Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

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Research Report No. 2023-41
December 2023

Seminar für Angewandte Mathematik
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Funding ERC: 770924

LINEARIZED PARTIAL DATA CALDERÓN PROBLEM FOR BIHARMONIC OPERATORS

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Abstract. We consider a linearized partial data Calderón problem for biharmonic operators extending the analogous result for harmonic operators [DSFKSU09b]. We construct special solutions and utilize Segal-Bargmann transform to recover lower order perturbations.

1. Introduction and main result

The Calderón problem, introduced by Calderón in 1980 [Cal80], aims to recover the electrical conductivity γ of a medium based on the Dirichlet to Neumann map (DN map) denoted as $\Lambda_\gamma := \gamma\partial_\nu u|_{\partial\Omega}$. Here, u represents the solution to the conductivity equation $\nabla\cdot(\gamma\nabla u) = 0$ with specified Dirichlet boundary conditions. As the mapping from γ to Λ_γ is nonlinear, it is valuable to investigate its linearization, known as the linearized Calderón problem. In [Cal80], he solved the linearized problem by proving the following result that if γ is a bounded function in Ω and satisfies the equation

$$\int_\Omega \gamma\nabla u\cdot\nabla v = 0 \quad \text{for all harmonic functions } u \text{ and } v \text{ in } \Omega, \text{ then } \gamma = 0 \text{ in } \Omega.$$

Sylvester and Uhlmann solved the Calderón problem for $C^2(\Omega)$ conductivities in [SU87]. Subsequently, various authors have investigated inverse problems for different types of partial differential equations. For more comprehensive results on the subject, we refer to the surveys [Uhl14, Uhl09].

A closely related inverse problem is for the Schrödinger equation $(-\Delta + q)u = 0$, which can be derived from the conductivity equation for C^2 conductivities [SU87]. For this reason the linearized problem for the Schrödinger is also an interesting question and it can be formulated in the following way: Let q be a bounded function in Ω and satisfies

$$\int_\Omega quv = 0 \quad \text{for all harmonic functions } u \text{ and } v \text{ in } \Omega, \text{ then } q = 0 \text{ in } \Omega.$$

Another closely related problem involves recovering the conductivity γ (or potential) from partial measurements. In [KSU07], the authors proved the unique determination of q when the Dirichlet and Neumann measurements are given on two complementary open subsets of the boundary (roughly speaking) in dimensions $n \geq 3$. However, the unique recovery of q, when the Dirichlet and Neumann data are prescribed on the same part of the boundary, remains an open question in dimensions $n \geq 3$. In two dimensions, this was solved by Imanuvilov, Uhlmann, and Yamamoto [IUY10]. Several partial answers are known either by dropping the support condition on the Dirichlet data or by assuming additional symmetry on the domain Ω; see [GU01, BU02, KSU07, Isa07, KS13, KS14]. Nevertheless, for an arbitrary domain Ω, a linearized version of this problem was addressed in [DSFKSU09b]. For a comprehensive overview of the Calderón problem with partial data, we recommend the survey article [KS14].

This article focuses on a linearized Calderón problem for biharmonic operators, inspired by [DSFKSU09b]. Let $\Omega \subset \mathbb{R}^n\,(n \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\Sigma \subseteq \partial\Omega$ be a non-empty open subset of $\partial\Omega$. We consider the following boundary value problem for biharmonic

2020 *Mathematics Subject Classification.* Primary 35R30, 31B20, 31B30, 35J40.

Key words and phrases. Calderón problem, biharmonic operator, Anisotropic perturbation, Segal-Bargmann transform.

operator $\mathcal{L}$ with lower order anisotropic perturbations up to order 3:

$$\begin{cases} \mathcal{L}(x,D) &= (-\Delta)^2 + Q(x,D) \quad \text{in} \quad \Omega \\ (u, \partial_\nu u) &= (f_1, f_2) \quad \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

where $Q(x,D) := \sum_{l=0}^{3} a^l_{i_1\cdots i_l}(x)\, D^{i_1\cdots i_l}$ is a differential operator of order 3 with $1 \le i_1, \cdots, i_l \le n$ and a^l is a smooth symmetric tensor field of order l in $\overline{\Omega}$. Here f_j are suitable functions on the boundary with support of $f_j \subset \Sigma$, for $j = 1, 2$. Einstein summation convention is assumed for repeated indices throughout the article.

Suppose 0 is not an eigenvalue of (1), then the boundary measurements associated to (1) can be encoded in terms of partial DN map as follows:

$$\Lambda_Q(f_1, f_2) := (\partial_\nu^2 u|_\Sigma, \partial_\nu^3 u|_\Sigma).$$

Alternatively, one can also prescribe the partial boundary measurements in terms of the Cauchy data set as follows:

$$\mathcal{C}_{Q,\Sigma} := \{(u, \partial_\nu u, \partial_\nu^2 u, \partial_\nu^3 u)|_\Sigma : u \in H^4(\Omega), \mathcal{L}(x,D)u = 0 \text{ in } \Omega\}.$$

The inverse problem we are interested in is to show that the Fréchet derivative of Λ_Q (evaluated at $Q = 0$) is injective. This result establishes local uniqueness for the linearized Calderón problem of biharmonic operators. To state our main result, let us define the set

$$\mathcal{E} := \{u \in C^\infty(\overline{\Omega}) : \Delta^2 u = 0 \text{ in } \Omega, \text{ and } (u, \partial_\nu u) = 0 \text{ on } \Gamma\}, \quad \text{where} \quad \Gamma := \partial\Omega \backslash \Sigma. \tag{2}$$

Our main result is

Theorem 1.1. *Let $a^2, a^0 \in C^\infty(\overline{\Omega})$ and $a^1 \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$. Suppose*

$$\int_\Omega \left(a^2 \Delta u + a^1 \cdot \nabla u + a^0 u\right) v \, \mathrm{d}x = 0 \quad \textit{holds for all} \quad u, v \in \mathcal{E}. \tag{3}$$

Then $a^j = 0$ in Ω, for $j = 0, 1, 2$.

Previous research has focused on inverse problems for polyharmonic operators (which are perturbations of $(-\Delta)^m$ when $m \ge 2$) of order 2m. Krupchyk, Lassas, and Uhlmann initiated the study of inverse problems for polyharmonic operators in their works [KLU14, KLU12]. More recently, several authors have investigated inverse problems for higher-order elliptic operators, such as biharmonic and polyharmonic operators; see [GK16, BG19, BG22]. These works have successfully solved inverse problems for polyharmonic operators which recovered functions, vector fields, or two tensor fields. In a recent work [BKS21] the authors solved an inverse problem for polyharmonic operators of order $2m$ with lower-order tensorial perturbations up to order m, utilizing momentum ray transforms. Furthermore, a linearized Calderón problem for polyharmonic operators of order $2m$ with lower order perturbation up-to $2m-1$ was considered in [SS23], also employing momentum ray transform techniques. However, when dealing with partial data, the lack of sufficient information to extract the momentum ray transform of unknown tensor fields poses a technical challenge. In this work, instead of relying on the momentum ray transform, the authors employ techniques of the Segal-Bargmann transform from the work [DSFKSU09b]. It appears that addressing linearized partial data inverse problems for polyharmonic operators with lower order anisotropic perturbations may require new techniques or tools, which the authors aim to explore in future research.

To the best of the authors' knowledge, this work is the first to consider the linearized problem for higher-order operators with partial data. Recent research has shown that solutions to the linearized problem can be used to solve inverse problems for nonlinear partial differential equations (PDEs) using higher-order linearization techniques, as demonstrated in previous works. We refer [LLLS21, LLST22, KU20, CFK$^+$21, KKU23] for nonlinear elliptic PDEs, [KLU18, KLOU22, HUZ22] for nonlinear hyperbolic PDEs to name a few. The authors hope that their work will pave the way for similar results in the future for higher-order elliptic operators.

The rest of the article is structured as follows. In Section 2, we start by presenting some preliminary results that are essential for the proof of our main result. The section is divided into four subsections, each dealing with a specific topic. In the first subsection, the integral identity is transformed to an equivalent integral identity through a conformal change of variables. The second subsection discusses the construction of special solutions of biharmonic equations, where their Dirichlet data vanishes in a part of the boundary (as shown in Lemma 2.1). The third subsection provides a brief introduction to the Segal-Bargmann transform and the fourth subsection presents a local uniqueness result (Proposition 2.2). Finally, Section 3 focuses on demonstrating the main result of the article which is Theorem 1.1. In appendix A we linearize Dirichlet to Neumann map, and in appendix B we present a decay estimate of special solutions used in Lemma 2.1.

