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BOUNDARY INTEGRAL EXTERIOR CALCULUS*

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Abstract. We report a surprising and deep structural property of first-kind boundary integral operators for Hodge–Dirac and Hodge–Laplace operators associated with de Rham Hilbert complexes on a bounded domain Ω in a Riemannian manifold. We show that from a variational perspective, those first-kind boundary integral operators are Hodge–Dirac and Hodge–Laplace operators as well, this time set in a trace de Rham Hilbert complex on the boundary $\partial\Omega$ whose underlying spaces of differential forms are equipped with non-local inner products defined through layer potentials. On the way to this main result we conduct a thorough analysis of layer potentials in operator-induced trace spaces and derive representation formulas.

Key words. De Rham complex, Hodge–Dirac operator, Hodge–Laplace operators, Newton potential, layer potentials, representation formula, jump relations, trace de Rham complex, first-kind boundary integral operators

AMS subject classifications. 45P05, 58J32, 58J10, 58A14, 35R01

1. Introduction.

1.1. Setting and notations.

We start from a smooth orientable N-dimensional Riemannian manifold $\mathcal{M}$ without boundary and with uniformly bounded curvature. We let $\Omega = \Omega^- \subset \mathcal{M}$ denote an open connected compact sub-manifold of the same dimension with a compatibly oriented Lipschitz boundary $\Gamma := \partial\Omega$ [18, Appendix A] and write $\Omega^+ = \mathcal{M}\setminus \operatorname{cl}\Omega$ for the open complement of Ω.

Following the notations of [2, Section 2], for some open N-dimensional submanifold ω of $\mathcal{M}$, we denote by $L^2\Lambda^\ell(\omega)$ the Hilbert space of square-integrable differential forms of degree $\ell \in \{0,\dots,N\}$ ("ℓ-forms") on ω. In general, we write $X\Lambda^\ell(\omega)$, for instance $C^\infty\Lambda^\ell(\omega)$, $L^\infty\Lambda^\ell(\omega)$, etc., for a space of ℓ-forms on ω with coefficients in the function space $X(\omega)$ and the inherited topology. In particular, $C_0^\infty\Lambda^\ell(\omega)$ is the space of "test ℓ-forms" and $(C_0^\infty\Lambda^\ell(\omega))'$ the corresponding dual space of distributions. If $\ell < 0$ or $\ell > N$ we adopt the convention that $X\Lambda^\ell(\omega) = \{0\}$ to simplify the treatment of special cases.

Recall the local and isometric Hodge star operators $\star_\ell : L^2\Lambda^\ell(\omega) \to L^2\Lambda^{N-\ell}(\omega)$, $0 \le \ell \le N$, induced by the Riemannian metric on $\mathcal{M}$ [16, Section 2.1]. The symmetric pairing

$$\langle U_\ell, V_\ell\rangle_\omega = \int_\omega U_\ell \wedge \star_\ell V_\ell\ , \quad U_\ell, V_\ell \in L^2\Lambda^\ell(\omega)\ , \tag{1.1}$$

is bilinear and is to be distinguished from the sesqui-linear inner product written $(U_\ell, V_\ell)_{L^2\Lambda^\ell(\omega)} := \langle U_\ell, \overline{V_\ell}\rangle_\Omega$, where the overline indicates complex conjugation. We will regularly rely on duality pairings that extend symmetric L^2-type pairings of the form (1.1). We use double angle brackets for those duality pairings, e.g. $\langle\!\langle \cdot,\cdot\rangle\!\rangle_\omega$.

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We will also need Sobolev spaces of ℓ-forms [23, Section 1.3], in particular

$$H^1\Lambda^\ell(\omega) := \left\{ v \in L^2\Lambda^\ell(\omega) \middle|\, \nabla_\ell v \in L^2(\Lambda^1(\omega) \otimes \Lambda^\ell(\omega)) \right\} , \tag{1.2}$$

where ∇_ℓ stands for the Levi-Civita connection [16, Section 9.2]. An equivalent definition can be given based on coordinate representations, see [2, Section 2.2]. Primes generally tag dual spaces or adjoint linear operators, but we also use the customary notation $H^{-1}\Lambda^\ell(\mathcal{M}) := \left(H^1\Lambda^\ell(\mathcal{M})\right)'$.

We recall the exterior derivative d_ℓ acting on ℓ–forms and its formal L^2-adjoint, the exterior co-derivative $\delta_{\ell+1} = (-1)^{\ell+1} \star_\ell^{-1} \mathrm{d}_{N-\ell-1} \star_{\ell+1}$, $0 \leq \ell \leq N$ [23, Definition 1.2.2]. Note the special cases $\mathrm{d}_k = 0$, $k \notin \{0, \dots, N-1\}$, and $\delta_m = 0$, $m \notin \{1, \dots, N\}$. Both d_ℓ and $\delta_{\ell+1}$ are viewed as closed densely defined unbounded operators $\mathrm{d}_\ell : L^2\Lambda^\ell(\omega) \to L^2\Lambda^{\ell+1}(\omega)$ and $\delta_{\ell+1} : L^2\Lambda^{\ell+1}(\omega) \to L^2\Lambda^\ell(\omega)$. As such they give rise to the (primal and dual) de Rham domain Hilbert complexes [3]

$$\begin{aligned}
&H\Lambda^0(\mathrm{d},\omega) \xrightarrow{\mathrm{d}_0} H\Lambda^1(\mathrm{d},\omega) \xrightarrow{\quad\mathrm{d}_1\quad} \dots \\
&\hookrightarrow \dots \xrightarrow{\quad\mathrm{d}_{\ell-1}\quad} H\Lambda^\ell(\mathrm{d},\omega) \xrightarrow{\mathrm{d}_\ell} H\Lambda^{\ell+1}(\mathrm{d},\omega) \xrightarrow{\quad\mathrm{d}_{\ell+1}\quad} \dots \\
&\qquad\qquad\hookrightarrow \dots \xrightarrow{\quad\mathrm{d}_{N-2}\quad} H\Lambda^{N-1}(\mathrm{d},\omega) \xrightarrow{\mathrm{d}_{N-1}} L^2\Lambda^N(\omega) \,,
\end{aligned} \tag{1.3a}$$

$$\begin{aligned}
&L^2\Lambda^0(\omega) \xleftarrow{\delta_1} H\Lambda^1(\delta,\omega) \xleftarrow{\quad\delta_2\quad} \dots \hookleftarrow \\
&\dots \xleftarrow{\quad\delta_\ell\quad} H\Lambda^\ell(\delta,\omega) \xleftarrow{\delta_{\ell+1}} H\Lambda^{\ell+1}(\delta,\omega) \xleftarrow{\quad\delta_{\ell+2}\quad} \dots \hookleftarrow \\
&\qquad\qquad \dots \xleftarrow{\quad\delta_{N-1}\quad} H\Lambda^{N-1}(\delta,\omega) \xleftarrow{\delta_N} H\Lambda^N(\delta,\omega) \,,
\end{aligned} \tag{1.3b}$$

which enjoy the Fredholm property, cf. [1, Chapters 4 and 6]. Here, we have used the notations

$$H\Lambda^\ell(\mathsf{Op},\omega) := \{ v \in L^2\Lambda^\ell(\omega) |\, \mathsf{Op}\, v \in L^2\Lambda^{\ell\pm1}(\omega) \} \,, \quad \mathsf{Op} \in \{\mathrm{d}_\ell, \delta_\ell\} \,, \tag{1.4}$$

for the domain spaces of the exterior (co-)derivatives. Those are Hilbert spaces when equipped with the natural graph norms.

We drop the degree superscript for product spaces related to the full Grassmann algebra, for example, $L^2\Lambda(\omega) = \bigotimes_{\ell=0}^N L^2\Lambda^\ell(\omega)$. We write in a bold font, e.g. $\boldsymbol{U} = (U_\ell)_{\ell=0}^N$, elements of those spaces. The exterior (co-)derivatives induce diffuse Fredholm–nilpotent operators $\mathbf{d} : L^2\Lambda(\omega) \to L^2\Lambda(\omega)$ and $\boldsymbol{\delta} : L^2\Lambda(\omega) \to L^2\Lambda(\omega)$. They are formally adjoint with respect to the Hermitian inner product $(\boldsymbol{U}, \boldsymbol{V})_{L^2\Lambda(\omega)} := \langle \boldsymbol{U}, \overline{\boldsymbol{V}} \rangle_\omega$ defined through the degree-wise bi-linear pairing $\langle \boldsymbol{U}, \boldsymbol{V} \rangle_\omega = \sum_{\ell=0}^N \langle U_\ell, V_\ell \rangle_\omega$, $\boldsymbol{U}, \boldsymbol{V} \in L^2\Lambda(\omega)$, cf. (1.1). As $(N+1) \times (N+1)$ operator matrices acting on vectors of differential forms of the form $\boldsymbol{U} = [U_0, \dots, U_N]^\top$, the "*full*" exterior

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derivative and co-derivative read

$$\mathbf{d} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \mathrm{d}_0 & 0 & 0 & \dots & 0 \\ 0 & \mathrm{d}_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \mathrm{d}_{N-1} & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\delta} = \begin{bmatrix} 0 & \delta_1 & 0 & \dots & 0 \\ 0 & 0 & \delta_2 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \delta_N \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{1.5}$$

Similarly, the full Hodge star on $L^2\Lambda(\omega)$ can be represented as

$$\star := \begin{bmatrix} & 0 & & \star_N \\ & & \star_{N-1} & \\ & \iddots & & \\ & \star_1 & 0 & \\ \star_0 & & & \end{bmatrix} : L^2\Lambda(\omega) \to L^2\Lambda(\omega) . \tag{1.6}$$

We also adapt the spaces defined in (1.4) to the Grassmann algebra setting and then write $H\Lambda(\mathbf{d}, \omega)$ and $H\Lambda(\boldsymbol{\delta}, \omega)$, with evident meanings.

1.2. Overview and Outline. Our goal is to understand the structural properties of *first-kind* boundary integral operators (BIOs) associated with boundary value problems (BVPs) for the

$$\text{Hodge-Dirac operator} \qquad \mathfrak{D} = \mathbf{d} + \boldsymbol{\delta} , \tag{1.7}$$

and the

$$\text{Hodge-Laplace operators} \qquad -\Delta_\ell = \mathrm{d}_{\ell-1}\delta_\ell + \delta_{\ell+1}\mathrm{d}_\ell , \quad 0 \le \ell \le N , \tag{1.8}$$

which are unbounded first and second-order differential operators on $L^2\Lambda(\Omega)$ and $L^2\Lambda^\ell(\Omega)$, respectively. This article is inspired by the "modern approach" to layer potentials and BIOs, analyzing them in "operator-induced trace spaces"; we refer to [8] and the monographs [14, 20] for a comprehensive presentation. It comprises the following steps, also outlined in Figure 1:

(I) Applying integration by parts (Green's formulas) to the operators Δ_ℓ, $\mathfrak{D}$ and related ones, we can identify appropriate boundary conditions and trace operators. Those are reviewed in Section 3.
(II) Solution operators on $\mathcal{M}$ (Newton potentials) in conjunction with Green's formulas yield representation formulas, which we derive in Section 4. Layer potentials form their main building blocks.
(III) BIOs emerge from applying traces to representation formulas, see Section 5. We focus on *first-kind* BIOs, which map between trace spaces that are in duality with respect to an L^2 pivot space.

Our key new finding, stated in Theorem 5.2, Theorem 5.3 and Theorem 5.4, is that

> the obtained first-kind BIOs are Hodge–Dirac and Hodge–Laplace operators themselves, but associated with trace de Rham complexes on Γ, which are based on non-local inner products defined through layer potentials.

This discovery also reveals the importance of the *trace de Rham complex* when studying boundary integral equations for Hodge-Dirac and Hodge-Laplace operators. We recall this key concept in Section 2.

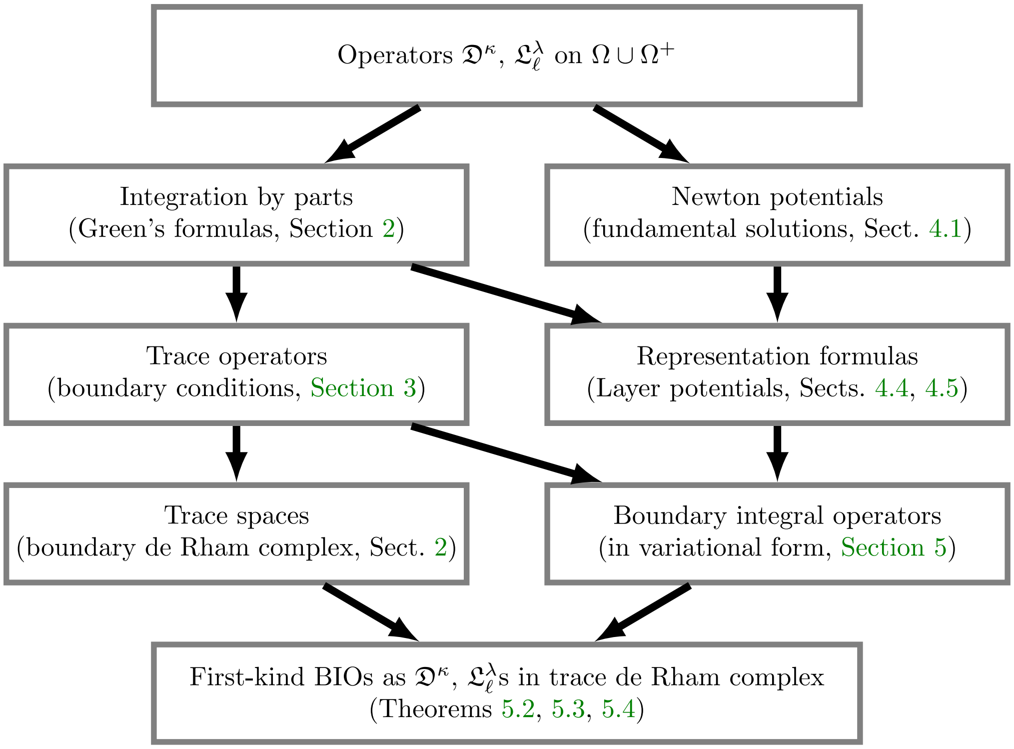

FIG. 1. *The road towards first-kind boundary integral operators induced by the Hodge-Laplacian and the Hodge-Dirac operator. Arrows indicate a "built-from" relationship.*

A difficulty arises at the outset of our program: $\mathfrak{D}$ and Δ_ℓ may not be injective even on $C^\infty\Lambda(\mathcal{M})$ or $C^\infty\Lambda^\ell(\mathcal{M})$, respectively, with nullspaces comprising smooth harmonic forms [16, Section 3.1]. Thus, to keep the presentation simple, we regularize the Hodge–Dirac and Hodge–Laplace operators by adding zero-order terms and work with *injective* modified operators of the form

$$\mathfrak{D}^\kappa := \mathfrak{D} + \imath\kappa\mathsf{Id}\,, \quad \kappa \in \mathbb{R}\backslash\{0\}\,, \qquad \text{and} \qquad \mathfrak{L}_\ell^\lambda := -\Delta_\ell + \lambda\mathsf{Id}, \qquad \lambda > 0, \tag{1.9}$$

which are related by the identity ($\boldsymbol{\Delta} := (\Delta_\ell)_{\ell=0}^N$)

$$(\mathfrak{D} - \imath\kappa)\,(\mathfrak{D} + \imath\kappa) = -\boldsymbol{\Delta} + \lambda\,, \quad \text{if} \quad \lambda = \kappa^2\,. \tag{1.10}$$

For the remainder of this article we fix $\lambda > 0$ and set $\kappa := \lambda^{1/2}$.

