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Abstract

In an open, bounded Lipschitz polygon $\Omega \subset \mathbb{R}^2$, we establish weighted analytic regularity for a semilinear, elliptic PDE with analytic nonlinearity and subject to a source term f which is analytic in Ω. The boundary conditions on each edge of $\partial\Omega$ are either homogeneous Dirichlet or homogeneous Neumann BCs.

The weighted analytic regularity implies exponential convergence of various approximation schemes: *hp*-Finite Elements, Reduced Order Models via Kolmogorov n-widths of solution sets in $H^1(\Omega)$, and certain deep neural networks.

1 Introduction

The efficient numerical approximation of solutions of elliptic PDEs in corner domains has received much attention in the past decades. It was motivated on the one hand by many applications in engineering and the sciences, by the development of the Finite Element Methods (FEM) and their analysis, and by the advance of elliptic regularity theory in corner domains. As is well known by now (see, e.g., [8, 18] and the references there), mathematical statements of high regularity of solutions in Sobolev spaces require either the use of the corner weights (as in [8, 18]) or the use of Besov-Triebel-Lizorkin spaces with summability indices $0 < p < 1$ as developed e.g. in [10] and the references there. The former regularity results facilitate the development of optimal order FEM approximations on so-called *graded meshes* whereas the latter are at the core of approximation classes for adaptive FEM (AFEM). See, e.g., [20] and the references there.

These developments pertained to the so-called h-FEM, which achieves convergence by mesh refinement, at fixed polynomial order of the elements. An alternative concept is furnished by the more general, so-called *hp*-FEM. There, mesh refinements and polynomial degree increase are combined. It has been proved in 80ies in a number of landmark papers by I. Babuška and B.Q. Guo that *hp*-FEM can achieve *exponential rates of convergence* for linear elliptic PDEs in polygonal domains, with analytic data (source term and inhomogeneous boundary data). A key ingredient in the theory is the *weighted analytic regularity* of solutions. Regularity results of this type in corner domains for linear elliptic PDEs appeared also in the 80ies. We mention only [5, 9, 2].

While analyticity of solutions of nonlinear, elliptic PDEs with analytic data (coefficients, nonlinearity, domain) are classical (e.g. [19, Chap. 5.8] and the references there), results on analytic regularity for *nonlinear elliptic PDE in corner domains and with singular nonlinearity* appeared recently in [15, 17, 11].

To establish *weighted analytic regularity and exponential convergence of hp-FE discretizations* for a class of semilinear, scalar elliptic PDEs in a polygon with analytic nonlinearity, and subject to analytic data, is the topic of the present paper. Its layout is as follows. In Sections 1.1-1.2 we provide a variational formulation of the semilinear elliptic boundary value problem, recapitulate notation and, in Section 1.3, definitions and basic properties of corner-weighted function spaces of Sobolev type. In Section 2, we recapitulate corner-weighted regularity shift results for the linear Poisson equation. Section 3 then contains the proof of the main result of the present paper: weighted analytic regularity for the weak solutions of the semilinear elliptic PDE with analytic in the polygon $\overline{\Omega}$ forcing. The final Section 4 then addresses some direct consequences from the analytic regularity results: exponential approximability of the weak solution by so-called *hp*-Finite Element Methods, and exponential bounds on Kolomogorov n-widths of the (nonlinear) solution manifold. Appendix A contains some auxiliary results supporting the proof of the main result.

1.1 Problem Formulation

Let $\Omega \subset \mathbb{R}^2$ be a polygon with $n \geq 3$ vertices c_i and n straight open edges Γ_i. We assume vertices and edges to be enumerated in clockwise order, with indexing modulo n, i.e. $c_i = c_{i+n}$ for all $i \in \mathbb{Z}$.

For $1 \leq i \leq n$, Γ_i connects c_i and c_{i+1} so that $\partial\Gamma_i = \{c_i, c_{i+1}\}$. We denote by $\omega_i \in (0, 2\pi)$ the internal angle at c_i. In particular, then, the polygon Ω has a Lipschitz boundary $\Gamma = \partial\Omega$ [8].

We study the analytic regularity of solutions of the following semilinear elliptic PDE in Ω

$$-\Delta u + \lambda u^{2k+1} = f \quad \text{in} \quad \Omega \,. \tag{1.1}$$

Here, $\lambda \geq 0$ and $k \in \mathbb{N}_0$, with the case $\lambda = 0$ corresponding to the linear Poisson equation, which was studied in [2] and the case $k = 0, \lambda > 0$ corresponding to a linear, reaction-diffusion boundary value problem.

The PDE (1.1) is completed by boundary conditions: on edge Γ_i, we impose either homogeneous Dirichlet or homogeneous Neumann BCs.

$$\gamma_0(u) = 0 \quad \text{or} \quad \gamma_1(u) = 0 \quad \text{on } \Gamma_i \,. \tag{1.2}$$

Here, γ_0 and γ_1 are the weak trace and normal derivative operators, respectively. We denote the BCs in (1.2) abstractly as $B(u) = 0$, with the boundary operator $B|_{\Gamma_i} \in \{\gamma_0, \gamma_1\}$ depending on whether Γ_i is a Dirichlet or Neumann edge. We collect the indices $i \in \{1, ..., n\}$ corresponding to Dirichlet edges in the index set $\mathcal{D}$, and the remaining indices in $\mathcal{N}$ (with membership in $\mathcal{D}$ or $\mathcal{N}$ again understood modulo n), so that $\mathcal{D}$ and $\mathcal{N}$ are a partition of $\{1, ..., n\}$, and $B|_{\Gamma_i} = \gamma_0$ for $i \in \mathcal{D}$. We assume throughout

$$\text{there is at least one edge } \Gamma_i \text{ where } \gamma_0(u) = 0 \,. \tag{1.3}$$

With these conventions, we set

$$H^1_D(\Omega) = \{v \in H^1(\Omega) : \gamma_0(v) = 0 \text{ on } \Gamma_i, i \in \mathcal{D}\} \,. \tag{1.4}$$

Due to (1.3), $\sum_{i\in\mathcal{D}} |\Gamma_i| > 0$. This implies the Poincaré inequality: there exists a constant $C > 0$ such that

$$\forall v \in H^1_D(\Omega) : \quad \|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)} \,. \tag{1.5}$$

In particular, therefore, on $H^1_D(\Omega)$, the expression $\|\nabla v\|_{L^2(\Omega)}$ is a norm.

We will show that given analytic data $f \in B^0_{\underline{\beta}}(\Omega) \cap L^2(\Omega)$, any *generalized* solution $u \in H^1_D(\Omega)$ to (1.1) will be contained in $B^2_{\underline{\beta}}(\Omega)$. Here the notion "*generalized solution*" refers to variational solutions which are defined as follows.

Given $f \in L^2(\Omega)$, we seek $u \in H^1_D(\Omega)$ such that for any $v \in H^1_D(\Omega)$ holds

$$\int_\Omega \nabla u \cdot \nabla v + \lambda u^{2k+1} v \; d\boldsymbol{x} = \int_\Omega f v \; d\boldsymbol{x} \tag{1.6}$$

It can be shown by following the proof of [22, Proposition 27.21] and using the property that the nonlinear term u^{2k+1} is strictly monotone (see, e.g., [22, Example 25.5]) that for every $f \in L^2(\Omega)$ there exists a unique *generalized* solution $u \in H^1_D(\Omega)$ of (1.6).

The proof that $u \in B^2_{\underline{\beta}}(\Omega)$ for $f \in B^0_{\underline{\beta}}(\Omega) \cap L^2(\Omega)$ generally follows the route in [11], where the incompressible stationary Navier-Stokes equation was considered. It will be based on a local regularity-shift result in a sector obtained for the linear Poisson problem in [2] and a weighted L^2-estimate of the derivatives of the nonlinearity λu^{2k+1}.

1.2 Notation

For any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$, we write $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}$, $\mathcal{D}^\alpha = \partial_r^{\alpha_1}\partial_\theta^{\alpha_2}$ and $|\alpha| = \alpha_1+\alpha_2$. Factorials $\alpha!$ are defined as $\alpha! = \alpha_1!\alpha_2!$ with the convention $0! := 1$. We denote with an underline n-dimensional tuples $\underline{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$. We suppose that for multi-indices and n-dimensional tuples, arithmetic operations and inequalities such as $\underline{\gamma} < \underline{\beta}$ are understood component-wise: e.g., $\underline{\beta}+k = (\beta_1+k, \dots, \beta_n+k)$ for all $k \in \mathbb{N}$; furthermore, we indicate, e.g., $\underline{\beta} > 0$ if $\beta_i > 0$ for all $i \in \{1, \dots, n\}$. For $a \in \mathbb{R}$, we denote its nonnegative real part as $[a]_+ = \max(0, a)$. For a nonnegative integer k, we denote by $\mathbb{N}_{>k} = \{n \in \mathbb{N} : n > k\}$ and by $\mathbb{N}_{\geq k} = \{n \in \mathbb{N} : n \geq k\}$.

For any $\alpha, \gamma \in \mathbb{N}_0^2$ or $i, j \in \mathbb{N}$, we denote by $\delta_{\alpha,\gamma}$ or $\delta_{i,j}$ the Kronecker function which equals to 1 if the two parameters are identical and vanishes otherwise.