2. Preliminaries

2.1. **Change of coordinates.** Let us first consider a change of coordinates for ease of calculations. Fix a point $x_0 \in \Sigma$ and choose an exterior ball to Ω at x_0, say $B(a,r)$ i.e. $\overline{\Omega} \cap \overline{B(a,r)} = \{x_0\}$. Consider the conformal change of variables

$$\psi : x \mapsto \frac{x-a}{|x-a|^2} r^2 + a$$

which fixes the point x_0 and maps Ω to the interior of the ball $B(a,r)$. Next, we observe that a function u is biharmonic if and only if $u^* = r^{n-4}|x-a|^{4-n} u \circ \psi$ is biharmonic. This is the analog of the Kelvin transform for the bilaplacian; see [Xu00].

A function u and normal derivative of u are zero on Γ if and only if u^* and normal derivative of u^* are zero on $\psi(\Gamma)$. Consequently, the integral identity becomes

$$\int\limits_{\psi(\Omega)} \left(\tilde{a}^2 \Delta u + \tilde{a}^1 \cdot \nabla u + \tilde{a}^0 u\right) v \, \mathrm{d}x = 0,$$

for all smooth biharmonic functions u, v in $\psi(\Omega)$ such that $(u, \frac{\partial u}{\partial \nu})|_{\psi(\Gamma)} = 0 = (v, \frac{\partial v}{\partial \nu})|_{\psi(\Gamma)}$. Moreover, a^0, a^1 and a^2 are zero near x_0 if and only if $\tilde{a}^0, \tilde{a}^1$ and $\tilde{a}^2$ are zero near x_0. After a rotation and translation we can bring x_0 to the origin. Thus our set-up will be as follows: $\Omega \subset B(-e_1, 1)$ and $\Gamma \subset \{x_1 < -2c\}$ for some $c > 0$, after a suitable translation and rotation. We want to show the following:

$$\int\limits_{\Omega} [a^2 \Delta u + a^1_i \partial_i u + a^0 u] v \, \mathrm{d}x = 0 \quad \text{for all} \quad u, v \in \mathcal{E} \implies a^2 = a^1 = a^0 = 0 \quad \text{in} \quad \Omega.$$

2.2. **Construction of special solutions.** In this section we carry out the construction of special solutions of biharmonic operators. To construct the solutions which vanish on part of the boundary, we use the method analogous to [DSFKSU09b]. To this end, let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a cut-off function which is 1 in a neighbourhood of Γ, and define $H_K(y) := \sup\limits_{x \in K} x \cdot y$, where $K = \operatorname{supp} \chi \cap \partial\Omega$ which can be taken to be subset of $\{x_1 < -c\}$.

Lemma 2.1. *Suppose $\xi \in \mathbb{C}^n$ such that $\xi \cdot \xi = 0$ and a be any smooth function satisfying*

$$\Delta a = \textit{constant} \quad \textit{and} \quad \sum_{i,j} \nabla^2_{ij} a \cdot \xi_i \xi_j = 0.$$

There exists $u \in \mathcal{E}$ of the form $u = e^{-\mathrm{i}x\cdot\xi/h} a(x) + r(x,h)$, where r satisfies

$$\|r\|_{H^2(\Omega)} \le C\left(1 + \frac{|\xi|^2}{h^2} + \frac{|\xi|^4}{h^4}\right)^{1/2} e^{\frac{1}{h} H_K(\operatorname{Im}\, \xi)},$$

where C is a constant independent of ξ, and h.

Proof. Fix a smooth function a satisfying the assumptions of the lemma. For such a and ξ, the function $ae^{-\mathrm{i}x\cdot\xi/h}$ is biharmonic in Ω. The term r serves as the correction term which forces the solution to vanish on the required part of the boundary. Let us choose r to be the solution of

$$\begin{cases} \Delta^2 r &= 0 \quad \text{in} \quad \Omega \\ (r, \partial_\nu r) &= (-(ae^{-\mathrm{i}x\cdot\xi/h})\chi, -\partial_\nu(ae^{-\mathrm{i}x\cdot\xi/h})\chi) \quad \text{on} \quad \partial\Omega. \end{cases}$$

Clearly, the function $u = e^{-\mathrm{i}x\cdot\xi/h}a(x) + r(x,h) \in \mathcal{E}$ and the bounds on r are obtained from Lemma B.1. $\square$

2.3. **Segal-Bargmann transform.** The Segal-Bargmann transform [DSFKSU09b] of a function f on $\mathbb{R}^n$ is defined for $z \in \mathbb{C}^n$ as

$$Tf(z) := \int e^{-\frac{1}{2h}(z-y)^2} f(y)\,\mathrm{d}y.$$

This is well-defined for $f \in L^\infty(\mathbb{R}^n)$ and one has

$$|Tf(z)| \leq \int |e^{-\frac{1}{2h}(z-y)^2}||f(y)|\,\mathrm{d}y \leq \int e^{-\frac{1}{2h}(|\mathrm{Re} z-y|^2-|\mathrm{Im} z|^2)}|f(y)|\,\mathrm{d}y \leq e^{\frac{1}{2h}|\mathrm{Im} z|^2}\|f\|_\infty(2\pi h)^{n/2}.$$

The property most important to us is that when $z \in \mathbb{C}^n$ is restricted to $x \in \mathbb{R}^n$, we obtain $\frac{1}{(2\pi h)^{n/2}}Tf(x) = f * G(x)$, which is convolution of f with Gaussian $G(x) = e^{-\frac{|x|^2}{2h}}$. Therefore, for a bounded function f with compact support,

$$\lim_{h\to 0} \frac{1}{(2\pi h)^{n/2}} Tf = f \quad \text{in } L^p(\mathbb{R}^n) \text{ for all } 1 \leq p < \infty.$$

Therefore, to prove that f vanishes in a region, we will prove that the limit on the left vanishes in that region. To do this, we will utilize the exponential bounds of special solutions constructed in the previous section; see Lemma 2.1. Notice that

$$e^{-\frac{1}{2h}(z-y)^2} = e^{-\frac{z^2}{2h}}(2\pi h)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{t^2}{2h}} e^{-\frac{\mathrm{i}}{h}y\cdot(t+\mathrm{i}z)}\mathrm{d}t.$$

Therefore interchanging the order of integration, we obtain

$$\begin{aligned} |Tf(z)| &\leq \frac{1}{(2\pi h)^{n/2}} e^{\frac{-1}{2h}(|\mathrm{Re} z|^2-|\mathrm{Im} z|^2)} \int_{\mathbb{R}^n} e^{-\frac{t^2}{2h}} \left| \int_{\mathbb{R}^n} e^{-\frac{\mathrm{i}}{h}y\cdot(t+\mathrm{i}z)} f(y)\mathrm{d}y \right| \mathrm{d}t \\ &\leq \frac{1}{(2\pi h)^{n/2}} e^{\frac{-1}{2h}(|\mathrm{Re} z|^2-|\mathrm{Im} z|^2)} \Big(\int_{|t|<\epsilon a} + \int_{|t|\geq \epsilon a} \Big) \left(e^{-\frac{t^2}{2h}} \left| \int_{\mathbb{R}^n} e^{-\frac{\mathrm{i}}{h}y\cdot(t+\mathrm{i}z)} f(y)\mathrm{d}y \right| \right). \end{aligned}$$

For functions f supported in the bounded set $\Omega \subset \{y_1 \leq 0\}$, the above estimate yields

$$|Tf(z)| \leq e^{\frac{-1}{2h}(|\mathrm{Re} z|^2-|\mathrm{Im} z|^2)} \left(\sup_{|t|<\epsilon a} \left| \int e^{-\frac{\mathrm{i}}{h}y\cdot(t+\mathrm{i}z)} f(y)\mathrm{d}y \right| + \sqrt{2} e^{\frac{1}{h}|\mathrm{Re} z'|} e^{-\frac{\epsilon^2 a^2}{4h}} \int_\Omega |f(y)|\mathrm{d}y \right). \tag{4}$$

2.4. **A local uniqueness result.**

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with smooth boundary. Let $x_0 \in \partial\Omega \setminus \Gamma$. Under the assumptions of Theorem 1.1, there exists $\delta > 0$ such that $a^j = 0$ in $B(x_0,\delta) \cap \Omega$, for $j = 0, 1, 2$.*

Proof. We start with following integral identity

$$\int_\Omega (a^2\Delta u + a_i^1\partial_i u + a^0 u)v\,\mathrm{d}x = 0, \tag{5}$$

where $\Omega \subset B(-e_1, 1)$ and $\Gamma \subset \{x_1 < -2c\}$ for some $c > 0$. The idea of the proof is as follows: we use the special solutions constructed in Lemma 2.1 to derive some identities, using which we will be able to establish the desired estimates. These estimates when invoked in (4) will then imply that the coefficients vanish locally. We will do this for each of the three coefficients in the following steps.

