ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

# Multilevel Domain Uncertainty Quantification in Computational Electromagnetics

R. Aylwin and C. Jerez-Hanckes and Ch. Schwab and J. Zech

Research Report No. 2022-47
December 2022

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

# MULTILEVEL DOMAIN UNCERTAINTY QUANTIFICATION IN COMPUTATIONAL ELECTROMAGNETICS

RUBÉN AYLWIN, CARLOS JEREZ-HANCKES, CHRISTOPH SCHWAB, AND JAKOB ZECH

Abstract. We continue our study [Domain Uncertainty Quantification in Computational Electromagnetics, JUQ (2020), **8**:301–341] of the numerical approximation of time-harmonic electromagnetic fields for the Maxwell lossy cavity problem for uncertain geometries. We adopt the same affine-parametric shape parametrization framework, mapping the physical domains to a nominal polygonal domain with piecewise smooth maps. The regularity of the pullback solutions on the nominal domain is characterized in piecewise Sobolev spaces. We prove error convergence rates and optimize the algorithmic steering of parameters for edge-element discretizations in the nominal domain combined with: (a) multilevel Monte Carlo sampling, and (b) multilevel, sparse-grid quadrature for computing the expectation of the solutions with respect to uncertain domain ensembles. In addition, we analyze sparse-grid interpolation to compute surrogates of the domain-to-solution mappings. All calculations are performed on the polyhedral nominal domain, which enables the use of standard simplicial finite element meshes. We provide a rigorous fully discrete error analysis and show, in all cases, that dimension-independent algebraic convergence is achieved. For the multilevel sparse-grid quadrature methods, we prove higher order convergence rates which are free from the so-called curse of dimensionality, i.e. independent of the number of parameters used to parametrize the admissible shapes. Numerical experiments confirm our theoretical results and verify the superiority of the sparse-grid methods.

## 1. Introduction

In recent years, *computational uncertainty quantification* (computational UQ for short) has emerged as a sub-discipline of computational science and engineering. A broad theme within computational UQ is the efficient numerical analysis of parametric partial differential equation models in science and engineering. They are usually based on parametric families of domains which are homeomorphic, in particular, to one fixed *nominal* reference domain via parametric families of diffeomorphisms. In the common case that the parametric dependence involves possibly infinitely many parameters, the solution families become likewise infinite-parametric. The numerical approximation of such parametric solution sets is costly due to the usually large number of numerical approximations of parametric solutions that must be generated to approximate the entire parametric solution family. In the presently considered time-harmonic Maxwell equation model, the physical domain is necessarily three-dimensional, which is an additional source of computational complexity.

The present paper addresses the formulation and error analysis of multilevel Monte-Carlo (MLMC) Finite Element (FE) schemes for efficient computational domain uncertainty quantification of time-harmonic, electromagnetic scattering from parametric families of lossy cavities in a bounded domain in three-dimensional space.

1.1. **Previous work.** The question of *shape recovery in time-harmonic acoustic and electromagnetic scattering* subject to noisy measurements of far-fields has received increasing attention in recent years. Classical shape calculus suggests that under certain conditions, the dependence of the

*Date*: **December 12, 2022.**

2010 *Mathematics Subject Classification.* 35A20, 35B30, 32D05, 35Q61.

*Key words and phrases.* Computational Electromagnetics, Uncertainty Quantification, Finite Elements, Shape Holomorphy, Smolyak Quadrature.

This work was supported in part by Fondecyt Regular 1171491, Conicyt-PFCHA/Doctorado Nacional/2017-21171791 and by the Swiss National Science Foundation under grant SNF149819.

*forward-map* from scatterer shape to far-field is continuous and differentiable in suitable topologies (cf. [17] and references therein). This continuous dependence implies strong measurability of solution families corresponding to random ensembles of admissible domains, thereby justifying the use of Monte-Carlo sampling to explore the solution manifold.

In the discipline of computational uncertainty quantification, one is interested in *numerically approximating parametric solution families* corresponding to parametric representations of admissible shapes, e.g., by Fourier-, wavelet- or Karhúnen-Loève -expansions. Naturally, many-parametric representations of shapes will imply many-parametric solution families, thereby mandating *sampling and interpolation of parametric solutions in high-dimensional parameter domains*. A broad class of forward data-to-solution maps which arise in shape uncertainty quantification have been shown to be holomorphic as maps between—possibly complexified—Banach spaces. We refer to [24, 26, 15] and the references there for proofs of this so-called *Shape Holomorphy* of PDEs. This property has been used in [2] for the numerical analysis of UQ in the time-harmonic propagation of electromagnetic waves in lossy cavities in $\mathbb{R}^3$. In this last reference, we developed a single-level algorithm whose convergence properties were analyzed; in particular, dimension-independent convergence rates of suitable sparse-grid interpolation and approximation algorithms for the many-parametric dependence were shown. Moreover, we showed that these algorithms outperform the corresponding Monte-Carlo sampling considerably, also for many-parametric shape representations.

The present paper develops the corresponding multilevel algorithms and their error analysis. As is well-known from previous work, e.g., [4, 14], for scalar, parametric PDEs, the analysis of the multilevel FE algorithms does require additional regularity of the parametric solution families. Specifically, we require holomorphy of the parametric, time-harmonic solutions for a Maxwell-like, non-homogeneous problem on a so-called *nominal*—reference—domain, with non-homogeneous coefficients resulting from the pullback of the parametric, physical domain to this nominal domain. The requited regularity results for solutions of Maxwell-like equations in the nominal domain, for *non-constant coefficients of low differentiability*, were recently obtained in [1]. The shape holomorphy results required for the error analysis of the multilevel Smolyak quadrature algorithms is developed in the present paper.

1.2. **Paper Layout.** The structure of this text is as follows. In Section 2, we present the governing equations and the problem formulation of Maxwell's equations on a lossy cavity in three space dimensions. Specifically, we focus on the parametric model of the cavity's uncertain shape, and of the variational form of the governing equations. We set in particular notation, and recapitulate basic or known results on the unique solvability of the forward model; particular attention is paid to *a priori* bounds which are uniform over all realizations of domains. The presented approach opts for the so-called *domain-mapping formulation*, whereby all realizations of domains which occur in numerical simulation are mathematically formulated on one fixed, so-called *nominal domain*. As we show here, these entails the need to numerically solve a potentially large number of Maxwell-like PDEs with variable, metric-dependent coefficients in the nominal domain.

Section 3 addresses the discretization of the parametric forward problem, in particular edge-elements in the nominal domain (cf. Section 3.2). Section 4 describes the mathematical setting of regularity of the parametric solution families. In comparison to the error analysis of the single-level Smolyak quadrature algorithms in [2], stronger parametric regularity results are required. Section 5.1 recapitulates the MLMC algorithm, and Section 5.2 the corresponding multilevel Smolyak quadrature one. Finally, Section 6 contains several numerical experiments to illustrate our theoretical results, confirming in particular the superior accuracy versus cost performance of the multilevel version compared to the single-level algorithms from [2].

1.3. **General notation.** Let $m \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and let $\mathrm{D} \subset \mathbb{R}^d$ be an open and bounded Lipschitz domain, then $\mathcal{C}^m(\mathrm{D})$ and $\mathcal{C}^m(\mathrm{D};\mathbb{C})$ denote the set of $m$-times continuously differentiable functions with real and complex values in $\overline{\mathrm{D}}$, respectively. Furthermore, $\mathcal{C}^\infty(\mathrm{D})$ denotes the space of infinitely differentiable functions in $\overline{\mathrm{D}}$ and we write $\mathcal{C}^m_0(\mathrm{D})$ for the set of elements of $\mathcal{C}^m(\mathrm{D})$ with compact support on D—analogous definitions apply for $\mathcal{C}^\infty(\mathrm{D};\mathbb{C})$ and $\mathcal{C}^m_0(\mathrm{D};\mathbb{C})$.

For $k \in \mathbb{N}_0$, we write $\mathbb{P}_k(\mathrm{D};\mathbb{C}^d)$ the space of functions from D to $\mathbb{C}^d$, such that each of their $d$ components is a polynomial with complex coefficients of total degree $k$ or smaller. $\widetilde{\mathbb{P}}_k(\mathrm{D};\mathbb{C}^d)$ represents the space of functions in $\mathbb{P}_k(\mathrm{D};\mathbb{C}^d)$ which have polynomials of degree exactly $k$ on each component.

The set of all bounded antilinear mappings from a Banach space $X$ to $\mathbb{C}$ is written as $X^*$ and is referred to as the dual space of $X$. The norm and duality product of a Banach space shall be denoted by the use of subscript ($\|\cdot\|_X$ and $\langle\cdot,\cdot\rangle_{X\times X^*}$, respectively). For a pair of Banach spaces $X$ and $Y$, the set of all linear mappings from $X$ to $Y$ is denoted $\mathcal{L}(X;Y)$. For $s \geq 0$ and $p \geq 1$, $L^p(\mathrm{D})$ denotes the Banach space of $p$-integrable functions over D, while $W^{s,p}(\mathrm{D})$ denotes the standard Sobolev spaces of order $s$ as defined in [29, Chap. 3], where we use the convention $W^{0,p}(\mathrm{D}) = L^p(\mathrm{D})$. If $p = 2$, we shall use the standard notation $H^s(\mathrm{D}) = W^{s,2}(\mathrm{D})$. Furthermore, the norm and semi-norm of $H^s(\mathrm{D})$ are denoted as $\|\cdot\|_{s,\mathrm{D}}$ and $|\cdot|_{s,\mathrm{D}}$, respectively.

Let $p \geq 1$ and let $(\Omega, \mathfrak{F}, \nu)$ be a probability space, where we take $\Omega$ to be a sample space, $\mathfrak{F}$ a $\sigma$-algebra on $\Omega$ and $\nu$ a probability measure. Given a separable Hilbert space $X$, we say that a function $f : \Omega \to X$ is measurable (or a random variable) if for every Borel set $B \subset X$ there holds that $\{\zeta \in \Omega \,:\, f(\zeta) \in B\} \in \mathfrak{F}$, we say that $f$ is strongly measurable if it is the pointwise limit of a sequence of simple functions $\{f_n\}_{n\in\mathbb{N}}$, and we say that $f$ is Bochner integrable if it is strongly measurable and $\lim_{n\to\infty} \int_\Omega \|f(\zeta) - f_n(\zeta)\|_X \,\mathrm{d}\nu(\zeta) = 0$. Moreover, if $f : \Omega \to X$ is Bochner integrable, its integral over $\Omega$ is defined as $\int_\Omega f(\zeta)\,\mathrm{d}\nu(\zeta) := \lim_{n\to\infty} \int_\Omega f_n(\zeta)\,\mathrm{d}\nu(\zeta)$. We denote the Bochner space of $p$ integrable, measurable mappings in $(\Omega, \mathfrak{F}, \nu)$ with values in $X$ as $L^p(\Omega, \mathfrak{F}, \nu; X)$. When the $\sigma$-algebra on $\Omega$ and the probability measure are clear from the context, we will denote the norm of $\phi \in L^p(\Omega, \mathfrak{F}, \nu; X)$ as $\|\phi\|_{L^p(\Omega;X)}$. For further details we refer to [16, 18, 25].

In general, boldface symbols will be used to differentiate the vector-valued counterparts of scalar functions and functional spaces, e.g., $\boldsymbol{L}^2(\mathrm{D})$ denotes the functional space of vector-valued functions with each of its $d$-components in $L^2(\mathrm{D})$. In particular, the $L^2(\mathrm{D})$-inner product is denoted $(\cdot,\cdot)_\mathrm{D}$. Functional spaces built of tensor quantities are to be identified by indicating the range of the elements it contains, eg., $\boldsymbol{L}^2(\mathrm{D};\mathbb{C}^{d\times d})$ represents the space of tensor-valued functions with each of their $d^2$ entries belonging to $L^2(\mathrm{D})$.

Euclidean norms in $\mathbb{R}^d$ are denoted by $\|\cdot\|_{\mathbb{R}^d}$, while the induced matrix norm is denoted by $\|\cdot\|_{\mathbb{R}^{d\times d}}$ with analogous versions when in $\mathbb{C}^d$ and $\mathbb{C}^{d\times d}$. The Jacobian of a differentiable function $\mathbf{U} : \mathrm{D} \to \mathbb{C}^d$ is written as $\mathrm{d}\mathbf{U} : \mathrm{D} \to \mathbb{C}^{d\times d}$. For a general square matrix $\mathbf{A} \in \mathbb{C}^{d\times d}$, we write the transpose matrix of $\mathbf{A}$ as $\mathbf{A}^\top$, its determinant as $\det(\mathbf{A})$ and its inverse as $\mathbf{A}^{-1}$, when it exists. Finally, the overline notation will be used to represent complex conjugation as well as the closure of a set.

## 2. Maxwell's equations on a lossy cavity

We begin this section by stating the time-harmonic Maxwell's lossy cavity problem and its functional framework.

### 2.1. Functional spaces for Maxwell's equations.

Let D be an open and bounded Lipschitz domain in $\mathbb{R}^3$ with simply connected boundary $\partial\mathrm{D}$, exterior $\mathrm{D}^c := \mathbb{R}^3 \setminus \overline{\mathrm{D}}$ and with exterior unit normal vector $\mathbf{n}$—pointing from D to $\mathrm{D}^c$. We recall the standard functional spaces required to formulate Maxwell problems:

$$\boldsymbol{H}(\mathbf{curl};\mathrm{D}) := \{\mathbf{U} \in \boldsymbol{L}^2(\mathrm{D}) \;:\; \mathbf{curl}\,\mathbf{U} \in \boldsymbol{L}^2(\mathrm{D})\},$$
$$\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl};\mathrm{D}) := \{\mathbf{U} \in \boldsymbol{H}(\mathbf{curl};\mathrm{D}) \;:\; \mathbf{curl}\,\mathbf{U} \in \boldsymbol{H}(\mathbf{curl};\mathrm{D})\},$$
$$\boldsymbol{H}(\mathrm{div};\mathrm{D}) := \{\mathbf{U} \in \boldsymbol{L}^2(\mathrm{D}) \;:\; \mathrm{div}\,\mathbf{U} \in L^2(\mathrm{D})\},$$

and introduce, for $m \in \mathbb{N}_0$ and $p \geq 1$, extensions of $\boldsymbol{H}(\mathbf{curl};\mathrm{D})$ and $\boldsymbol{H}(\mathrm{div};\mathrm{D})$ to spaces with additional regularity and arbitrary integrability,

$$\boldsymbol{W}^{m,p}(\mathbf{curl};\mathrm{D}) := \{\mathbf{U} \in \boldsymbol{W}^{m,p}(\mathrm{D}) \;:\; \mathbf{curl}\,\mathbf{U} \in \boldsymbol{W}^{m,p}(\mathrm{D})\}, \quad \boldsymbol{H}^m(\mathbf{curl};\mathrm{D}) := \boldsymbol{W}^{m,2}(\mathbf{curl};\mathrm{D}),$$
$$\boldsymbol{W}^{m,p}(\mathrm{div};\mathrm{D}) := \{\mathbf{U} \in \boldsymbol{W}^{m,p}(\mathrm{D}) \;:\; \mathrm{div}\,\mathbf{U} \in W^{m,p}(\mathrm{D})\}, \quad \boldsymbol{H}^m(\mathrm{div};\mathrm{D}) := \boldsymbol{W}^{m,2}(\mathrm{div};\mathrm{D}),$$

with associated norms

$$\|\mathbf{U}\|_{\boldsymbol{W}^{m,p}(\mathbf{curl};\mathrm{D})} := (\|\mathbf{U}\|^p_{\boldsymbol{W}^{m,p}(\mathrm{D})} + \|\mathbf{curl}\,\mathbf{U}\|^p_{\boldsymbol{W}^{m,p}(\mathrm{D})})^{\frac{1}{p}},$$
$$\|\mathbf{U}\|_{\boldsymbol{W}^{m,p}(\mathrm{div};\mathrm{D})} := (\|\mathbf{U}\|^p_{\boldsymbol{W}^{m,p}(\mathrm{D})} + \|\mathrm{div}\,\mathbf{U}\|^p_{W^{m,p}(\mathrm{D})})^{\frac{1}{p}},$$

for $p \in [1,\infty)$ and the usual modification for $p = \infty$. We point out that $\boldsymbol{W}^{0,2}(\mathbf{curl};\mathrm{D}) \equiv \boldsymbol{H}(\mathbf{curl};\mathrm{D})$ and $\boldsymbol{W}^{0,2}(\mathrm{div};\mathrm{D}) \equiv \boldsymbol{H}(\mathrm{div};\mathrm{D})$.

**Definition 2.1.** For $\mathbf{U} \in \boldsymbol{\mathcal{C}}^\infty(\mathrm{D})$ we define the following trace operators:

$$\gamma_D\mathbf{U} := \mathbf{n}\times(\mathbf{U}\times\mathbf{n})|_{\partial\mathrm{D}}, \quad \gamma_D^\times\mathbf{U} := (\mathbf{n}\times\mathbf{U})|_{\partial\mathrm{D}} \quad \text{and} \quad \gamma_N\mathbf{U} := (\mathbf{n}\times\mathbf{curl}\,\mathbf{U})|_{\partial\mathrm{D}},$$

as the Dirichlet trace, flipped Dirichlet trace and Neumann trace operators, respectively.

The trace operators in Definition 2.1 may be extended to continuous linear functionals from $\boldsymbol{H}(\mathbf{curl};\mathrm{D})$ and $\boldsymbol{H}(\mathbf{curl}\,\mathbf{curl};\mathrm{D})$ to subsets of $\boldsymbol{H}^{-\frac{1}{2}}(\partial\mathrm{D}) := \boldsymbol{H}^{\frac{1}{2}}(\partial\mathrm{D})^*$. Specifically, we consider the following trace spaces (cf. [6, 8]):

$$\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\partial\mathrm{D}) := \{\mathbf{U} \in (\mathbf{n}\times(\boldsymbol{H}^{\frac{1}{2}}(\partial\mathrm{D})\times\mathbf{n}))^* : \mathrm{div}_{\partial\mathrm{D}}\,\mathbf{U} \in H^{-\frac{1}{2}}(\partial\mathrm{D})\},$$
$$\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\partial\mathrm{D}) := \{\mathbf{U} \in (\boldsymbol{H}^{\frac{1}{2}}(\partial\mathrm{D})\times\mathbf{n})^* : \mathrm{curl}_{\partial\mathrm{D}}\,\mathbf{U} \in H^{-\frac{1}{2}}(\partial\mathrm{D})\},$$

where $\mathrm{div}_{\partial\mathrm{D}}$ and $\mathrm{curl}_{\partial\mathrm{D}}$ are, respectively, the surface divergence and surface scalar curl operators and $\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\partial\mathrm{D}) = \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\partial\mathrm{D})^*$ (cf. [7, Thm. 2] and [30, Rmk. 3.32]). Then, the operators in Definition 2.1 may be continuously extended as

$$\gamma_D : \boldsymbol{H}(\mathbf{curl};\mathrm{D}) \to \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\partial\mathrm{D}),$$
$$\gamma_D^\times : \boldsymbol{H}(\mathbf{curl};\mathrm{D}) \to \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\partial\mathrm{D}),$$
$$\gamma_N : \boldsymbol{H}(\mathbf{curl}\,\mathbf{curl};\mathrm{D}) \to \boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\partial\mathrm{D}).$$

We also introduce the space of functions with well defined curl and null flipped Dirichlet trace:

$$\boldsymbol{H}_0(\mathbf{curl};\mathrm{D}) := \{\mathbf{U} \in \boldsymbol{H}(\mathbf{curl};\mathrm{D}) \ : \ \gamma_{\mathrm{D}}^\times\mathbf{U} = \mathbf{0} \text{ on } \partial\mathrm{D}\},$$

which is a closed subspace of $\boldsymbol{H}(\mathbf{curl};\mathrm{D})$. Finally, for $\mathbf{U}$ and $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl};\mathrm{D})$ there holds the following integration by parts formula [6, Eq. (27)]:

$$(\mathbf{U}, \mathbf{curl}\,\mathbf{V})_{\mathrm{D}} - (\mathbf{curl}\,\mathbf{U}, \mathbf{V})_{\mathrm{D}} = -\langle\gamma_{\mathrm{D}}^\times\mathbf{U}, \gamma_{\mathrm{D}}\mathbf{V}\rangle_{\partial\mathrm{D}} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{\partial\mathrm{D}}$ denotes the duality between $\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{div}}(\partial\mathrm{D})$ and $\boldsymbol{H}^{-\frac{1}{2}}_{\mathrm{curl}}(\partial\mathrm{D})$.

2.2. **Lossy cavity problem.** We consider the EM cavity problem for a time-harmonic dependence $e^{\imath\omega t}$ with circular frequency $\omega > 0$ and $\imath^2 = -1$ on D. The electric permittivity is denoted $\varepsilon(\boldsymbol{x}) \in \boldsymbol{L}^\infty(\mathrm{D};\mathbb{C}^{3\times3})$ and the magnetic permeability is denoted as $\mu(\boldsymbol{x}) \in \boldsymbol{L}^\infty(\mathrm{D};\mathbb{C}^{3\times3})$, where losses are represented by their respective imaginary parts. With the current density $\mathbf{J} \in \boldsymbol{L}^2(\mathrm{D})$, Maxwell's equations in D read

$$\begin{aligned} \mathbf{curl}\,\mathbf{E} + \imath\omega\mu\mathbf{H} &= \mathbf{0} && \text{in D,} \\ \imath\omega\varepsilon\mathbf{E} - \mathbf{curl}\,\mathbf{H} &= -\mathbf{J} && \text{in D.} \end{aligned} \tag{2.2}$$

Assuming the pointwise inverse $\mu^{-1} \in \boldsymbol{L}^\infty(\mathrm{D};\mathbb{C}^{3\times3})$ to be well defined, the system in (2.2) can be reduced to

$$\mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{E} - \omega^2\varepsilon\mathbf{E} = -\imath\omega\mathbf{J} \qquad \text{in D.} \tag{2.3}$$

We further impose perfect electric conductor (PEC) boundary conditions on $\partial$D,

$$\gamma_{\mathrm{D}}^\times\mathbf{E} = \mathbf{0} \quad \text{on } \partial\mathrm{D}. \tag{2.4}$$

*Remark* 2.2. Under the assumption that the pointwise inverse $\mu^{-1} : \mathrm{D} \to \mathbb{C}^{3\times3}$ belongs to $\boldsymbol{L}^\infty(\mathrm{D};\mathbb{C}^{3\times3})$, any weak solution $\mathbf{E}$, $\mathbf{H} \in \boldsymbol{H}(\mathbf{curl};\mathrm{D})$ of (2.2) gives a weak solution of (2.3) and viceversa by setting

$$\mathbf{H} := \frac{\imath}{\omega}\mu^{-1}\,\mathbf{curl}\,\mathbf{E}.$$

2.3. **Existence and uniqueness of solutions.** By multiplying (2.3) by a test function $\mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})$ and integrating by parts, using (2.1), one derives the usual weak formulation of the Maxwell lossy cavity problem (2.3)–(2.4).

**Problem 2.3** (Maxwell cavity problem)**.** We seek $\mathbf{E} \in \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})$ such that, with

$$\begin{aligned} a(\mathbf{U}, \mathbf{V}) &:= \int_{\mathrm{D}} \mu^{-1} \,\mathbf{curl}\,\mathbf{U} \cdot \mathbf{curl}\,\overline{\mathbf{V}}\, \mathrm{d}\boldsymbol{x} - \int_{\mathrm{D}} \omega^2 \varepsilon \mathbf{U} \cdot \overline{\mathbf{V}}\, \mathrm{d}\boldsymbol{x}, \\ F(\mathbf{V}) &:= -\imath\omega \int_{\mathrm{D}} \mathbf{J} \cdot \overline{\mathbf{V}}\, \mathrm{d}\boldsymbol{x}, \end{aligned} \tag{2.5}$$

for all $\mathbf{U}$, $\mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})$, it holds that

$$a(\mathbf{E}, \mathbf{V}) = F(\mathbf{V}) \qquad \forall\, \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D}).$$

We will work under a positivity assumption for the parameters defining $a(\cdot, \cdot)$ (cf. [2, 21]).

