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Coupled Surface and Volume Integral Equations for Electromagnetism

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***Abstract*—We study frequency domain electromagnetic scattering at a bounded, penetrable, and inhomogeneous obstacle. By defining constant reference coefficients, a new representation formula for interior and exterior vector fields is proposed, based on the general form of the Stratton-Chu integral representation. The final integral equation system consists of surface integral operators arising from a Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) formulation and compact volume integral operators with weakly singular kernels. The problem is solved with a Galerkin approach with usual Curl-conforming and Div-conforming finite elements on the surface and in the volume. Compression techniques and special quadrature rules for singular integrands are required for an efficient and accurate solution. Numerical experiments provide evidence that our new formulation enjoys promising properties.**

***Index Terms*—volume integral equations, boundary integral operators, boundary elements, finite elements, electromagnetic waves**

I. INTRODUCTION

We study electromagnetic wave propagation in the frequency domain. A bounded, penetrable, and inhomogeneous obstacle is considered, with smoothly varying coefficients inside, i.e. jumps in the material properties across its surface are admitted. For the general case mentioned above, volume integral equations lead to a formulation where the problem reduces to finding electric and magnetic fields inside the domain of interest [4]. The integral equations involve strongly singular integral operators, which are not compact in the general case. This is a problem even for simple settings, such as dielectric materials. Alternatively, a popular option for problems posed on (exterior) unbounded domains is to use an expression for the Dirichlet-to-Neumann map. This is the foundation for coupling finite element methods (FEM) with boundary element methods (BEM) and it leads to stable discretizations for the transmission problem [7]. In the particular case of piecewise constant material properties, it is possible to completely reformulate the problem as a surface integral equation system [6], [11]. That system can then be discretized with BEM in various ways.

We propose a formulation that reduces to PMCHWT in the particular case of constant coefficients, and is coupled with compact volume integral operators in the case of spatially varying inhomogeneities. We have developed a rigorous analysis that proves well-posedness of the continuous and discrete systems. The approach is closely related to the one for acoustic scattering presented in [8].

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II. ELECTROMAGNETIC SCATTERING

We are interested in solving the frequency domain electromagnetic wave scattering problem in a medium that is homogeneous outside a bounded region $\Omega \subset \mathbb{R}^3$. We denote the exterior domain $\Omega^+ := \mathbb{R}^3 \setminus \bar{\Omega}$. Material properties are given by functions $\varepsilon \in L^\infty(\mathbb{R}^3)$ and $\mu \in L^\infty(\mathbb{R}^3)$ where

$$\varepsilon(\boldsymbol{x}) \equiv \varepsilon_0, \quad \mu(x) \equiv \mu_0 \quad \text{for } \boldsymbol{x} \in \Omega^+, \tag{1}$$

and $\varepsilon_{\text{max}} > \varepsilon(\boldsymbol{x}) > \varepsilon_{\text{min}} > 0$, $\mu_{\text{max}} > \mu(\boldsymbol{x}) > \mu_{\text{min}} > 0$ almost everywhere in $\mathbb{R}^3$.

The frequency-domain Maxwell equations governing the problem of finding the total electric field $\boldsymbol{u} := \boldsymbol{u}^s + \boldsymbol{u}^{\text{inc}}$ and total magnetic field $\boldsymbol{v} := \boldsymbol{v}^s + \boldsymbol{v}^{\text{inc}}$ in this inhomogeneous medium are

$$\mathbf{curl}(\boldsymbol{u}) - i\omega\mu(\boldsymbol{x})\boldsymbol{v} = 0, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \tag{2}$$
$$\mathbf{curl}(\boldsymbol{v}) + i\omega\varepsilon(\boldsymbol{x})\boldsymbol{u} = 0, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \tag{3}$$

where $\boldsymbol{u}^{\text{inc}}, \boldsymbol{v}^{\text{inc}}$ are the incident fields satisfying the Maxwell equations in the whole space,

$$\mathbf{curl}(\boldsymbol{u}^{\text{inc}}) - i\omega\mu_0\boldsymbol{v}^{\text{inc}} = 0, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \tag{4}$$
$$\mathbf{curl}(\boldsymbol{v}^{\text{inc}}) + i\omega\varepsilon_0\boldsymbol{u}^{\text{inc}} = 0, \quad \text{for } \boldsymbol{x} \in \mathbb{R}^3, \tag{5}$$

and $\boldsymbol{u}^s, \boldsymbol{v}^s$ satisfy Silver-Müller radiation conditions

$$\boldsymbol{v}^s \times \frac{\boldsymbol{x}}{r} - \boldsymbol{u}^s = \mathcal{O}\left(\tfrac{1}{r^2}\right), \quad r = |\boldsymbol{x}| \to +\infty. \tag{6}$$