Remark 1.1. In the case that $\mathcal{M}$ is the Euclidean space $\mathbb{R}^N$, with some modifications the developments of this article carry over to the case $\lambda = \kappa = 0$ and even $\lambda < 0$, $\kappa \in \imath\mathbb{R}$, provided that suitable decay or radiation conditions "at ∞" are imposed [6, Section 3.3], [14, Chapter 9], [10, Section 6]. We decided not to treat this technically more challenging setting in this article.

1.3. Related work. There is a plethora of literature on layer potentials and boundary integral equations concerned with Hodge-Laplace and Hodge-Dirac operators in the setting of this article, see [19], the monographs [18] and [16] and the numerous references therein. To the authors' astonishment, in these works all attention is directed at *second-kind* BIOs on Γ and those are mainly considered as operators in $L^p(\Gamma)$-type spaces. We could not find a single article devoted to the first-kind BIOs we are going to study in this work.

Of course, in Euclidean space $\mathcal{M} = \mathbb{R}^N$, $N = 2, 3$, first-kind BIOs play a prominent role as foundation for the boundary element discretization of BVPs for various strongly elliptic partial differential equations [20, Chapter 3], [14, Chapters 7-10], and even the time-harmonic Maxwell's equations [5]. The analysis of first-kind BIOs for the Hodge-Laplacian for Euclidean space $\mathcal{M} = \mathbb{R}^3$ was pursued by some of the authors in [6] and [7], at the time, entirely in the framework of classical vector calculus.

What can be seen as precursors of and sources of inspirations for the present paper are the works [12] and [22]. In the latter article we caught a first glimpse of the above-mentioned discovery for the Hodge-Dirac operator $\mathcal{D}$ in the special case $\mathcal{M} = \mathbb{R}^3$. The investigations were solely based on classical vector calculus. A crucial observation was that, although the Hodge–Dirac operator is only first-order, it is still amenable to arguments borrowed from the well-known theory of first-kind boundary integral equations for second-order elliptic operators in Euclidean space.

This article was motivated by the desire

(i) to generalize the results of [22] to arbitrary dimensions by translating them into the language of differential forms, and
(ii) to extend them to analogous results for the Hodge-Laplacian.

We fully succeeded in this endeavor and, thus, gleaned completely new deep insights into structural properties of BIOs.

Of course, also this research heavily draws on previously established theory, in particular as regards traces and energy trace spaces for differential forms. The important results of [17] and [26] on the existence and properties of surjective trace operators for spaces of differential forms in $\mathbb{R}^N$ proved instrumental in the development of boundary integral exterior calculus on boundaries of mere Lipschitz regularity. Abstract trace complexes are also studied in [11], where an alternative proof to that given in [17] is provided for the compactness property of the so-called trace de Rham complex.

List of notations. In this article we prefer "verbose" notations conveying maximum information about entities. We admit that this leads to lavishly adorned symbols, but enhanced precision is worth this price.

$\mathcal{M}$	"ambient" Riemannian manifold, Page 1
$\Omega = \Omega^-$	open, bounded domain $\subset \mathcal{M}$, Page 1
$\Omega^+ := \mathcal{M} \setminus \operatorname{cl}\Omega$	open complement of Ω, Page 1
$\Gamma := \partial\Omega$	boundary of Ω, Page 1
$L^2\Lambda^\ell(\omega)$	Hilbert space of square-integrable ℓ-forms on $\omega \subset \mathcal{M}$, Page 1
$X\Lambda^\ell(\omega)$	space of ℓ-forms with coefficients in $X(\omega)$, Page 1
$\langle\cdot,\cdot\rangle_\omega$	bilinear symmetric L^2-pairing, (1.1)
$\langle\!\langle\cdot,\cdot\rangle\!\rangle_\omega$	duality pairing extending an L^2-pairing
$\mathfrak{D}^\kappa$	regularized Hodge-Dirac operator, (1.9)
$\mathfrak{L}_\ell^\lambda$	regularized Hodge-Laplace operator acting on ℓ-forms, (1.9)
$\star_\ell^\Gamma$	Hodge operator on Γ, Page 6
$\imath_\Gamma^\mp$	inclusion map $\Gamma \subset \overline{\Omega}$, Page 6

$\mathsf{t}_\ell^{\mp}$	tangential trace of an ℓ-form from $\Omega\mp$, (2.1)
$\mathsf{n}_\ell^{\mp}$	normal trace of an ℓ-form on $\Omega^{\mp}$, (2.1)
$\left.\begin{matrix} H_{\parallel}^{\frac{1}{2}}\Lambda^\ell(\Gamma), \\ H_{\perp}^{\frac{1}{2}}\Lambda^\ell(\Gamma) \end{matrix}\right\}$ ...	regular trace spaces, (2.3)
d_ℓ^Γ, δ_ℓ^Γ	exterior (co-)derivative on Γ, Page 7
$H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}^\Gamma,\Gamma)$ ..	domain space of d_ℓ^Γ on $H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$
$H_{\parallel}^{-\frac{1}{2}}\Lambda^\ell(\delta^\Gamma,\Gamma)$ ..	domain space of δ_ℓ^Γ on $H_{\parallel}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$
$\langle\!\langle\cdot,\cdot\rangle\!\rangle$	L^2 duality pairing on Γ
$\mathbf{t}^{\mp}$, $\mathbf{n}^{\mp}$	traces on graded Grassmann algebra of differential forms, (2.10)
$\mathcal{A}_{\boldsymbol{\delta}}(\cdot,\cdot)$, $\mathcal{A}_{\mathbf{d}}(\cdot,\cdot)$	bilinear forms associated with the Hodge-Dirac operator, (3.2)
$H\Lambda^\ell(\Delta,\mathcal{M}\setminus\Gamma)$	domain space of Hodge-Laplacian acting on ℓ-forms, (3.5)
$\mathsf{T}_{\Delta,\ell}^{\mathsf{t},-}$, $\mathsf{T}_{\Delta,\ell}^{\mathsf{n},-}$	trace operators associated with the Hodge-Laplacian (3.6)
$H_\Delta^{\mathsf{t},\ell}(\Gamma)$, $H_\Delta^{\mathsf{n},\ell}(\Gamma)$	product trace spaces, (3.8)
$\mathfrak{M}_\ell^\lambda$, $\mathfrak{R}_\ell^\lambda$	mixed-order Hodge-Laplacians, (3.9), (3.14)
$\mathsf{T}_{\mathfrak{M}}^{\mathsf{t},-}$, $\mathsf{T}_{\mathfrak{M}}^{\mathsf{n},-}$	complementary trace operators for $\mathfrak{M}_\ell^\lambda$, (3.12)
$\mathsf{T}_{\mathfrak{R}}^{\mathsf{t},-}$, $\mathsf{T}_{\mathfrak{R}}^{\mathsf{n},-}$	complementary trace operators for $\mathfrak{R}_\ell^\lambda$, (3.17)
$\mathcal{B}_\delta^\lambda$, $\mathcal{B}_\mathrm{d}^\lambda$	mixed-order bilinear forms for $\mathfrak{L}_\ell^\lambda$, (3.16), (3.11)
N_ℓ^λ, $\mathbf{N}^\lambda$	Newton potentials, (4.4), (4.5)
S_ℓ^λ, D_ℓ^λ	basic layer potentials, (4.8a), (4.8b)
$\mathbf{S}^\lambda$, $\mathbf{D}^\lambda$	compound basic layer potentials, (4.17)
$\left.\begin{matrix} (u_\ell,v_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}}, \\ (w_\ell,z_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} \end{matrix}\right\}$	non-local inner products for $H_{\parallel}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$ and $H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$, (4.21)
$\mathbf{SL}^\kappa[\mathfrak{D}]$, $\mathbf{DL}^\kappa[\mathfrak{D}]$	layer potentials for the Hodge-Dirac operator, (4.33)
$\mathsf{SL}_\ell^\lambda[\Delta]$, $\mathsf{DL}_\ell^\lambda[\Delta]$	layer potentials for Hodge-Laplacian $\mathfrak{L}_\ell^\lambda$, (4.38)
$\mathsf{SL}_\ell^\lambda[\mathfrak{M}]$, $\mathsf{DL}_\ell^\lambda[\mathfrak{M}]$	layer potentials induced by mixed-order Hodge-Laplacian, (4.44)

2. Traces of Differential Forms and Trace De Rham Complexes. The boundary $\Gamma := \partial\Omega$ is an oriented Lipschitz sub-manifold of $\mathcal{M}$ of dimension $N-1$, cf. [14, Chapter 3], [15, Section 2], [18, Appendix A], [24, Section 1] and [26, Section 1]. As such it is also a Riemannian manifold, which inherits its metric as the restriction of the metric of $\mathcal{M}$ to the tangent bundle of Γ. Spaces of differential forms $L^2\Lambda^\ell(\Gamma)$ and Hodge operators $\star_\ell^\Gamma : L^2\Lambda^\ell(\Gamma) \to L^2\Lambda^{N-1-\ell}(\Gamma)$ can be defined as usual. They are non-trivial only for $0 \le \ell \le N-1$.

Suitable *trace operators* for $H\Lambda^\ell(\mathrm{d},\Omega^{\mp})$ and $H\Lambda^\ell(\delta,\Omega^{\mp})$ are obtained by extending the pullback and "rotated" pullback of differential forms, also called tangential and normal traces. Writing $\imath_\Gamma^{\mp} : \Gamma \to \Omega^{\mp}$ for the inclusion map, and $\imath_{\mp}^*$ for the induced pullback, those trace operators are defined for all smooth (up to the boundary) ℓ-forms $U_\ell^{\mp} \in C^\infty\Lambda^\ell(\overline{\Omega}^{\mp})$, $0 \le \ell \le N$, by

$$\mathsf{t}_\ell^{\mp}U_\ell^{\mp} := \imath_{\mp}^* U_\ell^{\mp} \in L^2\Lambda^\ell(\Gamma) \quad \text{and} \quad \mathsf{n}_\ell^{\mp}U_\ell^{\mp} := \left(\star_{\ell-1}^\Gamma\right)^{-1} \imath_{\mp}^* \star_\ell U_\ell^{\mp} \in L^2\Lambda^{\ell-1}(\Gamma)\,. \tag{2.1}$$

We drop the superscript "$-$" or "$+$" when we let trace operators act on functions defined everywhere on $\mathcal{M}$, for instance, $\mathsf{t}_\ell : C^\infty\Lambda^\ell(\mathcal{M}) \to L^2\Lambda^\ell(\Gamma)$.

Generalizing the notation of [12], we define the dual spaces

$$H_{\parallel}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) := (H_{\parallel}^{\frac{1}{2}}\Lambda^\ell(\Gamma))' \quad \text{and} \quad H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) := (H_{\perp}^{\frac{1}{2}}\Lambda^\ell(\Gamma))'\,, \tag{2.2}$$

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where the so-called *regular trace spaces* are given as the ranges

$$H_{\parallel}^{\frac{1}{2}}\Lambda^{\ell}(\Gamma) := \mathsf{t}_{\ell}^{\mp}\left(H^{1}\Lambda^{\ell}(\Omega^{\mp})\right) \quad \text{and} \quad H_{\perp}^{\frac{1}{2}}\Lambda^{\ell}(\Gamma) := \mathsf{n}_{\ell}^{\mp}\left(H^{1}\Lambda^{\ell+1}(\Omega^{\mp})\right) , \tag{2.3}$$

and endowed with the corresponding trace norms, *cf.* [26, Section 2]. They generalize the well-known fractional Sobolev space of Dirichlet traces $H^{\frac{1}{2}}\Lambda^{0}(\Gamma) = H^{\frac{1}{2}}(\Gamma)$ [14, Chapter 3], which has inspired the notation. The spaces introduced in (2.2) and (2.3) form Gelfand triples with pivot space $L^{2}\Lambda^{\ell}(\Gamma)$.

In the spirit of [11, Section 7], we view the exterior derivative and the co-derivative on the boundary Γ as the closed densely defined unbounded linear operators $\mathrm{d}_{\ell}^{\Gamma} : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \to H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell+1}(\Gamma)$ and $\delta_{\ell}^{\Gamma} : H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \to H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma)$, where δ_{ℓ}^{Γ} is the $L^{2}\Lambda^{\ell-1}(\Gamma)$- or $L^{2}\Lambda^{\ell}(\Gamma)$-adjoint, respectively, of $\mathrm{d}_{\ell-1}$.

Definition 2.1 (Trace de Rham complexes, [11, Theorem 7.1]). *The* Hilbert complexes *spawned by the closed unbounded operators* $\mathrm{d}_{\ell}^{\Gamma} : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \to H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell+1}(\Gamma)$ *and* $\delta_{\ell}^{\Gamma} : H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \to H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma)$ *are called* trace de Rham complexes.

From [11, Theorem 7.3] we learn that the trace de Rham complexes possess the Fredholm property. The associated domain complexes can be written as

$$\begin{aligned}
&H\Lambda^{0}(\mathrm{d}^{\Gamma},\Gamma) \xrightarrow{\mathrm{d}_{0}^{\Gamma}} H\Lambda^{1}(\mathrm{d}^{\Gamma},\Gamma) \xrightarrow{\quad\mathrm{d}_{1}^{\Gamma}\quad} \dots \\
&\quad \dots \xrightarrow{\quad\mathrm{d}_{\ell-1}^{\Gamma}\quad} H\Lambda^{\ell}(\mathrm{d}^{\Gamma},\Gamma) \xrightarrow{\mathrm{d}_{\ell}^{\Gamma}} H\Lambda^{\ell+1}(\mathrm{d}^{\Gamma},\Gamma) \xrightarrow{\quad\mathrm{d}_{\ell+1}^{\Gamma}\quad} \dots \\
&\qquad\qquad \dots \xrightarrow{\quad\mathrm{d}_{N-3}^{\Gamma}\quad} H\Lambda^{N-2}(\mathrm{d}^{\Gamma},\Gamma) \xrightarrow{\mathrm{d}_{N-2}^{\Gamma}} L^{2}\Lambda^{N-1}(\Gamma) ,
\end{aligned} \tag{2.4a}$$

and

$$\begin{aligned}
&L^{2}\Lambda^{0}(\Gamma) \xleftarrow{\delta_{1}^{\Gamma}} H\Lambda^{1}(\delta^{\Gamma},\Gamma) \xleftarrow{\quad\delta_{2}^{\Gamma}\quad} \dots \\
&\quad \dots \xleftarrow{\quad\delta_{\ell}^{\Gamma}\quad} H\Lambda^{\ell}(\delta^{\Gamma},\Gamma) \xleftarrow{\delta_{\ell+1}^{\Gamma}} H\Lambda^{\ell+1}(\delta^{\Gamma},\Gamma) \xleftarrow{\quad\delta_{\ell+2}^{\Gamma}\quad} \dots \\
&\qquad\qquad \dots \xleftarrow{\quad\delta_{N-2}^{\Gamma}\quad} H\Lambda^{N-2}(\delta^{\Gamma},\Gamma) \xleftarrow{\delta_{N-1}^{\Gamma}} H\Lambda^{N-1}(\delta^{\Gamma},\Gamma) .
\end{aligned} \tag{2.4b}$$

Here, $H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}^{\Gamma},\Gamma)$ and $H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\delta^{\Gamma},\Gamma)$ designate the domain spaces of $\mathrm{d}_{\ell}^{\Gamma}$ and δ_{ℓ}^{Γ}, respectively, equipped with the graph norms. Note that in general $H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}^{\Gamma},\Gamma) \not\subset L^{2}\Lambda^{\ell}(\Gamma)$ and $H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\delta^{\Gamma},\Gamma) \not\subset L^{2}\Lambda^{\ell}(\Gamma)$.

