Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

Weighted analytic regularity for the integral fractional Laplacian in polyhedra

M. Faustmann and C. Marcati and J. Melenk and Ch. Schwab

Research Report No. 2023-31
July 2023

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

WEIGHTED ANALYTIC REGULARITY FOR THE INTEGRAL FRACTIONAL LAPLACIAN IN POLYHEDRA

MARKUS FAUSTMANN*, CARLO MARCATI†, JENS MARKUS MELENK*, AND CHRISTOPH SCHWAB‡

Abstract. On polytopal domains in 3D, we prove weighted analytic regularity of solutions to the Dirichlet problem for the integral fractional Laplacian with analytic right-hand side. Employing the Caffarelli-Silvestre extension allows to localize the problem and to decompose the regularity estimates into results on vertex, edge, face, vertex-edge, vertex-face, edge-face and vertex-edge-face neighborhoods of the boundary. Using tangential differentiability of the extended solutions, a bootstrapping argument based on Caccioppoli inequalities on dyadic decompositions of the neighborhoods provides control of higher order derivatives.

Key word. fractional Laplacian, analytic regularity, corner domains, weighted Sobolev spaces

AMS subject classifications. 26A33, 35A20, 35B45, 35J70, 35R11.

1. Introduction.

On a bounded, polytopal domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$ comprising of (the closure of) a finite union of plane, open polygons, we consider the Dirichlet problem for the integral fractional Laplacian

$$(-\Delta)^s u = f \text{ on } \Omega, \qquad u = 0 \text{ on } \mathbb{R}^d \setminus \overline{\Omega}, \tag{1.1}$$

with $0 < s < 1$, subject to a source term f that is analytic in $\overline{\Omega}$.

As solutions to fractional PDEs typically exhibit a singular behaviour close to the whole boundary $\partial\Omega$ of the domain, the aim of this article is to capture this singular behaviour in Sobolev scales by introducing certain weight functions, which are powers of distances to vertices, edges or faces of the polytope and vanish on $\partial\Omega$. As such, we derive weighted analytic-type estimates for the variational solution u in Ω, which also extends the analysis of our previous work [FMMS22] (on 2D polygons) to the 3D-case.

Our analysis will, as in the two-dimensional setting [FMMS22], be based on *localization of* (1.1) *through a local, divergence form, elliptic degenerate operator in dimension* 4. Furthermore, the proof technique initiated in [BFM+23, FMMS22] will also be used here: we establish a base regularity shift of the variational solutions in Ω via the difference-quotient technique due to Savaré [Sav98], rather than by localization and Mellin-analysis as is customary in the regularity analysis of elliptic PDEs in corner domains (see, e.g., [MR10] and the references there). This allows, largely building upon the general results in [Sav98, FMMS22], for a more succinct proof of a small regularity shift in fractional order, non-weighted Sobolev spaces. Subsequently, this regularity is inductively bootstrapped to arbitrary order of regularity via local regularity estimates of Caccioppoli type on appropriately scaled balls in a Besicovitch covering of the domain. These local, analytic regularity estimates are subsequently assembled into a-priori bounds in weighted Sobolev spaces, with corner-, edge- and face-weight functions.

While structurally similar to our analysis of the two-dimensional case [FMMS22], the analysis in polyhedral domains brings additional technical difficulties: the coverings and local regularity estimates exhibit a certain "recursive by dimension of the singular set" structure, reminiscent to the "singular chains" of M. Dauge in the analysis of the singularities of the Laplacean in polytopal domain in $\mathbb{R}^d$ for general dimension $d \geq 2$ in [Dau88].

*Institut für Analysis und Scientific Computing, TU Wien, A-1040 Wien, Austria
†Dipartimento di Matematica "F. Casorati", Università di Pavia, I-27100 Pavia, Italy
‡Seminar for Applied Mathematics, ETH Zurich, CH-8092 Zürich, Switzerland

Funding: The research of JMM is funded by the Austrian Science Fund (FWF) by the special research program *Taming complexity in PDE systems* (grant SFB F65).

1.1. Relation to previous work. As mentioned, the present analysis extends our work [FMMS22] to polyhedral domains in $\mathbb{R}^3$, thereby being the first analytic regularity results for the integral fractional Laplacian in three space dimensions.

Previous, recent work [BN23a] establishes essentially optimal finite regularity shifts in (non-weighted) Besov spaces in general Lipschitz domains $\Omega \subset \mathbb{R}^d$ in arbitrary dimension $d \geq 2$, which are also applicable in the presently considered case. As compared with [BN23a], we consider a more restricted geometric setting of Lipschitz polyhedra $\Omega \subset \mathbb{R}^3$ with a finite number of faces. As in [BN23a] and in the two-dimensional case [FMMS22] we build the base regularity shift on the techniques of Savare [Sav98]. To obtain the analytic regularity shifts, however, we then employ coverings and local Caccioppoli-type estimates with inductive bootstrapping. This is distinct from the analysis in [GB97, BG88], which is based on inductive bootstrapping in finite-order, corner-weighted spaces of Kondrat'ev type. As in [FMMS22], we develop this regularity analysis for the four-dimensional, singular *local* elliptic divergence-form PDE related to (1.1) which was developed in [CS16] and the references there.

1.2. Impact on numerical methods. As is customary in the convergence rate analysis of Finite Element Methods and in line with other recent works (e.g. [BLN22] and the references there) on numerical approximation methods for the fractional Laplacean, sharp regularity for variational solutions of (1.1) will imply corresponding convergence rate estimates of Galerkin approximations. Similar to the two-dimensional case, where analytic regularity of solutions to (1.1) on bounded, polygonal domains Ω, which we obtained in [FMMS22], implied exponential convergence bounds for corresponding hp Finite Element Galerkin approximations in [FMMS23], the weighted analytic regularity estimates obtained in the present paper form the foundation for proving *exponential rates of convergence* of suitable families of hp-Finite Element Methods in polyhedral domains Ω in a forthcoming work.

1.3. Structure of this text. Upon fixing some notation in the next subsection, we establish the variational formulation of (1.1) in Section 2. We also introduce the scales of boundary-, edge- and vertex-weighted Sobolev spaces in which we subsequently will establish analytic regularity shifts. In Section 2.3, we state our main regularity result, Theorem 2.3. The proof of this theorem is developed in the remaining part of the paper. Section 4 recapitulates a global regularity shift and localized interior regularity estimates for the extension problem, which were proved in [FMMS22]. In Section 5, local regularity for various tangential derivatives of the solution of the extension problem, in a vicinity of (smooth parts of) the boundary will be considered. While the mathematical structure of the proofs is identical to the polygonal case in [FMMS22], the number of cases to be distinguished is larger than in the polygonal case: singular sets now have either dimension zero (vertices $\mathbf{v}$), one (edges $\mathbf{e}$) or two (faces $\mathbf{f}$). A somewhat larger number of combined cases (listed in Section 2.1) needs to be discussed item by item. These localized estimates are combined in Section 6 with covering arguments and scaling to establish the weighted analytic regularity. Section 7 gives a summary of our main results. Appendix A develops some elementary estimates related to fractional norms, which are used in some of the arguments in the main text.

1.4. Notation. The notation used here is largely consistent with our analysis in the polygonal setting in [FMMS22]. For open $\omega \subseteq \mathbb{R}^d$ and $t \in \mathbb{N}_0$, the spaces $H^t(\omega)$ are the classical Sobolev spaces of order t. For $t \in (0,1)$, fractional order Sobolev spaces are given in terms of the Aronstein-Slobodeckij seminorm $|\cdot|_{H^t(\omega)}$ and the full norm $\|\cdot\|_{H^t(\omega)}$ by

$$|v|^2_{H^t(\omega)} = \int_{x\in\omega}\int_{z\in\omega} \frac{|v(x)-v(z)|^2}{|x-z|^{d+2t}}\,dz\,dx, \qquad \|v\|^2_{H^t(\omega)} = \|v\|^2_{L^2(\omega)} + |v|^2_{H^t(\omega)}, \tag{1.2}$$

where we denote the Euclidean norm in $\mathbb{R}^d$ by $|\cdot|$.

For bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ and $t \in (0,1)$, we additionally introduce

$$\widetilde{H}^t(\Omega) := \left\{u \in H^t(\mathbb{R}^d) \,:\, u \equiv 0 \text{ on } \mathbb{R}^d \backslash \overline{\Omega}\right\}, \quad \|v\|^2_{\widetilde{H}^t(\Omega)} := \|v\|^2_{H^t(\Omega)} + \left\|v/r^t_{\partial\Omega}\right\|^2_{L^2(\Omega)},$$

where $r_{\partial\Omega}(x) := \operatorname{dist}(x, \partial\Omega)$ denotes the Euclidean distance of a point $x \in \Omega$ from the boundary $\partial\Omega$. On $\widetilde{H}^t(\Omega)$ we have, by combining [Gri11, Lemma 1.3.2.6] and [AB17, Proposition 2.3], the estimate

$$\forall u \in \widetilde{H}^t(\Omega)\colon \quad \|u\|_{\widetilde{H}^t(\Omega)} \leq C|\tilde{u}|_{H^t(\mathbb{R}^d)} \tag{1.3}$$

for some $C > 0$ depending only on t and Ω. For $t \in (0,1)\backslash\{\frac{1}{2}\}$, the norms $\|\cdot\|_{\widetilde{H}^t(\Omega)}$ and $\|\cdot\|_{H^t(\Omega)}$ are equivalent on $\widetilde{H}^t(\Omega)$, see, e.g., [Gri11, Sec. 1.4.4]. Furthermore, for $t > 0$, the space $H^{-t}(\Omega)$ denotes the dual space of $\widetilde{H}^t(\Omega)$, and we write $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ for the duality pairing that extends the $L^2(\Omega)$-inner product.

We denote by $\mathbb{R}_+$ the positive real numbers. For subsets $\omega \subset \mathbb{R}^d$, we will use the notation $\omega^+ := \omega \times \mathbb{R}_+$; in addition, for real $\mathcal{Y} > 0$, we write $\omega^{\mathcal{Y}} = \omega \times (0, \mathcal{Y})$. For any multi index $\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{N}_0^d$, we denote $\partial_x^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_d}^{\beta_d}$ and $|\beta| = \sum_{i=1}^d \beta_i$. We adhere to the convention that empty sums are null, i.e., $\sum_{j=a}^b c_j = 0$ when $b < a$; this even applies to the case where the terms c_j may not be defined. We also follow the standard convention $0^0 = 1$.

We use the notation $\lesssim$ to abbreviate $\leq$ up to a generic constant $C > 0$ that does not depend on critical parameters in our analysis.

2. Setting and Statement of the Main Result. There are several different ways to define the fractional Laplacian $(-\Delta)^s$ for $s \in (0,1)$. A classical definition on the full space $\mathbb{R}^d$ is in terms of the Fourier transformation $\mathcal{F}$, i.e., $(\mathcal{F}(-\Delta)^s u)(\xi) = |\xi|^{2s}(\mathcal{F}u)(\xi)$. Alternative, equivalent definitions of $(-\Delta)^s$ are, e.g., via spectral, semi-group, or operator theory, [Kwa17] or via singular integrals.

In the following, we consider the integral fractional Laplacian defined pointwise for sufficiently smooth functions u as the principal value integral

$$(-\Delta)^s u(x) := C(d,s) \text{ P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(z)}{|x-z|^{d+2s}}\, dz \quad \text{with} \quad C(d,s) := -2^{2s}\frac{\Gamma(s + d/2)}{\pi^{d/2}\Gamma(-s)}, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes the Gamma function. We investigate the fractional differential equation

$$(-\Delta)^s u = f \qquad \text{in } \Omega, \tag{2.2a}$$

$$u = 0 \qquad \text{in } \Omega^c := \mathbb{R}^d \backslash \overline{\Omega}, \tag{2.2b}$$

where $s \in (0,1)$ and $f \in H^{-s}(\Omega)$ is a given right-hand side. Equation (2.2) is understood in weak form: Find $u \in \widetilde{H}^s(\Omega)$ such that

$$a(u,v) := \langle(-\Delta)^s u, v\rangle_{L^2(\mathbb{R}^d)} = \langle f, v\rangle_{L^2(\Omega)} \qquad \forall v \in \widetilde{H}^s(\Omega). \tag{2.3}$$

The bilinear form $a(\cdot,\cdot)$ has the alternative representation

$$a(u,v) = \frac{C(s)}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\tilde{u}(x) - \tilde{u}(z))(\tilde{v}(x) - \tilde{v}(z))}{|x-z|^{d+2s}}\, dz\, dx \qquad \forall u, v \in \widetilde{H}^s(\Omega). \tag{2.4}$$

Observe that the domain of integration in the bilinear form $a(\cdot,\cdot)$ in (2.4) equals $(\Omega \times \mathbb{R}^d) \cup (\mathbb{R}^d \times \Omega)$. Existence and uniqueness of a weak solution $u \in \widetilde{H}^s(\Omega)$ of (2.3) follow from the Lax–Milgram Lemma for any $f \in H^{-s}(\Omega)$, upon the observation that the bilinear form $a(\cdot,\cdot) : \widetilde{H}^s(\Omega) \times \widetilde{H}^s(\Omega) \to \mathbb{R}$ is continuous and coercive (observing that coercivity with respect to the $\widetilde{H}^s(\Omega)$-norm follows from (1.3)).

The main result of this article asserts that, provided the data f is analytic in $\overline{\Omega}$, the variational solution u of (2.2) admits *weighted analytic regularity* in a scale of boundary-, edge- and corner-weighted Sobolev spaces in Ω. To state the result, we introduce some notation.

In the following, we consider $\Omega \subset \mathbb{R}^3$ a bounded, Lipschitz polyhedron with boundary $\partial\Omega$ comprised of finitely many vertices, and straight edges and plane faces. In $\overline{\Omega}$, we denote by $\mathcal{V}$ the set of vertices $\mathbf{v}$ and by $\mathcal{E}$ the set of the (open) edges $\mathbf{e}$, and by $\mathcal{F}$ the set of the (open) faces $\mathbf{f}$ of $\partial\Omega$. Evidently then, $\partial\Omega = \bigcup_{\mathcal{F}} \mathbf{f} \cup \bigcup_{\mathcal{E}} \mathbf{e} \cup \bigcup_{\mathcal{V}} \mathbf{v}$.

For $\mathbf{v} \in \mathcal{V}$, $\mathbf{e} \in \mathcal{E}$, and $\mathbf{f} \in \mathcal{F}$, we shall require the distance functions

$$r_{\mathbf{v}}(x) := |x - \mathbf{v}|, \qquad r_{\mathbf{e}}(x) := \inf_{y \in \mathbf{e}} |x - y|, \qquad r_{\mathbf{f}}(x) := \inf_{y \in \mathbf{f}} |x - y|, \quad x \in \Omega,$$

and corresponding (nondimensional) relative distances

$$\rho_{\mathbf{ve}}(x) := r_{\mathbf{e}}(x)/r_{\mathbf{v}}(x), \qquad \rho_{\mathbf{ef}}(x) := r_{\mathbf{f}}(x)/r_{\mathbf{e}}(x).$$

2.1. Partition of Ω. For each vertex $\mathbf{v} \in \mathcal{V}$, we denote by $\mathcal{E}_{\mathbf{v}} := \{\mathbf{e} \in \mathcal{E} : \mathbf{v} \in \overline{\mathbf{e}}\}$ the set of all edges that meet at $\mathbf{v}$, and $\mathcal{F}_{\mathbf{v}} := \{\mathbf{f} \in \mathcal{F} : \mathbf{f} \cap \overline{\mathbf{v}} \neq \emptyset\}$ the set of all faces abutting at the vertex $\mathbf{v}$. For any edge $\mathbf{e} \in \mathcal{E}$, we define $\mathcal{V}_{\mathbf{e}} := \{\mathbf{v} \in \mathcal{V} : \mathbf{v} \in \overline{\mathbf{e}}\} = \partial\mathbf{e}$, and $\mathcal{F}_{\mathbf{e}} := \{\mathbf{f} \in \mathcal{F} : \mathbf{f} \cap \overline{\mathbf{e}} \neq \emptyset\}$ as the set of faces sharing the edge $\mathbf{e}$.

For any face $\mathbf{f} \in \mathcal{F}$, $\mathcal{E}_{\mathbf{f}} := \{\mathbf{e} \in \mathcal{E} : \mathbf{e} \subset \partial\mathbf{f}\}$ is the set of edges abutting the face $\mathbf{f}$, and $\mathcal{V}_{\mathbf{f}} := \{\mathbf{v} \in \mathcal{V} : \mathbf{v} \subset \overline{\mathbf{f}}\}$ is the set of vertices contained in the face $\overline{\mathbf{f}}$.

For fixed, sufficiently small $\xi > 0$ and for $\mathbf{v} \in \mathcal{V}$, $\mathbf{e} \in \mathcal{E}$, $\mathbf{f} \in \mathcal{F}$, we decompose Ω into various neighborhoods defined as

$$\begin{aligned}
\omega^{\xi}_{\mathbf{vef}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) < \xi \quad \wedge \quad \rho_{\mathbf{ve}}(x) < \xi \quad \wedge \quad \rho_{\mathbf{ef}}(x) < \xi\},\\
\omega^{\xi}_{\mathbf{ve}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) < \xi \quad \wedge \quad \rho_{\mathbf{ve}}(x) < \xi \quad \wedge \quad \rho_{\mathbf{ef}}(x) \geq \xi \quad \forall \mathbf{f} \in \mathcal{F}_{\mathbf{e}}\},\\
\omega^{\xi}_{\mathbf{vf}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) < \xi \quad \wedge \quad \rho_{\mathbf{ve}}(x) \geq \xi \quad \wedge \quad \rho_{\mathbf{ef}}(x) < \xi \quad \forall \mathbf{e} \in \mathcal{E}_{\mathbf{v}} \cap \mathcal{E}_{\mathbf{f}}\},\\
\omega^{\xi}_{\mathbf{v}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) < \xi \quad \wedge \quad \rho_{\mathbf{ve}}(x) \geq \xi \quad \wedge \quad \rho_{\mathbf{ef}}(x) \geq \xi \quad \forall \mathbf{e} \in \mathcal{E}_{\mathbf{v}},\ \mathbf{f} \in \mathcal{F}_{\mathbf{v}}\},\\
\omega^{\xi}_{\mathbf{ef}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) \geq \xi \quad \wedge \quad r_{\mathbf{e}}(x) < \xi^2 \quad \wedge \quad \rho_{\mathbf{ef}}(x) < \xi \quad \forall \mathbf{v} \in \mathcal{V}_{\mathbf{e}}\},\\
\omega^{\xi}_{\mathbf{e}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) \geq \xi \quad \wedge \quad r_{\mathbf{e}}(x) < \xi^2 \quad \wedge \quad \rho_{\mathbf{ef}}(x) \geq \xi \quad \forall \mathbf{v} \in \mathcal{V}_{\mathbf{e}},\ \mathbf{f} \in \mathcal{F}_{\mathbf{e}}\},\\
\omega^{\xi}_{\mathbf{f}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) \geq \xi \quad \wedge \quad r_{\mathbf{e}}(x) \geq \xi^2 \quad \wedge \quad r_{\mathbf{f}}(x) < \xi^3 \quad \forall \mathbf{v} \in \mathcal{V}_{\mathbf{f}},\ \mathbf{e} \in \mathcal{E}_{\mathbf{f}}\},\\
\Omega^{\xi}_{\mathrm{int}} &:= \{x \in \Omega : r_{\mathbf{v}}(x) \geq \xi \quad \wedge \quad r_{\mathbf{e}}(x) \geq \xi^2 \quad \wedge \quad r_{\mathbf{f}}(x) \geq \xi^3 \quad \forall \mathbf{v}, \mathbf{e}, \mathbf{f}\}.
\end{aligned}$$

Figure 1 illustrates the neighborhoods near a vertex and Figure 2 shows the neighborhoods close to an edge but away from a vertex. We drop the superscript ξ unless strictly necessary.