Step 1: We prove that $a^2 = 0$ near origin.

To achieve this let us first choose $u(x,\xi,h) = u_\sharp(x,\xi,h) = |x|^2 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \mathbf{r}$ and $v(x,\eta,h) = v_0(x,\eta,h) = e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1$ and insert them into (5) to obtain

$$0 = \int_\Omega \Big[a^2\{2ne^{\frac{-\mathrm{i}x\cdot\xi}{h}} - 4\frac{\mathrm{i}}{h}x\cdot\xi e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \Delta\mathbf{r}\} + a_i^1\{2x_i e^{\frac{-\mathrm{i}x\cdot\xi}{h}} - \frac{\mathrm{i}}{h}\xi_i|x|^2 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \partial_i\mathbf{r}\} + a^0\{|x|^2 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \mathbf{r}\}\Big](e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1)\,\mathrm{d}x. \tag{6}$$

Let us again choose $u(x,\xi,h) = u_j(x,\xi,h) = x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \tilde{\mathbf{r}}_j$ and $v(x,\eta,h) = x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1$ and sum over j to conclude

$$0 = \int_\Omega \Big[a^2\{-\frac{2\mathrm{i}}{h}\xi_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \Delta\tilde{\mathbf{r}}_j\} + a_i^1\{\delta_{ij}e^{\frac{-\mathrm{i}x\cdot\xi}{h}} - \frac{\mathrm{i}}{h}\xi_i x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \partial_i\tilde{\mathbf{r}}_j\} + a^0\{x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \tilde{\mathbf{r}}_j\}\Big](x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1)\mathrm{d}x. \tag{7}$$

Subtracting (6) from twice of (7) we obtain

$$\begin{aligned}0 = \int_\Omega &[2na^2 e^{\frac{-\mathrm{i}x\cdot(\xi+\eta)}{h}} + 2na^2\mathbf{r}_1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} - 4\frac{\mathrm{i}}{h}x\cdot\xi a^2\mathbf{r}_1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + 4\frac{\mathrm{i}}{h}a^2\xi_j\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + a^2\Delta\mathbf{r}(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1)\\ &- 2a^2\Delta\tilde{\mathbf{r}}_j(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1) + 2a_j^1 x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\mathbf{r}_1 - 2a_j^1\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \frac{\mathrm{i}}{h}a_j^1\xi_j|x|^2 e^{\frac{-\mathrm{i}x\cdot(\xi+\eta)}{h}} - \frac{\mathrm{i}}{h}a_i^1\xi_i|x|^2\mathbf{r}_1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\\ &+ 2\frac{\mathrm{i}}{h}a_i^1\xi_i x_j\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + a_i^1\partial_i\mathbf{r}(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1) - 2a_i^1\partial_i\tilde{\mathbf{r}}_j(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1) - a^0|x|^2 e^{\frac{-\mathrm{i}x\cdot(\xi+\eta)}{h}}\\ &+ a^0\mathbf{r}_1|x|^2 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + a^0\mathbf{r}(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1) - 2a^0 x_j\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} - 2a^0\tilde{\mathbf{r}}_j(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1)]\,\mathrm{d}x.\end{aligned} \tag{8}$$

Finally, taking $u(x,\xi,h) = u_0(x,\xi,h) = e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \mathbf{r}_2$ and $v(x,\eta,h) = v_\sharp(x,\eta,h) = |x|^2 e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_3$, we conclude

$$0 = \int_\Omega [a^2\{\Delta\mathbf{r}_2\} + a_i^1\{-\frac{\mathrm{i}}{h}\xi_i e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \partial_i\mathbf{r}_2\} + a^0\{e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \mathbf{r}_2\}](|x|^2 e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_3)\,\mathrm{d}x. \tag{9}$$

We add (8) and (9) to derive

$$\begin{aligned}\int_\Omega a^2 e^{\frac{-\mathrm{i}x\cdot(\xi+\eta)}{h}}\mathrm{d}x = -\frac{1}{2n}\Bigg[\int_\Omega &2na^2\mathbf{r}_1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} - 4\frac{\mathrm{i}}{h}x\cdot\xi a^2\mathbf{r}_1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + 4\frac{\mathrm{i}}{h}a^2\xi_j\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + a^2\Delta\mathbf{r}(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1)\\ &+ a^2\Delta\mathbf{r}_2(|x|^2 e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_3) - 2a^2\Delta\tilde{\mathbf{r}}_j(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1) + 2a_j^1 x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\mathbf{r}_1 - 2a_j^1\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\\ &- \frac{\mathrm{i}}{h}a_j^1\xi_j e^{\frac{-\mathrm{i}x\cdot(\xi)}{h}}\mathbf{r}_3 + a_i^1\partial_i\mathbf{r}_2(|x|^2 e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_3) - \frac{\mathrm{i}}{h}a_i^1\xi_i|x|^2\mathbf{r}_1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + 2\frac{\mathrm{i}}{h}a_i^1\xi_i x_j\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\\ &+ a_i^1\partial_i\mathbf{r}(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1) - 2a_i^1\partial_i\tilde{\mathbf{r}}_j(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1) + a^0\mathbf{r}_3 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + a^0\mathbf{r}_2(|x|^2 e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_3)\\ &+ a^0\mathbf{r}_1|x|^2 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + a^0\mathbf{r}(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1) - 2a^0\, x_j\tilde{\mathbf{r}}_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}} - 2a^0\,\tilde{\mathbf{r}}_j(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1)\,\mathrm{d}x\Bigg].\end{aligned}$$

The goal of the exercise was that now on the right hand side, each summand contains at least one term which has a good decay, as stated in Lemma 2.1. Thus, we have the bound:

$$|\int_\Omega a^2 e^{\frac{-\mathrm{i}x\cdot(\xi+\eta)}{h}}\mathrm{d}x| \le C(\|a^2\|_\infty+\|a^1\|_\infty+\|a\|_\infty)(1+\frac{|\xi|^2}{h^2}+\frac{|\xi|^4}{h^4})^{1/2}(1+\frac{|\eta|^2}{h^2}+\frac{|\eta|^4}{h^4})^{1/2} \times e^{\frac{1}{h}H_K(\mathrm{Im}\ \xi)}e^{\frac{1}{h}H_K(\mathrm{Im}\ \eta)}. \tag{10}$$

We use the decomposition result, stated in [DSFKSU09a], which says that any $z\in\mathbb{C}^n$ such that $|z-2\mathrm{i}ae_1|<2\epsilon a$, for ϵ small enough, can be written as

$$z=\xi+\eta,\quad \xi\cdot\xi=0=\eta\cdot\eta,\quad |\xi-a(\mathrm{i}e_1+e_2)|<C\epsilon a,\quad |\eta+a(e_2-\mathrm{i}e_1)|<C\epsilon a.$$

Notice that for ϵ small enough, the vectors ξ and η can be chosen such that $\mathrm{Im}\xi_1,\mathrm{Im}\eta_1\ge\frac{a}{4}$. Keeping in mind that $\Omega\subset B(-e_1,1)$ and $K\subset\{x_1<-c\}$, this yields

$$H_K(\mathrm{Im}\xi)\le -c\mathrm{Im}\xi_1+|\mathrm{Im}(\xi')|,\quad \text{and}\quad H_K(\mathrm{Im}\eta)\le -c\mathrm{Im}\eta_1+|\mathrm{Im}(\eta')|.$$

This decomposition and estimate invoked in (10) leads to

$$|\int_\Omega a^2 e^{\frac{-\mathrm{i}x\cdot z}{h}}\mathrm{d}x|\le Ch^{-4}(\|a^2\|_\infty+\|a^1\|_\infty+\|a^0\|_\infty)e^{-\frac{ca}{2h}}e^{\frac{2C\epsilon a}{h}}, \tag{11}$$

for all $z\in\mathbb{C}^n$ such that $|z-2\mathrm{i}ae_1|<2\epsilon a$. When $|t|<\epsilon a$ and $|z-2ae_1|<\epsilon a$, we also have $|(t+\mathrm{i}z)-2\mathrm{i}ae_1|<2\epsilon a$. We next insert (11) in (4) to estimate Ta^2, the Segal-Bargmann transform of a^2 introduced in Section 2.3.