The problem can be formulated as the following transmission problem: find $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \mathbb{R}^d)$ such that

$$\mathbf{curl}(\boldsymbol{u}_0) - i\omega\mu_0\boldsymbol{v}_0 = 0, \qquad \text{in } \Omega^+, \tag{7a}$$
$$\mathbf{curl}(\boldsymbol{v}_0) + i\omega\varepsilon_0\boldsymbol{u}_0 = 0, \qquad \text{in } \Omega^+, \tag{7b}$$
$$\mathbf{curl}(\boldsymbol{u}) - i\omega\mu(\boldsymbol{x})\boldsymbol{v} = 0, \qquad \text{in } \Omega, \tag{7c}$$
$$\mathbf{curl}(\boldsymbol{v}) + i\omega\varepsilon(\boldsymbol{x})\boldsymbol{u} = 0, \qquad \text{in } \Omega, \tag{7d}$$
$$\gamma_\tau^+\boldsymbol{u}_0 - \gamma_\tau^-\boldsymbol{u} = -\gamma_\tau\boldsymbol{u}^{\text{inc}}, \qquad \text{on } \Gamma, \tag{7e}$$
$$\gamma_\tau^+\boldsymbol{v}_0 - \gamma_\tau^-\boldsymbol{v} = -\gamma_\tau\boldsymbol{v}^{\text{inc}}, \qquad \text{on } \Gamma, \tag{7f}$$
$$\boldsymbol{v}_0 \times \frac{\boldsymbol{x}}{r} - \boldsymbol{u}_0 = \mathcal{O}\left(\tfrac{1}{r^2}\right), \qquad r := |\boldsymbol{x}| \to +\infty, \tag{7g}$$

where $\gamma_\tau^\pm$ denotes the exterior/interior tangential trace operators: $\gamma_\tau\boldsymbol{e} = \boldsymbol{e}|_\Gamma \times \boldsymbol{n}$.

III. Preliminaries

We state the mathematical setting required for our formulations. Let $L^2(\Omega)$ be the Hilbert space of square-integrable functions in Ω, equipped with the usual inner-product

$$(u, v) := \int_\Omega u(\boldsymbol{x})\overline{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

We define $\mathbf{L}^2(\Omega)$ as the space of square-integrable vector fields in Ω. We denote as $\mathbf{H}(\mathbf{curl}, \Omega)$ the space of vector fields in $\mathbf{L}^2(\Omega)$, whose $\mathbf{curl}$ is also in $\mathbf{L}^2(\Omega)$. Similarly, $\mathbf{H}(\text{div}, \Omega)$ denotes the space of vector fields with divergence in $L^2(\Omega)$.

The trace operator γ_τ acts on elements of $\mathbf{H}(\mathbf{curl}, \Omega)$ and maps onto the space $\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)$ of tangential vector fields [5]. We define the duality product in $\mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)$ as an extension of

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle_\Gamma := \int_\Gamma \boldsymbol{\alpha}(\boldsymbol{x}) \cdot (\boldsymbol{n}(\boldsymbol{x}) \times \boldsymbol{\beta}(\boldsymbol{x}))\mathrm{d}s_x.$$

We will also make use of the normal trace $\gamma_{\mathbf{n}}$, which acts on elements of $\mathbf{H}(\text{div}, \Omega)$ and takes the normal component of the restriction of a vector field to the boundary: $\gamma_{\mathbf{n}}\boldsymbol{u} := \boldsymbol{n} \cdot \boldsymbol{u}|_\Gamma$.

The fundamental solution for the Helmholtz equation with wavenumber $\kappa_\star$ is denoted $G_\star(\boldsymbol{x}, \boldsymbol{y})$ and reads $G_\star(\boldsymbol{x}, \boldsymbol{y}) = \exp(i\kappa_\star|\boldsymbol{x} - \boldsymbol{y}|)/(4\pi|\boldsymbol{x} - \boldsymbol{y}|)$. We define the (scalar) single-layer potential $S_\star$ and Newton potential $\mathsf{N}_\star$ as follows:

$$(S_\star\varphi)(\boldsymbol{x}) := \int_\Gamma G_\star(\boldsymbol{x}, \boldsymbol{y})\varphi(\boldsymbol{y})\mathrm{d}s_y, \tag{8}$$

$$(\mathsf{N}_\star f)(\boldsymbol{x}) := \int_\Omega G_\star(\boldsymbol{x}, \boldsymbol{y})f(\boldsymbol{y})\mathrm{d}\boldsymbol{y}, \tag{9}$$

for $\boldsymbol{x} \in \mathbb{R}^3 \setminus \Gamma$.

Finally, we state a general version of the *Stratton-Chu integral representation* for general smooth vector fields: let $\boldsymbol{u}, \boldsymbol{v} \in C^2(\Omega)$ and $\kappa_\star = \omega\sqrt{\mu_\star\varepsilon_\star}$. Then