Decompositions: We decompose the Lipschitz polyhedron Ω into (possibly overlapping) sectorial neighborhoods of vertices $\mathbf{v}$, which are unions of vertex, vertex-edge, vertex-face, and vertex-edge-face neighborhoods (as depicted in Figure 1), wedge-shaped neighborhoods of edges $\mathbf{e}$ (that are bounded away from a vertex, but are unions of edge- and edge-face neighborhoods as depicted in Figure 2), neighborhoods of faces $\mathbf{f}$, and an interior Ω_{int}, i.e.,

$$(2.5) \qquad \Omega = \Omega_{\mathrm{int}} \cup \bigcup_{\mathbf{v} \in \mathcal{V}} \left(\omega_{\mathbf{v}} \cup \bigcup_{\mathbf{e} \in \mathcal{E}_{\mathbf{v}},\, \mathbf{f} \in \mathcal{F}_{\mathbf{v}}} \omega_{\mathbf{ve}} \cup \omega_{\mathbf{vf}} \cup \omega_{\mathbf{vef}} \right) \cup \bigcup_{\mathbf{e} \in \mathcal{E}} \left(\omega_{\mathbf{e}} \cup \bigcup_{\mathbf{f} \in \mathcal{F}_{\mathbf{e}}} \omega_{\mathbf{ef}} \right) \cup \bigcup_{\mathbf{f} \in \mathcal{F}} \omega_{\mathbf{f}}\ .$$

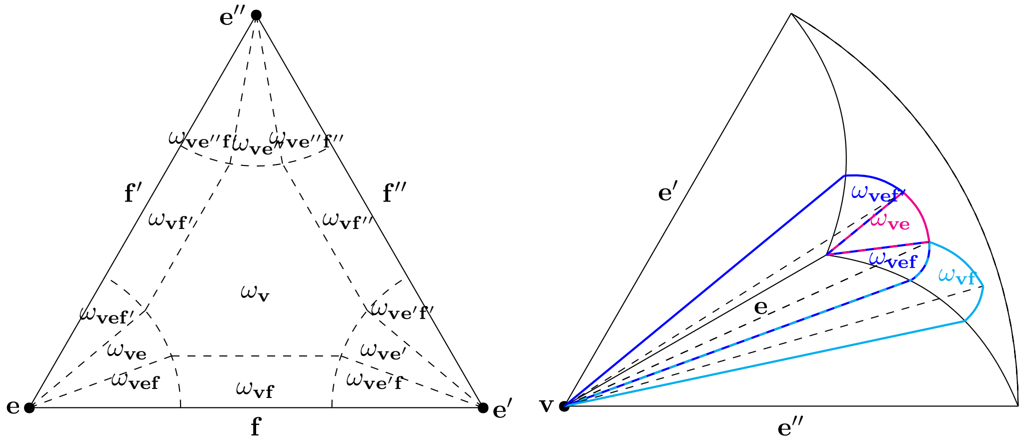

Fig. 1: Notation near a vertex $\mathbf{v}$, left: top view of the vertex cone (the vertex $\mathbf{v}$ is behind, on a straight line to the barycenter of the triangle), right: side view of the vertex cone.

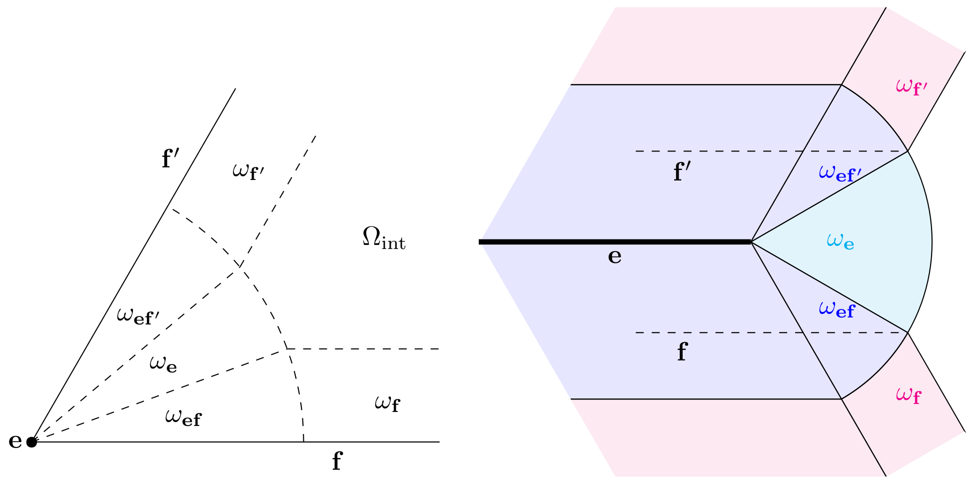

Fig. 2: Notation near an edge $\mathbf{e}$ with two faces $\mathbf{f}, \mathbf{f}'$ meeting at the edge and no vertex close by, left: front view (edge collapses to point), right: side view.

Each sectoral and edge neighborhood may have a different value ξ, but we assume that each $\omega_\bullet$ abutts at most at one vertex, one edge or one face of $\partial\Omega$. Since only finitely many distinct types of neighborhoods are needed to decompose the polygon, the interior $\Omega_{\mathrm{int}} \subset \Omega$ has a positive distance from the boundary.

2.2. Coordinates. To state the main result, and throughout the ensuing proof of analytic estimates, we require coordinates tangential resp. perpendicular to edges $\mathbf{e}$ and faces $\mathbf{f}$ in the local neighborhoods.

Definition 2.1. [*Co-ordinates and directional derivatives in neighborhoods of singular sets*]

1. *In* **face or vertex-face neighborhoods** $\omega_{\mathbf{f}}$, $\omega_{\mathbf{vf}}$, *we let* $\mathbf{f}_{i,\parallel}$, $i = 1, 2$ *and* $\mathbf{f}_\perp$ *be unit vectors such that* $\mathbf{f}_{i,\parallel}$ *are mutually orthogonal and span the tangential plane to* $\mathbf{f}$, *and* $\mathbf{f}_\perp$ *is normal to* $\mathbf{f} \in \mathcal{F}$. *We assume that* $\mathbf{f}_\perp$ *and* $\mathbf{f}_{i,\parallel}$ *are right-oriented.*
2. *In* **edge or vertex-edge neighborhoods** $\omega_{\mathbf{e}}$, $\omega_{\mathbf{ve}}$, *we let* $\mathbf{e}_\parallel$ *and* $\mathbf{e}_{1,\perp}$, $\mathbf{e}_{2,\perp}$ *be unit vectors such that* $\mathbf{e}_\parallel$ *is tangential to* $\mathbf{e}$ *and* $\mathbf{e}_{i,\perp}$ *are mutually orthogonal and span the plane transversal to* $\mathbf{e}$.
3. *In* **edge-face or vertex-edge-face neighborhoods** $\omega_{\mathbf{ef}}$, $\omega_{\mathbf{vef}}$, *we choose three linearly independent, right-oriented unit vectors* $\{\mathbf{g}_\parallel, \mathbf{g}_\vDash, \mathbf{g}_\perp\}$ *satisfying*
   - $\mathbf{g}_\parallel$ *is parallel to* $\mathbf{e}$ *and* $\mathbf{f}$;
   - $\mathbf{g}_\vDash$ *is perpendicular to* $\mathbf{e}$ *and parallel to* $\mathbf{f}$;
   - $\mathbf{g}_\perp$ *is perpendicular to* $\mathbf{e}$ *and* $\mathbf{f}$.

For $\mathbf{s} \in \{\mathbf{e}_{i,\perp}, \mathbf{e}_\parallel, \mathbf{f}_\perp, \mathbf{f}_{i,\parallel}, \mathbf{g}_\parallel, \mathbf{g}_\vDash, \mathbf{g}_\perp\}$ *we denote first order derivatives as* $D_{\mathbf{s}} v := \mathbf{s} \cdot \nabla_x v$. *For higher order derivatives, we set*

$$D^k_{\mathbf{s}} v := D_{\mathbf{s}}(D^{k-1}_{\mathbf{s}} v) \quad \textit{for } k > 1.$$

Finally, for $\beta = (\beta_1, \beta_2) \in \mathbb{N}^2_0$, *we write*

$$D^\beta_{\mathbf{e}_\perp} = D^{\beta_1}_{\mathbf{e}_{1,\perp}} D^{\beta_2}_{\mathbf{e}_{2,\perp}}, \qquad D^\beta_{\mathbf{f}_\parallel} = D^{\beta_1}_{\mathbf{f}_{1,\parallel}} D^{\beta_2}_{\mathbf{f}_{2,\parallel}}.$$

The coordinates introduced above can be written in a unified way. The following definition formalizes the notation used to write the statement of our main result and the proofs in a compact form.

Definition 2.2. *Let* $\omega \subset \Omega$ *be any connected set abutting at most one vertex* $\mathbf{v}$, *one edge* $\mathbf{e}$, *and one face* $\mathbf{f}$ *of* $\partial\Omega$. *We take* $(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\parallel)$ *to be linearly independent unit vectors in* $\mathbb{R}^3$ *that additionally satisfy*
- $\mathbf{g}_\perp$ *is perpendicular to* $\mathbf{f}$ *if* $\mathbf{f} \cap \partial\omega \neq \emptyset$ *and perpendicular to* $\mathbf{e}$ *if* $\mathbf{e} \cap \partial\omega \neq \emptyset$;
- $\mathbf{g}_\vDash$ *is parallel to* $\mathbf{f}$ *if* $\mathbf{f} \cap \partial\omega \neq \emptyset$ *and perpendicular to* $\mathbf{e}$ *if* $\mathbf{e} \cap \partial\omega \neq \emptyset$;
- $\mathbf{g}_\parallel$ *is parallel to* $\mathbf{f}$ *if* $\mathbf{f} \cap \partial\omega \neq \emptyset$ *and parallel to* $\mathbf{e}$ *if* $\mathbf{e} \cap \partial\omega \neq \emptyset$.

With these vectors and for $\beta = (\beta_\perp, \beta_\vDash, \beta_\parallel) \in \mathbb{N}^3_0$, *we introduce the derivative*

$$D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\parallel)} = D^{\beta_\perp}_{\mathbf{g}_\perp} D^{\beta_\vDash}_{\mathbf{g}_\vDash} D^{\beta_\parallel}_{\mathbf{g}_\parallel}.$$

2.3. Statement of the main result. The following statement is the main result of this work. It provides weighted analytic regularity in all neighborhoods used to decompose Ω.

Theorem 2.3. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded, open Lipschitz polyhedron whose boundary* $\partial\Omega$ *comprises finitely many vertices, straight edges and plane faces.*

Let the data $f \in C^\infty(\overline{\Omega})$ *satisfy with a constant* $\gamma_f > 0$

$$\forall j \in \mathbb{N}_0\colon \quad \sum_{|\beta|=j} \|\partial^\beta_x f\|_{L^2(\Omega)} \leq \gamma_f^{j+1} j^j. \tag{2.6}$$

Let u *be the weak solution of* (2.2).

Then, there exists $\gamma > 0$ *depending only on* γ_f, s, *and* Ω *such that for all* $t < 1/2$, *there exists* $C_t > 0$ *such that for all* $\beta = (\beta_\perp, \beta_\vDash, \beta_\parallel) \in \mathbb{N}^3_0$ *and all* $\omega \subset \Omega$ *as in Definition* 2.2, *it holds that*

$$\|r^{-t-s}_{\partial\Omega} r^{\beta_\parallel}_{\mathbf{v}} r^{\beta_\vDash}_{\mathbf{e}} r^{\beta_\perp}_{\mathbf{f}} D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\parallel)} u\|_{L^2(\omega)} \leq C_t \gamma^{|\beta|} |\beta|^{|\beta|}$$

with $\mathbf{v}$, $\mathbf{e}$, $\mathbf{f}$ *being the closest vertex, edge, face to* ω.

The rest of this paper will develop the proof of these bounds.

3. The Caffarelli-Silvestre extension.

3. The Caffarelli-Silvestre extension. Key to the present regularity analysis is a localization of the fractional Laplacian provided by the so-called *Caffarelli-Silvestre extension*, [CS07]: the nonlocal operator $(-\Delta)^s$ can be realized via a Dirichlet-to-Neumann map of a degenerate, *local* elliptic PDE on a half space in $\mathbb{R}^{d+1}$. Here, we shall be mainly interested in $d=3$.

3.1. Weighted spaces for the Caffarelli-Silvestre extension. We recapitulate from [FMMS22] certain weighted function spaces which will be used in the sequel. We distinguish the last component of points in $\mathbb{R}^{d+1}$ with the notation (x,y) where $x=(x_1,\dots,x_d)\in\mathbb{R}^d$, $y\in\mathbb{R}$ and we set

$$\alpha := 1-2s. \tag{3.1}$$

For open sets $D\subset\mathbb{R}^d\times\mathbb{R}_+$, the weighted L^2-norm $\|\cdot\|_{L^2_\alpha(D)}$ is defined via

$$\|U\|^2_{L^2_\alpha(D)} := \int_{(x,y)\in D} y^\alpha\,|U(x,y)|^2\,dx\,dy. \tag{3.2}$$

For the variational formulation of the CS extension, we require the space $L^2_\alpha(D)$ of functions on D that are square (Lebesgue-)integrable with respect to the weight y^α. With the weighted space $H^1_\alpha(D):=\{U\in L^2_\alpha(D):\ \nabla U\in L^2_\alpha(D)\}$ we introduce the Beppo-Levi space [DL54]

$$\mathrm{BL}^1_\alpha := \{U\in L^2_{loc}(\mathbb{R}^d\times\mathbb{R}_+)\,:\,\nabla U\in L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)\}. \tag{3.3}$$

Elements $U\in\mathrm{BL}^1_\alpha$ admit a trace at $y=0$, which we denote as $\operatorname{tr}U$. It holds that (e.g., [KM19, Lem. 3.8]) $\operatorname{tr}U\in H^s_{loc}(\mathbb{R}^d)$. Also, for $\operatorname{supp}\operatorname{tr}U\subset\overline{\Omega}$ for a bounded Lipschitz domain Ω, $\operatorname{tr}U\in\widetilde{H}^s(\Omega)$ and

$$\|\operatorname{tr}U\|_{\widetilde{H}^s(\Omega)} \overset{(1.3)}{\lesssim} |\operatorname{tr}U|_{H^s(\mathbb{R}^d)} \overset{\text{[KM19, Lem. 3.8]}}{\lesssim} \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \tag{3.4}$$

with implied constant depending on s and Ω.

3.2. Statement of the Caffarelli-Silvestre extension. Given $u\in\widetilde{H}^s(\Omega)$, let $U=U(x,y)$ denote the (unique in BL^1_α, see [FMMS22]) minimum norm extension of u to $\mathbb{R}^d\times\mathbb{R}_+$, i.e.,

$$U=\operatorname{argmin}\{\|\nabla U\|^2_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}\,|\,U\in\mathrm{BL}^1_\alpha,\ \operatorname{tr}U=u \text{ in } H^s(\mathbb{R}^d)\}.$$

The Euler-Lagrange equations corresponding to this extension problem read

$$\operatorname{div}(y^\alpha\nabla U)=0 \qquad \text{in } \mathbb{R}^d\times(0,\infty), \tag{3.5a}$$

$$U(\cdot,0)=u \qquad \text{in } \mathbb{R}^d. \tag{3.5b}$$

Henceforth, when referring to solutions of (3.5), we will additionally understand that $U\in\mathrm{BL}^1_\alpha$.

The relevance of (3.5) is due to the fact that the fractional Laplacian applied to $u\in\widetilde{H}^s(\Omega)$ can be recovered as distributional normal trace of the extension problem [CS07, Section 3], [CS16]:

$$(-\Delta)^s u = -d_s\lim_{y\to0^+} y^\alpha\partial_y U(x,y), \qquad d_s=2^{2s-1}\Gamma(s)/\Gamma(1-s). \tag{3.6}$$

3.3. Variational Formulation of the CS Extension. Fix $\mathcal{Y}>0$. Given $F\in L^2_{-\alpha}(\mathbb{R}^d\times(0,\mathcal{Y}))$ and $f\in H^{-s}(\Omega)$, consider the problem to find the minimizer $U=U(x,y)$ with $x\in\mathbb{R}^d$ and $y\in\mathbb{R}_+$ of

$$\text{minimize } \mathcal{F} \text{ on } \mathrm{BL}^1_{\alpha,0,\Omega} := \{U\in\mathrm{BL}^1_\alpha\,:\,\operatorname{tr}U=0 \ \text{ on } \Omega^c\}, \tag{3.7}$$

where

$$\mathcal{F}(U) := \frac{1}{2}b(U,U) - \int_{\mathbb{R}^d\times(0,\mathcal{Y})} FU\,dx\,dy - \int_\Omega f\,\mathrm{tr}\,U\,dx, \quad b(U,V) := \int_{\mathbb{R}^d\times\mathbb{R}_+} y^\alpha \nabla U\cdot\nabla V\,dx\,dy. \tag{3.8}$$

In virtue of a Poincaré inequality ([FMMS22, Lemma 3.1]), the map $\mathrm{BL}^1_{\alpha,0,\Omega} \ni U \mapsto \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}$ is a norm. The space $\mathrm{BL}^1_{\alpha,0,\Omega}$ endowed with this norm is a Hilbert space with corresponding inner-product given by the bilinear form $b(\cdot,\cdot)$ in (3.8). Hence, for every $\mathcal{Y}\in(0,\infty)$, there is $C_{\mathcal{Y},\alpha}>0$ such that

$$\forall U\in \mathrm{BL}^1_{\alpha,0,\Omega}: \quad \|U\|_{L^2_\alpha(\mathbb{R}^d\times(0,\mathcal{Y}))} \le C_{\mathcal{Y},\alpha}\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)}. \tag{3.9}$$

Details of the proof are in [FMMS22, Appendix B].

Existence and uniqueness of solutions of (3.7) follows from the Lax-Milgram Lemma since, for $F\in L^2_{-\alpha}(\mathbb{R}^d\times(0,\mathcal{Y}))$ and $f\in H^{-s}(\Omega)$, the map $U\mapsto \int_{\mathbb{R}^d\times(0,\mathcal{Y})} FU + \int_\Omega f\,\mathrm{tr}\,U$ in (3.8) extends to a bounded linear functional on $\mathrm{BL}^1_{\alpha,0,\Omega}$. In view of (3.9) and the trace estimate (3.4), the minimization problem (3.7) admits by Lax-Milgram a unique solution $U\in\mathrm{BL}^1_{\alpha,0,\Omega}$ with the *a priori* estimate

$$\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \le C\left[\|F\|_{L^2_{-\alpha}(\mathbb{R}^d\times(0,\mathcal{Y}))} + \|f\|_{H^{-s}(\Omega)}\right] \tag{3.10}$$

with constant C dependent on $s\in(0,1)$, $\mathcal{Y}>0$, and Ω.

The Euler-Lagrange equations formally satisfied by the solution U of (3.7) are:

$$-\mathrm{div}(y^\alpha\nabla U) = F \quad \text{in } \mathbb{R}^d\times(0,\infty), \tag{3.11a}$$

$$\partial_{n_\alpha} U(\cdot,0) = f \quad \text{in } \Omega, \tag{3.11b}$$

$$\mathrm{tr}\,U = 0 \quad \text{on } \Omega^c, \tag{3.11c}$$

where $\partial_{n_\alpha}U(x,0) = -d_s\lim_{y\to 0} y^\alpha\partial_y U(x,y)$ and we implicitly extended F to $\mathbb{R}^d\times\mathbb{R}_+$ by zero. In view of (3.6) together with the fractional PDE $(-\Delta)^s u = f$, this is a Neumann-type Caffarelli-Silvestre extension problem with an additional source F.

Remark 3.1. The system (3.11) is understood in a weak sense, i.e., to find $U\in\mathrm{BL}^1_{\alpha,0,\Omega}$ such that

$$\forall V\in\mathrm{BL}^1_{\alpha,0,\Omega}: \quad b(U,V) = \int_{\mathbb{R}^d\times\mathbb{R}_+} FV\,dx\,dy + \int_\Omega f\,\mathrm{tr}\,V\,dx. \tag{3.12}$$

Due to (3.9), the integral $\int_{\mathbb{R}^d\times\mathbb{R}_+} FV\,dx\,dy$ is well-defined.

∎

4. Solution regularity for the CS extension. As in [FMMS22], we prove analytic regularity of solutions of (1.1) in polyhedral $\Omega\subset\mathbb{R}^3$ via local (higher order) regularity results for solutions to the Caffarelli-Silvestre extension problem in Section 3.2. These were obtained in [FMMS22, Sec.3] for general space dimension $d\ge 2$. We re-state these for further reference for $d=3$.

4.1. Global regularity: a shift theorem. The following lemma provides additional regularity of the extension problem in the x–direction. Its proof is based on the difference quotient technique developed in [Sav98], and was already used in our analysis in two spatial variables

[FMMS22] and in [BN23a] to establish a regularity shift in Besov scales for the Dirichlet fractional Laplacian.

For functions U, F, f, it is convenient to introduce the abbreviation

$$N^2(U,F,f) := \|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} \left(\|\nabla U\|_{L^2_\alpha(\mathbb{R}^d\times\mathbb{R}_+)} + \|F\|_{L^2_{-\alpha}(\mathbb{R}^d\times(0,\mathcal{Y}))} + \|f\|_{H^{1-s}(\Omega)}\right). \tag{4.1}$$

In view of the *a priori* estimate (3.10), we have the simplified bound (with updated constant C)

$$N^2(U,F,f) \le C\left(\|f\|^2_{H^{1-s}(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^d\times(0,\mathcal{Y}))}\right). \tag{4.2}$$

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain, and let $B_{\widetilde{R}}\subset\mathbb{R}^3$ be a ball with $\Omega\subset B_{\widetilde{R}}$. For $t\in[0,1/2)$, there is $C_t>0$ (depending only on s, t, Ω, $\widetilde{R}$, and $\mathcal{Y}$) such that for $f\in C^\infty(\overline{\Omega})$, $F\in L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))$ the solution U of (3.7) satisfies*

$$\int_{\mathbb{R}_+} y^\alpha \|\nabla U(\cdot,y)\|^2_{H^t(B_{\widetilde{R}})}\,dy \le C_t N^2(U,F,f)$$

with $N^2(U,F,f)$ given by (4.1).

This is [FMMS22, Lemma 3.3] with $d=3$.