**Proposition 2.4.** *Assume that* $\mathbf{J} \in \boldsymbol{L}^2(\mathrm{D})$*, that* $\varepsilon,\ \mu^{-1} \in \boldsymbol{L}^\infty(\mathrm{D}; \mathbb{C}^{3\times 3})$ *and that there exist* $\theta \in \mathbb{R}$ *and* $\alpha > 0$ *such that*

$$\inf_{0 \neq \boldsymbol{\zeta} \in \mathbb{C}^3} \operatorname*{ess\,inf}_{\boldsymbol{x} \in \mathrm{D}} \min \left\{ \frac{\operatorname{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta} \mu(\boldsymbol{x})^{-1} \overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{-\operatorname{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta} \omega^2 \varepsilon(\boldsymbol{x}) \overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}} \right\} \geq \alpha, \tag{2.6}$$

*holds. Then, Problem 2.3 has a unique solution* $\mathbf{E} \in \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})$ *and*

$$\|\mathbf{E}\|_{\boldsymbol{H}(\mathbf{curl}; \mathrm{D})} \leq \frac{1}{\alpha} \|F\|_{\boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})^*}, \tag{2.7}$$

*for* $F$ *in* (2.5) *and* $\alpha > 0$ *as in* (2.6)*.*

*Proof.* Under our assumptions, the sesquilinear form $e^{\imath\theta} a(\cdot, \cdot)$ is coercive and continuous on the space $\boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})$, i.e.,

$$\begin{aligned} &\operatorname{Re}(e^{\imath\theta} a(\mathbf{U}, \mathbf{U})) \geq \alpha \|\mathbf{U}\|^2_{\boldsymbol{H}(\mathbf{curl}; \mathrm{D})} \\ &\operatorname{Re}(e^{\imath\theta} a(\mathbf{U}, \mathbf{V})) \leq C(\|\mu^{-1}\|_{\boldsymbol{L}^\infty(\mathrm{D}; \mathbb{C}^{3\times 3})} + \|\varepsilon\|_{\boldsymbol{L}^\infty(\mathrm{D}; \mathbb{C}^{3\times 3})}) \|\mathbf{U}\|_{\boldsymbol{H}(\mathbf{curl}; \mathrm{D})} \|\mathbf{V}\|_{\boldsymbol{H}(\mathbf{curl}; \mathrm{D})}, \end{aligned}$$

for all $\mathbf{U}$, $\mathbf{V}$ in $\boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})$, and with a constant $C > 0$ independent of the parameters $\mu^{-1}$ and $\varepsilon$. The complex Lax-Milgram lemma (see, e.g., [34, Chap. VI, Thm. 1.4]) implies

$$\mathbf{E} \mapsto e^{\imath\theta} a(\mathbf{E}, \cdot) : \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D}) \to \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})^*$$

to be an isomorphism, so that

$$\mathbf{E} \mapsto a(\mathbf{E}, \cdot) : \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D}) \to \boldsymbol{H}_0(\mathbf{curl}; \mathrm{D})^*$$

is an isomorphism as well. Additionally, the Lax-Milgram lemma gives the *a priori* bound on the solution in (2.7). □

Due to Remark 2.2, we may recover the magnetic field as $\mathbf{H} := \frac{\imath}{\omega} \mu^{-1} \,\mathbf{curl}\,\mathbf{E}$, from where the pair $\mathbf{E}$ and $\mathbf{H}$ belong to $\boldsymbol{H}(\mathbf{curl}; \mathrm{D})$ and solve (2.2).

2.4. **Domain perturbations.** We consider Maxwell's lossy cavity problem (Problem 2.3) on a family of domains given as perturbations of $\widehat{\mathrm{D}} \subset \mathbb{R}^3$, an open and bounded Lipschitz domain henceforth referred to as the *nominal domain*. The set of admissible domain perturbations is $\mathfrak{T}$ and we set $\mathrm{D}_{\mathbf{T}} := \mathbf{T}(\widehat{\mathrm{D}})$ for every $\mathbf{T} \in \mathfrak{T}$. In order to consider Maxwell's equations on the family of domains $\{\mathrm{D}_{\mathbf{T}}\}_{\mathbf{T} \in \mathfrak{T}}$, we will require suitable extensions of the data $\varepsilon$, $\mu$ and $\mathbf{J}$ to every perturbed domain, as well as assumptions on $\mathfrak{T}$. Furthermore, in order to prove uniform convergence rates of finite element solutions of Maxwell's equations on these perturbed domains we enforce smoothness conditions on $\widehat{\mathrm{D}} \subset \mathbb{R}^3$, the perturbations $\mathbf{T} \in \mathfrak{T}$ and the data $\varepsilon$, $\mu$ and $\mathbf{J}$.

*Assumption* 2.5. Fix $N \in \mathbb{N}$, $q > 3$, $\vartheta \in (0, 1)$, $\theta \in \mathbb{R}$ and $\alpha, \alpha_s > 0$. We assume the existence of an open, convex and bounded domain $\mathrm{D}_H \subset \mathbb{R}^3$ such that $\mathrm{D}_{\mathbf{T}} := \mathbf{T}(\widehat{\mathrm{D}}) \subset \mathrm{D}_H$ for all $\mathbf{T} \in \mathfrak{T}$ and assume that the following conditions hold:

(i) the *nominal domain* $\widehat{\mathrm{D}} \subset \mathbb{R}^3$ is a bounded and path connected domain of class $\mathcal{C}^{N,1}$,

(ii) the set of admissible domain perturbations $\mathfrak{T}$ is a compact subset of $\boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}})$ such that every $\mathbf{T} \in \mathfrak{T}$ is bijective and such that $\mathbf{T}^{-1} \in \boldsymbol{\mathcal{C}}^{N,1}(\mathrm{D}_{\mathbf{T}})$, $\det(\mathrm{d}\mathbf{T}) > 0$ everywhere on $\widehat{\mathrm{D}}$ and

$$\vartheta \leq \| \det \mathrm{d}\mathbf{T}\|_{L^\infty(\widehat{\mathrm{D}})}, \quad \| \det \mathrm{d}\mathbf{T}^{-1}\|_{L^\infty(\widehat{\mathrm{D}})}, \quad \|\mathbf{T}\|_{\boldsymbol{\mathcal{C}}^{0,1}(\widehat{\mathrm{D}})}, \quad \|\mathbf{T}^{-1}\|_{\boldsymbol{\mathcal{C}}^{0,1}(\mathrm{D}_{\mathbf{T}})} \leq \vartheta^{-1}, \tag{2.8}$$

(iii) the magnetic permeability is invertible everywhere on $\mathrm{D}_H$ and there holds that $\varepsilon$, $\mu$ and $\mu^{-1}$ belong to $\boldsymbol{W}^{N,\infty}(\mathrm{D}_H;\mathbb{C}^{3\times 3})$, and that $\mathbf{J}$ belongs to $\mathbf{W}^{N,q}(\mathrm{div};\mathrm{D}_H)$,

(iv) $\varepsilon$ and $\mu^{-1}$ satisfy

$$\inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3} \operatorname*{ess\,inf}_{\boldsymbol{x}\in\mathrm{D}_H} \min\left\{ \frac{\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta}\mu(\boldsymbol{x})^{-1}\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{-\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta}\omega^2\varepsilon(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}} \right\} \geq \alpha, \tag{2.9}$$

(v) $\varepsilon$ and $\mu$ satisfy

$$\inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3} \operatorname*{ess\,inf}_{\boldsymbol{x}\in\mathrm{D}_H} \min\left\{ \frac{\mathrm{Re}(\boldsymbol{\zeta}^\top \mu(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{\mathrm{Re}(\boldsymbol{\zeta}^\top \varepsilon(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}} \right\} \geq \alpha_s.$$

*Remark* 2.6. Note that item (iv) in Assumption 2.5 implies a rotated positivity property on the permittivity $\mu$ that could be used to replace item (v) in the same assumption. We choose, however, to include both conditions for brevity and simplicity, since they will be required for different purposes. Item (iv) allows us to ensure existence and uniqueness of Maxwell's equations on each one of the uncertain domains (*cf.* Proposition 2.4), while item (v) is required in [1] to ensure the unique solvability of an auxiliary problem and in order to prove the smoothness properties of solutions to Maxwell's equations (*cf.* Theorem 2.15 below).

*Remark* 2.7. We recall the identity, $\boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}}) = \boldsymbol{W}^{N+1}(\widehat{\mathrm{D}})$ (*cf.* [17, Sec. 2.6.4]), since we will often employ Sobolev norms of the transformations $\mathbf{T} \in \mathfrak{T}$.

We can then consider the following family of $\mathbf{T}$-dependent problems.

**Problem 2.8** (Maxwell cavity problem on perturbed domains)**.** For each $\mathbf{T} \in \mathfrak{T}$, we seek $\mathbf{E}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})$ such that, with

$$a_{\mathbf{T}}(\mathbf{U},\mathbf{V}) := \int_{\mathrm{D}_{\mathbf{T}}} \mu^{-1}\,\mathbf{curl}\,\mathbf{U}\cdot\mathbf{curl}\,\overline{\mathbf{V}} - \omega^2\varepsilon\mathbf{U}\cdot\overline{\mathbf{V}}\,\mathrm{d}\boldsymbol{x},$$
$$F_{\mathbf{T}}(\mathbf{V}) := -\imath\omega \int_{\mathrm{D}_{\mathbf{T}}} \mathbf{J}\cdot\overline{\mathbf{V}}\,\mathrm{d}\boldsymbol{x},$$

for all $\mathbf{U}$, $\mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})$, it holds that

$$a_{\mathbf{T}}(\mathbf{E}_{\mathbf{T}},\mathbf{V}) = F_{\mathbf{T}}(\mathbf{V}) \quad \forall\, \mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}}).$$

Under Assumption 2.5, and arguing as in Section 2.3, there exists a unique solution $\mathbf{E}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})$ to Problem 2.8 for each $\mathbf{T} \in \mathfrak{T}$ satisfying an *a priori* bound.

**Proposition 2.9.** *Under Assumption 2.5 and for each* $\mathbf{T} \in \mathfrak{T}$*, Problem 2.8 has a unique solution* $\mathbf{E}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})$ *satisfying*

$$\|\mathbf{E}_{\mathbf{T}}\|_{\boldsymbol{H}(\mathbf{curl};\mathrm{D}_{\mathbf{T}})} \leq \frac{\omega}{\alpha}\|\mathbf{J}\|_{\boldsymbol{L}^2(\mathrm{D}_{\mathbf{T}})}, \tag{2.10}$$

*for* $\alpha > 0$ *as in* (2.9).

*Proof.* The uniqueness and existence of a solution to Problem 2.8 follows from Assumption 2.5 and the Lax-Milgram lemma as in the proof of Proposition 2.4. Furthermore, the reasoning in the proof of Proposition 2.4 also yields

$$\|\mathbf{E}_{\mathbf{T}}\|_{\boldsymbol{H}(\mathbf{curl};\mathrm{D}_{\mathbf{T}})} \leq \frac{1}{\alpha}\|F_{\mathbf{J},\mathbf{T}}\|_{\boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})^*}.$$

The estimate (2.10) then follows from

$$\|F_{\mathbf{J},\mathbf{T}}\|_{\boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})^*} \leq \omega\|\mathbf{J}\|_{\boldsymbol{L}^2(\mathrm{D}_{\mathbf{T}})}$$

since $\mathbf{J} \in \boldsymbol{L}^2(\mathrm{D}_{\mathbf{T}})$ by assumption. □

2.5. **Pullback to the nominal domain.** As in [2, 26], rather than considering Maxwell's equations on each domain $\{\mathrm{D}_{\mathbf{T}}\}_{\mathbf{T}\in\mathfrak{T}}$, we pull back Problem 2.8 to a family of variational problems set on $\widehat{\mathrm{D}}$ through an appropriate curl-conforming pullback given, for any $\mathbf{T}\in\mathfrak{T}$, as the extension to $\boldsymbol{H}(\mathbf{curl};\mathrm{D}_{\mathbf{T}})$ of

$$\Phi_{\mathbf{T}}(\mathbf{U}) := \mathrm{d}\mathbf{T}^{\top}(\mathbf{U}\circ\mathbf{T}) \tag{2.11}$$

for $\mathbf{U}\in\boldsymbol{\mathcal{C}}^{\infty}(\mathrm{D}_{\mathbf{T}};\mathbb{C}^3)$.

**Lemma 2.10** (Lemma 2.2 in [26])**.** *Under Assumption 2.5, for each* $\mathbf{T}\in\mathfrak{T}$ *the mapping in* (2.11) *can be extended to an isomorphism* $\Phi_{\mathbf{T}}:\boldsymbol{H}(\mathbf{curl};\mathrm{D}_{\mathbf{T}})\to\boldsymbol{H}(\mathbf{curl};\widehat{\mathrm{D}})$*. In addition,* $\Phi_{\mathbf{T}}:\boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})\to\boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$ *is an isomorphism. Furthermore, for* $\mathbf{U}\in\boldsymbol{H}(\mathbf{curl};\mathrm{D}_{\mathbf{T}})$*, it holds that*

$$\mathbf{curl}\,\Phi_{\mathbf{T}}(\mathbf{U}) = \det(\mathrm{d}\mathbf{T})\,\mathrm{d}\mathbf{T}^{-1}\,\mathbf{curl}\,\mathbf{U}\circ\mathbf{T}$$

*in* $\boldsymbol{L}^2(\widehat{\mathrm{D}})$.

We introduce the following family of $\mathbf{T}$-dependent problems over $\boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$.

**Problem 2.11** (Nominal Maxwell cavity problem)**.** For each $\mathbf{T}\in\mathfrak{T}$, we seek $\widehat{\mathbf{E}}_{\mathbf{T}}\in\boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$ such that, with

$$\begin{aligned}\hat{a}_{\mathbf{T}}(\widehat{\mathbf{U}},\widehat{\mathbf{V}}) &:= \int_{\widehat{\mathrm{D}}}\left[\frac{\mu^{-1}\circ\mathbf{T}}{\det(\mathrm{d}\mathbf{T})}\mathrm{d}\mathbf{T}\,\mathbf{curl}\,\widehat{\mathbf{U}}\cdot\mathrm{d}\mathbf{T}\,\mathbf{curl}\,\overline{\widehat{\mathbf{V}}} - \omega^2(\varepsilon\circ\mathbf{T})\det(\mathrm{d}\mathbf{T})\,\mathrm{d}\mathbf{T}^{-\top}\widehat{\mathbf{U}}\cdot\mathrm{d}\mathbf{T}^{-\top}\overline{\widehat{\mathbf{V}}}\right]\mathrm{d}\widehat{\boldsymbol{x}},\\ \hat{F}_{\mathbf{T}}(\widehat{\mathbf{V}}) &:= -\imath\omega\int_{\widehat{\mathrm{D}}}\det(\mathrm{d}\mathbf{T})(\mathbf{J}\circ\mathbf{T})\cdot\mathrm{d}\mathbf{T}^{-\top}\overline{\widehat{\mathbf{V}}}\,\mathrm{d}\widehat{\boldsymbol{x}},\end{aligned} \tag{2.12}$$

for all $\widehat{\mathbf{U}}$, $\widehat{\mathbf{V}}\in\boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$, it holds that

$$\hat{a}_{\mathbf{T}}(\widehat{\mathbf{E}}_{\mathbf{T}},\widehat{\mathbf{V}}) = \hat{F}_{\mathbf{T}}(\widehat{\mathbf{V}})\quad\forall\,\widehat{\mathbf{V}}\in\boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}}). \tag{2.13}$$

*Remark* 2.12. Note that the sesquilinear and antilinear forms in (2.12) may be written as

$$\begin{aligned}\hat{a}_{\mathbf{T}}(\widehat{\mathbf{U}},\widehat{\mathbf{V}}) &= \int_{\widehat{\mathrm{D}}}\left[\mu_{\mathbf{T}}^{-1}\,\mathbf{curl}\,\widehat{\mathbf{U}}\cdot\mathbf{curl}\,\overline{\widehat{\mathbf{V}}} - \omega^2\varepsilon_{\mathbf{T}}\widehat{\mathbf{U}}\cdot\overline{\widehat{\mathbf{V}}}\right]\mathrm{d}\widehat{\boldsymbol{x}},\\ \hat{F}_{\mathbf{T}}(\widehat{\mathbf{V}}) &= -\imath\omega\int_{\widehat{\mathrm{D}}}\mathbf{J}_{\mathbf{T}}\cdot\overline{\widehat{\mathbf{V}}}\,\mathrm{d}\widehat{\boldsymbol{x}},\end{aligned}$$

where

$$\begin{aligned}\mu_{\mathbf{T}} &:= \det(\mathrm{d}\mathbf{T})\,\mathrm{d}\mathbf{T}^{-1}\,(\mu\circ\mathbf{T})\,\mathrm{d}\mathbf{T}^{-\top},\\ \varepsilon_{\mathbf{T}} &:= \det(\mathrm{d}\mathbf{T})\,\mathrm{d}\mathbf{T}^{-1}\,(\varepsilon\circ\mathbf{T})\,\mathrm{d}\mathbf{T}^{-\top},\\ \mathbf{J}_{\mathbf{T}} &:= \det(\mathrm{d}\mathbf{T})\,\mathrm{d}\mathbf{T}^{-1}\,(\mathbf{J}\circ\mathbf{T}).\end{aligned}$$

Also, Lemma 3.59 in [30] yields,

$$\operatorname{div}\mathbf{J}_{\mathbf{T}} = \det(\mathrm{d}\mathbf{T})\operatorname{div}(\mathbf{J}\circ\mathbf{T}),$$

whenever $\mathbf{J}\in\boldsymbol{H}(\mathrm{div};\mathrm{D}_H)$.

Due to Lemma 2.10, Problem 2.11 is equivalent to Problem 2.8, in the sense that, for a fixed $\mathbf{T}\in\mathfrak{T}$, $\widehat{\mathbf{E}}_{\mathbf{T}}\in\boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$ is a solution to Problem 2.11 if and only if $\Phi_{\mathbf{T}}^{-1}(\widehat{\mathbf{E}}_{\mathbf{T}})\in\boldsymbol{H}_0(\mathbf{curl};\mathrm{D}_{\mathbf{T}})$ is a solution to Problem 2.8; we refer to [2, 26] for more details.

**Theorem 2.13.** *Under Assumption 2.5 and for all* $\mathbf{T}\in\mathfrak{T}$*, Problem 2.11 has a unique solution* $\widehat{\mathbf{E}}_{\mathbf{T}}\in\boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$ *satisfying*

$$\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{H}(\mathbf{curl};\widehat{\mathrm{D}})} \le C\frac{\omega}{\alpha}\|\mathbf{J}\|_{\boldsymbol{L}^2(\mathrm{D}_{\mathbf{T}})}, \tag{2.14}$$

*where* $\alpha>0$ *is as in* (2.9) *and the constant* $C>0$ *is independent of* $\mathbf{T}\in\mathfrak{T}$.

*Proof.* Under our assumptions, Proposition 2.11 in [2] ensures that

$$\operatorname{Re}(e^{i\theta}\hat{a}_{\mathbf{T}}(\widehat{\mathbf{U}},\widehat{\mathbf{U}})) \geq \alpha\vartheta^3\|\widehat{\mathbf{U}}\|^2_{\boldsymbol{H}(\mathbf{curl};\widehat{\mathrm{D}})} \quad \text{and} \quad \left|\hat{a}_{\mathbf{T}}(\widehat{\mathbf{U}},\widehat{\mathbf{V}})\right| \leq C\|\widehat{\mathbf{U}}\|_{\boldsymbol{H}(\mathbf{curl};\widehat{\mathrm{D}})}\|\widehat{\mathbf{V}}\|_{\boldsymbol{H}(\mathbf{curl};\widehat{\mathrm{D}})}$$

for all $\mathbf{U}$, $\mathbf{V} \in \boldsymbol{H}(\mathbf{curl};\widehat{\mathrm{D}})$ and all $\mathbf{T} \in \mathfrak{T}$, where the positive continuity constant $C$ depends on $\vartheta$ but is independent of $\mathbf{T} \in \mathfrak{T}$. The complex Lax-Milgram Lemma then ensures the existence and uniqueness of the solution to Problem 2.11 for each $\mathbf{T} \in \mathfrak{T}$ and the *a priori* bound

$$\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{H}(\mathbf{curl};\widehat{\mathrm{D}})} \leq \vartheta^{-3}\frac{\omega}{\alpha}\|\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})},$$

where $\mathbf{J}_{\mathbf{T}}$ is as in Remark 2.12. Assumption 2.5 and a change of variables yield,

$$\|\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})} \leq \vartheta^{-2}\|\mathbf{J}\circ\mathbf{T}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})} \leq \vartheta^{-\frac{5}{2}}\|\mathbf{J}\|_{\boldsymbol{L}^2(\mathrm{D}_{\mathbf{T}})}, \tag{2.15}$$

and (2.14) follows. □

*Remark* 2.14. Note that we have not yet made use of the smoothness properties of the domain $\widehat{\mathrm{D}}$ nor of the parameters $\varepsilon$, $\mu$ and $\mathbf{J}$ nor of the transformations $\mathbf{T} \in \mathfrak{T}$ specified in Assumption 2.5. In fact, all of the results in Sections 2.4 and 2.5 hold with $N = 0$ in Assumption 2.5.

2.6. **Spatial regularity.** We continue by recalling a regularity statement for the solution of (2.2) from [1] (also, see [28, 35] for earlier but less sharp results establishing $\boldsymbol{H}^1$-regularity).

**Theorem 2.15** (Theorem 9 in [1])**.** *Fix $N \in \mathbb{N}$, $q > 3$ and $\alpha_s > 0$ and let $\mathrm{D} \subset \mathbb{R}^3$ be an open and bounded domain of class $\mathcal{C}^{N,1}$. Assume the parameters $\varepsilon$, $\mu$ and $\mathbf{J}$ to satisfy*

$$\varepsilon,\ \mu \in \boldsymbol{W}^{N,q}(\mathrm{D};\mathbb{C}^{3\times3}), \quad \mathbf{J} \in \mathbf{W}^{N-1,q}(\mathrm{div};\mathrm{D}),$$

$$\inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3}\ \operatorname*{ess\,inf}_{\boldsymbol{x}\in\mathrm{D}}\ \min\left\{\frac{\operatorname{Re}(\boldsymbol{\zeta}^\top\mu(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{\operatorname{Re}(\boldsymbol{\zeta}^\top\varepsilon(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}\right\} \geq \alpha_s,$$

*and that the imaginary parts of $\varepsilon$ and $\mu$ are symmetric and let $R > 0$ be such that*

$$R > \max(\|\varepsilon\|_{\boldsymbol{W}^{N,q}(\mathrm{D};\mathbb{C}^{3\times3})}, \|\mu\|_{\boldsymbol{W}^{N,q}(\mathrm{D};\mathbb{C}^{3\times3})}).$$

*Then, there exists a positive constant $C$ depending on $R$, $\omega$, $q$, $\alpha_s$ and $\mathrm{D}$ such that any weak solution pair $\mathbf{E}$, $\mathbf{H} \in \boldsymbol{H}(\mathbf{curl};\mathrm{D})$ of (2.2) belong to $\mathbf{W}^{N,q}(\mathrm{D})$ and satisfy*

$$\|\mathbf{E}\|_{\mathbf{W}^{N,q}(\mathrm{D})} + \|\mathbf{H}\|_{\mathbf{W}^{N,q}(\mathrm{D})} \leq C(\|\mathbf{E}\|_{\boldsymbol{L}^2(\mathrm{D})} + \|\mathbf{H}\|_{\boldsymbol{L}^2(\mathrm{D})} + \|\mathbf{J}\|_{\mathbf{W}^{N-1,q}(\mathrm{div};\mathrm{D})}).$$

*Remark* 2.16. The fact that the constant $C$ in Theorem 2.15 depends on $\mu$ and $\varepsilon$ only through $R > 0$ is not explicitly stated in [1], but follows from the proof of [1, Thm. 9] and the references therein.

We now adapt Theorem 2.15 to our setting and prove, under Assumption 2.5, a uniform (on $\mathbf{T} \in \mathfrak{T}$) smoothness result for the solution of Problem 2.11.

