}_0^{(\xi_p, \xi_q)}(\mu) = \xi_q, \quad \widehat{q}_1^{(\xi_p, \xi_q)}(\mu) = (2\mu + \beta)\xi_p + \frac{\xi_p - \xi_q}{\beta}, \quad (3.8)$$

$$\widehat{q}_{k+1}^{(\xi_p, \xi_q)}(\mu) = 2\mu\widehat{q}_k^{(\xi_p, \xi_q)}(\mu) - \widehat{q}_{k-1}^{(\xi_p, \xi_q)}(\mu),$$

where  $\beta = y\beta_2/\beta_1$ . Then we state the following result.





and thus,

$$x_4 = -\frac{\beta_2^2}{\beta_1^2}x_2 + x_3,$$

where  $\zeta = \frac{\beta_2^2}{\beta_1^2}$ . Continuing the process, we can easily verify that

$$x_{2k+1} = \frac{(\alpha - \lambda)^2}{\beta_1^2}x_{2k} - x_{2k-1}, \quad 1 \leq k \leq m, \quad (3.14)$$

$$x_{2k} = x_{2k-1} - \frac{\beta_2^2}{\beta_1^2}x_{2k-2}, \quad 2 \leq k \leq m. \quad (3.15)$$

Applying to (3.14) the same manipulations as those used in the proof of [3, Theorem 3.3], it is now easy to see that  $\mathbf{x}$  is of the form (3.9).

**Step 2.** We conclude the proof by considering the last  $2m$  rows. By the fact that  $\mathcal{C}$  is  $S = \text{antidiag}(1, \dots, 1) \in \mathbb{R}^{(4m+1) \times (4m+1)}$  symmetric, *i.e.*,  $SCS = \mathcal{C}$ , and  $\mathcal{C}$  has only simple eigenvalues [17] we can immediately conclude that  $S\mathbf{v} = \pm\mathbf{v}$  and thus obtain the desired result.  $\blacksquare$

## 4. Localised interface modes

In this section we will prove the existence of a spectral gap for a defectless structure of dimers and then show the existence of an eigenvalue in the gap for the defect structure. Furthermore, we will demonstrate that a direct relationship exists between an eigenvalue being within the spectral gap and the localisation of its corresponding eigenvector.

### 4.1. Asymptotic spectral gap

We first show the existence of a spectral gap for the capacitance matrix of an unperturbed structure of dimers.

By physical considerations we assume  $\beta_1, \beta_2 < 0$  and  $\beta_1 < \beta_2$ . It will often be useful to shift  $\mathcal{C}$  in order to have most diagonal entries zero, to that end we introduce  $z(x) := x - \alpha$ .

**DEFINITION 4.1** (Spectral bulk and gaps). Consider a finite structure of resonators. We define the *asymptotic spectral bulk*  $\Sigma$  and *asymptotic spectral gap*  $\Gamma$  of the structure as the spectral bulk and spectral gap of the associated infinite periodic system, respectively.

The spectral gap and spectral bulk of infinite periodic dimer systems have been computed in [2, Lemma 5.3].

**PROPOSITION 4.2.** Consider a system of repeated dimers (without defect) with  $N = 2m$  resonators. Denote by  $C_N$  the associated capacitance matrix and let  $\Sigma$  be the asymptotic spectral bulk. Then

$$\Sigma = \overline{\lim_{N \rightarrow \infty} \sigma(C_N)} = \left[0, \frac{2}{s_2}\right] \cup \left[\frac{2}{s_1}, \frac{2}{s_1} + \frac{2}{s_2}\right],$$

where  $\lim$  denotes the Hausdorff limit. Consequently, the asymptotic spectral gap is

$$\Gamma = \left(\frac{2}{s_2}, \frac{2}{s_1}\right) \subset \mathbb{R}.$$

*Proof.* From (3.6) we see that any eigenvalue  $\lambda$  of the capacitance matrix of a structure of dimer without defect — that is, a capacitance matrix of the form  $C = A_{2m}^{(\beta_2, \beta_2)}$  — satisfy

$$\begin{aligned} 0 &= P_m^*(\lambda) + (2\beta_2 z + 2\beta_2^2 + \beta_2^2)P_{m-1}^*(\lambda) + \beta_2^2 \beta_1^2 P_{m-2}^* \\ &= \beta_1^m \beta_2^m U_m(y) + (2\beta_2 z + 2\beta_2^2) \beta_1^{m-1} \beta_2^{m-1} U_{m-1}(y) + \beta_1^m \beta_2^m U_{m-2}(y), \end{aligned}$$

where for ease of notation we use  $z$  for  $z(\lambda)$  and  $y$  for  $y(z(\lambda))$ . Using the recurrence relation of the Chebyshev polynomials of the second kind we get that

$$\begin{aligned} 0 &= \beta_1^m \beta_2^m [2yU_{m-1}(y) - U_{m-2}(y)] \\ &\quad + (2\beta_2 z + 2\beta_2^2) \beta_1^{m-1} \beta_2^{m-1} U_{m-1}(y) + \beta_1^m \beta_2^m U_{m-2}(y) \\ \Leftrightarrow 0 &= U_{m-1}(y) [(2\beta_2 z + 2\beta_2^2) \beta_1^{m-1} \beta_2^{m-1} + \beta_1^m \beta_2^m 2y] \\ \Leftrightarrow 0 &= U_{m-1}(y) [(z + \beta_2) + \beta_1 y]. \end{aligned}$$