Given an angle $\omega \in (0, 2\pi)$ and a radius $\delta \in (0, +\infty]$ we define a model, reference plane sector $Q_{\delta,\omega}$

$$Q_{\delta,\omega} := \{(r, \theta) \in \mathbb{R}^2 | r \in (0, \delta), \theta \in (0, \omega)\}. \tag{1.7}$$

For any corner c_i and a radius $\delta \in (0, \min(\frac{1}{4}\min_{i,j\in\{1,2,\cdots,n\},i\neq j} d(c_i, c_j), 1))$, we set

$$Q_{\delta,\omega_i}(c_i) := c_i + \{(r, \theta) \in \mathbb{R}^2 | r \in (0, \delta), \theta \in (0, \omega_i)\}. \tag{1.8}$$

Here the polar coordinate system is assumed to be such that the half line $c_i + \{\theta = 0\}$ touches Γ_{i-1} (so that $c_i + \{\theta = \omega_i\}$ touches Γ_i).

1.3 Function spaces

For $x \in \Omega$ and for $i \in \{1, \dots, n\}$, let $r_i(x) := \mathrm{dist}(x, c_i)$. Following [2], we define the *corner weight function*

$$\Phi_{\underline{\beta}}(x) := \prod_{i=1}^{n} r_i^{\beta_i}(x), \quad x \in \Omega .$$

1.3.1 Weighted spaces in the whole domain Ω

For any $k, l \in \mathbb{N}_0$ with $k \geq l$ and for any $\underline{\beta} \in (0, 1)^n$, we introduce corner-weighted norms $\|v\|_{H^{k,l}_{\underline{\beta}}(\Omega)}$ by

$$\|v\|^2_{H^{k,l}_{\underline{\beta}}(\Omega)} := \|v\|^2_{H^{l-1}(\Omega)} + \sum_{|\alpha|=l}^{k} \|\Phi_{\underline{\beta}+|\alpha|-l}\partial^\alpha v\|^2_{L^2(\Omega)}, \tag{1.9}$$

where the term $\|v\|^2_{H^{l-1}(\Omega)}$ is dropped if $l = 0$. See [2, Sec.1.2].

We also define the following weighted analytic function classes

$$B^l_{\underline{\beta}}(\Omega) := \left\{ v \in \bigcap_{k\geq l} H^{k,l}_{\underline{\beta}}(\Omega) : \exists C, A > 0 \text{ s. t. } \|\Phi_{\underline{\beta}+|\alpha|-l}\partial^\alpha v\|_{L^2(\Omega)} \leq CA^{|\alpha|-l}(|\alpha|-l)!, \forall |\alpha| \geq l \right\}. \tag{1.10}$$

1.3.2 Weighted spaces in a sector

In a sector $Q_{\delta,\omega}$, we define, for all $k \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, the corner-weighted space $W^k_\beta(Q_{\delta,\omega})$ of functions v with finite norm given by

$$\|v\|^2_{W^k_\beta(Q_{\delta,\omega})} = \sum_{|\alpha|\leq k} \|r^{\beta-k+\alpha_1}\mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega})}. \tag{1.11}$$

For $k, l \in \mathbb{N}_0$ with $k \geq l$ and for $\beta \in \mathbb{R}$, $\mathcal{H}^{k,l}_\beta(Q_{\delta,\omega})$ denote the space of functions with finite norm

$$\|v\|^2_{\mathcal{H}^{k,l}_\beta(Q_{\delta,\omega})} := \|v\|^2_{H^{l-1}(Q_{\delta,\omega})} + \sum_{|\alpha|\geq l} \|r^{\alpha_1+\beta-l}\mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega})},$$

where the first term is dropped if $l = 0$.

For $l \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, the *weighted analytic class in polar coordinates* is defined by

$$\mathcal{B}^l_\beta(Q_{\delta,\omega}) = \left\{ v \in \bigcap_{k,l}^{\infty} \mathcal{H}^{k,l}_\beta(Q_{\delta,\omega}) : \exists C, A > 0 \text{ s. t. } \|r^{\alpha_1+\beta-l}\mathcal{D}^\alpha v\|_{L^2(Q_{\delta,\omega})} \le CA^{|\alpha|-l}(|\alpha|-l)!, \, \forall |\alpha| \ge l \right\}. \tag{1.12}$$

The definition of the spaces $H^{k,l}_\beta(Q_{\delta,\omega})$ and $B^l_\beta(Q_{\delta,\omega})$ follows from replacing $\Phi_{\underline{\beta}+|\alpha|-l}$ in (1.9) and (1.10) with $r^{\beta+|\alpha|-l}$.

We require the following two lemmas regarding the relation between those spaces in $Q_{\delta,\omega}$.

Lemma 1.1. *Let $0 < \delta \le 1$, $\omega \in (0, 2\pi)$, $\beta \in (0,1)$. Then the following equivalence relations hold for any $l \in \{0,1,2\}$ and $\mathbb{N}_0 \ni k \ge l$:*

$$v \in H^{k,l}_\beta(Q_{\delta,\omega}) \Leftrightarrow v \in \mathcal{H}^{k,l}_\beta(Q_{\delta,\omega}), \quad v \in B^l_\beta(Q_{\delta,\omega}) \Leftrightarrow v \in \mathcal{B}^l_\beta(Q_{\delta,\omega}), \quad v \in H^{1,1}_\beta(Q_{\delta,\omega}) \Leftrightarrow v \in W^1_\beta(Q_{\delta,\omega}).$$

For a proof we refer to [2, Theorem 1.1, Theorem 2.1, Lemma A.2].

Lemma 1.2. *Let $0 < \delta \le 1$, $\omega \in (0, 2\pi)$, $\beta \in (0,1)$. Then the following imbedding relations hold:*

1. *$W^2_\beta(Q_{\delta,\omega}) \subset H^{2,2}_\beta(Q_{\delta,\omega}) \subset C^0(\overline{Q_{\delta,\omega}})$.*
2. *If $v \in H^{2,2}_\beta(Q_{\delta,\omega})$ and $v((0,0)) = 0$, then $v \in W^2_\beta(Q_{\delta,\omega})$.*

For the proof of this lemma, see [2, Lemma 1.1, Lemma A.1, Lemma A.2] and [3, Section 2].

2 Poisson problem in a sector

The inductive proof of weighted, analytic regularity will require a W^2_β-regularity shift for the linear part of the problem (1.1). By localization, this estimate is only required locally, in the vicinity of each corner. Hence, consider the following Poisson problem in any $Q_{\delta,\omega}$ with $\delta < +\infty$,

$$-\Delta u = f \qquad \text{in } Q_{\delta,\omega}, \quad B(u) = 0 \quad \text{on } \partial Q_{\delta,\omega} \setminus \{r = \delta\}. \tag{2.1}$$

Following the proofs of [2, Lemma 2.2-2.8, Theorem 2.1] item by item, we have the following result.

Proposition 2.1. *Let $\beta \in (0,1)$ such that $\beta > 1 - \frac{\pi}{\omega}$ for either Dirichlet or Neumann BCs, i.e. if $B|_{\partial Q_{\delta,\omega}\setminus\{r=\delta\}} \in \{\gamma_0, \gamma_1\}$ and assume that $\beta > 1 - \frac{\pi}{2\omega}$ for mixed boundary conditions, i.e. if $B|_{\partial Q_{\delta,\omega}\cap\{\theta=0\}} = \gamma_0$ and $B|_{\partial Q_{\delta,\omega}\cap\{\theta=\gamma\}} = \gamma_1$. Furthermore let $f \in L_\beta(Q_{\delta,\omega})$.*

Then there exists a constant $C_{sec} > 1$ such that any weak solution u to (2.1) satisfies $u \in H^{2,2}_\beta(Q_{\delta,\omega})$ and there holds the a-priori estimate

$$\|u - u(0,0)\|_{W^2_\beta(Q_{\delta/2,\omega})} \le C_{sec}(\|f\|_{L_\beta(Q_{\delta,\omega})} + \|u\|_{H^1(Q_{\delta,\omega}\setminus Q_{\delta/2,\omega})}). \tag{2.2}$$

3 Weighted analytic regularity of the solution

Applying [19, Lemma 5.8.6, Lemma 5.8.6'] we derive the analyticity of u and of λu^{2k+1} in the interior of Ω and up to analytic parts of the boundary $\partial\Omega$.