**Theorem 2.17.** *Let Assumption 2.5 hold. Then, for each $\mathbf{T} \in \mathfrak{T}$, the solution $\widehat{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$ to Problem 2.11 belongs to $\boldsymbol{W}^{N,q}(\mathbf{curl};\widehat{\mathrm{D}})$, with the bound*

$$\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\mathbf{curl};\widehat{\mathrm{D}})} \leq C\|\mathbf{J}\|_{\boldsymbol{W}^{N-1,q}(\mathrm{div};\mathrm{D}_H)},$$

*where the constant $C > 0$ depends on $\mathfrak{T}$ but is independent of $\mathbf{T} \in \mathfrak{T}$.*

*Proof.* Theorem 2.13 yields the existence and uniqueness of a solution $\widehat{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\widehat{\mathrm{D}})$ to Problem 2.11 for each $\mathbf{T} \in \mathfrak{T}$. Due to Remark 2.2 and recalling the notation therein introduced, we have

$$\widehat{\mathbf{H}}_{\mathbf{T}} := \imath\omega^{-1}\mu_{\mathbf{T}}^{-1}\,\mathbf{curl}\,\widehat{\mathbf{E}}_{\mathbf{T}},$$

then $\widehat{\mathbf{E}}_{\mathbf{T}}$ and $\widehat{\mathbf{H}}_{\mathbf{T}}$ are weak solution pair to the system:

$$\begin{aligned} \mathbf{curl}\,\widehat{\mathbf{E}}_{\mathbf{T}} + \imath\omega\mu_{\mathbf{T}}\widehat{\mathbf{H}}_{\mathbf{T}} &= \mathbf{0} &&\text{in } \widehat{\mathrm{D}},\\ \imath\omega\varepsilon_{\mathbf{T}}\widehat{\mathbf{E}}_{\mathbf{T}} - \mathbf{curl}\,\widehat{\mathbf{H}}_{\mathbf{T}} &= -\mathbf{J}_{\mathbf{T}} &&\text{in } \widehat{\mathrm{D}}. \end{aligned}$$

The product rule then yields

$$\begin{aligned}\|\mu_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} &\leq C(\mathbf{T})\|\mu\circ\mathbf{T}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})},\\ \|\varepsilon_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} &\leq C(\mathbf{T})\|\varepsilon\circ\mathbf{T}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})},\\ \|\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{W}^{N-1,q}(\widehat{\mathrm{D}})} &\leq C(\mathbf{T})\|\mathbf{J}\circ\mathbf{T}\|_{\boldsymbol{W}^{N-1,q}(\widehat{\mathrm{D}})},\\ \|\operatorname{div}\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{W}^{N-1,q}(\widehat{\mathrm{D}})} &\leq C(\mathbf{T})\|\operatorname{div}\mathbf{J}_{\mathbf{T}}\circ\mathbf{T}\|_{\boldsymbol{W}^{N-1,q}(\widehat{\mathrm{D}})},\end{aligned}\tag{2.16}$$

where the constant $C(\mathbf{T})>0$ depends continuously on $\mathbf{T}\in\mathfrak{T}\Subset\boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}})$. Repeated application of the chain rule (*cf.* [11, Lemma 1] and [12, Lemma 3]) yields,

$$\begin{aligned}\|\varepsilon\circ\mathbf{T}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} &\leq C\left(1+\|\mathbf{T}\|_{\boldsymbol{W}^{N,\infty}(\widehat{\mathrm{D}})}\right)^N\|\det(\mathrm{d}\mathbf{T})^{-1}\|^{\frac{1}{q}}_{L^\infty(\widehat{\mathrm{D}})}\|\varepsilon\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_{\mathbf{T}})}\\ &\leq C\left(1+\|\mathbf{T}\|_{\boldsymbol{W}^{N,\infty}(\widehat{\mathrm{D}})}\right)^N\|\det(\mathrm{d}\mathbf{T})^{-1}\|^{\frac{1}{q}}_{L^\infty(\widehat{\mathrm{D}})}\|\varepsilon\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_H)},\end{aligned}\tag{2.17}$$

where the constant $C>0$ is independent of $\mathbf{T}\in\mathfrak{T}$. Analogously,

$$\begin{aligned}\|\mu\circ\mathbf{T}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} &\leq C\left(1+\|\mathbf{T}\|_{\boldsymbol{W}^{N,\infty}(\widehat{\mathrm{D}})}\right)^N\|\det(\mathrm{d}\mathbf{T})^{-1}\|^{\frac{1}{q}}_{L^\infty(\widehat{\mathrm{D}})}\|\mu\|_{\mathbf{W}^{N,q}(\mathrm{D}_H)},\\ \|\mathbf{J}\circ\mathbf{T}\|_{\boldsymbol{W}^{N-1,q}(\widehat{\mathrm{D}})} &\leq C\left(1+\|\mathbf{T}\|_{\boldsymbol{W}^{N-1,\infty}(\widehat{\mathrm{D}})}\right)^{N-1}\|\det(\mathrm{d}\mathbf{T})^{-1}\|^{\frac{1}{q}}_{L^\infty(\widehat{\mathrm{D}})}\|\mathbf{J}\|_{\mathbf{W}^{N-1,q}(\mathrm{D}_H)},\\ \|\operatorname{div}\mathbf{J}\circ\mathbf{T}\|_{\boldsymbol{W}^{N-1,q}(\widehat{\mathrm{D}})} &\leq C\left(1+\|\mathbf{T}\|_{\boldsymbol{W}^{N-1,\infty}(\widehat{\mathrm{D}})}\right)^{N-1}\|\det(\mathrm{d}\mathbf{T})^{-1}\|^{\frac{1}{q}}_{L^\infty(\widehat{\mathrm{D}})}\|\operatorname{div}\mathbf{J}\|_{\mathbf{W}^{N-1,q}(\mathrm{D}_H)}.\end{aligned}\tag{2.18}$$

The combination of the estimates in (2.16) with those in (2.17) and (2.18) imply that

$$\varepsilon_{\mathbf{T}},\ \mu_{\mathbf{T}}\in\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}};\mathbb{C}^{3\times3}),\quad \mathbf{J}_{\mathbf{T}}\in\mathbf{W}^{N-1,q}(\operatorname{div};\widehat{\mathrm{D}}).$$

Furthermore, for every $\boldsymbol{\zeta}\in\mathbb{C}^3$ and $\widehat{\boldsymbol{x}}\in\widehat{\mathrm{D}}$ we have, due to Assumption 2.5, that

$$\begin{aligned}\left|\boldsymbol{\zeta}^\top\left(\mathrm{d}\mathbf{T}^{-1}\left(\varepsilon\circ\mathbf{T}\right)\mathrm{d}\mathbf{T}^{-\top}\right)(\widehat{\boldsymbol{x}})\boldsymbol{\zeta}\right| &\geq \alpha_s\|\,\mathrm{d}\mathbf{T}^{-\top}(\widehat{\boldsymbol{x}})\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}\\ &\geq \alpha_s\|\,\mathrm{d}\mathbf{T}(\widehat{\boldsymbol{x}})\|^{-2}_{\mathbb{C}^{3\times3}}\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}\geq c\alpha_s\vartheta^2\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3},\end{aligned}\tag{2.19}$$

with an analogous computation for $\left|\boldsymbol{\zeta}^\top(\mathrm{d}\mathbf{T}^{-1}\mu\circ\mathbf{T}\,\mathrm{d}\mathbf{T}^{-\top})(\widehat{\boldsymbol{x}})\boldsymbol{\zeta}\right|$, where the constant $c>0$ follows from the equivalence of norms over finite dimensional spaces and is independent of $\mathbf{T}\in\mathfrak{T}$. Therefore, there holds that

$$\inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3}\operatorname*{ess\,inf}_{\boldsymbol{x}\in\widehat{\mathrm{D}}}\min\left\{\frac{\operatorname{Re}(\boldsymbol{\zeta}^\top\mu_{\mathbf{T}}(\widehat{\boldsymbol{x}})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}},\frac{\operatorname{Re}(\boldsymbol{\zeta}^\top\varepsilon_{\mathbf{T}}(\widehat{\boldsymbol{x}})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}\right\}\geq c\vartheta^3\alpha_s,$$

where the additional power in $\vartheta$ follows from the bound for $\det(\mathbf{T})$ in Assumption 2.5. Theorem 2.15 then ensures that $\widehat{\mathbf{E}}_{\mathbf{T}}$ and $\widehat{\mathbf{H}}_{\mathbf{T}}$ belong to $\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})$ together with the bound

$$\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})}+\|\widehat{\mathbf{H}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})}\leq C(\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})}+\|\widehat{\mathbf{H}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})}+\|\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{W}^{N-1,q}(\operatorname{div};\widehat{\mathrm{D}})}),\tag{2.20}$$

where the constant $C>0$ depends on $R>\max(\|\varepsilon_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}};\mathbb{C}^{3\times3})},\|\mu_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}};\mathbb{C}^{3\times3})})$, $\omega$, $q$, $\alpha_s$, $\vartheta$ and $\widehat{\mathrm{D}}$. The product rule and the previous estimates for $\mu_{\mathbf{T}}$ (2.16) then yield

$$\begin{aligned}\|\operatorname{\mathbf{curl}}\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} &= \omega\|\mu_{\mathbf{T}}\widehat{\mathbf{H}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})}\leq C(\mathbf{T})\|\widehat{\mathbf{H}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})},\\ \|\widehat{\mathbf{H}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})} &= \omega^{-1}\|\mu_{\mathbf{T}}^{-1}\operatorname{\mathbf{curl}}\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})}\leq c(\mathbf{T})\|\operatorname{\mathbf{curl}}\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})},\end{aligned}\tag{2.21}$$

where the constants $C(\mathbf{T}) > 0$ and $c(\mathbf{T}) > 0$ depend on $\mu \in \boldsymbol{W}^{N,q}(\mathrm{D}_H;\mathbb{C}^{3\times 3})$ and, continuously, on $\mathbf{T} \in \mathfrak{T} \Subset \boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}})$. A combination of the estimates in (2.20), (2.21) and (2.14) then yields

$$\begin{aligned}
&\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} + \|\,\mathbf{curl}\,\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} \\
&\leq (1+C(\mathbf{T}))\left(\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})} + \|\widehat{\mathbf{H}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\widehat{\mathrm{D}})}\right) \\
&\leq C(1+C(\mathbf{T}))\left(\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})} + \|\widehat{\mathbf{H}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})} + \|\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{W}^{N-1,q}(\mathrm{div};\widehat{\mathrm{D}})}\right) \\
&\leq C(1+c(\mathbf{T}))(1+C(\mathbf{T}))\left(\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})} + \|\,\mathbf{curl}\,\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{L}^2(\widehat{\mathrm{D}})} + \|\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{W}^{N-1,q}(\mathrm{div};\widehat{\mathrm{D}})}\right) \\
&\leq C(1+c(\mathbf{T}))(1+C(\mathbf{T}))(1+C_{\mathbf{J}}(\mathbf{T}))\frac{C_\vartheta}{\alpha}\|\mathbf{J}\|_{\boldsymbol{W}^{N-1,q}(\mathrm{div};\mathrm{D}_H)},
\end{aligned} \tag{2.22}$$

where the constant $C > 0$ is as in (2.20), $C(\mathbf{T}) > 0$ and $c(\mathbf{T}) > 0$ are as in (2.21), $C_\vartheta > 0$ and $\alpha > 0$ are as in (2.14), with the dependence on $\omega > 0$ has been absorbed by the constant $C$. $C_{\mathbf{J}}(\mathbf{T}) > 0$ follows from combining the estimates for $\mathbf{J}_{\mathbf{T}}$ and $\operatorname{div}\mathbf{J}_{\mathbf{T}}$ in (2.16) and (2.18). The compactness of $\mathfrak{T}$ in $\boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}})$ together with the continuous dependence of the constants on $\mathbf{T} \in \boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}})$ then ensures a uniform bound on $\mathbf{T} \in \mathfrak{T}$ in (2.22). □

Theorem 2.17 will ensure a uniform bound on the convergence rates of finite element approximations of the fields $\widehat{\mathbf{E}}_{\mathbf{T}}$ and will allow for the design of multilevel algorithms in the approximation of the expectation of the mapping $\mathbf{T} \mapsto \widehat{\mathbf{E}}_{\mathbf{T}}$. We continue our analysis by studying the approximation of solutions to Problem 2.11 by the finite element method.

## 3. Discrete solution

To compute a discrete finite element approximation to the solution of Problem 2.11, the test and trial space $\boldsymbol{H}_0(\mathbf{curl};\mathrm{D})$ in (2.13) is to be replaced with a finite dimensional subspace. Since the PDE coefficients defining $\hat{a}_{\mathbf{T}}(\cdot,\cdot)$ and $\hat{F}_{\mathbf{T}}(\cdot)$ in Problem 2.11 are not constant, the corresponding stiffness matrix must itself be approximated by means of numerical quadrature on each element of the mesh, which introduces a further source of error. Following [2, 3], in this section we discuss existence, uniqueness and the approximation properties of such a discrete solution. However, first we provide a framework to accommodate non-polyhedral domains.

### 3.1. Pullback to a polyhedral domain.

The regularity result in Theorem 2.17 requires the domain $\widehat{\mathrm{D}} \subset \mathbb{R}^3$ to possess a $\mathcal{C}^{N,1}$-boundary for some $N \in \mathbb{N}$, which precludes polyhedral domains and the usage of standard tetrahedral meshes. We circumvent this problem by pulling back the respective Maxwell problems to a polyhedral domain, henceforth referred to as the *computational domain*, satisfying the following assumption.

*Assumption* 3.1. Let $\widetilde{\mathrm{D}} \subset \mathbb{R}^3$ be a polyhedral domain, referred to as the *computational domain*. There exists a bijective bi-Lipschitz map $\widehat{\mathbf{T}}$ mapping $\widetilde{\mathrm{D}}$ onto $\widehat{\mathrm{D}}$. For $n \in \mathbb{N}$, there are two sets of pairwise disjoint subsets of $\widetilde{\mathrm{D}}$ and $\widehat{\mathrm{D}}$, $\{\widetilde{\mathrm{D}}_j\}_{j=1}^n$ and $\{\widehat{\mathrm{D}}_j\}_{j=1}^n$, respectively, such that the domains $\{\widetilde{\mathrm{D}}_j\}_{j=1}^n$ are polyhedral, the domains $\{\widehat{\mathrm{D}}_j\}_{j=1}^n$ are Lipschitz, and it holds that

$$\widehat{\mathrm{D}} = \operatorname{int}\left(\bigcup_{j=1}^n \overline{\widehat{\mathrm{D}}_j}\right), \qquad \widetilde{\mathrm{D}} = \operatorname{int}\left(\bigcup_{j=1}^n \overline{\widetilde{\mathrm{D}}_j}\right),$$

$$\widehat{\mathbf{T}}|_{\widetilde{\mathrm{D}}_j} : \widetilde{\mathrm{D}}_j \to \widehat{\mathrm{D}}_j, \quad \widehat{\mathbf{T}}|_{\widetilde{\mathrm{D}}_j} \in \boldsymbol{\mathcal{C}}^{N,1}(\widetilde{\mathrm{D}}_j), \quad \widehat{\mathbf{T}}^{-1}|_{\widehat{\mathrm{D}}_j} \in \boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}}_j) \quad \forall j \in \{1,\ldots,n\},$$

where $N \in \mathbb{N}$ is as in Assumption 2.5.

For each $\mathbf{T} \in \mathfrak{T}$, we introduce the mapping

$$\widetilde{\mathbf{T}} := \mathbf{T} \circ \widehat{\mathbf{T}} : \widetilde{\mathrm{D}} \to \mathrm{D}_{\mathbf{T}},$$

and the set of admissible computational perturbations $\widetilde{\mathfrak{T}} := \{\widetilde{\mathbf{T}} \,:\, \widetilde{\mathbf{T}} := \mathbf{T} \circ \widehat{\mathbf{T}}\ \forall\, \mathbf{T} \in \mathfrak{T}\}$. Figure 1 illustrates the setting of Assumption 3.1.

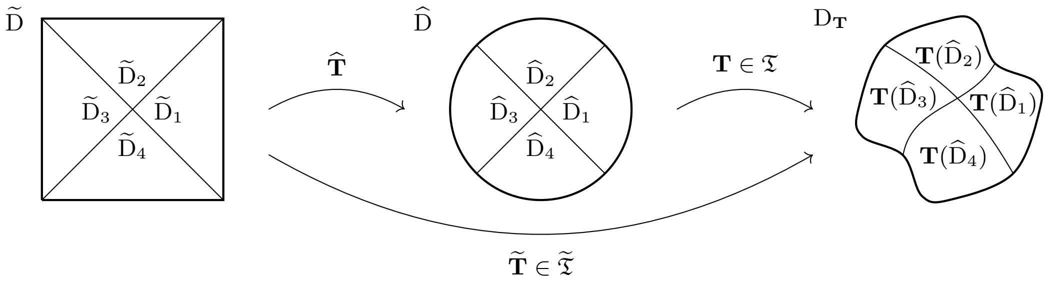



FIGURE 1. Setting for domain transformations. The domains $\widehat{\mathrm{D}}$ and $\widetilde{\mathrm{D}}$, as well as the transformation $\widehat{\mathbf{T}} : \widetilde{\mathrm{D}} \to \widehat{\mathrm{D}}$ are considered fixed. For a family of domain transformations $\mathfrak{T}$, the physical domains are given as $\mathrm{D}_{\mathbf{T}} = \mathbf{T}(\widehat{\mathrm{D}})$ for $\mathbf{T} \in \mathfrak{T}$, or equivalently as $\widetilde{\mathbf{T}}(\widetilde{\mathrm{D}})$ for $\widetilde{\mathbf{T}} = \mathbf{T} \circ \widehat{\mathbf{T}} \in \widetilde{\mathfrak{T}} := \mathfrak{T} \circ \widehat{\mathbf{T}}$. The pullback solution on the smooth nominal domain $\widehat{\mathrm{D}}$ can be shown to belong to a certain regularity class. This allows to deduce convergence rates for finite element approximations of the pullback solutions computed on the polyhedral domain $\widetilde{\mathrm{D}}$.

**Definition 3.2.** Let $\widetilde{\mathrm{D}}$ be as in Assumption 3.1. For $m \in \mathbb{N}$ and $p \in [1, \infty]$, we introduce

$$\boldsymbol{W}^{m,p}_{\mathrm{pw}}(\widetilde{\mathrm{D}}) := \{\mathbf{U} \in \boldsymbol{L}^p(\widetilde{\mathrm{D}}) \,:\, \mathbf{U}|_{\widetilde{\mathrm{D}}_j} \in \boldsymbol{W}^{m,q}(\widetilde{\mathrm{D}}_j) \;\forall j \,\in \{1, \dots, n\}\}$$

with

$$\|\mathbf{U}\|_{\boldsymbol{W}^{m,p}_{\mathrm{pw}}(\widetilde{\mathrm{D}})} := \left( \sum_{i=j}^{n} \|\mathbf{U}|_{\widetilde{\mathrm{D}}_j}\|^p_{\boldsymbol{W}^{m,p}(\widetilde{\mathrm{D}}_j)} \right)^{\frac{1}{p}},$$

if $p < \infty$ and the usual adjustment in case $p = \infty$.

**Definition 3.3.** Let $\widetilde{\mathrm{D}}$ be as in Assumption 3.1. For $m \in \mathbb{N}$ and $p \in [1, \infty]$, we introduce

$$\boldsymbol{W}^{m,p}_{\mathrm{pw}}(\mathbf{curl}; \widetilde{\mathrm{D}}) := \{\mathbf{U} \in \boldsymbol{W}^{m,p}_{\mathrm{pw}}(\widetilde{\mathrm{D}}) \,:\, \mathbf{curl}\,\mathbf{U} \in \boldsymbol{W}^{m,p}_{\mathrm{pw}}(\widetilde{\mathrm{D}})\},$$

with

$$\|\mathbf{U}\|_{\boldsymbol{W}^{m,p}_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})} := \left( \|\mathbf{U}\|^p_{\boldsymbol{W}^{m,p}_{\mathrm{pw}}(\widetilde{\mathrm{D}})} + \|\,\mathbf{curl}\,\mathbf{U}\|^p_{\boldsymbol{W}^{m,p}_{\mathrm{pw}}(\widetilde{\mathrm{D}})} \right)^{\frac{1}{p}},$$

if $p < \infty$ and the usual adjustment in case $p = \infty$. Furthermore, for any $m \in \mathbb{N}$, we set

$$\boldsymbol{H}^{m}_{\mathrm{pw}}(\mathbf{curl}; \widetilde{\mathrm{D}}) := \boldsymbol{W}^{m,2}_{\mathrm{pw}}(\mathbf{curl}; \widetilde{\mathrm{D}}).$$

*Remark* 3.4. Under Assumptions 2.5 and 3.1, compactness of $\mathfrak{T} \Subset \boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}})$ and continuity of $\mathbf{T} \mapsto \mathbf{T} \circ \widehat{\mathbf{T}} : \boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}}) \to \boldsymbol{W}^{N+1,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}})$ imply the compactness of $\widetilde{\mathfrak{T}} = \{\mathbf{T} \circ \widehat{\mathbf{T}} \,:\, \mathbf{T} \in \mathfrak{T}\} \subseteq \boldsymbol{W}^{N+1,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}})$.

**Lemma 3.5.** *Let Assumptions 3.1 hold and let* $\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}}) \cap \boldsymbol{W}^{m,p}(\mathbf{curl}; \widehat{\mathrm{D}})$ *for* $m \in \mathbb{N}$ *with* $m \leq N$*, where* $N \in \mathbb{N}$ *is as in Assumption 3.1, and* $p > 1$*. Then, with* $\widehat{\mathbf{T}} : \widetilde{\mathrm{D}} \to \widehat{\mathrm{D}}$ *as in Assumption 3.1 and* $\Phi_{\widehat{\mathbf{T}}} : \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}}) \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ *as in* (2.11) *and Lemma 2.10, it holds that* $\Phi_{\widehat{\mathbf{T}}}\mathbf{U}$ *belongs to* $\boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}}) \cap \boldsymbol{W}^{m,p}_{\mathrm{pw}}(\mathbf{curl}; \widetilde{\mathrm{D}})$*, with*

$$\|\Phi_{\widehat{\mathbf{T}}}\mathbf{U}\|_{\boldsymbol{W}^{m,p}_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C \|\mathbf{U}\|_{\boldsymbol{W}^{m,p}(\mathbf{curl};\widehat{\mathrm{D}})}, \tag{3.1}$$

*where the constant* $C > 0$ *is independent of* $\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}}) \cap \boldsymbol{W}^{m,p}(\mathbf{curl}; \widehat{\mathrm{D}})$.

*Proof.* Take an arbitrary $\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}}) \cap \boldsymbol{W}^{m,p}(\mathbf{curl}; \widehat{\mathrm{D}})$. Lemma 2.10 gives,

$$\Phi_{\widehat{\mathbf{T}}}\mathbf{U} = \mathrm{d}\widehat{\mathbf{T}}^\top \mathbf{U} \circ \widehat{\mathbf{T}} \quad \text{and} \quad \mathbf{curl}\,\Phi_{\widehat{\mathbf{T}}}\mathbf{U} = \det(\mathrm{d}\widehat{\mathbf{T}})\, \mathrm{d}\widehat{\mathbf{T}}^{-1}\, \mathbf{curl}\,\mathbf{U} \circ \widehat{\mathbf{T}},$$

and $\Phi_{\widehat{\mathbf{T}}}\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$. Fix $j \in \{1, \dots, n\}$. By Assumption 3.1, $\widehat{\mathbf{T}} \in \boldsymbol{W}^{N+1,\infty}(\widetilde{\mathrm{D}}_j)$, according to [11, Lemma 1]—also see [12, Lemma 3]—it holds that $\mathbf{U}|_{\widehat{\mathrm{D}}_j} \circ \widehat{\mathbf{T}}$, $\mathbf{curl}\,\mathbf{U}|_{\widehat{\mathrm{D}}_j} \circ \widehat{\mathbf{T}} \in \boldsymbol{W}^{m,p}(\widetilde{\mathrm{D}}_j)$.