So the eigenvalues are given by

$$\lambda = \alpha + (\beta_1 - \beta_2), \quad \lambda = \alpha + (-\beta_1 - \beta_2) \quad \text{and} \quad \lambda = \alpha \pm \sqrt{\beta_1^2 + 2\beta_1\beta_2 \cos\left(\frac{k\pi}{m}\right) + \beta_2^2}, \quad (4.1)$$

for  $1 \leq k \leq m-1$ . The sought result follows now directly from inserting the definition of  $\alpha, \beta_1, \beta_2$  in terms of  $s_1$  and  $s_2$  as in (2.7).  $\blacksquare$

#### 4.2. Existence and uniqueness of eigenvalues in the gap

In this subsection, we will be working with a dimer structure with defect as in Figure 2, whose subwavelength spectrum is given by the eigenvalues of  $\mathcal{C}$  from (2.6).

We start with a simple monotonicity result for Chebyshev polynomials of second kind.

LEMMA 4.3. *Let  $k \in \mathbb{N}$  then*

$$\frac{U_{k-1}(x)}{U_k(x)}$$

*is strictly decreasing for  $x \in (-\infty, -1) \cup (1, +\infty)$  for any  $k \in \mathbb{N}$ .*

*Proof.* Considering the derivative of  $\frac{U_{k-1}(x)}{U_k(x)}$ , we have

$$\left(\frac{U_{k-1}(x)}{U_k(x)}\right)' = \frac{U_k(x)U_{k-1}'(x) - U_k'(x)U_{k-1}(x)}{U_k^2(x)}. \quad (4.2)$$

A well-known property of Chebyshev polynomials is that

$$U_k'(x) = \frac{(k+1)T_{k+1}(x) - xU_k(x)}{x^2 - 1},$$

where  $T_k(x)$  are the Chebyshev polynomials of the first kind. Thus, to demonstrate the negativity of  $R_k'(x)$  for  $|x| > 1$ , we only need to show that

$$\begin{aligned} 0 &> U_k(x)[kT_k(x) - xU_{k-1}(x)] - [(k+1)T_{k+1}(x) - xU_k(x)]U_{k-1}(x) \\ &= U_k(x)kT_k(x) - (k+1)T_{k+1}(x)U_{k-1}(x). \end{aligned} \quad (4.3)$$

It is also well-known that

$$T_\ell(x)U_n(x) = \begin{cases} \frac{1}{2}(U_{\ell+n}(x) + U_{n-\ell}(x)), & \text{if } n \geq \ell - 1, \\ \frac{1}{2}(U_{\ell+n}(x) - U_{\ell-n-2}(x)), & \text{if } n \leq \ell - 2. \end{cases} \quad (4.4)$$

So that (4.3) becomes

$$\frac{k}{2}(U_{2k}(x) + U_0(x)) - \frac{k+1}{2}(U_{2k}(x) + U_0(x)) = -\frac{U_{2k}(x) + 1}{2}.$$

By  $U_{2k}(x) > 0, x \in (-\infty, -1) \cup (1, +\infty)$  and the proof is complete.  $\blacksquare$

In order to show the existence of a frequency in the asymptotic spectral gap  $\Gamma$ , we first consider the submatrix  $C_1$  from (2.6) and prove the following result.





with  $A = \frac{r^{1-m}(c_1r-c_2)}{r^2-1} = \mathcal{O}(\frac{1}{r^m})$  and  $B = \frac{r^m(c_2r-c_1)}{r^2-1} = \mathcal{O}(r^{m-1})$  as  $m \rightarrow \infty$  for  $c_1, c_2 \in \mathbb{R}$  independent of  $m$ . The same asymptotics (with a slight different formula) hold for  $\tilde{A}$  and  $\tilde{B}$ ;

if  $y(\lambda)^2 < 1$ :

$$\begin{aligned} v^{(|2m-2j|)} &= A \cos(j\theta) + B \sin(j\theta), \\ v^{(|2m-2j-1|)} &= \tilde{A} \cos(j\theta) + \tilde{B} \sin(j\theta), \end{aligned}$$

with  $r = e^{i\theta}$  and  $A, B, \tilde{A}, \tilde{B}$  bounded as  $m \rightarrow \infty$ ;

if  $y(\lambda)^2 = 1$ :  $r = \pm 1$  and

$$\begin{aligned} v^{(|2m-2j|)} &= Ar_1^j + Br_1^j \cdot j, \\ v^{(|2m-2j-1|)} &= \tilde{A}r_1^j + \tilde{B}r_1^j \cdot j, \end{aligned}$$

with  $A = \frac{r^{1-m}(c_1mr-c_1r-c_2m)}{mr^2-m-r^2}$  and  $B = \frac{r^m(c_2r-c_1)}{mr^2-m-r^2}$  as  $m \rightarrow \infty$  for  $c_1, c_2 \in \mathbb{R}$  independent of  $m$ . The same asymptotics (with a slight different formula) hold for  $\tilde{A}$  and  $\tilde{B}$ .