Further, the results of [11, Section 3] ensure that the operators

$$\mathsf{t}_{\ell}^{\mp} : H^{1}\Lambda^{\ell}(\mathcal{M}) \longrightarrow H_{\parallel}^{\frac{1}{2}}\Lambda^{\ell}(\Gamma) \qquad \text{and} \qquad \mathsf{n}_{\ell}^{\mp} : H^{1}\Lambda^{\ell}(\mathcal{M}) \longrightarrow H_{\perp}^{\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \tag{2.5}$$

can be extended to continuous and *surjective* mappings

$$\begin{aligned}
\mathsf{t}_{\ell}^{\mp} &: H\Lambda^{\ell}(\mathrm{d},\Omega^{\mp}) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}^{\Gamma},\Gamma) , \\
\mathsf{n}_{\ell}^{\mp} &: H\Lambda^{\ell}(\delta,\Omega^{\mp}) \longrightarrow H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta^{\Gamma},\Gamma) ,
\end{aligned} \tag{2.6}$$

such that the integration by parts formula ("Green's formula")

$$\langle \mathrm{d}_\ell\, U_\ell, V_{\ell+1}\rangle_{\Omega^\mp} = \langle U_\ell, \delta_{\ell+1}\, V_{\ell+1}\rangle_{\Omega^\mp} \pm \langle\!\langle \mathsf{t}_\ell^\mp U_\ell, \mathsf{n}_{\ell+1}^\mp V_{\ell+1}\rangle\!\rangle_\Gamma \tag{2.7}$$

holds for all $U_\ell \in H\Lambda^\ell(\mathrm{d},\Omega^\mp)$ and $V_{\ell+1} \in H\Lambda^{\ell+1}(\delta,\Omega^\mp)$. On the right-hand side of (2.7), the duality pairing on the boundary

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_\Gamma : H_\perp^{-\frac12}\Lambda^\ell(\mathrm{d}^\Gamma,\Gamma) \times H_\|^{-\frac12}\Lambda^\ell(\delta^\Gamma,\Gamma) \to \mathbb{C} \tag{2.8}$$

extends the $L^2\Lambda^\ell(\Gamma)$-pairing. That is, it puts $H_\perp^{-\frac12}\Lambda^\ell(\mathrm{d}^\Gamma,\Gamma)$ in duality with the trace space $H_\|^{-\frac12}\Lambda^{\ell-1}(\delta^\Gamma,\Gamma)$ using $L^2\Lambda^\ell(\Gamma)$ as a pivot space, see [11, Section 4].

Despite Γ being merely Lipschitz regular, the usual commutative relations

$$\mathsf{t}_\ell^\mp \circ \mathrm{d}_\ell = \mathrm{d}_\ell^\Gamma \circ \mathsf{t}_\ell^\mp \quad\text{and}\quad \mathsf{n}_{\ell-1}^\mp \circ \delta_\ell = -\delta_{\ell-1}^\Gamma \circ \mathsf{n}_\ell^\mp\ , \tag{2.9}$$

still hold for the trace operators. The first reflects the fact that pullback and exterior derivative commute, and the second identity can be obtained from the first:

$$\begin{aligned}\mathsf{n}_{\ell-1}^\mp \delta_\ell &= \left(\star_{\ell-2}^\Gamma\right)^{-1} \imath_\pm^* \star_{\ell-1}\left((-1)^\ell(\star_{\ell-1})^{-1}\mathrm{d}_{N-\ell}\star_\ell\right) = -(-1)^{\ell-1}(\star_{\ell-2}^\Gamma)^{-1}\mathrm{d}_{N-\ell}^\Gamma\,\imath_\mp^*\star_\ell \\ &= -\left((-1)^{\ell-1}(\star_{\ell-2}^\Gamma)^{-1}\mathrm{d}_{N-\ell}^\Gamma\star_{\ell-1}^\Gamma\right)\left(\star_{\ell-1}^\Gamma\right)^{-1}\imath_\mp^*\star_\ell = -\delta_{\ell-1}^\Gamma \circ \mathsf{n}_\ell^\mp\ .\end{aligned}$$

We use a bold font to denote traces acting on the full Grassmann algebra of forms in a component-wise manner, i.e.

$$\mathbf{t}^\mp \boldsymbol{U} := \boldsymbol{\imath}_\mp^* \boldsymbol{U} \quad\text{and}\quad \mathbf{n}^\mp \boldsymbol{V} := \left(\star^\Gamma\right)^{-1}\mathbf{t}_\mp \star \boldsymbol{V}\ . \tag{2.10}$$

Then, applying the integration by parts formula (2.7) to the components of $\boldsymbol{U}$ and $\boldsymbol{V}$ separately yields

$$\langle \mathbf{d}\,\boldsymbol{U}, \boldsymbol{V}\rangle_{\Omega^\mp} = \langle \boldsymbol{U}, \boldsymbol{\delta}\,\boldsymbol{V}\rangle_{\Omega^\mp} \pm \langle\!\langle \mathbf{t}^\mp\boldsymbol{U}, \mathbf{n}^\mp\boldsymbol{V}\rangle\!\rangle_\Gamma \tag{2.11}$$

for all $\boldsymbol{U} \in H\Lambda(\mathbf{d},\Omega^\mp)$ and $\boldsymbol{V} \in H\Lambda(\boldsymbol{\delta},\Omega^\mp)$.

3. Variational Boundary Value Problems (BVPs). We briefly review well-posed boundary value problems for the (regularized) Hodge-Dirac operator $\mathfrak{D}^\kappa$ and Hodge-Laplace operator $\mathfrak{L}_\ell^\lambda$, $\lambda > 0$, $\lambda = \kappa^2$, $l \in \{0,\dots,N\}$, see (1.9). Our focus is on variational formulations with natural boundary conditions.

3.1. BVPs for regularized Hodge-Dirac operators. Writing $H\Lambda(\mathfrak{D},\Omega) = H\Lambda(\mathbf{d},\Omega) \cap H\Lambda(\boldsymbol{\delta},\Omega)$ for the maximal domain space of $\mathfrak{D}$, the following two boundary value problems on Ω are associated with self-adjoint specializations of $\mathfrak{D}^\kappa$ with compact resolvent [16, Section 1.3]:

$$\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega): \quad \begin{cases} \mathfrak{D}^\kappa\boldsymbol{U} = \mathbf{0} & \text{in } \Omega \\ \mathbf{t}^-\,\boldsymbol{U} = \boldsymbol{g} & \text{on } \partial\Omega \end{cases}, \qquad \boldsymbol{g} \in H_\perp^{-\frac12}\Lambda(\mathbf{d}^\Gamma,\Gamma)\ , \tag{3.1a}$$

and

$$\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega): \quad \begin{cases} \mathfrak{D}^\kappa\boldsymbol{U} = \mathbf{0} & \text{in } \Omega \\ \mathbf{n}^-\,\boldsymbol{U} = \boldsymbol{h} & \text{on } \partial\Omega \end{cases}, \qquad \boldsymbol{h} \in H_\|^{-\frac12}\Lambda(\boldsymbol{\delta}^\Gamma,\Gamma)\ . \tag{3.1b}$$

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As discussed in [22, Section 3], the Hodge–Dirac operator induces two distinct fundamental symmetric bilinear forms

$$\mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U},\boldsymbol{V}) := \langle \boldsymbol{\delta}\boldsymbol{U},\boldsymbol{V}\rangle_\Omega + \langle \boldsymbol{U},\boldsymbol{\delta}\boldsymbol{V}\rangle_\Omega \qquad \forall \boldsymbol{U},\boldsymbol{V}\in H\Lambda(\boldsymbol{\delta},\Omega)\ , \tag{3.2a}$$

$$\mathcal{A}_{\mathbf{d}}(\boldsymbol{U},\boldsymbol{V}) := \langle \mathbf{d}\boldsymbol{U},\boldsymbol{V}\rangle_\Omega + \langle \boldsymbol{U},\mathbf{d}\boldsymbol{V}\rangle_\Omega \qquad \forall \boldsymbol{U},\boldsymbol{V}\in H\Lambda(\mathbf{d},\Omega)\ . \tag{3.2b}$$

They play a key role in the following "Dirac counterparts" of Green's first identity:

$$\langle \mathfrak{D}\boldsymbol{U},\boldsymbol{V}\rangle_\Omega = \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U},\boldsymbol{V}) + \langle\!\langle \mathbf{t}^-\boldsymbol{U},\mathbf{n}^-\boldsymbol{V}\rangle\!\rangle_\Gamma\ , \tag{3.3a}$$

$$\langle \mathfrak{D}\boldsymbol{U},\boldsymbol{V}\rangle_\Omega = \mathcal{A}_{\mathbf{d}}(\boldsymbol{U},\boldsymbol{V}) - \langle\!\langle \mathbf{n}^-\boldsymbol{U},\mathbf{t}^-\boldsymbol{V}\rangle\!\rangle_\Gamma\ , \tag{3.3b}$$

which hold for all $\boldsymbol{U},\boldsymbol{V}\in H\Lambda(\mathfrak{D},\Omega)$. They yield two variational problems representing a weak form of (3.1a) and (3.1b), respectively [13, Section 2.2]:

(3.4a)
$$\boldsymbol{U}\in H\Lambda(\boldsymbol{\delta},\Omega):\quad \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U},\boldsymbol{V}) + i\kappa\langle \boldsymbol{U},\boldsymbol{V}\rangle_\Omega = -\langle\!\langle \boldsymbol{g},\mathbf{n}^-\boldsymbol{V}\rangle\!\rangle_\Gamma \qquad \forall \boldsymbol{V}\in H\Lambda(\boldsymbol{\delta},\Omega)\ ,$$

(3.4b)
$$\boldsymbol{U}\in H\Lambda(\mathbf{d},\Omega):\quad \mathcal{A}_{\mathrm{d}}(\boldsymbol{U},\boldsymbol{V}) + i\kappa\langle \boldsymbol{U},\boldsymbol{V}\rangle_\Omega = \langle\!\langle \boldsymbol{h},\mathbf{t}^-\boldsymbol{V}\rangle\!\rangle_\Gamma \qquad \forall \boldsymbol{V}\in H\Lambda(\mathbf{d},\Omega)\ .$$

It is an easy exercise in integration by parts to verify by using suitable test functions that the variational problems (3.4a) and (3.4b) generalize the strong formulations (3.1a) and (3.1b), respectively. Moreover, for both (3.4a) and (3.4b), inf-sup conditions can be verified confirming existence and uniqueness of (weak) solutions [21, Section 5.2.1.2].

3.2. BVPs for regularized Hodge-Laplace operators. We fix $0\le \ell\le N$ and, in the spirit of (1.4), write

$$H\Lambda^\ell(\Delta,\mathcal{M}\setminus\Gamma) := \{v\in H^1\Lambda^\ell(\omega)\,|\,\Delta_\ell v\in L^2\Lambda^\ell(\omega)\}\ . \tag{3.5}$$

To state suitable boundary conditions for the (regularized) Hodge-Laplacian in second-order (strong) form $\mathfrak{L}_\ell^\lambda := -\Delta_\ell + \lambda : H\Lambda^\ell(\Delta,\Omega)\to L^2\Lambda^\ell(\Omega)$ we rely on the trace operators

$$\mathsf{T}^{\mathsf{t},-}_{\Delta,\ell}\,U_\ell := \begin{bmatrix}\mathsf{t}^-_{\ell-1}\delta_\ell U_\ell\\ \mathsf{t}^-_\ell U_\ell\end{bmatrix} \quad\text{and}\quad \mathsf{T}^{\mathsf{n},-}_{\Delta,\ell}\,U_\ell := \begin{bmatrix}\mathsf{n}^-_\ell U_\ell\\ \mathsf{n}^-_{\ell+1}\mathrm{d}_\ell U_\ell\end{bmatrix}\ . \tag{3.6}$$

They are related to self-adjoint specializations of $\mathfrak{L}_\ell^\lambda$ with compact resolvents and give rise to the well-posed boundary value problems [16, Section 1.1]

$$U_\ell\in H\Lambda^\ell(\Delta,\Omega):\quad \begin{cases}\mathfrak{L}_\ell^\lambda U_\ell = 0 & \text{in }\Omega\\ \mathsf{T}^{\mathsf{t},-}_{\Delta,\ell}\,U_\ell = \begin{bmatrix}g_{\ell-1}\\ g_\ell\end{bmatrix} & \text{on }\partial\Omega\end{cases}\ ,\quad \begin{bmatrix}g_{\ell-1}\\ g_\ell\end{bmatrix}\in H^{\mathsf{t},\ell}_\Delta(\Gamma)\ , \tag{3.7a}$$

and

$$U_\ell\in H\Lambda^\ell(\Delta,\Omega):\quad \begin{cases}\mathfrak{L}_\ell^\lambda U_\ell = 0 & \text{in }\Omega\\ \mathsf{T}^{\mathsf{n},-}_{\Delta,\ell}\,U_\ell = \begin{bmatrix}h_{\ell-1}\\ h_\ell\end{bmatrix} & \text{on }\partial\Omega\end{cases}\ ,\quad \begin{bmatrix}h_{\ell-1}\\ h_\ell\end{bmatrix}\in H^{\mathsf{n},\ell}_\Delta(\Gamma)\ , \tag{3.7b}$$

where the boundary values belong to the product trace spaces

$$\begin{aligned} H^{\mathsf{t},\ell}_{\Delta}(\Gamma) &:= H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d}^{\Gamma},\Gamma) \times H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}^{\Gamma},\Gamma)\,, \\ H^{\mathsf{n},\ell}_{\Delta}(\Gamma) &:= H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta^{\Gamma},\Gamma) \times H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\delta^{\Gamma},\Gamma)\,. \end{aligned} \tag{3.8}$$

Those are in duality with respect to the pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\Gamma}$.