4.2. Caccioppoli inequalities for the CS extension. Our regularity will be based on Caccioppoli inequalities for solutions to the extension problem (3.11). These inequalities were derived in [FMMS22], but we also require them for some more general cases of tangential derivatives. Roughly speaking, they imply quantitative control of second order derivatives of U on some local set (balls or sets introduced below) in terms of first order derivatives on a (slightly) enlarged set.

Definition 4.2 (Half ball, wedge). *We call the intersection between a ball and a half space whose boundary passes through the center of the ball a* half ball.

We call the intersection between a ball and two non-identical half spaces with boundaries passing through the center of the ball a wedge.

Lemma 4.3 (Caccioppoli inequalities). *Let $B_R(x_0)$ be an open ball with radius $R>0$ centered at $x_0\in\overline{\Omega}\setminus\mathcal{V}$. Let $R>0$ be so small that*

(i) $B_R(x_0)\subset\Omega$, if $x_0\in\Omega$;
(ii) $B_R(x_0)\cap\Omega$ is a half ball, if $x_0\in\mathbf{f}$;
(iii) $B_R(x_0)\cap\Omega$ is a wedge, if $x_0\in\mathbf{e}$.

For $\theta\in(0,\infty]$ and $c\in(0,1]$ denote by $B^\theta_{cR} := (B_{cR}(x_0)\cap\Omega)\times(0,\theta)\subset\mathbb{R}^3\times\mathbb{R}^+$ the corresponding concentrically scaled and extended ball/half-ball/wedge, respectively.

Let U satisfy (3.11) with given data f and F with $\operatorname{supp}(F)\subset\mathbb{R}^3\times[0,\mathcal{Y}]$ and let $\theta'>\theta$.

Then, for $\bullet\in\{x_i : i=1,2,3\}$ in case (i), $\bullet\in\{\mathbf{f}_{i,\|} : i=1,2\}$ in case (ii), and $\bullet=\mathbf{e}_\|$ in case (iii), there is $C_{\text{int}}>0$ independent of R and c,θ,θ' such that

$$\begin{aligned}\|D_\bullet(\nabla U)\|^2_{L^2_\alpha(B^\theta_{cR})} \le C^2_{\text{int}}\Big(&\left(((1-c)R)^{-2} + (\theta'-\theta)^{-2}\right)\|\nabla U\|^2_{L^2_\alpha(B^{\theta'}_R)} \\ &+ \|D_\bullet f\|^2_{L^2(B_R)} + \|F\|^2_{L^2_{-\alpha}(B^+_R)}\Big).\end{aligned} \tag{4.3}$$

Proof. We use a cut-off function $\zeta=\zeta(x,y)$ with $0\le\zeta\le1$ and product structure

$$\zeta(x,y)=\zeta_x(x)\zeta_y(y), \qquad \zeta_x\in C^\infty_0(B_R), \quad \zeta_y\in C^\infty_0(-\theta',\theta').$$

Here, ζ_x is such that $\zeta_x \equiv 1$ on B_{cR} as well as $\|\nabla\zeta_x\|_{L^\infty(B_R)} \leq C_\zeta((1-c)R)^{-1}$ for some $C_\zeta > 0$ independent of c, R. Similarly, ζ_y satisfies $\zeta_y \equiv 1$ on $(-\theta,\theta)$ as well as $\|\partial_y^j \zeta_y\|_{L^\infty(-\theta',\theta')} \leq C_\zeta(\theta'-\theta)^{-j}$ for $j \in \{0,1\}$ with a constant C_ζ independent of R, θ, θ'. Hence $\|\nabla\zeta\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}_+)} \lesssim ((1-c)R)^{-1} + (\theta'-\theta)^{-1}$.

Let $e_\bullet$ be the already defined unit vectors for $\bullet \in \{\mathbf{f}_{i,\|}, \mathbf{e}_\|\}$ and e_{x_i} be the unit vector in the x_i-coordinate. Let $\tau \in \mathbb{R}\backslash\{0\}$. We define the difference quotient $D_\bullet^\tau$ as the operator such that, for all $w : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$,

$$(D_\bullet^\tau w)(x,y) := \frac{w(x+\tau e_\bullet, y) - w(x,y)}{\tau}, \qquad \forall x \in \mathbb{R}^3, y \in \mathbb{R}^+.$$

We recall that by, e.g., [Eva98, Sec. 6.3], we have uniformly in τ

$$\|D_\bullet^\tau v\|_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)} \lesssim \|\nabla v\|_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)}. \tag{4.4}$$

For $|\tau|$ sufficiently small, consider the function $V = D_\bullet^{-\tau}(\zeta^2 D_\bullet^\tau U)$. We claim $V \in \mathrm{BL}^1_{\alpha,0,\Omega}$, i.e.,

$$\operatorname{tr} V = 0 \text{ on } \Omega^c, \qquad V \in L^2_{loc}(\mathbb{R}^3 \times \mathbb{R}_+), \qquad \nabla V \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+).$$

The first property is true as long as τ is small enough, due to the compact support of ζ_x in $B_R \subset \Omega$. The second property follows from $\zeta \in L^\infty(\mathbb{R}^3 \times \mathbb{R}_+)$ and $V \in L^2_{loc}(\mathbb{R}^3 \times \mathbb{R}_+)$. To show the third one, note that derivatives commute with the difference quotient operator. It follows that

$$\partial_y V = D_\bullet^{-\tau}(\zeta^2 D_\bullet^\tau \partial_y U)).$$

Hence, $\partial_y V \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$ since $\partial_y U \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$ and ζ is bounded.

Similarly, for any $j \in \{1,2,3\}$,

$$\partial_{x_j} V = 2D_\bullet^{-\tau}\big(\zeta(\partial_{x_j}\zeta)D_\bullet^\tau U\big) + D_\bullet^{-\tau}(\zeta^2 D_\bullet^\tau \partial_{x_j} U) =: (I) + (II).$$

We have

$$(I) = \frac{2}{\tau}\Big[\big(\zeta\partial_{x_j}\zeta\big)(x-\tau e_\bullet, y)D_\bullet^{-\tau}U + \big(\zeta\partial_{x_j}\zeta\big)(x,y)D_\bullet^\tau U\Big].$$

Using the boundedness of $\zeta\partial_{x_j}\zeta$ and since $D_\bullet^{-\tau}U \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$ and $D_\bullet^\tau U \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$ by (4.4), we obtain that $(I) \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$. In addition, by the boundedness of ζ and since $U \in \mathrm{BL}^1_{\alpha,0,\Omega}$ implies $\partial_{x_j} U \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$, we also obtain $(II) \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$. We conclude that $\nabla V \in L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)$. This implies $V \in \mathrm{BL}^1_{\alpha,0,\Omega}$.

We can therefore choose V as a test function in the weak formulation of (3.11) and calculate

$$\operatorname{tr} V = -\frac{1}{\tau^2}\Big(\zeta_x^2(x-\tau e_\bullet)(u(x) - u(x-\tau e_\bullet)) + \zeta_x^2(x)(u(x) - u(x+\tau e_\bullet))\Big) = D_\bullet^{-\tau}(\zeta_x^2 D_\bullet^\tau u).$$

Integration by parts in (3.11) tested with V over $\mathbb{R}^3 \times \mathbb{R}_+$ and using that the Neumann trace (up to the constant d_s from (3.6)) realizes the fractional Laplacian gives

$$\int_{\mathbb{R}^3\times\mathbb{R}_+} FV\,dx\,dy - \frac{1}{d_s}\int_{\mathbb{R}^3} (-\Delta)^s u \operatorname{tr} V\,dx = \int_{\mathbb{R}^3\times\mathbb{R}_+} y^\alpha \nabla U \cdot \nabla V\,dx\,dy$$

$$= \int_{\mathbb{R}^3\times\mathbb{R}_+} D_\bullet^\tau(y^\alpha \nabla U) \cdot \nabla(\zeta^2 D_\bullet^\tau U)\,dx\,dy$$

$$= \int_{B_R^+} y^\alpha D_\bullet^\tau(\nabla U) \cdot \left(\zeta^2 \nabla D_\bullet^\tau U + 2\zeta \nabla\zeta D_\bullet^\tau U\right) dx\, dy$$

$$= \int_{B_R^+} y^\alpha \zeta^2 D_\bullet^\tau(\nabla U) \cdot D_\bullet^\tau(\nabla U)\, dx\, dy + \int_{B_R^+} 2y^\alpha \zeta \nabla\zeta \cdot D_\bullet^\tau(\nabla U) D_\bullet^\tau U\, dx\, dy.$$

Using the equation $(-\Delta)^s u = f$ on Ω, Young's inequality, and the Poincaré inequality together with the trace estimate (3.4), we get the existence of constants $C_j > 0$, $j \in \{1, \ldots, 5\}$, such that

$$\begin{aligned}
\|\zeta D_\bullet^\tau(\nabla U)\|^2_{L^2_\alpha(B_R^+)} &\le C_1 \Bigg(\left| \int_{B_R^+} y^\alpha \zeta \nabla\zeta \cdot D_\bullet^\tau(\nabla U) D_\bullet^\tau U\, dx\, dy \right| + \left| \int_{\mathbb{R}^3 \times \mathbb{R}_+} F\, D_\bullet^{-\tau} \zeta^2 D_\bullet^\tau U\, dx\, dy \right| \\
&\quad + \left| \int_{\mathbb{R}^3} D_\bullet^\tau f \zeta_x^2 D_\bullet^\tau u\, dx \right| \Bigg) \\
&\le \frac{1}{4} \|\zeta D_\bullet^\tau(\nabla U)\|^2_{L^2_\alpha(B_R^+)} + C_2 \Big(\|\nabla\zeta\|^2_{L^\infty(B_R^+)} \|D_\bullet^\tau U\|^2_{L^2_\alpha(B_R^{\theta'})} \\
&\quad + \|F\|_{L^2_{-\alpha}(B_R^+)} \|\nabla(\zeta^2 D_\bullet^\tau U)\|_{L^2_\alpha(B_R^+)} + \|\zeta_x D_\bullet^\tau f\|_{H^{-s}(\Omega)} \|\zeta_x D_\bullet^\tau u\|_{H^s(\mathbb{R}^3)} \Big) \\
&\le \frac{1}{2} \|\zeta D_\bullet^\tau(\nabla U)\|^2_{L^2_\alpha(B_R^+)} + C_3 \Big(\|\nabla\zeta\|^2_{L^\infty(B_R^+)} \|\nabla U\|^2_{L^2_\alpha(B_R^{\theta'})} + \|F\|^2_{L^2_{-\alpha}(B_R^+)} \\
&\quad + \|\zeta_x D_\bullet^\tau f\|_{H^{-s}(\Omega)} |\zeta_x D_\bullet^\tau u|_{H^s(\mathbb{R}^3)} \Big) \\
&\overset{(3.4)}{\le} \frac{1}{2} \|\zeta D_\bullet^\tau(\nabla U)\|^2_{L^2_\alpha(B_R^+)} + C_4 \Big(\|\nabla\zeta\|^2_{L^\infty(B_R^+)} \|\nabla U\|^2_{L^2_\alpha(B_R^{\theta'})} + \|F\|^2_{L^2_{-\alpha}(B_R^+)} \\
&\quad + \|\zeta_x D_\bullet^\tau f\|_{H^{-s}(\Omega)} \|\nabla(\zeta D_\bullet^\tau U)\|_{L^2_\alpha(\mathbb{R}^3 \times \mathbb{R}_+)} \Big) \\
&\le \frac{3}{4} \|\zeta D_\bullet^\tau(\nabla U)\|^2_{L^2_\alpha(B_R^+)} \\
&\quad + C_5 \Big(\|\nabla\zeta\|^2_{L^\infty(B_R^+)} \|\nabla U\|^2_{L^2_\alpha(B_R^{\theta'})} + \|F\|^2_{L^2_{-\alpha}(B_R^+)} + \|\zeta_x D_\bullet^\tau f\|^2_{H^{-s}(\Omega)} \Big).
\end{aligned}$$

Absorbing the first term of the right-hand side in the left-hand side and taking the limit $\tau \to 0$, we obtain the sought inequality for the second derivatives since $\|\nabla\zeta\|_{L^\infty(B_R^+)} \lesssim ((1-c)R)^{-1} + (\theta'-\theta)^{-1}$. We conclude using $\|\zeta_x D_\bullet f\|_{H^{-s}(\Omega)} \le C_{\mathrm{loc}} \|D_\bullet f\|_{L^2(B_R)}$ for some $C_{\mathrm{loc}} > 0$ independent of R, c, and f. ∎

The Caccioppoli inequality in Lemma 4.3 can be iterated on concentric balls to provide control of higher order derivatives by lower order derivatives locally.

Corollary 4.4 (High order interior Caccioppoli inequality). *Let $B_R(x_0) \subset \Omega$ be an open ball with radius $R > 0$ centered at $x_0 \in \Omega$. For $\theta \in (0, \infty]$ and $c \in (0, 1]$ denote by $B_{cR}^\theta := B_{cR}(x_0) \times (0, \theta)$ the corresponding concentrically scaled and extended ball. Let U satisfy (3.11) with given data f and F with $\operatorname{supp}(F) \subset \mathbb{R}^3 \times [0, \mathcal{Y}]$ and let $\theta' > \theta$.*

Then, there is $\gamma > 0$ such that for all $\beta \in \mathbb{N}_0^3$ we have with $p = |\beta|$

$$\begin{aligned}
\text{(4.5)} \quad \left\|\partial_x^\beta \nabla U\right\|^2_{L^2_\alpha(B_{cR}^\theta)} &\le (\gamma p)^{2p} R^{-2p} \|\nabla U\|^2_{L^2_\alpha(B_R^{\theta'})} \\
&\quad + \sum_{j=1}^{p} (\gamma p)^{2(p-j)} R^{2(j-p)} \left(\max_{\substack{|\eta|=j \\ \eta \le \beta}} \|\partial_x^\eta f\|^2_{L^2(B_R)} + \max_{\substack{|\eta|=j-1 \\ \eta \le \beta}} \|\partial_x^\eta F\|^2_{L^2_{-\alpha}(B_R^+)} \right).
\end{aligned}$$

Proof. We start by noting that the case $p = 0$ is trivially true since empty sums are zero and $0^0 = 1$. For $p \geq 1$, we fix a multi index β such that $|\beta| = p$. As the x-derivatives commute with the differential operator in (3.11), we have that $\partial_x^\beta U$ solves equation (3.11) with data $\partial_x^\beta F$ and $\partial_x^\beta f$. For given $c > 0$ and $0 < \theta < \theta'$, let

$$c_i = c + (i-1)\frac{1-c}{p}, \quad \theta_i = \theta + (i-1)\frac{\theta' - \theta}{p}, \qquad i = 1, \ldots, p+1.$$

Then, we have $c_{i+1}R - c_i R = \frac{(1-c)R}{p}$, $c_1 R = cR$, and $c_{p+1}R = R$ as well as $\theta_{i+1} - \theta_i = \frac{\theta' - \theta}{p}$, $\theta_1 = \theta$, and $\theta_{p+1} = \theta'$. As $R \leq \operatorname{diam} \Omega$, we obtain

$$(\theta_{i+1} - \theta_i)^{-2} + (c_{i+1}R - c_i R)^{-2} \leq Cp^2 R^{-2}/(1-c)^2$$

with a constant $C > 0$ depending only on Ω, θ, θ'. For ease of notation and without loss of generality, we assume that $\beta_1 > 0$. Applying Lemma 4.3 iteratively on the sets $B^{\theta_i}_{c_i R}$ for $i > 1$ provides

$$\begin{aligned}
&\left\|\partial_x^\beta \nabla U\right\|^2_{L^2_\alpha(B^\theta_{cR})} \\
&\leq C_{\text{int}}^2 \left(\frac{p^2}{(1-c)^2} R^{-2} \left\|\partial_x^{(\beta_1 - 1, \beta_2)} \nabla U\right\|^2_{L^2_\alpha(B^{\theta_2}_{c_2 R})} + C_{\text{loc}}^2 \left\|\partial_x^\beta f\right\|^2_{L^2(B_{c_2 R})} + \left\|\partial_x^{(\beta_1 - 1, \beta_2)} F\right\|^2_{L^2_{-\alpha}(B^+_{c_2 R})} \right) \\
&\leq \left(\frac{C_{\text{int}} p}{(1-c)}\right)^{2p} R^{-2p} \left\|\nabla U\right\|^2_{L^2_\alpha(B^{\theta'}_R)} + C_{\text{loc}}^2 \sum_{j=1}^{p} \left(\frac{C_{\text{int}} p}{(1-c)}\right)^{2p-2j} R^{-2p+2j} \max_{|\eta| = j} \left\|\partial_x^\eta f\right\|^2_{L^2(B_{c_{p-j+2} R})} \\
&\quad + \sum_{j=0}^{p-1} \left(\frac{C_{\text{int}} p}{(1-c)}\right)^{2p-2j-2} R^{-2p+2j+2} \max_{|\eta| = j} \left\|\partial_x^\eta F\right\|^2_{L^2_{-\alpha}(B^+_{c_{p-j+1} R})}.
\end{aligned}$$

Choosing $\gamma = \max(C_{\text{loc}}^2, 1) C_{\text{int}}/(1-c)$ concludes the proof. ∎

The same arguments also apply to the other cases in the statement of Lemma 4.3 for sets near faces and edges.

Corollary 4.5 (High order boundary Caccioppoli inequality on $\mathbf{f}$). *Let $\mathbf{f} \in \mathcal{F}$ be an open face of $\partial\Omega$ and $x_0 \in \mathbf{f}$. For $R > 0$, let $B_R(x_0) \cap \Omega$ be an open half-ball. For $\theta \in (0, \infty]$ and $c \in (0, 1]$ denote by $B^\theta_{cR} := (B_{cR}(x_0) \cap \Omega) \times (0, \theta)$ the corresponding concentrically scaled and extended half-ball. Let U satisfy (3.11) with given data f and F with $\operatorname{supp}(F) \subset \mathbb{R}^3 \times [0, \mathcal{Y}]$ and let $\theta' > \theta$.*

Then, there is $\gamma > 0$ such that for every for all $\beta_\parallel \in \mathbb{N}_0^2$ with $p = |\beta_\parallel|$,

$$\begin{aligned}
(4.6) \quad \|D^{\beta_\parallel}_{\mathbf{f}_\parallel} \nabla U\|^2_{L^2_\alpha(B^\theta_{cR})} &\leq (\gamma p)^{2p} R^{-2p} \|\nabla U\|^2_{L^2_\alpha(B^{\theta'}_R)} \\
&\quad + \sum_{j=1}^{p} (\gamma p)^{2(p-j)} R^{2(j-p)} \left(\max_{\substack{|\eta| = j \\ \eta \leq \beta_\parallel}} \|D^\eta_{\mathbf{f}_\parallel} f\|^2_{L^2(B_R)} + \max_{\substack{|\eta| = j-1 \\ \eta \leq \beta_\parallel}} \|D^\eta_{\mathbf{f}_\parallel} F\|^2_{L^2_{-\alpha}(B^+_R)} \right).
\end{aligned}$$

Corollary 4.6 (High order boundary Caccioppoli inequality on $\mathbf{e}$). *Let $\mathbf{e} \in \mathcal{E}$ be an open edge of $\partial\Omega$ and $x_0 \in \mathbf{e}$. For $R > 0$, let $B_R(x_0) \cap \Omega$ be an open wedge. For $\theta \in (0, \infty]$ and $c \in (0, 1]$ denote by $B^\theta_{cR} := (B_{cR}(x_0) \cap \Omega) \times (0, \theta)$ the corresponding concentrically scaled and extended wedge. Let U satisfy (3.11) with given data f and F with $\operatorname{supp}(F) \subset \mathbb{R}^3 \times [0, \mathcal{Y}]$ and let $\theta' > \theta$.*

Then, there is $\gamma > 0$ such that for every $p \in \mathbb{N}_0$

$$\|D^p_{\mathbf{e}_\|}\nabla U\|^2_{L^2_\alpha(B^\theta_{cR})} \le (\gamma p)^{2p} R^{-2p} \|\nabla U\|^2_{L^2_\alpha(B^{\theta'}_R)} + \sum_{j=1}^{p} (\gamma p)^{2(p-j)} R^{2(j-p)} \left(\|D^j_{\mathbf{e}_\|} f\|^2_{L^2(B_R)} + \|D^{j-1}_{\mathbf{e}_\|} F\|^2_{L^2_{-\alpha}(B^+_R)} \right). \tag{4.7}$$

5. Local tangential regularity for the CS extension. Employing additional regularity of U, which was shown in Lemma 4.1, the term $\|\nabla U\|_{L^2_\alpha(B^+_R)}$ in (4.5) – (4.7) is small for $R \to 0$. This is the made precise in the following lemma, which is the exact analog of the corresponding statement in dimension $d = 2$ near edges [FMMS22, Lem. 4.3].

Lemma 5.1. *For $t \in [0, 1/2)$, there exists $C_{\mathrm{reg}} > 0$ (depending only on t and Ω) such that the solution U of (3.7) satisfies*

$$\|r^{-t}_{\partial\Omega}\nabla U\|^2_{L^2_\alpha(\Omega^+)} \le C_{\mathrm{reg}} C_t N^2(U, F, f) \tag{5.1}$$

with the constant $C_t > 0$ from Lemma 4.1 and $N^2(U, F, f)$ given by (4.1).