*Proof.* It is enough to consider the  $\hat{p}_k^{(\xi_p, \xi_q)}$  and  $\hat{q}_k^{(\xi_p, \xi_q)}$  part of the eigenvector in Proposition 3.2 as the rest is a multiplicative factor. We only consider  $\hat{p}_k^{(\xi_p, \xi_q)}$  as the procedure for  $\hat{q}_k^{(\xi_p, \xi_q)}$  is exactly the same. Let  $(\lambda, v)$  be an eigenpair and recall from (3.7) the recurrence formula

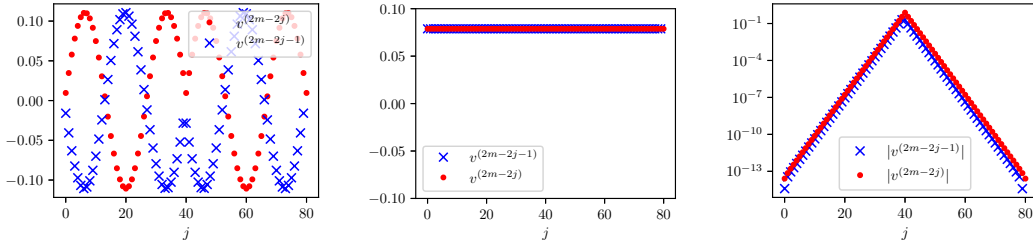
$$\hat{p}_{k+1}^{(\xi_p, \xi_q)}(\mu) = 2\mu\hat{p}_k^{(\xi_p, \xi_q)}(\mu) - \hat{p}_{k-1}^{(\xi_p, \xi_q)}(\mu),$$

where  $\mu = y(\lambda)$ . This is a two term recurrence formula with characteristic equation

$$X^2 - 2\mu X + 1 = 0 \quad (4.7)$$

having none, one or two roots  $r_1$  and  $r_2$  in exactly the cases delineated in the proposition. The formulas for the eigenvectors follow. By Vieta's formula, we know that the two roots of (4.7) satisfy  $r_1r_2 = 1$  so one of them must satisfy  $|r_i| \geq 1$ . The constants  $A, B, \tilde{A}, \tilde{B} \in \mathbb{R}$  follow from the initial conditions (3.7) with the values expressed in Proposition 3.3. ■

We remark that the three cases presented in Proposition 4.7 correspond to  $\lambda \in \Gamma$ ,  $\lambda \in \Sigma$  and  $\lambda \in \partial\Sigma$ , respectively. We show this behaviour in Figure 3.



(A) Eigenvector associated to an eigenvalue in the asymptotic spectral bulk. (B) Eigenvector associated to an eigenvalue in the boundary of the asymptotic spectral bulk. (C) Eigenvector associated to an eigenvalue in the asymptotic spectral gap.  $y$  axis in  $\log$ -scale.

FIGURE 4. Eigenvector behaviour based on the location of eigenvalue. Computation performed with  $N = 81$ ,  $s_1 = 1$ ,  $s_2 = 3$ .

**Theorem 4.8.** Consider a dimer structure with a defect as described by Figure 2 composed of  $4m + 1$  resonators with  $2m > \frac{s_2}{s_2 - s_1}$ . Then, there exists exactly one subwavelength interface mode.

*Proof.* From Proposition 4.5, we know that the capacitance matrix associated to the problem has exactly one eigenvalue in the spectral gap. Proposition 4.7 shows that an eigenvector presents exponential decay if and only if the corresponding eigenvalue lies in the gap. We can then transpose the result to asymptotic behaviour of eigenmodes via Proposition 2.2.  $\blacksquare$

## 5. Convergence of the resonant frequency in the gap

In this section, we analyse the behaviour as the size of the system grows, specifically the limiting behaviour as  $N \rightarrow \infty$ . We will show that the eigenvalue of  $\mathcal{C}$  lying in the asymptotic spectral gap converges exponentially fast to a value in the gap.

We start by remarking that by performing Laplace extension on the top block of rows  $1 \leq i \leq 2m + 1$ , we obtain that the determinant of  $\mathcal{C} \in \mathbb{R}^{(4m+1) \times (4m+1)}$  is given by

$$\begin{aligned} \det \mathcal{C} &= \det A_{2m+1}^{(a, \eta - \alpha)}(\alpha, \beta_1, \beta_2) \cdot \det A_{2m}^{(0, b)}(\alpha, \beta_1, \beta_2) \\ &\quad - \beta_2^2 \det A_{2m}^{(a, 0)}(\alpha, \beta_1, \beta_2) \cdot \det A_{2m-1}^{(0, b)}(\alpha, \beta_2, \beta_1), \end{aligned} \quad (5.1)$$

where  $a = b = \beta_2$ . We observe that the last term has  $\beta_1$  and  $\beta_2$  switched. Consequently, the characteristic polynomial  $p(x)$  of  $\mathcal{C}$  is given by

$$\begin{aligned} p(x) &= \left( [(x - \alpha - \beta_2 - (\beta_1 - \beta_2)) P_m^*(x) + (\beta_2(\beta_1 - \beta_2)(x - \alpha) - \beta_2\beta_1^2 - (\beta_1 - \beta_2)\beta_2^2) P_{m-1}^*(x)] \right. \\ &\quad \left. - \beta_2^2 [(x - \alpha - \beta_2) P_{m-1}^*(x) + (-\beta_2\beta_1^2) P_{m-2}^*(x)] \right) \chi_{A_{2m}^{(a, 0)}(\alpha, \beta_1, \beta_2)}(x), \end{aligned} \quad (5.2)$$

as one swiftly remarks that  $A_{2m}^{(0, b)}(\alpha, \beta_1, \beta_2)$  and  $A_{2m}^{(a, 0)}(\alpha, \beta_1, \beta_2)$  are similar matrices.

