A Stable and Jump-Aware Projection onto a Discrete Multi-Trace Space

M. Averseng

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Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

A STABLE AND JUMP-AWARE PROJECTION ONTO A DISCRETE MULTI-TRACE SPACE.

MARTIN AVERSENG*

Abstract. This work is concerned with boundary element methods on singular geometries, specifically, those falling in the framework of "multi-screens" by Claeys and Hiptmair. We construct a stable quasi-interpolant which preserves piecewise linear jumps on the multi-trace space. This operator is the boundary element analog of the Scott-Zhang quasi-interpolant used in the analysis of finite-element methods. More precisely, let Γ be a multi-screen resolved by a triangulation $(\mathcal{M}_{\Gamma,h})$, and let $\mathbb{V}_h(\Gamma)$ be the space of continuous piecewise-linear multi-traces on Γ. We construct a linear operator $\Pi_h : \mathbb{H}^{1/2}(\Gamma) \to \mathbb{V}_h(\Gamma)$ with the following properties: (i) $\|\Pi_h u\|_{\mathbb{H}^{1/2}} \leq C_h \|u\|_{\mathbb{H}^{1/2}(\Gamma)}$ for all $u \in \mathbb{H}^{1/2}(\Gamma)$, (ii) $\Pi_h u_h = u_h$ for $u_h \in \mathbb{V}_h(\Gamma)$ and, (iii) $[\Pi_h u] = 0$ for every single trace $u \in H^{1/2}([\Gamma])$. The stability constant C_h only depends on the aspect ratio of the elements of $\mathcal{M}_{\Omega,h}$, where $\mathcal{M}_{\Omega,h}$ is a tetrahedral mesh of Ω extending $\mathcal{M}_{\Gamma,h}$. We deduce uniform bounds for the stability of the discrete jump lifting, and the equivalence of the $\widetilde{H}^{1/2}$ norm with a discrete quotient norm.

Key words. Boundary Element Methods, Singularities, Interpolation

MSC codes. 65N12, 65N38

1. Introduction.

The motivation for this work is the numerical analysis of boundary element methods for applications involving geometric singularities. Integral equations, and their resolution by the boundary element methods, are by now well developed on Lipschitz domains, see e.g. [27, 31]. In the past 30 years, a lot of effort has been dedicated to extend the range of geometries that one may tackle, both theoretically and numerically. Initially "screens" and "cracks" were considered, [10, 35, 37] and more complex geometries were studied in recent works [12, 16, 17]. For a selection of real-life studies using such complex geometric models in various types of applications, see [2, 13, 21, 23, 26, 33, 38] and references therein.

The numerical analysis of boundary element methods in such singular geometries involves at least three main challenges, compared to the case of a Lipschitz regular obstacle. First, at the theory level, one has to give a suitable definition for the function spaces in which the boundary integral operators naturally act. For instance, when the obstacle Γ is an infinitely thin screen, the spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are no longer dual to each other, and the weakly singular operator (resp. hypersingular operator) act in $\widetilde{H}^{-1/2}(\Gamma)$ (resp. $\widetilde{H}^{1/2}(\Gamma)$). We refer to [11] for a survey on Sobolev spaces on "rough" (possibly fractal) sets. The second challenge, which is connected to the first one, is that second-kind formulations are difficult to design for singular obstacles, and, when it comes to first-kind integral equations, the known preconditioning methods, with Calderón preconditioning as a prominent example [14, 34], must be adapted to take the singularity into account [3, 4, 8, 9, 25, 29]. Finally, for the a priori analysis and convergence theory, one must develop a good understanding of the singularity of the solutions of the scattering problems (see e.g. [20, 24]). This knowledge is important in order to choose the proper mesh refinement method (e.g. h-p refinement [7, 36]), and to quantify the order of convergence of the numerical solutions.

In this work, we make a step in addressing those challenges for a class of singular geometric models called "multi-screens" [16, 17]. Essentially, multi-screens are arrangements of two-dimensional surfaces in $\mathbb{R}^3$, which may intersect each other in

*Department of Mathematical Sciences, University of Bath, BATH BA2 7AY, UK. (ma3092@bath.ac.uk, https://people.bath.ac.uk/ma3092)

complex ways. In particular, they combine two types of singularities: sharp edges (the boundary of the screen) and junction lines and points. Because of the latter, a multi-screen may fail to be a manifold at some locations. For such geometries, an adapted functional framework has been recently established by Claeys and Hiptmair [16, 17], and was applied with a lot of success in the context of domain decomposition methods [18]. In parallel, the first numerical implementation of a first-kind boundary element method for multi-screens has recently appeared in [15], and ideas for corresponding preconditioners are emerging [5, 19].

The aim of this paper is to construct a stable quasi-interpolant which preserves piecewise linear jumps on the multi-trace space. The properties of the operator that we construct are completely analogous to those of the celebrated Scott-Zhang quasi-interpolant of [32] (see also the recent work [22] on a related interpolant for discrete differential forms). The Scott-Zhang operator has a tremendous importance for the numerical analysis of strongly elliptic equations in Lipschitz domains. Its main use is to analyze the approximation of functions by piecewise polynomials (see [1, Thm 1.1]). It can also be used to derive uniform bounds on discrete jump liftings, see e.g. [28, Lemma 1.56], which are commonly used in domain decomposition methods. The operator we construct here (restricted to piecewise linear functions, and not general polynomial order as in [32]) similarly implies uniform bounds for a discrete jump lifting on the jump space $\widetilde{H}^{1/2}([\Gamma])$. Its properties are used to analyze some new preconditioners for the boundary element methods on multi-screens in [5, 19]. We also expect that it will be useful for the a priori analysis of the convergence of the boundary element solution to the true solution.

The remainder of this work is organized as follows. In Section 2, we introduce the necessary notation to state our main result. The construction of the quasi-interpolant involves a (primal) basis of the discrete multi-trace space, which is introduced in Section 4. This allows to give the definition of the quasi-interpolant Π_h and prove its properties in Section 5. A central role in those proofs is played by a set of "dual" basis functions, whose construction is presented in Section 6. This construction is the main novelty of this work.

2. Notations and main result.

Simplices and meshes. An *n-simplex* ($n = 2$ for a triangle or 3 for a tetrahedron) is the closed convex hull of $n+1$ affinely independent points in $\mathbb{R}^3$ called its vertices. A *face* of a simplex S is a $(n-1)$-simplex spanned by n vertices of S.

A n-dimensional *mesh* $\mathcal{M}$ is a finite set of n-simplices such that if $K, K' \in \mathcal{M}$, then the intersection $K \cap K'$ is either empty, or equal to a common subsimplex (i.e. a vertex, edge, or face) of both K and K'. The set of faces of a mesh $\mathcal{M}$, denoted by $\mathcal{F}(\mathcal{M})$, is the set of faces of the simplices of $\mathcal{M}$. The boundary $\partial\mathcal{M}$ of an n-dimensional mesh $\mathcal{M}$ is defined as the subset of $\mathcal{F}(\mathcal{M})$ whose elements are the face of exactly one n-simplex in $\mathcal{M}$. The *geometry* of $\mathcal{M}$, denoted by $|\mathcal{M}|$, is the union of all of its elements, i.e.

$$|\mathcal{M}| = \bigcup_{K \in \mathcal{M}} K\,.$$

Given $\gamma > 0$, we say that a mesh $\mathcal{M}$ is *γ-shape-regular* if it satisfies

$$\frac{h_K}{\rho_K} \geq \gamma\,, \quad \forall K \in \mathcal{M}\,,$$

where h_K is the diameter of K and ρ_K is the radius of the largest ball contained in K. A mesh $\mathcal{M}$ is *regular* if its geometry is an n-dimensional (piecewise linear) manifold.