$$\begin{aligned}\boldsymbol{u} = {} & \mathbf{curl}\mathsf{N}_\star(\mathbf{curl}(\boldsymbol{u}) - i\omega\mu_\star\boldsymbol{v}) \\ & + i\omega\mu_\star\mathsf{N}_\star(\mathbf{curl}(\boldsymbol{v}) + i\omega\varepsilon_\star\boldsymbol{u}) - \nabla\mathsf{N}_\star(\text{div}(\boldsymbol{u})) \\ & + \mathbf{curl}S_\star(\gamma_\tau\boldsymbol{u}) + \nabla S_\star(\gamma_{\mathbf{n}}\boldsymbol{u}) + i\omega\mu S_\star(\gamma_\tau\boldsymbol{v}),\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{v} = {} & \mathbf{curl}\mathsf{N}_\star(\mathbf{curl}(\boldsymbol{v}) + i\omega\varepsilon_\star\boldsymbol{u}) \\ & - i\omega\varepsilon_\star\mathsf{N}_\star(\mathbf{curl}(\boldsymbol{u}) - i\omega\mu_\star\boldsymbol{v}) - \nabla\mathsf{N}_\star(\text{div}(\boldsymbol{v})) \\ & + \mathbf{curl}S_\star(\gamma_\tau\boldsymbol{v}) + \nabla S_\star(\gamma_{\mathbf{n}}\boldsymbol{v}) - i\omega\varepsilon_\star S_\star(\gamma_\tau\boldsymbol{u}),\end{aligned}$$

hold as integral representations. When $\boldsymbol{u}$ and $\boldsymbol{v}$ are solutions of Maxwell's equations with constant coefficients, the representations simplify to mere surface integrals depending on tangential traces of the fields.

IV. Formulation for Homogeneous Medium

For the case of constant coefficients, integral representations of the solutions of (7) for the exterior and interior domain are

$$\begin{aligned}\boldsymbol{u}_0 &= -\mathbf{curl}S_0(\gamma_\tau^+\boldsymbol{u}_0) - \nabla S_0(\gamma_{\mathbf{n}}^+\boldsymbol{u}_0) - i\omega\mu_0 S_0(\gamma_\tau^+\boldsymbol{v}_0), \\ \boldsymbol{v}_0 &= -\mathbf{curl}S_0(\gamma_\tau^+\boldsymbol{v}_0) - \nabla S_0(\gamma_{\mathbf{n}}^+\boldsymbol{v}_0) + i\omega\varepsilon_0 S_\kappa(\gamma_\tau^+\boldsymbol{u}_0), \\ \boldsymbol{u} &= \ \mathbf{curl}S_1(\gamma_\tau^-\boldsymbol{u}) + \nabla S_1(\gamma_{\mathbf{n}}^-\boldsymbol{u}) + i\omega\mu_1 S_1(\gamma_\tau^-\boldsymbol{v}), \\ \boldsymbol{v} &= \ \mathbf{curl}S_1(\gamma_\tau^-\boldsymbol{v}) + \nabla S_1(\gamma_{\mathbf{n}}^-\boldsymbol{v}) - i\omega\varepsilon_1 S_1(\gamma_\tau^-\boldsymbol{u}).\end{aligned}$$

Using the identities

$$\gamma_{\mathbf{n}}^+\boldsymbol{u}_0 = -\tfrac{1}{i\omega\varepsilon_0}\,\text{div}_\Gamma(\gamma_\tau^+\boldsymbol{v}_0), \quad \gamma_{\mathbf{n}}^+\boldsymbol{v}_0 = \tfrac{1}{i\omega\mu_0}\,\text{div}_\Gamma(\gamma_\tau^+\boldsymbol{u}_0),$$
$$\gamma_{\mathbf{n}}^-\boldsymbol{u} = -\tfrac{1}{i\omega\varepsilon_1}\,\text{div}_\Gamma(\gamma_\tau^-\boldsymbol{v}), \quad \gamma_{\mathbf{n}}^-\boldsymbol{v} = \tfrac{1}{i\omega\mu_1}\,\text{div}_\Gamma(\gamma_\tau^-\boldsymbol{u}),$$

and denoting

$$\boldsymbol{\alpha}^+ := \gamma_\tau^+\boldsymbol{u}_0, \quad \boldsymbol{\beta}^+ := i\omega\mu_0\gamma_\tau^+\boldsymbol{v}_0 \tag{11a}$$
$$\boldsymbol{\alpha} := \gamma_\tau^-\boldsymbol{u}, \quad \boldsymbol{\beta} := i\omega\mu_1\gamma_\tau^-\boldsymbol{v} \tag{11b}$$

the representation formula can be written as

$$\boldsymbol{u}_0 = -\mathsf{DL}_0(\boldsymbol{\alpha}^+) - \mathsf{SL}_0(\boldsymbol{\beta}^+), \tag{12a}$$
$$i\omega\mu_0\boldsymbol{v}_0 = -\kappa_0^2\mathsf{SL}_0(\boldsymbol{\alpha}^+) - \mathsf{DL}_0(\boldsymbol{\beta}^+), \tag{12b}$$
$$\boldsymbol{u} = \mathsf{DL}_1(\boldsymbol{\alpha}) + \mathsf{SL}_1(\boldsymbol{\beta}), \tag{12c}$$
$$i\omega\mu_1\boldsymbol{v} = \kappa_1^2\mathsf{SL}_1(\boldsymbol{\alpha}) + \mathsf{DL}_1(\boldsymbol{\beta}), \tag{12d}$$

where the *Maxwell layer potentials* are defined as

$$\mathsf{SL}_\star := \tfrac{1}{\kappa_\star^2}\nabla S_\star\text{div}_\Gamma + S_\star, \tag{13}$$
$$\mathsf{DL}_\star := \mathbf{curl}S_\star. \tag{14}$$