For the sake of brevity, we rewrite

$$p(x) = \chi_{A_{2m}^{(a, 0)}(\alpha, \beta_1, \beta_2)}(x) \left( [AP_m^*(x) + BP_{m-1}^*(x)] - \beta_2^2 [EP_{m-1}^*(x) + FP_{m-2}^*(x)] \right),$$

where  $A, B, E, F$  do not depend on  $m$ . Thus,  $\lambda \in \Gamma$  is an eigenvalue if and only if

$$\begin{aligned} [AP_m^*(\lambda) + BP_{m-1}^*(\lambda)] &= \beta_2^2 [EP_{m-1}^*(\lambda) + FP_{m-2}^*(\lambda)] \\ \Leftrightarrow P_m^*(\lambda) \left[ A + B \frac{P_{m-1}^*(\lambda)}{P_m^*(\lambda)} \right] &= \beta_2^2 P_{m-1}^*(\lambda) \left[ E + F \frac{P_{m-2}^*(\lambda)}{P_{m-1}^*(\lambda)} \right]. \end{aligned}$$

In the above step we were able to divide by  $\chi_{A_{2m}^{(a, 0)}(\lambda)}$  and  $P_{m-2}^*(\lambda)$  as we only consider  $\lambda \in \Gamma$  so the latter term is non-zero because the Chebyshev polynomials only have roots in  $(-1, 1)$  and the former term was shown to be non-zero in  $\Gamma$  in Proposition 4.5. Let now  $L := \lim_{m \rightarrow \infty} \frac{P_{m-1}^*(\lambda)}{P_m^*(\lambda)}$ . Then,

$$\begin{aligned} [A + BL] &= \beta_2^2 L [E + FL] \\ \Leftrightarrow L^2(\beta_2^2 F) + L(\beta_2^2 E - B) - A &= 0, \end{aligned}$$

from which we get the condition

$$L = \frac{B - E\beta_2^2 \pm \sqrt{4AF\beta_2^2 + B^2 - 2BE\beta_2^2 + E^2\beta_2^4}}{2F\beta_2^2}. \quad (5.3)$$

On the other hand, by the recurrence formula of  $U_m$ , for  $\lambda \in \Gamma$ ,

$$L = \lim_{m \rightarrow \infty} \frac{P_{m-1}^*(\lambda)}{P_m^*(\lambda)} = \frac{y + \sqrt{y^2 - 1}}{\beta_1\beta_2} = -\frac{-2\beta_1\beta_2 \sqrt{\frac{(\beta_1^2 + \beta_2^2 - z^2)^2}{4\beta_1^2\beta_2^2}} - 1 + \beta_1^2 + \beta_2^2 - z^2}{2\beta_1^2\beta_2^2}, \quad (5.4)$$

by using again the shorthand  $z$  for  $z(\lambda)$  and  $y$  for  $y(z(\lambda))$ . Now, an algebraic manipulation shows that conditions (5.3) and (5.4) have exactly one common solution in the gap given by

$$z_0 = \frac{1}{2} \left( -\sqrt{9\beta_1^2 - 14\beta_1\beta_2 + 9\beta_2^2} - \beta_1 - \beta_2 \right). \quad (5.5)$$

We thus have shown the following result.

PROPOSITION 5.1. *Consider a perturbed structure of dimers as illustrated in Figure 2. Then the unique interface mode has frequency converging to*

$$\omega_i = v_b \sqrt{\delta \frac{1}{2} \left( -\sqrt{\frac{9}{s_1^2} - \frac{14}{s_1 s_2} + \frac{9}{s_2^2}} + \frac{3}{s_1} + \frac{3}{s_2} \right)} \quad (5.6)$$

as  $N \rightarrow \infty$ . Furthermore, the convergence is exponential, i.e., if  $\omega_i^{(N)}$  denotes the interface frequency for a structure with  $N$  resonators for  $N$  big enough, then

$$|\omega_i - \omega_i^{(N)}| < A e^{-BN}, \quad (5.7)$$

for some  $A, B > 0$  independent of  $N$ .

*Proof.* The only part left to prove is the convergence rate. Denote by  $0_d \in \mathbb{R}^d$  the zero element of that vector space and by  $v_{i,N}$  the  $l^2$ -normalised localised mode associated to  $\omega_i^{(N)}$ . Then, we remark that

$$\left\| (\mathcal{C}_{N+4} - \lambda_i^{(N)}) \begin{pmatrix} 0_2 \\ v_{i,N} \\ 0_2 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} 0 \\ \beta_2 v_{i,N}^{(1)} \\ -v_{i,N}^{(1)} \beta_2 \\ 0_{N-6} \\ -v_{i,N}^{(N)} \beta_2 \\ \beta_2 v_{i,N}^{(N)} \\ 0 \end{pmatrix} \right\|_2 = 4\beta_2^2 (v_{i,N}^{(1)})^2 =: \varepsilon_N. \quad (5.8)$$