Repeatedly applying the chain rule—as in the proof of Theorem 2.17—yields the existence of $C > 0$, independent of $\mathbf{U} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}})$, such that

$$\|\mathbf{U} \circ \widehat{\mathbf{T}}\|_{\boldsymbol{W}^{m,p}(\widetilde{\mathrm{D}}_j)} \leq C \left(1 + \|\widehat{\mathbf{T}}\|_{\boldsymbol{W}^{m,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}})}\right)^m \|\det(\mathrm{d}\widehat{\mathbf{T}})^{-1}\|^{\frac{1}{p}}_{L^\infty(\widetilde{\mathrm{D}})} \|\mathbf{U}\|_{\boldsymbol{W}^{m,p}(\widehat{\mathrm{D}}_j)},$$
$$\|\,\mathbf{curl}\,\mathbf{U} \circ \widehat{\mathbf{T}}\|_{\boldsymbol{W}^{m,p}(\widetilde{\mathrm{D}}_j)} \leq C \left(1 + \|\widehat{\mathbf{T}}\|_{\boldsymbol{W}^{m,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}})}\right)^m \|\det(\mathrm{d}\widehat{\mathbf{T}})^{-1}\|^{\frac{1}{p}}_{L^\infty(\widetilde{\mathrm{D}})} \|\,\mathbf{curl}\,\mathbf{U}\|_{\boldsymbol{W}^{m,p}(\widehat{\mathrm{D}}_j)}.$$

Furthermore $\mathrm{d}\widehat{\mathbf{T}} \in \boldsymbol{W}^{N,\infty}(\widetilde{\mathrm{D}}_j; \mathbb{C}^{3\times 3})$ and therefore $\Phi_{\widehat{\mathbf{T}}}\mathbf{U}|_{\widetilde{\mathrm{D}}_j} \in \boldsymbol{W}^{m,p}(\widetilde{\mathrm{D}}_j)$ for all $j \in \{1, \dots, n\}$. Analogously, we have that $\det(\mathrm{d}\widehat{\mathbf{T}})\, \mathrm{d}\widehat{\mathbf{T}}^{-1} \in \boldsymbol{W}^{N,\infty}(\widetilde{\mathrm{D}}_j; \mathbb{C}^{3\times 3})$ (upon recalling that $\det(\mathrm{d}\widehat{\mathbf{T}})\, \mathrm{d}\widehat{\mathbf{T}}^{-1}$ is the cofactor matrix of $\mathrm{d}\widehat{\mathbf{T}}$) and therefore $\mathbf{curl}\,\Phi_{\widehat{\mathbf{T}}}\mathbf{U}|_{\widetilde{\mathrm{D}}_j} \in \boldsymbol{W}^{m,p}(\widetilde{\mathrm{D}}_j)$ for all $j \in \{1, \dots, n\}$. The estimate (3.1) then follows by the product rule. □

**Problem 3.6** (Computational Maxwell cavity problem). For each $\mathbf{T} \in \mathfrak{T}$, we seek $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ such that with

$$\tilde{a}_{\mathbf{T}}(\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}}) := \int_{\widetilde{\mathrm{D}}} \left[\mu^{-1}_{\widetilde{\mathbf{T}}}\, \mathbf{curl}\, \widetilde{\mathbf{U}} \cdot \mathbf{curl}\, \overline{\widetilde{\mathbf{V}}} - \omega^2 \varepsilon_{\widetilde{\mathbf{T}}} \widetilde{\mathbf{U}} \cdot \overline{\widetilde{\mathbf{V}}}\right] \mathrm{d}\tilde{\boldsymbol{x}}$$
$$\tilde{F}_{\mathbf{T}}(\widetilde{\mathbf{V}}) := -\imath\omega \int_{\widetilde{\mathrm{D}}} \mathbf{J}_{\widetilde{\mathbf{T}}} \cdot \overline{\widetilde{\mathbf{V}}}\, \mathrm{d}\tilde{\boldsymbol{x}},$$

for all $\widetilde{\mathbf{U}}$, $\widetilde{\mathbf{V}} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$, it holds that

$$\tilde{a}_{\mathbf{T}}(\widetilde{\mathbf{E}}_{\mathbf{T}}, \widetilde{\mathbf{V}}) = \tilde{F}_{\mathbf{T}}(\widetilde{\mathbf{V}}) \quad \forall\, \widetilde{\mathbf{V}} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}}),$$

where $\widetilde{\mathbf{T}} := \mathbf{T} \circ \widehat{\mathbf{T}}$, $\widehat{\mathbf{T}} : \widetilde{\mathrm{D}} \to \widehat{\mathrm{D}}$ is as in Assumption 3.1 and $\mu_{\widetilde{\mathbf{T}}}$, $\varepsilon_{\widetilde{\mathbf{T}}}$ and $\mathbf{J}_{\widetilde{\mathbf{T}}}$ are as in Remark 2.12.

**Theorem 3.7.** *Let Assumptions 2.5 and 3.1 hold. Then, for each* $\mathbf{T} \in \mathfrak{T}$*, there is a unique solution* $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ *to Problem 3.6 that satisfies* $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}}) \cap \boldsymbol{W}^{N,q}_{\mathrm{pw}}(\widetilde{\mathrm{D}})$*, where* $N \in \mathbb{N}$ *and* $q \geq 3$ *are as in Assumption 2.5, with the bound*

$$\|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C \|\mathbf{J}\|_{\boldsymbol{W}^{N-1,q}(\mathrm{div};\mathrm{D}_H)}, \tag{3.2}$$

*where the constant* $C > 0$ *depends on* $\mathfrak{T}$ *but is independent of* $\mathbf{T} \in \mathfrak{T}$.

*Proof.* Under our assumptions, we may repeat our analysis in Sections 2.4 through 2.6 on $\widetilde{\mathrm{D}}$ instead of on $\widehat{\mathrm{D}}$, so that for each $\widetilde{\mathbf{T}} \in \widetilde{\mathfrak{T}}$ there is a unique $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ that solves Problem 3.6. Moreover, as before, it holds that $\widetilde{\mathbf{E}}_{\mathbf{T}} \equiv \Phi_{\widehat{\mathbf{T}}}\widehat{\mathbf{E}}_{\mathbf{T}}$, where $\widehat{\mathbf{T}} : \widetilde{\mathrm{D}} \to \widehat{\mathrm{D}}$ is as in Assumption 3.1, $\Phi_{\widehat{\mathbf{T}}} : \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}}) \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ is as in Lemma 2.10 and $\widehat{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}})$ is the solution of Problem 2.11 (*cf.* [2, 26] for more details). Then, Theorem 2.17 yields $\widehat{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{W}^{N,q}(\mathbf{curl}; \widehat{\mathrm{D}})$ with the bound

$$\|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\mathbf{curl};\widehat{\mathrm{D}})} \leq C \|\mathbf{J}\|_{\boldsymbol{W}^{N-1,q}(\mathrm{div};\mathrm{D}_H)},$$

where the positive constant $C$ depends on $\widetilde{\mathfrak{T}}$ but is independent of $\widetilde{\mathbf{T}} \in \widetilde{\mathfrak{T}}$. Lemma 3.5 then yields a positive constant $C$ such that,

$$\|\Phi_{\widehat{\mathbf{T}}}\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C \|\widehat{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,q}(\mathbf{curl};\widehat{\mathrm{D}})},$$

for each $\mathbf{T} \in \mathfrak{T}$, so that the result then follows from the equivalence $\widetilde{\mathbf{E}}_{\mathbf{T}} \equiv \Phi_{\widehat{\mathbf{T}}}\widehat{\mathbf{E}}_{\mathbf{T}}$. □

3.2. **Finite elements.** We now introduce discretization spaces for Problem 3.6. We shall consider a sequence of affine meshes $\{\tau_{h_i}\}_{i\in\mathbb{N}}$, indexed by their positive mesh-sizes, on the computational domain $\widetilde{\mathrm{D}}$.

*Assumption* 3.8. Let $\widetilde{\mathrm{D}}$ be as in Assumption 3.1. There exist constants $s \in (0, 1)$, $C_1 > 0$, $C_2 > 0$ and a sequence of meshes $\{\tau_{h_i}\}_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ the following conditions hold:

(i) $\tau_{h_i}$ is a set of pairwise disjoint tetrahedrons generally denoted $K$ such that

$$\widetilde{\mathrm{D}} = \mathrm{int}\left(\bigcup_{K\in\tau_{h_i}} \overline{K}\right),$$

(ii) there exists a partition of $\tau_{h_i} = \bigcup_{j=1}^{n} \tau_{h_i,j}$ such that,

$$\widetilde{\mathrm{D}}_j = \operatorname{int}\left( \bigcup_{K\in\tau_{h_i,j}} \overline{K} \right),$$

for all $j \in \{1,\dots,n\}$,

(iii) $\tau_{h_i}$ is a shape-regular and quasi-uniform mesh (*cf.* [20, Chap. 1]),

(iv)

$$C_1 s^i \leq h_i \leq C_2 s^i. \tag{3.3}$$

We denote an arbitrary mesh on the sequence $\{\tau_{h_i}\}_{i\in\mathbb{N}}$ as $\tau_h$. Note that condition (iv) in Assumption 3.8 implies $\lim_{i\to\infty} h_i = 0$.

*Remark* 3.9. Equation (3.3) ensures that the cardinality $|\tau_{h_i}|$ of the mesh $\tau_{h_i}$ increases. Specifically, it holds that

$$C_{s,3} s^{-3i} \leq \dim(\boldsymbol{P}_k^c(\tau_{h_i})) \leq C_{s,4} s^{-3i}, \tag{3.4}$$

for a second pair of positive constants $C_{s,3}$ and $C_{s,4}$. This will be relevant for the multilevel results presented in Section 5.

In the following, we assume given a reference tetrahedron $\breve{K} \subset \mathbb{R}^3$ such that for every $K \in \tau_h$ there is an affine bijective map $\boldsymbol{T}_K : \breve{K} \mapsto K$. For an arbitrary tetrahedron $K$, we shall make use of the following space of polynomial functions of degree $k \in \mathbb{N}$,

$$\boldsymbol{P}_k^c(K) := \mathbb{P}_{k-1}\left(K;\mathbb{C}^3\right) \oplus \{\boldsymbol{p} \in \widetilde{\mathbb{P}}_k\left(K,\mathbb{C}^3\right) \,:\, \boldsymbol{x}\cdot\boldsymbol{p}(\boldsymbol{x}) = 0 \quad \forall\, \boldsymbol{x} \in K\}.$$

The curl-conforming edge finite element (FE) on a tetrahedron $K$ is given by the triple

$$(K, \boldsymbol{P}_k^c(K), \Sigma_k^c(K)),$$

where $\Sigma_K^c$ is a set of uni-solvent linear functionals over $\boldsymbol{P}_k^c(K)$ (*cf.* [30, Sec. 5.5]). The curl-conforming FE space—satisfying the PEC boundary condition (2.4)—on an affine mesh $\tau_h \in \{\tau_{h_i}\}_{i\in\mathbb{N}}$ is then built as follows

$$\boldsymbol{P}_k^c(\tau_h) := \{\mathbf{V} \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}}) \,:\, \mathbf{V}|_K \in \boldsymbol{P}_k^c(K) \quad \forall K \in \tau_h\}.$$

For the sake of brevity, we avoid specifying all properties satisfied by the mappings $\boldsymbol{T}_K$ as well as those satisfied by the space $\boldsymbol{P}_k^c(\tau_h)$ (see [20] and [30] and references therein).

### 3.3. Discrete problem, Quadrature error and Strang's Lemma.

We continue by stating the fully discrete version of Problem 2.11 and briefly comment on the conditions required of the quadrature rules used to approximate the integrals defining $\tilde{a}_{\mathbf{T}}(\cdot,\cdot)$ and $\tilde{F}_{\mathbf{T}}(\cdot)$ in Problem 3.6 to ensure convergence rates of the solution to the fully discrete problem. For a more detailed analysis we refer to [2, 3].

3.3.1. *Numerical quadrature.* On the fixed reference tetrahedron $\breve{K}$, we define a quadrature rule $Q : \mathcal{C}^0(\breve{K};\mathbb{C}) \to \mathbb{C}$ as

$$Q(f) := \sum_{l=1}^{L} \breve{w}_l f(\breve{\boldsymbol{b}}_l),$$

for certain quadrature nodes $(\breve{\boldsymbol{b}}_l)_{l=1}^L \subseteq \breve{K}$ and quadrature weights $(\breve{w}_l)_{l=1}^L \subseteq \mathbb{R}\backslash\{0\}$. Given a (nondegenerate) tetrahedron $K$ and the affine bijective element map $\boldsymbol{T}_K : \breve{K} \to K$ we obtain a transformed quadrature rule $Q_K : \mathcal{C}(K;\mathbb{C}) \to \mathbb{C}$ on $K$ via

$$Q_K(f) := \sum_{l=1}^{L} w_{l,K} f(\boldsymbol{b}_{l,K}) \qquad \text{where} \qquad w_{l,K} := |\det(\mathrm{d}\boldsymbol{T}_K)|\, \breve{w}_l, \quad \boldsymbol{b}_{l,K} := \boldsymbol{T}_K(\breve{\boldsymbol{b}}_l). \tag{3.5}$$

3.3.2. *Discrete variational formulation.* Approximating all the integrals in Problem 3.6 with quadratures $Q_K^{\bullet}$ as in (3.5)—on each element $K$ of the mesh $\tau_h$—leads to the following sesquilinear and antilinear forms:

$$\tilde{a}_{h;\mathbf{T}}(\widetilde{\mathbf{U}}_h, \widetilde{\mathbf{V}}_h) := \sum_{K\in\tau_h} Q_K^1\left(\mu_{\widetilde{\mathbf{T}}}^{-1}\,\mathbf{curl}\,\widetilde{\mathbf{U}}_h \cdot \mathbf{curl}\,\overline{\widetilde{\mathbf{V}}_h}\right) - \omega^2 Q_K^2\left(\varepsilon_{\widetilde{\mathbf{T}}}\widetilde{\mathbf{U}}_h \cdot \overline{\widetilde{\mathbf{V}}_h}\right), \tag{3.6}$$

and

$$\tilde{F}_{h;\mathbf{T}}(\widetilde{\mathbf{V}}_h) := -\imath\omega \sum_{K\in\tau_h} Q_K^2\left(\mathbf{J}_{\widetilde{\mathbf{T}}} \cdot \overline{\widetilde{\mathbf{V}}_h}\right), \tag{3.7}$$

for all $\widetilde{\mathbf{U}}_h$, $\widetilde{\mathbf{V}}_h \in \boldsymbol{P}_k^c(\tau_h)$, where we have used the same notation as in the statement of Problem 3.6, $Q_K^1$ and $Q_K^2$ are two different quadrature rules on each $K \in \tau_h$, constructed from two different quadrature rules $Q^1$ and $Q^2$ over $\breve{K}$ as indicated in equation (3.5). Since the quadrature rules require pointwise function evaluations to be well-defined, here $\mu^{-1} : \mathrm{D}_H \to \mathbb{C}^{3\times3}$, $\varepsilon : \mathrm{D}_H \to \mathbb{C}^{3\times3}$ and $\mathbf{J} : \mathrm{D}_H \to \mathbb{C}^3$ are required to be continuous in each element $K \in \tau_h$. Function evaluations on the boundary of an element $K$ are understood with respect to the interior limit on the element $K$. With the previous definitions at hand, we arrive at the fully discrete variational problem.

**Problem 3.10** (Fully discrete computational Maxwell cavity problem)**.** For each $\mathbf{T} \in \mathfrak{T}$, we seek $\widetilde{\mathbf{E}}_{\mathbf{T},h} \in \boldsymbol{P}_k^c(\tau_h)$ such that

$$\tilde{a}_{h;\mathbf{T}}(\widetilde{\mathbf{E}}_{\mathbf{T},h}, \widetilde{\mathbf{V}}_h) = \tilde{F}_{h;\mathbf{T}}(\widetilde{\mathbf{V}}_h) \qquad \forall\, \widetilde{\mathbf{V}}_h \in \boldsymbol{P}_k^c(\tau_h). \tag{3.8}$$

**Theorem 3.11.** *Let Assumptions 2.5, 3.1 and 3.8 hold and assume that the weights of the quadratures $Q^1$, $Q^2$ are positive and at least one of the following two conditions:*

(i) *The nodes defining $Q^1$ and $Q^2$ are $\mathbb{P}_{k-1}(\breve{K};\mathbb{C})$ and $\mathbb{P}_k(\breve{K};\mathbb{C})$-unisolvent, respectively.*
(ii) *$Q^1$ and $Q^2$ are exact on $\mathbb{P}_{2k-2}(\breve{K};\mathbb{C})$ and $\mathbb{P}_{2k}(\breve{K};\mathbb{C})$, respectively.*

*Then, there exists a unique solution $\widetilde{\mathbf{E}}_{\mathbf{T},h} \in \boldsymbol{P}_k^c(\tau_h)$ of Problem 3.10 and it holds that*

$$\|\widetilde{\mathbf{E}}_{\mathbf{T},h}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C\frac{\omega}{\alpha}\|\mathbf{J}\|_{\boldsymbol{L}^2(\mathrm{D}_\mathbf{T})}, \tag{3.9}$$

*where $\alpha > 0$ is as in (2.9) and the constant $C > 0$ is independent of the mesh-size and of $\mathbf{T} \in \mathfrak{T}$, but depends on $\vartheta$ in (2.8).*

*Proof.* The discrete coercivity

$$\left|\tilde{a}_{h;\mathbf{T}}(\widetilde{\mathbf{U}}_h, \widetilde{\mathbf{U}}_h)\right| \geq C\alpha\|\widetilde{\mathbf{U}}_h\|^2_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \quad \forall\, \widetilde{\mathbf{U}}_h \in \boldsymbol{P}_0^c(\tau_h),$$

where $\alpha > 0$ is as in (2.9) and $C > 0$ is independent of both the mesh-size and $\mathbf{T} \in \mathfrak{T}$, but depends on $\vartheta$ in (2.8), was shown in [2, Thm. 3.13]. The continuity

$$\left|\tilde{a}_{h;\mathbf{T}}(\widetilde{\mathbf{U}}_h, \widetilde{\mathbf{V}}_h)\right| \leq C\|\widetilde{\mathbf{U}}_h\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})}\|\widetilde{\mathbf{V}}_h\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \quad \forall\, \widetilde{\mathbf{U}}_h,\, \widetilde{\mathbf{V}}_h \in \boldsymbol{P}_0^c(\tau_h),$$

on the other hand, follows from [2, Lem. 3.12], where $C > 0$ is as before and not necessarily the same in each appearance. Moreover, by an application of [2, Lem. 3.12 (i)] we have that

$$\left|\tilde{F}_{h;\mathbf{T}}(\widetilde{\mathbf{V}}_h)\right| = \left|\omega \sum_{K\in\tau_h} Q_K^2\left(\mathbf{J}_{\widetilde{\mathbf{T}}} \cdot \overline{\widetilde{\mathbf{V}}_h}\right)\right| \leq C\omega \sum_{K\in\tau_h} \|\mathbf{J}_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{L}^2(K)}\|\widetilde{\mathbf{V}}_h\|_{\boldsymbol{L}^2(K)} \quad \forall\, \widetilde{\mathbf{V}}_h \in \boldsymbol{P}_0^c(\tau_h),$$

for $C > 0$ as before. Together with our assumptions, the Cauchy-Schwartz inequality and (2.15), the complex Lax-Milgram Lemma then ensures the existence and uniqueness of the solution to Problem 3.10 for each $\mathbf{T} \in \mathfrak{T}$ and the *a priori* bound in (3.9). □

*Remark* 3.12. Much like the results in Sections 2.4 and 2.5, Theorem 3.11 makes no use of the smoothness of $\widehat{\mathrm{D}}$, the parameters $\varepsilon$, $\mu$ and $\mathbf{J}$ or of the transformations $\mathbf{T} \in \mathfrak{T}$, and still hold true when $N = 0$ in Assumption 2.5.

3.3.3. *Discretization error.* The discretization error $\|\widetilde{\mathbf{E}}_{\mathbf{T}} - \widetilde{\mathbf{E}}_{\mathbf{T},h}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})}$ may be bounded with the help of Strang's Lemma. The following results—studied first in [2, Sec. 3] and later generalized in [3]— will give sufficient conditions on the quadrature rules defining $\tilde{a}_{h;\mathbf{T}}(\cdot,\cdot)$ and $\tilde{F}_{h;\mathbf{T}}(\cdot)$ to ensure bounds on the discretization error with respect to the mesh-size. We begin by stating Strang's Lemma (*cf.* [13]).

For $\widetilde{\mathbf{U}} \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})$ and arbitrary $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$, set

$$\mathrm{A}_{1,\mathbf{T},i}(\widetilde{\mathbf{U}}) := \inf_{\widetilde{\mathbf{U}}_{h_i} \in \boldsymbol{P}_k^c(\tau_{h_i})} \|\widetilde{\mathbf{U}} - \widetilde{\mathbf{U}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} + \sup_{\substack{\widetilde{\mathbf{V}}_{h_i} \in \boldsymbol{P}_k^c(\tau_{h_i}) \\ \widetilde{\mathbf{V}}_{h_i} \neq 0}} \frac{|\tilde{a}_{\mathbf{T}}(\widetilde{\mathbf{U}}_{h_i}, \widetilde{\mathbf{V}}_{h_i}) - \tilde{a}_{h;\mathbf{T}}(\widetilde{\mathbf{U}}_{h_i}, \widetilde{\mathbf{V}}_{h_i})|}{\|\widetilde{\mathbf{V}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})}},$$

$$\mathrm{A}_{2,\mathbf{T},i} := \sup_{\substack{\widetilde{\mathbf{V}}_{h_i} \in \boldsymbol{P}_k^c(\tau_{h_i}) \\ \widetilde{\mathbf{V}}_{h_i} \neq 0}} \frac{|\tilde{F}_{\mathbf{T}}(\widetilde{\mathbf{V}}_{h_i}) - \tilde{F}_{h;\mathbf{T}}(\widetilde{\mathbf{V}}_{h_i})|}{\|\widetilde{\mathbf{V}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})}}.$$

**Lemma 3.13.** *Under the assumptions of Theorems 2.13 and 3.11, there exist unique solutions* $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})$ *and* $\widetilde{\mathbf{E}}_{\mathbf{T},h_i} \in \boldsymbol{P}_k^c(\tau_{h_i})$ *of Problems 3.6 and 3.10, respectively, and* $c > 0$ *independent of the mesh-size and of* $\mathbf{T} \in \mathfrak{T}$ *such that*

$$\|\widetilde{\mathbf{E}}_{\mathbf{T}} - \widetilde{\mathbf{E}}_{\mathbf{T},h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq c(\mathrm{A}_{1,\mathbf{T},i}(\widetilde{\mathbf{E}}_{\mathbf{T}}) + \mathrm{A}_{2,\mathbf{T},i}),$$

*holds for every* $i \in \mathbb{N}$ *and* $\mathbf{T} \in \mathfrak{T}$.

We continue by stating relevant consistency error estimates that will permit a later application of Strang's Lemma. They correspond to adaptations to our context of Theorems 2 and 3 in [3] (*cf.* Lemmas 3.15 and 3.16 in [2]).

**Lemma 3.14.** *Let Assumptions 2.5, 3.1 and 3.8 hold. If the quadrature rule* $Q^2$ *on* $\breve{K}$ *is exact on polynomials of degree* $k+N-1$*, where* $N \in \mathbb{N}$ *is as in Assumption 2.5, then, for any sequence* $\{\widetilde{\mathbf{V}}_{h_i}\}_{i\in\mathbb{N}}$ *with* $\widetilde{\mathbf{V}}_{h_i} \in \boldsymbol{P}_k^c(\tau_{h_i})$ *for all* $i \in \mathbb{N}$*, it holds that*

$$|\tilde{F}_{\mathbf{T}}(\widetilde{\mathbf{V}}_{h_i}) - \tilde{F}_{h;\mathbf{T}}(\widetilde{\mathbf{V}}_{h_i})| \leq C h_i^N |\widetilde{\mathrm{D}}|^{\frac{1}{2}-\frac{1}{q}} \|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_H)} \|\widetilde{\mathbf{V}}_{h_i}\|_{0,\widetilde{\mathrm{D}}},$$

*for each* $\mathbf{T} \in \mathfrak{T}$*, where* $q > 3$ *is as in Assumption 2.5 and the constant* $C > 0$ *is independent of* $i \in \mathbb{N}$ *and* $\mathbf{T} \in \mathfrak{T}$.