We notice that the boundary data for both BVPs lie in product trace spaces. Anticipating the considerations of Section 5 this means that the BIOs linked to the Hodge-Laplacian will have to be analyzed in the framework of those product trace spaces. In light of our main result outlined in Subsection 1.2, Page 3, this suggests that we also examine *mixed-order formulations* of (3.7a) and (3.7b), which are posed on product spaces. For instance, by introducing an auxiliary variable $U_{\ell-1} = \delta_\ell U_\ell \in H\Lambda^{\ell-1}(\mathrm{d},\Omega)$ we obtain one possible mixed-order form [2, Section 7.1]

$$\mathfrak{L}^{\lambda}_{\ell} U_\ell = 0 \quad\Leftrightarrow\quad \mathfrak{M}^{\lambda}_{\ell}\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} = \mathbf{0} \quad\text{with}\quad \mathfrak{M}^{\lambda}_{\ell} := \begin{bmatrix} -\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \delta_{\ell+1}\mathrm{d}_\ell + \lambda\mathrm{Id} \end{bmatrix}. \tag{3.9}$$

Resorting to the integration-by-parts formula (2.7), we find that for all $U_\ell, V_\ell \in C^\infty\Lambda^\ell(\overline{\Omega})$ and $U_{\ell-1}, V_{\ell-1} \in C^\infty\Lambda^{\ell-1}(\overline{\Omega})$

$$\begin{aligned} &\left\langle \mathfrak{M}^{\lambda}_{\ell}\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix}\right\rangle_{\Omega} \\ &\quad= \mathcal{B}^{\lambda}_{\mathrm{d}}\left(\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix}\right) - \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{n},-}_{\mathfrak{M},\ell}\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}^{\mathsf{t},-}_{\mathfrak{M},\ell}\begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix}\right\rangle\!\!\right\rangle_{\Gamma} \\ &\quad= \left\langle \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathfrak{M}^{\lambda}_{\ell}\begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix}\right\rangle_{\Omega} - \\ &\qquad \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{n},-}_{\mathfrak{M},\ell}\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}^{\mathsf{t},-}_{\mathfrak{M},\ell}\begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix}\right\rangle\!\!\right\rangle_{\Gamma} + \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{t},-}_{\mathfrak{M},\ell}\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}^{\mathsf{n},-}_{\mathfrak{M},\ell}\begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix}\right\rangle\!\!\right\rangle_{\Gamma}, \end{aligned} \tag{3.10}$$

with the bilinear form

$$\boxed{\begin{aligned} \mathcal{B}^{\lambda}_{\mathrm{d}}\left(\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix}\right) &= (\mathrm{d}_\ell U_\ell, \mathrm{d}_\ell V_\ell)_\Omega + \lambda\,(U_\ell, V_\ell)_\Omega + (\mathrm{d}_{\ell-1}U_{\ell-1}, V_\ell)_\Omega \\ &\quad + (U_\ell, \mathrm{d}_\ell V_{\ell-1})_\Omega - (U_{\ell-1}, V_{\ell-1})_\Omega\,, \end{aligned}} \tag{3.11}$$

and complementary trace operators

$$\mathsf{T}^{\mathsf{t},-}_{\mathfrak{M},\ell}\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} := \begin{bmatrix} \mathsf{t}^{-}_{\ell-1}U_{\ell-1} \\ \mathsf{t}^{-}_{\ell}U_\ell \end{bmatrix} \text{ and } \mathsf{T}^{\mathsf{n},-}_{\mathfrak{M},\ell}\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} := \mathsf{T}^{n,-}_{\Delta,\ell}U_\ell = \begin{bmatrix} \mathsf{n}^{-}_{\ell}U_\ell \\ \mathsf{n}^{-}_{\ell+1}\,\mathrm{d}_\ell\,U_\ell \end{bmatrix}. \tag{3.12}$$

Those supply bounded linear operators

$$\mathsf{T}^{\mathsf{t},\mp}_{\mathfrak{M},\ell} : H\Lambda^{\ell-1}(\mathrm{d},\Omega^{\mp}) \times H\Lambda^{\ell}(\mathrm{d},\Omega^{\mp}) \to H^{\mathsf{t},\ell}_{\Delta}(\Gamma)\,, \tag{3.13a}$$

$$\mathsf{T}^{\mathsf{n},\mp}_{\mathfrak{M},\ell} : L^2\Lambda^{\ell-1}(\Omega^{\mp}) \times H\Lambda^{\ell}(\delta\,\mathrm{d},\Omega^{\mp}) \to H^{\mathsf{n},\ell}_{\Delta}(\Gamma)\,. \tag{3.13b}$$

Alternatively we can choose as auxiliary variable $U_{\ell+1} = \mathrm{d}_\ell U_\ell \in H\Lambda^\ell(\delta,\Omega)$, which gives us another mixed-order formulation

$$\mathfrak{L}^{\lambda}_{\ell} U_\ell = 0 \quad\Leftrightarrow\quad \mathfrak{R}^{\lambda}_{\ell}\begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix} = \mathbf{0} \quad\text{with}\quad \mathfrak{R}^{\lambda}_{\ell} := \begin{bmatrix} \mathrm{d}_{\ell-1}\delta_\ell + \lambda\mathrm{Id} & \delta_{\ell+1} \\ \mathrm{d}_\ell & -\mathrm{Id} \end{bmatrix}. \tag{3.14}$$

The operator $\mathfrak{R}^\lambda_\ell$ can also be cast in weak form by means of integration by parts:

$$\left\langle \mathfrak{R}^\lambda_\ell \begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix}, \begin{bmatrix} V_\ell \\ V_{\ell+1} \end{bmatrix} \right\rangle_\Omega \tag{3.15}$$
$$= \mathcal{B}^\lambda_\delta \left(\begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix}, \begin{bmatrix} V_\ell \\ V_{\ell+1} \end{bmatrix} \right) + \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{t},-}_{\mathfrak{R},\ell} \begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix}, \mathsf{T}^{\mathsf{n},-}_{\mathfrak{R},\ell} \begin{bmatrix} V_\ell \\ V_{\ell+1} \end{bmatrix} \right\rangle\!\!\right\rangle$$
$$= \left\langle \begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix}, \mathfrak{R}^\lambda_\ell \begin{bmatrix} V_\ell \\ V_{\ell+1} \end{bmatrix} \right\rangle_\Omega$$
$$+ \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{t},-}_{\mathfrak{R},\ell} \begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix}, \mathsf{T}^{\mathsf{n},-}_{\mathfrak{R},\ell} \begin{bmatrix} V_\ell \\ V_{\ell+1} \end{bmatrix} \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{n},-}_{\mathfrak{R},\ell} \begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix}, \mathsf{T}^{\mathsf{t},-}_{\mathfrak{R},\ell} \begin{bmatrix} V_\ell \\ V_{\ell+1} \end{bmatrix} \right\rangle\!\!\right\rangle ,$$

which involves the bilinear form

$$\boxed{\begin{aligned} \mathcal{B}^\lambda_\delta \left(\begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix}, \begin{bmatrix} V_\ell \\ V_{\ell+1} \end{bmatrix} \right) &:= \langle \delta_\ell U_\ell, \delta_\ell V_\ell \rangle_\Omega + \lambda \langle U_\ell, V_\ell \rangle_\Omega + \langle \delta_{\ell+1} U_{\ell+1}, V_\ell \rangle_\Omega + \\ &\langle U_\ell, \delta_{\ell+1} V_{\ell+1} \rangle_\Omega - \langle U_{\ell+1}, V_{\ell+1} \rangle_\Omega \end{aligned}} \tag{3.16}$$

and the complementary trace operators

$$\mathsf{T}^{\mathsf{t},-}_{\mathfrak{R},\ell} \begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix} := \begin{bmatrix} \mathsf{t}^-_{\ell-1} \delta_\ell U_\ell \\ \mathsf{t}^-_\ell U_\ell \end{bmatrix} = \mathsf{T}^{\mathsf{t},-}_{\Delta,\ell} U_\ell \quad \text{and} \quad \mathsf{T}^{\mathsf{n},-}_{\mathfrak{R},\ell} \begin{bmatrix} U_\ell \\ U_{\ell+1} \end{bmatrix} = \begin{bmatrix} \mathsf{n}^-_\ell U_\ell \\ \mathsf{n}^-_{\ell+1} U_{\ell+1} \end{bmatrix} . \tag{3.17}$$

They map continuously

$$\mathsf{T}^{\mathsf{t},\mp}_{\mathfrak{R},\ell} : H\Lambda^\ell(\mathrm{d}\,\delta, \Omega^\mp) \times L^2\Lambda^{\ell+1}(\Omega^\mp) \to H^{\mathsf{t},\ell}_\Delta(\Gamma) , \tag{3.18a}$$
$$\mathsf{T}^{\mathsf{n},\mp}_{\mathfrak{R},\ell} : H\Lambda^\ell(\delta, \Omega^\mp) \times H\Lambda^{\ell+1}(\delta, \Omega^\mp) \to H^{\mathsf{n},\ell}_\Delta(\Gamma) , \tag{3.18b}$$

and their co-domain spaces are in duality with respect to the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\Gamma$.

Both bilinear forms $\mathcal{B}^\lambda_{\mathrm{d}}$ and $\mathcal{B}^\lambda_\delta$ and are continuous on the relevant spaces of differential forms $H\Lambda^{\ell-1}(\mathrm{d}, \Omega) \times H\Lambda^\ell(\mathrm{d}, \Omega)$ and $H\Lambda^\ell(\delta, \Omega) \times H\Lambda^{\ell+1}(\delta, \Omega)$, respectively. They occur in two different mixed-order variational formulations of the boundary value problems (3.7a) and (3.7b) for $\mathfrak{L}^\lambda_\ell$ [21, Section 5.2.3]. Existence and uniqueness of solutions of the resulting variational problems can be shown by establishing corresponding inf-sup conditions for the bilinear forms $\mathcal{B}^\lambda_{\mathrm{d}}$ and $\mathcal{B}^\lambda_\delta$ [2, Section 7.1].

4. Layer Potentials. Our main tool to derive first-kind BIOs for Hodge–Dirac and Hodge–Laplace operators is a *calculus of layer potentials*. The two layer potentials defined in this section are the elementary building blocks from which all the other layer potentials appearing in this work are obtained via differentiation. They are also the crucial components entering (i) the definitions of the non-local inner products with which we equip the trace de Rham complexes and (ii) representation formulas.

4.1. Newton potential. As $\lambda > 0$ the results of [16, Section 3.1] confirm that the regularized Hodge–Laplace operator $\mathfrak{L}^\lambda_\ell := -\Delta_\ell + \lambda$ is invertible as an operator $H^1\Lambda^\ell(\mathcal{M}) \to H^{-1}\Lambda^\ell(\mathcal{M})$. We owe this to the uniformly bounded curvature of $\mathcal{M}$, which ensures the equivalence of the norms of $H^1\Lambda^\ell(\mathcal{M})$ and $H\Lambda^\ell(\mathrm{d}, \mathcal{M}) \cap H\Lambda^\ell(\delta, \mathcal{M})$, also known as *Gaffney inequality*. As a consequence, the function spaces can be identified: $H\Lambda^\ell(\mathrm{d}, \mathcal{M}) \cap H\Lambda^\ell(\delta, \mathcal{M}) = H^1\Lambda^\ell(\mathcal{M})$. Thus $(\mathfrak{L}^\lambda_\ell)^{-1} F_\ell$, $F_\ell \in H^{-1}\Lambda^\ell(\mathcal{M})$, can be defined as the unique solution of the linear variational problem: seek $(\mathfrak{L}_\ell^\lambda)^{-1}F \in H^1\Lambda^\ell(\mathcal{M})$ such that

$$\left\langle \mathrm{d}_\ell(\mathfrak{L}_\ell^\lambda)^{-1}F_\ell, \mathrm{d}_\ell V_\ell \right\rangle_{\mathcal{M}} + \left\langle \delta_\ell(\mathfrak{L}_\ell^\lambda)^{-1}F_\ell, \delta_\ell V_\ell \right\rangle_{\mathcal{M}} + \lambda \left\langle (\mathfrak{L}_\ell^\lambda)^{-1}F_\ell, V_\ell \right\rangle_{\mathcal{M}} = \langle\!\langle F_\ell, V_\ell \rangle\!\rangle_{\mathcal{M}} \tag{4.1}$$

for all $V_\ell \in H\Lambda^\ell(\mathrm{d},\mathcal{M}) \cap H\Lambda^\ell(\delta,\mathcal{M}) = H^1\Lambda^\ell(\mathcal{M})$. As $\mathcal{M}$ is smooth and $\partial\mathcal{M} = \emptyset$, pseudo-differential calculus [18, Chapter 3] also shows that $(\mathfrak{L}_\ell^\lambda)^{-1} : H^{s-1}\Lambda^\ell(\mathcal{M}) \to H^{s+1}\Lambda^\ell(\mathcal{M})$ continuously for all $s \geq 0$, and, as elaborated in [20, Section 3.1.1] and [14, Chapter 6], by duality we can also extend $(\mathfrak{L}_\ell^\lambda)^{-1}$ to a continuous operator between spaces of distribution $(\mathfrak{L}_\ell^\lambda)^{-1} : (C_0^\infty\Lambda^\ell(\mathcal{M}))' \to (C^\infty\Lambda^\ell(\mathcal{M}))'$.

The Schwartz kernel of the continuous inverse $(\mathfrak{L}_\ell^\lambda)^{-1} : H^{-1}\Lambda^\ell(\mathcal{M}) \to H^1\Lambda^\ell(\mathcal{M})$ is a double form $\mathcal{G}_\ell^\lambda(x,y)$ of bi-degree (ℓ,ℓ) with an integrable singularity at $x = y$ and smooth everywhere else [18, Chapter 6]. It satisfies

$$\mathrm{d}_{\ell,x}\,\mathcal{G}_\ell^\lambda(x,y) = \delta_{\ell+1,y}\,\mathcal{G}_{\ell+1}^\lambda(x,y) \quad \text{and} \quad \delta_{\ell,x}\,\mathcal{G}_\ell^\lambda(x,y) = \mathrm{d}_{\ell-1,y}\,\mathcal{G}_{\ell-1}^\lambda(x,y) \tag{4.2}$$

for $x \neq y$, cf. [12, Lemma 3] and [16, (3.1.44)], and

$$\star_{\ell,y} \star_{\ell,x} \mathcal{G}_\ell^\lambda = \star_{\ell,x} \star_{\ell,y} \mathcal{G}_\ell^\lambda = \mathcal{G}_{N-\ell}^\lambda \,, \tag{4.3}$$

cf. [16, (3.1.23)] and [12, Lemma 1]. The associated integral transformation

$$(\mathsf{N}_\ell^\lambda\, U_\ell)(x) := \langle\, \mathcal{G}_\ell^\lambda(x,\cdot), U_\ell(\cdot)\,\rangle_{\mathcal{M}}, \qquad U_\ell \in C_0^\infty\Lambda^\ell(\mathcal{M}) \,, \tag{4.4}$$

can be extended to the Sobolev scale and then provides an alternative representation of the inverse $(\mathfrak{L}_\ell^\lambda)^{-1} : H^{-1}\Lambda^\ell(\mathcal{M}) \to H^1\Lambda^\ell(\mathcal{M})$, in this form known as *Newton potential operator* N_ℓ^λ, cf. [9, Chapters 12 and 16], [12, Sections 2.2 and 2.3], [14, Chapter 6], [16, Chapter 3], [18, Chapter 2], [20, Chapter 3] and [25, Section 3].