In what follows, we fix a Lipschitz polyhedron $\Omega \subset \mathbb{R}^3$, that is, a connected open set such that there holds $\overline{\Omega} = |\mathcal{M}_\Omega|$, for some regular tetrahedral mesh $\mathcal{M}_\Omega$. We also fix a set $\Gamma = |\mathcal{M}_\Gamma|$, where $\mathcal{M}_\Gamma$ is a triangular mesh satisfying

$$\mathcal{M}_\Gamma \subset \mathcal{F}(\mathcal{M}_\Omega) \setminus \partial \mathcal{M}_\Omega \, .$$

We do not require $\mathcal{M}_\Gamma$ to be regular, however, we impose that Γ be a *multi-screen*, in the sense of Claeys and Hiptmair [16]. For instance, $\mathcal{M}_\Gamma$ may be as in Figure 2.1.

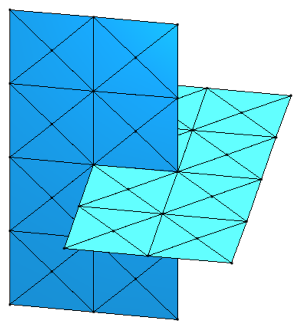

FIGURE 2.1. *Possible choice of mesh $\mathcal{M}_\Gamma$.*

Function spaces. For an open set U, let $C_c^\infty(U)$ be the set of real-valued functions u that are infinitely differentiable and compactly supported on U. We denote by $H^1(U)$ the Sobolev space of real-valued functions u which are square-integrable on U and such that there exists a square-integrable vector field $\boldsymbol{p} \in (L^2(U))^3$ satisfying

$$\int_U u \operatorname{div} \boldsymbol{\phi} = -\int_U \boldsymbol{p} \cdot \boldsymbol{\phi} \, , \quad \forall \boldsymbol{\phi} \in (C_c^\infty(U))^3 \, .$$

Writing $\nabla u := \boldsymbol{p}$ the *weak gradient* of u on U, a norm on $H^1(U)$ is defined by

$$\|u\|_{H^1(U)}^2 := \|u\|_{L^2(U)}^2 + \|\nabla u\|_{L^2(U)}^2 \, .$$

Let $H^1_{0,\Gamma}(\Omega)$ be the closure of $C_c^\infty(\Omega \setminus \Gamma)$ in $H^1(\Omega)$. The *multi-trace space* $\mathbb{H}^{1/2}(\Gamma)$ (see [16]) is the Hilbert space defined by the quotient

$$\mathbb{H}^{1/2}(\Gamma) := H^1(\Omega \setminus \Gamma)/H^1_{0,\Gamma}(\Omega) \, .$$

Let $\pi_D : H^1(\Omega \setminus \Gamma)$ the corresponding canonical surjection, and $H^{1/2}([\Gamma])$ the *single-trace space*, which is the closed subspace of $\mathbb{H}^{1/2}(\Gamma)$ defined by

$$H^{1/2}([\Gamma]) := \pi_D(H^1(\Omega)) \, .$$

In turn, the *jump space* $\widetilde{H}^{1/2}(\Gamma)$ is the Hilbert space defined by the quotient

$$\widetilde{H}^{1/2}(\Gamma) := \mathbb{H}^{1/2}(\Gamma)/H^{1/2}([\Gamma]) \, ,$$

and $[\cdot]$ will denote the corresponding canonical surjection.

Finite-dimensional subspaces. If $\mathcal{M}_{\Omega,h}$ and $\mathcal{M}_{\Gamma,h}$ are meshes of Ω and Γ (possibly different from $\mathcal{M}_\Omega$ and $\mathcal{M}_\Gamma$), we say that the pair $(\mathcal{M}_{\Omega,h}, \mathcal{M}_{\Gamma,h})$ is *trace-compatible* if $\mathcal{M}_{\Gamma,h} \subset \mathcal{F}(\mathcal{M}_{\Omega,h}) \backslash \partial\mathcal{M}_{\Omega,h}$. Given a trace-compatible pair $(\mathcal{M}_{\Omega,h}, \mathcal{M}_{\Gamma,h})$, let $V_h(\Omega\backslash\Gamma)$ be the finite-dimensional subspace of $H^1(\Omega\backslash\Gamma)$ consisting of piecewise linear functions on $\mathcal{M}_\Omega$, that is,

$$V_h(\Omega \setminus \Gamma) := \left\{ u \in \mathrm{H}^1(\Omega \setminus \Gamma) \;\middle|\; u_{|K} \text{ is affine } \forall K \in \mathcal{M}_\Omega \right\} .$$

Finally, define

$$\mathbb{V}_h(\Gamma) := \pi_D(V_h(\Omega \setminus \Gamma)) \,, \quad \widetilde{V}_h(\Gamma) := [\mathbb{V}_h(\Gamma)] \,.$$

Main result. The goal of this work is to prove the following result.

THEOREM 2.1. *For each $\gamma_0 > 0$, there exists a constant $C(\gamma_0) > 0$ such that the following holds. Suppose that $(\mathcal{M}_{\Omega,h}, \mathcal{M}_{\Gamma,h})$ is a trace-compatible pair of meshes of Ω and Γ, and assume that $\mathcal{M}_{\Omega,h}$ and $\mathcal{M}_{\Gamma,h}$ are γ_0-shape-regular. Then there exists a linear operator $\Pi_h : \mathbb{H}^{1/2}(\Gamma) \to \mathbb{V}_h(\Gamma)$ such that*
(i) $\|\Pi_h u\|_{\mathbb{H}^{1/2}} \leq C(\gamma_0) \|u\|_{\mathbb{H}^{1/2}}$ *for all* $u \in \mathbb{H}^{1/2}(\Gamma)$,
(ii) $\Pi_h u_h = u_h$ *for all* $u_h \in \mathbb{V}_h(\Gamma)$, *that is,* Π_h *is a projection,*
(iii) $u \in H^{1/2}([\Gamma]) \implies \Pi_h u \in H^{1/2}([\Gamma])$.

Notice the analogy with [32]: $\mathbb{H}^{1/2}(\Gamma)$ plays the role of $H^1(\Omega)$, and jumps play the role of boundary values. The definition of Π_h is given in Definition 5.1. The proof of Theorem 2.1 is also inspired by [32], but the essential difficulty is the construction of suitable "dual basis functions".

Remark 2.2. The stability constant depends on the aspect ratio of the elements of the mesh $\mathcal{M}_{\Omega,h}$, and not just $\mathcal{M}_{\Gamma,h}$. Although our proof requires this condition, we conjecture that the result still holds if we only assume that $\mathcal{M}_{\Gamma,h}$ is γ_0-shape-regular and that its elements are sufficiently small.

3. Applications. Theorem 2.1 has the following important consequences. First, it gives a uniform bound for the stability of the *discrete jump lifting* $\Phi_h : \widetilde{V}_h(\Gamma) \to \mathbb{V}_h(\Gamma)$, defined by

$$\Phi_h : [w_h] \mapsto u_h$$

where u_h is the unique minimizer of the $\mathbb{H}^{1/2}$ norm over the set of $w_h + H^{1/2}([\Gamma]) \cap \mathbb{V}_h(\Gamma)$.