$$|Ta^2(z)|\le Ch^{-4}(\|a^2\|_\infty+\|a^1\|_\infty+\|a^0\|_\infty)e^{\frac{-1}{2h}(|\mathrm{Re}z|^2-|\mathrm{Im}z|^2)}(e^{-\frac{ca}{2h}}e^{\frac{2C\epsilon a}{h}}+e^{-\frac{\epsilon^2a^2}{4h}}e^{\frac{\epsilon a}{h}}),$$

whenever $|z-2ae_1|<\epsilon a$. Now choosing $\epsilon<c/8C$ and $a>(c+4\epsilon)/\epsilon^2$, we deduce

$$|Ta^2(z)|\le Ch^{-4}(\|a^2\|_\infty+\|a^1\|_\infty+\|a^0\|_\infty)e^{\frac{1}{2h}(|\mathrm{Im}z|^2-|\mathrm{Re}z|^2-\frac{ca}{2})}.$$

Therefore, we end up with the following bound, as in [DSFKSU09b]:

$$e^{-\frac{\Phi(z_1)}{2h}}|Ta^2(z_1,x')|\le Ch^{-4}(\|a^2\|_\infty+\|a^1\|_\infty+\|a^0\|_\infty)\times\begin{cases}1, & \text{when}\quad z_1\in\mathbb{C}\\ e^{-\frac{ca}{4h}}, & \text{when } |z_1-2a|\le\frac{\epsilon a}{2}, |x'|<\frac{\epsilon a}{2}\end{cases}$$

where $x'\in\mathbb{R}^{n-1}$ and the weight Φ is defined as:

$$\Phi(z_1)=\begin{cases}(\mathrm{Im}z_1)^2, & \text{when}\,\mathrm{Re}z_1\le 0\\ (\mathrm{Im}z_1)^2-(\mathrm{Re}z_1)^2, & \text{when}\,\mathrm{Re}z_1\ge 0.\end{cases}$$

Now we are in a position to invoke [DSFKSU09b, Lemma 4.1] for the function

$$F(s)=\frac{h^4\,Ta^2(s,x')}{C(\|a^2\|_\infty+\|a^1\|_\infty+\|a^0\|_\infty)},$$

to obtain that there exists $c'>0$ such that

$$|Ta^2(x)|\le Ch^{-4}(\|a^2\|_\infty+\|a^1\|_\infty+\|a^0\|_\infty)e^{-\frac{c'}{2h}},$$

for all $x\in\Omega$ such that $|x_1|\le\delta$ for δ small enough. Multiplying this estimate by $(2\pi h)^{-n/2}$ and letting $h\to 0$, we obtain $a^2(x)=0$ for $x\in\Omega$, $|x_1|<\delta$. This implies $a^2=0$ near origin.

Step 2: We show that $a^1=0$ near origin.

Now we have the identity

$$\int_\Omega(a^2\Delta u+a_i^1\partial_i u+a^0u)v\,\mathrm{d}x=0,\quad \text{with } a^2(x)=0\quad \text{for } x\in\Omega, \text{ such that } |x_1|<\delta.$$

For fixed j, let us now choose the solutions $u(x,\xi,h) = u_j(x,\xi,h) = x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \tilde{\mathbf{r}}_j$ and $v(x,\eta,h) = v_0(x,\eta,h) = e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1$ in the above identity to obtain

$$\int_\Omega \Big[a^2\{\frac{-2\mathrm{i}\xi_j}{h}e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \Delta\tilde{\mathbf{r}}_j\} + a_i^1\{(\delta_{ij} - \frac{\mathrm{i}x_j\xi_i}{h})e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \partial_i\tilde{\mathbf{r}}_j\} + a^0\{x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \tilde{\mathbf{r}}_j\}\Big](e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1)\,\mathrm{d}x = 0. \tag{12}$$

Now, for the same j as above, we again choose $u(x,\xi,h) = u_0(x,\xi,h) = e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \mathbf{r}_2$ and $v(x,\eta,h) = v_j(x,\eta,h) = x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1$ to obtain

$$\int_\Omega \Big[a^2\Delta\mathbf{r}_2 + a_i^1\{-\frac{\mathrm{i}\xi_i}{h}e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \partial_i\mathbf{r}_2\} + a^0\{e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \mathbf{r}_2\}\Big](x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1)\,\mathrm{d}x = 0. \tag{13}$$

Subtracting equation (13) from (12), we deduce

$$\begin{aligned}
&\int_\Omega a_j^1 e^{\frac{-\mathrm{i}x\cdot(\xi+\eta)}{h}} \\
&= \int_\Omega a^2(\frac{2\mathrm{i}\xi_j}{h} - \Delta\tilde{\mathbf{r}}_j)(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1) + a^2\Delta\mathbf{r}_2(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1) + \int_\Omega -a_i^1(\partial_i\tilde{\mathbf{r}}_j)(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1) \\
&\quad + \int_\Omega a_i^1(\partial_i\mathbf{r}_2)(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1) - a_j^1 e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\mathbf{r}_1 + \int_\Omega a_i^1\frac{\mathrm{i}x_j\xi_i}{h}\mathbf{r}_1 - a_i^1\frac{\mathrm{i}\xi_i}{h}e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\tilde{\mathbf{r}}_j^1\,\mathrm{d}x - a^0\tilde{\mathbf{r}}_j(e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1) \\
&\quad + \int_\Omega a^0\mathbf{r}_2(x_j e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \tilde{\mathbf{r}}_j^1) - a^0 x_j e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\mathbf{r}_1 + a^0 e^{\frac{-\mathrm{i}x\cdot\xi}{h}}\tilde{\mathbf{r}}_j^1\,.
\end{aligned}$$

Notice that except for the first term in the integrand, every term can be bounded using the decay bound on the remainder terms. To bound the first term, we use the fact that $a^2(x) = 0$ for $x_1 > -\delta$, and obtain

$$\left|\int_\Omega a^2\frac{2\mathrm{i}\xi_j}{h}e^{\frac{-\mathrm{i}x\cdot\eta}{h}}\,\mathrm{d}x\right| = \left|\int_{\Omega\cap\{x_1\le-\delta\}} a^2\frac{2\mathrm{i}\xi_j}{h}e^{\frac{-\mathrm{i}x\cdot\eta}{h}}\,\mathrm{d}x\right| \le C\frac{\|a^2\|_\infty}{h}\times e^{-\frac{\delta a}{4h}}\times e^{|\mathrm{Im}\eta'|/h}.$$

Now estimating as in **Step 1**, we obtain

$$|\int_\Omega a_j^1 e^{\frac{-\mathrm{i}x\cdot z}{h}}\mathrm{d}x| \le Ch^{-4}(\|a^2\|_\infty + \|a^1\|_\infty + \|a^0\|_\infty)e^{-\frac{\delta a}{2h}}e^{\frac{2C\epsilon a}{h}},$$

for all $z \in \mathbb{C}^n$ such that $|z - 2\mathrm{i}ae_1| < 2\epsilon a$. Carrying out exactly as in **Step 1**, we find that $a_j^1(x) = 0$ for $x \in \Omega$ such that $|x_1| < \delta'$.

Step 3: Finally we show that $a^0 = 0$ near origin.

We have the identity

$$\int_\Omega (a^2\Delta u + a_i^1\partial_i u + a^0 u)v\,\mathrm{d}x = 0,$$

with $a^2(x) = 0 = a^1(x) \quad \text{for } x \in \Omega, |x_1| < \delta'$. Now using the solutions $u(x,\xi,h) = u_0(x,\xi,h) = e^{\frac{-\mathrm{i}x\cdot\xi}{h}} + \mathbf{r}_2$ and $v(x,\eta,h) = v_0(x,\eta,h) = e^{\frac{-\mathrm{i}x\cdot\eta}{h}} + \mathbf{r}_1$, and estimating as above, we arrive at

$$|\int_\Omega a^0 e^{\frac{-\mathrm{i}x\cdot z}{h}}\mathrm{d}x| \le Ch^{-4}(\|a^2\|_\infty + \|a^1\|_\infty + \|a^0\|_\infty)e^{-\frac{\delta' a}{2h}}e^{\frac{2C\epsilon a}{h}},$$

for all $z \in \mathbb{C}^n$ such that $|z - 2\mathrm{i}ae_1| < 2\epsilon a$.