Remark also that the same estimation holds if we replace  $\mathcal{C}_{N+4}$  in (5.8) with  $\mathcal{C}_{N+4k}$  for any  $k \in \mathbb{N}$ . This shows that  $\lambda_i^{(N)}$  is an  $\varepsilon$ -pseudo-eigenvalue of  $\mathcal{C}_{N+4k}$  for any  $k \in \mathbb{N}$  for every  $\varepsilon > \varepsilon_N$ . Thus,  $|\omega_i - \omega_i^{(N)}| \leq \varepsilon_N$ . The formulae of Proposition 4.7 yield

$$v_{i,N}^{(1)} = \mathcal{O}(1) \quad \|v_{i,N}\|_2 = \mathcal{O}(e^{C_1 N}) \quad \text{as } N \rightarrow \infty \quad (5.9)$$

for some  $C_1 > 0$ . So if  $v_{i,N}$  needs to be normalised  $v_{i,N}^{(1)}$  decays exponentially.  $\blacksquare$

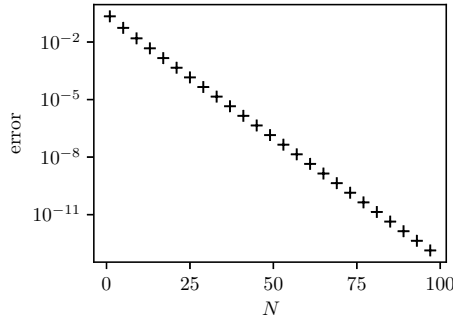


FIGURE 5. Convergence of the eigenvalue in the gap ( $y$ -axis in  $\log$  scale). We display the left-hand side of (5.7) for a structure with  $s_1 = 1$  and  $s_2 = 2$ .

## 6. Stability analysis

Interface modes of SSH-like structures are well known to be stable, *i.e.* perturbations of the system affect them only slightly. In this section we show that perturbation in the geometry have limited effect on both the resonant frequencies and the associated modes and we quantify this effect.

To this end we consider a system of  $N = 4m + 1$  resonators as shown in Figure 2 but the spacings  $s_i$  are now perturbed:

$$s_i = \begin{cases} s_1 + \tilde{\varepsilon}_i, & 1 \leq i \leq 2m, \text{ } i \text{ odd,} \\ s_2 + \tilde{\varepsilon}_i, & 1 \leq i \leq 2m, \text{ } i \text{ even,} \\ s_1 + \tilde{\varepsilon}_i, & 2m + 1 \leq i \leq 4m, \text{ } i \text{ even,} \\ s_2 + \tilde{\varepsilon}_i, & 2m + 1 \leq i \leq 4m, \text{ } i \text{ odd.} \end{cases} \quad (6.1)$$

Furthermore, we denote

$$\varepsilon_i = \begin{cases} -\frac{\tilde{\varepsilon}_i}{s_1(s_1 + \tilde{\varepsilon}_i)}, & 1 \leq i \leq 2m, \text{ } i \text{ odd,} \\ -\frac{\tilde{\varepsilon}_i}{s_2(s_2 + \tilde{\varepsilon}_i)}, & 1 \leq i \leq 2m, \text{ } i \text{ even,} \\ -\frac{\tilde{\varepsilon}_i}{s_1(s_1 + \tilde{\varepsilon}_i)}, & 2m + 1 \leq i \leq 4m, \text{ } i \text{ even,} \\ -\frac{\tilde{\varepsilon}_i}{s_2(s_2 + \tilde{\varepsilon}_i)}, & 2m + 1 \leq i \leq 4m, \text{ } i \text{ odd.} \end{cases} \quad (6.2)$$

The following proposition handles stability of the eigenvalues and is a direct application of the well-known Weyl theorem.

**PROPOSITION 6.1.** *Let  $\widehat{\mathcal{C}}$  be the capacitance matrix associated to the structure described in (6.1) and let*

$$\varepsilon := \max_{1 \leq i \leq N-2} |\varepsilon_i| + |\varepsilon_{i+1}|. \quad (6.3)$$

*Then, the eigenvalues  $\widehat{\lambda}_k$  (sorted increasingly) satisfy*

$$|\widehat{\lambda}_k - \lambda_k| \leq 2\varepsilon, \quad 1 \leq k \leq N, \quad (6.4)$$

*where  $\lambda_k$  are the eigenvalues of  $\mathcal{C}$ .*

*Proof.* Weyl's theorem states the same result but with the bound of (6.4) replaced by  $\|\widehat{\mathcal{C}} - \mathcal{C}\|$ . However, for a tridiagonal matrix  $M$  with  $\mathbf{1}$  in its kernel,  $\|M\| \leq 2 \max_i |M_{i(i-1)}| + |M_{i(i+1)}|$  by the Gershgorin circle theorem so we obtain the result.  $\blacksquare$

Applying Proposition 6.1 to our system of dimers where the perturbations  $|\tilde{\varepsilon}_i| \leq \eta$  are in the interval  $(-\eta, \eta)$  for some  $\eta > 0$ , we obtain that the eigenvalue perturbation is bounded by

$$\frac{2\eta}{s_1(s_1 - \eta)} + \frac{2\eta}{s_2(s_2 - \eta)} = 2\eta \left( \frac{1}{s_1^2} + \frac{1}{s_2^2} \right) + \mathcal{O}(\eta^2) \quad \text{as } \eta \rightarrow 0.$$

In order to analyse the stability of the eigenvectors, we will use the Davis–Kahan theorem [13], which needs some preliminary introduction. Let  $E$  and  $F$  be  $d \times r$  matrices with orthonormal columns such that  $\text{span}(E) = \mathcal{E}$  and  $\text{span}(F) = \mathcal{F}$ . The *canonical angles* between  $\mathcal{E}$  and  $\mathcal{F}$  are defined as

$$\theta_j = \cos^{-1} \sigma_j \quad 1 \leq j \leq r,$$

where  $\sigma_j$  are the  $r$  singular values of  $E^T F$ . We denote by

$$\Theta(\mathcal{E}, \mathcal{F}) = \text{diag}(\theta_1, \dots, \theta_r)$$

the canonical angles' matrix.