Lemma 4.1 provides global regularity for the solution U of (3.11). For all $R, \mathcal{Y} > 0$ and $x_0 \in \mathbb{R}^3$, let $B^{\mathcal{Y}}_R(x_0) := B_R(x_0) \times (0, \mathcal{Y})$. We introduce, for any set $B^{\mathcal{Y}}_R \subset \mathbb{R}^3 \times \mathbb{R}_+$ and any $p \in \mathbb{N}_0$,

$$\widetilde{N}^{(p)}_{B^{\mathcal{Y}}_R}(F, f) := \sum_{j=1}^{p+1} (\gamma p)^{-2j} \left(3^j \max_{|\beta|=j} \|\partial^\beta_x f\|^2_{L^2(B_R)} + 3^{j-1} \max_{|\beta|=j-1} \|\partial^\beta_x F\|^2_{L^2_{-\alpha}(B^{\mathcal{Y}}_R)} \right). \tag{5.2}$$

We derive localized versions of Lemma 4.1 for tangential derivatives of U at the boundary. Their proofs are minor variations of arguments in the proof of [FMMS22, Lemma 4.4]; we present the details here for completeness.

Lemma 5.2 (High order localized shift theorem near a face or an edge). *Let U be the solution of (3.7). Let $\mathbf{s} \in \mathcal{E} \cup \mathcal{F}$. Let $x_0 \in \mathbf{s}$. Let $R \in (0, 1/2]$, $c \in (0, 1)$, and assume that $B_R(x_0) \cap \Omega$ is a half ball (if $\mathbf{s} \in \mathcal{F}$) or a wedge (if $\mathbf{s} \in \mathcal{E}$).*

Then, for $t \in [0, 1/2)$, there is $C > 0$ independent of R and x_0 such that, for all $\beta \in \mathbb{N}$ (if $\mathbf{s} \in \mathcal{E}$) or $\beta \in \mathbb{N}^2_0$ (if $\mathbf{s} \in \mathcal{F}$), with $|\beta| =: p \in \mathbb{N}_0$,

$$\|r^{-t}_{\partial\Omega} D^\beta_{\mathbf{s}_\|}\nabla U\|^2_{L^2_\alpha(B^{\mathcal{Y}/2}_{cR})} \le C R^{-2p-1} (\gamma p)^{2p} (1 + \gamma p) \left(\|\nabla U\|^2_{L^2_\alpha(B^{\mathcal{Y}}_R)} + R^{s+1} \widetilde{N}^{(p)}_{B^{\mathcal{Y}}_R}(F, f) \right), \tag{5.3}$$

where γ is the constant in Corollary 4.6 or 4.5.

Proof. Let $\tilde{c} = (c+1)/2 \in (c, 1)$. Let $\eta_x \in C^\infty_0(B_{\tilde{c}R}(x_0))$ with $\eta_x \equiv 1$ on $B_{cR}(x_0)$, $\eta_y \in C^\infty_0(-\mathcal{Y}, \mathcal{Y})$ with $\eta_y \equiv 1$ on $(-\mathcal{Y}/2, \mathcal{Y}/2)$ and $\|\nabla^j \eta_x\|_{L^\infty(B_R(x_0))} \le C_\eta R^{-j}$, $j \in \{0, 1, 2\}$ as well as $\|\partial^j_y \eta_y\|_{L^\infty(-\mathcal{Y},\mathcal{Y})} \le C_\eta \mathcal{Y}^{-j}$, $j \in \{0, 1, 2\}$, with a constant $C_\eta > 0$ independent of R and $\mathcal{Y}$. Let $\eta(x, y) := \eta_x(x)\eta_y(y)$.

We denote $\kappa = 1$ if $\mathbf{s} \in \mathcal{E}$ and $\kappa = 2$ if $\mathbf{s} \in \mathcal{F}$ (so that $\beta \in \mathbb{N}^\kappa_0$). We abbreviate $U^{(\beta)}_\| := D^\beta_{\mathbf{s}_\|} U$, $\widetilde{U}^{(\beta)} := \eta D^\beta_{\mathbf{s}_\|} U$, $F^{(\beta)}_\| = D^\beta_{\mathbf{s}_\|} F$, and $f^{(\beta)}_\| = D^\beta_{\mathbf{s}_\|} f$. Throughout the proof we will use the fact that, for all $j \in \mathbb{N}$ and all sufficiently smooth functions v, we have

$$\max_{|\eta|=j} |D^\eta_{\mathbf{s}_\|} v| \le 3^{j/2} \max_{|\beta|=j} |\partial^\beta_x v|.$$

We also note that the assumptions on $\eta(x, y) = \eta_x(x)\eta_y(y)$ imply the existence of $\tilde{C}_\eta > 0$ (which absorbs the dependence on $\mathcal{Y}$ and c that we do not further track) such that

$$\|\nabla^j_x \partial^{j'}_y \eta\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R})} \le \tilde{C}_\eta R^{-j}, \qquad j \in \{0, 1, 2\}, j' \in \{0, 1, 2\}. \tag{5.4}$$

Step 1. (Localization of the equation). Using that U solves the extension problem (3.11), we obtain that the function $\widetilde{U}^{(\beta)} = \eta U_{\|}^{(\beta)}$ satisfies in $\Omega \times (0,\infty)$ the equation

$$
\begin{aligned}
\widetilde{F}^{(\beta)} &:= \operatorname{div}(y^\alpha \nabla \widetilde{U}^{(\beta)}) \\
&= y^\alpha \operatorname{div}_x(\nabla_x \widetilde{U}^{(\beta)}) + \partial_y(y^\alpha \partial_y \widetilde{U}^{(\beta)}) \\
&= y^\alpha \left((\Delta_x \eta) U_{\|}^{(\beta)} + 2\nabla_x \eta \cdot \nabla_x U_{\|}^{(\beta)} + \eta \Delta_x U_{\|}^{(\beta)} \right) + \eta \partial_y (y^\alpha \partial_y U_{\|}^{(\beta)}) \\
&\qquad + \partial_y (y^\alpha U_{\|}^{(\beta)} \partial_y \eta) + y^\alpha \partial_y U_{\|}^{(\beta)} \partial_y \eta \\
&= y^\alpha \left((\Delta_x \eta) U_{\|}^{(\beta)} + 2\nabla_x \eta \cdot \nabla_x U_{\|}^{(\beta)} \right) + \partial_y (y^\alpha U_{\|}^{(\beta)} \partial_y \eta) + y^\alpha \partial_y U_{\|}^{(\beta)} \partial_y \eta + \eta \operatorname{div}(y^\alpha \nabla U_{\|}^{(\beta)}) \\
&= y^\alpha \left((\Delta_x \eta) U_{\|}^{(\beta)} + 2\nabla_x \eta \cdot \nabla_x U_{\|}^{(\beta)} \right) + \partial_y (y^\alpha U_{\|}^{(\beta)} \partial_y \eta) + y^\alpha \partial_y U_{\|}^{(\beta)} \partial_y \eta + \eta F_{\|}^{(\beta)}
\end{aligned}
$$

as well as the boundary conditions

$$
\begin{aligned}
\partial_{n_\alpha} \widetilde{U}^{(\beta)}(\cdot,0) &= \eta(\cdot,0) D_{\mathbf{s}_\|}^{\beta} f =: \widetilde{f}^{(\beta)} && \text{on } \Omega, \\
\operatorname{tr} \widetilde{U}^{(\beta)} &= 0 && \text{on } \Omega^c.
\end{aligned}
$$

By the support properties of the cut-off function η, we have $\operatorname{supp} \widetilde{F}^{(\beta)} \subset \overline{B_{\tilde{c}R}}(x_0) \times [0,\mathcal{Y}]$. Using Lemma 4.1, for all $t \in [0,1/2)$, there is a $C_t > 0$ such that

$$
\int_{\mathbb{R}_+} y^\alpha \|\nabla \widetilde{U}^{(\beta)}(\cdot,y)\|^2_{H^t(B_{\widetilde{R}})} dy \le C_t N^2(\widetilde{U}^{(\beta)}, \widetilde{F}^{(\beta)}, \widetilde{f}^{(\beta)}), \tag{5.5}
$$

where $B_{\widetilde{R}}$ is a ball containing $\overline{\Omega}$. By (4.1), we must bound $N^2(\widetilde{U}^{(\beta)}, \widetilde{F}^{(\beta)}, \widetilde{f}^{(\beta)})$, i.e., the quantities $\|\nabla \widetilde{U}^{(\beta)}\|_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)}$, $\|\widetilde{F}^{(\beta)}\|_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))}$, and $\|\widetilde{f}^{(\beta)}\|_{H^{1-s}(\Omega)}$. In the following, γ is the constant introduced in Corollary 4.6 or 4.5.

Step 2. (Estimate of $\|\nabla \widetilde{U}^{(\beta)}\|_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)}$). Let $\widetilde{\beta} \in \mathbb{N}_0^\kappa$ be any (multi-)index such that $|\widetilde{\beta}| = p-1$. We write

$$
\begin{aligned}
\|\nabla \widetilde{U}^{(\beta)}\|^2_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)} &\le 2\|\nabla \eta\|^2_{L^\infty(B_R^{\mathcal{Y}})} \|\nabla_x U_{\|}^{(\widetilde{\beta})}\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})} + 2\|\eta\|^2_{L^\infty(B^{\mathcal{Y}}_{\tilde{c}R})} \|\nabla U_{\|}^{(\beta)}\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})} \\
&\le 2\widetilde{C}_\eta^2 \left(R^{-2} \|\nabla U_{\|}^{(\widetilde{\beta})}\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})} + \|\nabla U_{\|}^{(\beta)}\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})} \right).
\end{aligned} \tag{5.6}
$$

We employ Corollary 4.6 or 4.5 (with $\tilde{c}$ instead of c) to obtain for all $\beta \in \mathbb{N}_0^\kappa$

(5.7)

$$
\begin{aligned}
\|\nabla U_{\|}^{(\beta)}\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})} &\le R^{-2p} (\gamma p)^{2p} \Bigg(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} \\
&\qquad + \sum_{j=1}^{p} R^{2j} (\gamma p)^{-2j} \Big(\max_{\substack{|\eta|=j \\ \eta\le\beta}} \|D^\eta_{\mathbf{s}_\|} f\|^2_{L^2(B_R)} + \max_{\substack{|\eta|=j-1 \\ \eta\le\beta}} \|D^\eta_{\mathbf{s}_\|} F\|^2_{L^2_{-\alpha}(B_R^{\mathcal{Y}})} \Big) \Bigg) \\
&\le R^{-2p} (\gamma p)^{2p} \Bigg(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} \\
&\qquad + R^2 \sum_{j=1}^{p} R^{2(j-1)} (\gamma p)^{-2j} \Big(3^j \max_{|\beta|=j} \|\partial_x^\beta f\|^2_{L^2(B_R)} + 3^{j-1} \max_{|\beta|=j-1} \|\partial_x^\beta F\|^2_{L^2_{-\alpha}(B_R^{\mathcal{Y}})} \Big) \Bigg) \\
&\le R^{-2p} (\gamma p)^{2p} \left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^2 \widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f) \right).
\end{aligned}
$$

For $p \in \mathbb{N}$, we apply (5.7) to the $\widetilde{\beta}$-derivative and exploit the estimate $(\gamma(p-1))^{-2} \leq \max\{1,\gamma^{-2}\}$ for $p > 1$ to bound $(\gamma(p-1))^{2p-2}\widetilde{N}^{(p-1)}_{B_R^{\mathcal{Y}}}(F,f) \lesssim \max\{1,\gamma^{-2}\}(\gamma p)^{2p}\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)$. Consequently, we obtain the existence of a constant $C > 0$ such that for all $p \in \mathbb{N}$ it holds that (recall $|\widetilde{\beta}| = p-1$)

$$\|\nabla U_\|^{(\widetilde{\beta})}\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})} \leq C\max\{1,\gamma^{-2}\}R^{-2p+2}(\gamma p)^{2p}\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^2\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right) . \tag{5.8}$$

Inserting (5.7) and (5.8) into (5.6) provides the estimate

$$\|\nabla\widetilde{U}^{(\beta)}\|^2_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)} \leq CR^{-2p}(\gamma p)^{2p}\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^2\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right)$$

with a constant $C > 0$ depending only on the constants $\tilde{C}_\eta$, c, and γ.

Step 3. (Estimate of $\|\widetilde{F}^{(\beta)}\|_{L^2_{-\alpha}(\mathbb{R}^3\times\mathbb{R}_+)}$). We treat the five terms appearing in $\|\widetilde{F}^{(\beta)}\|_{L^2_{-\alpha}(\mathbb{R}^3\times\mathbb{R}_+)}$ separately. With (5.7), we obtain

$$\begin{aligned}\left\|y^\alpha\nabla_x\eta\cdot\nabla_x U_\|^{(\beta)}\right\|^2_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))} &= \left\|\nabla_x\eta\cdot\nabla_x U_\|^{(\beta)}\right\|^2_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)} \leq C_\eta^2\frac{1}{R^2}\left\|\nabla_x U_\|^{(\beta)}\right\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})}\\ &\overset{(5.7)}{\leq} CR^{-2p-2}(\gamma p)^{2p}\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^2\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right) .\end{aligned}$$

Similarly, we get (with $|\widetilde{\beta}| = p-1$ again)

$$\begin{aligned}\left\|y^\alpha(\Delta_x\eta)U_\|^{(\beta)}\right\|^2_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))} &= \left\|(\Delta_x\eta)U_\|^{(\beta)}\right\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})} \leq C_\eta^2\frac{1}{R^4}\left\|\nabla U_\|^{(\widetilde{\beta})}\right\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})}\\ &\overset{(5.8)}{\leq} CR^{-2p-2}(\gamma p)^{2p}\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^2\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right) .\end{aligned}$$

Next, we estimate

$$\|\eta F_\|^{(\beta)}\|^2_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))} \leq \|F_\|^{(\beta)}\|^2_{L^2_{-\alpha}(B^{\mathcal{Y}}_{\tilde{c}R})} \leq 3^p\max_{|\beta|=p}\|\partial_x^\beta F\|^2_{L^2_{-\alpha}(B^{\mathcal{Y}}_{\tilde{c}R})} \leq (\gamma p)^{2p+2}\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f).$$

Finally, for the term $\partial_y(y^\alpha U_\|^{(\beta)}\partial_y\eta) + y^\alpha\partial_y U_\|^{(\beta)}\partial_y\eta$, we observe that $\partial_y\eta$ vanishes near $y = 0$ so that the weight y^α does not come into play as it can be bounded from above and below by positive constants depending only on $\mathcal{Y}$. We arrive at

$$\begin{aligned}&\left\|\partial_y(y^\alpha U_\|^{(\beta)}\partial_y\eta) + y^\alpha\partial_y U_\|^{(\beta)}\partial_y\eta\right\|^2_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))}\\ &\qquad\leq C\left(\mathcal{Y}^{-2}\|U_\|^{(\beta)}\|^2_{L^2_\alpha(B_{\tilde{c}R}\times(0,\mathcal{Y}))} + \mathcal{Y}^{-1}\|\nabla U_\|^{(\beta)}\|^2_{L^2_\alpha(B^{\mathcal{Y}}_{\tilde{c}R})}\right)\\ &\qquad\overset{(5.7),(5.8)}{\leq} C_{\mathcal{Y}}R^{-2p-2}(\gamma p)^{2p}\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^2\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right)\end{aligned}$$

for suitable $C_{\mathcal{Y}} > 0$ depending on $\mathcal{Y}$.

Step 4. (Estimate of $\|\widetilde{f}^{(\beta)}\|_{H^{1-s}(\Omega)}$.) Here, we use Lemma A.1 and $R < 1/2$ together with $s < 1$ to obtain

$$\begin{aligned}\|\widetilde{f}^{(\beta)}\|^2_{H^{1-s}(\Omega)} &\leq 2C^2_{\mathrm{loc},2}C^2_\eta\left(9R^{2s-2}\|D^\beta_{\mathbf{s}_\|}f\|^2_{L^2(B_R)} + |D^\beta_{\mathbf{s}_\|}f|^2_{H^{1-s}(B_R)}\right)\\ &\leq CC^2_{\mathrm{loc},2}C^2_\eta R^{2s-2}\left(3^p\max_{|\beta|=p}\|\partial_x^\beta f\|^2_{L^2(B_R)} + 3^{p+1}\max_{|\beta|=p+1}\|\partial_x^\beta f\|^2_{L^2(B_R)}\right)\\ &\leq CC^2_{\mathrm{loc},2}C^2_\eta R^{2s-2}(\gamma p)^{2p}(1+(\gamma p)^2)\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\end{aligned}$$

with a constant $C > 0$ depending only on Ω, s, and c.

Step 5. (Putting everything together.) Combining the above estimates, we obtain that there exists a constant $C > 0$ depending only on $\tilde{C}_\eta$, $C_{\mathrm{loc},2}$, $\mathcal{Y}$, γ, Ω, s, and c such that

$$
\begin{aligned}
&N^2(\widetilde{U}^{(\beta)}, \widetilde{F}^{(\beta)}, \widetilde{f}^{(\beta)}) \\
&\quad = \|\nabla \widetilde{U}^{(\beta)}\|^2_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)} + \|\nabla \widetilde{U}^{(\beta)}\|_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)}\|\widetilde{F}^{(\beta)}\|_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))} + \|\nabla \widetilde{U}^{(\beta)}\|_{L^2_\alpha(\mathbb{R}^3\times\mathbb{R}_+)}\|\widetilde{f}^{(\beta)}\|_{H^{1-s}(\Omega)} \\
&\quad \le C\left(1+\gamma p R^{-1} + R^{-1}(1+\gamma p)\right) R^{-2p}(\gamma p)^{2p}\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^{s+1}\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right).
\end{aligned}
$$

Inserting this estimate in (5.5) we conclude that

$$
\int_{\mathbb{R}_+} y^\alpha \left\|\nabla \widetilde{U}^{(\beta)}(\cdot,y)\right\|^2_{H^t(\Omega)} dy \le C\,(1+\gamma p)\, R^{-2p-1}(\gamma p)^{2p}\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^{s+1}\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right).
$$

Step 6. The estimate (5.3) follows from [Gri11, Thm. 1.4.4.3], which gives

$$
\int_{\mathbb{R}_+} y^\alpha \|r_{\partial\Omega}^{-t}\nabla \widetilde{U}^{(p)}(\cdot,y)\|^2_{L^2(\Omega)}\,dy \le C\int_{\mathbb{R}_+} y^\alpha \|\nabla \widetilde{U}^{(p)}(\cdot,y)\|^2_{H^t(\Omega)}\,dy
$$

and from $\widetilde{U}^{(\beta)} = D^p_{\mathbf{s}_\parallel} U$ on $B_{cR}\times(0,\mathcal{Y}/2)$ by the definition of η. ∎

The following lemma is the same of the above, but in the interior of the domain.

Lemma 5.3 (High order localized shift theorem in the interior). *Let U be the solution of (3.7). Let $x_0 \in \Omega$. Let $R \in (0, 1/2]$, $c \in (0,1)$, and assume that $B_R(x_0) \subset \Omega$.*

Then, for $t \in [0, 1/2)$, there is $C > 0$ independent of R and x_0 such that, for all $\beta \in \mathbb{N}_0^3$, with $p = |\beta| \in \mathbb{N}_0$,

$$
\|r_{\partial\Omega}^{-t}\partial_x^\beta \nabla U\|^2_{L^2_\alpha(B_{cR}^{\mathcal{Y}/2})} \le C R^{-2p-1}(\gamma p)^{2p}(1+\gamma p)\left(\|\nabla U\|^2_{L^2_\alpha(B_R^{\mathcal{Y}})} + R^{s+1}\widetilde{N}^{(p)}_{B_R^{\mathcal{Y}}}(F,f)\right). \tag{5.9}
$$

Proof. The proof is the same as that of Lemma 5.2, with Corollary 4.4 replacing Corollary 4.6 or 4.5. ∎

6. Weighted H^p-estimates in polyhedra. In this section, we derive higher order weighted regularity results, at first for the extension problem and finally for the fractional PDE. The strategy is as in the two-dimensional case: we first introduce suitable countable, locally finite coverings of the various neighborhoods in Section 6.1. We then obtain in each of the neighborhoods local, Caccioppoli-type regularity shifts for the solution U of the CS extension defined in Section 3.2, in Section 6.2. Finally, in Section 6.3, we deduce from the estimates on U the analytic regularity results for the solution u of (2.3).

6.1. Coverings. As in space dimension $d = 2$, [FMMS22], a main ingredient in the proofs of a-priori estimates are suitable localizations of all the geometric neighborhoods in the partition (2.5) of the polyhedron Ω.

This is achieved by covering such neighborhoods by balls, half-balls or wedges with the following two properties: a) their diameter is proportional to the distance to lower-dimensional singular supports, i.e., vertices, edges and faces, and b) scaled versions of the balls/cut-balls satisfy a locally finite overlap property.

The general procedure in our construction of suitable localized coverings of all neighborhoods is hierarchic with respect to the dimension of the singular support set: if $\omega_\bullet$ is close to only one singular component, i.e., to either one vertex, edge or face (i.e. $\bullet \in \{\mathbf{v}, \mathbf{e}, \mathbf{f}\}$), we use balls inscribed in Ω with radii proportional to the distance to $\partial\Omega$.