Again, proceeding as in **Step 1 and Step 2**, we conclude $a^0(x) = 0$ for $x \in \Omega, |x_1| < \delta''$ for $\delta'' > 0$ small enough. Thus combining all the steps, we obtain that a^2, a^1 and a^0 vanish in a small enough neighbourhood of the origin. □

3. Proof of main result

Our objective in this section is to demonstrate that a^2, a^1 and a^0 vanish identically. To achieve this, we first present the following density result.

Lemma 3.1. *Let $\Omega_1 \subset \Omega_2$ be two bounded open sets with smooth boundaries. Let G_2 be the Green kernel for the biharmonic operator associated to the open set Ω_2 i.e.,*

$$-\Delta_y^2 G_2(x,y) = \delta(x-y) \quad \text{for all} \quad x, y \in \Omega_2$$
$$\left(G_2(x,\cdot), \frac{\partial G_2}{\partial \nu}(x,\cdot)\right) = 0 \qquad\qquad \text{on} \quad \partial\Omega_2.$$

Then the set $\mathcal{A}$ is dense in the subspace of all biharmonic functions $u \in C^\infty(\overline{\Omega}_1)$ such that $u|_{\partial\Omega_1\cap\partial\Omega_2} = 0 = \frac{\partial u}{\partial \nu}|_{\partial\Omega_1\cap\partial\Omega_2}$, equipped with $L^2(\Omega_1)$ topology, where

$$\mathcal{A} := \left\{ \int_{\Omega_2} G_2(\cdot,y)a(y)\,\mathrm{d}y : a \in C^\infty(\overline{\Omega}_2), \operatorname{supp} a \subset \Omega_2 \setminus \overline{\Omega}_1 \right\}. \tag{14}$$

Proof. Let $v \in L^2(\Omega_1)$ be a function which is orthogonal to subspace (14), then by Fubini's theorem we have

$$\int_{\Omega_2} a(y) \left(\int_{\Omega_1} G_2(x,y)v(x)\,\mathrm{d}x \right) \mathrm{d}y = 0,$$

for all $a \in C^\infty(\overline{\Omega}_2)$ supported in $\Omega_2 \setminus \overline{\Omega}_1$, therefore $\int_{\Omega_1} G_2(x,y)v(x)\,\mathrm{d}x = 0$ for all $y \in \Omega_2 \setminus \overline{\Omega}_1$. We want to show that v is orthogonal to any biharmonic function $u \in C^\infty(\overline{\Omega}_1)$ satisfying $u|_{\partial\Omega_1\cap\partial\Omega_2} = 0 = \frac{\partial u}{\partial\nu}|_{\partial\Omega_1\cap\partial\Omega_2}$. Let us consider the function $w(y) = \int_{\Omega_1} G_2(x,y)v(x)\,\mathrm{d}x$. Clearly, $w \in H^4(\Omega_2)$, and it satisfies $\Delta^2 w = v$ in Ω_1. Then, using the fact that $\Delta^2 w = v$ in Ω_1, we obtain

$$\begin{aligned}
\int_{\Omega_1} uv\,\mathrm{d}x &= \int_{\Omega_1} u\Delta^2 w - \int_{\Omega_1} w\Delta^2 u \\
&= -\int_{\partial\Omega_1} \partial_\nu u \Delta w + \int_{\partial\Omega_1} u\partial_\nu(\Delta w) + \int_{\partial\Omega_1} \partial_\nu w\Delta u - \int_{\partial\Omega_1} w\partial_\nu(\Delta u) \\
&= \int_{\partial\Omega_1\setminus\partial\Omega_2} -\partial_\nu u\Delta w + u\partial_\nu(\Delta w) + \partial_\nu w\Delta u - w\partial_\nu(\Delta u).
\end{aligned}$$

Since $w(y) = \int_{\Omega_1} G_2(x,y)v(x)\,\mathrm{d}x = 0$ for all $y \in \Omega_2\setminus\overline{\Omega}_1$ and $w \in H^4(\Omega_2)$, this implies $\partial_\nu^k w = 0$ on $\partial\Omega_1\setminus\partial\Omega_2$ for $k = 0, \cdots, 3$. Hence we conclude $\int_{\Omega_1} uv\,\mathrm{d}x = 0$, for all u such that $\Delta^2 u = 0, u|_{\partial\Omega_1\cap\partial\Omega_2} = 0 = \frac{\partial u}{\partial\nu}|_{\partial\Omega_1\cap\partial\Omega_2}$. This completes the proof. □

We now present the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix a point $x_1 \in \Omega$, and let $\Theta : [0,1] \to \overline{\Omega}$ be a smooth curve joining $x_0 \in \partial\Omega\setminus\Gamma$ to x_1. The curve Θ satisfies $\Theta(0) = x_0$, $\Theta'(0)$ is the interior normal to $\partial\Omega$ at x_0, and $\Theta(t) \in \Omega$ for all $t \in (0,1]$. By local result (Proposition 2.2), there exists $\epsilon_0 > 0$ such that $(a^0, a^1, a^2) = 0$ in $B_{\epsilon_0}(x_0)$. To this end, we consider the closed neighbourhood of the curve ending at $\Theta(t)$ defined as:

$$\Theta_\epsilon(t) := \{x \in \overline{\Omega} : \operatorname{dist}(x, \Theta([0,t])) \leq \epsilon\}.$$

Moreover, we define the set

$$I = \{t \in [0,1] : a^0 = a^1 = a^2 = 0 \text{ a.e. on } \Theta_\epsilon(t) \cap \Omega\}.$$

Clearly, I is a closed subset of $[0,1]$. Also, I is non-empty for ϵ small enough, by the local result. Let us now prove that I is also open. WLOG, assume ϵ small enough such that $\Theta_\epsilon(1) \subset\subset \overline{\Omega}$.

Let $t_0 \in I$ (which ensures $[0,t_0] \subset I$). Let us smoothen $\Omega\backslash\Theta_\epsilon(t_0)$ into an open subset Ω_1 of Ω with smooth boundary, such that $\Omega_1 \supset \Omega\backslash\Theta_\epsilon(t_0)$. For $\epsilon < \epsilon_0$ we have that $\partial\Theta_\epsilon(t_0)\cap\partial\Omega \subset B_{\epsilon_0}(x_0)\cap\partial\Omega \subset \partial\Omega\backslash\Gamma$. This implies $\partial\Omega_1 \cap \partial\Omega \supset \Gamma$. Let us also smoothen out $\Omega \cup B_{\tilde{\epsilon}}(x_0)$ ($\tilde{\epsilon} < \epsilon < \epsilon_0$) into an open subset Ω_2 with smooth boundary. Hence,

$$\Omega \cup B_{\tilde{\epsilon}}(x_0) \subset \Omega_2, \quad \text{and} \quad \Gamma \subset \partial\Omega_1 \cap \partial\Omega \subset \partial\Omega_2 \cap \partial\Omega.$$

For each $x \in \Omega_2$, let $G_2(x,\cdot)$ be the Green kernel associated to the open set Ω_2:

$$\begin{cases} \Delta_y^2 G_2(x,y) = \delta(x-y) \\ \left(G_2(x,\cdot)|_{\partial\Omega_2}, \frac{\partial G_2}{\partial\nu(\cdot)}(x,\cdot)|_{\partial\Omega_2}\right) = 0. \end{cases}$$

To proceed further, we next define a new function G as follows:

$$G(x,y) := a^2(y)\,\Delta_y G_2(x,y) + \langle\, a^1(y), \nabla_y G_2(x,y)\,\rangle \; + a^0(y)\, G_2(x,y) \quad \text{where } x \in \Omega_2\backslash\overline{\Omega}_1, y \in \Omega_1.$$