*Proof.* Fix $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$ and recall $\widetilde{\mathbf{T}} := \mathbf{T} \circ \widehat{\mathbf{T}}$ for $\widehat{\mathbf{T}} : \widetilde{\mathrm{D}} \to \widehat{\mathrm{D}}$ as in Assumption 3.1. An application of the chain and product rules as in the proof of Theorem 2.17 yields, together with Assumption 3.1, for any $j \in \{1, \dots, n\}$ with $n \in \mathbb{N}$ as in Assumption 3.1, that

$$\|\mathbf{J}_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,q}(\widetilde{\mathrm{D}}_j)} \leq C\|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathbf{T}(\widehat{\mathrm{D}}_j))}.$$

The constant $C$ may be chosen independently of $\mathbf{T} \in \mathfrak{T}$, so that $\mathbf{J}_{\widetilde{\mathbf{T}}} \in \boldsymbol{W}^{N,q}_{\mathrm{pw}}(\widetilde{\mathrm{D}})$ and

$$\|\mathbf{J}_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,q}_{\mathrm{pw}}(\widetilde{\mathrm{D}})} \leq C\|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_{\mathbf{T}})} \leq C\|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_H)}, \tag{3.10}$$

for $C > 0$ not necessarily the same in each appearance. Then, and for any $\widetilde{\mathbf{V}}_{h_i} \in \boldsymbol{P}_k^c(\tau_{h_i})$, Lemma 7 in [3] yields

$$\left|\tilde{F}_{\mathbf{T}}(\widetilde{\mathbf{V}}_{h_i}) - \tilde{F}_{h;\mathbf{T}}(\widetilde{\mathbf{V}}_{h_i})\right| \leq C h_i^N \sum_{K\in\tau_{h_i}} |K|^{\frac{1}{2}-\frac{1}{p}} \|\mathbf{J}_{\mathbf{T}}\|_{\boldsymbol{W}^{N,p}(K)} \|\widetilde{\mathbf{V}}_{h_i}\|_{0,K}, \tag{3.11}$$

where the positive constant is independent of $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$. Then, using that $\mathbf{J}_{\widetilde{\mathbf{T}}} \in \boldsymbol{W}^{N,p}_{\mathrm{pw}}(\widetilde{\mathrm{D}})$ with $p > 2$ and Assumption 3.8, Hölder's inequality yields

$$\sum_{K\in\tau_{h_i}} |K|^{\frac{1}{2}-\frac{1}{q}} \|\mathbf{J}_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,q}(K)} \|\widetilde{\mathbf{V}}_{h_i}\|_{0,K} \leq |\widetilde{\mathrm{D}}|^{\frac{1}{2}-\frac{1}{q}} \|\mathbf{J}_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,q}_{\mathrm{pw}}(\widetilde{\mathrm{D}})} \|\widetilde{\mathbf{V}}_{h_i}\|_{0,\widetilde{\mathrm{D}}}. \tag{3.12}$$

Combining the estimates in (3.10), (3.11) and (3.12) yields the required result. □

**Lemma 3.15.** *Let Assumptions 2.5, 3.1 and 3.8 hold. If the quadrature rules $Q^1$ and $Q^2$ on $\breve{K}$ are exact on polynomials of degree $k+N-2$ and $k+N-1$, respectively, where $N \in \mathbb{N}$ is as in Assumption 2.5, then, for any pair of sequences $\{\widetilde{\mathbf{U}}_{h_i}\}_{i\in\mathbb{N}}$ and $\{\widetilde{\mathbf{V}}_{h_i}\}_{i\in\mathbb{N}}$ with $\widetilde{\mathbf{U}}_{h_i}$, $\widetilde{\mathbf{V}}_{h_i} \in \boldsymbol{P}_k^c(\tau_{h_i})$ for all $i \in \mathbb{N}$, it holds that*

$$|\tilde{a}_{\mathbf{T}}(\widetilde{\mathbf{U}}_{h_i}, \widetilde{\mathbf{V}}_{h_i}) - \tilde{a}_{h;\mathbf{T}}(\widetilde{\mathbf{U}}_{h_i}, \widetilde{\mathbf{V}}_{h_i})| \leq h_i^N C \|\widetilde{\mathbf{U}}_{h_i}\|_{\boldsymbol{H}^N_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})} \|\widetilde{\mathbf{V}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})},$$

*for each $\mathbf{T} \in \mathfrak{T}$, where the constant $C > 0$ is independent of $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$.*

*Proof.* An analogous reasoning as that in the proof of Lemma 3.14 shows that there exists a constant $C > 0$ such that

$$\begin{aligned} \|\mu_{\widetilde{\mathbf{T}}}^{-1}\|_{\boldsymbol{W}^{N,\infty}(\widetilde{\mathrm{D}}_j;\mathbb{C}^{3\times3})} &\leq C\|\mu^{-1}\|_{\boldsymbol{W}^{N,\infty}(\mathbf{T}(\widehat{\mathrm{D}}_j);\mathbb{C}^{3\times3})}, \\ \|\varepsilon_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,\infty}(\widetilde{\mathrm{D}}_j;\mathbb{C}^{3\times3})} &\leq C\|\varepsilon\|_{\boldsymbol{W}^{N,\infty}(\mathbf{T}(\widehat{\mathrm{D}}_j);\mathbb{C}^{3\times3})}, \end{aligned}$$

for any $\mathbf{T} \in \mathfrak{T}$ and $j \in \{1,\ldots,n\}$—$n \in \mathbb{N}$ is as in Assumption 3.1. Therefore, we have that $\varepsilon_{\widetilde{\mathbf{T}}}$, $\mu_{\widetilde{\mathbf{T}}}^{-1} \in \boldsymbol{W}^{N,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}};\mathbb{C}^{3\times3})$, with

$$\begin{aligned} \|\mu_{\widetilde{\mathbf{T}}}^{-1}\|_{\boldsymbol{W}^{N,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}};\mathbb{C}^{3\times3})} &\leq C\|\mu^{-1}\|_{\boldsymbol{W}^{N,\infty}(\mathrm{D}_{\mathbf{T}};\mathbb{C}^{3\times3})}, \\ \|\varepsilon_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,\infty}_{\mathrm{pw}}(\widehat{\mathrm{D}};\mathbb{C}^{3\times3})} &\leq C\|\varepsilon\|_{\boldsymbol{W}^{N,\infty}(\mathrm{D}_{\mathbf{T}};\mathbb{C}^{3\times3})}, \end{aligned} \tag{3.13}$$

for $C > 0$ as before.

Fix $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$. For any pair $\widetilde{\mathbf{U}}_{h_i}$, $\widetilde{\mathbf{V}}_{h_i} \in \boldsymbol{P}_k^c(\tau_{h_i})$ [3, Lemma 6]—also [3, Thm. 2]—yields

$$\begin{aligned} &|\tilde{a}_{\mathbf{T}}(\widetilde{\mathbf{U}}_{h_i}, \widetilde{\mathbf{V}}_{h_i}) - \tilde{a}_{h;\mathbf{T}}(\widetilde{\mathbf{U}}_{h_i}, \widetilde{\mathbf{V}}_{h_i})| \\ &\leq C h_i^N \sum_{K\in\tau_{h_i}} C_{\varepsilon_{\widetilde{\mathbf{T}}},K}\|\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\widetilde{\mathbf{V}}_{h_i}\|_{0,K} + C_{\mu_{\widetilde{\mathbf{T}}}^{-1},K}\|\mathbf{curl}\,\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\mathbf{curl}\,\widetilde{\mathbf{V}}_{h_i}\|_{0,K}, \end{aligned} \tag{3.14}$$

where, for each $K \in \tau_{h_i}$, we have defined

$$C_{\varepsilon_{\widetilde{\mathbf{T}}},K} := \omega^2\|\varepsilon_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,\infty}(K)} \quad \text{and} \quad C_{\mu_{\widetilde{\mathbf{T}}}^{-1},K} := \|\mu_{\widetilde{\mathbf{T}}}^{-1}\|_{\boldsymbol{W}^{N,\infty}(K)},$$

and the constant $C > 0$ is independent of $\varepsilon, \mu \in \boldsymbol{W}^{N,\infty}(\mathrm{D}_H;\mathbb{C}^{3\times3})$, $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$. Since $\varepsilon_{\widetilde{\mathbf{T}}}$, $\mu_{\widetilde{\mathbf{T}}}^{-1} \in \boldsymbol{W}^{N,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}};\mathbb{C}^{3\times3})$, we have that

$$\begin{aligned} &\sum_{K\in\tau_{h_i}} C_{\varepsilon_{\widetilde{\mathbf{T}}},K}\|\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\widetilde{\mathbf{V}}_{h_i}\|_{0,K} + C_{\mu_{\widetilde{\mathbf{T}}}^{-1},K}\|\mathbf{curl}\,\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\mathbf{curl}\,\widetilde{\mathbf{V}}_{h_i}\|_{0,K} \\ &\leq C_{\varepsilon_{\widetilde{\mathbf{T}}}} \sum_{K\in\tau_{h_i}} \|\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\widetilde{\mathbf{V}}_{h_i}\|_{0,K} + C_{\mu_{\widetilde{\mathbf{T}}}^{-1}} \sum_{K\in\tau_{h_i}} \|\mathbf{curl}\,\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\mathbf{curl}\,\widetilde{\mathbf{V}}_{h_i}\|_{0,K} \\ &\leq \max(C_{\varepsilon_{\widetilde{\mathbf{T}}}}, C_{\mu_{\widetilde{\mathbf{T}}}^{-1}}) \sum_{K\in\tau_{h_i}} \|\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\widetilde{\mathbf{V}}_{h_i}\|_{0,K} + \|\mathbf{curl}\,\widetilde{\mathbf{U}}_{h_i}\|_{N,K}\|\mathbf{curl}\,\widetilde{\mathbf{V}}_{h_i}\|_{0,K} \\ &\leq \max(C_{\varepsilon_{\widetilde{\mathbf{T}}}}, C_{\mu_{\widetilde{\mathbf{T}}}^{-1}}) \sum_{K\in\tau_{h_i}} \|\widetilde{\mathbf{U}}_{h_i}\|_{\boldsymbol{H}^N(\mathbf{curl};K)}\|\widetilde{\mathbf{V}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};K)} \\ &\leq \max(C_{\varepsilon_{\widetilde{\mathbf{T}}}}, C_{\mu_{\widetilde{\mathbf{T}}}^{-1}})\|\widetilde{\mathbf{U}}_{h_i}\|_{\boldsymbol{H}^N_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})}\|\widetilde{\mathbf{V}}_{h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})}, \end{aligned} \tag{3.15}$$

where

$$C_{\varepsilon_{\widetilde{\mathbf{T}}}} := \omega^2\|\varepsilon_{\widetilde{\mathbf{T}}}\|_{\boldsymbol{W}^{N,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}})} \quad \text{and} \quad C_{\mu_{\widetilde{\mathbf{T}}}^{-1}} := \|\mu_{\widetilde{\mathbf{T}}}^{-1}\|_{\boldsymbol{W}^{N,\infty}_{\mathrm{pw}}(\widetilde{\mathrm{D}})}.$$

Combining the estimates in (3.13), (3.14) and (3.15) yields the required result. □

In virtue of the requirements of Lemmas 3.14 and 3.15, we continue under the next assumption on the data $\varepsilon$, $\mu$ and $\mathbf{J}$ as well as on the quadrature rules used to construct the sesquilinear and antilinear forms in (3.6) and (3.7).

*Assumption* 3.16. Recall $k \in \mathbb{N}$ as the polynomial degree of the finite element spaces $\boldsymbol{P}_k^c(\tau_h)$ and let $N \in \mathbb{N}$ be as in Assumption 2.5. We assume that $k \leq N$, that the weights of the quadratures $Q^1$ and $Q^2$ are positive, that $Q^1$ and $Q^2$ are exact on polynomials of degree $k+N-2$ and $k+N-1$, respectively, and at least one of the following two conditions is satisfied

(i) The nodes defining $Q^1$ and $Q^2$ are $\mathbb{P}_{k-1}(\breve{K};\mathbb{C})$ and $\mathbb{P}_k(\breve{K};\mathbb{C})$-unisolvent, respectively.
(ii) $Q^1$ and $Q^2$ are exact on $\mathbb{P}_{2k-2}(\breve{K};\mathbb{C})$ and $\mathbb{P}_{2k}(\breve{K};\mathbb{C})$, respectively.

The combination of Lemmas 3.14 and 3.15 together with Strang's Lemma (Lemma 3.13) yields the following estimate for the convergence rate of $\widetilde{\mathbf{E}}_{\mathbf{T},h}$ to $\widetilde{\mathbf{E}}_{\mathbf{T}}$, solutions to Problems 3.10 and 3.6, respectively.

**Theorem 3.17.** *Let Assumptions 2.5, 3.1, 3.8 and 3.16 hold. Then, for any $\mathbf{T} \in \mathfrak{T}$, there exists a unique solution of Problem 3.10, $\widetilde{\mathbf{E}}_{\mathbf{T},h_i} \in \boldsymbol{P}_k^c(\tau_{h_i})$, which satisfies*

$$\|\widetilde{\mathbf{E}}_{\mathbf{T}} - \widetilde{\mathbf{E}}_{\mathbf{T},h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C h_i^k \|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_H)},$$

*where $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})$ is the solution of Problem 3.6, $N \in \mathbb{N}$ and $q > 3$ are as in Assumption 3.16 and the constant $C > 0$ is independent of $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$.*

*Proof.* Fix $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$. Theorems 3.7 and 3.11 ensure the existence of unique solutions $\widetilde{\mathbf{E}}_{\mathbf{T}}$ and $\widetilde{\mathbf{E}}_{\mathbf{T},h_i}$ for each $i \in \mathbb{N}$ of Problems 3.6 and 3.10, respectively. Furthermore, Theorem 3.7 also states the piecewise smoothness of the solution of Problem 3.6, i.e., $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{W}_{\mathrm{pw}}^{N,q}(\mathbf{curl};\widetilde{\mathrm{D}})$, with the bound

$$\|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}_{\mathrm{pw}}^{N,q}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C\|\mathbf{J}\|_{\boldsymbol{W}^{N-1,q}(\mathrm{div};\mathrm{D}_H)}, \tag{3.16}$$

where the constant $C > 0$ is independent of $\mathbf{T} \in \mathfrak{T}$. Note that our assumption that $\mathbf{J} \in \boldsymbol{W}^{N,q}(\mathrm{D}_H)$ in Assumption 3.16 implies that $\mathbf{J} \in \boldsymbol{W}^{N-1,q}(\mathrm{div};\mathrm{D}_H)$. Moreover, since $q > 3$ we have that $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_{\mathrm{pw}}^N(\mathbf{curl};\widetilde{\mathrm{D}})$ with

$$\|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{H}_{\mathrm{pw}}^{N}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C\|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{W}_{\mathrm{pw}}^{N,q}(\mathbf{curl};\widetilde{\mathrm{D}})}, \tag{3.17}$$

for $C > 0$ independent of $\mathbf{T} \in \mathfrak{T}$. Now, let $\mathcal{I}_k^c : \boldsymbol{H}^N(\mathbf{curl};\widetilde{\mathrm{D}}) \to \boldsymbol{P}_k^c(\tau_{h_i})$ be the canonical curl-conforming interpolation operator (*cf.* [30, Sec. 5.5]), which is a bounded operator in the $\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})$-norm in $\boldsymbol{P}_k^c(\tau_{h_i})$ for any $N \in \mathbb{N}$ (see [30, Lemma 5.38]). Lemmas 3.13 (Strang's Lemma), 3.14 and 3.15 then yield,

$$\begin{aligned}
&\|\widetilde{\mathbf{E}}_{\mathbf{T}} - \widetilde{\mathbf{E}}_{\mathbf{T},h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \\
&\leq C\left[\|\widetilde{\mathbf{E}}_{\mathbf{T}} - \mathcal{I}_k^c(\widetilde{\mathbf{E}}_{\mathbf{T}})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} + h_i^N\left(\|\mathcal{I}_k^c(\widetilde{\mathbf{E}}_{\mathbf{T}})\|_{\boldsymbol{H}_{\mathrm{pw}}^N(\mathbf{curl};\widetilde{\mathrm{D}})} + |\widetilde{\mathrm{D}}|^{\frac{1}{2}-\frac{1}{q}}\|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_H)}\right)\right],
\end{aligned} \tag{3.18}$$

where $C > 0$ may be chosen to be independent of both $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$. Since the approximation and continuity properties of $\mathcal{I}_k^c$ hold on each mesh element $K \in \tau_{h_i}$ (*cf.* [30, Lem. 5.48, Thm. 5.41 & Rmk. 5.42]), they also hold on $\widehat{\mathrm{D}}_j$ for each $j \in \{1,\ldots,n\}$ by virtue of Assumption 3.1, so that

$$\begin{aligned}
\|\widetilde{\mathbf{E}}_{\mathbf{T}} - \mathcal{I}_k^c(\widetilde{\mathbf{E}}_{\mathbf{T}})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} &\leq c h_i^k \|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{H}_{\mathrm{pw}}^k(\mathbf{curl};\widetilde{\mathrm{D}})}, \\
\|\mathcal{I}_k^c(\widetilde{\mathbf{E}}_{\mathbf{T}})\|_{\boldsymbol{H}_{\mathrm{pw}}^N(\mathbf{curl};\widetilde{\mathrm{D}})} &\leq c\|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{H}_{\mathrm{pw}}^N(\mathbf{curl};\widetilde{\mathrm{D}})},
\end{aligned} \tag{3.19}$$

where $c > 0$ is, once again, independent of $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$. A combination of the estimates in (3.16), (3.17), (3.18) and (3.19) yields,

$$\begin{aligned}
&\|\widetilde{\mathbf{E}}_{\mathbf{T}} - \widetilde{\mathbf{E}}_{\mathbf{T},h_i}\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \\
&\leq C\left[h_i^k\|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{H}_{\mathrm{pw}}^k(\mathbf{curl};\widetilde{\mathrm{D}})} + h_i^N\left(\|\widetilde{\mathbf{E}}_{\mathbf{T}}\|_{\boldsymbol{H}_{\mathrm{pw}}^N(\mathbf{curl};\widetilde{\mathrm{D}})} + |\widetilde{\mathrm{D}}|^{\frac{1}{2}-\frac{1}{q}}\|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_H)}\right)\right] \\
&\leq C(1 + |\widehat{\mathrm{D}}|^{\frac{1}{2}-\frac{1}{q}})h_i^k\|\mathbf{J}\|_{\boldsymbol{W}^{N,q}(\mathrm{D}_H)}
\end{aligned}$$

where the constant $C > 0$ is not necessarily the same in each appearance but is always independent of $i \in \mathbb{N}$ and $\mathbf{T} \in \mathfrak{T}$. ∎

# 4. Parametric solutions

We now introduce a rigorous framework to treat domain uncertainties described by infinite-dimensional parametrizations of admissible perturbations $\mathfrak{T}$. Proving convergence rates for the approximation of the parametric solution will require smoothness results with respect to the parameters. In particular, we recall shape holomorphy results [26] and also discuss higher-order spatial regularity of these extensions. As such, the current section serves as preparation for the subsequent convergence results.

## 4.1. **Admissible parameters.**

We shall allow the perturbations $\mathbf{T} \in \mathfrak{T}$ to be given through a random variable with values in the compact set $\mathfrak{T} \Subset \boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}})$ (see Assumption 2.5). Let, in the following,

$$Z_N := \boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}}; \mathbb{C}^3), \tag{4.1}$$

and

$$Z := \boldsymbol{\mathcal{C}}^{0,1}(\widehat{\mathrm{D}}; \mathbb{C}^3). \tag{4.2}$$

Note that we have continuous embedding $Z_N \hookrightarrow Z$. Throughout, elements of $Z_N$ will always additionally be interpreted to belong to $Z$ without distinction in the notation. Moreover, and as in [2], we will also require the data $\varepsilon$, $\mu$ and $\mathbf{J}$ to possess holomorphic extensions to an open set in $\mathbb{C}^3$ containing the hold-all domain $\mathrm{D}_H$. We shall work under the following assumption on $\mathfrak{T}$ and on the data.

*Assumption* 4.1. There exists an open set $O_{\mathrm{D}_H} \subset \mathbb{C}^3$, such that $\mathrm{D}_H \subset O_{\mathrm{D}_H}$, and holomorphic extensions of $\epsilon$, $\mu$ and $\mathbf{J}$ (for which we use the same notation) to $O_{\mathrm{D}_H}$ satisfying, for some $\theta \in \mathbb{R}$, $\alpha > 0$ and $\alpha_s > 0$, the following bounds:

$$\inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3} \operatorname*{ess\,inf}_{\boldsymbol{x}\in O_{\mathrm{D}_H}} \min\left\{ \frac{\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta}\mu(\boldsymbol{x})^{-1}\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{-\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta}\omega^2\varepsilon(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}} \right\} \geq \alpha,$$

$$\inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3} \operatorname*{ess\,inf}_{\boldsymbol{x}\in O_{\mathrm{D}_H}} \min\left\{ \frac{\mathrm{Re}(\boldsymbol{\zeta}^\top \mu(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{\mathrm{Re}(\boldsymbol{\zeta}^\top \varepsilon(\boldsymbol{x})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}} \right\} \geq \alpha_s.$$

## 4.2. **Holomorphic extension in $\boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$.**

We now show that the solution map possesses certain holomorphic extensions. The term *solution map* here refers to the function mapping each perturbation $\mathbf{T} \in \mathfrak{T}$ to the solution of either Problem 3.6 or 3.10. By *holomorphic* we mean that this map is complex Fréchet differentiable as a function between two complex Banach spaces.

Before proceeding, we rewrite the $\mathbf{T} \in \mathfrak{T}$-dependent quantities in the sesquilinear and antilinear forms defining Problems 3.6 and 3.10 so that the dependence on $\mathbf{T} \in \mathfrak{T}$ is made explicit. Then, with the notation introduced in Remark 2.12, it holds that

$$\begin{aligned}
\mu_{\widetilde{\mathbf{T}}} &= \det(\mathrm{d}\widehat{\mathbf{T}})\,\mathrm{d}\widehat{\mathbf{T}}^{-1}\det(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})^{-1}(\mu\circ\mathbf{T}\circ\widehat{\mathbf{T}})(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})^{-\top}\,\mathrm{d}\widehat{\mathbf{T}}^{-\top},\\
\varepsilon_{\widetilde{\mathbf{T}}} &= \det(\mathrm{d}\widehat{\mathbf{T}})\,\mathrm{d}\widehat{\mathbf{T}}^{-1}\det(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})^{-1}(\varepsilon\circ\mathbf{T}\circ\widehat{\mathbf{T}})(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})^{-\top}\,\mathrm{d}\widehat{\mathbf{T}}^{-\top},\\
\mathbf{J}_{\widetilde{\mathbf{T}}} &= \det(\mathrm{d}\widehat{\mathbf{T}})\,\mathrm{d}\widehat{\mathbf{T}}^{-1}\det(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})(\mathrm{d}\mathbf{T}\circ\widehat{\mathbf{T}})^{-1}(\mathbf{J}\circ\mathbf{T}\circ\widehat{\mathbf{T}}).
\end{aligned} \tag{4.3}$$

The structure of the coefficients in (4.3) is only slightly different from the structure of the coefficients considered in [2, Sec. 4]. Specifically, the coefficients only differ by the composition with the fixed transformation $\widehat{\mathbf{T}} : \widetilde{\mathrm{D}} \to \widehat{\mathrm{D}}$ and by the product with $\widehat{\mathbf{T}}$-depending quantities. Therefore, the proofs of Theorems 4.2 and 4.3, establishing holomorphic extensions of the continuous and discrete solution maps. Indeed, mapping each $\mathbf{T} \in \mathfrak{T}$ to the solutions of Problems 3.6 and 3.10 are only slight variations of the proofs of Theorems 4.5 and 4.8 in [2] and are omitted for brevity.

### 4.2.1. *Exact solution.*

**Theorem 4.2.** *Let Assumptions 2.5, 3.1 and 4.1 hold. Then, there exists an open set $O_{\mathfrak{T}} \subseteq Z$, with $\mathfrak{T} \subseteq O_{\mathfrak{T}}$, and a holomorphic function $\widetilde{\mathfrak{E}} : O_{\mathfrak{T}} \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ such that, for every $\mathbf{T} \in O_{\mathfrak{T}}$,* *there exists a unique solution* $\widetilde{\mathbf{E}}_{\mathbf{T}} \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})$ *of Problem 3.6 and* $\widetilde{\mathbf{E}}_{\mathbf{T}} = \widetilde{\mathfrak{E}}(\mathbf{T})$*. Moreover, it holds that*

$$\sup_{\mathbf{T}\in O_{\mathfrak{T}}} \|\widetilde{\mathfrak{E}}(\mathbf{T})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} < \infty. \tag{4.4}$$

The bound in (4.4) is not stated explicitly in [2, Thm. 4.5], but can be achieved by choosing the open superset $O_{\mathfrak{T}}$ of the compact set $\mathfrak{T}$ small enough (as in [2, Thm. 4.8]).

4.2.2. *Discrete solution.* Using the discrete holomorphy result [2, Thm. 4.8], we obtain a discrete version of Theorem 4.2. Recall that $k$ denotes the fixed polynomial degree of our approximation spaces introduced in Section 3.2.