Proposition 3.1. *For any $0 < \delta \le \frac{1}{4}\min_{i,j\in\{1,2,\cdots,n\},i\neq j} d(c_i, c_j)$, any solution u to (1.1) and, for this u, λu^{2k+1} for $k \in \mathbb{N}$ are analytic in $\overline{\Omega \setminus (\cup_{i=1}^n Q_{\delta/2,\omega_i}(c_i))}$.*

By the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ valid for any $q \in (1, +\infty)$ and by the Hölder inequality, one obtains that for any $u \in H^1(\Omega)$ and any $\underline{\beta} \in (0,1)^n$, $\lambda u^{2k+1} \in L^2(\Omega) \subset L_{\underline{\beta}}(\Omega)$. Therefore, we can move λu^{2k+1} to the right-hand side in (1.1) and consider in Ω

$$-\Delta u = f - \lambda u^{2k+1} \text{ in } \Omega, \quad B(u) = 0 \text{ on } \partial\Omega \, . \tag{3.1}$$

Now Lemma 1.2 and Proposition 2.1 imply

Lemma 3.2. *Let $\delta \in (0, \frac{1}{4} \min_{i,j\in\{1,2,\cdots,n\}, i\neq j} d(c_i, c_j))$ and let $f \in L_{\underline{\beta}}(\Omega)$ where $\underline{\beta} \in (0,1)^n$ satisfies that for any $i \in \{1, 2, \cdots, n\}$, $\beta_i > 1 - \frac{\pi}{\omega_i}$ if $\{i-1, i\} \subset \mathcal{D}$ or $\{i-1, i\} \subset \mathcal{N}$ and $\beta_i > 1 - \frac{\pi}{2\omega_i}$ otherwise.*

Then any solution $u \in H^1_D(\Omega)$ to (1.1) satisfies $u|_{Q_{\delta,\omega_i}(c_i)} \in \mathcal{H}^{2,2}_{\beta_i}(Q_{\delta,\omega_i}(c_i)) \subset C^0(\overline{Q_{\delta,\omega_i}(c_i)})$ for $i \in \{1, 2, \cdots, n\}$.

3.1 Analytic estimates on the nonlinearity

In this subsection we examine the estimate on derivatives of λu^{2k+1}. For this purpose we need to study the structure of $\mathcal{D}^\alpha(\lambda u^{2k+1})$.

The case $k = 0$ is straightforward. If $k > 0$, then by Generalized Faà di Bruno formula[6, 14], the derivatives of u will take a complicated form. To describe it, we introduce the concept of *decomposition of a multi-index* $\alpha \in \mathbb{N}_0^2$. We say that $\alpha \in \mathbb{N}_0^2$ is decomposed into a finite number s of nonzero parts $p^1, \cdots, p^s \in \mathbb{N}_0^2$ with multiplicities $m_1, \cdots, m_s \in \mathbb{N}$ if

$$\alpha = \sum_{i=1}^{s} m_i p^i$$

holds and all p^i are distinct. Set $\underline{P} = (p^1, \ldots, p^s)$ and $\boldsymbol{M} = \{m_1, \ldots, m_s\}$, we call the triple $(s, \underline{P}, \boldsymbol{M})$ a *decomposition* of α. The *total multiplicity* of $\boldsymbol{M}$ is $m := \sum_{i=1}^{s} m_i$.

The generalized Faà di Bruno formula states that for any $\alpha \in \mathbb{N}_0^2$ and, for any function $g(\cdot) : \mathbb{R} \to \mathbb{R}$ and for any $u = u(r, \theta)$ with sufficient smoothness, $\mathcal{D}^\alpha g(u)$ takes the form

$$\mathcal{D}^\alpha g(u) = \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha} C_{(s,\underline{P},\boldsymbol{M})} \frac{d^m g(u)}{du^m} \prod_{i=1}^{s} (\mathcal{D}^{p^i} u)^{m_i}. \tag{3.2}$$

Here $\mathscr{D}_\alpha$ is the set of all possible decompositions of α and

$$C_{(s,\underline{P},\boldsymbol{M})} = \alpha! \prod_{i=1}^{s} \left(\frac{1}{m_i!} \left(\frac{1}{p^i!}\right)^{m_i}\right) > 0,$$

which depends only on the specific triple $(s, \underline{P}, \boldsymbol{M})$.

In the presently considered case $g(u) = \lambda u^{2k+1}$, so it suffices to consider decompositions satisfying $m \le 2k+1$.

Lemma 3.2 implies L^∞-boundedness of $\frac{d^m u^{2k+1}}{du^m}$ for any $m \in \mathbb{N}$ in $Q_{\delta,\omega_i}(c_i)$ for $i \in \{1, \ldots, n\}$. To estimate the weighted-L^2 norm of $\mathcal{D}^\alpha(\lambda u^{2k+1})$ based on (3.2) near each corner, we bound all individual terms $\prod_{i=1}^{s} (\mathcal{D}^{p^i} u)^{m_i}$ and the combinatorial constants $C_{(s,\underline{P},\boldsymbol{M})}$. For the first step, we need the following two lemmas which provide weighted interpolation estimates in a sector. The proofs of these lemmas are along the lines proposed in [11, Lemma 4.2], and are based on dyadic decomposition of the sector and scaling of an interpolation inequality in domains satisfying a cone condition(see [1]). These techniques are useful in treating singularities in a corner.

Lemma 3.3. *Assume given $\delta \in (0, +\infty)$, $\omega \in (0, 2\pi]$, $k \in \mathbb{N}$ and $\beta \in (0, 1)$. Then, there exists a constant $C_{int} = C_{int}(\delta, \omega, k, \beta) > 0$ such that for any function $\phi : Q_{\delta,\omega} \to \mathbb{R}$ for which there exists $\alpha \in \mathbb{N}_0^2$ satisfying, for any $l \in \mathbb{N}$ with $2 \le l \le 2k+1$,*

$$\max_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \mathcal{D}^{\alpha+\eta} \phi\|_{L^2(Q_{\delta,\omega})} < \infty \, ,$$

there holds the following bound

$$\|r^{\frac{\beta}{l}+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(Q_{\delta,\omega})} \le C_{int}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|^{\frac{1}{l}}_{L^2(Q_{\delta,\omega})}$$
$$\cdot\,(\sum_{|\eta|\le 1}\|r^{\beta-2+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\phi\|^{\frac{l-1}{l}}_{L^2(Q_{\delta,\omega})}+\alpha_1^{\frac{l-1}{l}}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|^{\frac{l-1}{l}}_{L^2(Q_{\delta,\omega})}).$$

The proof of this lemma is given in Appendix A.

Lemma 3.4. *Let $\delta\in(0,+\infty)$, $\omega\in(0,2\pi]$, $k\in\mathbb{N}$ and $\beta\in(0,1)$.*

Then there exists a constant $C_t = C_t(\delta,\omega,k,\beta) > 1$ such that for all $\phi\in\mathcal{H}^{2,2}_\beta(Q_{\delta,\omega})$ with $\|\phi-\phi(0,0)\|_{W^2_\beta(Q_{\delta,\omega})} < 1$ and such that there exist $A,E>1$ and $i\in\mathbb{N}$ satisfying

$$\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{\delta,\omega})} \le A^{|\alpha|-2}E^{\alpha_2}(|\alpha|-2)!,\quad \forall\alpha\in\mathbb{N}_0^2: 2\le|\alpha|\le i+1,$$

it holds for any $1\le|\alpha|\le i$ and any $2\le l\le 2k+1$ that,

$$\|r^{\beta/l+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(Q_{\delta,\omega})} \le C_tA^{|\alpha|-1}E^{\alpha_2+1}(|\alpha|-1)!.$$

Proof. We fix δ,ω,k,β and any ϕ satisfying the conditions in this lemma with some $A,E>1$ and $i\in\mathbb{N}$.

By lemma 3.3, there exists $C_{int}>0$ depending on δ,ω,k,β such that for any $1\le|\alpha|\le i$ and any $2\le l\le 2k+1$,

$$\begin{aligned}
&\|r^{\frac{\beta}{l}+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(Q_{\delta,\omega})} \le C_{int}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|^{\frac{1}{l}}_{L^2(Q_{\delta,\omega})}\\
&\cdot\,(\sum_{|\eta|\le 1}\|r^{\beta-2+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\phi\|^{\frac{l-1}{l}}_{L^2(Q_{\delta,\omega})}+\alpha_1^{\frac{l-1}{l}}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|^{\frac{l-1}{l}}_{L^2(Q_{\delta,\omega})})\\
&\le C_{int}\max\left((A^{|\alpha|-2}E^{\alpha_2}(|\alpha|-2)!,1\right)^{\frac{1}{l}}\cdot\left(3(A^{|\alpha|-1}E^{\alpha_2+1}(|\alpha|-1)!\right)^{\frac{l-1}{l}}+\max\left((A^{|\alpha|-2}E^{\alpha_2}(|\alpha|-2)!,1\right)^{\frac{l-1}{l}}\Big)\\
&\le 4C_{int}A^{|\alpha|-1}E^{\alpha_2+1}(|\alpha|-1)!.
\end{aligned}$$

This implies that $C_t := 4C_{int}$ satisfies all conditions of this lemma. □

We investigate the constant $C_{(s,\underline{P},\boldsymbol{M})}$ in (3.2). In the estimation of higher-order derivatives of the quadratic nonlinearity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ for the Navier-Stokes equation in [11, Lemma 4.5], another kind of combinatorial constant $\binom{\alpha}{\eta}$ for $\alpha,\eta\in\mathbb{N}_0^2$ appears in the expansion of higher-order derivatives. Their control with respect to the differentiation order is achieved in [11] using an elementary combinatorial identity.

Here, however, we do not derive a particular combinatorial identity that is best suited to bound the nonlinearity in problem (1.1). Instead, the following lemma provides sufficient control of $C_{(s,\underline{P},\boldsymbol{M})}$.

Lemma 3.5. *Let $A>E>0$. Then for any $|\alpha|\ge 1$ and any $|\eta|=1$,*

$$\begin{aligned}
&\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_{\alpha+\eta},m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^{s}(A^{|p^i|-1}E^{p^i_2+1}(|p^i|-1)!)^{m_i}\\
&\le(|\alpha|+1)AE^{\eta_2}\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_{\alpha},m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^{s}(A^{|p^i|-1}E^{p^i_2+1}(|p^i|-1)!)^{m_i}.
\end{aligned}$$

See Appendix A for the proof.