COROLLARY 3.1 (Uniform bound for the discrete jump lifting). *Let the assumptions of Theorem 2.1 be satisfied. Then there holds*

$$\|\Phi_h \widetilde{\varphi}_h\|_{\mathbb{H}^{1/2}} \leq C(\gamma_0) \|\widetilde{\varphi}_h\|_{\widetilde{H}^{1/2}} \,, \quad \forall \widetilde{\varphi}_h \in \widetilde{V}_h(\Gamma) \,. \tag{3.1}$$

Proof. By definition of the quotient norm on $\widetilde{H}^{1/2}(\Gamma)$, and due to the Hilbert structure on this space, there exists a *continuous harmonic lifting* Φ, that is, an isometry

$$\Phi : \widetilde{H}^{1/2}(\Gamma) \to \mathbb{H}^{1/2}(\Gamma)$$

such that $[\Phi\widetilde{\varphi}] = \widetilde{\varphi}$ and

$$\|\Phi\widetilde{\varphi}\|_{\mathbb{H}^{1/2}} = \|\widetilde{\varphi}\|_{\widetilde{H}^{1/2}} \,, \quad \forall \widetilde{\varphi} \in \widetilde{H}^{1/2}(\Gamma) \,.$$

We claim that the operator $\Psi_h : \widetilde{V}_h(\Gamma) \to \mathbb{H}^{1/2}(\Gamma)$, defined by $\Psi_h := \Pi_h \circ \Phi$, satisfies the property

$$\forall \widetilde{\varphi}_h \in \widetilde{V}_h(\Gamma) \,, \quad [\Psi_h \widetilde{\varphi}_h] = \widetilde{\varphi}_h \,. \tag{3.2}$$

In other words, Ψ_h is a right-inverse of the jump operator on $\widetilde{V}_h(\Gamma)$. Provided that this holds, we have by the minimization property of Φ_h, and using Theorem 2.1 property (i):

$$\forall \widetilde{\varphi}_h \in \widetilde{V}_h(\Gamma)\,, \quad \|\Phi_h \widetilde{\varphi}_h\|_{\mathbb{H}^{1/2}(\Gamma)} \leq \|\Psi_h \widetilde{\varphi}_h\|_{\mathbb{H}^{1/2}} \\ \leq C(\gamma_0)\,\|\Phi \widetilde{\varphi}_h\|_{\mathbb{H}^{1/2}} = C(\gamma_0)\,\|\widetilde{\varphi}_h\|_{\widetilde{H}^{1/2}}\,.$$

It remains to show the property eq.(3.2). For this, fix $\widetilde{\varphi}_h \in \widetilde{V}_h(\Gamma)$ and let

$$v_h := \Psi_h \widetilde{\varphi}_h\,.$$

Furthermore, pick $w_h \in \mathbb{V}_h(\Gamma)$ such that $[w_h] = \widetilde{\varphi}_h$. Since $[\Phi\widetilde{\varphi}_h - w_h] = 0$, it follows by Theorem 2.1 (ii) and (iii) that

$$0 = [\Pi_h(\Phi\widetilde{\varphi}_h - w_h)] = [\Pi_h(\Phi\widetilde{\varphi}_h) - \Pi_h w_h] = [v_h] - [w_h] = [v_h] - \widetilde{\varphi}_h\,.$$

This shows that $[v_h] = \widetilde{\varphi}_h$, concluding the proof. ∎

Theorem 2.1 can also be used to prove the equivalence on $\widetilde{V}_h(\Gamma)$ of the $\widetilde{H}^{1/2}$ norm and the *discrete quotient norm* N_h, defined by

$$\forall \widetilde{\varphi}_h \in \widetilde{V}_h(\Gamma)\,, \quad N_h(\widetilde{\varphi}_h) := \inf\left\{\|u_h\|_{\mathbb{H}^{1/2}} \mid u_h \in \mathbb{V}_h(\Gamma) \text{ s.t. } [u_h] = \widetilde{\varphi}_h\right\}\,.$$

Corollary 3.2. *Let the assumptions of Theorem 2.1 be satisfied. Then for all $\widetilde{\varphi}_h \in \widetilde{V}_h(\Gamma)$, there holds*

$$\inf\left\{\|v_h\|_{\mathbb{H}^{1/2}} \mid v_h \in \mathbb{V}_h(\Gamma) \text{ s.t. } [v_h] = \widetilde{\varphi}_h\right\} \\ \leq C(\gamma_0) \inf\left\{\|v\|_{\mathbb{H}^{1/2}} \;\middle|\; v \in \mathbb{H}^{1/2}(\Gamma) \text{ s.t. } [v] = \widetilde{\varphi}_h\right\}\,,$$

or in other words, one has the norm equivalence

$$\frac{1}{C(\gamma_0)} N_h(\widetilde{\varphi}_h) \leq \|\widetilde{\varphi}_h\|_{\widetilde{H}^{1/2}} \leq N_h(\widetilde{\varphi}_h)\,. \tag{3.3}$$

Proof. The right inequality in eq. (3.3) follows from the definition of the quotient norm on $\widetilde{H}^{1/2}(\Gamma)$. Reciprocally, if $\widetilde{\varphi}_h \in \widetilde{V}_h(\Gamma)$, then by Corollary 3.1, we have

$$\inf\left\{\|v_h\|_{\mathbb{H}^{1/2}(\Gamma)} \;\middle|\; v_h \in \mathbb{V}_h(\Gamma) \text{ s.t. } [v_h] = \widetilde{\varphi}_h\right\} \leq \|\Phi_h \widetilde{\varphi}_h\|_{\mathbb{H}^{1/2}(\Gamma)} \leq C_h \|\widetilde{\varphi}_h\|_{\widetilde{H}^{1/2}}\,,$$

giving the left inequality. ∎

The rest of this work is dedicated to prove Theorem 2.1. From now on, we fix a constant $\gamma_0 > 0$ and a γ_0-regular pair of trace-compatible meshes $\mathcal{M}_{\Gamma,h}$ and $\mathcal{M}_{\Omega,h}$. The dependence in h will be omitted from some of the notation. We shall also use the letter C to denote a generic positive constants which only depends on Γ, Ω and γ_0, but is, in particular, independent on the size of the elements of $\mathcal{M}_{\Gamma,h}$ and $\mathcal{M}_{\Omega,h}$.

4. Basis of the discrete multi-trace space. In order to construct the operator Π_h of Theorem 2.1, we first construct a basis of $\mathbb{V}_h(\Gamma)$. The main ideas are from [6]. Let us denote by $\{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ be the vertices of $\mathcal{M}_{\Omega,h}$ with the common vertices of $\mathcal{M}_{\Omega,h}$ and $\mathcal{M}_{\Gamma,h}$ given by $\boldsymbol{x}_1, \dots, \boldsymbol{x}_M$. Let $V_h(\Omega)$ be the space of functions in $V_h(\Omega \setminus \Gamma)$, which are continuous. Notice that

$$V_h(\Omega) = H^1(\Omega) \cap V_h(\Omega \setminus \Gamma)\,.$$

Let $\{\phi_i\}_{1 \leq i \leq N}$ be the nodal basis of $V_h(\Omega)$, that is, the set of elements of $V_h(\Omega)$ defined by

$$\phi_i(\boldsymbol{x}_{i'}) = \delta_{i,i'}\,, \quad 1 \leq i, i' \leq N\,.$$

For each $i \in \{1, \dots, N\}$, the *star* of $\boldsymbol{x}_i$, denoted by $\mathrm{st}(\boldsymbol{x}_i, \mathcal{M}_{\Omega,h})$, is the set of tetrahedra $K \in \mathcal{M}_{\Omega,h}$ containing $\boldsymbol{x}_i$ as a vertex. We define a graph $\mathcal{G}(\boldsymbol{x}_i)$ with

- **Nodes:** The elements of $\mathrm{st}(\boldsymbol{x}_i, \mathcal{M}_{\Omega,h})$
- **Edges:** The pairs $\{K, K'\} \subset \mathrm{st}(\boldsymbol{x}_i, \mathcal{M}_\Omega)$ such that K and K' share a face not contained in $\mathcal{M}_\Gamma$.