For $\omega_\bullet$ close to two singular components of $\partial\Omega$, i.e., $\bullet \in \{\mathbf{ve}, \mathbf{vf}, \mathbf{ef}\}$, we localize at first with half-balls (in case of neighborhoods close to faces) centered on $\mathbf{f}$ in direction of the edge/vertex or wedges (in case of $\omega_{\mathbf{ve}}$) in direction of the vertex. Then, the half-balls/wedges are localized again using balls centered in Ω in direction of the face/edge (implicitly done in Lemma 6.8 and Lemma 6.11).

For $\omega_\bullet$ situated simultaneously close to three singular components of $\partial\Omega$, i.e. belonging to vertex-edge-face-neighborhoods, we first localize with wedges centered on the edge in direction of the vertex, then with half-balls centered on the face in direction of the edge, and finally with balls centered in Ω in direction of the face.

As in the two-dimensional case [FMMS22, Lemma 5.1], we work with local estimates obtained from Besicovitch's Covering Theorem.

Lemma 6.1 ([MW12, Lem. A.1], [HMW13, Lem. A.1]). *Let $\omega \subset \mathbb{R}^d$ be bounded, open and let $M \subset \partial\omega$ be closed, and nonempty. Fix c, $\zeta \in (0, 1)$ such that $1 - c(1 + \zeta) =: c_0 > 0$. For each $x \in \omega$, let $B_x := \overline{B}_{c\,\mathrm{dist}(x,M)}(x)$ be the closed ball of radius $c\,\mathrm{dist}(x, M)$ centered at x, and let $\widehat{B}_x := \overline{B}_{(1+\zeta)c\,\mathrm{dist}(x,M)}(x)$ be the scaled closed ball of radius $(1 + \zeta)c\,\mathrm{dist}(x, M)$ centered at x.*

Then, there is a countable set $(x_i)_{i\in\mathcal{I}} \subset \omega$ (for some suitable index set $\mathcal{I} \subset \mathbb{N}$) and a number $N \in \mathbb{N}$ depending solely on d, c, ζ with the following properties:

1. *(covering property) $\bigcup_i B_{x_i} \supset \omega$.*
2. *(finite overlap) $\mathrm{card}\{i \,|\, x \in \widehat{B}_{x_i}\} \leq N$ for all $x \in \mathbb{R}^d$.*

6.1.1. Covering of $\omega_{\mathbf{v}}$, $\omega_{\mathbf{e}}$, and $\omega_{\mathbf{f}}$. We start with coverings of vertex, edge and face neighborhoods and provide coverings using balls insribed in Ω whose size is proportional to their distance to the vertex, edge or face, respectively.

Lemma 6.2 (covering of $\omega_\bullet$, $\bullet \in \{\mathbf{v}, \mathbf{e}, \mathbf{f}\}$). *Given $\bullet \in \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ and $\xi > 0$, there are parameters $0 < c < \widehat{c} < 1$ as well as points $(x_i)_{i\in\mathbb{N}} \subset \omega_\bullet = \omega_\bullet^\xi$ such that:*

(i) The collection $\mathcal{B} := \{B_i := B_{c\,\mathrm{dist}(x_i,\bullet)}(x_i) \,|\, i \in \mathbb{N}\}$ of open balls covers $\omega_\bullet$.

(ii) The collection $\widehat{\mathcal{B}} := \{\widehat{B}_i := B_{\widehat{c}\,\mathrm{dist}(x_i,\bullet)}(x_i) \,|\, i \in \mathbb{N}\}$ of open balls satisfies a finite overlap property, i.e., there is an integer $N > 0$ depending only on the spatial dimension $d = 3$ and the parameters c, $\hat{c}$ such that $\mathrm{card}\{i \,|\, x \in \widehat{B}_i\} \leq N$ for all $x \in \mathbb{R}^3$. The balls from $\widehat{\mathcal{B}}$ are contained in Ω.

Proof. Apply Lemma 6.1 with $M = \{\bullet\}$ and sufficiently small parameters $c, \zeta > 0$. Observe that by possibly slightly increasing the parameter c, one can ensure that the open balls rather than the closed balls given by Lemma 6.1 cover $\omega_\bullet$. Also, since $c < 1$, the index set $\mathcal{I}$ of Lemma 6.1 cannot be finite so that we may assume $\mathcal{I} = \mathbb{N}$. ∎

6.1.2. Covering of $\omega_{\mathbf{ef}}$. We now introduce a covering of edge-face neighborhoods $\omega_{\mathbf{ef}}$. We start by a covering of half-balls resting on the face $\mathbf{f}$ and with size proportional to the distance from the edge.

Lemma 6.3. *Given $\mathbf{e} \in \mathcal{E}$, $\mathbf{f} \in \mathcal{F}_{\mathbf{e}}$, there are $\xi > 0$ and parameters $0 < c < \widehat{c} < 1$ as well as points $(x_i)_{i\in\mathbb{N}} \subset \mathbf{f}$ such that, denoting $R_i = c\,\mathrm{dist}(x_i, \mathbf{e})$ and $\widehat{R}_i = \hat{c}\,\mathrm{dist}(x_i, \mathbf{e})$:*

- (*i*) *The sets $H_i := B_{R_i}(x_i) \cap \Omega$ are half-balls and the collection $\mathcal{B} := \{H_i \mid i \in \mathbb{N}\}$ covers $\omega_{\mathbf{ef}} = \omega^\xi_{\mathbf{ef}}$.*
- (*ii*) *The collection $\widehat{\mathcal{B}} := \{\widehat{H}_i := B_{\widehat{R}_i}(x_i) \cap \Omega\}$ is a collection of half-balls and satisfies a finite overlap property, i.e., there is $N > 0$ depending only on the spatial dimension $d = 3$ and the parameters c, $\hat{c}$ such that $\mathrm{card}\{i \,|\, x \in \widehat{H}_i\} \le N$ for all $x \in \mathbb{R}^3$.*

Proof. Let $\widetilde{\mathbf{f}}$ be the (infinite) plane containing $\mathbf{f}$. We apply Lemma 6.1 to the 2D plane surface $\mathbf{f} \cap \partial\omega^\xi_{\mathbf{ef}}$ (for some sufficiently small ξ) and $M := \{\mathbf{e}\}$ and the parameter c sufficiently small so that $B_{2c\,\mathrm{dist}(x,\mathbf{e})}(x) \cap \Omega$ is a half-ball for all $x \in \mathbf{f} \cap \partial\omega^\xi_{\mathbf{ef}}$. Lemma 6.1 provides a collection $(x_i)_{i\in\mathbb{N}} \subset \mathbf{f}$ such that the balls $B_i := B_{R_i}(x_i) \subset \mathbb{R}^3$ and the scaled balls $\widehat{B}_i := B_{c(1+\zeta)\,\mathrm{dist}(x_i,\mathbf{e})}(x_i) \subset \mathbb{R}^3$ (for suitable, sufficiently small ζ) satisfy the following: the 2D balls $\{B_i \cap \widetilde{\mathbf{f}} \,|\, i \in \mathbb{N}\}$ cover $\partial\omega^\xi_{\mathbf{ef}} \cap \mathbf{f}$, and the 2D balls $\{\widehat{B}_i \cap \widetilde{\mathbf{f}} \,|\, i \in \mathbb{N}\}$ satisfy a finite overlap condition on $\widetilde{\mathbf{f}}$. By possibly slightly increasing the parameter c (e.g., by replacing c with $c(1 + \zeta/2)$), the newly defined balls B_i then cover a set $\omega^\xi_{\mathbf{ef}}$ for a possibly reduced ξ. It remains to see that the balls $\widehat{B}_i$ satisfy a finite overlap condition on $\mathbb{R}^2$: given $x \in \widehat{B}_i$, its projection $x_{\mathbf{f}}$ onto $\widetilde{\mathbf{f}}$ satisfies $x_{\mathbf{f}} \in \widehat{B}_i \cap \widetilde{\mathbf{f}}$ since $x_i \in \mathbf{f} \subset \widetilde{\mathbf{f}}$. This implies that the overlap constants of the 3D balls $\widehat{B}_i$ in $\mathbb{R}^3$ is the same as the overlap constant of the 2D balls $\widehat{B}_i \cap \widetilde{\mathbf{f}}$ in $\widetilde{\mathbf{f}}$. The half-balls $H_i := B_i \cap \Omega$ and $\widehat{H}_i := \widehat{B}_i \cap \Omega$ have the stated properties. ∎

6.1.3. Covering of $\omega_{\mathbf{vf}}$. Similarly, we provide a covering of the vertex-face neighborhoods $\omega_{\mathbf{vf}}$ using half-balls centered on the face f.

Lemma 6.4. *Given $\mathbf{v} \in \mathcal{V}$, $\mathbf{f} \in \mathcal{F}_{\mathbf{v}}$, there are $\xi > 0$ and parameters $0 < c < \widehat{c} < 1$ as well as points $(x_i)_{i\in\mathbb{N}} \subset \mathbf{f}$ such that, denoting $R_i = c\,\mathrm{dist}(x_i, \mathbf{v})$ and $\widehat{R}_i = \hat{c}\,\mathrm{dist}(x_i, \mathbf{v})$:*

- (*i*) *The sets $H_i := B_{R_i}(x_i) \cap \Omega$ are half-balls and the collection $\mathcal{B} := \{H_i \mid i \in \mathbb{N}\}$ covers $\omega_{\mathbf{vf}} = \omega^\xi_{\mathbf{vf}}$.*
- (*ii*) *The collection $\widehat{\mathcal{B}} := \{\widehat{H}_i := B_{\widehat{R}_i}(x_i) \cap \Omega\}$ is a collection of half-balls and satisfies a finite overlap property, i.e., there is $N > 0$ depending only on the spatial dimension $d = 3$ and the parameters c, $\hat{c}$ such that $\mathrm{card}\{i \,|\, x \in \widehat{H}_i\} \le N$ for all $x \in \mathbb{R}^3$.*

Proof. The proof is the same as the proof of Lemma 6.3. ∎

6.1.4. Covering of $\omega_{\mathbf{ve}}$. For the vertex-edge neighborhoods $\omega_{\mathbf{ve}}$, we introduce a covering using wedges centered on the edge with size proportional to the distance to the vertex.

Lemma 6.5. *Given $\mathbf{v} \in \mathcal{V}$, $\mathbf{e} \in \mathcal{E}_{\mathbf{v}}$, there are $\xi > 0$ and parameters $0 < c < \widehat{c} < 1$ as well as points $(x_i)_{i\in\mathbb{N}} \subset \mathbf{e}$ such that, denoting $R_i = c\,\mathrm{dist}(x_i, \mathbf{v})$ and $\widehat{R}_i = \hat{c}\,\mathrm{dist}(x_i, \mathbf{v})$:*

- (*i*) *The collection of wedges $\mathcal{B} := \{W_i \subset B_{R_i}(x_i) \cap \Omega\}_{i\in\mathbb{N}}$ covers $\omega_{\mathbf{ve}} = \omega^\xi_{\mathbf{ve}}$.*
- (*ii*) *The collection of wedges $\widehat{\mathcal{B}} := \{\widehat{W}_i \subset B_{\widehat{R}_i}(x_i) \cap \Omega\}_{i\in\mathbb{N}}$ satisfies $W_i \subset \widehat{W}_i$ and a finite overlap property, i.e., there is $N > 0$ depending only on the spatial dimension $d = 3$ and the parameters c, $\hat{c}$ such that $\mathrm{card}\{i \,|\, x \in \widehat{W}_i\} \le N$ for all $x \in \mathbb{R}^3$.*

Proof. Let $\widetilde{\mathbf{e}}$ be the (infinite) line containing $\mathbf{e}$. We apply Lemma 6.1 to the intervals $\mathbf{e} \cap \partial\omega^\xi_{\mathbf{ve}}$ (for some sufficiently small ξ) and $M := \{\mathbf{v}\}$ and the parameter c sufficiently small so that

$B_{2c\,\mathrm{dist}(x,\mathbf{e})}(x) \cap \Omega$ is a wedge for all $x \in \mathbf{e} \cap \partial\omega^{\xi}_{\mathbf{ve}}$. Lemma 6.1 provides a collection $(x_i)_{i\in\mathbb{N}} \subset \mathbf{e}$ such that the balls $B_i := B_{R_i}(x_i) \subset \mathbb{R}^3$ and the scaled balls $\widehat{B}_i := B_{c(1+\zeta)\,\mathrm{dist}(x_i,\mathbf{v})}(x_i) \subset \mathbb{R}^3$ (for suitable, sufficiently small ζ) satisfy the following: the intervals $\{B_i \cap \widetilde{\mathbf{e}} \,|\, i \in \mathbb{N}\}$ cover $\partial\omega^{\xi}_{\mathbf{ve}} \cap \mathbf{e}$, and the intervals $\{\widehat{B}_i \cap \widetilde{\mathbf{e}} \,|\, i \in \mathbb{N}\}$ satisfy a finite overlap condition on $\widetilde{\mathbf{e}}$. Upon increasing the parameter c (e.g., by replacing c with $c(1+\zeta/2)$), the newly defined balls B_i then cover a set $\omega^{\xi}_{\mathbf{ve}}$ for a possibly reduced ξ. It remains to see that the balls $\widehat{B}_i$ satisfy a finite overlap condition on $\mathbb{R}^2$: given $x \in \widehat{B}_i$, its projection $x_{\mathbf{e}}$ onto $\widetilde{\mathbf{e}}$ satisfies $x_{\mathbf{e}} \in \widehat{B}_i \cap \widetilde{\mathbf{e}}$ since $x_i \in \mathbf{e} \subset \widetilde{\mathbf{e}}$. This implies that the overlap constants of the balls $\widehat{B}_i$ in $\mathbb{R}^3$ is the same as the overlap constant of the intervals $\widehat{B}_i \cap \widetilde{\mathbf{e}}$ in $\widetilde{\mathbf{e}}$. The wedges $W_i := B_i \cap \Omega$ and $\widehat{W}_i := \widehat{B}_i \cap \Omega$ have the stated properties. ∎

6.1.5. Covering of $\omega_{\mathbf{vef}}$. In the same way, we obtain a covering of the vertex-edge-face neighborhoods $\omega_{\mathbf{vef}}$.

LEMMA 6.6. *Given $\mathbf{v} \in \mathcal{V}$, $\mathbf{e} \in \mathcal{E}_{\mathbf{v}}$, and $\mathbf{f} \in \mathcal{F}_{\mathbf{e}} \cap \mathcal{F}_{\mathbf{v}}$, there are $\xi > 0$ and parameters $0 < c < \widehat{c} < 1$ as well as points $(x_i)_{i\in\mathbb{N}} \subset \mathbf{e}$ such that, denoting $R_i = c\,\mathrm{dist}(x_i,\mathbf{v})$ and $\widehat{R}_i = \hat{c}\,\mathrm{dist}(x_i,\mathbf{v})$:*

- *(i) The sets $W_i := B_{R_i}(x_i) \cap \Omega$ are wedges and the collection $\mathcal{B} := \{W_i \,|\, i \in \mathbb{N}\}$ covers $\omega_{\mathbf{vef}} = \omega^{\xi}_{\mathbf{vef}}$.*
- *(ii) The collection $\widehat{\mathcal{B}} := \{\widehat{W}_i := B_{\widehat{R}_i}(x_i) \cap \Omega\}$ is a collection of wedges and satisfies a finite overlap property, i.e., there is $N > 0$ depending only on the spatial dimension $d = 3$ and the parameters c, $\hat{c}$ such that $\mathrm{card}\{i \,|\, x \in \widehat{W}_i\} \le N$ for all $x \in \mathbb{R}^3$.*

Proof. The proof is the same as that of Lemma 6.5, with $\omega_{\mathbf{vef}}$ replacing $\omega_{\mathbf{ve}}$. ∎

6.2. Weighted H^p-regularity for the CS extension. In the following, we provide separate weighted analytic regularity estimates on extensions of each neighborhood $\omega_{\bullet}$ used to decompose Ω in (2.5). Hereby, for any set $\omega \subset \mathbb{R}^3$ and $\mathcal{Y} > 0$, define $\omega^{\mathcal{Y}} := \omega \times (0, \mathcal{Y})$.

6.2.1. Vertex neighborhoods $\omega_{\mathbf{v}}$. We have

$$r_{\mathbf{f}} \sim r_{\mathbf{e}} \sim r_{\mathbf{v}} \qquad \text{on } \omega_{\mathbf{v}}.$$

The following lemma provides higher order regularity estimates in vertex-weighted norms for solutions to the Caffarelli-Silvestre extension problem with smooth data.

LEMMA 6.7 (Weighted H^p-regularity in $\omega_{\mathbf{v}}$). *Let $\omega_{\mathbf{v}} = \omega^{\xi}_{\mathbf{v}}$ be given for some $\xi > 0$ and $\mathbf{v} \in \mathcal{V}$. Let U be the solution of (3.7). There is $\gamma > 0$ depending only on s, Ω, $\omega_{\mathbf{v}}$, and $\mathcal{Y}$, and for every $\varepsilon \in (0, 1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $\beta \in \mathbb{N}^3_0$, with $p = |\beta|$,*

$$\|r_{\mathbf{v}}^{p-1/2+\varepsilon}\partial_x^{\beta}\nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{v}}\times(0,\mathcal{Y}))} \le C_\varepsilon \gamma^{2p+1} p^{2p} \Bigg[\|f\|^2_{H^1(\Omega)} + \|F\|^2_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))} + \sum_{j=1}^{p} p^{-2j} \left(\max_{|\eta|=j} \|\partial_x^{\eta} f\|^2_{L^2(\Omega)} + \max_{|\eta|=j-1} \|\partial_x^{\eta} F\|^2_{L^2_{-\alpha}(\mathbb{R}^3\times(0,\mathcal{Y}))} \right) \Bigg].$$

Proof. The case $p = 0$ follows from Lemma 5.1 and the estimates (4.1), (4.2).

We therefore assume in the remainder of this proof that $p \in \mathbb{N}$. Lemma 6.2 gives the covering $\bigcup_i B_i \supset \omega_{\mathbf{v}}$ with scaled balls $B_i = B_{cr_{\mathbf{v}}(x_i)}(x_i)$ and scaled balls $\widehat{B}_i = B_{\hat{c}r_{\mathbf{v}}(x_i)}(x_i)$. We denote $R_i := \hat{c}\,\mathrm{dist}(x_i,\mathbf{v})$ the radius of the ball $\widehat{B}_i$ and note that, for some $C_B > 1$,

$$\forall i \in \mathbb{N} \quad \forall x \in \widehat{B}_i \qquad C_B^{-1} R_i \le r_{\mathbf{v}}(x) \le C_B R_i. \tag{6.1}$$

We assume (for convenience) that $R_i \le 1$ for all i.

For any multi index β, with $p = |\beta|$,

$$\begin{aligned}
\|r_{\mathbf{v}}^{p-1/2+\varepsilon}\partial_x^\beta \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{v}}^{\mathcal{Y}/2})} &\overset{\text{L. 6.2}}{\leq} \sum_{i\in\mathbb{N}} \|r_{\mathbf{v}}^{p-1/2+\varepsilon}\partial_x^\beta \nabla U\|^2_{L^2_\alpha(B_i^{\mathcal{Y}/2})} \\
&\overset{(6.1)}{\leq} \sum_{i\in\mathbb{N}} (C_B R_i)^{2p+\varepsilon} \|r_{\mathbf{v}}^{-1/2+\varepsilon/2}\partial_x^\beta \nabla U\|^2_{L^2_\alpha(B_i^{\mathcal{Y}/2})} \\
&\overset{\text{C. 5.3}}{\lesssim} \sum_{i\in\mathbb{N}} (C_B R_i)^{2p+\varepsilon}(\gamma_1 p)^{2p+1} R_i^{-2p-1}\left[\|\nabla U\|^2_{L^2_\alpha(\widehat{B}_i^{\mathcal{Y}})} + R_i^{s+1}\widetilde{N}^{(p)}_{\widehat{B}_i^{\mathcal{Y}}}(F,f)\right] \\
&\leq C_B^{2p}(\gamma_1 p)^{2p+1} \sum_{i\in\mathbb{N}} \left[C_B \|r_{\mathbf{v}}^{-1/2+\varepsilon/2}\nabla U\|^2_{L^2_\alpha(\widehat{B}_i^{\mathcal{Y}})} + R_i^{s+\varepsilon}\widetilde{N}^{(p)}_{\widehat{B}_i^{\mathcal{Y}}}(F,f)\right] \\
&\lesssim C_B^{2p}(\gamma_1 p)^{2p+1}\left[C_B \|r_{\mathbf{v}}^{-1/2+\varepsilon/2}\nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{v}}^{\hat{\xi}}\times(0,\mathcal{Y}))} + \widetilde{N}^{(p)}_{\Omega^+}(F,f)\right].
\end{aligned}$$

We conclude by using that in $\omega_{\mathbf{v}}$, $r_{\mathbf{v}} \simeq r_{\partial\Omega}$ and using Lemma 5.1, Lemma 4.1 and (4.2). ∎

6.2.2. Edge-neighborhoods $\omega_{\mathbf{e}}$. We have

$$r_{\mathbf{f}} \sim r_{\mathbf{e}} \qquad \text{on } \omega_{\mathbf{e}}.$$

We start with a weighted regularity estimate on arbitrary wedges centered on an edge $\mathbf{e}$.