**Theorem 4.3** (Theorem 4.8 in [2])**.** *Let the assumptions of Theorem 4.2, as well as Assumptions 3.8 and 3.16 hold. Then, there exists an open set* $O_{\mathfrak{T}} \subseteq Z$ *independent of the mesh* $\tau_h \in \{\tau_{h_i}\}_{i\in\mathbb{N}}$*, with* $\mathfrak{T} \subseteq O_{\mathfrak{T}}$*, and holomorphic functions* $\widetilde{\mathfrak{E}}_h : O_{\mathfrak{T}} \to \boldsymbol{P}^c_k(\tau_h)$ *such that, for every* $\mathbf{T} \in O_{\mathfrak{T}}$*, there exists a unique solution* $\widetilde{\mathbf{E}}_{\mathbf{T},h} \in \boldsymbol{P}^c_k(\tau_h)$ *of Problem 3.10 and* $\widetilde{\mathbf{E}}_{\mathbf{T},h} = \widetilde{\mathfrak{E}}_h(\mathbf{T})$*. Moreover,*

$$\sup_{\mathbf{T}\in O_{\mathfrak{T}}} \|\widetilde{\mathfrak{E}}_h(\mathbf{T})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} < \infty.$$

*Remark* 4.4. The fact that we may take the open set $O_{\mathfrak{T}}$ as a subset of $Z$ in Theorems 4.2 and 4.3 follows from Remarks 2.14 and 3.12.

4.3. **Extension in** $\boldsymbol{W}^{N,q}_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})$**.** In addition to establishing existence of holomorphic extensions in $\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})$ with quantified bounds on the size of the holomorphy domains, proving dimension-independent convergence rate bounds of multilevel algorithms requires higher order spatial regularity. For our analysis it will suffice that the solution map allows an extension as a mapping to $\boldsymbol{W}^{N,q}_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})$, but we shall not require holomorphy with respect to this topology.

4.3.1. *Exact Solution.*

**Proposition 4.5.** *Let Assumptions 2.5, 3.1 and 4.1 hold. Then, with* $O_{\mathfrak{T}} \subseteq Z$ *and* $\widetilde{\mathfrak{E}}$ *as in Theorem 4.2, there exists an open set* $O_{N,\mathfrak{T}} \subset Z_N$ *with* $\mathfrak{T} \subseteq O_{N,\mathfrak{T}} \subseteq O_{\mathfrak{T}}$ *such that for every* $\mathbf{T} \in O_{N,\mathfrak{T}}$ *it holds that* $\widetilde{\mathfrak{E}}(\mathbf{T}) \in \boldsymbol{W}^{N,q}_{\mathrm{pw}}(\mathbf{curl};\widehat{\mathrm{D}})$*, where* $N \in \mathbb{N}$ *and* $q > 3$ *are as in Assumption 3.16. In addition, one has*

$$\sup_{\mathbf{T}\in O_{N,\mathfrak{T}}} \|\widetilde{\mathfrak{E}}(\mathbf{T})\|_{\boldsymbol{W}^{N,q}_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})} < \infty. \tag{4.5}$$

*Proof.* In the proof of Theorem 2.17 (see (2.19)), we showed that for every $\mathbf{T} \in \mathfrak{T}$ and $\boldsymbol{x} \in \mathrm{D}_H$ there holds that

$$\begin{aligned}
\mathrm{Re}(\boldsymbol{\zeta}^\top \varepsilon_{\mathbf{T}}(\boldsymbol{x})\boldsymbol{\zeta}) &\geq \alpha_s \|\,\mathrm{d}\mathbf{T}^{-\top}(\boldsymbol{x})\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq \alpha_s \|\,\mathrm{d}\mathbf{T}(\boldsymbol{x})\|^{-2}_{\mathbb{C}^{3\times 3}} \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq c\alpha_s\vartheta^2 \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3},\\
\mathrm{Re}(\boldsymbol{\zeta}^\top \mu_{\mathbf{T}}(\boldsymbol{x})\boldsymbol{\zeta}) &\geq \alpha_s \|\,\mathrm{d}\mathbf{T}^{-\top}(\boldsymbol{x})\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq \alpha_s \|\,\mathrm{d}\mathbf{T}(\boldsymbol{x})\|^{-2}_{\mathbb{C}^{3\times 3}} \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq c\alpha_s\vartheta^2 \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3},
\end{aligned}$$

for all $\mathbf{T} \in \mathfrak{T}$ and $x \in \mathrm{D}_H$, where $\alpha_s > 0$ is as in Assumption 4.1, $\vartheta \in (0,1)$ as in Assumption 2.5, and $c > 0$ is a constant independent of $\mathbf{T} \in \mathfrak{T}$. An analogous computation shows that

$$\begin{aligned}
-\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta} \varepsilon_{\mathbf{T}}(\boldsymbol{x})\boldsymbol{\zeta}) &\geq \alpha \|\,\mathrm{d}\mathbf{T}^{-\top}(\boldsymbol{x})\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq \alpha \|\,\mathrm{d}\mathbf{T}(\boldsymbol{x})\|^{-2}_{\mathbb{C}^{3\times 3}} \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq c\alpha\vartheta^2 \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3},\\
\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta} \mu^{-1}_{\mathbf{T}}(\boldsymbol{x})\boldsymbol{\zeta}) &\geq \alpha \|\,\mathrm{d}\mathbf{T}^{-\top}(\boldsymbol{x})\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq \alpha \|\,\mathrm{d}\mathbf{T}(\boldsymbol{x})\|^{-2}_{\mathbb{C}^{3\times 3}} \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3} \geq c\alpha\vartheta^2 \|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3},
\end{aligned}$$

where $\alpha > 0$ and $\theta \in \mathbb{R}$ are as in Assumption 4.1 and $\vartheta \in (0,1)$ and $c > 0$ are as before. Under our assumptions, we can find a bounded open set around each $\mathbf{T} \in \mathfrak{T}$, denoted $N_{\mathbf{T}}$, such that,

$$\begin{aligned}
\inf_{\mathbf{L}\in N_{\mathbf{T}}} \inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3} \operatorname*{ess\,inf}_{\boldsymbol{x}\in\widehat{\mathrm{D}}} \min\left\{ \frac{\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta}\mu_{\mathbf{L}}(\widehat{\boldsymbol{x}})^{-1}\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{-\mathrm{Re}(\boldsymbol{\zeta}^\top e^{\imath\theta}\varepsilon_{\mathbf{L}}(\widehat{\boldsymbol{x}})\overline{\boldsymbol{\zeta}})}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}} \right\} &\geq \tilde{\alpha} := \frac{c\alpha\vartheta^3}{2},\\
\inf_{\mathbf{L}\in N_{\mathbf{T}}} \inf_{0\neq\boldsymbol{\zeta}\in\mathbb{C}^3} \operatorname*{ess\,inf}_{\boldsymbol{x}\in\widehat{\mathrm{D}}} \min\left\{ \frac{\boldsymbol{\zeta}^\top\mu_{\mathbf{L}}(\widehat{\boldsymbol{x}})\overline{\boldsymbol{\zeta}}}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}}, \frac{\boldsymbol{\zeta}^\top\varepsilon_{\mathbf{L}}(\widehat{\boldsymbol{x}})\overline{\boldsymbol{\zeta}}}{\|\boldsymbol{\zeta}\|^2_{\mathbb{C}^3}} \right\} &\geq \tilde{\alpha}_s := \frac{c\alpha_s\vartheta^3}{2},
\end{aligned} \tag{4.6}$$

where, for $\mathbf{L} \in N_{\mathbf{T}}$, $\mu_{\mathbf{L}}$ and $\varepsilon_{\mathbf{L}}$ are as in Remark 2.12. The compactness of $\mathfrak{T}$ implies that we can cover $\mathfrak{T}$ by a finite number of such sets $N_{\mathbf{T}}$, whose union yields an open and bounded set denoted $O_{N,\mathfrak{T}}$ on which there hold the uniform coercivity conditions in (4.6). Decreasing the open set $O_{N,\mathfrak{T}}$ if necessary, together with an application of Theorem 3.7 yields the uniform bound in (4.5) (*cf.* Theorem 2.17 for the dependence of the constant $C > 0$ in (3.2) on $\mathbf{T} \in \mathfrak{T}$. □

4.3.2. *Discrete Solution.* For the discrete solution, Proposition 4.5 and Theorem 3.17 imply the following uniform approximation result. Its proof results from arguments analogous to those in the proof of Proposition 4.5.

**Proposition 4.6.** *Let Assumptions 2.5 through 4.1 hold. Then, with the holomorphic mappings* $\widetilde{\mathfrak{E}} : \mathfrak{T} \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ *and* $\widetilde{\mathfrak{E}}_h : \mathfrak{T} \to \boldsymbol{P}_k^c(\tau_h)$ *introduced in Theorems 4.2 and 4.3, respectively, there holds that*

$$\sup_{\mathbf{T}\in O_{N,\mathfrak{T}}} \|\widetilde{\mathfrak{E}}(\mathbf{T}) - \widetilde{\mathfrak{E}}_{h_i}(\mathbf{T})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C B_{N,\mathfrak{T},\mathbf{J}} h_i^k,$$

*with*

$$B_{N,\mathfrak{T},\mathbf{J}} := \sup_{\mathbf{T}\in O_{N,\mathfrak{T}}} \|\widetilde{\mathfrak{E}}(\mathbf{T})\|_{\boldsymbol{H}^N_{\mathrm{pw}}(\mathbf{curl};\widetilde{\mathrm{D}})} + \|\mathbf{J}\|_{\boldsymbol{\mathcal{C}}^N(O_{\mathrm{D}_H})},$$

*for each* $i \in \mathbb{N}$*, where* $N \in \mathbb{N}$ *and* $q > 3$ *are as in Assumption 2.5,* $k \leq N$ *is as in Assumption 3.16,* $O_{N,\mathfrak{T}} \subset Z_N$ *is as in Proposition 4.5 and the constant* $C > 0$ *is independent of the mesh-size.*

4.4. **Parameter discretization.** Numerical computations require to suitably discretize the parameter set $\mathfrak{T}$ in Assumption 4.1. To this end, set $U := [-1, 1]^{\mathbb{N}}$ and assume that

$$\{\widetilde{\mathbf{T}}_j\}_{j\in\mathbb{N}_0} \subset Z_N$$

is a summable sequence in $Z_N$ (see (4.1)), i.e.,

$$\{\|\widetilde{\mathbf{T}}_j\|_{Z_N}\}_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0).$$

For $\boldsymbol{y} \in U$, as in [2] we define

$$\sigma(\boldsymbol{y}) := \mathbf{T}_0 + \sum_{j\in\mathbb{N}} y_j \mathbf{T}_j \in Z_N \hookrightarrow Z, \tag{4.7}$$

and we consider the set of admissible domain perturbations to be given as

$$\mathfrak{T} := \{\sigma(\boldsymbol{y}) \,:\, \boldsymbol{y} \in U\}. \tag{4.8}$$

The continuity of $\sigma : U \to \mathfrak{T}$ and the compactness of $U$, with the product topology (see [31, Sec. 12.2]) yields the compactness of $\mathfrak{T}$ in $Z_N$. Hence, for every $\boldsymbol{y} \in U$, $\sigma(\boldsymbol{y}) \in \mathfrak{T}$ is an expansion of a perturbation $\mathbf{T} \in Z_N$ in terms of the sequence $\boldsymbol{y}$. Throughout what follows, we interpret item (ii) in Assumption 2.5 as an Assumption on the sequence $\{\|\widetilde{\mathbf{T}}_j\|_{Z_N}\}_{j\in\mathbb{N}_0}$, i.e., we assume that $\{\|\widetilde{\mathbf{T}}_j\|_{Z_N}\}_{j\in\mathbb{N}_0}$ is such that item (ii) in Assumption 2.5 holds. Moreover, introducing the infinite product probability measure $\mu = \otimes_{j\in\mathbb{N}} \frac{\lambda}{2}$, where $\lambda$ denotes the Lebesgue measure on $[-1, 1]$, we can consider $\sigma$ in (4.7) as a random variable on the probability space $(U, \mathfrak{B}, \mu)$, where $\mathfrak{B}$ is the infinite product Borel $\sigma$-algebra on $U$. Under Assumption 4.1, Problem 3.6 has a unique solution for each $\boldsymbol{y} \in U$, which we shall denote by $\widetilde{\mathbf{E}}(\boldsymbol{y}) \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$, more precisely, with the solution map $\widetilde{\mathfrak{E}}$ from Theorem 4.2,

$$\widetilde{\mathbf{E}}(\boldsymbol{y}) := \widetilde{\mathfrak{E}}(\sigma(\boldsymbol{y})) \in \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}}). \tag{4.9}$$

Similarly, the discrete problem (Problem 3.10) possesses a unique solution

$$\widetilde{\mathbf{E}}_h(\boldsymbol{y}) := \widetilde{\mathfrak{E}}_h(\sigma(\boldsymbol{y})) \in \boldsymbol{P}_k^c(\tau_h), \tag{4.10}$$

where $\widetilde{\mathfrak{E}}_h$ is as in Theorem 4.3. Thus, our goal is to approximate the expected values of the mappings $\widetilde{\mathbf{E}} : U \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ and $\widetilde{\mathbf{E}}_h : U \to \boldsymbol{P}_k^c(\tau_h)$.

Furthermore, Theorems 4.2 and 4.3, together with the continuity of $\sigma : U \to \mathfrak{T}$ in (4.7) and the compactness of $U$ with the product topology (*cf.* [31, Sec. 12.2]) yield the following result.

**Proposition 4.7.** *Let Assumptions 2.5 through 4.1 hold. Further assume that, for* $\sigma : U \to Z_N$ *as in* (4.7)*, it holds that* $\{\|\mathbf{T}_j\|_{Z_N}\}_{j\in\mathbb{N}}$ *belongs to* $\ell^1(\mathbb{N})$*. Then, the mappings* $\widetilde{\mathbf{E}}$ *and* $\widetilde{\mathbf{E}}_h$ *in* (4.9) *and*

(4.10), *respectively, are such that* $\widetilde{\mathbf{E}} \in L^2(U, \mathfrak{B}, \upmu; \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}}))$ *and* $\widetilde{\mathbf{E}}_h \in L^2(U, \mathfrak{B}, \upmu; \boldsymbol{P}^c_k(\tau_h))$. *Moreover, under the assumptions of Proposition 4.6, there holds that*

$$\|\mathbb{E}(\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}_{h_i})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C B_{N,\mathfrak{T},\mathbf{J}} h_i^k,$$
$$\|\widetilde{\mathbf{E}}_{h_{i+1}} - \widetilde{\mathbf{E}}_{h_i}\|_{L^2(U;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} \leq C B_{N,\mathfrak{T},\mathbf{J}} h_{i+1}^k,$$

*for each* $i \in \mathbb{N}$*, where the constant* $C > 0$ *is independent of the mesh-size,* $N \in \mathbb{N}$*,* $q > 3$ *and* $k \leq N$ *are as in Assumption 3.16, and* $O_{N,\mathfrak{T}} \subset Z_N$ *is as in Proposition 4.5.*

*Proof.* From our assumption that $\{\|\mathbf{T}_j\|_{Z_N}\}_{j\in\mathbb{N}}$ belongs to $\ell^1(\mathbb{N})$ there follows that the mapping $\sigma : U \to Z_N$ is continuous, while Theorems 4.2 and 4.3 imply the continuity of the mappings $\widetilde{\mathfrak{E}} : \mathfrak{T} \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ and $\widetilde{\mathfrak{E}}_h : \mathfrak{T} \to \boldsymbol{P}^c_k(\tau_h)$, respectively. Since, the composition $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{E}}_h$ are defined as the compositions of $\widetilde{\mathfrak{E}}$ and $\widetilde{\mathfrak{E}}_h$, respectively, with the continuous mapping $\sigma$, there holds that they are continuous as well. An application of Corollary A.2.3 in [36] then gives the Bochner integrability of $\widetilde{\mathbf{E}} : U \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ and $\widetilde{\mathbf{E}}_h : U \to \boldsymbol{P}^c_k(\tau_h)$, with

$$\left(\int_U \|\widetilde{\mathbf{E}}(\boldsymbol{y})\|^2_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \,\mathrm{d}\upmu(\boldsymbol{y})\right)^{\frac{1}{2}} \leq \sup_{\mathbf{T}\in O_{\mathfrak{T}}} \|\widetilde{\mathfrak{E}}(\mathbf{T})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} < \infty,$$
$$\left(\int_U \|\widetilde{\mathbf{E}}_h(\boldsymbol{y})\|^2_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \,\mathrm{d}\upmu(\boldsymbol{y})\right)^{\frac{1}{2}} \leq \sup_{\mathbf{T}\in O_{\mathfrak{T}}} \|\widetilde{\mathfrak{E}}_h(\mathbf{T})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} < \infty,$$

where $O_{\mathfrak{T}} \subset Z$ is as in Theorems 4.2 and 4.3. Moreover, by an application of Proposition 4.6 and for each $i \in \mathbb{N}$, we have that

$$\|\mathbb{E}(\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}_{h_i})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \leq \int_U \|\widetilde{\mathbf{E}}(\boldsymbol{y}) - \widetilde{\mathbf{E}}_{h_i}(\boldsymbol{y})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} \,\mathrm{d}\upmu(\boldsymbol{y}) \leq C B_{N,\mathfrak{T},\mathbf{J}} h_i^k,$$

and, for each $i \in \mathbb{N}$,

$$\begin{aligned}
\|\widetilde{\mathbf{E}}_{h_{i+1}} - \widetilde{\mathbf{E}}_{h_i}\|_{L^2(U;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} &\leq \|\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}_{h_{i+1}}\|_{L^2(U;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} + \|\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}_{h_i}\|_{L^2(U;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} \\
&\leq C B_{N,\mathfrak{T},\mathbf{J}} (h_{i+1}^k + h_i^k) \\
&\leq C B_{N,\mathfrak{T},\mathbf{J}} \left(1 + \frac{C_2}{s C_1}\right) h_{i+1}^k,
\end{aligned}$$

where $O_{N,\mathfrak{T}} \subset Z_N$ is as in Proposition 4.5, $N \in \mathbb{N}$ is as in Assumptions 2.5 and 3.1, $q > 3$ is as in Assumption 3.16, $C > 0$ is as in Proposition 4.6 and $0 < C_1 \leq C_2$ are as in Assumption 3.8. □

## 5. Multilevel Approximation

### 5.1. **Multilevel Monte Carlo.** 

To present the Multilevel Monte Carlo (MLMC) method, we briefly recall some useful definitions and results. For further details, we refer to [4, 14, 22] and the references therein.

#### 5.1.1. *Single level Monte Carlo method.* 

Let $X$ be a separable Hilbert space. For a Bochner integrable random variable $f : U \to X$, the Monte Carlo (MC) method attempts to approximate the expected value of $f$ by its mean over a finite set of $\mathcal{N} \in \mathbb{N}$ independent and identically distributed sample evaluations over $U$,

$$Q^{\mathrm{MC}}_{\mathcal{N}}(f) := \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} f(\boldsymbol{y}_{(i)}),$$

where $\{\boldsymbol{y}_{(i)}\}_{i=1}^{\mathcal{N}}$ are independent uniform random variables on a probability space $(\Omega, \mathfrak{F}, \nu)$ with values in $U$ (so that $Q^{\mathrm{MC}}_{\mathcal{N}}(f) : \Omega \to X$ is, in itself, a random variable on $(\Omega, \mathfrak{F}, \nu)$ with values in $X$). If $f$ belongs to $L^2(U, \mathfrak{B}, \upmu; X)$, then the following convergence estimate for the MC method holds (*cf.* [4, Lemma 4.1]),

$$\|Q^{\mathrm{MC}}_{\mathcal{N}}(f) - \mathbb{E}(f)\|_{L^2(\Omega;X)} \leq \mathcal{N}^{-\frac{1}{2}} \|f\|_{L^2(U;X)}.$$

Moreover, if each evaluation $f(\boldsymbol{y}_{(i)}(\zeta))$, for $\zeta \in \Omega$, cannot be computed exactly but can only be approximated by a numerical method yielding an approximation to the random variable $f$, denoted $f_h$—where the subindex $h$ signifies, as before, the precision of the method—, then it holds that (*cf.* [14, Section 2.1])

$$\begin{aligned}\|Q^{\mathrm{MC}}_{\mathcal{N}}(f_h) - \mathbb{E}(f)\|_{L^2(\Omega;X)} &\leq \|Q^{\mathrm{MC}}_{\mathcal{N}}(f_h) - \mathbb{E}(f_h)\|_{L^2(\Omega;X)} + \|\mathbb{E}(f_h - f)\|_{L^2(\Omega;X)} \\ &\leq \mathcal{N}^{-\frac{1}{2}}\|f_h\|_{L^2(U;X)} + \|\mathbb{E}(f_h - f)\|_X.\end{aligned}$$

The previous estimate implies that the number of samples $\mathcal{N}$ needs to be chosen proportional to $\|\mathbb{E}(f_h - f)\|_X^{-2}$—assuming $\|f_h\|_{L^2(U;X)}$ remains bounded—to balance the error contributions in the upper bound.

5.1.2. *Multilevel Monte Carlo method.* The MLMC method differs from the standard single level MC Galerkin discretization in that it employs simultaneously different discretization levels of the random variable $f$, namely $\{f_{h_i}\}_{i=1}^L$ for $L \in \mathbb{N}$, to approximate $\mathbb{E}(f)$:

$$Q^{\mathrm{MLMC}}_L(f) := \sum_{i=1}^{L} Q^{\mathrm{MC}}_{\mathcal{N}_{L,i}}(f_{h_i} - f_{h_{i-1}}), \tag{5.1}$$

where $\{\mathcal{N}_{L,i}\}_{i=1}^L \subset \mathbb{N}$ corresponds to the number of samples at each level, $f_{h_0} \equiv 0$ and we assume that $\|\mathbb{E}(f_{h_i} - f)\|_X$ decreases as the sub-index $i$ increases, i.e. the precision of the method increases with $i$. Then, under the same assumptions as before,

$$\begin{aligned}&\|Q^{\mathrm{MLMC}}_L(f) - \mathbb{E}(f)\|_{L^2(\Omega;X)} \\ &\leq \|Q^{\mathrm{MLMC}}_L(f) - \sum_{i=1}^{L}\mathbb{E}(f_{h_i} - f_{h_i-1})\|_{L^2(\Omega;X)} + \|\mathbb{E}(f_{h_L} - f)\|_{L^2(\Omega;X)} \\ &\leq \sum_{i=1}^{L}\mathcal{N}_{L,i}^{-\frac{1}{2}}\|f_{h_i} - f_{h_{i-1}}\|_{L^2(\Omega;X)} + \|\mathbb{E}(f_{h_L} - f)\|_X.\end{aligned} \tag{5.2}$$

5.1.3. *Multilevel Monte Carlo for Problem 2.11.* We now return to our specific setting and compute convergence estimates for the approximation, via the MLMC method, of the expected value of $\widetilde{\mathbf{E}} : U \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$, as in (4.9), through the approximations given by $\widetilde{\mathbf{E}}_h : U \to \boldsymbol{P}^c_k(\tau_h)$ as in (4.10). Under its respective assumptions, Proposition 4.7 ensures that both $\widetilde{\mathbf{E}}(\mathbf{y})$ and $\widetilde{\mathbf{E}}_h(\mathbf{y})$ belong to $L^2(U, \mathfrak{B}, \mu; \boldsymbol{H}_0(\mathbf{curl}; \widehat{\mathrm{D}}))$ as well as the error estimates required to effectively bound the MLMC error. The number of samples $\mathcal{N}_{L,i}$ (5.2) of each level will be chosen so that the convergence rate of the MLMC method is the same as that of the FE approximation of $\widetilde{\mathbf{E}}$, namely $h^k$. To estimate the total work of the method, we see that the computation of the MLMC estimator at level $L$ requires us to solve for $\widetilde{\mathbf{E}}_{h_i} - \widetilde{\mathbf{E}}_{h_{i-1}}$ at $\mathcal{N}_{L,i}$ random points in $U$, corresponding to $[\dim(\boldsymbol{P}^c_k(\tau_{h_i})) + \dim(\boldsymbol{P}^c_k(\tau_{h_{i-1}}))]\mathcal{N}_{L,i}$ degrees of freedom[1]. Hence, we define the following quantity as an estimate of the computational complexity of the method

$$\mathrm{work}(Q^{\mathrm{MLMC}}_L) := \sum_{i=1}^{L}\mathcal{N}_{L,i}\left[\dim(\boldsymbol{P}^c_k(\tau_{h_i})) + \dim(\boldsymbol{P}^c_k(\tau_{h_{i-1}}))\right]. \tag{5.3}$$

**Theorem 5.1.** *Let Assumptions 2.5 through 4.1 hold and let* $\widetilde{\mathbf{E}} : U \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$ *be as in* (4.9). *Then, tere is a choice of* $\{\mathcal{N}_{L,i}\}_{i=1}^L$ *for each* $L \in \mathbb{N}$, *such that*

$$\|Q^{\mathrm{MLMC}}_L(\widetilde{\mathbf{E}}) - \mathbb{E}(\widetilde{\mathbf{E}})\|_{L^2(\Omega;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} \leq C B_{N,\mathfrak{T},\mathbf{J}} h^k_L, \tag{5.4}$$

*where the constant* $C > 0$ *is independent of* $L$*—hence, of the mesh-size—and* $N \in \mathbb{N}$, $q > 3$ *and* $k \leq N$ *are as in Assumption 3.16. Furthermore, the total work is bounded by*

$$\mathit{work}(Q^{\mathrm{MLMC}}_L) = \begin{cases}\mathcal{O}\left(\dim(\boldsymbol{P}^c_k(\tau_{h_L}))\log\left(\dim(\boldsymbol{P}^c_k(\tau_{h_L}))\right)^{2+2\delta}\right) & \text{if } k = 1, \\ \mathcal{O}\left(\dim(\boldsymbol{P}^c_k(\tau_{h_L}))^{\frac{2}{3}k}\right) & \text{if } k > 1,\end{cases}$$

*for any* $\delta > 0$.