Let $\gamma_{i,1}, \dots, \gamma_{i,q_i}$ be the connected components of $\mathcal{G}(\boldsymbol{x}_i)$, and let

$$|\gamma_{i,j}| := \bigcup_{K \in \gamma_{i,j}} \overline{K}\,.$$

Let us write

$$\mathcal{H}(\Omega) := \left\{(i,j) \in \mathbb{N}^2 \;\middle|\; 1 \leq i \leq N\,,\ 1 \leq j \leq q_i\right\}\,.$$

For $(i,j) \in \mathcal{H}(\Omega)$, we denote by $\phi_{i,j}$ the *split basis function* of $V_h(\Omega \setminus \Gamma)$ on $\gamma_{i,j}$, which is defined by

$$\phi_{i,j}(\boldsymbol{x}) := \begin{cases} \phi_i(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \mathrm{int}(|\gamma_{i,j}|)\,, \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathrm{int}(|\gamma_{i,j}|)$ is the interior of $|\gamma_{i,j}|$.

LEMMA 4.1. *The split basis functions $\{\phi_{i,j}\}_{(i,j)\in\mathcal{H}(\Omega)}$ form a basis of $V_h(\Omega \setminus \Gamma)$.*

Proof. Suppose that there exist real coefficients $\lambda_{i,j}$ such that

$$\forall \boldsymbol{x} \in \Omega\,, \qquad \sum_{(i,j)\in\mathcal{H}(\Omega)} \lambda_{i,j}\phi_{i,j}(\boldsymbol{x}) = 0\,. \tag{4.1}$$

Fix $(i_0, j_0) \in \mathcal{H}(\Omega)$ and let $K \in \gamma_{i,j}$. Let $(\boldsymbol{y}_n)_{n\in\mathbb{N}}$ be a sequence of points in the interior of K, such that

$$\lim_{n\to\infty} \boldsymbol{y}_n = \boldsymbol{x}_{i_0}\,.$$

It is easy to show that $\lim_{n\to\infty} \phi_{i,j}(\boldsymbol{y}_n) \to \delta_{i,i_0}\delta_{j,j_0}$. Therefore, applying eq. (4.1) to $\boldsymbol{y}_n$ and passing to the limit, we conclude that $\lambda_{i_0,j_0} = 0$. This proves that the functions $\phi_{i,j}$ are linearly independent.

Next, we consider $u_h \in V_h(\Omega \setminus \Gamma)$. For each $(i,j) \in \mathcal{H}(\Omega)$, we fix a tetrahedron $K_{i,j} \in \gamma_{i,j}$ and a sequence $(\boldsymbol{y}_n^{i,j})_{n\in\mathbb{N}}$ converging to $\boldsymbol{x}_i$ from $K_{i,j}$, as above. Let $\lambda_{i,j} = \lim_{n\to\infty} u_h(\boldsymbol{y}_n^{i,j})$. We prove that

$$u_h = \sum_{(i,j)\in\mathcal{H}(\Omega)} \lambda_{i,j}\phi_{i,j}\,, \tag{4.2}$$

by showing that this equality holds on the interior of each tetrahedron $K \in \mathcal{M}_{\Omega,h}$. Suppose that the vertices of K are $\boldsymbol{x}_{i_0}, \dots, \boldsymbol{x}_{i_3}$, and for each $p \in \{0, \dots, 3\}$, let j_p be such that $K \in \gamma_{i_p,j_p}$. Let $\boldsymbol{y}_n^p$ be a sequence of points in the interior of K converging to $\boldsymbol{x}_{i_p}$. We claim that

$$\lim_{n\to\infty} \phi_{i,j}(\boldsymbol{y}_n^p) = \delta_{i,i_p}\delta_{j,j_p}\,, \quad \lim_{n\to\infty} u_h(\boldsymbol{y}_n^p) = \lambda_{i_p,j_p}\,. \tag{4.3}$$

As before, the first limit can be shown easily. The second one is obvious if $K = K_{i_p,j_p}$. If K shares a face $F \notin \mathcal{M}_\Gamma$ with K_{i_p,j_p}, then it is a consequence of the well-known property that for $u \in H^1(\Omega \setminus \Gamma)$, the traces of u of F from K and K_{i_p,j_p} must agree. Otherwise, by definition of $\mathcal{G}(\boldsymbol{x}_i)$, one can consider a face-connected path of tetrahedra from K to K_{i_p,j_p}, and the desired limit is established by repeating the previous argument for each pair of consecutive tetrahedra in this path.

Having shown the property (4.3), we conclude that the linear functions defined on K by each side of eq. (4.2) have a common limit at 4 affinely independent points, thus they are equal on K. This concludes the proof of the lemma. ∎

For $u_h \in V_h(\Omega \setminus \Gamma)$, we will denote by $u_h(\gamma_{i,j})$ the coefficient of u_h on the split basis function $\phi_{i,j}$. On $V_h(\Omega \setminus \Gamma)$, we introduce the discrete l^2 scalar product

$$\forall (u,v) \in V_h(\Omega \setminus \Gamma) \times V_h(\Omega \setminus \Gamma)\,, \quad [u,v]_{l^2} := \sum_{i=1}^{N} \sum_{j=1}^{q_i} u(\gamma_{i,j}) v(\gamma_{i,j})\,,$$

Let $Y_h(\Omega)$ be the $[\cdot,\cdot]_{l^2}$ orthogonal complement of $V_h(\Omega)$. Let

$$\widetilde{\mathcal{H}} := \{(i,j) \in \mathcal{H}(\Omega) \mid q_i > 1 \text{ and } j \leq q_i - 1\}\,,$$

and for $(i,j) \in \widetilde{\mathcal{H}}$, define

$$\widetilde{\phi}_{i,j} := \phi_{i,j} - \phi_{i,q_i}\,.$$

Using that $\{\phi_i\}_{1\leq i \leq N}$ is a basis of $V_h(\Omega)$, together with the property

$$\phi_i = \sum_{j=1}^{q_i} \phi_{i,j}\,,$$

and Lemma 4.1, a simple algebraic reasoning shows that $\{\widetilde{\phi}_{i,j}\}_{(i,j)\in\widetilde{\mathcal{H}}}$ is a basis of $Y_h(\Omega)$. Define

$$\eta_i := \pi_D(\phi_i)\,, \quad \widetilde{\eta}_{i,j} := \pi_D(\widetilde{\phi}_{i,j})\,.$$

Let $V_h([\Gamma]) := \mathbb{V}_h \cap H^{1/2}([\Gamma])$ and $Y_h(\Gamma) := \pi_D(Y_h(\Omega))$. In summary, we have the following result:

Lemma 4.2. *There holds*

$$\mathbb{V}_h(\Gamma) = V_h([\Gamma]) \oplus Y_h(\Gamma)\,.$$

Moreover, $\{\eta_i\}_{1\leq i\leq M}$ is a basis of $V_h([\Gamma])$ and $\{\widetilde{\eta}_{i,j}\}_{(i,j)\in\widetilde{\mathcal{H}}}$ is a basis of $Y_h(\Gamma)$.

5. Definition of the quasi-interpolant. We now define the quasi-interpolant $\Pi_h : \mathbb{H}^{1/2}(\Gamma) \to \mathbb{V}_h(\Gamma)$, much in the same way as in [32]. We refer to [16] for the definition of the space $\mathbb{H}^{-1/2}(\Gamma)$ and the duality pairing $\langle\!\langle \cdot\,,\cdot \rangle\!\rangle : \mathbb{H}^{1/2}(\Gamma) \times \mathbb{H}^{-1/2}(\Gamma) \to \mathbb{R}$. Also recall the operator $\pi_N : H(\mathrm{div}, \Omega \setminus \Gamma) \to \mathbb{H}^{-1/2}(\Gamma)$.