Surface integral equations are obtained by applying the tangential trace to (12):

$$\boldsymbol{\alpha}^+ = (\tfrac{1}{2}\mathbf{I} - \mathbf{K}_0)\boldsymbol{\alpha}^+ - \mathbf{V}_0\boldsymbol{\beta}^+, \tag{15a}$$
$$\boldsymbol{\beta}^+ = -\kappa_0^2\mathbf{V}_0\boldsymbol{\alpha}^+ + (\tfrac{1}{2}\mathbf{I} - \mathbf{K}_0)\boldsymbol{\beta}^+, \tag{15b}$$
$$\boldsymbol{\alpha} = (\tfrac{1}{2}\mathbf{I} + \mathbf{K}_1)\boldsymbol{\alpha} + \mathbf{V}_1\boldsymbol{\beta}, \tag{15c}$$
$$\boldsymbol{\beta} = \kappa_1^2\mathbf{V}_1\boldsymbol{\alpha} + (\tfrac{1}{2}\mathbf{I} + \mathbf{K}_1)\boldsymbol{\beta}, \tag{15d}$$

where

$$\mathbf{V}_\star := \tfrac{1}{\kappa_\star^2}\mathbf{curl}_\Gamma \circ V_\star \circ \text{div}_\Gamma - \boldsymbol{n} \times V_\star,$$
$$\mathbf{K}_\star := \tfrac{1}{2}(\gamma_\tau^+ + \gamma_\tau^-)\mathbf{curl}S_\star,$$

are the usual surface integral operators for Maxwell's equations [6].

Combining (15) with the transmission conditions, we arrive at the PMCHWT formulation: find $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{H}^{-1/2}(\text{div}_\Gamma, \Gamma)$ such that

$$(\mathbf{M}^{-1}\mathbf{A}_0\mathbf{M} + \mathbf{A}_1)\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = -\mathbf{M}^{-1}\begin{pmatrix}\boldsymbol{\alpha}^{\text{inc}}\\ \boldsymbol{\beta}^{\text{inc}}\end{pmatrix}, \tag{16}$$

where

$$\mathbf{A}_\star := \begin{pmatrix}\mathbf{K}_\star & \mathbf{V}_\star\\ \kappa_\star^2\mathbf{V}_\star & \mathbf{K}_\star\end{pmatrix}, \quad \mathbf{M} := \begin{pmatrix}1 & 0\\ 0 & \frac{\mu_0}{\mu_1}\end{pmatrix}. \tag{17}$$

V. FEM-BEM COUPLING

For a setting with varying coefficients, we start by writing the curl-curl problem for the electric field:

$$\mathbf{curl}\tfrac{1}{\mu}\mathbf{curl}(\boldsymbol{u}) - \omega^2\varepsilon\boldsymbol{u} = 0 \quad \text{in } \Omega. \tag{18}$$

A variational formulation for this problem reads as follows: find $\boldsymbol{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that

$$(\tfrac{1}{\mu}\mathbf{curl}(\boldsymbol{u}), \mathbf{curl}(\boldsymbol{w})) - \omega^2(\varepsilon\boldsymbol{u}, \boldsymbol{w}) \tag{19a}$$
$$+ \langle \tfrac{1}{\mu}\gamma_\tau^-\mathbf{curl}(\boldsymbol{u}), \gamma_\tau^-\boldsymbol{w}\rangle_\Gamma = 0, \tag{19b}$$

holds for all $\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}, \Omega)$. From the transmission conditions we know

$$\gamma_\tau^-\boldsymbol{u} - \gamma_\tau\boldsymbol{u}^{\text{inc}} = \gamma_\tau^+\boldsymbol{u}_0, \tag{20a}$$
$$\tfrac{1}{\mu}\gamma_\tau^-\mathbf{curl}(\boldsymbol{u}) - \tfrac{1}{\mu_0}\gamma_\tau\mathbf{curl}(\boldsymbol{u}^{\text{inc}}) = \tfrac{1}{\mu_0}\gamma_\tau^+\mathbf{curl}(\boldsymbol{u}_0), \tag{20b}$$

and from an integral representation in the exterior domain we get

$$\boldsymbol{\alpha}^+ := \gamma_\tau^+\boldsymbol{u}_0 = (\tfrac{1}{2}\mathbf{I} - \mathbf{K}_0)\boldsymbol{\alpha}^+ - \mathbf{V}_0\boldsymbol{\beta}^+. \tag{21}$$