LEMMA 6.8 (Weighted H^p-regularity in a wedge). *Let $\mathbf{e} \in \mathcal{E}$, $x_0 \in \mathbf{e}$, $R > 0$, $\zeta > 0$ and let*

$$W_R = B_R(x_0) \cap \{x \in \Omega : \rho_{\mathbf{ef}}(x) > \zeta \ \forall \mathbf{f} \in \mathcal{F}_{\mathbf{e}}\}$$

be a wedge either in $\omega_{\mathbf{e}}$ or $\omega_{\mathbf{ve}}$. Let $c \in (0,1)$ and let U be the solution of (3.7).

Then, there exists $\gamma > 0$ depending only on s, Ω, ζ and $\mathcal{Y}$, and for every $\varepsilon \in (0, 1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $\beta_\perp = (\beta_{\perp,1}, \beta_{\perp,2}) \in \mathbb{N}_0^2$ and all $\beta_\parallel \in \mathbb{N}_0$, with $p_\perp = \beta_{\perp,1} + \beta_{\perp,2}$, $p_\parallel = \beta_\parallel$, and $p = p_\perp + p_\parallel$, it holds that

$$\begin{aligned}
(6.2)\quad \|r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} D^{\beta_\perp}_{\mathbf{e}_\perp} D^{\beta_\parallel}_{\mathbf{e}_\parallel}\nabla U\|^2_{L^2_\alpha(W_{cR}^{\mathcal{Y}/4})} \leq C_\varepsilon \gamma^{2p+1} p^{2p}\Bigg[R^{-2p_\parallel-1}\Big(\|\nabla U\|^2_{L^2_\alpha(W_R^{\mathcal{Y}})} \\
+ R^{s+1}\widetilde{N}^{(p_\parallel)}_{W_R^{\mathcal{Y}}}(F,f)\Big) + \widetilde{N}^{(p_\perp)}_{W_R^{\mathcal{Y}/2}}(D^{p_\parallel}_{\mathbf{e}_\parallel}F, D^{p_\parallel}_{\mathbf{e}_\parallel}f)\Bigg]
\end{aligned}$$

where $D^{\beta_\perp}_{\mathbf{e}_\perp} = D^{\beta_{\perp,1}}_{\mathbf{e}_{1,\perp}} D^{\beta_{\perp,2}}_{\mathbf{e}_{2,\perp}}$.

Proof. The case $p_\perp = 0$ follows from Lemma 5.2 and from the estimates (4.1), (4.2).

We therefore assume in the following that $p_\perp \in \mathbb{N}$. Denote $\tilde{c} = (c+1)/2 \in (c, 1)$.

We observe that the argument of Lemma 6.2 also gives a covering $\bigcup_i B_i \supset W_{cR}$ with balls $B_i = B_{c_1 r_{\mathbf{e}}(x_i)}(x_i)$ and scaled balls $\widehat{B}_i = B_{\hat{c}_1 r_{\mathbf{e}}(x_i)}(x_i)$ such that $\bigcup_i \widehat{B}_i \subset W_{\tilde{c}R}$, provided one chooses the parameters $c_1, \hat{c}_1 > 1$ small enough.

We denote $R_i := \hat{c}_1 \operatorname{dist}(x_i, \mathbf{e})$ the radius of the ball $\widehat{B}_i$ and note that, for some $C_B > 1$,

$$(6.3) \qquad \forall i \in \mathbb{N} \quad \forall x \in \widehat{B}_i \qquad C_B^{-1} R_i \leq r_{\mathbf{e}}(x) \simeq r_{\partial\Omega}(x) \leq C_B R_i.$$

We assume (for convenience) that $R_i \leq 1$ for all i.

We apply Lemma 5.3 to the function $D^{p_\parallel}_{\mathbf{e}_\parallel} U$ (noting that this function satisfies (3.11) with data $D^{p_\parallel}_{\mathbf{e}_\parallel} f$, $D^{p_\parallel}_{\mathbf{e}_\parallel} F$) with the pair $(B_i, \widehat{B}_i)$ of concentric balls, with $\mathcal{Y}/2$ instead of $\mathcal{Y}$, and with constant denoted $\gamma_1 \geq 1$. For any $\beta_\perp = (\beta_{\perp,1}, \beta_{\perp,2}) \in \mathbb{N}_0^2$ and $\beta_\parallel \in \mathbb{N}_0$, with $p_\perp = |\beta_\perp| \in \mathbb{N}$ and $p_\parallel = \beta_\parallel$, it holds that

$$
\begin{aligned}
&\|r_{\mathbf{e}}^{p_\perp - 1/2+\varepsilon} D_{\mathbf{e}_\perp}^{\beta_\perp} D_{\mathbf{e}_\parallel}^{p_\parallel} \nabla U\|^2_{L^2_\alpha(W_{cR}^{\mathcal{Y}/4})} \\
&\overset{\text{L. 6.2}}{\leq} \sum_{i\in\mathbb{N}} \|r_{\mathbf{e}}^{p_\perp - 1/2+\varepsilon} D_{\mathbf{e}_\perp}^{\beta_\perp} D_{\mathbf{e}_\parallel}^{p_\parallel} \nabla U\|^2_{L^2_\alpha(B_i^{\mathcal{Y}/4})} \\
&\overset{(6.3)}{\leq} \sum_{i\in\mathbb{N}} (C_B R_i)^{2p_\perp+\varepsilon} \|r_{\mathbf{e}}^{-1/2+\varepsilon/2} D_{\mathbf{e}_\perp}^{\beta_\perp} D_{\mathbf{e}_\parallel}^{p_\parallel} \nabla U\|^2_{L^2_\alpha(B_i^{\mathcal{Y}/4})} \\
&\overset{\text{L. 5.3}}{\leq} \sum_{i\in\mathbb{N}} (C_B R_i)^{2p_\perp+\varepsilon} (\gamma_1 p_\perp)^{2p_\perp+1} R_i^{-2p_\perp-1} \left[\|D_{\mathbf{e}_\parallel}^{p_\parallel} \nabla U\|^2_{L^2_\alpha(\widehat{B}_i^{\mathcal{Y}/2})} + R_i^{s+1} \widetilde{N}^{(p_\perp)}_{\widehat{B}_i^{\mathcal{Y}/2}}(D_{\mathbf{e}_\parallel}^{p_\parallel} F, D_{\mathbf{e}_\parallel}^{p_\parallel} f)\right] \\
&\overset{(6.3)}{\lesssim} C_B^{2p_\perp+1} (\gamma_1 p_\perp)^{2p_\perp+1} \sum_{i\in\mathbb{N}} \left[\|r_{\mathbf{e}}^{-1/2+\varepsilon/2} D_{\mathbf{e}_\parallel}^{p_\parallel} \nabla U\|^2_{L^2_\alpha(\widehat{B}_i^{\mathcal{Y}/2})} + R^{s+\varepsilon} \widetilde{N}^{(p_\perp)}_{\widehat{B}_i^{\mathcal{Y}/2}}(D_{\mathbf{e}_\parallel}^{p_\parallel} F, D_{\mathbf{e}_\parallel}^{p_\parallel} f)\right] \\
&\lesssim C_B^{2p_\perp+1} (\gamma_1 p_\perp)^{2p_\perp+1} \left[\|r_{\mathbf{e}}^{-1/2+\varepsilon/2} D_{\mathbf{e}_\parallel}^{p_\parallel} \nabla U\|^2_{L^2_\alpha(W_{\tilde{c}R}^{\mathcal{Y}/2})} + \widetilde{N}^{(p_\perp)}_{W_{\tilde{c}R}^{\mathcal{Y}/2}}(D_{\mathbf{e}_\parallel}^{p_\parallel} F, D_{\mathbf{e}_\parallel}^{p_\parallel} f)\right] \\
&\overset{\text{L. 5.2}}{\leq} C_B^{2p_\perp+1} (\gamma_1 p_\perp)^{2p_\perp+1} (\gamma_2 p_\parallel)^{2p_\parallel+1} \left[R^{-2p_\parallel-1} \left(\|\nabla U\|^2_{L^2_\alpha(W_R^{\mathcal{Y}})} + R^{s+1} \widetilde{N}^{(p_\parallel)}_{W_R^{\mathcal{Y}}}(F, f)\right) \right. \\
&\qquad\qquad \left. + \widetilde{N}^{(p_\perp)}_{W_R^{\mathcal{Y}/2}}(D_{\mathbf{e}_\parallel}^{p_\parallel} F, D_{\mathbf{e}_\parallel}^{p_\parallel} f)\right],
\end{aligned}
$$

where we have used Lemma 5.2 in the last step. ∎

Corollary 6.9. *Let $\mathbf{e} \in \mathcal{E}$ and $\mathcal{Y} > 0$. Let U be the solution of (3.7).*

Then, there exists $\gamma > 0$ depending only on s, Ω, ζ and $\mathcal{Y}$, and, for every $\varepsilon \in (0, 1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $\beta_\perp = (\beta_{\perp,1}, \beta_{\perp,2}) \in \mathbb{N}_0^2$ and all $\beta_\parallel \in \mathbb{N}_0$, with $p_\perp = \beta_{\perp,1} + \beta_{\perp,2}$, $p_\parallel = \beta_\parallel$, and $p = p_\perp + p_\parallel$, it holds that

$$\|r_{\mathbf{e}}^{p_\perp - 1/2+\varepsilon} D_{\mathbf{e}_\perp}^{\beta_\perp} D_{\mathbf{e}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{e}}^{\mathcal{Y}/4})} \leq C_\varepsilon \gamma^{2p+1} p^{2p} \widetilde{N}^{(p)}_{\Omega^{\mathcal{Y}}}(F, f). \tag{6.4}$$

Proof. This follows directly from Lemma 6.8 with $R \simeq 1$ and from (4.2). ∎

6.2.3. Vertex-edge neighborhoods $\omega_{\mathbf{ve}}$.

We have

$$r_{\mathbf{f}} \sim r_{\mathbf{e}} \qquad \text{and} \qquad r_{\mathbf{e}} \leq r_{\mathbf{v}} \qquad \text{on } \omega_{\mathbf{ve}}.$$

Lemma 6.10 (Weighted H^p-regularity in $\omega_{\mathbf{ve}}$). *Let U be the solution of (3.7). There is $\gamma > 0$ depending only on s, Ω, and $\mathcal{Y}$, and for every $\varepsilon \in (0, 1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $\beta_\perp = (\beta_{\perp,1}, \beta_{\perp,2}) \in \mathbb{N}_0^2$ and $\beta_\parallel \in \mathbb{N}_0$, with $p_\perp = \beta_{\perp,1} + \beta_{\perp,2}$, $p_\parallel = \beta_\parallel$, and $p = p_\perp + p_\parallel$, it holds that*

$$\|r_{\mathbf{v}}^{p_\parallel+\varepsilon} r_{\mathbf{e}}^{p_\perp - 1/2+\varepsilon} D_{\mathbf{e}_\perp}^{\beta_\perp} D_{\mathbf{e}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{ve}}^{\mathcal{Y}/4})} \leq C_\varepsilon \gamma^{2p+1} p^{2p} \widetilde{N}^{(p)}_{\Omega^{\mathcal{Y}}}(F, f), \tag{6.5}$$

where $D_{\mathbf{e}_\perp}^{\beta_\perp} = D_{\mathbf{e}_{1,\perp}}^{\beta_{\perp,1}} D_{\mathbf{e}_{2,\perp}}^{\beta_{\perp,2}}$.

Proof. We use the covering of wedges $W_i \subset B_{cR_i}(x_i)$ with $\widehat{W}_i \subset B_{R_i}(x_i)$ given by Lemma 6.5. We have, for a constant $C_W > 1$,

$$\forall i \in \mathbb{N} \quad \forall x \in \widehat{W}_i \qquad C_W^{-1} R_i \leq r_{\mathbf{v}}(x) \leq C_W R_i.$$

Using this and Lemma 6.8,

$$
\begin{aligned}
&\|r_{\mathbf{v}}^{p_\parallel+\varepsilon} r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{e}_\perp}^{\beta_\perp} D_{\mathbf{e}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{ve}}^{\mathcal{Y}/4})} \\
&\qquad \leq \sum_{i\in\mathbb{N}} (C_W R_i)^{2p_\parallel+2\varepsilon} \|r_{\mathbf{e}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{e}_\perp}^{\beta_\perp} D_{\mathbf{e}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(W_i^{\mathcal{Y}/4})} \\
&\qquad \leq \sum_{i\in\mathbb{N}} (C_W R_i)^{2p_\parallel+2\varepsilon} \gamma_1^{2p+1} p^{2p} \Bigg[R_i^{-2p_\parallel} \left(R_i^{-1} \|\nabla U\|^2_{L^2_\alpha(\widehat{W}_i^{\mathcal{Y}})} + R_i^s \widetilde{N}^{(p_\parallel)}_{\widehat{W}_i^{\mathcal{Y}}}(F,f) \right) \\
&\qquad\qquad + \widetilde{N}^{(p_\perp)}_{\widehat{W}_i^{\mathcal{Y}/2}}(D_{\mathbf{e}_\parallel}^{p_\parallel} F, D_{\mathbf{e}_\parallel}^{p_\parallel} f) \Bigg] \\
&\qquad \leq \gamma^{2p+1} p^{2p} \sum_{i\in\mathbb{N}} \left[\|r_{\mathbf{v}}^{-1/2+\varepsilon} \nabla U\|^2_{L^2_\alpha(\widehat{W}_i^{\mathcal{Y}})} + \widetilde{N}^{(p_\parallel)}_{\widehat{W}_i^{\mathcal{Y}}}(F,f) + \widetilde{N}^{(p_\perp)}_{\widehat{W}_i^{\mathcal{Y}/2}}(D_{\mathbf{e}_\parallel}^{p_\parallel} F, D_{\mathbf{e}_\parallel}^{p_\parallel} f) \right].
\end{aligned}
$$

The bound $r_{\mathbf{v}}(x) \geq r_{\partial\Omega}(x)$, the finite overlap of the wedges $\widehat{W}_i$, Lemma 5.1, and (4.2) conclude the proof. ∎

6.2.4. Face neighborhoods $\omega_{\mathbf{f}}$. We write $H_R^{\mathcal{Y}} := H_R \times (0,\mathcal{Y})$ and start with a weighted regularity estimate on arbitrary half-balls centered on a face $\mathbf{f}$.

Lemma 6.11 (Weighted H^p-regularity in a half-ball). *Let $\mathbf{f} \in \mathcal{F}$, $x_0 \in \mathbf{f}$, $R > 0$, $\zeta > 0$ and let*

$$H_R = B_R(x_0) \cap \Omega$$

be a half-ball. Let $c \in (0,1)$ and let U be the solution of (3.7). There is $\gamma > 0$ depending only on s, Ω, ζ and $\mathcal{Y}$, and for every $\varepsilon \in (0,1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $\beta_\parallel = (\beta_{\parallel,1}, \beta_{\parallel,2}) \in \mathbb{N}_0^2$ and $\beta_\perp \in \mathbb{N}_0$, with $p_\parallel = \beta_{\parallel,1} + \beta_{\parallel,2}$, $p_\perp = \beta_\perp$, and $p = p_\parallel + p_\perp$, it holds that

$$
\begin{aligned}
(6.6)\quad \|r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(H_{cR}^{\mathcal{Y}/4})} &\leq C_\varepsilon \gamma^{2p+1} p^{2p} \Bigg[R^{-2p_\parallel-1} \Bigg(\|\nabla U\|^2_{L^2_\alpha(H_R^{\mathcal{Y}})} \\
&\quad + R^{s+1} \widetilde{N}^{(p_\parallel)}_{W_R^{\mathcal{Y}}}(F,f) \Bigg) + \widetilde{N}^{(p_\perp)}_{H_R^{\mathcal{Y}/2}}(D_{\mathbf{f}_\parallel}^{p_\parallel} F, D_{\mathbf{f}_\parallel}^{p_\parallel} f) \Bigg]
\end{aligned}
$$

where $D_{\mathbf{f}_\parallel}^{\beta_\parallel} = D_{\mathbf{f}_{1,\parallel}}^{\beta_{\parallel,1}} D_{\mathbf{f}_{2,\parallel}}^{\beta_{\parallel,2}}$.

Proof. The case $p_\perp = 0$ follows from Lemma 5.2 and the estimates (4.1), (4.2). We therefore assume $p_\perp \in \mathbb{N}$.

Denote $\tilde{c} = (c+1)/2 \in (c,1)$. The arguments of Lemma 6.2 give a covering $\bigcup_i B_i \supset H_{cR}$ with balls $B_i = B_{c_1 r_{\mathbf{f}}(x_i)}(x_i)$ and scaled balls $\widehat{B}_i = B_{\hat{c}_1 r_{\mathbf{f}}(x_i)}(x_i)$ such that $\bigcup_i \widehat{B}_i \subset H_{\tilde{c}R}$, if one chooses the parameters $c_1, \hat{c}_1 > 1$ small enough.

We denote $R_i := \hat{c}_1 \operatorname{dist}(x_i, \mathbf{f})$ the radius of the ball $\widehat{B}_i$ and note that, for some $C_B > 1$,

$$(6.7)\qquad \forall i \in \mathbb{N} \quad \forall x \in \widehat{B}_i \qquad C_B^{-1} R_i \leq r_{\mathbf{f}}(x) = r_{\partial\Omega}(x) \leq C_B R_i.$$

We assume (for convenience) that $R_i \leq 1$ for all i.

We apply Lemma 5.3 to the function $D_{\mathbf{f}_\parallel}^{\beta_\parallel} U$ (noting that this function satisfies (3.11) with data $D_{\mathbf{f}_\parallel}^{\beta_\parallel} f$, $D_{\mathbf{f}_\parallel}^{\beta_\parallel} F$) with the pair $(B_i, \widehat{B}_i)$ of concentric balls, with $\mathcal{Y}/2$ instead of $\mathcal{Y}$, and with constant denoted $\gamma_1 \geq 1$. For any $\beta_\parallel = (\beta_{\parallel,1}, \beta_{\parallel,2}) \in \mathbb{N}_0^2$ and $\beta_\perp \in \mathbb{N}_0$, with $p_\parallel = |\beta_\parallel| \in \mathbb{N}$ and $p_\perp = \beta_\perp$, it holds that

$$
\begin{aligned}
&\|r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\|}^{p_\|}\nabla U\|^2_{L^2_\alpha(H^{\mathcal{Y}/4}_{cR})} \\
&\overset{\text{L. 6.2}}{\le} \sum_{i\in\mathbb{N}} \|r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\|}^{\beta_\|}\nabla U\|^2_{L^2_\alpha(B_i^{\mathcal{Y}/4})}\\
&\overset{(6.7)}{\le} \sum_{i\in\mathbb{N}} (C_B R_i)^{2p_\perp+\varepsilon}\|r_{\mathbf{f}}^{-1/2+\varepsilon/2} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\|}^{\beta_\|}\nabla U\|^2_{L^2_\alpha(B_i^{\mathcal{Y}/4})}\\
&\overset{\text{L. 5.3}}{\le} \sum_{i\in\mathbb{N}} (C_BR_i)^{2p_\perp+\varepsilon}(\gamma_1 p_\perp)^{2p_\perp+1} R_i^{-2p_\perp-1}\Big[\|D_{\mathbf{f}_\|}^{\beta_\|}\nabla U\|^2_{L^2_\alpha(\widehat{B}_i^{\mathcal{Y}/2})} \\
&\qquad\qquad + R_i^{s+1}\widetilde{N}^{(p_\perp)}_{\widehat{B}_i^{\mathcal{Y}/2}}(D_{\mathbf{f}_\|}^{\beta_\|} F, D_{\mathbf{f}_\|}^{\beta_\|} f)\Big]\\
&\overset{(6.3)}{\lesssim} C_B^{2p_\perp}(\gamma_1p_\perp)^{2p_\perp+1}\sum_{i\in\mathbb{N}}\Big[\|r_{\mathbf{f}}^{-1/2+\varepsilon/2} D_{\mathbf{f}_\|}^{\beta_\|} \nabla U\|^2_{L^2_\alpha(\widehat B_i^{\mathcal{Y}/2})} + R^{s+\varepsilon}\widetilde N^{(p_\perp)}_{\widehat B_i^{\mathcal{Y}/2}}(D_{\mathbf{f}_\|}^{\beta_\|} F, D_{\mathbf{f}_\|}^{\beta_\|} f)\Big]\\
&\lesssim C_B^{2p_\perp}(\gamma_1p_\perp)^{2p_\perp+1}\Big[\|r_{\mathbf{f}}^{-1/2+\varepsilon/2} D_{\mathbf{f}_\|}^{\beta_\|}\nabla U\|^2_{L^2_\alpha(H^{\mathcal{Y}/2}_{\tilde cR})} + \widetilde N^{(p_\perp)}_{H^{\mathcal{Y}/2}_{\tilde cR}}(D_{\mathbf{f}_\|}^{\beta_\|} F, D_{\mathbf{f}_\|}^{\beta_\|} f)\Big]\\
&\overset{\text{L. 5.2}}{\le} C_B^{2p_\perp}(\gamma_1p_\perp)^{2p_\perp+1}(\gamma_2p_\|)^{2p_\|+1}\Big[R^{-2p_\|-1}\Big(\|\nabla U\|^2_{L^2_\alpha(H_R^{\mathcal{Y}})} + R^{s+1}\widetilde N^{(p_\|)}_{H_R^{\mathcal{Y}}}(F,f)\Big) \\
&\qquad\qquad + \widetilde N^{(p_\perp)}_{H_R^{\mathcal{Y}/2}}(D_{\mathbf{f}_\|}^{\beta_\|} F, D_{\mathbf{f}_\|}^{\beta_\|} f)\Big],
\end{aligned}
$$

where we have used Lemma 5.2 in the last step. ∎

Corollary 6.12. *Let $\mathbf{f} \in \mathcal{F}$ and $\mathcal{Y} > 0$. Let U be the solution of (3.7). Then, there exists $\gamma > 0$ depending only on s, Ω, ζ and $\mathcal{Y}$, and for every $\varepsilon \in (0, 1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $\beta_\| = (\beta_{\|,1}, \beta_{\|,2}) \in \mathbb{N}_0^2$ and $\beta_\perp \in \mathbb{N}_0$, with $p_\| = \beta_{\|,1} + \beta_{\|,2}$, $p_\perp = \beta_\perp$, and $p = p_\| + p_\perp$, it holds that*

$$
(6.8)\qquad \|r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\|}^{\beta_\|}\nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{f}}^{\mathcal{Y}/4})} \le C_\varepsilon \gamma^{2p+1} p^{2p}\widetilde N^{(p)}_{\Omega^{\mathcal{Y}}}(F, f).
$$