Definition 5.1. *Let $\Pi_h : \mathbb{H}^{1/2}(\Gamma) \to \mathbb{V}_h(\Gamma)$ be the operator defined by*

$$\forall u \in \mathbb{H}^{1/2}(\Gamma)\,, \quad \Pi_h u := \sum_{k=1}^{M} \langle\!\langle u\,, \boldsymbol{\psi}_k \rangle\!\rangle \eta_k \ + \sum_{(i,j)\in\widetilde{\mathcal{H}}} \langle\!\langle u\,, \boldsymbol{\psi}_{i,j} \rangle\!\rangle \widetilde{\eta}_{i,j}\,.$$

where $\boldsymbol{\psi}_k = \pi_N(\boldsymbol{w}_k)$, $\boldsymbol{\psi}_{i,j} = \pi_N(\boldsymbol{w}_{i,j})$, and the vector fields $\boldsymbol{w}_k$ and $\boldsymbol{w}_{i,j}$ are given in Proposition 5.2.

Proposition 5.2 (Dual basis functions). *There exists two sets of vector fields $\{\boldsymbol{w}_k\}_{1\leq k\leq N}$ and $\{\boldsymbol{w}_{i,j}\}_{(i,j)\in\widetilde{\mathcal{H}}}$ satisfying the following properties:*

$$\boldsymbol{w}_k \in H(\operatorname{div}, \Omega\setminus\Gamma)\,, \quad \boldsymbol{w}_{i,j} \in H(\operatorname{div},\Omega)\,,$$

$$\operatorname{supp}(\boldsymbol{w}_k) \subset |\operatorname{st}(\boldsymbol{x}_k, \mathcal{M}_{\Omega,h})|\,, \quad \operatorname{supp}(\boldsymbol{w}_{i,j}) \subset |\operatorname{st}(\boldsymbol{x}_i, \mathcal{M}_{\Omega,h})|\,,$$

$$\|\boldsymbol{w}_k\|_\infty \leq Ch_k^{-2}\,, \quad \|\operatorname{div}\boldsymbol{w}_k\|_\infty \leq Ch_k^{-3}\,,$$

$$\|\boldsymbol{w}_{i,j}\|_\infty \leq Ch_i^{-2}\,, \quad \|\operatorname{div}\boldsymbol{w}_{i,j}\|_\infty \leq Ch_i^{-3}\,,$$

and the orthogonality relations

$$\begin{cases} \displaystyle\int_{\Omega\setminus\Gamma} \nabla\phi_k\cdot\boldsymbol{w}_{k'} + \phi_k \operatorname{div}\boldsymbol{w}_{k'} = \delta_{k,k'}\,, & \displaystyle\int_{\Omega\setminus\Gamma} \nabla\widetilde{\phi}_{i,j}\cdot\boldsymbol{w}_{k'} + \phi_{i,j}\operatorname{div}\boldsymbol{w}_{k'} = 0 \\ \displaystyle\int_{\Omega\setminus\Gamma} \nabla\phi_k\cdot\boldsymbol{w}_{i',j'} + \phi_k\operatorname{div}\boldsymbol{w}_{i',j'} = 0\,, & \displaystyle\int_{\Omega\setminus\Gamma} \nabla\widetilde{\phi}_{i,j}\cdot\boldsymbol{w}_{i',j'} + \widetilde{\phi}_{i,j}\operatorname{div}\boldsymbol{w}_{i',j'} = \delta_{i,i'}\delta_{j,j'} \end{cases} \tag{5.1}$$

for all $(k,k') \in \{1,\ldots,N\}^2$ and $((i,j),(i',j')) \in \widetilde{\mathcal{H}}^2$, where h_i is the diameter of $|\operatorname{st}(\boldsymbol{x}_i,\mathcal{M}_{\Omega,h})|$.

The proof of this proposition is the object of the next section. From the relations (5.1), it follows that

$$\begin{cases} \langle\!\langle \eta_k\,, \boldsymbol{\psi}_{k'}\rangle\!\rangle = \delta_{k,k'}\,, & \langle\!\langle \eta_k\,, \boldsymbol{\psi}_{i',j'}\rangle\!\rangle = 0\,, \\ \langle\!\langle \widetilde{\eta}_{i,j}\,, \boldsymbol{\psi}_{k'}\rangle\!\rangle = 0\,, & \langle\!\langle \widetilde{\eta}_{i,j}\,, \boldsymbol{\psi}_{i',j'}\rangle\!\rangle = \delta_{i,i'}\delta_{j,j'}\,, \end{cases}$$

which implies that Π_h is indeed a projection on $\mathbb{V}_h(\Gamma)$. Moreover, since $\boldsymbol{w}_{i,j} \in H(\operatorname{div},\Omega)$, we have $\boldsymbol{\psi}_{i,j} \in H^{-1/2}([\Gamma])$ (where $H^{-1/2}([\Gamma])$ is the Neumann single trace space, cf. [16]), ensuring that the property

$$\forall u \in H^{1/2}([\Gamma])\,, \quad \Pi_h u \in V_h([\Gamma])\,,$$

is satisfied, due to the polarity of $H^{1/2}([\Gamma])$ and $H^{-1/2}([\Gamma])$ [16, Prop. 6.3]. This proves properties (ii) and (iii) of Theorem 2.1, so it remains to prove the uniform $\mathbb{H}^{1/2}$ stability of Π_h. For this, we mostly follow the method of [32]. We consider another projection $\mathcal{Z}_h : H^1(\Omega\setminus\Gamma) \to V_h(\Omega\setminus\Gamma)$ with the property that

$$\pi_D \circ \mathcal{Z}_h = \Pi_h \circ \pi_D\,. \tag{5.2}$$

We shall need yet another set of dual basis functions $(\psi_i)_{M+1\leq i\leq N}$, which are defined as in [32]. Namely, for each $i \in \{M+1,\ldots,N\}$, we pick a tetrahedron K_i such that $\boldsymbol{x}_i$

is a vertex of K_i. Then, ψ_i is defined as the L^2 dual affine function on K_i associated to $\boldsymbol{x}_i$. Let

$$\mathcal{Z}_h f := \sum_{i=M+1}^{N} \left(\int_{K_i} \psi_i(x) f(x) dx \right) \phi_i + \sum_{i=1}^{M} \langle\!\langle \pi_D(f)\,, \boldsymbol{\psi}_i \rangle\!\rangle \phi_i + \sum_{(i,j)\in\widetilde{\mathcal{H}}} \langle\!\langle \pi_D(f)\,, \boldsymbol{\psi}_{i,j} \rangle\!\rangle \widetilde{\phi}_{i,j}\,.$$

It is immediate that $\mathcal{Z}_h$ is a projection and that eq. (5.2) is satisfied. For a tetrahedron $K \in \mathcal{M}_{\Omega,h}$, we denote by $\mathrm{st}(K, \mathcal{M}_{\Omega,h})$ the set of tetrahedra of $\mathcal{M}_{\Omega,h}$ sharing at least a vertex with K. The estimate of [32, Thm 3.1] carries over for $\mathcal{Z}_h$ with minor modifications as we show now.

LEMMA 5.3. *For $f \in \mathrm{H}^1(\Omega \setminus \Gamma)$ and $K \in \mathcal{M}_{\Omega,h}$, there holds*

$$\|\mathcal{Z}_h f\|_{H^1(K)} \leq C \left(h_K^{-1} \|f\|_{L^2(S_K)} + \|\nabla f\|_{L^2(S_K)} \right)\,,$$

where $S_K := |\mathrm{st}(K, \mathcal{M}_{\Omega,h})| \setminus \Gamma$.