Replacing (20) and (21) into (19), we obtain

$$(\tfrac{1}{\mu}\mathbf{curl}(\boldsymbol{u}), \mathbf{curl}(\boldsymbol{w})) - \omega^2(\varepsilon\boldsymbol{u}, \boldsymbol{w}) \tag{22a}$$
$$+ \langle \tfrac{1}{\mu_0}\boldsymbol{\beta}^+, \gamma_\tau^-\boldsymbol{w}\rangle_\Gamma = \langle \tfrac{1}{\mu_0}\boldsymbol{\beta}^{\text{inc}}, \gamma_\tau^-\boldsymbol{w}\rangle_\Gamma, \tag{22b}$$
$$(\tfrac{1}{2}\mathbf{I} + \mathbf{K}_0)\gamma_\tau^-\boldsymbol{u} + \mathbf{V}_0\boldsymbol{\beta}^+ = -(\tfrac{1}{2}\mathbf{I} + \mathbf{K}_0)\boldsymbol{\alpha}^{\text{inc}}. \tag{22c}$$

VI. COMBINED SURFACE-VOLUME INTEGRAL EQUATIONS

We define constant reference coefficients in a way such that the support of inhomogeneities can be reduced, such as

$$\varepsilon_1 = \frac{1}{|\Omega|}\int_\Omega \varepsilon(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \quad \mu_1 = \frac{1}{|\Omega|}\int_\Omega \mu(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{23}$$

where ε_1 and μ_1 will be used for the constant coefficient part of the formulation, i.e. the Green's function in the interior domain.

We write the transmission problem (7) as follows

$$\mathbf{curl}(\boldsymbol{u}_0) - i\omega\mu_0\boldsymbol{v}_0 = 0, \quad \text{in } \Omega^+, \tag{24a}$$
$$\mathbf{curl}(\boldsymbol{v}_0) + i\omega\varepsilon_0\boldsymbol{u}_0 = 0, \quad \text{in } \Omega^+, \tag{24b}$$
$$\mathbf{curl}(\boldsymbol{u}) - i\omega\mu_1\boldsymbol{v} = \boldsymbol{f}_1, \quad \text{in } \Omega, \tag{24c}$$
$$\mathbf{curl}(\boldsymbol{v}) + i\omega\varepsilon_1\boldsymbol{u} = \boldsymbol{f}_2, \quad \text{in } \Omega, \tag{24d}$$
$$\gamma_\tau^+\boldsymbol{u}_0 - \gamma_\tau^-\boldsymbol{u} = -\gamma_\tau\boldsymbol{u}^{\text{inc}}, \quad \text{on } \Gamma, \tag{24e}$$
$$\gamma_\tau^+\boldsymbol{v}_0 - \gamma_\tau^-\boldsymbol{v} = -\gamma_\tau\boldsymbol{v}^{\text{inc}}, \quad \text{on } \Gamma, \tag{24f}$$
$$\boldsymbol{v}_0 \times \frac{\boldsymbol{x}}{r} - \boldsymbol{u}_0 = \mathcal{O}\left(\frac{1}{r^2}\right), \quad r = |\boldsymbol{x}| \to +\infty, \tag{24g}$$

where

$$\boldsymbol{f}_1(\boldsymbol{x}) := -i\omega\mu_1 p_m(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}), \qquad p_m(\boldsymbol{x}) := 1 - \frac{\mu(\boldsymbol{x})}{\mu_1},$$
$$\boldsymbol{f}_2(\boldsymbol{x}) := i\omega\varepsilon_1 p_e(\boldsymbol{x})\boldsymbol{u}(\boldsymbol{x}), \qquad p_e(\boldsymbol{x}) := 1 - \frac{\varepsilon(\boldsymbol{x})}{\varepsilon_1},$$

and $\varepsilon_1, \mu_1 \in \mathbb{R}_+$ are conveniently chosen parameters as in (23). The representation formula in Ω now reads

$$\boldsymbol{u} = -i\omega\mu_1\mathbf{curl}\mathsf{N}_1(p_m\boldsymbol{v}) - \kappa_1^2\mathsf{N}_1(p_e\boldsymbol{u}) - \nabla\mathsf{N}_1(\mathrm{div}(\boldsymbol{u})) + \mathbf{curl}S_1(\gamma_\tau^-\boldsymbol{u}) + \nabla S_1(\gamma_{\mathbf{n}}^-\boldsymbol{u}) + i\omega\mu_1 S_1\left(\gamma_\tau^-\boldsymbol{v}\right),$$

and

$$\boldsymbol{v} = i\omega\varepsilon_1\mathbf{curl}\mathsf{N}_1(p_e\boldsymbol{u}) + \kappa_1^2\mathsf{N}_1(p_m\boldsymbol{v}) - \nabla\mathsf{N}_1(\mathrm{div}(\boldsymbol{v})) + \mathbf{curl}S_1(\gamma_\tau^-\boldsymbol{v}) + \nabla S_1(\gamma_{\mathbf{n}}^-\boldsymbol{v}) - i\omega\varepsilon_1 S_1\left(\gamma_\tau^-\boldsymbol{u}\right).$$