At the level of the full algebra of differential forms, the Newton potentials can be combined into the block-diagonal $(N+1) \times (N+1)$ operator matrix

$$\mathbf{N}^\lambda := \begin{bmatrix} \mathsf{N}_0^\lambda & & & & \\ & \mathsf{N}_1^\lambda & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & \mathsf{N}_{N-1}^\lambda & \\ & & & & \mathsf{N}_N^\lambda \end{bmatrix}, \tag{4.5}$$

and the above identities (4.2) and (4.3) satisfied by the kernel $\boldsymbol{\mathcal{G}}^\lambda = (\mathcal{G}_\ell^\lambda)_{\ell=1}^N$ of $\mathbf{N}^\lambda$ translate to

$$\mathbf{d}_x\,\boldsymbol{\mathcal{G}}^\lambda = \boldsymbol{\delta}_y\,\boldsymbol{\mathcal{G}}^\lambda \quad , \quad \boldsymbol{\delta}_x\,\boldsymbol{\mathcal{G}}^\lambda = \mathbf{d}_y\,\boldsymbol{\mathcal{G}}^\lambda \,, \tag{4.6}$$

and

$$\boldsymbol{\star}_y \boldsymbol{\star}_x \boldsymbol{\mathcal{G}}^\lambda = \boldsymbol{\star}_x \boldsymbol{\star}_y \boldsymbol{\mathcal{G}}^\lambda = \boldsymbol{\mathcal{G}}^\lambda \,. \tag{4.7}$$

4.2. Basic layer potentials. We define the basic layer potentials

$$\mathsf{S}_\ell^\lambda := \mathsf{N}_\ell^\lambda\, \mathsf{t}_\ell' : H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) \longrightarrow H^1\Lambda^\ell(\mathcal{M}) \,, \tag{4.8a}$$

$$\mathsf{D}_\ell^\lambda := \mathsf{N}_\ell^\lambda\, \mathsf{n}_{\ell-1}' : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \longrightarrow H^1\Lambda^\ell(\mathcal{M}) \,, \tag{4.8b}$$

where the bounded operators

$$\mathsf{t}'_\ell : H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) \to H^{-1}\Lambda^\ell(\mathcal{M}) \quad \text{and} \quad \mathsf{n}'_\ell : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \to H^{-1}\Lambda^\ell(\mathcal{M}) \tag{4.9}$$

are the adjoints of the trace mappings from (2.5), cf. [6], [8] and [22]. Notice that, if $v_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$, then from (4.1) we conclude that the global ℓ-form $\mathsf{S}^\lambda_\ell v_\ell \in H^1\Lambda^\ell(\mathcal{M})$ is the unique solution to the variational equation

$$\left\langle \mathrm{d}_\ell \mathsf{S}^\lambda_\ell v_\ell, \mathrm{d}_\ell V_\ell \right\rangle_{\mathcal{M}} + \left\langle \delta_\ell \mathsf{S}^\lambda_\ell v_\ell, \delta_\ell V_\ell \right\rangle_{\mathcal{M}} + \lambda \left\langle \mathsf{S}^\lambda_\ell v_\ell, V_\ell \right\rangle_{\mathcal{M}} = \langle\!\langle v_\ell, \mathsf{t}_\ell V_\ell \rangle\!\rangle_\Gamma \tag{4.10}$$

for all $V_\ell \in H^1\Lambda^\ell(\mathcal{M})$. A similar variational characterization holds for D^λ_ℓ: Given $u_{\ell-1} \in H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma)$, we find that $\mathsf{D}^\lambda_\ell u_{\ell-1} \in H^1\Lambda^\ell(\mathcal{M})$ satisfies

$$\begin{aligned}\left\langle \mathrm{d}_\ell \mathsf{D}^\lambda_\ell u_{\ell-1}, \mathrm{d}_\ell V_\ell \right\rangle_{\mathcal{M}} + \left\langle \delta_\ell \mathsf{D}^\lambda_\ell u_{\ell-1}, \delta_\ell V_\ell \right\rangle_{\mathcal{M}} + \lambda \left\langle \mathsf{D}^\lambda_\ell u_{\ell-1}, V_\ell \right\rangle_{\mathcal{M}} \\ = \langle\!\langle u_{\ell-1}, \mathsf{n}_\ell V_\ell \rangle\!\rangle_\Gamma\end{aligned} \tag{4.11}$$

for all $V_\ell \in H^1\Lambda^\ell(\mathcal{M})$. From the variational characterization of the basic layer potentials we draw two immediate and important conclusions.

LEMMA 4.1. *For $\ell \in \{0, \dots, N-1\}$ the basic layer potentials give rise to continuous mappings*

$$\mathsf{S}^\lambda_\ell : H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\delta^\Gamma, \Gamma) \longrightarrow H^1\Lambda^\ell(\mathcal{M}) \cap H\Lambda^\ell(\Delta, \mathcal{M}\setminus\Gamma)\ , \tag{4.12a}$$

$$\mathsf{D}^\lambda_{\ell+1} : H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}^\Gamma, \Gamma) \longrightarrow H^1\Lambda^{\ell+1}(\mathcal{M}) \cap H\Lambda^{\ell+1}(\Delta, \mathcal{M}\setminus\Gamma)\ , \tag{4.12b}$$

satisfying for all $u_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta^\Gamma, \Gamma)$ and $w_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}^\Gamma, \Gamma)$

$$\begin{aligned} \mathfrak{L}^\lambda_\ell \mathsf{S}^\lambda_\ell(u_\ell) = 0 \quad &\textit{in} \quad H^{-1}\Lambda^{\ell+1}(\mathcal{M}\setminus\Gamma) \\ \textit{and} \quad \mathfrak{L}^\lambda_{\ell+1} \mathsf{D}^\lambda_{\ell+1}(w_\ell) = 0 \quad &\textit{in} \quad H^{-1}\Lambda^\ell(\mathcal{M}\setminus\Gamma)\ . \end{aligned} \tag{4.13}$$

Proof. The property (4.13) is immediate from (4.10) and (4.11) when testing with smooth forms compactly supported in $\Omega^- \cup \Omega^+$. For the first identity in (4.13), we also refer to [16, Eq. 3.2.5] and [12, Lem. 3 (ii)]. The second can also be obtained from (4.16), because the Hodge star commutes with the Hodge–Laplacian [16, Lem. 2.8]. ∎

Next, denote the jump of a trace across Γ by $[\![\bullet]\!] = \bullet^+ - \bullet^-$, where $\bullet = \mathsf{t}$ or n.

THEOREM 4.2 (Jump relations for basic potentials). *The basic layer potentials in the interpretation of (4.12) fulfill*

$$[\![\mathsf{t}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0\ , \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0\ , \qquad [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0\ , \tag{4.14a}$$

$$[\![\mathsf{n}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0\ , \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{S}^\lambda_\ell = -\mathrm{Id}\ , \qquad [\![\mathsf{n}_{\ell-1}\delta_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0\ , \tag{4.14b}$$

$$[\![\mathsf{t}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0\ , \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0\ , \qquad [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\,\mathsf{D}^\lambda_\ell = \mathrm{Id}\ , \tag{4.14c}$$

$$[\![\mathsf{n}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0\ , \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0\ , \qquad [\![\mathsf{n}_{\ell-1}\delta_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0\ . \tag{4.14d}$$

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Proof. We test (4.10) and (4.11) with $V_\ell \in C^\infty\Lambda^\ell(\mathcal{M})$ and perform integration by parts locally on Ω^- and Ω^+. This yields jump terms, because of the opposite relative orientation of Γ with respect to Ω^- and Ω^+. Combining these jump terms with the right-hand-side functionals in (4.10) and (4.11) yields the jump relations asserted by the lemma. ∎

If $u_\ell \in L^1\Lambda^\ell(\Gamma)$, $\ell \in \{0,\dots,N-1\}$, it follows by symmetry of the fundamental solution that for $x \notin \Gamma$ they admit the integral representations

$$(\mathsf{S}^\lambda_\ell u_\ell)(x) = \langle u_\ell, \mathsf{t}_\ell\, \mathcal{G}^\lambda_\ell(x,\cdot)\rangle_\Gamma \quad \text{and} \quad (\mathsf{D}^\lambda_{\ell+1} u_\ell)(x) = \langle u_\ell, \mathsf{n}_{\ell+1}\mathcal{G}^\lambda_{\ell+1}(x,\cdot)\rangle_\Gamma\ , \tag{4.15}$$

which generalizes [14, Theorem 6.10] and [20, Theorem 3.1.6].

Since Hodge star operators are L^2-isometric, we observe, using (4.3), that away from Γ ($x \notin \Gamma$)

$$\begin{aligned}\star^{-1}_{\ell+1,x}\, &\langle \star^\Gamma_\ell\, u_\ell, \mathsf{t}_{N-\ell-1}\, \mathcal{G}^\lambda_{N-\ell-1}(x,\cdot)\rangle_\Gamma \\ &= \langle\, u_\ell, (\star^\Gamma_\ell)^{-1}\mathsf{t}_{N-\ell-1}\star_{\ell+1}\mathcal{G}^\lambda_{\ell+1}(x,\cdot)\rangle_\Gamma = \langle\, u_\ell, \mathsf{n}_{\ell+1}\mathcal{G}^\lambda_{\ell+1}(x,\cdot)\rangle_\Gamma\ .\end{aligned}$$

Therefore, a density argument eventually shows that for $1 \le \ell \le N$,

$$\star^{-1}_\ell\, \mathsf{S}^\lambda_{N-\ell} \star^\Gamma_{\ell-1} = \mathsf{D}^\lambda_\ell \quad \text{and} \quad \star^{-1}\, \mathbf{S}^\lambda \star^\Gamma = \mathbf{D}^\lambda, \tag{4.16}$$

where we have introduced the rectangular $(N+1)\times N$ block operator matrices of boundary potentials

$$\mathbf{S}^\lambda := \begin{bmatrix} \mathsf{S}^\lambda_0 & & & 0 \\ & \mathsf{S}^\lambda_1 & & \\ & & \ddots & \\ & & & \mathsf{S}^\lambda_{N-1} \\ 0 & & \dots & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{D}^\lambda := \begin{bmatrix} 0 & & \dots & 0 \\ \mathsf{D}^\lambda_1 & & & \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \mathsf{D}^\lambda_N \end{bmatrix}, \tag{4.17}$$

acting on the spaces $H_{\|}^{-\frac{1}{2}}\Lambda(\Gamma)$, $H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma)$ related to the Grassmann algebra of differential forms on Γ.

The basic layer potentials have a special relationship with exterior differentiation, expressed in the next lemma.

Lemma 4.3. *For all $v_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\delta^\Gamma,\Gamma)$, $u_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}^\Gamma,\Gamma)$, $\ell \in \{0,\dots,N-1\}$, hold*

$$\delta_\ell\, \mathsf{S}^\lambda_\ell(v_\ell) = \mathsf{S}^\lambda_{\ell-1}(\delta^\Gamma_\ell\, v_\ell) \quad \text{and} \quad \mathrm{d}_{\ell+1}\, \mathsf{D}^\lambda_{\ell+1}(u_\ell) = -\mathsf{D}^\lambda_{\ell+2}(\mathrm{d}^\Gamma_\ell\, u_\ell)\ . \tag{4.18}$$

Proof. The proof relies on techniques used to show [12, Lemma 3] and [16, (3.2.41)]. We elaborate this for the second identity asserted by the lemma. The first can then be concluded as a consequence of (4.16).

Let $u_\ell \in L^\infty\Lambda^\ell(\Gamma) \cap H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d}^\Gamma,\Gamma)$ be the tangential trace of a smooth ℓ-form on $\mathcal{M}$. Then, for $x \notin \Gamma$, we can evaluate directly, using (4.2) and the integral representation (4.15) of the boundary potential, that

$$(\mathrm{d}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}u_\ell)(x) = \int_\Gamma u_\ell \wedge \imath^* \star_{\ell+1} (\mathrm{d}_{\ell+1,x}\, \mathcal{G}^{\lambda}_{\ell+1,\ell+1})(x,\cdot)$$

$$= \int_\Gamma u_\ell \wedge \imath^* \star_{\ell+1} \delta_{\ell+2}\, (\mathcal{G}^{\lambda}_{\ell+2,\ell+2}(x,\cdot)) \tag{4.19a}$$

$$= (-1)^{\ell+2} \int_\Gamma u_\ell \wedge \mathrm{d}^\Gamma_{N-\ell-2}\, \imath^* \star_{\ell+2} (\mathcal{G}^{\lambda}_{\ell+2,\ell+2}(x,\cdot)) \tag{4.19b}$$

$$= -(-1)^{\ell}(-1)^{\ell+2} \int_\Gamma \mathrm{d}^\Gamma_\ell\, u_\ell \wedge \imath^* \star_{\ell+2} (G^{\lambda}_{\ell+2,\ell+2}(x,\cdot)) \tag{4.19c}$$

$$= -\langle \mathrm{d}^\Gamma_\ell\, u_\ell, \mathsf{n}_{\ell+2}\mathcal{G}^{\lambda}_{\ell+2,\ell+2}(x,\cdot)\rangle_\Gamma\ ,$$

where (4.19a) is obtained by using (4.2), (4.19b) holds because the exterior derivative commutes with pullbacks, and (4.19c) follows by integration by parts. ∎

COROLLARY 4.4. *For all* $\boldsymbol{v} \in H_{\|}^{-\frac{1}{2}}\Lambda(\delta^\Gamma,\Gamma)$ *and* $\boldsymbol{u} \in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathrm{d}^\Gamma,\Gamma)$ *holds true*

$$\boldsymbol{\delta}\mathbf{S}^\lambda(\boldsymbol{v}) = \mathbf{S}^\lambda(\boldsymbol{\delta}^\Gamma\boldsymbol{v}) \quad \text{and} \quad \mathbf{dD}^\lambda(\boldsymbol{u}) = -\mathbf{D}^\lambda(\mathbf{d}^\Gamma\boldsymbol{u})\ . \tag{4.20}$$

4.3. Non-local inner products on trace spaces. We generalize to manifold the theory presented in [22, Section 8]. Similar results can be found in [11] and [17].

The key observation is that the continuous sesquilinear forms

$$(u_\ell, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}} := \langle\!\langle u_\ell, \mathsf{t}_\ell \mathsf{S}^\lambda_\ell(\overline{v}_\ell)\rangle\!\rangle_\Gamma\ , \qquad u_\ell, v_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)\ , \tag{4.21a}$$

$$(w_\ell, z_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} := \langle\!\langle w_\ell, \mathsf{n}_{\ell+1} \mathsf{D}^\lambda_{\ell+1}(\overline{z}_\ell)\rangle\!\rangle_\Gamma\ , \qquad w_\ell, z_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)\ , \tag{4.21b}$$

define *non-local inner products* on the spaces $H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$ and $H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$. For the sake of brevity we introduce the notation

$$\|U_\ell\|^2_{\lambda,\omega} := \lambda\, \|U_\ell\|^2_{L^2\Lambda^\ell(\omega)} + \|\mathrm{d}_\ell U_\ell\|^2_{L^2\Lambda^{\ell+1}(\omega)} + \|\delta_\ell U_\ell\|^2_{L^2\Lambda^{\ell-1}(\omega)}\ , \tag{4.22}$$

$U_\ell \in H\Lambda^\ell(\mathrm{d},\omega) \cap H\Lambda^\ell(\delta,\omega)$, $\ell \in \{0,\dots,N\}$, where ω is an open N-dimensional sub-manifold of $\mathcal{M}$. We also write $(\cdot,\cdot)_{\lambda,\omega}$ for the inner product inducing that norm and draw attention to the connection between $(\cdot,\cdot)_{\lambda,\mathcal{M}}$ and the left-hand sides of the variational definitions (4.10) and (4.11).