We are ready to present the following *corner-weighted regularity estimate on the nonlinearity.*

Lemma 3.6 (weighted regularity estimate on the nonlinearity)**.**
Fix $\delta\in(0,1)$, $\omega\in(0,2\pi]$, $\lambda\in\mathbb{R}$, $k\in\mathbb{N}_0$ and $\beta\in(0,1)$.

There exists a constant $C_{non} = C_{non}(\delta,\omega,\beta,\lambda,k) > 0$ *such that for any* $\phi \in \mathcal{H}^{2,2}_\beta(Q_{\delta,\omega})$ *with* $\|\phi\|_{\mathcal{H}^{2,2}_\beta(Q_{\delta,\omega})} < 1$ *and* $\|\phi-\phi(0,0)\|_{W^2_\beta(Q_{\delta,\omega})} < 1$ *for which there exist* $i \in \mathbb{N}$ *and constants* $A > E > 1$ *such that, for* $2 \le |\alpha| \le i+1$*, hold the bounds*

$$\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{\delta,\omega})} \le A^{|\alpha|-2}E^{\alpha_2}(|\alpha|-2)!\ ,$$

there holds, for $1 \le |\alpha| \le i$*,*

$$\|r^{\beta+\alpha_1}\mathcal{D}^\alpha(\lambda\phi^{2k+1})\|_{L^2(Q_{\delta,\omega})} \le C_{non}A^{|\alpha|-1}E^{\alpha_2+1}|\alpha|!\ . \tag{3.3}$$

We remark that due to Lemma 1.2, the value of ϕ at the point $\{r=0\}$ is well-defined.

Proof. Without loss of generality we assume that $\lambda > 0$. With the condition $\|\phi\|_{\mathcal{H}^{2,2}_\beta(Q_{\delta,\omega})} < 1$ and Lemma 1.2 we may assume $\max_{0\le j\le 2k+1}\|\frac{\partial^j\phi^{2k+1}}{\partial\phi^j}\|_{L^\infty(Q_{\delta,\omega})} \le K$ for some $K = K(\delta,\omega,\beta,k) > 0$. The assumptions $\delta < 1$ and $\|\phi-\phi(0,0)\|_{W^2_\beta(Q_{\delta,\omega})} < 1$ imply, for any $1 \le |\alpha| \le i+1$,

$$\|r^{\beta+\alpha_1}\mathcal{D}^\alpha(\lambda\phi)\|_{L^2(Q_{\delta,\omega})} \le \lambda\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{\delta,\omega})} \le \lambda\max(A^{|\alpha|-2}E^{\alpha_2}(|\alpha|-2)!,1) \le \lambda A^{|\alpha|-1}E^{\alpha_2+1}(|\alpha|-1)!,$$

so the case $k=0$ is verified for any $C_{non} \ge \lambda$.

Consider the case $k > 0$. By (3.2), the generalized Hölder inequality and Lemma 3.4, for any $1 \le |\alpha| \le i$ we have,

$$\begin{aligned}
&\|r^{\beta+\alpha_1}\mathcal{D}^\alpha(\lambda\phi^{2k+1})\|_{L^2(Q_{\delta,\omega})}\\
&\quad\le \lambda\|\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\frac{\partial^m\phi^{2k+1}}{\partial\phi^m}\prod_{i=1}^s(r^{\frac{\beta}{m}+p_1^i}\mathcal{D}^{p^i}\phi)^{m_i}\|_{L^2(Q_{\delta,\omega})}\\
&\quad\le \lambda K\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\|\prod_{i=1}^s(r^{\frac{\beta}{m}+p_1^i}\mathcal{D}^{p^i}\phi)^{m_i}\|_{L^2(Q_{\delta,\omega})}\\
&\quad\le \lambda K\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^s\|r^{\frac{\beta}{m}+p_1^i}\mathcal{D}^{p^i}\phi\|^{m_i}_{L^{2m}(Q_{\delta,\omega})}\\
&\quad\le \lambda KC_t^{2k+1}\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^s(A^{|p^i|-1}E^{p_2^i+1}(|p^i|-1)!)^{m_i}.
\end{aligned}$$

It suffices to show that there exists a constant $C_{non} > 0$ such that for any $1 \le |\alpha| \le i$

$$\lambda KC_t^{2k+1}\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^s(A^{|p^i|-1}E^{p_2^i+1}(|p^i|-1)!)^{m_i} \le C_{non}A^{|\alpha|-1}E^{\alpha_2+1}|\alpha|!. \tag{3.4}$$

It is easy to verify that the only possible decomposition for α with $|\alpha|=1$ is $\alpha = 1\cdot\alpha$ and $C_{(1,\{\alpha\},\{1\})} = 1$: it holds $\mathcal{D}^\alpha\phi^{2k+1} = \frac{\partial\phi^{2k+1}}{\partial\phi}\cdot\mathcal{D}^\alpha\phi$. Therefore, for $|\alpha|=1$ and for any $C_{non} \ge \lambda KC_t^{2k+1}$,

$$\begin{aligned}
&\lambda KC_t^{2k+1}\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^s(A^{|p^i|-1}E^{p_2^i+1}(|p^i|-1)!)^{m_i}\\
&= \lambda KC_t^{2k+1}C_{(1,\{\alpha\},\{1\})}A^0E^{\alpha_2+1}(0!) = \lambda KC_t^{2k+1}E^{\alpha_2+1} \le C_{non}E^{\alpha_2+1}.
\end{aligned}$$

We show now that for any $C_{non} \ge \lambda KC_t^{2k+1}$, (3.4) holds for $1 \le |\alpha| \le i$. The case $|\alpha|=1$ is already checked from the above equality and we consider all α such that $1 \le |\alpha| \le i$ by mathematical induction.

Assume now that (3.4) is true for $1 \le |\alpha| \le j < i$. For any $|\alpha| = j+1$, we select $|\eta| = 1$ such that $\alpha - \eta \in \mathbb{N}_0^2$. Then Lemma 3.5 implies that

$$\begin{aligned}
&\lambda K C_t^{2k+1} \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \prod_{i=1}^{s} (A^{|p^i|-1} E^{p_2^i+1}(|p^i|-1)!)^{m_i} \\
&\quad \le \lambda K C_t^{2k+1}(j+1) A E^{\eta_2} \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_{\alpha-\eta}, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \prod_{i=1}^{s} (A^{|p^i|-1} E^{p_2^i+1}(|p^i|-1)!)^{m_i} \\
&\quad \le (j+1) A E^{\eta_2} (\lambda K C_t^{2k+1} \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_{\alpha-\eta}, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \prod_{i=1}^{s} (A^{|p^i|-1} E^{p_2^i+1}(|p^i|-1)!)^{m_i}) \\
&\quad \le (j+1) A E^{\eta_2} \times C_{non} A^j E^{(\alpha_2-\eta_2)+1} j! \\
&\quad \le C_{non} A^{j+1} E^{\alpha_2+1} (j+1)!.
\end{aligned}$$

Therefore (3.4) holds for $|\alpha| = j+1$. Repeating this step validates (3.4) for $1 \le |\alpha| \le i$. Summarizing the case $k > 0$ and $k = 0$ shows that we could set $C_{non} := \lambda \max(K C_t^{2k+1}, 1)$ to satisfy the requirement of this lemma. □

3.2 Weighted analytic regularity near corners

By Lemma 1.2 and Lemma 3.2, we may fix $\delta \in (0, \min(\frac{1}{4}\min_{i,j\in\{1,2,\cdots,n\}, i\ne j} d(c_i, c_j), 1))$ such that $\|u - u(c_i)\|_{W_{\underline{\beta}}^2(Q_{\delta,\omega_i}(c_i))} < 1$ at each corner.

We are now in position to prove local weighted analytic regularity estimates near all corners. The inductive claim used here is similar to the one shown in the proof of [11, Lemma 4.7].