The Davis–Kahan theorem states the following.

**Theorem 6.2.** *Let  $S$  and  $\widetilde{S}$  be two  $d \times d$  symmetric matrices with eigenvalues*

$$\begin{aligned} \lambda_1 &\geq \lambda_2 \geq \dots \geq \lambda_d, \\ \widetilde{\lambda}_1 &\geq \widetilde{\lambda}_2 \geq \dots \geq \widetilde{\lambda}_d, \end{aligned}$$



respectively. Fix  $1 \leq r \leq s \leq d$  and let  $V$  and  $\tilde{V}$  be the matrices having as columns the eigenvectors corresponding to  $\lambda_j$  and  $\tilde{\lambda}_j$  for  $r \leq j \leq s$ . Let  $\text{span}(V) = \mathcal{V}$  and  $\text{span}(\tilde{V}) = \tilde{\mathcal{V}}$ . Define

$$\delta := \inf\{|\lambda - \tilde{\lambda}| : \lambda \in [\lambda_s, \lambda_r], \tilde{\lambda} \in (-\infty, \tilde{\lambda}_{s+1}] \cup [\tilde{\lambda}_{r-1}, \infty)\}.$$

If  $\delta > 0$ , then

$$\|\sin \Theta(\mathcal{V}, \tilde{\mathcal{V}})\|_2 \leq \frac{\|S - \tilde{S}\|_2}{\delta},$$

where  $\sin \Theta(\mathcal{V}, \tilde{\mathcal{V}})_{ii} = \sin(\Theta(\mathcal{V}, \tilde{\mathcal{V}})_{ii})$ .

As a direct consequence of Theorem 6.2, we have the following theorem for the stability of the interface eigemodes.

**Theorem 6.3.** Let  $\varepsilon < \frac{1}{2} \left( \frac{1}{s_1} - \frac{1}{s_2} \right)$  in (6.3). Let  $\mathbf{v}$  and  $\hat{\mathbf{v}}$  be the eigenvectors corresponding to the eigenvalues  $\lambda_i$  and  $\hat{\lambda}_i$  in the gap of  $\mathcal{C}$  and  $\hat{\mathcal{C}}$ , respectively. Then

$$\|\mathbf{v} - \hat{\mathbf{v}}\|_2 \leq \frac{2\sqrt{2}\varepsilon}{\delta} \tag{6.5}$$

$$\leq \frac{2\sqrt{2}\varepsilon}{\delta_0 - 2\varepsilon}, \tag{6.6}$$

where  $\delta := \min\{|\lambda_i - \hat{\lambda}_{i+1}|, |\lambda_i - \hat{\lambda}_{i-1}|\}$  and  $\delta_0 = \min\{|\lambda_i - \lambda_{i+1}|, |\lambda_i - \lambda_{i-1}|\}$ . The a priori estimate (6.6) holds for  $\delta_0 > 2\varepsilon$ .

Remark that for  $s_1 = 1$  and  $s_2 = 2$  and  $m$  large we have  $\delta_0 \approx 0.219$  so that the a priori estimate (6.6) holds for  $\varepsilon < 0.1$ . This a priori estimate is, however, suboptimal as Figure 6b shows.

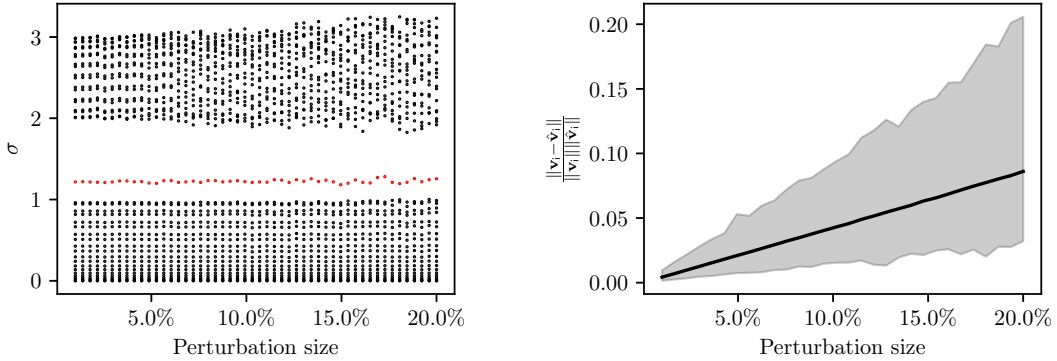
*Proof of Theorem 6.3.* The proof immediately follows from Theorem 6.2 with  $r = s$  by using the bound  $\|\mathbf{v} - \hat{\mathbf{v}}\|_2 \leq \sqrt{2} \sin \Theta(\mathbf{v}, \hat{\mathbf{v}})$  [27]. ■