We will repeatedly make use of the product rule

$$\mathbf{curl}\,(f\boldsymbol{F}) = \nabla f \times \boldsymbol{F} + f\mathbf{curl}(\boldsymbol{F}),$$

for $f \in C^1(\Omega), \boldsymbol{F} \in [C^1(\Omega)]^3$, and an integration by parts result on Newton potentials

$$\mathbf{curl}\mathsf{N}_\ell(\boldsymbol{F}) = \mathsf{N}_\ell(\mathbf{curl}\boldsymbol{F}) + S_\ell(\gamma_\tau\boldsymbol{F}). \tag{25}$$

Solutions of (24) also satisfy

$$\begin{aligned}
&\mathrm{div}(\varepsilon\boldsymbol{u}) = \nabla\varepsilon\cdot\boldsymbol{u} + \varepsilon\mathrm{div}(\boldsymbol{u}) = 0,\\
\Rightarrow\quad &\mathrm{div}(\boldsymbol{u}) = -\frac{\nabla\varepsilon}{\varepsilon}\cdot\boldsymbol{u} =: -\boldsymbol{\tau}_e\cdot\boldsymbol{u},\\
&\mathrm{div}(\mu\boldsymbol{v}) = \nabla\mu\cdot\boldsymbol{v} + \mu\mathrm{div}(\boldsymbol{v}) = 0,\\
\Rightarrow\quad &\mathrm{div}(\boldsymbol{v}) = -\frac{\nabla\mu}{\mu}\cdot\boldsymbol{v} =: -\boldsymbol{\tau}_m\cdot\boldsymbol{v},
\end{aligned}$$

and

$$\gamma_{\mathbf{n}}\boldsymbol{u} = -\frac{1}{i\omega\varepsilon_1\tilde{\varepsilon}}\ \mathrm{div}_\Gamma(\gamma_\tau\boldsymbol{v}), \quad \gamma_{\mathbf{n}}\boldsymbol{v} = \frac{1}{i\omega\mu_1\tilde{\mu}}\ \mathrm{div}_\Gamma(\gamma_\tau\boldsymbol{u}),$$

where

$$\tilde{\varepsilon}(\boldsymbol{x}) := \frac{\varepsilon(\boldsymbol{x})}{\varepsilon_1},\ \tilde{\mu}(\boldsymbol{x}) := \frac{\mu(\boldsymbol{x})}{\mu_1}$$

for all $\boldsymbol{x} \in \Omega$. We denote $\tilde{\boldsymbol{v}} := i\omega\mu_1\boldsymbol{v}$ and obtain a *new integral representation*

$$\boxed{\begin{aligned}\boldsymbol{u} &= \mathcal{K}^m\tilde{\boldsymbol{v}} + \mathcal{A}^e\boldsymbol{u} + \mathsf{DL}_1(\gamma_\tau^-\boldsymbol{u}) + \mathsf{SL}_1^{\tilde{\varepsilon},\tilde{\mu}}(\gamma_\tau^-\tilde{\boldsymbol{v}}), \quad (26)\\ \tilde{\boldsymbol{v}} &= \mathcal{K}^e\boldsymbol{u} + \mathcal{A}^m\tilde{\boldsymbol{v}} + \mathsf{DL}_1(\gamma_\tau^-\tilde{\boldsymbol{v}}) + \kappa_1^2\mathsf{SL}_1^{\tilde{\mu},\tilde{\varepsilon}}(\gamma_\tau^-\boldsymbol{u}), \quad (27)\end{aligned}}$$

where we defined the *volume integral operators*

$$\mathcal{K}^e\boldsymbol{u} := -\kappa_1^2\mathsf{N}_1(\mathbf{curl}(p_e\boldsymbol{u})), \tag{28a}$$
$$\mathcal{K}^m\boldsymbol{v} := -\mathsf{N}_1(\mathbf{curl}(p_m\boldsymbol{v})), \tag{28b}$$
$$\mathcal{A}^e\boldsymbol{u} := -\kappa_1^2\mathsf{N}_1(p_e\boldsymbol{u}) + \nabla\mathsf{N}_1(\boldsymbol{\tau}_e\cdot\boldsymbol{u}), \tag{28c}$$
$$\mathcal{A}^m\boldsymbol{v} := \kappa_1^2\mathsf{N}_1(p_m\boldsymbol{v}) + \nabla\mathsf{N}_1(\boldsymbol{\tau}_m\cdot\boldsymbol{v}). \tag{28d}$$

and the (scaled) layer potential

$$\mathsf{SL}_1^{a,b}(\boldsymbol{\beta}) := \tfrac{1}{\kappa_1^2}\nabla S_1(\tfrac{1}{a}\mathrm{div}_\Gamma(\boldsymbol{\beta})) + S_1(b\boldsymbol{\beta}). \tag{29}$$

Finally, we arrive at

- two integral equations on the interface Γ, of PMCHWT type, obtained by taking tangential traces of (26) and (27), and
- two equations in the volume, directly from (26) and (27).