Then the function

$$(z,x) \ni \Omega_2\backslash\overline{\Omega}_1 \times \Omega_2\backslash\overline{\Omega}_1 \longmapsto \int\limits_{\Omega_1} G(x,y)\, G_2(z,y)\,\mathrm{d}y,$$

is biharmonic viewed as a function of z and x variables. Moreover, it satisfies

$$\int\limits_{\Omega_1} G(x,y)\, G_2(z,y)\mathrm{d}y = \int\limits_{\Omega} G(x,y)\, G_2(z,y)\mathrm{d}y, \tag{15}$$

since (a^0,a^1,a^2) vanish on $\Theta_\epsilon(t_0)\cap\Omega$, and $\Omega\backslash\overline{\Omega}_1 \subset \Theta_\epsilon(t_0)\cap\Omega$. We see that the functions $y \mapsto G_2(z,y)$ and $y \mapsto G_2(x,y)$ are in $C^\infty(\overline{\Omega})$, biharmonic in Ω, and vanish on $\Gamma \subset \partial\Omega_2$, when $z,x \in \Omega_2\backslash\overline{\Omega}$. Thus $y \mapsto G_2(z,y)$ and $y \mapsto G_2(x,y)$ are in $\mathcal{E}$, see definition (2). Next inserting them into the integral identity (3) we obtain $\int_\Omega G(x,y)\, G_2(z,y)\,\mathrm{d}y = 0$ for all $z,x \in \Omega_2 \setminus \overline{\Omega}$. Note that, biharmonic functions are real analytic, and any real analytic function that vanishes on some open set must be zero everywhere. The combination of this along with (15), and $\int_\Omega G(x,y)\, G_2(z,y)\,\mathrm{d}y = 0$ for all $z,x \in \Omega_2 \setminus \overline{\Omega}$ implies $\int_{\Omega_1} G(x,y)\, G_2(z,y)\,\mathrm{d}y = 0$ for all $z,x \in \Omega_2 \setminus \overline{\Omega_1} = 0$. Next, multiplying $\int_{\Omega_1} G(x,y)\, G_2(z,y)\,\mathrm{d}y$ by $b_1 = b_1(z) \in C_c^\infty(\Omega_2\backslash\overline{\Omega}_1)$, and then taking integrating over Ω_2 we obtain

$$\int\limits_{\Omega_2}\int\limits_{\Omega_1} G(x,y)\, G_2(z,y)\,\mathrm{d}y\; b_1(z)\,\mathrm{d}z = 0, \text{ for all } x \in \Omega_2\backslash\overline{\Omega}_1.$$

Using Fubini theorem, this further entails

$$\int\limits_{\Omega_1} G(x,y) \left(\int\limits_{\Omega_2} G_2(z,y)\, b_1(z)\mathrm{d}z \right) \mathrm{d}y = 0, \text{ for all } x \in \Omega_2\backslash\overline{\Omega}_1.$$

We next use the density result from Lemma 3.1 to approximation any biharmonic function u in $\overline{\Omega_1}$ with $u|_{\partial\Omega_1\cap\partial\Omega_2} = 0 = \frac{\partial u}{\partial\nu}|_{\partial\Omega_1\cap\partial\Omega_2}$, utilizing the integral of the form $\int_{\Omega_2} G_2(z,y)\, b_1(z)\mathrm{d}z$. This gives

$$\int\limits_{\Omega_1} u(y)\, G(x,y)\,\mathrm{d}y = \int\limits_{\Omega_1} u(y)[a^2\Delta_y G_2(x,y) + \langle\, a^1(y), \nabla_y G_2(x,y)\,\rangle + a^0(y) G_2(x,y)]\,\mathrm{d}y = 0,$$

for all $x \in \Omega_2 \backslash \overline{\Omega}_1$. Integration by parts then gives

$$0 = \int_{\Omega_1} \Delta_y(a^2(y)u(y))G_2(x,y) - \langle \nabla u(y), a^1(y)\rangle G_2(x,y) - \mathrm{div}(a^1)(y)\, G_2(x,y)\, u(y) \;+\; a^0(y)u(y)G_2(x,y)\, \mathrm{d}y$$
$$+ \int_{\partial\Omega_1} a^2(y)u(y)\frac{\partial G_2}{\partial \nu}(x,y) - G_2(x,y)\frac{\partial(a^2 u(y))}{\partial \nu}\, \mathrm{d}\sigma(y) + \int_{\partial\Omega_1} u(y)\langle a^1, \nu\rangle G_2(x,y)\, \mathrm{d}\sigma(y).$$

Since $a^1 = 0$ on $\partial\Omega_1 \cap \Omega$, $a^2 = 0$ on $\Omega_1^c \cap \Omega$ and $G_2(x,\cdot) = 0$ on $\partial\Omega_1 \cap \partial\Omega_2$, for all $x \in \Omega_2 \backslash \overline{\Omega}_1$, this implies

$$0 = \int_{\Omega_1} [a^2 \Delta u(y) + u(y)\Delta a^2 + 2\langle \nabla a^2, \nabla u(y)\rangle] G_2(x,y) - \langle \nabla u(y), a^1(y)\rangle G_2(x,y) \mathrm{d}y$$
$$- \int_{\Omega_1} (\mathrm{div}(a^1)\, G_2(x,y)\, u(y) - a^0(y)u(y)G_2(x,y))\, \mathrm{d}y.$$

Multiplying this by a function $b_2 = b_2(x) \in C_c^\infty(\Omega_2 \backslash \overline{\Omega}_1)$ and integrating over Ω_2,

$$\int_{\Omega_2}\int_{\Omega_1} [a^2 \Delta u(y) + \langle \nabla u(y), 2\nabla a^2 - a^1(y)\rangle + \left(\Delta a^2 - \mathrm{div}(a^1) + a^0(y)\right) u(y)] G_2(x,y)\, \mathrm{d}y b_2(x) \mathrm{d}x = 0.$$

Again, we use Fubini theorem to interchange the order of the integral and invoke Lemma 3.1 to obtain

$$\int_{\Omega_1} [a^2 \Delta u(y) + \langle \nabla u(y), 2\nabla a^2 - a^1(y)\rangle + \left(\Delta a^2 - \mathrm{div}(a^1) + a^0(y)\right) u(y)] v(y)\, \mathrm{d}y = 0.$$

This gives $a^2 = 0, 2\nabla a^2 - a^1 = 0$, and $\Delta a^2 - \mathrm{div}\, a^1 + a^0 = 0$ locally at each point of $\partial\Omega_1 \cap \Omega$ using the local uniqueness result from Proposition 2.2 replacing Ω by Ω_1. Now, for each $x \in \partial\Theta_\epsilon(t_0) \setminus B_{\epsilon_0}(x_0)$, Ω_1 can be chosen such that $\partial\Omega_1 \cap \partial\Theta_\epsilon(t_0) = \{x\}$. Thus, a^0, a^1 and a^2 vanish locally at each point of $\partial\Theta_\epsilon(t_0)$. This proves that I is open. Since I is a non-empty connected set that is both open and closed. This implies $I = [0,1]$. Since x_1 was an arbitrary point, this implies $a^j = 0$ everywhere in Ω for $j = 0,1,2$. This completes the proof. □

Appendix A. Linearization and derivation of the integral identity

In the section, we linearize the Dirichlet-to-Neumann map by computing its Fréchet derivative, when both Dirichlet and Neumann data are known only on a non-empty open set of the boundary. Let $\Sigma \subset \partial\Omega$ be non-empty and open and let $H^s_\Sigma(\partial\Omega)$ denote the space of functions $f \in H^s(\partial\Omega)$ such that $\mathrm{supp} f \subset \Sigma$. Recall the operator (1) together with its Dirichlet boundary conditions:

$$\begin{cases} \mathcal{L}_Q(x,D) &= (-\Delta)^2 + Q(x,D) \quad \text{in} \quad \Omega \\ (u, \partial_\nu u) &= (f_1, f_2) \qquad\qquad \text{on} \quad \partial\Omega \end{cases} \tag{16}$$

where $Q(x,D) := \sum_{l=0}^3 a^l_{i_1\cdots i_l}(x)\, D^{i_1\cdots i_l}$ is a differential operator of order 3 with $1 \le i_1, \cdots, i_l \le n$ and a^l is a smooth symmetric tensor field of order l in $\overline{\Omega}$ and $(f_0, f_1) \in H^{7/2}_\Sigma(\partial\Omega) \times H^{5/2}_\Sigma(\partial\Omega)$. In what follows, we identify the tensor fields $(a^l)_{l=0}^3$ to the associated differential operator $\sum_{l=0}^3 a^l_{i_1\cdots i_l}(x)\, D^{i_1\cdots i_l}$ and let $\mathcal{S}$ denote the space of all such bounded smooth tensor fields, equipped with the L^∞ norm. Then, the partial DN map is defined as:

$$\Lambda : \mathcal{S} \to B\left(H^{7/2}_\Sigma(\partial\Omega) \times H^{5/2}_\Sigma(\partial\Omega), \left(H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)\right)\big|_\Sigma\right), \quad \Lambda_Q(f_1, f_2) := (\partial^2_\nu u|_\Sigma, \partial^3_\nu u|_\Sigma).$$

where $\left(H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)\right)\big|_\Sigma$ denotes the restriction of the corresponding functions to Σ. The following lemma gives an expression for the Fréchet derivative of the map $Q \mapsto \Lambda_Q$.