LEMMA 4.5. *For* $\ell \in \{0,\dots,N-1\}$ *and all* $h_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$ *and* $g_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$, *we have*

$$\|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}} = \langle\!\langle \overline{h}_\ell, \mathsf{t}_\ell \mathsf{S}^\lambda_\ell(h_\ell)\rangle\!\rangle_\Gamma = \left\|\mathsf{S}^\lambda_\ell h_\ell\right\|^2_{\lambda,\mathcal{M}}\ , \tag{4.23a}$$

$$\|g_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{n}} = \langle\!\langle \overline{g}_\ell, \mathsf{n}_{\ell+1} \mathsf{D}^\lambda_{\ell+1}(g_\ell)\rangle\!\rangle_\Gamma = \left\|\mathsf{D}^\lambda_{\ell+1} g_\ell\right\|^2_{\lambda,\mathcal{M}}\ . \tag{4.23b}$$

Proof. Fixing $h_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$, let us abbreviate $\Psi := \mathsf{S}^\lambda_\ell h_\ell$. Integrating by parts the first term on the right-hand side of $\|\Psi\|^2_{\lambda,\mathcal{M}} = \|\Psi\|^2_{\lambda,\Omega^-} + \|\Psi\|^2_{\lambda,\Omega^+}$, we obtain

$$\begin{aligned}\|\Psi\|^2_{\lambda,\Omega^-} &= (\mathrm{d}_\ell\Psi, \mathrm{d}_\ell\Psi)_{\Omega^-} + (\delta_\ell\Psi, \delta_\ell\Psi)_{\Omega^-} + \lambda\|\Psi\|^2_{L^2\Lambda^\ell(\Omega^-)}\\ &= (-\Delta_\ell\Psi, \Psi)_{\Omega^-} + \langle\!\langle \mathsf{n}^-_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma - \langle\!\langle \mathsf{t}^-_{\ell-1}\delta_\ell\Psi, \mathsf{n}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma + \lambda\|\Psi\|^2_{L^2\Lambda^\ell(\Omega^-)}\\ &= \langle\!\langle \mathsf{n}^-_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma - \langle\!\langle \mathsf{t}^-_{\ell-1}\delta_\ell\Psi, \mathsf{n}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma\ ,\end{aligned}$$

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where we have used the fact that Ψ satisfies the equation $-\Delta_\ell\Psi = -\lambda\Psi$ in Ω^-, i.e. $(-\Delta_\ell\Psi,\Psi)_{\Omega^-} = -\lambda(\Psi,\Psi)_{\Omega^-} = -\lambda\|\Psi\|^2_{L^2\Lambda^\ell(\Omega^-)}$. Similarly, we find in Ω^+ that

$$\|\Psi\|^2_{\lambda,\Omega^+} = -\langle\!\langle \mathsf{n}^+_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^+_\ell\overline{\Psi}\rangle\!\rangle_\Gamma + \langle\!\langle \mathsf{t}^+_{\ell-1}\delta_\ell\Psi, \mathsf{n}^+_\ell\overline{\Psi}\rangle\!\rangle_\Gamma. \tag{4.24}$$

Summing these equalities and using the jump relations from Theorem 4.2 yields

$$\|\Psi\|^2_{\lambda,\mathcal{M}} = \langle\!\langle -[\![\, \mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\Psi, \mathsf{t}_\ell\overline{\Psi}\rangle\!\rangle_\Gamma = \langle\!\langle \overline{h}_\ell, \mathsf{t}_\ell\mathsf{S}^\lambda_\ell h_\ell\rangle\!\rangle_\Gamma = \|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}} \;.$$

The proof of (4.23b) employs the same arguments and we skip it here. ∎

The next result generalizes [4, Thm. 4] to arbitrary dimensions. Inequalities that hold up to a positive constant only depending on Ω, λ and ℓ are denoted by $\lesssim$.

THEOREM 4.6. *For $\ell \in \{0,\dots,N-1\}$ we have*

$$\|h_\ell\|^2_{H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}} \qquad \forall h_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)\;, \tag{4.25a}$$

$$\|g_\ell\|^2_{H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \|g_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{n}} \qquad \forall w_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)\;. \tag{4.25b}$$

Proof. We focus on the first inequality. The second can be obtained using analogous arguments. Let $\mathsf{t}^\dagger_\ell : H_{\|}^{\frac{1}{2}}\Lambda^\ell(\Gamma) \to H^1\Lambda^\ell(\Omega)$ be a bounded right-inverse for the tangential (pullback) trace t^-_ℓ and $\mathsf{E}_\ell : H^1\Lambda^\ell(\Omega) \to H^1\Lambda^\ell(\mathcal{M})$ be a continuous *extension operator* such that $(\mathsf{E}_\ell\, U_\ell)\big|_\Omega = U_\ell$ for all $U_\ell \in H^1\Lambda(\Omega)$ [17, Proposition 3.1].

Given $h_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$, we start from the definition of $H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$ to estimate

$$\begin{aligned}
\|h_\ell\|_{H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} &= \sup_{g_\ell\in H_{\|}^{\frac{1}{2}}\Lambda^\ell(\Gamma)} \frac{\left|\langle\!\langle h_\ell, g_\ell\rangle\!\rangle_\Gamma\right|}{\|g_\ell\|_{H_{\|}^{\frac{1}{2}}\Lambda^\ell(\Gamma)}} \lesssim \sup_{g_\ell\in H_{\|}^{\frac{1}{2}}\Lambda^\ell(\Gamma)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell\mathsf{t}^\dagger_\ell g_\ell\rangle\!\rangle_\Gamma\right|}{\left\|\mathsf{t}^\dagger_\ell g_\ell\right\|_{H^1\Lambda^\ell(\Omega)}} \\
&\le \sup_{W_\ell\in H^1\Lambda^\ell(\Omega)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell W_\ell\rangle\!\rangle_\Gamma\right|}{\|W_\ell\|_{H^1\Lambda^\ell(\Omega)}} \lesssim \sup_{W_\ell\in H^1\Lambda^\ell(\Omega)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell\mathsf{E}_\ell W_\ell\rangle\!\rangle_\Gamma\right|}{\|\mathsf{E}_\ell W_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}} \\
&\lesssim \sup_{V_\ell\in H^1\Lambda^\ell(\mathcal{M})} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell V_\ell\rangle\!\rangle_\Gamma\right|}{\|V_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}} \;.
\end{aligned} \tag{4.26}$$

Recalling (4.10), we introduce $|\langle\!\langle h_\ell, \mathsf{t}_\ell V_\ell\rangle\!\rangle_\Gamma| = |\left(\mathsf{S}^\lambda_\ell h_\ell, V_\ell\right)_{\lambda,\mathcal{M}}|$ in (4.26) to obtain

$$\|h_\ell\|_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \sup_{V_\ell\in H^1\Lambda^\ell(\mathcal{M})} \frac{|\left(\mathsf{S}^\ell_\ell h_\ell, V_\ell\right)_{\lambda,\mathcal{M}}|}{\|V_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}} \;. \tag{4.27}$$

Then, since, obviously, $\|V_\ell\|_{\lambda,\mathcal{M}} \lesssim \|V_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}$ for all $V_\ell \in H^1\Lambda^\ell(\Omega)$, we find

$$\|h_\ell\|_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \sup_{V_\ell\in H^1\Lambda^\ell(\mathcal{M})} \frac{|\left(\mathsf{S}^\ell_\ell h_\ell, V_\ell\right)_{\lambda,\mathcal{M}}|}{\|V_\ell\|_{\lambda,\mathcal{M}}} = \left\|\mathsf{S}^\lambda_\ell h_\ell\right\|_{\lambda,\mathcal{M}} \;,$$

by applying the Cauchy-Schwarz inequality. Lemma 4.5 concludes the proof. ∎

4.4. Representation formula for the regularized Hodge-Dirac operator. We generalize [22, Section 4.4]. To begin with, integrating by parts after using the commutative relations (4.6) eventually verifies that

$$\mathsf{N}^{\lambda}\,\mathfrak{D} = \mathfrak{D}\,\mathsf{N}^{\lambda} \tag{4.28}$$

in the sense of distributions. Recalling (1.10) with $\lambda = \kappa^2$, we find that

$$(\mathfrak{D} - i\kappa)\,\mathsf{N}^{\lambda}\,(\mathfrak{D} + i\kappa) = (-\boldsymbol{\Delta} + \kappa^2)\mathsf{N}^{\lambda} = \mathrm{Id}\,, \tag{4.29}$$

which shows that $(\mathfrak{D} - i\kappa)\,\mathsf{N}^{\lambda}$ is a Newton-potential-type operator for the regularized Dirac operator $\mathfrak{D}^{\kappa} = (\mathfrak{D}+i\kappa)$, and that its Schwartz kernel $(\mathfrak{D} - i\kappa)\,\boldsymbol{\mathcal{G}}^{\lambda}$ can be viewed as the corresponding fundamental solution for $\mathfrak{D}^{\kappa}$.

Proposition 4.7 (Representation formula for $\mathfrak{D}^{\kappa}$). *If $\boldsymbol{U} \in L^2\Lambda(\mathcal{M})$ and there exists $\boldsymbol{F} \in L^2\Lambda(\mathcal{M})$ such that $\boldsymbol{F}|_{\Omega^-} = \mathfrak{D}^{\kappa}\boldsymbol{U}|_{\Omega^-}$ and $\boldsymbol{F}|_{\Omega^+} = \mathfrak{D}^{\kappa}\boldsymbol{U}|_{\Omega^+}$, then*

$$\boldsymbol{U} = (\mathfrak{D} - i\kappa)\left(\mathsf{N}^{\lambda}\boldsymbol{F} - \mathsf{S}^{\lambda}[\![\mathsf{n}\boldsymbol{U}]\!] + \mathsf{D}^{\lambda}[\![\mathsf{t}\boldsymbol{U}]\!]\right)\,. \tag{4.30}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
\langle\!\langle(\mathfrak{D} + i\kappa)\,\boldsymbol{U}, \boldsymbol{V}\rangle\!\rangle_{\mathcal{M}} &= \langle\boldsymbol{U}, (\mathfrak{D} + i\kappa)\boldsymbol{V}\rangle_{\Omega^-} + \langle\boldsymbol{U}, (\mathfrak{D} + i\kappa)\boldsymbol{V}\rangle_{\Omega^+}\\
&= \langle\boldsymbol{F}, \boldsymbol{V}\rangle_{\Omega^-} + \langle\!\langle\mathbf{t}^-\boldsymbol{V}, \mathbf{n}^-\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} - \langle\!\langle\mathbf{t}^-\,\boldsymbol{U}, \mathbf{n}^-\boldsymbol{V}\rangle\!\rangle_{\Gamma} +\\
&\qquad \langle\boldsymbol{F}, \boldsymbol{V}\rangle_{\Omega^+} - \langle\!\langle\mathbf{t}^+\boldsymbol{V}, \mathbf{n}^+\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} + \langle\!\langle\mathbf{t}^+\,\boldsymbol{U}, \mathbf{n}^+\boldsymbol{V}\rangle\!\rangle_{\Gamma}\\
&= \langle\boldsymbol{F}, \boldsymbol{V}\rangle_{\mathcal{M}} - \langle\!\langle\mathbf{t}\,\boldsymbol{V}, [\![\mathbf{n}]\!]\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} + \langle\!\langle[\![\mathbf{t}]\!]\boldsymbol{U}, \mathbf{n}\boldsymbol{V}\rangle\!\rangle_{\Gamma}
\end{aligned}$$

for all $\boldsymbol{V} \in C_0^{\infty}\Lambda(\mathcal{M})$. The regularity assumption on $\boldsymbol{U}$ guarantees that the traces are well-defined. We have also used the fact that $\boldsymbol{V}$ is smooth across the boundary Γ to obtain the last equality, because that global smoothness implies that $\mathbf{t}^+\boldsymbol{V} = \mathbf{t}^-\boldsymbol{V}$ and $\mathbf{n}^+\boldsymbol{V} = \mathbf{n}^-\boldsymbol{V}$, i.e. the jumps vanish on Γ. Hence,

$$(\mathfrak{D} + i\kappa)\,\boldsymbol{U} = \boldsymbol{F} - \mathbf{t}'\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathbf{n}'\,[\![\mathbf{t}\,\boldsymbol{U}]\!] \quad \text{in} \quad H^{-1}\Lambda(\mathcal{M})\,. \tag{4.31}$$

Applying the Newton potential operator $\mathbf{N}_{\boldsymbol{\lambda}}$ on both sides of this equation and inserting in the definitions of the basic layer potentials from (4.8) yields

$$\begin{aligned}
\mathsf{N}^{\lambda}(\mathfrak{D} + i\kappa)\,\boldsymbol{U} &= \mathsf{N}^{\lambda}\boldsymbol{F} - \mathsf{N}^{\lambda}\mathbf{t}'\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathsf{N}^{\lambda}\mathbf{n}'\,[\![\mathbf{t}\,\boldsymbol{U}]\!]\\
&= \mathsf{N}^{\lambda}\boldsymbol{F} - \mathsf{S}^{\lambda}\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathsf{D}^{\lambda}\,[\![\mathbf{t}\,\boldsymbol{U}]\!]\,.
\end{aligned} \tag{4.32}$$

Since $\mathfrak{D}^{\kappa}\boldsymbol{U}$ is square-integrable, the mapping properties of the Newton potential $\mathsf{N}^{\lambda} : L^2\Lambda(\mathcal{M}) \to H^1\Lambda(\mathcal{M})$ ensure that the left-hand side in this identity belongs in the domain of the Hodge–Dirac operator. Moreover, from Lemma 4.1 we know that the images of the basic layer potentials belong to $H^1\Lambda(\mathcal{M})$. Therefore, we can apply $\mathfrak{D} - i\kappa$ on both sides of (4.32) and use the commutation relation (4.28) plus (4.29) to arrive at (4.30). ∎

We can identify two separate layer potentials contributing to the representation formula (4.30). They can be distinguished by the type of traces they act upon:

$$\mathsf{SL}^{\kappa}[\mathfrak{D}] := (\mathfrak{D} - i\kappa)\,\mathsf{S}^{\lambda} : H_{\parallel}^{-\frac{1}{2}}\Lambda(\delta^{\Gamma}, \Gamma) \longrightarrow H\Lambda(\mathfrak{D}, \mathcal{M}\setminus\Gamma)\,, \tag{4.33a}$$

$$\mathsf{DL}^{\kappa}[\mathfrak{D}] := (\mathfrak{D} - i\kappa)\,\mathsf{D}^{\lambda} : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathrm{d}^{\Gamma}, \Gamma) \longrightarrow H\Lambda(\mathfrak{D}, \mathcal{M}\setminus\Gamma)\,. \tag{4.33b}$$

It follows immediately from Theorem 4.2 that these layer potentials satisfy jump relations. For example,

$$\llbracket \mathbf{n} \rrbracket \mathbf{SL}^{\kappa}[\mathfrak{D}] = \llbracket \mathbf{n} \rrbracket \left(\mathfrak{D} - i\kappa\right) \mathbf{S}^{\lambda} = \llbracket \mathbf{nd} \rrbracket \mathbf{S}^{\lambda} + \llbracket \mathbf{n}\delta \rrbracket \mathbf{S}^{\lambda} - i\kappa \llbracket \mathbf{n} \rrbracket \mathbf{S}^{\lambda} = -\mathrm{Id}\,. \tag{4.34}$$

Similar manipulations prove four jump relations and we collect them in the next proposition.