The resulting linear integral operator equation system reads as follows: find $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$\begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{pmatrix} \begin{pmatrix} U \\ \Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \Lambda^{\text{inc}} \end{pmatrix}, \tag{30}$$

holds, where

$$\begin{aligned} U &:= (\boldsymbol{u}, \tilde{\boldsymbol{v}}), \qquad \tilde{\boldsymbol{v}} := i\omega\mu_1 \boldsymbol{v}, \\ \Lambda &:= (\boldsymbol{\alpha}, \boldsymbol{\beta}), \qquad \Lambda^{\text{inc}} := (\boldsymbol{\alpha}^{\text{inc}}, \boldsymbol{\beta}^{\text{inc}}), \\ \mathbb{A} &:= \mathbf{M}^{-1}\mathbf{A}_0\mathbf{M} + \mathbf{A}_1^{\tilde{\varepsilon},\tilde{\mu}}, \\ \mathbb{B} &:= \begin{pmatrix} \mathbf{T}_e & \mathbf{T}_m \end{pmatrix}, \\ \mathbb{C} &:= \begin{pmatrix} -\mathsf{DL}_1 & -\mathsf{SL}_1^{\tilde{\varepsilon},\tilde{\mu}} \\ -\kappa_1^2 \mathsf{SL}_1^{\tilde{\mu},\tilde{\varepsilon}} & -\mathsf{DL}_1 \end{pmatrix}, \\ \mathbb{D} &:= \begin{pmatrix} \mathsf{I} - \mathcal{A}^e & -\mathcal{K}^m \\ -\mathcal{K}^e & \mathsf{I} - \mathcal{A}^m \end{pmatrix}. \end{aligned}$$

where we defined

$$\mathbf{A}_1^{\tilde{\varepsilon},\tilde{\mu}} := \begin{pmatrix} \mathbf{K}_1 & \mathbf{V}_1^{\tilde{\varepsilon},\tilde{\mu}} \\ \kappa_1^2 \mathbf{V}_1^{\tilde{\mu},\tilde{\varepsilon}} & \mathbf{K}_1 \end{pmatrix}, \ \mathbf{T}_e = \begin{pmatrix} \gamma_\tau^- \mathcal{A}^e \\ \gamma_\tau^- \mathcal{K}^e \end{pmatrix}, \ \mathbf{T}_m = \begin{pmatrix} \gamma_\tau^- \mathcal{K}^m \\ \gamma_\tau^- \mathcal{A}^m \end{pmatrix}.$$

VII. Galerkin Discretization

Let $\{\mathcal{T}_h\}_{h>0}$ be a globally quasi-uniform and shape-regular family of triangular meshes of Ω. Let $\{\Sigma_h\}_{h>0}$ be the induced family of meshes on Γ. We choose finite element spaces:

- $N_h := N_h(\mathcal{T}_h) \subset \mathbf{H}(\mathbf{curl}, \Omega)$ of Nédélec edge elements (in the volume).
- $W_h := W_h(\Sigma_h) \subset \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ of Rao-Wilton-Glisson face elements (on the boundary).

We will also use N_h as a conforming subspace of the dual space of $\mathbf{H}(\mathbf{curl}, \Omega)$. This leads to a stable discretization of the duality product in $\mathbf{H}(\mathbf{curl}, \Omega)$.

The implementation relies on standard techniques for numerical integration of weakly singular kernels [12, Chapter 5], extended to the case of interactions between tetrahedron-triangle and tetrahedron-tetrahedron [3], [10]. Moreover, as the linear system involves large dense blocks arising from volume integral operators, matrix compression such as $\mathcal{H}-$matrix with ACA [2] becomes crucial. We use Castor [1], a C++ linear algebra library that provides easy to integrate $\mathcal{H}-$matrix routines.

VIII. Numerical Experiments

We study the transmission problem (7) with an incident field given by a plane-wave

$$\boldsymbol{u}^{\text{inc}}(\boldsymbol{x}) := \hat{\mathbf{j}} \exp(i\kappa_0 \boldsymbol{x} \cdot \hat{\mathbf{k}}).$$

The domain corresponds to a cube of unit length $\Omega := [0,1]^3$. We set $\varepsilon_0 = \mu_0 \equiv 1$ and $\varepsilon_1 = 2, \mu_1 = 1$. The variable coefficients ε and μ are set to

$$\begin{aligned} \varepsilon(\boldsymbol{x}) &:= \varepsilon_1 + x_1(1-x_1)x_2(1-x_2)x_3(1-x_3), \\ \mu(\boldsymbol{x}) &:= \mu_1, \end{aligned}$$

for $\boldsymbol{x} \in \Omega$. We find Galerkin solutions for both (22) and (30), then measure the errors in the electric field, given as

$$\text{error}_{\mathbf{L}^2(\Omega)} := \frac{\|\boldsymbol{u}_h^\star - \boldsymbol{u}_h\|_{\mathbf{L}^2(\Omega)}}{\|\boldsymbol{u}_h^\star\|_{\mathbf{L}^2(\Omega)}},$$

$$\text{error}_{\mathbf{H}(\mathbf{curl},\Omega)} := \frac{\|\boldsymbol{u}_h^\star - \boldsymbol{u}_h\|_{\mathbf{H}(\mathbf{curl},\Omega)}}{\|\boldsymbol{u}_h^\star\|_{\mathbf{H}(\mathbf{curl},\Omega)}},$$

where $\boldsymbol{u}_h^\star$ is a solution obtained from a highly refined mesh. Information about the meshes used in our experiments can be found in Table I.