Proof. We focus on the case $K \cap \Gamma \neq \emptyset$, because $K \cap \Gamma = \emptyset$ is settled by [32, Thm. 3.1]. Given a tetrahedron K incident to a face F of $\mathcal{M}_{\Gamma,h}$, we can write

$$\begin{aligned} \|\mathcal{Z}_h f\|_{H^1(K)} \leq & \sum_{i=M+1}^{N} \left| \int_{K_i} \psi_i(x) f(x) dx \right| \|\phi_i\|_{H^1(K)} \\ & + \sum_{i=1}^{M} |\langle\!\langle \pi_D(f)\,, \pi_N(\boldsymbol{w}_i) \rangle\!\rangle|\, \|\phi_i\|_{H^1(K)} \\ & + \sum_{(i,j)\in\widetilde{\mathcal{H}}} |\langle\!\langle \pi_D(f)\,, \pi_N(\boldsymbol{w}_{i,j}) \rangle\!\rangle|\, \left\| \widetilde{\phi}_{i,j} \right\|_{H^1(K)} \end{aligned}$$

by the triangular inequality. The first line of the right hand side is estimated as in [32, Thm. 3.1]. We furthermore recall the classical inequality

$$\|\phi_i\|_{H^1(K)} \leq C h_K^{1/2}\,,$$

using the shape regularity assumption. A similar estimate holds for $\widetilde{\phi}_{i,j}$, since $\phi_{i,j}$ is either equal to 0 or ϕ_i on K. On the other hand, by definition of $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$, we have the expression

$$\langle\!\langle \pi_D(f)\,, \boldsymbol{\psi}_i \rangle\!\rangle = \int_{\Omega} \nabla f \cdot \boldsymbol{w}_i + \mathrm{div}(\boldsymbol{w}_i) f = \int_{S_K} \nabla f \cdot \boldsymbol{w}_i + \mathrm{div}(\boldsymbol{w}_i) f\,,$$

since $\boldsymbol{w}_i$ is zero outside $\mathrm{st}(\boldsymbol{x}_i, \mathcal{M}_{\Omega,h}) \subset S_K$. By the previous estimates on $\boldsymbol{w}_i$, using the shape-regularity of $\mathcal{M}_{\Omega,h}$ and some elementary inequalities, this leads to

$$|\langle\!\langle u\,, \pi_N(\boldsymbol{w}_i) \rangle\!\rangle| \leq C \left(h_K^{1/2} \|\nabla f\|_{L^2(S_K)} + h_K^{-1/2} \|f\|_{L^2(S_K)} \right)\,.$$

The coefficients involving $\boldsymbol{w}_{i,j}$ are treated similarly. The rest of the proof follows is as in [32, Thm. 3.1]. ∎

Using now exactly the same arguments as in [32, Thm 4.1] and [32, Cor 4.1], we deduce

$$\|\mathcal{Z}_h f\|_{H^1(\Omega\setminus\Gamma)} \leq C \|f\|_{H^1(\Omega\setminus\Gamma)}\,. \tag{5.3}$$

Remark 5.4. The main tool, to derive (5.3) from Lemma 5.3, is a Bramble-Hilbert inequality in domains U consisting of face-connected and shape-regular unions of tetrahedra. To apply this result, a crucial requirement is that the restrictions of elements $V_h(\Omega \setminus \Gamma)$ to each tetrahedron $K \in \mathcal{M}_\Omega$ span all polynomial functions of degree 1 in K.

COROLLARY 5.5. *There exists a constant $C > 0$ only depending on shape-regularity such that*

$$\forall u \in \mathbb{H}^{1/2}(\Gamma)\,, \quad \|\Pi_h u\|_{\mathbb{H}^{1/2}} \leq C\, \|u\|_{\mathbb{H}^{1/2}}$$

Proof. Fix $f \in H^1(\Omega \setminus \Gamma)$ such that $u = \pi_D(f)$. Then we have $\Pi_h u = \pi_D(\mathcal{Z}_h f)$. Therefore, it holds that $\|\Pi_h u\|_{\mathbb{H}^{1/2}(\Gamma)} \leq \|\mathcal{Z}_h f\|_{\mathrm{H}^1(\Omega\setminus\Gamma)}$ by definition of the $\mathbb{H}^{1/2}$ norm. By the previous lemma, this gives

$$\|\Pi_h u\|_{\mathbb{H}^{1/2}} \leq C\, \|f\|_{H^1(\Omega\setminus\Gamma)}\ .$$

This holds for any $f \in \mathrm{H}^1(\Omega\setminus\Gamma)$ such that $\pi_D(f) = u$. Hence, passing to the infimum, we conclude

$$\forall u \in \mathbb{H}^{1/2}(\Gamma)\,, \quad \|\Pi_h u\|_{\mathbb{H}^{1/2}} \leq C\, \|u\|_{\mathbb{H}^{1/2}}\ ,$$

which proves the claim. ◻

We address the construction of the "dual basis functions" $\boldsymbol{w}_k$ and $\boldsymbol{w}_{i,j}$ of Proposition 5.2 in the next section.

6. Construction of the dual basis functions. The main tool for constructing the required vector fields is the set of bridge functions that we define now. Let $i \in \{1,\dots,N\}$ and $1 \leq j,k \leq q_i$. We say that $\{j,k\}$, with $j \neq k$, is a *bridge* around $\boldsymbol{x}_i$ if there exist two tetrahedra $K \in \gamma_{i,j}$, $K' \in \gamma_{i,k}$ such that K and K' have a common face $F \in \mathcal{M}_{\Gamma,h}$. The set of bridges around $\boldsymbol{x}_i$ is denoted by $\mathcal{B}(i)$.

LEMMA 6.1 (Bridge functions). *For each bridge $\{j,k\} \in \mathcal{B}(i)$ with $j < k$, there exists a vector field $\boldsymbol{b}_{i,\{j,k\}} \in H(\mathrm{div},\Omega)$ such that*

$$\mathrm{supp}(\boldsymbol{b}_{i,\{j,k\}}) \subset |\mathrm{st}(\boldsymbol{x}_i, \mathcal{M}_{\Omega,h})|\ ,$$

$$\left\|\boldsymbol{b}_{i,\{j,k\}}\right\| \leq C h_i^{-2}\,, \quad \left\|\mathrm{div}\, \boldsymbol{b}_{i,\{j,k\}}\right\| \leq C h_i^{-3}\,, \tag{6.1}$$

and

$$\forall u_h \in V_h(\Omega\setminus\Gamma)\,, \quad \int_{\Omega\setminus\Gamma} \boldsymbol{b}_{i,\{j,k\}} \nabla u_h + \mathrm{div}(\boldsymbol{b}_{i,\{j,k\}})\, u_h = u_h(\gamma_{i,j}) - u_h(\gamma_{i,k})\,. \tag{6.2}$$

Proof. Given a tetrahedron K, a face F of K and an affine function ψ defined on F, we define the vector field $\boldsymbol{b}_{K,F,\psi}$ on K by

$$\boldsymbol{b}_{K,F,\psi}(\boldsymbol{y}) := \frac{1}{h(K,F)} \sum_{m=0}^{2} \psi(\boldsymbol{x}_m)\lambda_m(\boldsymbol{y})(\boldsymbol{x}_m - \boldsymbol{x}_3)\,, \quad \boldsymbol{y} \in K\,,$$

where $\{\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}_2\}$ are the vertices of F and λ_m is the barycentric coordinate of K associated to $\boldsymbol{x}_m$, and $h(K,F)$ is the height of K from F. Given a face F' of K, we denote by $\boldsymbol{n}_{K,F'}$ the outward pointing normal vector on F'. With this definition, we have $h(K,F) = \boldsymbol{n}_{K,F} \cdot (\boldsymbol{x}_m - \boldsymbol{x}_3)$, where m is any number in $\{0,1,2\}$ and $\boldsymbol{x}_3$ is the vertex of K opposite to F. One can check that

$$\boldsymbol{b}_{K,F,\psi} \cdot \boldsymbol{n}_{K,F'} = \begin{cases} \psi & \text{if } F' = F\,. \\ 0 & \text{otherwise} \end{cases}$$

If $K \in \mathcal{M}_{\Omega_h}$, we denote the extension of $\boldsymbol{b}_{K,F,\psi}$ by 0 outside K again by $\boldsymbol{b}_{K,F,\psi}$. By what precedes, $\boldsymbol{b}_{K,F,\psi} \in H(\mathrm{div}, \Omega \setminus \Gamma)$ if $F \in \mathcal{M}_{\Gamma,h}$.