[1] Here, we take $\widetilde{\mathbf{E}}_{h_0} \equiv \mathbf{0}$ and $\dim(\boldsymbol{P}^c_k(\tau_{h_0})) = 0$, as before.

*Proof.* From (5.2) and Proposition 4.7 it follows that

$$\begin{aligned}
&\|Q_L^{\mathrm{MLMC}}(\widetilde{\mathbf{E}}) - \mathbb{E}(\widetilde{\mathbf{E}})\|_{L^2(\Omega;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} \\
&\leq \|\mathbb{E}(\widetilde{\mathbf{E}} - \widetilde{\mathbf{E}}_{h_L})\|_{\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})} + \sum_{i=1}^{L} \mathcal{N}_{L,i}^{-\frac{1}{2}} \|\widetilde{\mathbf{E}}_{h_i} - \widetilde{\mathbf{E}}_{h_{i-1}}\|_{L^2(U;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} \\
&\leq C B_{N,\mathfrak{T},\mathbf{J}} \left( h_L^k + \sum_{i=1}^{L} \mathcal{N}_{L,i}^{-\frac{1}{2}} h_i^k \right),
\end{aligned}$$

where $C > 0$ is as in Proposition 4.7. We take, for each $L \in \mathbb{N}$ and $i \in \{1, \ldots, L\}$, $\mathcal{N}_{L,i} = \mathcal{O}((h_i/h_L)^{2k} i^{2+2\delta})$ for arbitrary $\delta > 0$, so that

$$\sum_{i=1}^{L} \mathcal{N}_{L,i}^{-\frac{1}{2}} h_i^k \leq C h_L^k \sum_{i=1}^{L} i^{-(1+\delta)},$$

where $C > 0$ is independent of $L \in \mathbb{N}$, and (5.4) follows upon noticing that the last sum is bounded for all $L \in \mathbb{N}$ and $\delta > 0$. We continue with the bounding the total work of the MLMC method. From (5.3) and Remark 3.9, it follows that

$$\mathrm{work}(Q_L^{\mathrm{MLMC}}) \leq C_{s,4}(1+s^3) \sum_{i=1}^{L} \mathcal{N}_{L,i} s^{-3i},$$

where $s \in (0,1)$ is as in Assumption 3.8. Recalling item (iv) from Assumption 3.8, we may express our choice for $\mathcal{N}_{L,i}$ as $\mathcal{N}_{L,i} = O(s^{2k(i-L)} i^{2+2\delta})$ so that

$$\begin{aligned}
\mathrm{work}(Q_L^{\mathrm{MLMC}}) &\leq C_{s,4}(1+s^3) \sum_{i=1}^{L} i^{2+2\delta} s^{2k(i-L)-3i} \\
&\leq C_{s,4}(1+s^3) s^{-2kL} \sum_{i=1}^{L} i^{2+2\delta} s^{(2k-3)i}.
\end{aligned}$$

If $k > 1$, then $2k - 3 > 0$—recall $k \in \mathbb{N}$—and the claimed bound on the total work follows from (3.4). If $k = 1$,

$$\begin{aligned}
\mathrm{work}(Q_L^{\mathrm{MLMC}}) &\leq C_{s,4}(1+s^3) s^{-2kL} \sum_{i=1}^{L} i^{2+2\delta} s^{(2k-3)i} \\
&= C_{s,4}(1+s^3) s^{-3L} \sum_{i=1}^{L} i^{2+2\delta} s^{(3-2k)(L-i)} \\
&\leq C_{s,4}(1+s^3) s^{-3L} \sum_{j=0}^{L-1} (L-j)^{2+2\delta} s^{(3-2k)j} \\
&\leq C_{s,4}(1+s^3) s^{-3L} L^{2+2\delta} \sum_{j=0}^{L-1} s^{(3-2k)j},
\end{aligned}$$

where the last sum is bounded for all $L \in \mathbb{N}$ and the bound on the total work follows, again, from (3.4). ☐

We may then use Theorem 5.1 to bound the approximation error of the MLMC method with respect to the total required work.

**Corollary 5.2.** *Under the assumptions of Theorem 5.1, there is a choice of* $\{\mathcal{N}_{L,i}\}_{i=1}^{L}$ *such that*

$$\|Q_L^{\mathrm{MLMC}}(\widetilde{\mathbf{E}}) - \mathbb{E}(\widetilde{\mathbf{E}})\|_{L^2(U;\boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}}))} \leq C B_{N,\mathfrak{T},\mathbf{J}}\, \mathit{work}(Q_L^{\mathrm{MLMC}})^{-\kappa(k)},$$

*where $k \in \mathbb{N}$ is the corresponding convergence rate in Proposition 4.6, $C > 0$ is as in Theorem 5.1, and*

$$\kappa(k) := \begin{cases} \frac{1}{3+\delta} & \text{if } k = 1, \\ \frac{1}{2} & \text{if } k > 1, \end{cases}$$

*for arbitrary $\delta > 0$.*

5.2. **Multilevel Smolyak.** The Smolyak algorithm provides a method for multi-dimensional polynomial interpolation of functions on *sparse-grids*. Recalling the well-known construction requires to introduce some standard notation first. The set of *finitely supported multi-indices* is denoted by

$$\mathcal{F} := \{\boldsymbol{\nu} = (\nu_j)_{j\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}} \,:\, |\boldsymbol{\nu}| < \infty\},$$

where $|\boldsymbol{\nu}| := \sum_{j\in\mathbb{N}} \nu_j$. For two multi-indices $\boldsymbol{\nu} = (\nu_j)_{j\in\mathbb{N}}$, $\boldsymbol{\mu} = (\mu_j)_{j\in\mathbb{N}} \in \mathcal{F}$ we will write $\boldsymbol{\mu} \le \boldsymbol{\nu}$ if $\mu_j \le \nu_j$ for all $j \in \mathbb{N}$. A set $\Lambda \subseteq \mathcal{F}$ will be called *downward closed* if

$$(\boldsymbol{\nu} \in \Lambda \text{ and } \boldsymbol{\mu} \le \boldsymbol{\nu}) \qquad \Rightarrow \qquad \boldsymbol{\mu} \in \Lambda$$

and it will be called *finite* in case it is of finite cardinality.

5.2.1. *Smolyak interpolation and quadrature.* Let $(\chi_j)_{j\in\mathbb{N}_0}$ be a sequence of so-called $\mathbb{R}$-*Leja points*, as constructed for example in [10]. In particular, the $(\chi_j)_{j\in\mathbb{N}_0}$ are distinct and $\chi_j \in [-1, 1]$ for all $j \in \mathbb{N}_0$. These points will represent, in the following, one-dimensional interpolation and quadrature points. In principle, other points could also be used, however this sequence has a known construction and the favourable property that the Lebesgue constant of $\{\chi_j\}_{j=0}^n$ grows at most polynomially as $n \to \infty$ (see [10]).

For $n \in \mathbb{N}_0$ let $I_n : \mathcal{C}^0([-1, 1]) \to \mathbb{P}_n$ be the polynomial interpolation operator interpolating a function in $(\chi_j)_{j=0}^n$, i.e.

$$(I_n f)(y) = \sum_{j=1}^{n} f(\chi_j) \prod_{\substack{i=0\\ i\neq j}}^{n} \frac{y - \chi_i}{\chi_j - \chi_i} \qquad y \in [-1, 1],$$

where empty products are understood to equal to one. For a multi-index $\boldsymbol{\nu} \in \mathcal{F}$, we set $I_{\boldsymbol{\nu}} := \bigotimes_{j\in\mathbb{N}} I_{\nu_j}$, meaning that

$$I_{\boldsymbol{\nu}} f(\boldsymbol{y}) = \sum_{\boldsymbol{\mu}\le\boldsymbol{\nu}} f((\chi_{\mu_j})_{j\in\mathbb{N}}) \prod_{j\in\mathbb{N}} \prod_{i\neq\mu_j}^{\nu_j} (y_j - \chi_i)/(\chi_{\mu_j} - \chi_i),$$

for every $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}} \in U$. For a finite downward closed set $\Lambda \subseteq \mathcal{F}$ the *Smolyak interpolant* $I_\Lambda : \mathcal{C}^0(U) \to \mathcal{C}^0(U)$ is defined with the so-called *combination formula* as

$$(I_\Lambda f)(\boldsymbol{y}) := \sum_{\boldsymbol{\nu}\in\mathcal{F}} c_{\Lambda,\boldsymbol{\nu}} I_{\boldsymbol{\nu}}(\boldsymbol{y}), \qquad c_{\Lambda,\boldsymbol{\nu}} := \sum_{\{\boldsymbol{e}\in\{0,1\}^{\mathbb{N}} \,:\, \boldsymbol{\nu}+\boldsymbol{e}\in\Lambda\}} (-1)^{|\boldsymbol{e}|}.$$

This interpolation operator satisfies $I_\Lambda f = f$ for all $f \in \operatorname{span}\{\boldsymbol{y}^{\boldsymbol{\nu}} \,:\, \boldsymbol{\nu} \in \Lambda\}$ (*cf.* [36, Lemma 1.3.3]). Similarly, with the numerical integration $Q_{\boldsymbol{\nu}} f := \int_U I_{\boldsymbol{\nu}} f$, we set

$$Q_\Lambda f := \sum_{\boldsymbol{\nu}\in\mathcal{F}} c_{\Lambda,\boldsymbol{\nu}} Q_{\boldsymbol{\nu}}(\boldsymbol{y}),$$

which gives a quadrature rule for which $Q_\Lambda f = \int_U f(\boldsymbol{y}) \,\mathrm{d}\mu(\boldsymbol{y})$ for all $f \in \operatorname{span}\{\boldsymbol{y}^{\boldsymbol{\nu}} \,:\, \boldsymbol{\nu} \in \Lambda\}$. For more details about the construction and the properties of $I_\Lambda$ and $Q_\Lambda$ we refer for example to [5, 27, 9, 36].

5.2.2. *Multilevel algorithm.* To define a multilevel algorithm, we first associate to every multi-index $\boldsymbol{\nu}$ a work level $w_{\boldsymbol{\nu}}$, or for short $\boldsymbol{w} = (w_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$. With $(\tau_{h_i})_{i\in\mathbb{N}}$ as in Assumption 3.8, we shall assume for all $\boldsymbol{\nu} \in \mathcal{F}$ that

$$w_{\boldsymbol{\nu}} \in \{\dim(\boldsymbol{P}_k^c(\tau_{h_i})) \,:\, i \in \mathbb{N}\} \cup \{0\},$$

so that $w_{\boldsymbol{\nu}}$ corresponds to the dimension of a FEM space. Furthermore $|\boldsymbol{w}| := \sum_{\boldsymbol{\nu}\in\mathcal{F}} w_{\boldsymbol{\nu}} < \infty$. For every $i \in \mathbb{N}$ this then yields a finite multi-index set

$$\Gamma_j(\boldsymbol{w}) := \{\boldsymbol{\nu} \in \mathcal{F} \,:\, w_{\boldsymbol{\nu}} \geq \dim(\boldsymbol{P}_k^c(\tau_{h_i}))\}. \tag{5.5}$$

The multilevel interpolation operator is defined as

$$I_{\boldsymbol{w}}^{\mathrm{ML}}\widetilde{\mathbf{E}} := \sum_{j\in\mathbb{N}} I_{\Gamma_i(\boldsymbol{w})}(\widetilde{\mathbf{E}}_{h_i} - \widetilde{\mathbf{E}}_{h_{i-1}}) \tag{5.6}$$

and the multilevel quadrature operator as

$$Q_{\boldsymbol{w}}^{\mathrm{ML}}\widetilde{\mathbf{E}} := \sum_{j\in\mathbb{N}} Q_{\Gamma_i(\boldsymbol{w})}(\widetilde{\mathbf{E}}_{h_i} - \widetilde{\mathbf{E}}_{h_{ji1}}), \tag{5.7}$$

where $\widetilde{\mathbf{E}}_{h_j} \in \boldsymbol{P}_k^c(\tau_{h_j})$, as earlier, is the discrete solution of (3.8). Here it is assumed that $\Gamma_j(\boldsymbol{w})$ is downward closed for every $j \in \mathbb{N}$, so that $I_{\Gamma_j(\boldsymbol{w})}$ and $Q_{\Gamma_j(\boldsymbol{w})}$ are well defined. Furthermore, we point out again our convention $\widetilde{\mathbf{E}}_0 \equiv 0$.

As a measure of the work required to compute $I_{\boldsymbol{w}}^{\mathrm{ML}}\widetilde{\mathbf{E}}$ and $Q_{\boldsymbol{w}}^{\mathrm{ML}}\widetilde{\mathbf{E}}$ we consider the total number of degrees of freedom of all required function approximations. For example, computing $I_{\Gamma_j(\boldsymbol{w})}\widetilde{\mathbf{E}}_{h_j} - \widetilde{\mathbf{E}}_{h_{j-1}})$ or $Q_{\Gamma_j(\boldsymbol{w})}(\widetilde{\mathbf{E}}_{h_j} - \widetilde{\mathbf{E}}_{h_{j-1}})$ requires to evaluate both approximations $\widetilde{\mathbf{E}}_{h_j} : U \to \boldsymbol{P}_k^c(\tau_{h_j})$ and $\widetilde{\mathbf{E}}_{h_{j-1}} : U \to \boldsymbol{P}_k^c(\tau_{h_{j-1}})$ to $\mathbf{E} : U \to \boldsymbol{H}_0(\mathbf{curl}; \widetilde{\mathrm{D}})$, at $|\Gamma_j(\boldsymbol{w})|$ points in $U$. This corresponds to $(\dim(\boldsymbol{P}_k^c(\tau_{h_j})) + \dim(\boldsymbol{P}_k^c(\tau_{h_{j-1}})))|\Gamma_j(\boldsymbol{w})|$ degrees of freedom. In total

$$\mathrm{work}(\boldsymbol{w}) := \sum_{j\in\mathbb{N}} (\dim(\boldsymbol{P}_k^c(\tau_{h_j})) + \dim(\boldsymbol{P}_k^c(\tau_{h_{j-1}})))|\Gamma_j(\boldsymbol{w})| \tag{5.8}$$

counts the degrees of freedom of all FE approximations required for the computation of the multilevel interpolant/quadrature.

5.2.3. *Abstract convergence theory.* We now recall an approximation result for multilevel Smolyak interpolation and quadrature from [36], also see [37]. It provides a statement about the algebraic convergence rate that is achievable in terms of $\mathrm{work}(\boldsymbol{w})$ in (5.8). Implementing the method requires to determine the work levels $\boldsymbol{w} = (w_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ a priori. We comment on possible choices when presenting numerical experiments in Section 6. We first recall the assumptions under which the subsequent convergence results are valid.

*Assumption* 5.3. The spaces $X$, $Z$ and $Z_N$ are complex Banach spaces and there holds the continuous embedding $Z_N \hookrightarrow Z$. For a summable sequence $(\psi_j)_{j\in\mathbb{N}} \subseteq Z_N$ denote $\sigma(\boldsymbol{y}) = \sum_{j\in\mathbb{N}} y_j\psi_j \in Z$ for $\boldsymbol{y} \in U$ and set $\mathcal{P} := \{\sigma(\boldsymbol{y}) \,:\, \boldsymbol{y} \in U\} \subseteq Z$. There exists a constant $M > 0$, two summability exponents $p \in (0,1)$, $p_N \in [p,1)$ and a FE method convergence rate $\alpha > 0$ such that, with $(\tau_{h_j})_{j\in\mathbb{N}}$ as in Assumption 3.8, the following is satisfied

(i) $(\|\psi_j\|_Z)_{j\in\mathbb{N}_0} \in \ell^p(\mathbb{N}_0)$ and $(\|\psi_j\|_{Z_N})_{j\in\mathbb{N}_0} \in \ell^{p_N}(\mathbb{N}_0)$,

(ii) there exists an open set $O_{\mathcal{P}} \subseteq Z$ and a Fréchet differentiable function $\widetilde{\mathfrak{E}} : O_{\mathcal{P}} \to X$ such that $\mathcal{P} \subseteq O_{\mathcal{P}}$ and $\sup_{\xi\in O_{\mathcal{P}}} \|\widetilde{\mathfrak{E}}(\xi)\|_X \leq M$,

(iii) there exists an open set $O_{\mathcal{P},N} \subseteq Z_N$ and Fréchet differentiable functions $\widetilde{\mathfrak{E}}_{h_j} : O_{\mathcal{P}} \to X$ for every $j \in \mathbb{N}$ such that $\mathcal{P} \subseteq O_{\mathcal{P},N}$ and for every $j \in \mathbb{N}$

$$\sup_{\xi\in O_{\mathcal{P}}} \|\widetilde{\mathfrak{E}}(\xi) - \widetilde{\mathfrak{E}}_{h_j}(\xi)\|_X \leq M \qquad \text{and} \qquad \sup_{\xi\in O_{\mathcal{P},N}} \|\widetilde{\mathfrak{E}}(\xi) - \widetilde{\mathfrak{E}}_{h_j}(\xi)\|_X \leq M \dim(\boldsymbol{P}_k^c(\tau_{h_j}))^{-\alpha}.$$

The functions $\widetilde{\mathbf{E}} : U \to X$ and $\widetilde{\mathbf{E}}_{h_j} : U \to X$ are given by $\widetilde{\mathbf{E}}(\boldsymbol{y}) = \widetilde{\mathfrak{E}}(\sigma(\boldsymbol{y}))$ and $\widetilde{\mathbf{E}}_{h_j}(\boldsymbol{y}) = \widetilde{\mathfrak{E}}_{h_j}(\sigma(\boldsymbol{y}))$ for all $\boldsymbol{y} \in U$ and $j \in \mathbb{N}$.

The following theorem follows [36, Thm. 3.2.11] and [36, Thm. 3.2.12]. These bounds rely on the fact that the sequence $(\dim(\boldsymbol{P}_k^c(\tau_{h_j})))_{j\in\mathbb{N}}$ of FE degrees of freedom increases exponentially in the sense of Remark 3.9.

**Theorem 5.4.** *Let $(\tau_{h_j})_{j\in\mathbb{N}}$ satisfy Assumption 3.8. Let $p\in(0,1)$, $p_N\in[p,1)$ and $\alpha>0$ be such that $\hat{\mathbf{E}}$ and $(\hat{\mathbf{E}}_{\mathfrak{w}})_{\mathfrak{w}\in\mathfrak{W}}$ satisfy Assumption 5.3. Then, there exists $C<\infty$ and*

(i) *a sequence $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ of sequences of work levels, such that $|\boldsymbol{w}_n|\to\infty$ as $n\to\infty$ and*

$$\|\widetilde{\mathbf{E}}-I^{\mathrm{ML}}_{\boldsymbol{w}_n}\widetilde{\mathbf{E}}\|_{\boldsymbol{C}^0(U;X)}\le C\mathrm{work}(\boldsymbol{w}_n)^{-r_I},\qquad r_I:=\min\left\{\alpha,\frac{\alpha(p^{-1}-1)}{\alpha+p^{-1}-{p_N}^{-1}}\right\},$$

(ii) *a sequence $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ of sequences of work levels, such that $|\boldsymbol{w}_n|\to\infty$ as $n\to\infty$ and*

$$\left\|\int_U\widetilde{\mathbf{E}}(\boldsymbol{y})\,\mathrm{d}\mu(\boldsymbol{y})-Q^{\mathrm{ML}}_{\boldsymbol{w}_n}\widetilde{\mathbf{E}}\right\|_X\le C\mathrm{work}(\boldsymbol{w}_n)^{-r_Q},\qquad r_Q:=\min\left\{\alpha,\frac{\alpha(2p^{-1}-1)}{\alpha+2p^{-1}-2{p_N}^{-1}}\right\}.$$

*Moreover, in both cases $\Gamma_j(\boldsymbol{w}_n)$ in (5.5) is finite and downward closed for all $j\in\mathbb{N}$ and all $n\in\mathbb{N}$.*

5.2.4. *Multilevel Interpolation.* Applying Thm. 5.4 in our setting we obtain the following theorem for sparse-grid approximation.

**Theorem 5.5.** *Fix $N\in\mathbb{N}$ and $q>3$. Let $k\in\mathbb{N}$ be less or equal to $N$ where $k$ denotes the polynomial degree of the FEM ansatz space. For some $p\in(0,1)$, $p_N\in[p,1)$ let (cp. (4.1), (4.2))*

$$(\|\widetilde{\mathbf{T}}_j\|_Z)_{j\in\mathbb{N}_0}\in\ell^p(\mathbb{N}_0),\qquad(\|\widetilde{\mathbf{T}}_j\|_{Z_N})_{j\in\mathbb{N}_0}\in\ell^{p_s}(\mathbb{N}_0).$$

*Suppose that with $\mathfrak{T}=\{\sigma(\boldsymbol{y}):\boldsymbol{y}\in U\}$ as in (4.7)–(4.8), Assumptions 2.5 through 4.1 are satisfied. Then, there exists $C<\infty$ and a sequence $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ of sequences of work levels, such that, $|\boldsymbol{w}_n|\to\infty$ as $n\to\infty$ and for all $n\in\mathbb{N}$, the solution $\widetilde{\mathbf{E}}(\boldsymbol{y}):=\widetilde{\mathbf{E}}_{\sigma(\boldsymbol{y})}$ of Problem 3.6 satisfies*

$$\|\widetilde{\mathbf{E}}(\cdot)-(I^{\mathrm{ML}}_{\boldsymbol{w}_n}\widetilde{\mathbf{E}})(\cdot)\|_{\boldsymbol{C}^0(U;X)}\le C\mathrm{work}(\boldsymbol{w}_n)^{-r_I}\quad\text{with}\quad r_I=\min\left\{\frac{k}{3},\frac{\frac{k}{3}(p^{-1}-1)}{\alpha+p^{-1}-{p_N}^{-1}}\right\},$$

*and where $I^{\mathrm{ML}}_{\boldsymbol{w}_n}\widetilde{\mathbf{E}}$ in (5.6) is defined with the discrete solutions $\widetilde{\mathbf{E}}_{h_j}\in\boldsymbol{P}_k^c(\tau_{h_j})$ of (3.8). Moreover, $\Gamma_j(\boldsymbol{w}_n)$ in (5.5) is finite and downward closed for all $j\in\mathbb{N}$ and all $n\in\mathbb{N}$.*

*Proof.* We need to verify that Assumption 5.3 holds with $\alpha:=\frac{k}{3}$ and the spaces

$$X:=\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}}),\qquad Z:=\boldsymbol{\mathcal{C}}^{0,1}(\widehat{\mathrm{D}};\mathbb{C}^3)\qquad\text{and}\qquad Z_N:=\boldsymbol{\mathcal{C}}^{N,1}(\widehat{\mathrm{D}};\mathbb{C}^3).$$

Then the statement is an immediate consequence of Thm. 5.4. In the following we choose the constant $k\in\mathbb{N}$ equal to $N$.