Lemma 3.7. *Let $\underline{\beta} \in (0,1)^n$ such that that for any $i \in \{1,2,\cdots,n\}$, $\beta_i > 1 - \frac{\pi}{\omega_i}$ if $\{i-1, i\} \subset \mathcal{D}$ or $\{i-1,i\} \subset \mathcal{N}$ and $\beta_i > 1 - \frac{\pi}{2\omega_i}$ otherwise. Furthermore, let $u \in H_D^1(\Omega)$ be the weak solution to (1.1) with right hand side $f \in B_{\underline{\beta}}^0(\Omega) \cap L^2(\Omega)$.*

Then there exists $A_u, E_u > 1$ such that for all $i \in \{1,2,\cdots,n\}$,

$$\|r^{\beta_i-2+\alpha_1}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \le A_u^{|\alpha|-2} E_u^{[\alpha_2-2,0]_+}(|\alpha|-2)! \qquad \forall \alpha \in \mathbb{N}_0^2 : |\alpha| \ge 2.$$

Proof. In each sector $Q_{\delta,\omega_i}(c_i)$, we rewrite (1.1) as

$$-(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)u = f - \lambda u^{2k+1} \quad \text{in } Q_{\delta,\omega_i}(c_i), \qquad B(u) = 0 \quad \text{on } \widehat{\Gamma}_i, \tag{3.5}$$

where $\widehat{\Gamma}_i = \partial Q_{\delta,\omega_i}(c_i) \cap \partial\Omega$, $f \in B_{\underline{\beta}}^0(\Omega)$. Lemma 1.1 and Proposition 3.1 then imply that there exists $A_1 > 1$ (depending on λ) such that for any $\alpha \in \mathbb{N}_0^2$,

$$\|r^{\beta_i+\alpha_1}\mathcal{D}^\alpha(\lambda u^{2k+1})\|_{L^2(Q_{\delta,\omega_i}(c_i)\setminus Q_{\delta/2,\omega_i}(c_i))} \le A_1^{|\alpha|}|\alpha|!, \tag{3.6a}$$

$$\|r^{\beta_i-2+\alpha_1}\mathcal{D}^\alpha(r^2 f)\|_{L^2(Q_{\delta,\omega_i}(c_i))} \le A_1^{|\alpha|}|\alpha|!, \tag{3.6b}$$

and, for all $j \in \mathbb{N}_0$,

$$\|r^j \partial_r^j u\|_{H^1(Q_{\delta,\omega_i}(c_i)\setminus Q_{\delta/2,\omega_i}(c_i))} \le A_1^j j!. \tag{3.6c}$$

Define the constants

$$A_u = \max(4C_{sec}A_1, 108(C_{sec}C_{non}+1), 162C_{non}). \tag{3.7a}$$

and

$$E_u = 18, \tag{3.7b}$$

Our proof will be based on the following induction assumption.

Induction assumption For $j_1 \in \mathbb{N}_{\geq 2}$ and $j_2 \in \mathbb{N}$ with $j_2 \leq j_1$, we say H_{j_1,j_2} holds if

$$\|r^{\beta_i-2+\alpha_1}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \leq A_u^{|\alpha|-2}E_u^{[\alpha_2-2,0]_+}(|\alpha|-2)! \quad \forall \alpha \in \mathbb{N}_0^2 : \begin{cases} 2 \leq |\alpha| \leq j_1 - 1, \\ \quad \text{or} \\ |\alpha| = j_1 \text{ and } \alpha_2 \leq j_2. \end{cases} \tag{3.8}$$

Here A_u and E_u are the constants in (3.7a) and (3.7b).

Then $H_{2,2}$ holds since $\|u - u(c_i)\|_{W^2_{\beta_i}(Q_{\delta,\omega_i}(c_i))} < 1$.

Strategy of the proof The proof consists of two steps:

1. We will show that for any $j \in \mathbb{N}_{\geq 2}$,

$$H_{j,j} \implies H_{j+1,2}. \tag{3.9}$$

2. We show

$$\forall j \in \mathbb{N}_{\geq 3} \ \ \forall l \in \mathbb{N}, 2 \leq l < j : \quad H_{j,l} \implies H_{j,l+1}. \tag{3.10}$$

Combining (3.9) and (3.10), we obtain that

$$H_{j,j} \implies H_{j+1,j+1}, \tag{3.11}$$

We infer from (3.11) that $H_{j,j}$ is verified for all $j \in \mathbb{N}_{\geq 2}$. This will conclude the proof.

Step 1: verification of (3.9) We will show equivalently that for any $j \in \mathbb{N}$,

$$H_{j+1,j+1} \implies H_{j+2,2}.$$

If $k > 1$, by Lemma 3.6 exists $C_{non} > 1$ such that for any $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \leq j$

$$\|r^{\beta_i+\alpha_1}\mathcal{D}^\alpha(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \leq C_{non}A_u^{j-1}E_u^{\alpha_2+1}j!,$$

and if $k = 1$, then

$$\|r^{\beta_i+\alpha_1}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \leq \|r^{\beta_i-2+\alpha_1}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \leq A_u^{j-2}E_u^{[\alpha_2-2]_+}j! \leq C_{non}A_u^{j-1}E_u^{\alpha_2+1}j!.$$

Define $v = r^j\partial_r^j u$. Then v solves the boundary value problem

$$\begin{array}{rcll} -(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2)v &=& r^{j-2}\partial_r^j(r^2(f - \lambda u^{2k+1})) & \text{in } Q_{\delta,\omega_i}(c_i), \\ B(v) &=& 0 \quad \text{on } \widehat{\Gamma}_i. & \end{array} \tag{3.12}$$

Proposition 2.1 and (3.6a)-(3.6c) now imply

$$\begin{aligned}
&\sum_{|\eta|=2}\|r^{\beta_i-2+\eta_1}\mathcal{D}^\eta v\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \leq \|v - v(0,0)\|_{W^2_{\beta_i}(Q_{\delta/2,\omega_i}(c_i))} \\
&\leq C_{sec}(\|r^{\beta_i+j-2}\partial_r^j(r^2(f-\lambda u^{2k+1}))\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + \|v\|_{H^1(Q_{\delta,\omega_i}(c_i)\setminus Q_{\delta/2,\omega_i}(c_i))}) \\
&\leq C_{sec}(\|r^{\beta_i+j-2}\partial_r^j(r^2 f)\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + \|r^{\beta_i+j}\partial_r^j(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + j\|r^{\beta_i+j-1}\partial_r^{j-1}(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&+ j(j-1)\|r^{\beta_i+j-2}\partial_r^{j-2}(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + \|v\|_{H^1(Q_{\delta,\omega_i}(c_i)\setminus Q_{\delta/2,\omega_i}(c_i))}) \\
&\leq C_{sec}(A_1^j j! + 3C_{non}A_u^{j-1}E_u j! + A_1^j j!) \\
&\leq C_{sec}(2A_1^j j! + 3C_{non}A_u^{j-1}E_u j!).
\end{aligned}$$

For all $\eta \in \mathbb{N}_0^2$ with $|\eta| = 2$, it holds

$$\mathcal{D}^\eta v = r^j\partial_r^j\mathcal{D}^\eta u + \eta_1 j r^{j-1}\partial_r^{j+\eta_1-1}\partial_\theta^{\eta_2}u + [\eta_1 - 1]_+ j(j-1)r^{j-2}\partial_r^j u.$$

Therefore, for all $\eta \in \mathbb{N}_0^2$ with $|\eta| = 2$,

$$
\begin{aligned}
&\|r^{\beta_i-2+j+\eta_1}\mathcal{D}^\eta\partial_r^j u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&\leq \sum_{|\eta|=2}\|r^{\beta_i-2+\eta_1}\mathcal{D}^\eta v\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + 2j\|r^{\beta_i-2+j+\eta_1}\partial_r^{j+\eta_1-1}\partial_\theta^{\eta_2}u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&+ j(j-1)\|r^{\beta_i-2+j}\partial_r^j u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&\leq C_{sec}(2A_1^j j! + 3C_{non}A_u^{j-1}E_u j!) + 3A_u^{j-1}j! \\
&\leq 2C_{sec}A_1^j j! + (3C_{sec}C_{non}+3)A_u^{j-1}E_u j! \\
&\leq A_u^j j!,
\end{aligned}
$$

which validates (3.9).

Step 2: proof of (3.10) We now fix $l \in \{2, \dots, j-1\}$ and assume that $H_{j,l}$ holds true. This implies, as before, that for any $1 \leq |\alpha| \leq j-2$

$$\|r^{\beta_i+\alpha_1}\mathcal{D}^\alpha(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \leq C_{non}A_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!.$$

So we have

$$
\begin{aligned}
&\|r^{\beta_i-2+(j-l-1)}\partial_r^{j-l-1}\partial_\theta^{l-1}(r^2\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&\leq \|r^{\beta_i+(j-l-1)}\partial_r^{j-l-1}\partial_\theta^{l-1}(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + 2(j-l-1)\|r^{\beta_i+(j-l-2)}\partial_r^{j-l-2}\partial_\theta^{l-1}(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&+ (j-l-1)(j-l-2)\|r^{\beta_i+(j-l-3)}\partial_r^{j-l-3}\partial_\theta^{l-1}(\lambda u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&\leq 3C_{non}A_u^{j-3}E_u^l(j-2)!.
\end{aligned}
$$