Convergence results can be found in Fig.1, and a snapshot of the solution in Fig.2. We observe convergence of the Galerkin solutions for both the coupled surface-volume formulation (denoted as STF-VIE for single-trace volume integral equations) and for the FEM-BEM coupling, which validates our formulation.

TABLE I
Meshes used in numerical experiments

Meshes			
Elements	**Nodes**	**Edges**	**Mesh size**
24	14	49	1/2
192	63	302	1/4
1536	365	2092	1/8
12288	2457	15512	1/16
98304	17969	119344	1/32

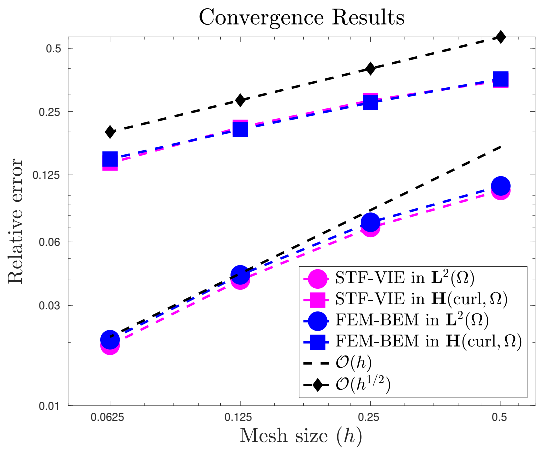

Fig. 1. Convergence of the electric field, measured with respect to a solution obtained from a highly refined mesh.

IX. Conclusion

In this work, we presented a new formulation for the Maxwell transmission problem, consisting of a coupling between surface and volume integral equations, based on modified integral representations. Our numerical experiments show convergence of solutions, validated by well-established variational formulations. Future work considers exploring the high-frequency behaviour of solutions to these integral equations, and compare it to the well-known pollution effect that affects solutions of standard finite element discretizations.

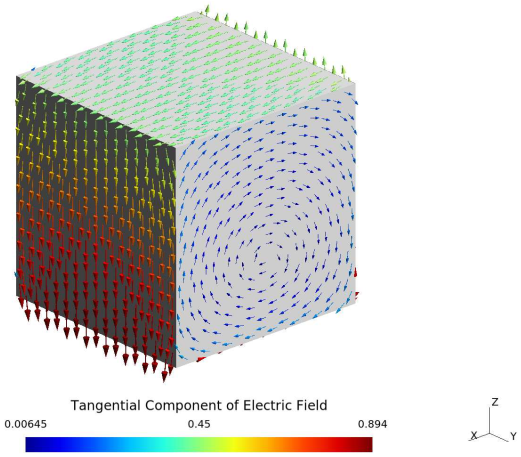

Fig. 2. Snapshot of the tangential component of the electric field.

REFERENCES

[1] M. Aussal, M. Bakry, and L. Series. “Castor: A C++ library to code “à la Matlab”,” Journal of Open Source Software, 7(71), 3965, https://doi.org/10.21105/joss.03965

[2] M. Bebendorf. “Hierarchical matrices,” Springer Berlin, Heidelberg, 2008.

[3] B. Feist and M. Bebendorf. “Fractional laplacian quadrature rules for singular double integrals in 3D,” Computational Methods in Applied Mathematics, 2023.

[4] M. M. Botha, “Solving the volume integral equations of electromagnetic scattering,” Journal of Computational Physics,2006, vol.218(1), 141-158.

[5] A. Buffa, M. Costabel, and D. Sheen, “On traces for H(curl, Omega) in Lipschitz domains,” Journal of mathematical analysis and applications,2002, vol.276(2), 845-867.

[6] A. Buffa and R. Hiptmair, “Galerkin boundary element methods for electromagnetic scattering,” In Topic in computational wave propagation, pages 83-124. Springer, 2003.

[7] R. Hiptmair. “Coupling of finite elements and boundary elements in electromagnetic scattering.” SIAM Journal on Numerical Analysis, 2003, vol. 41(3), 919-944.

[8] I. Labarca and R. Hiptmair, “Volume integral equations and single-trace formulations for acoustic wave scattering in an inhomogeneous medium,” Computational Methods in Applied Mathematics, 2023. https://doi.org/10.1515/cmam-2022-0119.

[9] P. Monk, “Finite element methods for Maxwell’s equations,” Oxford University Press, 2003.

[10] C. Münger and K. Cools, “Efficient and kernel-independent evaluation of singular integrals in volume integral equations,” 2021 IEEE International Conference on Microwaves, Antennas, Comunnications and Electronic Systems (COMCAS), Tel Aviv, Israel, 2021, pp. 188-192.

[11] A.J. Poggio and E.K. Miller, “Integral equation solutions of three-dimensional scattering problems.” In Computer Techniques for Electromagnetics, Oxford, U.K., 1973.

[12] S.A. Sauter and C. Schwab. “Boundary element methods,” Springer, 2010.