Assumption 5.3 (i) holds by assumption. The existence of an open set $O_{\mathfrak{T}}\subseteq Z$ containing $\mathfrak{T}$ and a bounded holomorphic function $\widetilde{\mathfrak{E}}:O_{\mathfrak{T}}\to X$ such that $\widetilde{\mathbf{E}}(\boldsymbol{y})=\widetilde{\mathfrak{E}}(\sigma(\boldsymbol{y}))$ for all $\boldsymbol{y}\in U$ follows by Theorem 4.2. This shows Assumption 5.3 (ii). Finally, Prop. 4.6 implies the existence of a suitable open set $O_{N,\mathfrak{T}}\subseteq Z_N$ containing $\mathfrak{T}$, and constant $M$, and bounded holomorphic maps $\widetilde{\mathfrak{E}}:O_{\mathfrak{T}}\to X$ such that

$$\sup_{\xi\in O_{\mathfrak{T}}}\|\widetilde{\mathfrak{E}}(\xi)-\widetilde{\mathfrak{E}}_{h_j}(\xi)\|_{\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})}\le M$$

and

$$\sup_{\xi\in O_{\mathcal{P},N}}\|\widetilde{\mathfrak{E}}(\xi)-\widetilde{\mathfrak{E}}_{h_j}(\xi)\|_{\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})}\le Ch_i^k\le C\dim(\boldsymbol{P}_k^c(\tau_{h_j}))^{-\frac{k}{3}}.$$

This shows that Assumption 5.3 (iii) is satisfied. □

5.2.5. *Multilevel Quadrature.* In the same fashion, we obtain a result for multilevel quadrature.

**Theorem 5.6.** *Let the assumptions of Theorem 5.5 be satisfied. Then there exists $C < \infty$ and a sequence $(\boldsymbol{w}_n)_{n\in\mathbb{N}}$ of sequences of work levels, such that $|\boldsymbol{w}_n| \to \infty$ as $n \to \infty$ and for all $n \in \mathbb{N}$ the solution $\widetilde{\mathbf{E}}(\boldsymbol{y})$ of Problem 3.6 satisfies*

$$\|\mathbb{E}(\widetilde{\mathbf{E}}(\cdot)) - (Q^{\mathrm{ML}}_{\boldsymbol{w}_n}\widetilde{\mathbf{E}})(\cdot)\|_{\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})} \leq C\mathrm{work}(\boldsymbol{w}_n)^{-r_Q} \quad \text{where} \quad r_Q = \min\left\{\frac{k}{3}, \frac{\frac{k}{3}(2p^{-1}-1)}{\alpha + 2p^{-1} - 2p_N{}^{-1}}\right\},$$

*where $Q^{\mathrm{ML}}_{\boldsymbol{w}_n}\widetilde{\mathbf{E}}$ in (5.7) is defined with the discrete solutions $\widetilde{\mathbf{E}}_{h_j} \in \boldsymbol{P}^c_k(\tau_{h_j})$ of (3.8). Moreover $\Gamma_j(\boldsymbol{w}_n)$ in (5.5) is finite and downward closed for all $j \in \mathbb{N}$ and all $n \in \mathbb{N}$.*

*Proof.* Assumption 5.3 holds by the same arguments as in the proof of Thm. 5.5. The statement thus follows by Thm. 5.4. □

## 6. Numerical experiments

We now present a numerical experiment in order to confirm our results in Theorems 5.1, 5.4, 5.5 and 5.6. Our numerical implementation of Problem 2.11 was carried out through the open source softwares GetDP [19] and GMSH [23].

6.1. **Problem Setting.** For $N_c \in \mathbb{N}$, we consider $\widetilde{\mathrm{D}} := [-1,1]^3$ and parametric transformations given by,

$$\sigma(\boldsymbol{y}) := \mathbf{T}_0 + \Theta\sum_{j=1}^{N_c} y_j\mathbf{T}_j, \quad \mathbf{T}_0 = \mathsf{I}, \quad \mathbf{T}_j = j^{-\rho-1}x_3\sin(2\pi j x_1)\hat{\boldsymbol{e}}_3,$$

where $\Theta \in (0, \frac{1}{2})$ is a scale parameter and $\rho > 1$ determines the decay properties of the sequences $(\|\mathbf{T}_j\|_Z)_{j\in\mathbb{N}_0}$ and $(\|\mathbf{T}_j\|_{Z_N})_{j\in\mathbb{N}_0}$. Specifically, with $Z_N$ and $N \in \mathbb{N}$ as in (4.1) and $\rho > N$, we impose

$$(\|\mathbf{T}_j\|_Z)_{j\in\mathbb{N}_0} \in \ell^p, \quad \forall\, 1 > p > \frac{1}{\rho},$$
$$(\|\mathbf{T}_j\|_{Z_N})_{j\in\mathbb{N}_0} \in \ell^{p_N}, \quad \forall\, 1 > p_N > \frac{1}{\rho - N}.$$

We fix the current $\mathbf{J}$ as a polynomial of first degree on $\mathrm{D}_H := [-2,2]^3$ and choose $\epsilon$ and $\mu$ as factors of the identity matrix. The quadratures $Q^1_{\breve{K}}$ and $Q^2_{\breve{K}}$ used to build the sesquilinear and antilinear forms in (3.6) and (3.7), respectively, are 5-points Gaussian quadrature rules—exact on polynomials of degree 3— and we consider only first order Nédélec elements for the discretization of $\boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})$, i.e., $k = 1$ in Section 3.2. Theorem 3.17 then requires $N \geq 1$ to ensure a convergence rate up to $\rho = 1$ with respect to the mesh-size $h$—or $\frac{1}{3}$ with respect to the dimension of $\boldsymbol{P}^c_1(\tau_h)$.

Numerical experiments—for brevity, not presented here—verify the convergence rate of order $N = 1$ with respect to the mesh size. Moreover, for linear functionals of the electric fields $G \in \boldsymbol{H}_0(\mathbf{curl};\widetilde{\mathrm{D}})^*$ we may prove, through Theorem 4.2.14 in [32], that $G(\widetilde{\mathfrak{E}}_h(\mathbf{T}))$ converges to $G(\widetilde{\mathfrak{E}}(\mathbf{T}))$ at twice the rate with respect to the mesh-size ($\rho = 2$ in our context) at which $\widetilde{\mathfrak{E}}_h(\mathbf{T})$ converges to $\widetilde{\mathfrak{E}}(\mathbf{T})$. We choose $G$ as

$$G(\mathbf{U}) := \int_{\widetilde{\mathrm{D}}} \boldsymbol{g}(\boldsymbol{x}) \cdot \overline{\mathbf{U}(\boldsymbol{x})}\,\mathrm{d}\boldsymbol{x},$$

for all $\mathbf{U} \in \boldsymbol{H}(\mathbf{curl};\widetilde{\mathrm{D}})$, where $\boldsymbol{g}$ is chosen as a polynomial in $\mathbb{P}_2(\mathrm{D}_H;\mathbb{R}^3)$. Hence, for any $\mathbf{U} \in \boldsymbol{P}^c_1(\tau_h)$, we may compute $G(\mathbf{U}_h)$ exactly through the use of appropriate Gaussian quadrature rules. Henceforth, we concern ourselves only with the approximation of $G(\widetilde{\mathbf{E}}(\boldsymbol{y}))$ for $\boldsymbol{y} \in U$ and of its expected value over $U$. Moreover, for a fair comparison between the MLMC method and the multilevel Smolyak algorithm, we truncate the dimension of the sample space, so that we consider $U := [-1,1]^{50}$.

6.2. **Number of samples for the MLMC method.** For our multilevel algorithms, we employ five meshes of the domain $\widetilde{\mathrm{D}}$, with 323, 3′208, 16′009, 117′370 and 707′141 degrees of freedom. Table 1 indicates the number of samples taken on each mesh for the multilevel Monte Carlo method. The expected value of the squared error is then computed as the average over 6 realizations of the method.

| DoF | $L=1$ | $L=2$ | $L=3$ | $L=4$ | $L=5$ |
|---|---|---|---|---|---|
| 323 | 1 | 9 | 69 | 1025 | 11271 |
| 3’208 | - | 2 | 13 | 203 | 2273 |
| 16’009 | - | - | 4 | 48 | 537 |
| 11’7370 | - | - | - | 6 | 69 |
| 707’141 | - | - | - | - | 9 |

TABLE 1. Number of samples per mesh (identified with their corresponding degrees of freedom) used in each realization of the MLMC method. We remark that these do not correspond to the $\{\mathcal{N}_{L,i}\}_{i=1}^{L}$ in (5.1), but to the total number of computations carried on each mesh (e.g., for $L=3$ we take $\mathcal{N}_{3,1}=60$, $\mathcal{N}_{3,2}=9$ and $\mathcal{N}_{3,3}=4$).

6.3. **Interpolation and quadrature results.** Figure 2 displays the interpolation error:

$$\|G(\widetilde{\mathbf{E}}(\cdot)) - (I^{\mathrm{ML}}_{\boldsymbol{w}_n} G(\widetilde{\mathbf{E}}))(\cdot)\|_{\mathcal{C}^0(U,\mathbb{C})}, \tag{6.1}$$

with respect to the total work of the Smolyak algorithm (as in (5.8)), where the supremum in the computation of the $\mathcal{C}^0(U)$-norm in (6.1) is approximated by taking the maximum of $G(\widehat{\mathbf{E}}(\boldsymbol{y})) - (I^{\mathrm{ML}}_{\boldsymbol{w}_n} G(\widehat{\mathbf{E}}))(\boldsymbol{y})$ on random points in $U$. Figure 3, on the other hand, displays the quadrature errors of the multilevel Smolyak algorithm

$$\left\|Q^{\mathrm{ML}}_{\boldsymbol{w}_n} G(\widetilde{\mathbf{E}}) - \mathbb{E}\left(G(\widetilde{\mathbf{E}})\right)\right\|_{L^2(\Omega,\mathbb{C})},$$

against the total work of the algorithm (as in (5.8)), and of the MLMC method,

$$\|Q^{\mathrm{MLMC}}_L(G(\widetilde{\mathbf{E}})) - \mathbb{E}\left(G(\widetilde{\mathbf{E}})\right)\|_{L^2(\Omega,\mathbb{C})},$$

against its corresponding total work (as in (5.3)), where $\mathbb{E}(G(\widetilde{\mathbf{E}}))$ is estimated through an overkill computation of the multilevel Smolyak algorithm. The figures display only the results computed with the first four meshes, so that the comparison against the overkill computation shows both the meshing error—coming from the finite element discretization—and the quadrature error—arising from both algorithms.

For $\rho = 2$, we have that for $p \in (1/2, 1)$

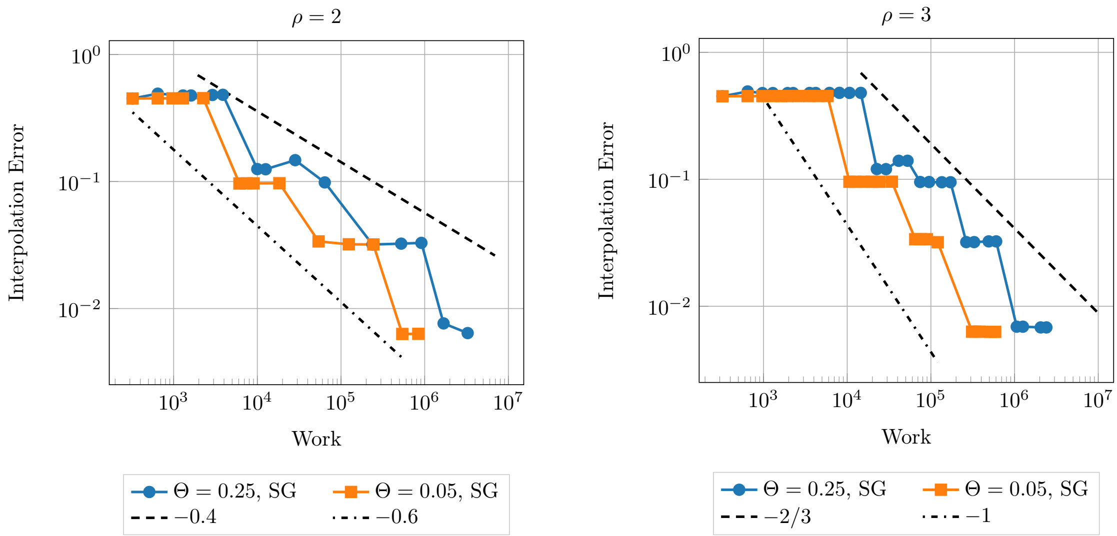
$$(\|\mathbf{T}_j\|_Z)_{j\in\mathbb{N}_0} \in \ell^p.$$

However, we cannot prove a summability property of $(\|\mathbf{T}_j\|_Z)_{j\in\mathbb{N}_0}$ (recall $N=1$). Considering, however, $\rho = 2+\epsilon$ for small $\epsilon > 0$ yields,

$$(\|\mathbf{T}_j\|_Z)_{j\in\mathbb{N}_0} \in \ell^p, \quad \forall\, 1 > p > \frac{1}{2+\epsilon},$$
$$(\|\mathbf{T}_j\|_{Z_N})_{j\in\mathbb{N}_0} \in \ell^{p_N}, \quad \forall\, 1 > p_N > \frac{1}{1+\epsilon},$$

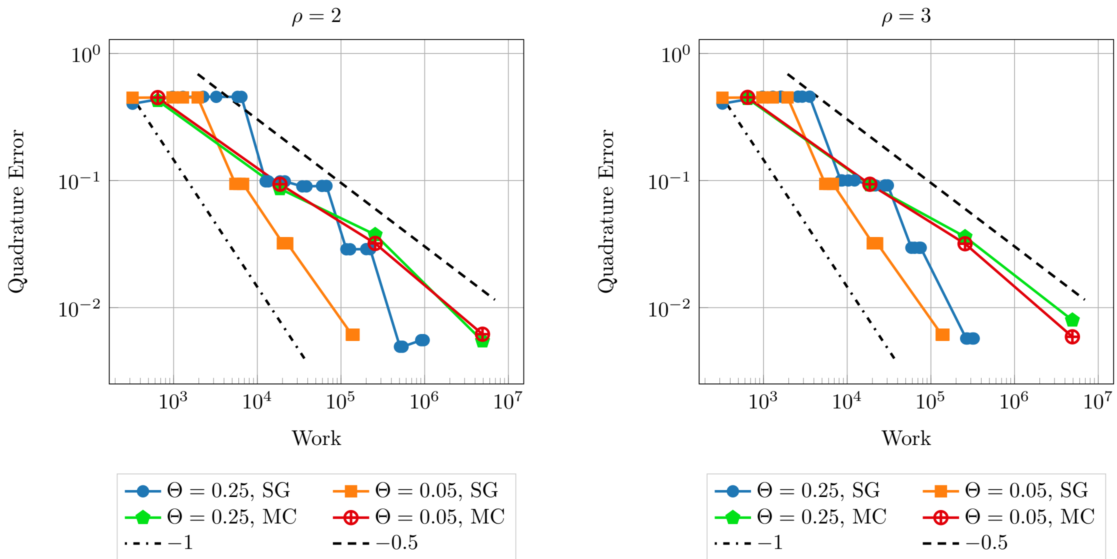
and the convergence rate of the multilevel Smolyak interpolation operator is given by (*cf.* Theorem 5.4),

$$\min\left\{\frac{2}{3}, \frac{\frac{2}{3}(2+\frac{1}{2}\epsilon-1)}{\frac{2}{3}+2+\frac{1}{2}\epsilon-1-\frac{1}{2}\epsilon}\right\} = \min\left\{\frac{2}{3}, \frac{\frac{2}{3}(1+\frac{1}{2}\epsilon)}{\frac{5}{3}}\right\} = \frac{2}{5}\left(1+\frac{1}{2}\epsilon\right).$$




(A) $\rho = 2$ (B) $\rho = 3$

FIGURE 2. Interpolation error. The theoretical asymptotic convergence rates are $0.4 - \epsilon$ for $\rho = 2$ and $\frac{2}{3}$ for $\rho = 3$.




(A) $\rho = 2$ (B) $\rho = 3$

FIGURE 3. Quadrature error. Theoretical asymptotic convergence rates are $\frac{1}{2}$ for MLMC and $\frac{2}{3}$ for ML Smolyak quadrature.

On the other hand, the convergence rate for the multilevel Smolyak quadrature will be given by (*cf.* Theorem 5.6),

$$\min\left\{\frac{2}{3}, \frac{\frac{2}{3}(4+\epsilon-1)}{\frac{2}{3}+4+\epsilon-2-\epsilon}\right\} = \min\left\{\frac{2}{3}, \frac{\frac{2}{3}(3+\epsilon)}{\frac{8}{3}}\right\} = \min\left\{\frac{2}{3}, \frac{3}{4}\left(1+\frac{1}{3}\epsilon\right)\right\} = \frac{2}{3}.$$

An analogous computation for the case $\rho = 3$ yields the convergence rate of $\kappa = \frac{2}{3}$ for both the multilevel Smolyak interpolation and quadrature operators.

## 7. Conclusions and future work

We have extended our original work [2] concerning shape UQ for Maxwell's lossy cavity problem to multilevel versions of MC and Smolyak quadrature and interpolation. Theoretically, regularity results for pullback solutions on the nominal domain are required in suitable Sobolev spaces. Algorithmically, we have then shown much better convergence rates and computational costs of parametric implementations of edge FE in the nominal domain. Our numerical experiments confirm our theoretical findings and pave the way for other EM applications or other approximation methods for the approximation of parametric solution manifolds such as deep neural networks, see e.g. [33].

## References


[1] Giovanni S. Alberti and Yves Capdeboscq. Elliptic regularity theory applied to time harmonic anisotropic Maxwell's equations with less than Lipschitz complex coefficients. *SIAM J. Math. Anal.*, 46(1):998–1016, 2014.

[2] R. Aylwin, C. Jerez-Hanckes, C. Schwab, and J. Zech. Domain uncertainty quantification in computational electromagnetics. *SIAM/ASA Journal on Uncertainty Quantification*, 8(1):301–341, jan 2020.

[3] Rubén Aylwin and Carlos Jerez-Hanckes. The effect of quadrature rules on finite element solutions of Maxwell variational problems. *Numerische Mathematik*, 147(4):903–936, 2021.

[4] Andrea Barth, Christoph Schwab, and Nathaniel Zollinger. Multi-level Monte Carlo Finite Element method for elliptic PDEs with stochastic coefficients. *Numerische Mathematik*, 119(1):123–161, 2011.

[5] Volker Barthelmann, Erich Novak, and Klaus Ritter. High dimensional polynomial interpolation on sparse grids. *Adv. Comput. Math.*, 12(4):273–288, 2000. Multivariate polynomial interpolation.

[6] A. Buffa, M. Costabel, and D. Sheen. On traces for $\mathbf{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.*, 276(2):845–867, 2002.

[7] Annalisa Buffa and Ralf Hiptmair. Galerkin boundary element methods for electromagnetic scattering. In *Topics in computational wave propagation*, volume 31 of *Lect. Notes Comput. Sci. Eng.*, pages 83–124. Springer, Berlin, 2003.

[8] Annalisa Buffa, Ralf Hiptmair, Tobias von Petersdorff, and Christoph Schwab. Boundary element methods for Maxwell transmission problems in Lipschitz domains. *Numer. Math.*, 95(3):459–485, 2003.

[9] Abdellah Chkifa, Albert Cohen, and Christoph Schwab. High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs. *Journ. Found. Comp. Math.*, 14(4):601–633, 2013.

[10] Moulay Abdellah Chkifa. On the Lebesgue constant of Leja sequences for the complex unit disk and of their real projection. *J. Approx. Theory*, 166:176–200, 2013.

[11] Philippe G Ciarlet and P-A Raviart. The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. In A.K. Aziz, editor, *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, pages 409 – 474. Academic Press, 1972.

[12] Philippe G Ciarlet and P-A Raviart. Interpolation theory over curved elements, with applications to finite element methods. *Comput. Methods Appl. Mech. Engrg.*, 1(2):217–249, 1972.

[13] Phillipe G. Ciarlet. *The finite element method for elliptic problems.* Society for Industrial and Applied Mathematics, 2002.

[14] K. A. Cliffe, M. B. Giles, R. Scheichl, and A. L. Teckentrup. Multilevel monte carlo methods and applications to elliptic pdes with random coefficients. *Computing and Visualization in Science*, 14(1):3, 2011.

[15] Albert Cohen, Christoph Schwab, and Jakob Zech. Shape holomorphy of the stationary Navier-Stokes equations. *SIAM J. Math. Anal.*, 50(2):1720–1752, 2018.

[16] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic Equations in Infinite Dimensions.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2 edition, 2014.

[17] M. C. Delfour and J.-P. Zolésio. *Shapes and geometries*, volume 22 of *Advances in Design and Control.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2011. Metrics, analysis, differential calculus, and optimization.

[18] J. Diestel and J. Uhl. *Vector Measures.* American Mathematical Society, jun 1977.

[19] P. Dular and C. Geuzaine. GetDP reference manual: the documentation for GetDP, a general environment for the treatment of discrete problems. http://getdp.info.

[20] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*, volume 159. Springer Science & Business Media, 2004.

[21] Alexandre Ern and Jean-Luc Guermond. Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions. *Computers & Mathematics with Applications*, 75(3):918–932, 2018.

[22] Robert Nicholas Gantner. *Computational Higher-Order Quasi-Monte Carlo for Random Partial Differential Equations.* PhD thesis, ETH Zurich, 2017.

[23] Christophe Geuzaine and Jean-François Remacle. Gmsh: A 3-d finite element mesh generator with built-in pre- and post-processing facilities. *International journal for numerical methods in engineering*, 79(11):1309–1331, 2009.

[24] R. Hiptmair, L. Scarabosio, C. Schillings, and Ch. Schwab. Large deformation shape uncertainty quantification in acoustic scattering. *Adv. Comput. Math.*, 44(5):1475–1518, 2018.

[25] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach spaces. Vol. I. Martingales and Littlewood-Paley theory*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Cham, 2016.

[26] Carlos Jerez-Hanckes, Christoph Schwab, and Jakob Zech. Electromagnetic wave scattering by random surfaces: Shape holomorphy. *Mathematical Models and Methods in Applied Sciences*, 27(12):2229–2259, 2017.

[27] Andreas Klimke. *Uncertainty Modeling using Fuzzy Arithmetic and Sparse Grids*. PhD thesis, Universität Stuttgart, 2005.

[28] Rolf Leis. Zur Theorie elektromagnetischer Schwingungen in anisotropen inhomogenen Medien. *Math. Z.*, 106:213–224, 1968.

[29] William McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[30] Peter Monk. *Finite element methods for Maxwell's equations*. Oxford University Press, 2003.

[31] HL Royden and Patrick Fitzpatrick. Real analysis (4th edtion). *New Jersey: Printice-Hall Inc*, 2010.

[32] Stefan A. Sauter and Christoph Schwab. *Boundary element methods*, volume 39 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2011. Translated and expanded from the 2004 German original.

[33] Christoph Schwab and Jakob Zech. Deep learning in high dimension: Neural network expression rates for generalized polynomial chaos expansions in UQ. *Analysis and Applications, Singapore*, 17(1):19–55, 2019.

[34] Angus Ellis Taylor and David C. Lay. *Introduction to functional analysis*. John Wiley & Sons, New York-Chichester-Brisbane, second edition, 1980.

[35] C. Weber. Regularity theorems for Maxwell's equations. *Math. Methods Appl. Sci.*, 3(4):523–536, 1981.

[36] Jakob Zech. *Sparse-Grid Approximation of High-Dimensional Parametric PDEs*. Dissertation, ETH Zürich, 2018.

[37] Jakob Zech, Dinh Dũng, and Christoph Schwab. Multilevel approximation of parametric and stochastic PDEs. *Math. Models Methods Appl. Sci.*, 29(9):1753–1817, 2019.

Faculty of Engineering and Sciences, Universidad Adolfo Ibáañez, 7941169 Santiago, Chile
*Email address*: rbayl1993@gmail.com

Faculty of Engineering and Sciences, Universidad Adolfo Ibáñez, 7941169 Santiago, Chile
*Email address*: carlos.jerez@uai.cl

Seminar for Applied Mathematics, ETH Zürich, 8092 Zürich, Switzerland
*Email address*: schwab@math.ethz.ch

Interdisciplinary Center for Scientific Computing (IWR), Universität Heidelberg, 69120 Heidelberg, Germany
*Email address*: jakob.zech@uni-heidelberg.de