Fix a bridge $\{j,k\} \in \mathcal{B}(i)$, and let $K_j \in \gamma_{i,j}$ and $K_k \in \gamma_{i,k}$ be two tetrahedra sharing a face $F_{jk} \in \mathcal{M}_{\Gamma,h}$. The face F_{jk} contains the vertex $\boldsymbol{x}_i$. Following [32], we define an affine function ψ such that, for any affine function ρ,

$$\int_F \psi(\boldsymbol{y})\rho(\boldsymbol{y})\,d\boldsymbol{y} = \rho(\boldsymbol{x}_i)\,.$$

Using pullback and scaling arguments (cf. [32, Lemma 3.1]), one has

$$\max_{\boldsymbol{y}\in F_{jk}} |\psi(\boldsymbol{y})| \leq C h_{F_{jk}}^{-2}\,, \tag{6.3}$$

where h_F is the diameter of F_{jk}. With this choice of function ψ, let

$$\boldsymbol{b}_{i,\{j,k\}} := \boldsymbol{b}_{K_j,F_{jk},\psi} - \boldsymbol{b}_{K_k,F_{jk},\psi}\,.$$

This time, $\boldsymbol{b}_{i,\{j,k\}} \in H(\mathrm{div}, \Omega)$. It is also clear that $\mathrm{supp}(\boldsymbol{b}_{i,\{j,k\}}) \subset |\mathrm{st}(\boldsymbol{x}_i, \mathcal{M}_{\Omega,h})|$. The estimate of the L^∞ norm of $\boldsymbol{b}_{i,\{j,k\}}$ in eq. (6.1) is obtained from the bound on $|\psi|$ (6.3), the bound γ_0 on the aspect ratio of K_j and K_k, together with the simple estimate $|\lambda_m|_\infty \leq 1$. The bound on $\left\|\mathrm{div}\left(b_{i,\{j,k\}}\right)\right\|_\infty$ follows from the previous one, since $\boldsymbol{b}_{i,\{j,k\}}$ is linear on each tetrahedron K_j and K_k (using again the control over the aspect ratio of K_j and K_k).

Fix $u_h \in V_h(\Omega \setminus \Gamma)$. We have

$$\begin{aligned}\int_{\Omega\setminus\Gamma} &\boldsymbol{b}_{i,\{j,k\}} \nabla u_h + \mathrm{div}(\boldsymbol{b}_{i,\{j,k\}})\, u_h \\ &= \sum_{(i',j')\in\mathcal{H}(\Omega)} u_h(\gamma_{i',j'}) \int_{K_j} \boldsymbol{b}_{K_j,F_{jk},\psi} \nabla \phi_{i',j'} + \mathrm{div}(\boldsymbol{b}_{K_j,F_{jk},\psi})\, \phi_{i',j'} \\ &\quad - \sum_{(i',j')\in\mathcal{H}(\Omega)} u_h(\gamma_{i',j'}) \int_{K_k} \boldsymbol{b}_{K_k,F_{jk},\psi} \nabla \phi_{i',j'} + \mathrm{div}(\boldsymbol{b}_{K_k,F_{jk},\psi})\, \phi_{i',j'}\,.\end{aligned} \tag{6.4}$$

Let $(i',j') \in \mathcal{H}(\Omega)$ and let $r_{i',j'}$ be the trace of $\phi_{i',j'}$ on F_{jk} from K_j. There holds

$$\int_{K_j} \boldsymbol{b}_{K_j,F_{jk},\psi} \nabla \phi_{i',j'} + \mathrm{div}(\boldsymbol{b}_{K_j,F_{jk},\psi})\, \phi_{i',j'} = \int_{F_{jk}} \psi(\boldsymbol{y}) r_{i',j'}(\boldsymbol{y}) d\boldsymbol{y} = r_{i',j'}(\boldsymbol{x}_i)\,.$$

If $i' \neq i$, then $r_{i',j'}(\boldsymbol{x}_i) = 0$. On the other hand, if $i' = i$ but $j' \neq j$, then $\phi_{i,j'}$ is identically 0 on $\gamma_{i,j}$, and in particular on K_j. Finally, if $i = i'$ and $j = j'$, one has $r_{i,j}(\boldsymbol{x}_i) = 1$. In summary,

$$\int_{K_j} \boldsymbol{b}_{K_j,F_{jk},\psi} \nabla \phi_{i',j'} + \mathrm{div}(\boldsymbol{b}_{K_j,F_{jk},\psi})\, \phi_{i',j'} = \delta_{i,i'}\delta_{j,j'}$$

and by the same reasoning,

$$\int_{K_k} \boldsymbol{b}_{K_k,F_{jk},\psi} \nabla \phi_{i',j'} + \operatorname{div}(\boldsymbol{b}_{K_k,F_{jk},\psi})\, \phi_{i',j'} = \delta_{i,i'}\delta_{k,j'}\,.$$

The identity (6.2) follows by injecting the two relations above in eq. (6.4). ◻

If $q_i = 2$, there is a unique bridge function $\boldsymbol{b}_{i,1}$ at $\boldsymbol{x}_i$, and one can immediately see that $\boldsymbol{w}_{i,1} := \boldsymbol{b}_{i,1}$ fulfills the requirements of Proposition 5.2. In general, the idea is that one can construct the vector fields $\boldsymbol{w}_{i,j}$ of Proposition 5.2 as a suitable linear combination of the bridge functions at $\boldsymbol{x}_i$. To see this, we fix $i \in \{1,\dots,N\}$ and introduce the vector spaces

$$F_i := \operatorname{Span}(\{\widetilde{\phi}_{i,j}\}_{1\leq j\leq q_i-1})\,,\quad G_i := \operatorname{Span}(\{\boldsymbol{b}_{i,\{j,k\}}\}_{\{j,k\}\in\mathcal{B}(i)})\,.$$

Note that $G_i \subset H(\operatorname{div},\Omega)$. Let F_i^* be the dual of F_i, i.e. the set of linear forms on F_i and define the operator $\mathcal{A} : G_i \to F_i^*$ by

$$\forall(b,u_h)\in G_i\times F_i\,,\quad (\mathcal{A}\boldsymbol{b})(u_h) := \int_{\Omega\setminus\Gamma} \nabla u_h\cdot\boldsymbol{b} + u_h \operatorname{div}\boldsymbol{b}\,.$$

Lemma 6.2. *The operator $\mathcal{A} : G_i \to F_i^*$ is surjective.*

Proof. We start by introducing the graph $\mathcal{G}^*(\boldsymbol{x}_i)$ defined by

- **Nodes**: the numbers $1,\dots,q_i$,
- **Edges**: the bridges $\{k,k'\}$.