Proof. This follows directly from Lemma 6.11 with $R \simeq 1$ and from (4.2). ∎

6.2.5. Vertex-face neighborhoods $\omega_{\mathbf{vf}}$. We have

$$
r_{\mathbf{v}} \sim r_{\mathbf{e}} \qquad \text{and} \qquad r_{\mathbf{f}} \le r_{\mathbf{e}} \qquad \text{on } \omega_{\mathbf{vf}}.
$$

Lemma 6.13 (Weighted H^p-regularity in $\omega_{\mathbf{vf}}$). *Let U be the solution of (3.7). There is $\gamma > 0$ depending only on s, Ω, and $\mathcal{Y}$, and for every $\varepsilon \in (0, 1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $\beta_\| = (\beta_{\|,1}, \beta_{\|,2}) \in \mathbb{N}_0^2$ and $\beta_\perp \in \mathbb{N}_0$, with $p_\| = \beta_{\|,1} + \beta_{\|,2}$, $p_\perp = \beta_\perp$, and $p = p_\| + p_\perp$, it holds that*

$$
(6.9)\qquad \|r_{\mathbf{v}}^{p_\|+\varepsilon} r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\|}^{\beta_\|}\nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{vf}}^{\mathcal{Y}/4})} \le C_\varepsilon \gamma^{2p+1} p^{2p}\widetilde N^{(p)}_{\Omega^{\mathcal{Y}}}(F, f),
$$

where $D_{\mathbf{f}_\|}^{\beta_\|} = D_{\mathbf{f}_{1,\|}}^{\beta_{\|,1}} D_{\mathbf{f}_{2,\|}}^{\beta_{\|,2}}$.

Proof. We use the covering of scaled half-balls $H_i = B_{cR_i}(x_i) \cap \Omega$ with $\widehat{H}_i = B_{R_i}(x_i) \cap \Omega$ given by Lemma 6.4. We have, for some constant $C_{\mathcal{Y}} > 1$,

$$
\forall i \in \mathbb{N} \quad \forall x \in \widehat{H}_i \qquad C_{\mathcal{Y}}^{-1} R_i \le r_{\mathbf{v}}(x) \le C_{\mathcal{Y}} R_i.
$$

Using this and Lemma 6.11, we obtain

$$
\begin{aligned}
&\|r_{\mathbf{v}}^{p_\parallel+\varepsilon} r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{vf}}^{\mathcal{Y}/4})} \\
&\qquad\qquad \leq \sum_{i\in\mathbb{N}} (C_{\mathcal{Y}} R_i)^{2p_\parallel+2\varepsilon} \|r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{f}_\perp}^{\beta_\perp} D_{\mathbf{f}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(H_i^{\mathcal{Y}/4})} \\
&\qquad\qquad \leq \sum_{i\in\mathbb{N}} (C_{\mathcal{Y}} R_i)^{2p_\parallel+2\varepsilon} \gamma_1^{2p+1} p^{2p} \Bigg[R_i^{-2p_\parallel} \left(R_i^{-1} \|\nabla U\|^2_{L^2_\alpha(\widehat{H}_i^{\mathcal{Y}})} + R_i^s \widetilde{N}^{(p_\parallel)}_{\widehat{H}_i^{\mathcal{Y}}}(F,f) \right) \\
&\qquad\qquad\qquad + \widetilde{N}^{(p_\perp)}_{\widehat{H}_i^{\mathcal{Y}/2}} (D_{\mathbf{f}_\parallel}^{\beta_\parallel} F, D_{\mathbf{f}_\parallel}^{\beta_\parallel} f) \Bigg] \\
&\qquad\qquad \leq \gamma^{2p+1} p^{2p} \sum_{i\in\mathbb{N}} \left[\|r_{\mathbf{v}}^{-1/2+\varepsilon} \nabla U\|^2_{L^2_\alpha(\widehat{H}_i^{\mathcal{Y}})} + \widetilde{N}^{(p_\parallel)}_{\widehat{H}_i^{\mathcal{Y}}}(F,f) + \widetilde{N}^{(p_\perp)}_{\widehat{H}_i^{\mathcal{Y}/2}} (D_{\mathbf{f}_\parallel}^{\beta_\parallel} F, D_{\mathbf{f}_\parallel}^{\beta_\parallel} f) \right].
\end{aligned}
$$

The bound $r_{\mathbf{v}}(x) \geq r_{\partial\Omega}(x)$, the finite overlap of the half-balls $\widehat{H}_i$, Lemma 5.1, and (4.2) conclude the proof. ∎

6.2.6. Edge-face neighborhoods $\omega_{\mathbf{ef}}$. We have

$$r_{\mathbf{f}} \leq r_{\mathbf{e}} \qquad \text{on } \omega_{\mathbf{ef}}.$$

We recall the directional coordinates in Def. 2.2.

Lemma 6.14 (Weighted H^p-regularity in $\omega_{\mathbf{ef}}$). *Let U be the solution of (3.7). There is $\gamma > 0$ depending only on s, Ω, and $\mathcal{Y}$, such that for every $\varepsilon \in (0,1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $(\beta_\parallel, \beta_\vDash, \beta_\perp) \in \mathbb{N}_0^3$, $p = \beta_\parallel + \beta_\vDash + \beta_\perp \in \mathbb{N}_0$,*

$$\|r_{\mathbf{e}}^{\beta_\vDash+\varepsilon} r_{\mathbf{f}}^{\beta_\perp-1/2+\varepsilon} D_{\mathbf{g}_\perp}^{\beta_\perp} D_{\mathbf{g}_\vDash}^{\beta_\vDash} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{ef}}^{\mathcal{Y}/8})} \leq C_\varepsilon \gamma^{2p+1} p^{2p} \widetilde{N}^{(p)}_{\Omega^{\mathcal{Y}}}(F,f)\ . \tag{6.10}$$

Proof. We write interchangeably $p_\bullet$ and $\beta_\bullet$, for $\bullet \in \{\vDash, \parallel, \perp\}$. We use the covering of scaled half-balls $H_i = B_{cR_i}(x_i) \cap \Omega$ with $\widehat{H}_i = B_{R_i}(x_i) \cap \Omega$ given by Lemma 6.4. We have, for some constant $C_{\mathcal{Y}} > 1$,

$$\forall i \in \mathbb{N} \quad \forall x \in \widehat{H}_i \qquad C_{\mathcal{Y}}^{-1} R_i \leq r_{\mathbf{e}}(x) \leq C_{\mathcal{Y}} R_i.$$

Applying Lemma 6.11 to the function $D_{\mathbf{g}_\parallel}^{\beta_\parallel} U$, which solves (3.7) with data $D_{\mathbf{g}_\parallel}^{\beta_\parallel} F$, $D_{\mathbf{g}_\parallel}^{\beta_\parallel} f$, and remarking that $\mathbf{g}_\vDash$ is parallel to $\mathbf{f}$,

$$
\begin{aligned}
&\|r_{\mathbf{e}}^{p_\vDash+\varepsilon} r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{g}_\perp}^{\beta_\perp} D_{\mathbf{g}_\vDash}^{\beta_\vDash} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{ef}}^{\mathcal{Y}/8})} \\
&\quad \leq \sum_{i\in\mathbb{N}} (C_{\mathcal{Y}} R_i)^{2p_\vDash+2\varepsilon} \|r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{g}_\perp}^{\beta_\perp} D_{\mathbf{g}_\vDash}^{\beta_\vDash} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(H_i^{\mathcal{Y}/8})} \\
&\quad \leq \sum_{i\in\mathbb{N}} (C_{\mathcal{Y}} R_i)^{2p_\vDash+2\varepsilon} \gamma_1^{2(p_\perp+p_\vDash)+1} (p_\perp+p_\vDash)^{2(p_\perp+p_\vDash)} \Bigg[R_i^{-2p_\vDash} \Bigg(R_i^{-1} \|D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\widehat{H}_i^{\mathcal{Y}/2})} + \\
&\qquad\qquad R_i^s \widetilde{N}^{(p_\vDash)}_{\widehat{H}_i^{\mathcal{Y}/2}} (D_{\mathbf{g}_\parallel}^{\beta_\parallel} F, D_{\mathbf{g}_\parallel}^{\beta_\parallel} f) \Bigg) + \widetilde{N}^{(p_\perp)}_{\widehat{H}_i^{\mathcal{Y}/4}} (D_{\mathbf{g}_\parallel}^{\beta_\parallel} D_{\mathbf{g}_\vDash}^{\beta_\vDash} F, D_{\mathbf{g}_\parallel}^{\beta_\parallel} D_{\mathbf{g}_\vDash}^{\beta_\vDash} f) \Bigg] \\
&\quad \leq \gamma^{2(p_\perp+p_\vDash)+1} (p_\perp+p_\vDash)^{2(p_\perp+p_\vDash)} \sum_{i\in\mathbb{N}} \Bigg[\|r_{\mathbf{e}}^{-1/2+\varepsilon} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\widehat{H}_i^{\mathcal{Y}/2})} + \\
&\qquad\qquad \widetilde{N}^{(p_\vDash+p_\parallel)}_{\widehat{H}_i^{\mathcal{Y}/2}}(F,f) + \widetilde{N}^{(p_\perp)}_{\widehat{H}_i^{\mathcal{Y}/4}} (D_{\mathbf{f}_\parallel}^{\beta_\parallel} F, D_{\mathbf{f}_\parallel}^{\beta_\parallel} f) \Bigg].
\end{aligned}
$$

The bound $r_{\mathbf{e}}(x) \geq r_{\partial\Omega}(x)$ and the finite overlap of the half-balls $\widehat{H}_i$ imply that we can apply Lemma 5.2 to obtain, for a constant $C > 0$ that depends on ξ and on the covering of half-balls,

$$\sum_{i\in\mathbb{N}} \|r_{\mathbf{e}}^{-1/2+\varepsilon} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\widehat{H}_i^{\mathcal{Y}/2})} \leq C R^{-2p_\parallel - 1}(\gamma p_\parallel)^{2p_\parallel}(1+\gamma p_\parallel)\left(\|\nabla U\|^2_{L^2_\alpha(\widetilde{\omega}_{\mathbf{ef}}^{\mathcal{Y}})} + R^{s+1}\widetilde{N}^{(p_\parallel)}_{\widetilde{\omega}_{\mathbf{ef}}^{\mathcal{Y}}}(F,f)\right),$$

where $\widetilde{\omega}_{\mathbf{ef}}^{\mathcal{Y}}$ is a domain that contains the union of the half-balls $\widehat{H}_i$ and where we can choose $R \simeq 1$. Equation (4.2) concludes the proof. □

6.2.7. Vertex-edge-face neighborhoods $\omega_{\mathbf{vef}}$. We have

$$r_{\mathbf{f}} \leq r_{\mathbf{e}} \leq r_{\mathbf{v}} \qquad \text{on } \omega_{\mathbf{vef}}.$$

We recall the directional coordinates in Def. 2.2.

LEMMA 6.15 (Weighted H^p-regularity in $\omega_{\mathbf{vef}}$). *Let U be the solution of (3.7). There is $\gamma > 0$ depending only on s, Ω, and $\mathcal{Y}$, and for every $\varepsilon \in (0, 1/2)$, there exists $C_\varepsilon > 0$ depending additionally on ε such that for all $(\beta_\parallel, \beta_\vDash, \beta_\perp) \in \mathbb{N}_0^3$, $p = \beta_\parallel + \beta_\vDash + \beta_\perp \in \mathbb{N}_0$,*

$$\|r_{\mathbf{v}}^{\beta_\parallel+\varepsilon} r_{\mathbf{e}}^{\beta_\vDash+\varepsilon} r_{\mathbf{f}}^{\beta_\perp-1/2+\varepsilon} D_{\mathbf{g}_\perp}^{\beta_\perp} D_{\mathbf{g}_\vDash}^{\beta_\vDash} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{vef}}^{\mathcal{Y}/8})} \leq C_\varepsilon \gamma^{2p+1} p^{2p} \widetilde{N}^{(p)}_{\Omega^{\mathcal{Y}}}(F,f)\,. \tag{6.11}$$

Proof. We write interchangeably $p_\bullet$ and $\beta_\bullet$, for $\bullet \in \{\vDash, \parallel, \perp\}$. We use the covering of wedges W_i, $\widehat{W}_i$ given by Lemma 6.6. We have, for some constant $C_W > 1$,

$$\forall i \in \mathbb{N} \quad \forall x \in \widehat{W}_i \qquad C_W^{-1} R_i \leq r_{\mathbf{v}}(x) \leq C_W R_i.$$

The arguments of Lemma 6.3 give a covering $\bigcup_j H_j \supset W_i$ with half-balls $H_j = B_{c_1 r_{\mathbf{f}}(x_j)}(x_j) \cap \Omega$, $x_j \in \mathbf{f}$ and scaled half-balls $\widehat{H}_j = B_{\hat{c}_1 r_{\mathbf{f}}(x_j)}(x_j) \cap \Omega$ such that $\bigcup_j \widehat{H}_j \subset \widehat{W}_i$, provided one chooses the parameters $c_1, \hat{c}_1 > 1$ small enough.

Consequently, as in the proof of Lemma 6.14, we have

$$\begin{aligned}
\|r_{\mathbf{e}}^{p_\vDash+\varepsilon} r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{g}_\perp}^{\beta_\perp} D_{\mathbf{g}_\vDash}^{\beta_\vDash} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(W_i^{\mathcal{Y}/8})} &\lesssim (\gamma_1 p)^{2p+1}\Bigg[R_i^{-2p_\parallel-1}\Bigg(\|\nabla U\|^2_{L^2_\alpha(\widehat{W}_i^{\mathcal{Y}})} \\
&\quad + R_i^{s+1}\widetilde{N}^{(p_\parallel)}_{\widehat{W}_i^{\mathcal{Y}}}(F,f)\Bigg) + \widetilde{N}^{(p_\vDash+p_\parallel)}_{\widehat{W}_i^{\mathcal{Y}}}(F,f) + \widetilde{N}^{(p_\perp)}_{\widehat{W}_i^{\mathcal{Y}/2}}(D_{\mathbf{f}_\parallel}^{\beta_\parallel}F, D_{\mathbf{f}_\parallel}^{\beta_\parallel}f)\Bigg].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|r_{\mathbf{v}}^{p_\parallel+\varepsilon} r_{\mathbf{e}}^{p_\vDash+\varepsilon} r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{g}_\perp}^{\beta_\perp} D_{\mathbf{g}_\vDash}^{\beta_\vDash} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(\omega_{\mathbf{vef}}^{\mathcal{Y}/8})} \\
&\qquad \leq \sum_{i\in\mathbb{N}} (C_W R_i)^{2p_\parallel+2\varepsilon} \|r_{\mathbf{e}}^{p_\vDash+\varepsilon} r_{\mathbf{f}}^{p_\perp-1/2+\varepsilon} D_{\mathbf{g}_\perp}^{\beta_\perp} D_{\mathbf{g}_\vDash}^{\beta_\vDash} D_{\mathbf{g}_\parallel}^{\beta_\parallel} \nabla U\|^2_{L^2_\alpha(W_i^{\mathcal{Y}/8})} \\
&\qquad \lesssim (\gamma p)^{2p+1} \sum_{i\in\mathbb{N}} \Bigg[\|r_{\mathbf{v}}^{-1/2+\varepsilon} \nabla U\|^2_{L^2_\alpha(\widehat{W}_i^{\mathcal{Y}})} \\
&\qquad\quad + R_i^{s+\varepsilon}\widetilde{N}^{p_\parallel}_{\widehat{W}_i^{\mathcal{Y}}}(F,f) + \widetilde{N}^{(p_\vDash+p_\parallel)}_{\widehat{W}_i^{\mathcal{Y}}}(F,f) + \widetilde{N}^{(p_\perp)}_{\widehat{W}_i^{\mathcal{Y}/2}}(D_{\mathbf{f}_\parallel}^{\beta_\parallel}F, D_{\mathbf{f}_\parallel}^{\beta_\parallel}f)\Bigg].
\end{aligned}$$

The finite overlap of the wedges $\widehat{W}_i$, Lemma 5.1, and equation (4.2) conclude the proof. □

6.2.8. Unified weighted analytic regularity bounds for U. We unify the bounds in all neighborhoods in the following statement.

PROPOSITION 6.16. *Let $\omega \subset \Omega$ be any set whose boundary intersect at most one $\mathbf{v} \in \mathcal{V}$, one $\mathbf{e} \in \mathcal{E}$, and one $\mathbf{f} \in \mathcal{F}$. Let $(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\|)$ be linearly independent unit vectors as in Def. 2.2. Then, there exists $\gamma > 0$ such that for all $t < 1/2$, there exists $C_t > 0$ such that for all $\beta = (\beta_\perp, \beta_\vDash, \beta_\|) \in \mathbb{N}_0^3$ with $\beta_{\mathbf{e}_\perp} = (\beta_\perp, \beta_\vDash)$,*

$$\|r_{\partial\Omega}^{-t} r_{\mathbf{v}}^{|\beta|} \rho_{\mathbf{ve}}^{|\beta_{\mathbf{e}_\perp}|} \rho_{\mathbf{ef}}^{\beta_\perp} D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\|)} \nabla U\|_{L^2_\alpha(\omega^{\mathcal{Y}/4})} \le C_t \gamma^{2|\beta|+1} |\beta|^{2|\beta|} \widetilde{N}^{(|\beta|)}_{\Omega\mathcal{Y}}(F, f).$$

6.3. H^p-regularity for the solution u in the polyhedron Ω. The preceding analytic regularity bounds on the solution U of the CS extension (3.11) imply corresponding weighted, analytic regularity on the weak solution u of the integral fractional Laplacian in the polyhedron Ω ie. (2.3) via (3.5b). Quantitative control of u in terms of U is achieved via the multiplicative trace estimate given in the next lemma.

LEMMA 6.17. *Let $\mathcal{Y} > 0$. There exists $C_{\mathrm{tr},\mathcal{Y}} > 0$ such that, for all $V : \Omega \times (0, \mathcal{Y}) \to \mathbb{R}$ with $V(x, \cdot) \in H^1_\alpha((0, \mathcal{Y}))$ for all $x \in \Omega$, it holds that*

$$|V(x,0)|^2 \le C_{\mathrm{tr},\mathcal{Y}} \left(\|V(x,\cdot)\|^{1-\alpha}_{L^2_\alpha((0,\mathcal{Y}))} \|\partial_y V(x,\cdot)\|^{1+\alpha}_{L^2_\alpha((0,\mathcal{Y}))} + \|V(x,\cdot)\|^2_{L^2_\alpha((0,\mathcal{Y}))} \right), \tag{6.12}$$

where, for a function $v : \mathbb{R}_+ \to \mathbb{R}$, we write $\|v\|^2_{L^2_\alpha((0,\mathcal{Y}))} := \int_0^{\mathcal{Y}} y^\alpha |v(y)|^2 \, dy$.