Multiply the first equation of (3.5) by r^2 and differentiate the product by $\partial_r^{j-l-1}\partial_\theta^{l-1}$ to obtain

$$
\begin{aligned}
&-(r^2\partial_r^{j-l+1}\partial_\theta^{l-1} + 2(j-l-1)\partial_r^{j-l}\partial_\theta^{l-1} + (j-l-1)(j-l-2)\partial_r^{j-l-1}\partial_\theta^{l-1} \\
&+ r\partial_r^{j-l}\partial_\theta^{l-1} + (j-l-1)\partial_r^{j-l-2}\partial_\theta^{l-1} + \partial_r^{j-l-1}\partial_\theta^{l+1})u = \partial_r^{j-l-1}\partial_\theta^{l-1}(r^2(f-(\lambda u^{2k+1}))).
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
&\partial_r^{j-l-1}\partial_\theta^{l+1}u = \\
&-(r^2\partial_r^{j-l+1}\partial_\theta^{l-1} + 2(j-l-1)\partial_r^{j-l}\partial_\theta^{l-1} + (j-l-1)(j-l-2)\partial_r^{j-l-1}\partial_\theta^{l-1} \\
&+ r\partial_r^{j-l}\partial_\theta^{l-1} + (j-l-1)\partial_r^{j-l-2}\partial_\theta^{l-1})u - \partial_r^{j-l-1}\partial_\theta^{l-1}(r^2(f-(\lambda u^{2k+1}))).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\|r^{\beta_i-2+(j-l-1)}\partial_r^{j-l-1}\partial_\theta^{l+1}u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&\leq \|r^{\beta_i-2+(j-l+1)}\partial_r^{j-l+1}\partial_\theta^{l-1}u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + 2(j-l-1)\|r^{\beta_i-2+(j-l)}\partial_r^{j-l}\partial_\theta^{l-1}u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&+ (j-l-1)(j-l-2)\|r^{\beta_i-2+(j-l-1)}\partial_r^{j-l-1}\partial_\theta^{l-1}u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + \|r^{\beta_i-2+(j-l-1)}\partial_r^{j-l}\partial_\theta^{l-1}u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&+ (j-l-1)\|r^{\beta_i-2+(j-l-1)}\partial_r^{j-l-1}\partial_\theta^{l-1}u\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} + \|r^{\beta_i-2+(j-l-1)}\partial_r^{j-l-1}\partial_\theta^{l-1}(r^2f)\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&+ \|r^{\beta_i-2+(j-l-1)}\partial_r^{j-l-1}\partial_\theta^{l-1}(\lambda r^2u^{2k+1})\|_{L^2(Q_{\delta/2,\omega_i}(c_i))} \\
&\leq A_u^{j-2}E_u^{[l-3]_+}(j-2)! + 2A_u^{j-3}E_u^{[l-3]_+}(j-2)! + A_u^{j-4}E_u^{[l-3]_+}(j-2)! + A_u^{j-3}E_u^{[l-3]_+}(j-3)! \\
&+ A_u^{j-4}E_u^{[l-3]_+}(j-3)! + A_1^{j-2}(j-2)! + 3C_{non}A_u^{j-3}E_u^l(j-2)! \\
&\leq 6A_u^{j-2}E_u^{[l-3]_+}(j-2)! + A_1^{j-2}(j-2)! + 3C_{non}A_u^{j-3}E_u^l(j-2)! \\
&\leq A_u^{j-2}E_u^{l-1}(j-2)! = A_u^{j-2}E_u^{[l-1]_+}(j-2)!\ .
\end{aligned}
$$

Therefore (3.10) holds true. The proof is completed by applying the strategy to show (3.11). □

3.3 Weighted analytic regularity in the polygon

The main result of this paper is now a straightforward consequence of the corner-weighted, analytic estimates of solutions and classical results on interior and boundary regularity.

Theorem 3.8. *Let $\underline{\beta} \in (0,1)^n$ such that that for any $i \in \{1, 2, \cdots, n\}$, $\beta_i > 1 - \frac{\pi}{\omega_i}$ if $\{i-1, i\} \subset \mathcal{D}$ or $\{i-1, i\} \subset \mathcal{N}$ and $\beta_i > 1 - \frac{\pi}{2\omega_i}$ otherwise. Furthermore, let $u \in H^1_D(\Omega)$ be the weak solution to* (1.1) *with right hand side $f \in B^0_{\underline{\beta}}(\Omega) \cap L^2(\Omega)$. Then $u \in B^2_{\underline{\beta}}(\Omega)$.*

Proof. We have analyticity of u in the interior and up to analytic parts of the boundary. In addition, Lemma 3.2 and Lemma 3.7 show that $u \in \mathcal{B}^2_\beta(Q_{\delta/2,\omega_i}(c_i))$ at each corner c_i. Using Lemma 1.1 and combining these two claims we conclude the proof. □

4 Exponential approximability

The weighted, analytic regularity of solutions in Theorem 3.8 implies, via well-known results on approximation properties of *hp*-FEM in [9, 21, 7] *exponential approximability* by finite-dimensional spaces of continuous, piecewise polynomial functions of the solution u of (1.1) with data $f \in B^0_{\underline{\beta}}(\Omega) \cap L^2(\Omega)$. Exponential approximability also holds for several other approximation methods: for *Reduced Basis* and for *Model Order Reduction* methods, as the *Kolmogorov n-width* in $H^1(\Omega)$ of the solution set of (1.1), (1.2) for data $f \in B^0_{\underline{\beta}}(\Omega) \cap L^2(\Omega)$ decreases exponentially as $n \to \infty$, and for deep neural networks and tensor-structured approximation schemes.

Theorem 4.1. *Assume that Ω is a polygon with $n \geq 3$ straight sides. Consider the nonlinear, elliptic PDE* (1.1), (1.2) *for analytic data*

$$f \in A := \{f \in B^0_{\underline{\beta}}(\Omega) \cap L^2(\Omega) : \|f\|_{L^2(\Omega)} \leq 1\},$$

with the corner-weight parameters β_i as in Theorem 3.8. Denote by S the solution map of (1.1), (1.2).

Then, there exists a sequence $\{V_p\}_{p\geq 1}$ of so-called hp-Finite Element subspaces of continous, piecewise polynomial functions v_p of total degree at most p on a sequence of nested, regular, simplicial partitions $\mathcal{T}_p$ of Ω which are obtained from $O(p)$ steps of geometric mesh refinement towards the corners of Ω, such that

$$\forall u \in S(A) : \quad \inf_{v_p \in V_P} \|u - v_p\|_{H^1(\Omega)} \leq C \exp(-b(\mathrm{dim} V_p)^{1/3}) \ ,$$

for certain constants $b, C > 0$ depending on A.

Furthermore, for every $n \in \mathbb{N}$, the Kolmogorov n-width d_n of the solution set $S(A)$ in $H^1(\Omega)$ is exponentially small: there holds

$$d_n(S(A); H^1(\Omega)) \leq C \exp(-bn^{1/3}) \ .$$

In addition, for each $u \in S(A)$, there exists a collection of feedforward neural networks $\{\Phi_{\varepsilon,u}\}_\varepsilon$ with ReLU activation that can represent solutions $u \in S(A)$ of (1.1) *with data $f \in B^0_{\underline{\beta}}(\Omega) \cap L^2(\Omega)$ with exponential expressivity in terms of the neural network size $M(\Phi_{\varepsilon,u})$ and depth $L(\Phi_{\varepsilon,u})$ to accuracy $\varepsilon > 0$ in $H^1(\Omega)$, i.e. their function-realizations $\mathrm{R}(\Phi_{\varepsilon,u})$ satisfy*

$$\|u - \mathrm{R}(\Phi_{\varepsilon,u})\|_{H^1(\Omega)} \leq \varepsilon, \ \ M(\Phi_{\varepsilon,u}) \leq C|\log(\varepsilon)|^5, \ \ L(\Phi_{\varepsilon,u}) \leq C|\log(\varepsilon)\log(|\log(\varepsilon)|)|.$$

Proof. By Theorem 3.8, $S(A) \subset B^2_{\underline{\beta}}(\Omega)$. Then, there exists a sequence $\{V_p\}_{p\geq 1}$ of *hp*-Finite Element spaces of continous, piecewise polynomial functions v_p of total degree at most p on a sequence of nested, regular, simplicial partitions $\mathcal{T}_p$ of Ω which are geometrically refined towards the corners of Ω such that there exists a constant $c > 0$ so that for all $p \in \mathbb{N}$ holds

(i) $\#(\mathcal{T}_p) \leq cp$,

(ii) $n_p = \dim(V_p) \leq cp^3$,

(iii) $\sup_{f\in B} \inf_{v_p \in V_p} \|S(f) - v_p\|_{H^1(\Omega)} \leq c\exp(-bp)$.

We refer, e.g., to [7] for a self-contained proof. This proves the first assertion.

With this (hp-FEM convergence) result in hand, we may bound the Kolmogorov n-width of the set $S(A) \subset\subset H^1(\Omega)$ as

$$\begin{aligned} d_n(S(A), H^1(\Omega)) &= \inf_{W_n \subset H^1(\Omega):\dim(W_n)=n} \sup_{u\in S(A)} \inf_{v_n\in V_n} \|u - v_n\|_{H^1(\Omega)} \\ &\leq \sup_{u\in S(A)} \inf_{v_p\in V_p} \|u - v_p\|_{H^1(\Omega)} \leq C\exp(-bp) \, . \end{aligned}$$

Here, the infimum in the definition of d_n is taken over all subspaces of $H^1_D(\Omega)$ of finite dimension not larger than n, and we used that $S(A) \subset B^2_{\underline{\beta}}(\Omega)$, and property (iii) of the hp-FEM.

The assertion then follows with property (ii) of the hp-FEM.

The final statement on the expression rates of deep ReLU neural networks follows once more from the inclusion $S(A) \subset B^2_{\underline{\beta}}(\Omega)$ with [16, Theorem 5.6]. □

The exponential bound on the Kolmogorov n-width in $H^1(\Omega)$ of the solution manifold $S(A)$ implies corresponding convergence rates of so-called *reduced basis approximations* which are generated by greedy searches. We refer to [12] and to the references there.