In Figure 6 we show numerically the high stability of the interface modes. We consider perturbations of the type  $\tilde{\varepsilon}_i \sim U(-\eta, \eta)$  where we call  $\eta$  the perturbation size, we display the latter as percentage relative to the resonator's size. Figure 6a shows that the interface eigenfrequency (lying in the gap) is only minimally perturbed even by perturbation in the size of 20%. We remark that numerically the bound of (6.4) can even be sharpened to  $|\hat{\lambda}_k - \lambda_k| \leq \frac{3}{2}\varepsilon$ . Figure 6b shows  $\frac{\|\mathbf{v} - \hat{\mathbf{v}}\|}{\|\mathbf{v}\| \|\hat{\mathbf{v}}\|}$  for various perturbation sizes. The black lines shows the average over  $10^4$  runs while the gray area encloses the range from the minimum to the maximum value of  $\frac{\|\mathbf{v} - \hat{\mathbf{v}}\|_2}{\|\mathbf{v}\|_2 \|\hat{\mathbf{v}}\|_2}$ .

## 7. Concluding remarks

In this paper, we have quantitatively characterised interface eigenmodes in finite, dimer systems of subwavelength resonators with a geometric defect and proved the exponential decay of their associated eigenmodes. Our characterisation is based on (broken) translation invariance properties of the associated capacitance matrix together with properties of Chebyshev polynomials. The 3-term recurrence relation satisfied by the Chebyshev polynomials is shown to be useful for analysing spectra of tridiagonal (perturbed) 2-Toeplitz matrices.

Following this line of research, it would be very interesting to generalise the results obtained in this paper to extended (known also as multi-band) SSH models, exhibiting exponentially localised interface eigenmodes with corresponding eigenfrequencies inside the multiple subwavelength band gaps of the structure [21, 26]. Another highly interesting direction would be to extend the current results to finite dimer systems of three-dimensional



(A) Spectrum of the capacitance matrix with perturbations given by  $\tilde{\varepsilon}_i \sim U(-\eta, \eta)$ . For every perturbation size the spectrum of one realisation is shown. The eigenvalue in red corresponds to the localised interface mode. (B) Stability of the interface mode. The solid black line shows the average dislocation over  $10^4$  runs, while the gray area encloses the range from the minimum to the maximum dislocation observed over these realisations.

FIGURE 6. The interface eigenvalue and the corresponding interface mode are very stable also in presence of big perturbations. Simulations in a system of  $N = 41$  resonators with  $s_1 = 1$  and  $s_2 = 2$ . Perturbations are uniformly distributed in  $(-\eta, \eta)$  where we call  $\eta$  the perturbation size, expressed relatively to the resonators' sizes.

subwavelength resonators. In the case of three-dimensional systems of subwavelength resonators, the main difficulty occurs from the long-range interactions between the resonators, which lead to a slow decay of the off-diagonal terms of the corresponding capacitance matrix. Nevertheless, in view of the recent results on  $k$ -banded approximations of three-dimensional capacitance matrices [1] together with the convergence spectral results as the size of the structure goes to infinity in [3, 7], it may be possible to prove existence and uniqueness of localised eigenmodes in finite chains of subwavelength resonators in three dimensions, as numerically shown in [8]. Another very interesting problem is to relate the localisation effect at the interface to the statistics of the eigenvalues of the capacitance matrix under random perturbations in the parameters of the system. This would extend the well-known Thouless localisation criterion [14] to interface eigenmodes.

### Data Availability

The data that support the findings of this study are openly available at <https://zenodo.org/doi/10.5281/zenodo.10361315>.

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## Appendix A. Pseudo-spectrum of a normal matrix

DEFINITION A.1.  $\sigma_{\varepsilon}(\mathbf{A})$  is the set of  $z \in \mathbb{C}$  such that

$$\|(z - \mathbf{A})\mathbf{v}\| < \varepsilon,$$

for some  $\mathbf{v} \in \mathbb{C}^N$  with  $\|\mathbf{v}\| = 1$ .

The next theorem expresses these facts with the aid of the following notation for an open  $\varepsilon$ -ball:

$$\Delta_\varepsilon = \{z \in \mathbb{C} : |z| < \varepsilon\}.$$

In this theorem, a sum of sets has the usual meaning:

$$\sigma(\mathbf{A}) + \Delta_\varepsilon = \{z : z = z_1 + z_2, z_1 \in \sigma(\mathbf{A}), z_2 \in \Delta_\varepsilon\},$$

which is equal to  $\{z : \text{dist}(z, \sigma(\mathbf{A})) < \varepsilon\}$ .

**Theorem A.2** (Pseudo-spectrum of a normal matrix). *For any  $\mathbf{A} \in \mathbb{C}^{N \times N}$ ,*

$$\sigma_\varepsilon(\mathbf{A}) \supseteq \sigma(\mathbf{A}) + \Delta_\varepsilon \quad \forall \varepsilon > 0,$$

*and if  $\mathbf{A}$  is normal and  $\|\cdot\| = \|\cdot\|_2$ , then*

$$\sigma_\varepsilon(\mathbf{A}) = \sigma(\mathbf{A}) + \Delta_\varepsilon \quad \forall \varepsilon > 0.$$

*Conversely, if  $\|\cdot\| = \|\cdot\|_2$ , then (2.17) implies that  $\mathbf{A}$  is normal.*