Lemma A.1. *Suppose* 0 *is not an eigen value of* (16) *and let* $P_Q : (H^{7/2}_\Sigma(\partial\Omega)\times H^{5/2}_\Sigma(\partial\Omega)) \to H^4(\Omega)$ *denote the solution operator for the problem* (16)*. Let* $G_Q : L^2(\Omega) \to \mathcal{D}(\mathcal{L}_Q)$ *be the Green operator satisfying*

$$\begin{cases} \mathcal{L}_Q(G_QF) & = F \quad in \quad \Omega \\ (G_QF, \partial_\nu(G_QF)) & = 0 \quad on \quad \partial\Omega. \end{cases}$$

The Fréchet derivative of Λ *is given as*

$$(\mathrm{d}\Lambda)_Q = B_Q : \mathcal{S} \to B\left(H^{7/2}_\Sigma(\partial\Omega)\times H^{5/2}_\Sigma(\partial\Omega), \left(H^{3/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)\right)|_\Sigma\right)$$

defined for $H \in \mathcal{S}$ *as*

$$(B_QH)(f) = \left(\partial_\nu^2 G_Q(-HP_Qf), \partial_\nu^3 G_Q(-HP_Qf)\right)\Big|_\Sigma, \quad for \quad f = (f_0, f_1) \in (H^{7/2}_\Sigma(\partial\Omega)\times H^{5/2}_\Sigma(\partial\Omega)).$$

Proof. Let $\|H\|_{L^\infty(\Omega)}$ be small enough such that Λ_{Q+H} is well defined, where H lives on the same space as Q. Given $f = (f_0, f_1) \in H^{7/2}_\Sigma(\partial\Omega)\times H^{5/2}_\Sigma(\partial\Omega)$, we have

$$\Lambda_{Q+H}f - \Lambda_Q f = \left(\partial_\nu^2(P_{Q+H}f - P_Qf), \partial_\nu^3(P_{Q+H}f - P_Qf)\right)\Big|_\Sigma.$$

The function $w \coloneqq P_{Q+H}f - P_Qf$ satisfies the following partial differential equation.

$$\begin{aligned} ((-\Delta)^2 + Q(x,D))w = -H\,w - H\,P_Qf \quad & \text{in} \quad \Omega \\ (w, \partial_\nu w) = 0 \quad & \text{on} \quad \partial\Omega. \end{aligned}$$

Thus, we can write $w = G_Q(-H\,w) + G_Q(-HP_Qf)$ and $w \in \mathcal{D}(\mathcal{L}_{Q+H})$. Utilizing the continuity of the Green operator $G_Q(H\,w)$ we obtain

$$\|G_Q(H\,w)\|_{H^4(\Omega)} \le c\|H\,w\|_{L^2(\Omega)} \le \frac{1}{2}\|w\|_{H^4(\Omega)} \quad \text{if} \quad \|H\|_{L^\infty(\Omega)} \text{ is small.}$$

We next estimate $\|w\|_{H^4(\Omega)} = \|G_Q(-H\,w) + G_Q(-HP_Qf)\|_{H^4(\Omega)}$. The combination of this with triangle inequality and last displayed relation implies $\|w\|_{H^4(\Omega)} \le \|H\|_{L^\infty}\,\|f\|_{H^{\frac{7}{2}}(\partial\Omega)\times H^{\frac{5}{2}}(\partial\Omega)}$. Next we observe that,

$$\begin{aligned} &\left(\Lambda_{Q+H}(f) - \Lambda_Q(f) - \left(\partial_\nu^2 G_Q(-HP_Qf), \partial_\nu^3 G_Q(-HP_Qf)\right)|_\Sigma\right) \\ &= \left(\partial_\nu^2(w - G_QHP_Qf), \partial_\nu^3(w - G_QHP_Qf)\right)\Big|_\Sigma = \left(\partial_\nu^2 G_Q(-Hw), \partial_\nu^3 G_Q(-Hw)\right)\Big|_\Sigma. \end{aligned}$$

Combining trace theorem, continuity of G_Q and $\|w\|_{H^4(\Omega)} \le \|H\|_{L^\infty(\Omega)}\,\|f\|_{H^{\frac{7}{2}}(\partial\Omega)\times H^{\frac{5}{2}}(\partial\Omega)}$ we obtain from above

$$\begin{aligned} &\|(\partial_\nu^2 G_Q(-Hw), \partial_\nu^3 G_Q(-Hw))\|_{H^{\frac{3}{2}}(\Sigma)\times H^{\frac{1}{2}}(\Sigma)} \le \|(\partial_\nu^2 G_Q(-Hw), \partial_\nu^3 G_Q(-Hw))\|_{H^{\frac{3}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)} \\ &\le \|G_Q(H\,w)\|_{H^4(\Omega)} \le \|H\|_{L^\infty(\Omega)}\,\|w\|_{H^4(\Omega)} \le \|H\|^2_{L^\infty(\Omega)}\,\|f\|_{H^{\frac{7}{2}}(\partial\Omega)\times H^{\frac{5}{2}}(\partial\Omega)}. \end{aligned}$$

This proves that the Fréchet derivative of $Q \mapsto \Lambda_Q$ at Q is B_Q. □

We are interested in studying the injectivity of $\mathrm{d}\Lambda|_{Q=0}$. This reduces to

Lemma A.2. *Let* $\mathrm{d}\Lambda|_{Q=0} = 0$*. Then for any* $H = (a^{(l)})_{l=0}^3 \in \mathcal{S}$*, the following integral identity holds*

$$\int_\Omega \sum_{l=0}^{3} a^{(l)}_{i_1\ldots i_l}\partial_{i_1\ldots i_l} uv\,\mathrm{d}x = 0,$$

for all biharmonic functions u *and* v *in* Ω *whose Dirichlet data is supported in* Σ*.*

Proof. Let $g = (g_0, g_1) \in H^{7/2}_{\Sigma}(\partial\Omega) \times H^{5/2}_{\Sigma}(\partial\Omega)$. The function $P_0 g \in H^4(\Omega)$ satisfies

$$\begin{cases} (-\Delta)^2 P_0 g = 0 & \text{in} \quad \Omega \\ (P_0 g, \partial_\nu P_0 g) = g & \text{on} \quad \partial\Omega. \end{cases}$$

Also, for $f = (f_0, f_1) \in H^{7/2}_{\Sigma}(\partial\Omega) \times H^{5/2}_{\Sigma}(\partial\Omega)$, $G_0(-HP_0 f)$ solves

$$\begin{cases} (-\Delta)^2 G_0(-HP_0 f) = -HP_0 f & \text{in} \quad \Omega \\ (G_0(-HP_0 f), \partial_\nu G_0(-HP_0 f)) = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Multiplying $P_0 g$ to the last equation and using Green's identities, we get

$$\begin{aligned} -\int_\Omega (HP_0 f) P_0 g \, \mathrm{d}x &= \int_\Omega (-\Delta)^2 G_0(-HP_0 f) P_0 g - G_0(-HP_0 f)(-\Delta)^2 P_0 g \\ &= \int_{\partial\Omega} \Big[-\partial_\nu P_0 g (\Delta G_0(-HP_0 f)) + P_0 g \partial_\nu (\Delta G_0(-HP_0 f)) \\ &\qquad + \Delta(P_0 g) \partial_\nu (G_0(-HP_0 f)) - G_0(-HP_0 f) \partial_\nu (\Delta P_0 g) \Big] \mathrm{d}S. \end{aligned}$$

The last two terms vanish due to the properties of the Green's function on the boundary. By definition of P_0, we have $(P_0 g, \partial_\nu P_0 g)|_{\partial\Omega} = (g_0, g_1)$ which vanishes outside Σ. This yields

$$-\int_\Omega (HP_0 f) P_0 g \, \mathrm{d}x = \int_\Sigma \Big[-\partial_\nu P_0 g (\Delta G_0(-HP_0 f)) + P_0 g \partial_\nu (\Delta G_0(-HP_0 f)) \Big] \mathrm{d}S.$$

Since $\mathrm{d}\Lambda|_0 = 0$, using the expression for $\mathrm{d}\Lambda|_0$ from the previous lemma, we have

$$\left(\partial_\nu^2 (G_0(-HP_0 f)), \partial_\nu^3 (G_0(-HP_0 f))\right)|_\Sigma = 0.$$