5 Conclusion

We summarize the main results of the present work, and indication directions for further research. Given analytic data f and g in (1.1), we established the analytic regularity of the solution u for the semilinear elliptic equation (1.1) in a polygon with homogeneous Dirichlet and Neumann boundary conditions. The analytic regularity shifts are shown in scales of corner-weighted spaces of Kondrat'ev type.

The analysis developed here is also capable of dealing with other similar semilinear elliptic problems. As an example, it is possible to study the analytic regularity of the solution to (1.1) with λu^{2k+1} replaced by any polynomial $g(u)$. For this we only need to modify Lemma 3.6 and the corresponding proof so that they are suitable for any polynomial $g(\phi)$ rather than $\lambda\phi^{2k+1}$. Another possibility would be studying the solution of (1.1) in a curvilinear domain or with $-\Delta u$ replaced by a general linear, divergence form second order elliptic operator $L(\cdot)$ defined by $L(u) = -\nabla\cdot(A(x)\nabla u) + b(x)\cdot\nabla u$ with analytic in $\overline{\Omega}$ coefficient matrix $A(x)$ and advection field $b(x)$. The analytic regularity of the solution u also reveals the potential to develop exponentially convergent numerical approximation methods such as hp-FEM, or reduced basis approximations based on subspace sequences obtained via greedy algorithms [4]. It also implies the exponential convergence of quantized, tensor-formatted approximations [13].

A Proof of Lemma 3.3

We firstly set $\delta = 1$. Consider the dyadic partition given by the sets

$$S^j := \left\{x \in Q_{1,\omega} : 2^{-j-1} < r(x) < 2^{-j}\right\}, \qquad j \in \mathbb{N}_0,$$

and denote the linear maps $\Psi_j : S^j \to S^0$ representing homothetic scaling. Denote $\widehat{\phi}_j := \phi\circ\Psi_j^{-1} : S^0 \to \mathbb{R}$ and write $\widehat{\mathcal{D}}^\alpha$ for derivation with respect to polar coordinates (r,θ) in S^0. Then, by scaling, for any $q \in [1,\infty)$ and any $\gamma \in \mathbb{R}$,

$$\|r^{\gamma+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^q(S^j)} = 2^{-j(\gamma+2/q)}\|r^{\gamma+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\phi}\|_{L^q(S^0)}. \tag{A.1}$$

Furthermore, the following interpolation inequality holds in S^0 [1, Theorem 3]: there exists $C_0 > 0$ depending on k such that for any $2 \le l \le 2k+1$, it holds that

$$\forall v \in H^1(S^0): \quad \|v\|_{L^{2l}(S^0)} \le C_0 \|v\|_{H^1(S^0)}^{1-1/l} \|v\|_{L^2(S^0)}^{1/l} . \tag{A.2}$$

Moreover, there also holds for all $v \in H^1(S^0)$,

$$\|v\|_{H^1(S^0)} \le 4\left(\|v\|_{L^2(S^0)} + \|\partial_r v\|_{L^2(S^0)} + \|\partial_\theta v\|_{L^2(S^0)}\right). \tag{A.3}$$

To check this inequality, we have, by elementary Calculus,

$$\partial_{x_1} = \cos\theta \partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \partial_{x_2} = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta.$$

These relations yield the following bounds:

$$\|\partial_{x_1} v\|_{L^2(S^0)} \le \|\cos\theta\partial_r v\|_{L^2(S^0)} + \|\frac{\sin\theta}{r}\partial_\theta v\|_{L^2(S^0)} \le \|\partial_r v\|_{L^2(S^0)} + 2\|\partial_\theta v\|_{L^2(S^0)},$$

and

$$\|\partial_{x_2} v\|_{L^2(S^0)} \le \|\sin\theta\partial_r v\|_{L^2(S^0)} + \|\frac{\cos\theta}{r}\partial_\theta v\|_{L^2(S^0)} \le \|\partial_r v\|_{L^2(S^0)} + 2\|\partial_\theta v\|_{L^2(S^0)}.$$

Therefore,

$$\begin{aligned}
\|v\|_{H^1(S^0)}^2 &\le \|v\|_{L^2(S^0)}^2 + \|\partial_{x_1} v\|_{L^2(S^0)}^2 + \|\partial_{x_2} v\|_{L^2(S^0)}^2 \\
&\le \|v\|_{L^2(S^0)}^2 + \left(\|\partial_r v\|_{L^2(S^0)} + 2\|\partial_\theta v\|_{L^2(S^0)}\right)^2 + \left(\|\partial_r v\|_{L^2(S^0)} + 2\|\partial_\theta v\|_{L^2(S^0)}\right)^2 \\
&\le \|v\|_{L^2(S^0)}^2 + 2\|\partial_r v\|_{L^2(S^0)}^2 + 8\|\partial_\theta v\|_{L^2(S^0)}^2 + 8\|\partial_r v\|_{L^2(S^0)} \cdot \|\partial_\theta v\|_{L^2(S^0)} \\
&\le \|v\|_{L^2(S^0)}^2 + 6\|\partial_r v\|_{L^2(S^0)}^2 + 12\|\partial_\theta v\|_{L^2(S^0)}^2 \\
&\le 16\left(\|v\|_{L^2(S^0)} + \|\partial_r v\|_{L^2(S^0)} + \|\partial_\theta v\|_{L^2(S^0)}\right)^2.
\end{aligned}$$

Taking the square root on both sides leads to (A.3). Combining (A.2) and (A.3) and choosing $v = r^{\alpha_1}\mathcal{D}^\alpha\phi$ give

$$\begin{aligned}
&\|r^{\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(S^0)} \\
&\quad\le 2C_0\|r^{\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(S^0)}^{1/l}\left(\sum_{|\eta|\le 1}\|\mathcal{D}^\eta(r^{\alpha_1}\mathcal{D}^\alpha\phi)\|_{L^2(S^0)}\right)^{1-1/l} \\
&\quad\le 2C_0\|r^{\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(S^0)}^{1/l}\left(\sum_{|\eta|\le 1}\|r^{\alpha_1}\mathcal{D}^{\alpha+\eta}\phi\|_{L^2(S^0)} + \alpha_1\|r^{\alpha_1-1}\mathcal{D}^\alpha\phi\|_{L^2(S^0)}\right)^{1-1/l}.
\end{aligned}$$

Therefore, using the bound $2^{-|a|} \le r(x)^a \le 2^{|a|}$ valid for all $x \in S^0$ and all $a \in \mathbb{R}$,

$$\begin{aligned}
\|r^{\beta/l+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(S^0)} &\le 2^{\beta/l+2-\beta}2C_0\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(S^0)}^{1/l} \\
&\quad\times\left(\sum_{|\eta|\le 1}\|r^{\beta-2+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\phi\|_{L^2(S^0)} + \alpha_1\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(S^0)}\right)^{1-1/l}.
\end{aligned}$$

Set $C_1 := 2^{(\beta-2)/(2k+1)+3-\beta}C_0$. Now by scaling back to S^j and using (A.1),

$$\|r^{\beta/l+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(S^j)} = 2^{-j(\frac{\beta}{l}+1/l)}\|r^{\beta/l+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\phi}\|_{L^{2l}(S^0)}$$

$$
\begin{aligned}
&\le 2^{-j(\beta/l+1/l)} C_1 \|r^{\beta-2+\alpha_1} \widehat{\mathcal{D}}^\alpha \widehat{\phi}\|_{L^2(S^0)}^{1/l} \\
&\quad \times \left(\sum_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \widehat{\mathcal{D}}^{\alpha+\eta}\widehat{\phi}\|_{L^2(S^0)} + \alpha_1 \|r^{\beta-2+\alpha_1}\widehat{\mathcal{D}}^\alpha \widehat{\phi}\|_{L^2(S^0)} \right)^{1-1/l} \\
&\le 2^{-j(\beta/l+1/l)} 4C_1 \|r^{\beta-2+\alpha_1} \widehat{\mathcal{D}}^\alpha \widehat{\phi}\|_{L^2(S^0)}^{1/l} \\
&\quad \times \left(\sum_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \widehat{\mathcal{D}}^{\alpha+\eta}\widehat{\phi}\|_{L^2(S^0)}^2 + \alpha_1^2 \|r^{\beta-2+\alpha_1}\widehat{\mathcal{D}}^\alpha \widehat{\phi}\|_{L^2(S^0)}^2 \right)^{1/2-1/2l} \\
&\le 2^{-j(\beta/l+1/l-(\beta-2+1))} 4C_1 \|r^{\beta-2+\alpha_1} \mathcal{D}^\alpha \phi\|_{L^2(S^j)}^{1/l} \\
&\quad \times \left(\sum_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \mathcal{D}^{\alpha+\eta}\phi\|_{L^2(S^j)}^2 + \alpha_1^2 \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha \phi\|_{L^2(S^j)}^2 \right)^{1/2-1/2l} .
\end{aligned}
$$