Proposition 4.8 (Jump relations for layer potentials induced by $\mathfrak{D}^{\kappa}$). *The layer potentials introduced in (4.33) satisfy in* $H_{\|}^{-\frac{1}{2}}\Lambda(\delta^{\Gamma},\Gamma)$ *and* $H_{\perp}^{-\frac{1}{2}}\Lambda(\mathrm{d}^{\Gamma},\Gamma)$, *respectively,*

$$\llbracket \mathbf{t} \rrbracket \mathbf{SL}^{\kappa}[\mathfrak{D}] = \mathbf{0}\,, \qquad \llbracket \mathbf{t} \rrbracket \mathbf{DL}^{\kappa}[\mathfrak{D}] = \mathrm{Id}\,, \tag{4.35a}$$

$$\llbracket \mathbf{n} \rrbracket \mathbf{SL}^{\kappa}[\mathfrak{D}] = -\mathrm{Id}\,, \qquad \llbracket \mathbf{n} \rrbracket \mathbf{DL}^{\kappa}[\mathfrak{D}] = \mathbf{0}\,. \tag{4.35b}$$

4.5. Representation formulas for the regularized Hodge-Laplace operators. Generalizing [6, Section 4], we apply the approach pursued in Subsection 4.4 to the regularized Hodge–Laplace operator $\mathfrak{L}_{\ell}^{\lambda}$, $\ell \in \{0,\dots,N\}$ fixed throughout the remainder of this section.

4.5.1. Second-order form of the Hodge-Laplacian. In Subsection 4.1 we have seen that a Newton potential is readily available for the Hodge–Laplacian in strong second-order formulation. From it we obtain a representation formula:

Proposition 4.9 (Representation formula for $\mathfrak{L}_{\ell}^{\lambda}$). *If* $\ell \in \{0,\dots,N\}$, $U_\ell \in L^2\Lambda^{\ell}(\mathcal{M})$, *and there exists* $F_\ell \in L^2\Lambda^{\ell}(\mathcal{M})$ *such that* $F_\ell|_{\Omega^-} = \mathfrak{L}_{\ell}^{\lambda} U_\ell|_{\Omega^-}$ *and* $F_\ell|_{\Omega^+} = \mathfrak{L}_{\ell}^{\lambda} U_\ell|_{\Omega^+}$, *then*

$$U_\ell = \mathsf{N}_{\ell}^{\lambda} F_\ell - \begin{bmatrix}\mathrm{d}_{\ell-1} & \mathrm{Id}\end{bmatrix} \begin{bmatrix} \mathsf{S}_{\ell-1}^{\lambda}\llbracket \mathsf{n}_\ell U_\ell \rrbracket \\ \mathsf{S}_{\ell}^{\lambda}\llbracket \mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell \rrbracket \end{bmatrix} + \begin{bmatrix}\mathrm{Id} & \delta_{\ell+1}\end{bmatrix} \begin{bmatrix} \mathsf{D}_{\ell}^{\lambda}\llbracket \mathsf{t}_{\ell-1}\delta_\ell U_\ell \rrbracket \\ \mathsf{D}_{\ell+1}^{\lambda}\llbracket \mathsf{t}_\ell U_\ell \rrbracket \end{bmatrix}.$$

Proof. The arguments are similar to those in the proof of Proposition 4.7. From Green's second formula

$$\begin{aligned} &\left\langle \mathfrak{L}_{\ell}^{\lambda} U_\ell, V_\ell \right\rangle_{\Omega^{\mp}} - \left\langle U_\ell, \mathfrak{L}_{\ell}^{\lambda} V_\ell \right\rangle_{\Omega^{\mp}} \\ &\quad = \pm \langle\!\langle \mathsf{T}_{\Delta,\ell}^{\mathsf{t},\mp} U_\ell, \mathsf{T}_{\Delta,\ell}^{\mathsf{n},\mp} V_\ell \rangle\!\rangle_{\Gamma} \mp \langle\!\langle \mathsf{T}_{\Delta,\ell}^{\mathsf{n},\mp} U_\ell, \mathsf{T}_{\Delta,\ell}^{\mathsf{t},\mp} V_\ell \rangle\!\rangle_{\Gamma}\,, \quad U_\ell, V_\ell \in H\Lambda^{\ell}(\Delta, \Omega^{\pm})\,, \end{aligned} \tag{4.36}$$

and the distributional interpretation of $\mathfrak{L}_{\ell}^{\lambda}$ we infer

$$\begin{aligned} \langle\!\langle \mathfrak{L}_{\ell}^{\lambda} U_\ell, V_\ell \rangle\!\rangle_{\mathcal{M}} &= \langle U_\ell, \mathfrak{L}_{\ell}^{\lambda} V_\ell \rangle_{\Omega^-} + \langle U_\ell, \mathfrak{L}_{\ell}^{\lambda} V_\ell \rangle_{\Omega^+} \\ &= \langle F_\ell, V_\ell \rangle_{\Omega^-} - \langle\!\langle \mathsf{T}_{\Delta,\ell}^{\mathsf{t},-} U_\ell, \mathsf{T}_{\Delta,\ell}^{\mathsf{n},-} V_\ell \rangle\!\rangle_{\Gamma} + \langle\!\langle \mathsf{T}_{\Delta,\ell}^{\mathsf{n},-} U_\ell, \mathsf{T}_{\Delta,\ell}^{\mathsf{t},-} V_\ell \rangle\!\rangle_{\Gamma} + \\ &\qquad \langle F_\ell, V_\ell \rangle_{\Omega^+} + \langle\!\langle \mathsf{T}_{\Delta,\ell}^{\mathsf{t},+} U_\ell, \mathsf{T}_{\Delta_\ell}^{\mathsf{n},+} V_\ell \rangle\!\rangle_{\Gamma} - \langle\!\langle \mathsf{T}_{\Delta,\ell}^{\mathsf{n},+} U_\ell, \mathsf{T}_{\Delta,\ell}^{\mathsf{t},+} V_\ell \rangle\!\rangle_{\Gamma} \\ &= \langle F_\ell, V_\ell \rangle_{\mathcal{M}} + \langle\!\langle \llbracket \mathsf{T}_{\Delta,\ell}^{\mathsf{t}} U_\ell \rrbracket, \mathsf{T}_{\Delta,\ell}^{\mathsf{n}} V_\ell \rangle\!\rangle_{\Gamma} - \langle\!\langle \llbracket \mathsf{T}_{\Delta,\ell}^{\mathsf{n}} U_\ell \rrbracket, \mathsf{T}_{\Delta,\ell}^{\mathsf{t}} V_\ell \rangle\!\rangle_{\Gamma} \end{aligned}$$

for all test functions $V_\ell \in C_0^{\infty}\Lambda^{\ell}(\mathcal{M})$. Hence, in the sense of distributions, we have

$$\mathfrak{L}_{\ell}^{\lambda} U_\ell = F_\ell + \left(\mathsf{T}_{\Delta,\ell}^{\mathsf{n}}\right)' \llbracket \mathsf{T}_{\Delta,\ell}^{\mathsf{t}} U_\ell \rrbracket - \left(\mathsf{T}_{\Delta,\ell}^{\mathsf{t}}\right)' \llbracket \mathsf{T}_{\Delta,\ell}^{\mathsf{n}} U_\ell \rrbracket\,.$$

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Applying the Newton potential operator N_ℓ^λ converts this into

$$U_\ell = \mathsf{N}_\ell^\lambda(-\Delta_\ell + \lambda \mathrm{Id})U_\ell = \mathsf{N}_\ell^\lambda F_\ell + \mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{n}}_{\Delta,\ell}\right)' [\![\mathsf{T}^{\mathsf{t}}_{\Delta,\ell} U_\ell]\!] - \mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{t}}_{\Delta,\ell}\right)' [\![\mathsf{T}^{\mathsf{n}}_{\Delta,\ell} U_\ell]\!] \, .$$

Explicitly, we appeal to the integral representations provided in (4.15) to evaluate for $x \notin \Gamma$

$$\begin{aligned}
\mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{t}}_{\Delta,\ell}\right)' \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix}(x) &= \langle\!\langle h_{\ell-1}, \mathsf{t}_{\ell-1}\delta_\ell \, \mathcal{G}_\ell^\lambda(x,\cdot)\rangle\!\rangle_\Gamma + \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}_\ell^\lambda(x,\cdot)\rangle\!\rangle_\Gamma \\
&= \mathrm{d}_{\ell-1,x} \langle\!\langle h_{\ell-1}, \mathsf{t}_{\ell-1} \mathcal{G}_{\ell-1}^\lambda(x,\cdot)\rangle\!\rangle_\Gamma + \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}_\ell^\lambda(x,\cdot)\rangle\!\rangle_\Gamma \\
&= \big(\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1})\big)(x) + \mathsf{S}_\ell^\lambda(h_\ell)(x) \, ,
\end{aligned} \tag{4.37a}$$

$$\begin{aligned}
\mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{n}}_{\Delta,\ell}\right)' \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix}(x) &= \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell \mathcal{G}_\ell^\lambda(x,\cdot)\rangle\!\rangle_\Gamma + \langle\!\langle g_\ell, \mathsf{n}_{\ell+1}\mathrm{d}_\ell \mathcal{G}_\ell^\lambda(x,\cdot)\rangle\!\rangle_\Gamma \\
&= \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell \mathcal{G}_\ell^\lambda(x,\cdot)\rangle\!\rangle_\Gamma + \delta_{\ell+1,x}\langle\!\langle g_\ell, \mathsf{n}_{\ell+1} \mathcal{G}_{\ell+1}^\lambda(x,\cdot)\rangle\!\rangle_\Gamma \\
&= \mathsf{D}_\ell^\lambda(g_{\ell-1})(x) + \big(\delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda(g_\ell)\big)(x) \, ,
\end{aligned} \tag{4.37b}$$

where we have used the identities stated in (4.2). Thus, we have arrived at the representation formula

$$U_\ell = \mathsf{N}_\ell^\lambda F_\ell - \mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda [\![\mathsf{n}_\ell U_\ell]\!] - \mathsf{S}_\ell^\lambda [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell]\!] + \mathsf{D}_\ell^\lambda [\![\mathsf{t}_{\ell-1}\delta_\ell U_\ell]\!] + \delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda [\![\mathsf{t}_\ell U_\ell]\!] \, . \qquad \square$$

In the representation formula of Proposition 4.9, we can identify two layer potentials

$$\begin{aligned}
\mathsf{SL}_\ell^\lambda[\Delta] \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix} &:= \mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + \mathsf{S}_\ell^\lambda(h_\ell) \, , \quad & \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix} &\in H_\Delta^{\mathsf{n},\ell}(\Gamma) \, , \\
\mathsf{DL}_\ell^\lambda[\Delta] \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix} &:= \mathsf{D}_\ell^\lambda(g_{\ell-1}) + \delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda(g_\ell) \, , \quad & \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix} &\in H_\Delta^{\mathsf{t},\ell}(\Gamma) \, ,
\end{aligned} \tag{4.38}$$

one taking as arguments traces of the form $[\![\mathsf{T}^{\mathsf{n}}_{\Delta,\ell} U_\ell]\!]$ and the other $[\![\mathsf{T}^{\mathsf{t}}_{\Delta,\ell} U_\ell]\!]$. Appealing to the mapping properties of the basic layer potentials, they provide continuous operators

$$\mathsf{SL}_\ell^\lambda[\Delta] : H_\Delta^{\mathsf{n},\ell}(\Gamma) \to H\Lambda^\ell(\Delta, \mathcal{M}\setminus\Gamma) \, , \tag{4.39a}$$

$$\mathsf{DL}_\ell^\lambda[\Delta] : H_\Delta^{\mathsf{t},\ell}(\Gamma) \to H\Lambda^\ell(\Delta, \mathcal{M}\setminus\Gamma) \, . \tag{4.39b}$$

Once again, jump relations for these potentials are obtained from Theorem 4.2. However, unlike for the Hodge–Dirac operator, for which the calculations were direct, we now also need to appeal to Lemma 4.1. For example, while

$$[\![\mathsf{T}^{\mathsf{n}}_{\Delta,\ell}]\!]\mathsf{SL}_\ell^\lambda[\Delta] \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix} = \begin{bmatrix} [\![\mathsf{n}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{n}_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell) \\ [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell) \end{bmatrix} = - \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix}$$

simply follows from Theorem 4.2 because $\mathrm{d}_\ell \, \mathrm{d}_{\ell-1} = 0$, we must evaluate in

$$[\![\mathsf{T}^{\mathsf{t}}_{\Delta,\ell}]\!]\mathsf{SL}_\ell^\lambda[\Delta] \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix} = \begin{bmatrix} [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell) \\ [\![\mathsf{t}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{t}_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell) \end{bmatrix}$$

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the jump $[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}$, which we have not encountered before. To show that it vanishes, we use the assertion of Lemma 4.1 that the basic layer potential satisfies the regularized Hodge–Laplace equation in the interior and exterior domains to compute

$$[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1} = -\mathrm{d}_{\ell-2}[\![\mathsf{t}_{\ell-1}\delta_{\ell-1}]\!]\mathsf{S}^\lambda_{\ell-1} - \lambda[\![\mathsf{t}_{\ell-1}]\!]\mathsf{S}^\lambda_{\ell-1} = 0. \tag{4.40}$$

The other jump relations are obtained similarly and the following proposition summarizes them:

Proposition 4.10 (Jump relations for layer potentials induced by $\mathfrak{L}^\lambda_\ell$). *The following identities hold in the product trace spaces $H^{\mathsf{n},\ell}_\Delta(\Gamma)$ and $H^{\mathsf{t},\ell}_\Delta(\Gamma)$, respectively,*

$$[\![\mathsf{T}^{\mathsf{t}}_{\Delta,\ell}]\!]\mathsf{SL}^\lambda_\ell[\Delta] = \mathbf{0}\,, \qquad [\![\mathsf{T}^{\mathsf{t}}_{\Delta,\ell}]\!]\mathsf{DL}^\lambda_\ell[\Delta] = \mathrm{Id}\,, \tag{4.41a}$$

$$[\![\mathsf{T}^{\mathsf{n}}_{\Delta,\ell}]\!]\mathsf{SL}^\lambda_\ell[\Delta] = -\mathrm{Id}\,, \qquad [\![\mathsf{T}^{\mathsf{n}}_{\Delta,\ell}]\!]\mathsf{DL}^\lambda_\ell[\Delta] = \mathbf{0}\,. \tag{4.41b}$$

4.5.2. Mixed-order form of $\mathfrak{L}^\lambda_\ell$. Similarly as for the Hodge–Dirac operator, we can build a Newton-potential type solution operator for the mixed-order Hodge–Laplacian $\mathfrak{M}^\lambda_\ell$ using the one available for the Hodge–Laplacian in strong formulation. Notice that

$$\begin{aligned}\begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id}\end{bmatrix}\mathfrak{M}^\lambda_\ell &= \begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id}\end{bmatrix}\begin{bmatrix} -\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \delta_{\ell+1}\mathrm{d}_\ell + \lambda\,\mathrm{Id}\end{bmatrix} \\ &= \begin{bmatrix} \mathfrak{L}^\lambda_{\ell-1} & 0 \\ 0 & \mathfrak{L}^\lambda_\ell\end{bmatrix},\end{aligned}$$

when acting on $C^\infty_0\Lambda^{\ell-1}(\mathcal{M}) \times C^\infty_0\Lambda^\ell(\mathcal{M})$ and, consequently, also in the sense of distributions. Moreover, integrating by parts after using the commutative relations (4.2) verifies the commutation property

$$\begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id}\end{bmatrix}\begin{bmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell\end{bmatrix} = \begin{bmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell\end{bmatrix}\begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id}\end{bmatrix},$$

which also holds in the sense of distributions. We conclude that

$$\begin{aligned}\begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id}\end{bmatrix}\begin{bmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell\end{bmatrix} &: C^\infty_0\Lambda^{\ell-1}(\mathcal{M}) \times C^\infty_0\Lambda^\ell(\mathcal{M}) \\ &\to C^\infty\Lambda^{\ell-1}(\mathcal{M}) \times C^\infty\Lambda^\ell(\mathcal{M})\end{aligned} \tag{4.42}$$

yields an inverse on $\mathcal{M}$ for the Hodge–Laplacian $\mathfrak{M}^\lambda_\ell$ in mixed-order formulation. By duality, it also provides an inverse in the sense of distributions.