## Appendix B. Topological origin

Infinite SSH structures have long been known for their topological nature. In this section we show that a topological invariant can be defined also for finite structures in such a way that when the size of the system grows they converge to the infinite invariants. For one-dimensional crystals, the *Zak phase* has been shown to be related through an *if and only if* condition to the existence of interface modes, see *e.g.* [11, Theorem 1] and also [16, 19]. Denoting by  $u_j^\alpha$  a family of eigenmodes depending piecewise smoothly on the quasiperiodicity  $\alpha$ , the Zak phase is defined as

$$\varphi_j^{\text{zak}} := \mathbf{i} \int_{Y^*} \left\langle u_j^\alpha, \frac{\partial}{\partial \alpha} u_j^\alpha \right\rangle d\alpha, \quad (\text{B.1})$$

where  $Y^*$  denotes the first Brillouin zone and  $\langle \cdot, \cdot \rangle$  the usual  $L^2$  inner product. The Zak phase is also related to the symmetries of the eigenfunctions [11]. Specifically, denoting by  $u_j^+$  the eigenfunction associated to the quasifrequency that maximises the  $j$ -th band function  $\omega_j^\alpha$  we may define bulk topological index of the  $j$ -th band gap as

$$\mathcal{J}_j := \begin{cases} +1, & u_j^+(x) = \mathcal{P}(u_j^+), \\ -1, & u_j^+(x) = -\mathcal{P}(u_j^+), \end{cases} \quad (\text{B.2})$$

for  $j = 1, 2$ , where  $\mathcal{P}$  denotes the mirroring of the unit cell  $Y$  about its center. Then, the Zak phase is related to  $\mathcal{J}_j$  via

$$\mathcal{J}_j = (-1)^{j-1} \prod_{k=1}^j e^{\mathbf{i}\varphi_k^{\text{zak}}}. \quad (\text{B.3})$$

Previous work [2, Proposition 5.5] has shown that the Zak phase of a periodic system of dimers is quantised and depends on the inter- and intra-spacing between the cells. In particular,

$$\varphi_j^{\text{zak}} = \begin{cases} \pi, & s_1 \geq s_2, \\ 0, & s_1 < s_2, \end{cases}$$

and thus

$$\mathcal{J}_1 = \begin{cases} -1, & s_1 \geq s_2, \\ 1, & s_1 < s_2. \end{cases}$$

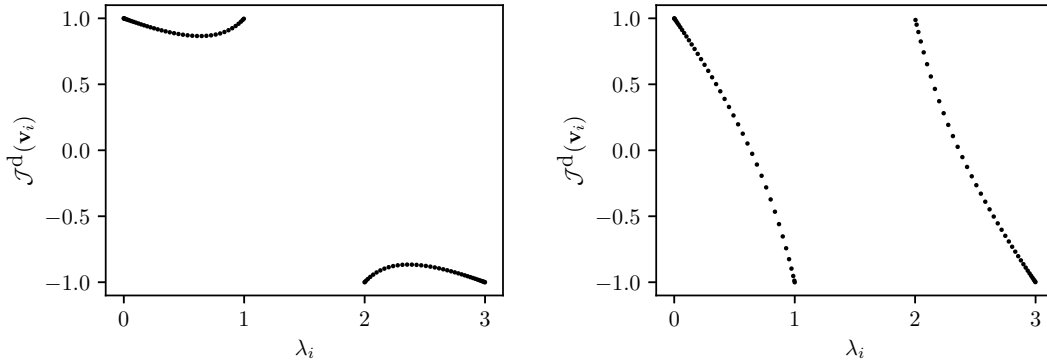
We will show now that (B.2) lends itself well to a reformulation in the finite case. We denote by  $\mathcal{P}^{\text{d}}$  the discrete equivalent of  $\mathcal{P}$  that is, for  $\mathbf{v} \in \mathbb{R}^{2k}$ ,

$$\mathcal{P}^{\text{d}}(\mathbf{v})^{(j)} := \begin{cases} \mathbf{v}^{(j-1)}, & 1 \leq j \leq 2k, j \text{ even}, \\ \mathbf{v}^{(j+1)}, & 1 \leq j \leq 2k, j \text{ odd}, \end{cases}$$

and define the discrete equivalent of (B.2) by

$$\mathcal{J}^d(\mathbf{v}) := \frac{1}{\|\mathbf{u}\|_2^2} \langle \mathbf{v}, \mathcal{P}^d(\mathbf{v}) \rangle. \quad (\text{B.4})$$

In Figure 7 we show the values of  $\mathcal{J}^d(\mathbf{v})$  for two structures composed of 40 dimers. Figure 7a shows the case  $s_1 = 1$  and  $s_2 = 2$  while Figure 7b shows the case  $s_1 = 2$  and  $s_2 = 1$ . We observe a very different behaviour, but in particular the value of interest in view of (B.2) is the one at  $\lambda = 1$ , where the two structures take values  $\approx +1$  and  $\approx -1$  respectively showing that through (B.4) we may generalise (B.2) to discrete structures.



(A) Discrete indicator  $\mathcal{J}^d(\mathbf{v})$   $s_1 = 1$  and  $s_2 = 2$ . (B) Discrete indicator  $\mathcal{J}^d(\mathbf{v})$   $s_1 = 2$  and  $s_2 = 1$ .

FIGURE 7. The discrete indicator function of (B.4) have very different behaviour for  $s_1 < s_2$  and  $s_1 > s_2$ . The values at the end of the bands approximate accurately the values of the infinite periodic case of (B.2).

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