Proof. From the proof of [KM19, Lem. 3.7], we have, for all $W(x, \cdot) \in H^1_\alpha(\mathbb{R}_+)$,

$$|W(x,0)|^2 \le C_{\mathrm{tr}} \left(\|W(x,\cdot)\|^{1-\alpha}_{L^2_\alpha(\mathbb{R}_+)} \|\partial_y W(x,\cdot)\|^{1+\alpha}_{L^2_\alpha(\mathbb{R}_+)} + \|W(x,\cdot)\|^2_{L^2_\alpha(\mathbb{R}_+)} \right). \tag{6.13}$$

Let then $\eta \in C_0^\infty(-\mathcal{Y}, \mathcal{Y})$ with $\eta(0) = 1$ and $\|\eta\|_{L^\infty(\mathbb{R})} + \|\eta'\|_{L^\infty(\mathbb{R})} \le C_\eta$. Choose $W = \eta V$ in (6.13). We obtain

$$\begin{aligned}
|V(x,0)|^2 &= |(\eta V)(x,0)|^2 \\
&\le C_{\mathrm{tr}} \left(\|(\eta V)(x,\cdot)\|^{1-\alpha}_{L^2_\alpha(\mathbb{R}_+)} \|(\partial_y(\eta V))(x,\cdot)\|^{1+\alpha}_{L^2_\alpha(\mathbb{R}_+)} + \|(\eta V)(x,\cdot)\|^2_{L^2_\alpha(\mathbb{R}_+)} \right) \\
&\le C_{\mathrm{tr}} C_\eta^2 \left(2\|V(x,\cdot)\|^{1-\alpha}_{L^2_\alpha((0,\mathcal{Y}))} \|(\partial_y V)(x,\cdot)\|^{1+\alpha}_{L^2_\alpha((0,\mathcal{Y}))} + 3\|V(x,\cdot)\|^2_{L^2_\alpha((0,\mathcal{Y}))} \right),
\end{aligned}$$

where we have also used that $(a+b)^{1+\alpha} \le 2(a^{1+\alpha} + b^{1+\alpha})$ for all $\alpha \in (-1, 1)$ and all non negative a, b. ∎

Proof of Thm. 2.3. Assume $|\beta| \ge 1$. Using $V = D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\|)} U$ in (6.12) together with multiplication by $r_{\partial\Omega}^{-2t-2s} r_{\mathbf{v}}^{2|\beta|} \rho_{\mathbf{ve}}^{2|\beta_{\mathbf{e}_\perp}|} \rho_{\mathbf{ef}}^{2\beta_\perp}$ and integration over ω leads to

$$\begin{aligned}
&\left\| r_{\partial\Omega}^{-t-s} r_{\mathbf{v}}^{\beta_\|} r_{\mathbf{e}}^{\beta_\vDash} r_{\mathbf{f}}^{\beta_\perp} D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\|)} u \right\|^2_{L^2(\omega)} \\
&\quad \le C_{\mathrm{tr},\mathcal{Y}} \left\| r_{\partial\Omega}^{-t-1} r_{\mathbf{v}}^{\beta_\|} r_{\mathbf{e}}^{\beta_\vDash} r_{\mathbf{f}}^{\beta_\perp} D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\|)} U \right\|^{1-\alpha}_{L^2_\alpha(\omega^{\mathcal{Y}/4})} \left\| r_{\partial\Omega}^{-t} r_{\mathbf{v}}^{\beta_\|} r_{\mathbf{e}}^{\beta_\vDash} r_{\mathbf{f}}^{\beta_\perp} D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\|)} \nabla U \right\|^{1+\alpha}_{L^2_\alpha(\omega^{\mathcal{Y}/4})} \\
&\qquad + C_{\mathrm{tr},\mathcal{Y}} \left\| r_{\partial\Omega}^{-t-s} r_{\mathbf{v}}^{\beta_\|} r_{\mathbf{e}}^{\beta_\vDash} r_{\mathbf{f}}^{\beta_\perp} D^\beta_{(\mathbf{g}_\perp, \mathbf{g}_\vDash, \mathbf{g}_\|)} U \right\|^2_{L^2_\alpha(\omega^{\mathcal{Y}/4})}.
\end{aligned}$$

On each neighborhood ω, it either holds that $r_{\partial\Omega} \simeq r_{\mathbf{v}}$ (when $\partial\omega$ does not intersect with any face or edge of the boundary), $r_{\partial\Omega} \simeq r_{\mathbf{e}}$ (when $\partial\omega$ intersects with an edge but no face of the boundary), or $r_{\partial\Omega} = r_{\mathbf{f}}$. Consequently, as $|\beta| \geq 1$, there is a suitable $\widetilde{\beta} \in \mathbb{N}_0^3$ with $|\widetilde{\beta}| = |\beta| - 1 \geq 0$ such that

$$\left\| r_{\partial\Omega}^{-t-1} r_{\mathbf{v}}^{\beta_{\|}} r_{\mathbf{e}}^{\beta_{\vDash}} r_{\mathbf{f}}^{\beta_{\perp}} D^{\beta}_{(\mathbf{g}_{\perp}, \mathbf{g}_{\vDash}, \mathbf{g}_{\|})} U \right\|_{L^2_\alpha(\omega^{\mathcal{Y}/4})} \leq \left\| r_{\partial\Omega}^{-t} r_{\mathbf{v}}^{\widetilde{\beta}_{\|}} r_{\mathbf{e}}^{\widetilde{\beta}_{\vDash}} r_{\mathbf{f}}^{\widetilde{\beta}_{\perp}} D^{\widetilde{\beta}}_{(\mathbf{g}_{\perp}, \mathbf{g}_{\vDash}, \mathbf{g}_{\|})} \nabla U \right\|_{L^2_\alpha(\omega^{\mathcal{Y}/4})}.$$

Now, the statement follows from Proposition 6.16.

The case $|\beta| = 0$ essentially follows from a 1D weighted Hardy inequality similarly as in [FMMS22]. Here, we illustrate the argument for the vertex-edge-face case $\omega = \omega_{\mathbf{vef}}$, noting that the remaining cases correspond verbatim to discussions in [FMMS22].

We use the coordinates $\{\mathbf{g}_{\|}, \mathbf{g}_{\vDash}, \mathbf{g}_{\perp}\}$ introduced in Definition 2.1 and – by rotation and translation – assume that the local orthogonal coordinate system coincides with the canonical coordinates in $\mathbb{R}^3$. We introduce the equivalent vertex-edge-face neighborhood

$$\widetilde{\omega}^{\mu,\xi}_{\mathbf{vef}} := \{x \in \Omega : x_1 \in (0, \mu), x_2 \in (0, \xi x_1), x_3 \in (0, \xi x_2)\}$$

and drop the superscripts in the following. We denote by $\widetilde{u}$ the function u in the coordinate system in $\widetilde{\omega}_{\mathbf{vef}}$. We remark that there exists $c \geq 1$ such that in $\widetilde{\omega}_{\mathbf{vef}}$ holds

$$x_1 \leq r_{\mathbf{v}}(x) \leq c x_1, \qquad x_2 \leq r_{\mathbf{e}}(x) \leq c x_2 \tag{6.14}$$

and we observe also $r_{\mathbf{f}}(x) = x_3 = r_{\partial\Omega}(x)$. Hence, for almost all $x_1 \in (0, \mu)$ and $x_2 \in (0, \xi x_1)$, it holds that

$$\left(x_3 \mapsto r_{\mathbf{f}}^{1-t-s} (D_{\mathbf{g}_{\perp}} \widetilde{u})(x) \right) \in L^2((0, \xi x_2)). \tag{6.15}$$

Now, the fundamental theorem of calculus, the Cauchy-Schwarz inequality, and (6.15) imply Hölder continuity of $\widetilde{u}(x_1, x_2, \cdot)$ for almost all x_1, x_2. As $u \in \widetilde{H}^s(\Omega)$, we can therefore employ the Hardy inequality of [KMR97, Lem. 7.1.3], which gives

$$\| r_{\mathbf{f}}^{-t-s} \widetilde{u}(x_1, x_2, \cdot) \|_{L^2((0,\xi x_2))} \leq C \| r_{\mathbf{f}}^{1-t-s} (D_{\mathbf{g}_{\perp}} \widetilde{u})(x_1, x_2, \cdot) \|_{L^2((0,\xi x_2))},$$

with a constant C independent of x_1, x_2. Squaring, integrating in turn over $x_2 \in (0, \xi x_1)$ and $x_1 \in (0, \mu)$, and using (6.14), we obtain

$$\| r_{\partial\Omega}^{-t-s} \widetilde{u} \|_{L^2(\widetilde{\omega}_{\mathbf{vef}})} = \| r_{\mathbf{f}}^{-t-s} \widetilde{u} \|_{L^2(\widetilde{\omega}_{\mathbf{vef}})} \leq C \| r_{\partial\Omega}^{-t-s} r_{\mathbf{f}} D_{\mathbf{g}_{\perp}} \widetilde{u} \|_{L^2(\widetilde{\omega}_{\mathbf{vef}})}.$$

The term in the right-hand side of the above inequality has been bounded in the first part of this proof; this completes the proof except for the fact that the region $\omega_{\mathbf{vef}} \setminus \widetilde{\omega}_{\mathbf{vef}}$ is not covered yet. This region can be treated with modifying the parameter ξ, exactly as in [FMMS22, Rem. 5.8].□

7. Conclusion. For the Dirichlet integral fractional Laplacian $(-\Delta)^s$ in a bounded, polytopal domain $\Omega \subset \mathbb{R}^3$, subject to a source term f which is analytic in $\overline{\Omega}$, we proved weighted, analytic regularity of weak solutions. The analysis and the result extends the theory in polygons $\Omega \subset \mathbb{R}^2$, developed in our previous work [FMMS22], to dimension $d = 3$.

As is well known from the numerical analysis of Galerkin approximations of solutions for elliptic PDEs, weighted Sobolev regularity of solutions has direct consequences for the approximation rate theory of numerical methods: boundary weighted Sobolev regularity and Besov regularity has recently been used to investigate the convergence rates of first order Galerkin FE discretizations on boundary-graded, shape-regular meshes in [BN23b]. The (boundary- and corner-) weighted analytic regularity proved in [FMMS22] is the basis of *exponential convergence rate bounds* for *hp*-FEM in space dimensions $d = 1, 2$ [BFM+23, FMMS23].

Directions for natural extensions of the present results in three space dimensions suggest themselves: first, the presently developed proof and the geometric structure of the weights in Ω should facilitate analogous weighted analytic regularity results for integral fractional diffusion such as $(-\nabla \cdot A(x)\nabla)^s$, with an anisotropic diffusion coefficient $A(\cdot)$ being a uniformly positive definite $d \times d$ matrix, again with analytic in $\overline{\Omega}$ entries. Likewise, the exponential convergence rate bound established in [FMMS23] in the two-dimensional setting will generalize to the presently considered, polyhedral setting, albeit with rate given by $C\exp(-bN^{1/6})$, with N denoting the number of the degrees of freedom of the hp-FE subspace, and with constants $b, C > 0$ depending on Ω, f but not on N. Here, the larger number of geometric situations for ≥ 3 edges meeting in one, common vertex of $\partial\Omega$ will mandate significant extensions and additional technical issues as compared to the proof in [FMMS23]. Details will be developed elsewhere.

Appendix A. Localization of fractional norms. The following lemma is a slightly improved version of [FMMS22, Lemma A.1]

Lemma A.1. *Let $R > 0$ such that $B_R \subset \Omega$, $c \in (0,1)$, $\eta \in C_0^\infty(B_{cR})$, and $s \in (0,1)$. Then,*

$$\|\eta f\|_{H^{-s}(\Omega)} \leq C_{\mathrm{loc}} \|\eta\|_{L^\infty(B_{cR})} \|f\|_{L^2(B_{cR})}, \tag{A.1}$$

$$\|\eta f\|_{H^{1-s}(\Omega)} \leq C_{\mathrm{loc},2}\Big[\big(R^s \|\nabla\eta\|_{L^\infty(B_{cR})} + (R^{s-1}+1)\|\eta\|_{L^\infty(B_{cR})}\big) \|f\|_{L^2(B_R)} + \|\eta\|_{L^\infty(B_{cR})} |f|_{H^{1-s}(B_R)}\Big], \tag{A.2}$$

where C_{loc} depends only on Ω and s, and $C_{\mathrm{loc},2}$ depends additionally on c.

Proof. (A.1) follows directly from the embedding $L^2 \subset H^{-s}$. For (A.2), we start from the definition of the Slobodecki semi-norm

$$|\eta f|^2_{H^{1-s}(\Omega)} = \int_\Omega \int_\Omega \frac{|\eta(x)f(x) - \eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx$$

We denote the intermediate radius between R and cR as $\widetilde{R} = \frac{1+c}{2}R$ and write $\tilde{c} = \frac{1-c}{2}$ so that $R - \widetilde{R} = \widetilde{R} - cR = \tilde{c}R$. We split the integration over $\Omega \times \Omega$ into four subsets,

- $B_{\widetilde{R}} \times B_R$,
- $B_{\widetilde{R}} \times B^c_R \cap \Omega$,
- $B^c_{\widetilde{R}} \cap \Omega \times B_{cR}$,
- $B^c_{\widetilde{R}} \cap \Omega \times B^c_{cR} \cap \Omega$.

For the last case, i.e., for all $(x,z) \in B^c_{\widetilde{R}} \cap \Omega \times B^c_{cR} \cap \Omega$, we have that $\eta(x) = \eta(z) = 0$ and the integral is zero. Then, for all $(x,z) \in B_{\widetilde{R}} \times B^c_R \cap \Omega$, we have $|x-z| \geq \tilde{c}R$. Hence, using polar coordinates centered at x,

$$\begin{aligned}
\int_{B_{\widetilde{R}}} \int_{B^c_R \cap \Omega} \frac{|\eta(x)f(x) - \eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx &= \int_{B_{\widetilde{R}}} \int_{B^c_R \cap \Omega} \frac{|\eta(x)f(x)|^2}{|x-z|^{d+2-2s}}\, dz\, dx \\
&\leq \int_{B_{\widetilde{R}}} |\eta(x)f(x)|^2 \int_{B^c_R} \frac{1}{|x-z|^{d+2-2s}} dz\, dx \lesssim \int_{B_{\widetilde{R}}} |\eta(x)f(x)|^2 \int_{\tilde{c}R}^\infty r^{-3+2s} dr\, dx \\
&\lesssim (\tilde{c}R)^{-2+2s} \|\eta\|^2_{L^\infty(B_{cR})} \int_{B_{\widetilde{R}}} |f(x)|^2\, dx \lesssim R^{-2+2s} \|\eta\|^2_{L^\infty(B_{cR})} \|f\|^2_{L^2(B_{\widetilde{R}})}.
\end{aligned}$$

For the integration over $B^c_{\widetilde{R}} \cap \Omega \times B_{cR}$, we write using polar coordinates (centered at z)

$$\int_{B^c_{\widetilde{R}} \cap \Omega} \int_{B_{cR}} \frac{|\eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\, dz\, dx = \int_{B_{cR}} |\eta(z)f(z)|^2 \int_{B^c_{\widetilde{R}} \cap \Omega} \frac{1}{|x-z|^{d+2-2s}}\, dx\, dz$$

$$\lesssim \int_{B_{cR}} |\eta(z)f(z)|^2 \int_{\tilde{c}R}^{\infty} \frac{1}{r^{3-2s}}\,dr\,dz \lesssim R^{2s-2}\,\|\eta\|^2_{L^\infty(B_{cR})}\,\|f\|^2_{L^2(B_{cR})}\,.$$

Finally, for the integration over $B_{\widetilde{R}} \times B_R$, we use the triangle inequality

$$\int_{B_{\widetilde{R}}}\int_{B_R} \frac{|\eta(x)f(x)-\eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\,dz\,dx$$
$$\lesssim \int_{B_{\widetilde{R}}}\int_{B_R} \frac{|\eta(x)f(x)-\eta(x)f(z)|^2}{|x-z|^{d+2-2s}}\,dz\,dx + \int_{B_{\widetilde{R}}}\int_{B_R} \frac{|\eta(x)f(z)-\eta(z)f(z)|^2}{|x-z|^{d+2-2s}}\,dz\,dx$$
$$=: (I) + (II)$$

We have

$$(I) \le \|\eta\|_{L^\infty(B_{cR})} \int_{B_{\widetilde{R}}}\int_{B_R} \frac{|f(x)-f(z)|^2}{|x-z|^{d+2-2s}}\,dz\,dx \le \|\eta\|_{L^\infty(B_{cR})}|f|_{H^{1-s}(B_R)}.$$

Since $|\eta(x)-\eta(z)| \le \|\nabla\eta\|_{L^\infty(B_{cR})}\,|x-z|$ and using polar coordinates (centered at z) we estimate

$$(II) \le \|\nabla\eta\|^2_{L^\infty(B_{cR})} \int_{B_R} |f(z)|^2 \int_{B_{\widetilde{R}}} \frac{1}{|x-z|^{d-2s}}\,dx\,dz$$
$$\lesssim \|\nabla\eta\|^2_{L^\infty(B_{cR})} \int_{B_R} |f(z)|^2 \int_0^{2R} r^{-1+2s}\,dr\,dz \lesssim \|\nabla\eta\|^2_{L^\infty(B_{cR})}\,\|f\|^2_{L^2(B_R)}\,R^{2s}.$$

The straightforward bound $\|\eta f\|_{L^2(\Omega)} \le \|\eta\|_{L^\infty(B_{cR})}\|f\|_{L^2(B_{cR})}$ concludes the proof. ∎

REFERENCES

[AB17] G. Acosta and J.P. Borthagaray, *A fractional Laplace equation: regularity of solutions and finite element approximations*, SIAM J. Numer. Anal. **55** (2017), no. 2, 472–495. MR 3620141

[BFM+23] B. Bahr, M. Faustmann, C. Marcati, J.M. Melenk, and Ch. Schwab, *Exponential convergence of hp−FEM for the integral fractional Laplacian in 1D*, Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2020+1 (J.M. Melenk, I. Perugia, J. Schöberl, and C. Schwab, eds.), Lecture Notes in Computational Science and Engineering, vol. 137, Springer Verlag, 2023, pp. 291–306.

[BG88] I. Babuška and B.Q. Guo, *Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order*, SIAM J. Math. Anal. **19** (1988), no. 1, 172–203. MR 924554

[BLN22] Juan Pablo Borthagaray, Wenbo Li, and Ricardo H. Nochetto, *Fractional Elliptic Problems on Lipschitz Domains: Regularity and Approximation*, 2022.

[BN23a] Juan Pablo Borthagaray and Ricardo H. Nochetto, *Besov regularity for the Dirichlet integral fractional Laplacian in Lipschitz domains*, J. Funct. Anal. **284** (2023), no. 6, Paper No. 109829, 33. MR 4530901

[BN23b] ———, *Constructive approximation on graded meshes for the integral fractional Laplacian*, Constr. Approx. **57** (2023), no. 2, 463–487. MR 4577390

[CS07] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260.

[CS16] L.A. Caffarelli and P.R. Stinga, *Fractional elliptic equations, Caccioppoli estimates and regularity*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 3, 767–807. MR 3489634

[Dau88] Monique Dauge, *Elliptic boundary value problems on corner domains*, Lecture Notes in Mathematics, vol. 1341, Springer-Verlag, Berlin, 1988, Smoothness and asymptotics of solutions. MR 961439

[DL54] J. Deny and J. L. Lions, *Les espaces du type de Beppo Levi*, Ann. Inst. Fourier (Grenoble) **5** (1954), 305–370. MR 74787

[Eva98] L.C. Evans, *Partial differential equations*, American Mathematical Society, 1998.

[FMMS22] M. Faustmann, C. Marcati, J.M. Melenk, and Ch. Schwab, *Weighted analytic regularity for the integral fractional Laplacian in polygons*, SIAM J. Math. Anal. **54** (2022), no. 6, 6323–6357.

[FMMS23] M. Faustmann, C. Marcati, J. Melenk, and Ch. Schwab, *Exponential Convergence of hp-FEM for the Integral Fractional Laplacian in Polygons*, SIAM Journ. Numer. Analysis (to appear) (2023).

[GB97] B. Guo and I. Babuška, *Regularity of the solutions for elliptic problems on nonsmooth domains in $\mathbf{R}^3$. II. Regularity in neighbourhoods of edges*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), no. 3, 517–545. MR 1453280

[Gri11] P. Grisvard, *Elliptic problems in nonsmooth domains*, Classics in Applied Mathematics, vol. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[HMW13] T. Horger, J.M. Melenk, and B. Wohlmuth, *On optimal L^2- and surface flux convergence in FEM*, Comput. Vis. Sci. **16** (2013), no. 5, 231–246.

[KM19] M. Karkulik and J.M. Melenk, *$\mathcal{H}$-matrix approximability of inverses of discretizations of the fractional Laplacian*, Adv. Comput. Math. **45** (2019), no. 5-6, 2893–2919.

[KMR97] V.A. Kozlov, V.G. Maz'ya, and J. Rossmann, *Elliptic boundary value problems in domains with point singularities*, Mathematical Surveys and Monographs, vol. 52, American Mathematical Society, Providence, RI, 1997.

[Kwa17] M. Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fract. Calc. Appl. Anal. **20** (2017), no. 1, 7–51.

[MR10] V. Maz'ya and J. Rossmann, *Elliptic equations in polyhedral domains*, Mathematical Surveys and Monographs, vol. 162, American Mathematical Society, Providence, RI, 2010. MR 2641539

[MW12] J.M. Melenk and B. Wohlmuth, *Quasi-optimal approximation of surface based Lagrange multipliers in finite element methods*, SIAM J. Numer. Anal. **50** (2012), no. 4, 2064–2087. MR 3022210

[Sav98] G. Savaré, *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal. **152** (1998), no. 1, 176–201.