Now using the equivalence of $\left(u, \partial_\nu u, \partial_\nu^2 u, \partial_\nu^3 u\right)|_\Sigma = 0 \iff (u, \partial_\nu u, \Delta u, \partial_\nu(\Delta u))|_\Sigma = 0$, where $\Sigma \subset \partial\Omega$. This completes the proof. ☐

Appendix B. Decay estimates

Lemma B.1. *Let $\mathbf{r}_2$ solves the following Dirichlet boundary value problem*

$$\begin{aligned} \Delta^2 \mathbf{r}_2 = 0 & \quad \text{in} \quad \Omega \\ (\mathbf{r}_2, \partial_\nu \mathbf{r}_2) = (w_0, \partial_\nu w_0) & \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Where $w_0 = \chi(x) e^{\frac{-ix\cdot\xi}{h}}$ with $\chi \in C_c^\infty(\mathbb{R}^n)$ and $\xi \in \mathbb{C}^n$ are from Lemma 2.1. Then there exists a constant C independent of ξ and h such that

$$\|\mathbf{r}_2\|_{H^2(\Omega)} \le C(1 + \frac{|\xi|^2}{h^2} + \frac{|\xi|^4}{h^4})^{1/2} e^{\frac{1}{h} H_K(\mathrm{Im}\ \xi)}, \quad \text{where} \quad H_K(y) = \sup_{x \in K} x \cdot y, \quad y \in \mathbb{R}^n.$$

Proof. From the well-posedness of the above Dirichlet problem [GGS10, Theorem 2.20], we obtain

$$\|\mathbf{r}_2\|_{H^2(\Omega)} \le C(\|w_0\|_{H^{3/2}(\partial\Omega)} + \|\frac{\partial w_0}{\partial\nu}\|_{H^{1/2}(\partial\Omega)}),$$

where $w_0(x) = \chi(x) e^{\frac{-ix\cdot\xi}{h}}$. Since $\partial\Omega$ is a compact Riemannian manifold of dimension $n-1$, we can choose a finite number of coordinate neighborhood system $\{(U_i, \phi_i)\}_{i=1}^{m_0}$, where

$$\phi_i : U_i \to V_i \subset\subset \mathbb{R}^{n-1}$$

is a diffeomorphism from U_i onto an open subset V_i contained in $\mathbb{R}^{n-1}$. Let $\{\psi_i\}_{i=1}^{m_0}$ be a partition of unity subordinate to $\{U_i\}_{i=1}^{m_0}$. The H^s-norm of $g \in C^\infty(\partial\Omega)$, for $s \in \mathbb{R}$, by

$$||g||_s^2 = \sum_{i=1}^{m_0} ||(\psi_i g) \circ \phi_i^{-1}||_s^2.$$

Let $w_{0,j}(x) = (\psi_j w_0) \circ \phi_j^{-1} = \psi_j \circ \phi_j^{-1} \cdot w_0 \circ \phi_j^{-1}$, $1 \le j \le m_0$. The $H^{3/2}(\partial\Omega)$ norm of w is defined as: $||w||_{H^{3/2}(\partial\Omega)}^2 := \sum_{j=1}^{m_0} ||w_{0,j}||_{3/2}^2$, where

$$||w_{0,j}||_{3/2}^2 := ||w_{0,j}||^2 + ||\nabla w_{0,j}||^2 + \int_{V_i}\int_{V_i} \frac{|\nabla w_{0,j}(x) - \nabla w_{0,j}(y)|^2}{|x-y|^n} \,\mathrm{d}x\,\mathrm{d}y.$$

It is enough to estimate $||w_{0,j}||_{3/2}^2$. We will estimate $||w_{0,j}||_{3/2}^2$ component-wise. To this end, we consider the L^2 norm of $w_{0,j}$ and observe that

$$||w_{0,j}||^2 \le C \int_{V_j} |w_0 \circ \phi_j^{-1}|^2 \,\mathrm{d}x \le C \sup_{\partial\Omega \cap \operatorname{supp}\chi} |w_{0,j}|^2 \le C e^{\frac{2}{h} H_K(\mathrm{Im}\xi)}. \tag{17}$$

Since $\nabla w_{0,j} = \nabla(\psi_j \circ \phi_j^{-1}) \cdot w_0 \circ \phi_j^{-1} + \psi_j \circ \phi_j^{-1} \cdot \nabla(w_0 \circ \phi_j^{-1}) = \nabla(\psi_j \circ \phi_j^{-1}) \cdot w_0 \circ \phi_j^{-1} + \psi_j \circ \phi_j^{-1} \cdot \nabla(w_0) \circ \phi_j^{-1} \cdot J(\phi_j^{-1})$, this implies

$$||\nabla w_{0,j}||^2 \le C\left(1 + \frac{|\xi|^2}{h^2}\right) e^{\frac{2}{h} H_K(\mathrm{Im}\xi)}. \tag{18}$$

Next, we consider

$$\begin{aligned}
&\int_{V_i}\int_{V_i} \frac{|\nabla w_{0,j}(x) - \nabla w_{0,j}(y)|^2}{|x-y|^n} \\
&\le 2\int_{V_i}\int_{V_i} \frac{|\nabla(\psi_j \circ \phi_j^{-1})(x) w_0 \circ \phi_j^{-1}(x) - \nabla(\psi_j \circ \phi_j^{-1})(x) w_0 \circ \phi_j^{-1}(y)|^2}{|x-y|^n} \\
&\quad + \int_{V_i}\int_{V_i} \frac{|\psi_j \circ \phi_j^{-1}(x) \cdot \nabla(w_0 \circ \phi_j^{-1})(x) - \psi_j \circ \phi_j^{-1}(y) \cdot \nabla(w_0 \circ \phi_j^{-1})(y)|^2}{|x-y|^n}.
\end{aligned}$$

Now using mean value theorem on each integrand of the above right hand side inequality and then using the property of w_0, we obtain

$$\int_{V_i}\int_{V_i} \frac{|\nabla w_{0,j}(x) - \nabla w_{0,j}(y)|^2}{|x-y|^n} \,\mathrm{d}x\mathrm{d}y \le C\left(1 + \frac{|\xi|^2}{h^2} + \frac{|\xi|^4}{h^4}\right) e^{\frac{2}{h} H_K(\mathrm{Im}\,\xi)}.$$

The combination of this along with (17) and (18) gives

$$||w_{0,j}||_{3/2}^2 \le C\left(1 + \frac{|\xi|^2}{h^2} + \frac{|\xi|^4}{h^4}\right) e^{\frac{2}{h} H_K(\mathrm{Im}\,\xi)}, \quad \text{for each } 1 \le j \le m_0.$$

This further entails

$$||w_0||_{H^{3/2}(\partial\Omega)} \le C\left(1 + \frac{|\xi|^2}{h^2} + \frac{|\xi|^4}{h^4}\right)^{1/2} e^{\frac{1}{h} H_K(\mathrm{Im}\,\xi)}.$$

In a very similar fashion, one can also obtain

$$\|\frac{\partial w_0}{\partial \nu}\|_{H^{1/2}(\partial\Omega)} \le C\left(1 + \frac{|\xi|^2}{h^2} + \frac{|\xi|^4}{h^4}\right)^{1/2} e^{\frac{1}{h} H_K(\mathrm{Im}\,\xi)}.$$

We next combine preceding estimates and conclude

$$\|\mathbf{r}_2\|_{H^2(\Omega)} \leq C(\|w_0\|_{H^{3/2}(\partial\Omega)} + \|\frac{\partial w_0}{\partial \nu}\|_{H^{1/2}(\partial\Omega)}) \leq C(1 + \frac{|\xi|^2}{h^2} + \frac{|\xi|^4}{h^4})^{1/2} e^{\frac{1}{h} H_K(\mathrm{Im}\ \xi)},$$

for all $\xi \in \mathbb{C}^n$ such that $\xi \cdot \xi = 0$, and the constant C is independent of ξ. □

Acknowledgements

S. K. S was partly supported by the Academy of Finland (Centre of Excellence in Inverse Modelling and Imaging, grant 284715) and by the European Research Council under Horizon 2020 (ERC CoG 770924). S. K. S would like to express his gratitude to M. Salo for their fruitful conversation on this subject. The authors thank Venky Krishnan and Sivaguru Ravisankar for several fruitful discussions which led to the improvement of the manuscript.

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