Setting $C_2 := (\beta+1)/l - (\beta-2+1) > 1-\beta > 0$, we have

$$
\begin{aligned}
&\sum_{j\in\mathbb{N}_0} \|r^{\beta/l+\alpha_1} \mathcal{D}^\alpha \phi\|_{L^{2l}(S^j)}^{2l} \\
&\le \frac{1}{1-2^{-2lC_2}} (4C_1)^{2l} \sum_{j\in\mathbb{N}_0} \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(S^j)}^2 \\
&\quad \times \sum_{j\in\mathbb{N}_0} \left(\sum_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \mathcal{D}^{\alpha+\eta}\phi\|_{L^2(S^j)}^2 + \alpha_1^2 \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha \phi\|_{L^2(S^j)}^2 \right)^{l-1} \\
&\le \frac{1}{1-2^{-2lC_2}} (4C_1)^{2l} \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{1,\omega}(c))}^2 \\
&\quad \times \left(\sum_{j\in\mathbb{N}_0} (\sum_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \mathcal{D}^{\alpha+\eta}\phi\|_{L^2(S^j)}^2 + \alpha_1^2 \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha \phi\|_{L^2(S^j)}^2) \right)^{l-1} \\
&\le \frac{1}{1-2^{-2lC_2}} (4C_1)^{2l} \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{1,\omega}(c))}^2 \\
&\quad \times \left(\sum_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \mathcal{D}^{\alpha+\eta}\phi\|_{L^2(Q_{1,\omega}(c))}^2 + \alpha_1^2 \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha \phi\|_{L^2(Q_{1,\omega}(c))}^2 \right)^{l-1} \\
&\le \frac{1}{1-2^{-4(1-\beta)}} (4C_1)^{2l} \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{1,\omega}(c))}^2 \\
&\quad \times \left(\sum_{|\eta|\le 1} \|r^{\beta-2+\alpha_1+\eta_1} \mathcal{D}^{\alpha+\eta}\phi\|_{L^2(Q_{1,\omega}(c))}^2 + \alpha_1^2 \|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha \phi\|_{L^2(Q_{1,\omega}(c))}^2 \right)^{l-1} .
\end{aligned}
$$

Here we used the fact that $\|\cdot\|_{l^p} \le \|\cdot\|_{l^1}$ for any $p \ge 1$ where $\|\cdot\|_{l^p}$ denotes the l^p-norm of a sequence. Taking $2l$-th root on both sides concludes the proof of the bound in $Q_{1,\omega}(c)$ with $C_{int} = (\frac{1}{1-2^{-4(1-\beta)}})^{\frac{1}{2l}}(4C_1)$. To deal with the case $\delta \ne 1$, we define $\Phi_\delta : Q_{\delta,\omega} \to Q_{1,\omega}(c)$ as a homothetic mapping, denote $\widetilde{\phi} := \phi \circ \Phi_\delta^{-1} : Q_{1,\omega}(c) \to \mathbb{R}$ and write $\widetilde{\mathcal{D}}^\alpha$ as the differentiation with respect to polar coordinates (r,θ) in $Q_{\delta,\omega}$. We observe that, for any $\phi : Q_{\delta,\omega} \to \mathbb{R}$, $2 \le l \le 2k+1$ and any $\alpha \in \mathbb{N}_0^2$,

$$
\|r^{\frac{\beta}{l}+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(Q_{\delta,\omega})} = \delta^{\frac{\beta}{l}+\frac{1}{l}} \|r^{\frac{\beta}{l}+\alpha_1}\widetilde{\mathcal{D}}^\alpha\widetilde{\phi}\|_{L^{2l}(Q_{1,\omega}(c))}
$$

and

$$
\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{\delta,\omega})} = \delta^{\beta-2+\frac{1}{l}} \|r^{\beta-2+\alpha_1}\widetilde{\mathcal{D}}^\alpha\widetilde{\phi}\|_{L^2(Q_{1,\omega}(c))}.
$$

By applying the scaling to ϕ and using our result for $\delta = 1$, we have, for general δ,

$$\begin{aligned}\|r^{\frac{\beta}{l}+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^{2l}(Q_{\delta,\omega})} &\le C_{int}\cdot\delta^{\frac{\beta}{l}-\beta+2}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{\delta,\omega})}^{\frac{1}{l}}\\ &\cdot\Big(\sum_{|\eta|\le 1}\|r^{\beta-2+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\phi\|_{L^2(Q_{\delta,\omega})}^{\frac{l-1}{l}}+\alpha_1^{\frac{l-1}{l}}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha\phi\|_{L^2(Q_{\delta,\omega})}^{\frac{l-1}{l}}\Big).\end{aligned}$$

Therefore we could set in general $C_{int} = \delta^{\frac{\beta}{l}-\beta+2}(\frac{1}{1-2^{-4(1-\beta)}})^{\frac{1}{2l}}(4C_1)$ to finish the proof. □

B Proof of Lemma 3.5

We fix α and η. There exists a polynomial $\psi(r,\theta)$ of at most order $|\alpha|+|\eta|$ such that for any $|\gamma|\ge 1$ with $\gamma\le\alpha+\eta$

$$\mathcal{D}^\gamma\psi(0,0) = A^{|\gamma|-1}E^{\gamma_2+1}(|\gamma|-1)!, \tag{B.1}$$

and there exists a polynomial $g(\psi)$ of at most order $2k+1$ such that

$$\frac{\partial^i g(\psi)}{\partial\psi^i}|_{\psi=\psi(0,0)} = 1$$

for any $0 < i \le 2k+1$. The construction of above polynomials is based on Hermite interpolation: we take $\psi(r,\theta)$ as an example to explain it.

For any $\gamma\le\alpha+\eta$, the following polynomial of order $|\gamma|$

$$\psi_\gamma(r,\theta) = \frac{1}{\gamma!}r^{\gamma_1}\theta^{\gamma_2}$$

satisfies for any $\alpha\in\mathbb{N}_0^2$,

$$\mathcal{D}^\alpha\psi(0,0) = \delta_{\alpha,\gamma}.$$

Therefore

$$\psi = \sum_{\gamma\le\alpha+\eta,|\gamma|\ge 1} A^{|\gamma|-1}E^{\gamma_2+1}(|\gamma|-1)!\psi_\gamma$$

will be a satisfactory choice for the required polynomial.

The identity (3.2) yields

$$\mathcal{D}^\alpha g(\psi)|_{(r,\theta)=(0,0)} = \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^s(A^{|p^i|-1}E^{p_2^i+1}(|p^i|-1)!)^{m_i}.$$

and

$$\mathcal{D}^{\alpha+\eta} g(\psi)|_{(r,\theta)=(0,0)} = \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_{\alpha+\eta}, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\prod_{i=1}^s(A^{|p^i|-1}E^{p_2^i+1}(|p^i|-1)!)^{m_i}.$$

Now we examine $\mathcal{D}^{\alpha+\eta}g(\psi)|_{(r,\theta)=(0,0)}$ in a different way: Note that

$$\begin{aligned}\mathcal{D}^{\alpha+\eta}g(\psi)|_{(r,\theta)=(0,0)} &= \mathcal{D}^\eta(\mathcal{D}^\alpha g(\psi))|_{(r,\theta)=(0,0)}\\ &= \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}(\mathcal{D}^\eta\prod_{i=1}^s(\mathcal{D}^{p^i}\psi)^{m_i})|_{(r,\theta)=(0,0)}\\ &\quad+\sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\mathcal{D}^\eta\psi|_{(r,\theta)=(0,0)}\cdot\prod_{i=1}^s(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \sum_{i=1}^{s}(m_i \prod_{j=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_j-\delta_{i,j}} \cdot \mathcal{D}^{p^i+\eta}\psi|_{(r,\theta)=(0,0)}) \\
&\quad + \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\mathcal{D}^{\eta}\psi|_{(r,\theta)=(0,0)} \cdot \prod_{i=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i} \\
&= \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \sum_{i=1}^{s}(m_i \prod_{j=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_j} \cdot (\frac{\mathcal{D}^{p^i+\eta}\psi}{\mathcal{D}^{p^i}\psi})|_{(r,\theta)=(0,0)}) \\
&\quad + \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\mathcal{D}^{\eta}\psi|_{(r,\theta)=(0,0)} \cdot \prod_{i=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i} \\
&= \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \prod_{j=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_j} \sum_{i=1}^{s}(m_i \cdot (\frac{\mathcal{D}^{p^i+\eta}\psi}{\mathcal{D}^{p^i}\psi})|_{(r,\theta)=(0,0)}) \\
&\quad + \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})}\mathcal{D}^{\eta}\psi|_{(r,\theta)=(0,0)} \cdot \prod_{i=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i}.
\end{aligned}
$$

Therefore, by using (B.1) and noting that $A > E > 1$ we have:

$$
\begin{aligned}
&\mathcal{D}^{\alpha+\eta}g(\psi)|_{(r,\theta)=(0,0)} \\
&\quad \le \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} AE^{\eta_2}\sum_{i=1}^{s}(m_i|p^i|)C_{(s,\underline{P},\boldsymbol{M})} \prod_{j=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i} \\
&\qquad + \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} E^{\eta_2+1}C_{(s,\underline{P},\boldsymbol{M})} \prod_{j=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i} \\
&\quad = (|\alpha|AE^{\eta_2} + E^{\eta_2+1}) \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \prod_{j=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i} \\
&\quad \le (|\alpha|+1)AE^{\eta_2} \sum_{(s,\underline{P},\boldsymbol{M})\in\mathscr{D}_\alpha, m\le 2k+1} C_{(s,\underline{P},\boldsymbol{M})} \prod_{j=1}^{s}(\mathcal{D}^{p^i}\psi|_{(r,\theta)=(0,0)})^{m_i},
\end{aligned}
$$

which is exactly what we want to prove. □

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