The key point is that $\mathcal{G}^*(\boldsymbol{x}_i)$ is connected (see e.g. [6, Lemma 1.5]). By [30, Lemma 3.9], to prove the lemma, it suffices to show that if $u_h \in F_i$ is such that

$$(\mathcal{A}\boldsymbol{b}_{i,\{j,k\}})u_h = 0 \quad \forall\{j,k\}\in\mathcal{B}(i)\,, \tag{6.5}$$

then $u_h = 0$. Every element v_h of F_i satisfies

$$\sum_{j=1}^{q_i} v_h(\gamma_{i,j}) = 0\,.$$

Hence, it remains to show that for $1\leq j,k\leq N$,

$$u_h(\gamma_{i,j}) = u_h(\gamma_{i,k})\,. \tag{6.6}$$

When $\{j,k\}\in\mathcal{B}(i)$, this relation is a consequence of the assumption (6.5). In general, we see that (6.6) holds by considering a path from j to k in $\mathcal{G}^*(\boldsymbol{x}_i)$. ◻

Let $\{w_{i,j}^*\}_{1\leq j\leq q_i-1}$ be the basis of F_i^* defined by the relations

$$w_{i,j}^*(\widetilde{\phi}_{i,j'}) = \delta_{j,j'}\,.$$

By the previous lemma, there exists an element $\boldsymbol{w}_{i,j}\in G_i$ such that $\mathcal{A}(\boldsymbol{w}_{i,j}) = w_{i,j}^*$. One has the properties

$$\boldsymbol{w}_{i,j}\in H(\operatorname{div},\Omega)\,,\quad \operatorname{supp}(\boldsymbol{w}_{i,j})\subset\left|\operatorname{st}(\boldsymbol{x}_i,\mathcal{M}_{\Omega,h})\right|\,,$$

by linearity since those properties hold for all bridge functions at $\boldsymbol{x}_i$. The same argument allows to check that for all $i'\in\{1,\dots,N\}$ and $(i'',j'')\in\widetilde{\mathcal{H}}$ with $i''\neq i$, there holds

$$\int_{\Omega\setminus\Gamma}\nabla\phi_{i'}\cdot\boldsymbol{w}_{i,j} + \phi_{i'}\operatorname{div}\boldsymbol{w}_{i,j} = 0\,,\quad \int_{\Omega\setminus\Gamma}\nabla\widetilde{\phi}_{i'',j''}\cdot\boldsymbol{w}_{i,j} + \widetilde{\phi}_{i'',j''}\operatorname{div}\boldsymbol{w}_{i,j} = 0\,.$$

Finally, we have

$$\int_{\Omega\setminus\Gamma} \nabla\widetilde{\phi}_{i,j'} \cdot \boldsymbol{w}_{i,j} + \widetilde{\phi}_{i,j'} \operatorname{div} \boldsymbol{w}_{i,j} = \delta_{j,j'}\,, \tag{6.7}$$

by construction. This proves the relations (5.1) for $\boldsymbol{w}_{i,j}$, and it remains to estimate its L^∞ norm as well as that of its divergence. This is addressed in the following lemma.

LEMMA 6.3. *One can choose $\boldsymbol{w}_{i,j}$ such that*

$$\|\boldsymbol{w}_{i,j}\|_\infty \leq C h_i^{-2}\,, \quad \|\operatorname{div} \boldsymbol{w}_{i,j}\|_\infty \leq C h_i^{-3}\,.$$

Proof. First, we remark that the number q_i is bounded independently of the meshes $\mathcal{M}_{\Omega,h}$ and $\mathcal{M}_{\Gamma,h}$. Indeed, one can check that q_i is equal to the number of connected components of $B \setminus \Gamma$, where B is a small enough ball centered at $\boldsymbol{x}_i$. We shall not prove this assertion in detail, for the sake of conciseness. Let us denote by Q an upper bound for $q_i - 1$.

Next, since $\mathcal{A}$ is surjective and since $\dim(F_i^*) = \dim(F_i) = q_i - 1$, one can find a set of pairs $\{\{j_p, k_p\}\}_{1\leq p\leq q_i-1}$ such that $\{\mathcal{A}\boldsymbol{b}_{i,\{j_p,k_p\}}\}_{1\leq p\leq q_i-1}$ is a basis of F_i^*. In this basis, $w_{i,j}^*$ is expressed by

$$w_{i,j}^* = \sum_{p=1}^{q_i-1} \lambda_p (\mathcal{A}\boldsymbol{b}_{i,\{j_p,k_p\}})\,.$$

The coefficients λ_p are found by solving the linear system

$$\mathbf{A} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{q_i-1} \end{bmatrix} = \boldsymbol{e}_j\,,$$

where $\boldsymbol{e}_j$ is the j-th vector of the canonical basis of $\mathbb{R}^{q_i-1}$ and the matrix coefficients

$$(\mathbf{A})_{p,p'} := \widetilde{\phi}_{i,p'}(\gamma_{i,j_p}) - \widetilde{\phi}_{i,p'}(\gamma_{i,k_p})$$

are integer between -2 and 2. The set $\mathcal{S}_Q$ of invertible square matrices $\mathbf{A}$ of size at most Q with coefficients in $\{-2,-1,0,1,2\}$ is finite. Therefore, we can write

$$|\lambda_p| \leq \sup_{\mathbf{A}\in\mathcal{S}_Q} |||\mathbf{A}^{-1}|||_\infty\,, \quad \forall p \in \{1\,,\ldots\,,q_i-1\}\,, \tag{6.8}$$

where, for an element v of $\mathbb{R}^r$ and a square $r\times r$ matrix M, we have denoted

$$\|v\|_\infty := \max_{1\leq q\leq r} |v_q|\,, \quad |||M|||_\infty := \sup_{v\in R^r} \frac{\|Mv\|_\infty}{\|v\|_\infty}\,.$$

The bound (6.8) only depends on $\mathcal{M}_\Gamma$ (through the number Q). The proof is concluded by using the triangular inequality and the bounds (6.1) from Lemma 6.1. ∎

The construction of the vector fields $\{\boldsymbol{w}_k\}_{1\leq k\leq M}$ of Proposition 5.2 is easier; we present this now. For each $(i,j) \in \mathcal{H}(\Omega)$, one can construct a vector field $\boldsymbol{c}_{i,j} \in H(\operatorname{div}, \Omega\setminus\Gamma)$ such that

$$\forall u_h \in V_h(\Omega\setminus\Gamma)\,, \quad \int_{\Omega\setminus\Gamma} \nabla u_h \cdot \boldsymbol{c}_{i,j} + u_h \operatorname{div}(\boldsymbol{c}_{i,j}) = u_h(\gamma_{i,j})\,.$$

With the notation of the proof of Lemma 6.1, it suffices to pick any tetrahedron K in $\gamma_{i,j}$ with a face F in $\mathcal{M}_{\Gamma,h}$. We then let $\boldsymbol{c}_{i,j} := \boldsymbol{b}_{K,F,\psi}$, where ψ is again the L^2 dual linear function associated to $\boldsymbol{x}_i$. Then, for $k \in \{1, \ldots, M\}$, we set

$$\boldsymbol{w}_k := \frac{1}{q_k} \sum_{l=1}^{q_k} \boldsymbol{c}_{k,l}\,,$$

and this definition fulfills the requirements of Proposition 5.2.

We have thus constructed the vector fields $\boldsymbol{w}_k$ and $\boldsymbol{w}_{i,j}$ and shown that they satisfy the properties stated in Proposition 5.2, and hence the proof of Theorem 2.1 is concluded.

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