A similar inverse can be built for $\mathfrak{R}^\lambda_\ell$, but since the following development is mirrored for the mixed formulation involving $\mathfrak{R}^\lambda_\ell$, we will focus our attention on $\mathfrak{M}^\lambda_\ell$.

PROPOSITION 4.11 (Representation formula for mixed-order Hodge-Laplacian). *If* $[U_{\ell-1}, U_\ell]^\top \in L^2\Lambda^{\ell-1}(\mathcal{M}) \times L^2\Lambda^\ell(\mathcal{M})$ *and there exists* $[F_{\ell-1}, F_\ell]^\top \in L^2\Lambda^{\ell-1}(\mathcal{M}) \times L^2\Lambda^\ell(\mathcal{M})$ *such that* $[F_{\ell-1}, F_\ell]^\top|_{\Omega^-} = \mathfrak{M}_\ell^\lambda [U_{\ell-1}, U_\ell]^\top|_{\Omega^-}$ *and* $[F_{\ell-1}, F_\ell]^\top|_{\Omega^+} = \mathfrak{M}_\ell^\lambda [U_{\ell-1}, U_\ell]^\top|_{\Omega^+}$, *then*

$$\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} = \begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{bmatrix} \left(\begin{bmatrix} \mathsf{N}_{\ell-1}^\lambda F_{\ell-1} \\ \mathsf{N}_\ell^\lambda F_\ell \end{bmatrix} - \begin{bmatrix} \mathsf{S}_{\ell-1}^\lambda \left[\!\left[\mathsf{n}_\ell U_\ell \right]\!\right] \\ \mathsf{S}_\ell^\lambda \left[\!\left[\mathsf{n}_{\ell+1} \mathrm{d}_\ell U_\ell \right]\!\right] \end{bmatrix} + \begin{bmatrix} 0 \\ \mathsf{D}_\ell^\lambda \left[\!\left[\mathsf{t}_{\ell-1} U_{\ell-1} \right]\!\right] + \delta_{\ell+1} \mathsf{D}_{\ell+1}^\lambda \left[\!\left[\mathsf{t}_\ell U_\ell \right]\!\right] \end{bmatrix} \right).$$

Proof. Using the same strategy as in Proposition 4.7 and Proposition 4.9, we obtain from Green's second formula (3.10) for the mixed-order Hodge–Laplacian that

$$\begin{aligned}
\left\langle\!\!\left\langle \mathfrak{M}_\ell^\lambda \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_{\mathcal{M}} &= \left\langle \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathfrak{M}_\ell^\lambda \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle_{\Omega^-} + \left\langle \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathfrak{M}_\ell^\lambda \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle_{\Omega^+} \\
&= \left\langle \begin{bmatrix} F_{\ell-1} \\ F_\ell \end{bmatrix}, \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle_{\Omega^-} + \left\langle \begin{bmatrix} F_{\ell-1} \\ F_\ell \end{bmatrix}, \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle_{\Omega^+} + \\
&\quad \left\langle\!\!\left\langle \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n},-} \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t},-} \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_\Gamma - \left\langle\!\!\left\langle \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t},-} \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n},-} \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_\Gamma - \\
&\quad \left\langle\!\!\left\langle \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n},+} \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t},+} \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_\Gamma + \left\langle\!\!\left\langle \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t},+} \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n},+} \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_\Gamma \\
&= \left\langle \begin{bmatrix} F_{\ell-1} \\ F_\ell \end{bmatrix}, \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle_{\mathcal{M}} - \\
&\quad \left\langle\!\!\left\langle \left[\!\left[\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n}} \right]\!\right] \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t}} \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_\Gamma + \left\langle\!\!\left\langle \left[\!\left[\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t}} \right]\!\right] \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix}, \mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n}} \begin{bmatrix} V_{\ell-1} \\ V_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_\Gamma
\end{aligned}$$

for all $[V_{\ell-1}, V_\ell]^\top \in C_0^\infty\Lambda^{\ell-1}(\mathcal{M}) \times C_0^\infty\Lambda^\ell(\mathcal{M})$ and conclude that

$$\mathfrak{M}_\ell^\lambda \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} = \begin{bmatrix} F_{\ell-1} \\ F_\ell \end{bmatrix} - (\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t}})' \left[\!\left[\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n}} \right]\!\right] \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} + (\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n}})' \left[\!\left[\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t}} \right]\!\right] \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} \tag{4.43}$$

in the sense of distributions. Applying the inverse provided in (4.42) yields

$$\begin{aligned}
\begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} = \begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{bmatrix} \Bigg(& \begin{bmatrix} \mathsf{N}_{\ell-1}^\lambda F_{\ell-1} \\ \mathsf{N}_\ell^\lambda F_\ell \end{bmatrix} + \\
& \begin{bmatrix} \mathsf{N}_{\ell-1}^\lambda & 0 \\ 0 & \mathsf{N}_\ell^\lambda \end{bmatrix} \left(\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n}} \right)' \left[\!\left[\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t}} \right]\!\right] \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} - \\
& \begin{bmatrix} \mathsf{N}_{\ell-1}^\lambda & 0 \\ 0 & \mathsf{N}_\ell^\lambda \end{bmatrix} \left(\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{t}} \right)' \left[\!\left[\mathsf{T}_{\mathfrak{M},\ell}^{\mathsf{n}} \right]\!\right] \begin{bmatrix} U_{\ell-1} \\ U_\ell \end{bmatrix} \Bigg).
\end{aligned}$$

Explicitly, we evaluate for $x \in \mathcal{M} \setminus \Gamma$

$$\left(\begin{bmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{bmatrix} \left(\mathsf{T}^{\mathsf{n}}_{\mathfrak{M},\ell}\right)' \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix} \right)(x) = \begin{bmatrix} 0 \\ \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell \mathcal{G}^\lambda_\ell(x,\cdot)\rangle\!\rangle_\Gamma + \langle\!\langle g_\ell, \mathsf{n}_{\ell+1} \mathrm{d}_\ell \mathcal{G}^\lambda_\ell(x,\cdot)\rangle\!\rangle_\Gamma \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ \left(\mathsf{D}^\lambda_\ell g_{\ell-1}\right)(x) + \left(\delta_{\ell+1} \mathsf{D}^\lambda_{\ell+1} g_\ell\right)(x) \end{bmatrix}$$

and

$$\left(\begin{bmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{bmatrix} \left(\mathsf{T}^{\mathsf{t}}_{\mathfrak{M},\ell}\right)' \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix} \right)(x) = \begin{bmatrix} \langle\!\langle h_{\ell-1}, \mathsf{t}_{\ell-1} \mathcal{G}^\lambda_{\ell-1}(x,\cdot)\rangle\!\rangle_\Gamma \\ \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}^\lambda_\ell(x,\cdot)\rangle\!\rangle_\Gamma \end{bmatrix}$$
$$= \begin{bmatrix} \left(\mathsf{S}^\lambda_{\ell-1} h_{\ell-1}\right)(x) \\ \left(\mathsf{S}^\lambda_\ell h_\ell\right)(x) \end{bmatrix}. \qquad \square$$

In the representation formula of Proposition 4.11, we recognize two layer potentials

$$\mathsf{SL}^\lambda_\ell[\mathfrak{M}] : H^{\mathsf{n},\ell}_\Delta(\Gamma) \to \mathrm{dom}(\mathfrak{M}^\lambda_\ell) \quad \text{and} \quad \mathsf{DL}^\lambda_\ell[\mathfrak{M}] : H^{\mathsf{t},\ell}_\Delta(\Gamma) \to \mathrm{dom}(\mathfrak{M}^\lambda_\ell) \,, \tag{4.44}$$

defined by

$$\mathsf{SL}^\lambda_\ell[\mathfrak{M}] \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix} := \begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{bmatrix} \begin{bmatrix} \mathsf{S}^\lambda_{\ell-1} h_{\ell-1} \\ \mathsf{S}^\lambda_\ell h_\ell \end{bmatrix}$$
$$= \begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1}\mathsf{S}^\lambda_{\ell-1} h_{\ell-1} - \lambda \mathsf{S}^\lambda_{\ell-1} h_{\ell-1} + \delta_\ell \mathsf{S}^\lambda_\ell h_\ell \\ \mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1} h_{\ell-1} + \mathsf{S}^\lambda_\ell h_\ell \end{bmatrix},$$
$$\mathsf{DL}^\lambda_\ell[\mathfrak{M}] \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix} := \begin{bmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{bmatrix} \begin{bmatrix} 0 \\ \mathsf{D}^\lambda_\ell g_{\ell-1} + \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \end{bmatrix}$$
$$= \begin{bmatrix} \delta_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} \\ \mathsf{D}^\lambda_\ell g_{\ell-1} + \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \end{bmatrix}.$$

Jump relations for these boundary potentials are obtained using the same techniques as in the previous sections:

Proposition 4.12 (Jump relations for layer potentials related to $\mathfrak{M}^\lambda_\ell$). *In the product trace spaces $H^{\mathsf{n},\ell}_\Delta(\Gamma)$ and $H^{\mathsf{t},\ell}_\Delta(\Gamma)$, respectively, holds*

$$[\![\mathsf{T}^{\mathsf{t}}_{\mathfrak{M},\ell}]\!]\mathsf{SL}^\lambda_\ell[\mathfrak{M}] = \mathbf{0} \,, \qquad [\![\mathsf{T}^{\mathsf{t}}_{\mathfrak{M},\ell}]\!]\mathsf{DL}^\lambda_\ell[\mathfrak{M}] = \mathrm{Id} \,, \tag{4.45a}$$
$$[\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{M},\ell}]\!]\mathsf{SL}^\lambda_\ell[\mathfrak{M}] = -\mathrm{Id} \,, \qquad [\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{M},\ell}]\!]\mathsf{DL}^\lambda_\ell[\mathfrak{M}] = \mathbf{0} \,. \tag{4.45b}$$

Proof. Using the third jump relation provided in (4.14c), we can evaluate directly

$$[\![\mathsf{T}^{\mathsf{t}}_{\mathfrak{M},\ell}]\!]\mathsf{DL}^\lambda_\ell[\mathfrak{M}] \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix} = \begin{bmatrix} [\![\mathsf{t}_{\ell-1}]\!]\delta_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} \\ [\![\mathsf{t}_\ell]\!]\mathsf{D}^\lambda_\ell g_{\ell-1} + [\![\mathsf{t}_\ell]\!]\delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \end{bmatrix} = \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix}. \tag{4.46}$$

Using the second jump relation in (4.14b) and the complex property $\mathrm{d}_\ell\,\mathrm{d}_{\ell=1} \equiv 0$, we obtain

$$(4.47)\qquad [\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{M},\ell}]\!]\mathsf{SL}^\lambda_\ell[\mathfrak{M}] = \begin{bmatrix} [\![\mathsf{n}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + [\![\mathsf{n}_\ell]\!]\mathsf{S}^\lambda_\ell h_\ell \\ [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathsf{S}^\lambda_\ell h_\ell \end{bmatrix} = -\begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix}.$$

The other jump relations are similarly obtained by applying the jump relations from Theorem 4.2, after combining the commutativity properties of Lemma 4.3 with (4.13) (see (4.40)). ∎

5. First-kind Boundary Integral Operators (BIOs). Boundary integral operators (BIOs) provide linear mappings between trace spaces. Generically, we obtain them by letting trace operators act on representation formulas, remember Figure 1. We now pursue this policy for the representation formulas found in Subsection 4.4 and Subsection 4.5, and the relevant traces from Subsection 3.1 and Subsection 3.2, respectively. We confine ourselves to so-called *first-kind BIOs* characterized as those BIOs that are bounded linear operators between trace spaces that can be put in duality with $L^2(\Gamma)$ pivot spaces.

Throughout this section the closed, densely defined unbounded operators

$$(5.1a)\qquad (\delta^\Gamma_\ell)^* : H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma)\,,$$

$$(5.1b)\qquad (\mathrm{d}^\Gamma_\ell)^* : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell+1}(\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma)\,,$$

designate the Hilbert space adjoints in $H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma)$ or $H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell+1}(\Gamma)$, respectively, of the closed and densely defined unbounded operators

$$(5.2a)\qquad \delta^\Gamma_\ell : H_{\|}^{-\frac{1}{2}}\Lambda^{\ell}(\delta^\Gamma,\Gamma) \subset H_{\|}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma)\,,$$

$$(5.2b)\qquad \mathrm{d}^\Gamma_\ell : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}^\Gamma,\Gamma) \subset H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell+1}(\Gamma)\,,$$

introduced in Section 2, which underlie the trace de Rham Hilbert complexes (2.4a) and (2.4b). We suppose that spaces $H_{\|}^{-\frac{1}{2}}\Lambda(\Gamma)$ and $H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma)$ are equipped with the non-local inner products $(\cdot,\cdot)_{-\frac{1}{2},\lambda,\mathsf{t}}$ and $(\cdot,\cdot)_{-\frac{1}{2},\lambda,\mathsf{n}}$ defined in Subsection 4.3, (4.21a) and (4.21b).

Remark 5.1. Since the trace de Rham complexes (2.4a) and (2.4b) are equipped with the non-local inner products (4.21a) and (4.21b), it goes without saying that neither the pair $(\delta^\Gamma_\ell, (\delta^\Gamma_\ell)^*)$ nor $(\mathrm{d}^\Gamma_\ell, (\mathrm{d}^\Gamma_\ell)^*)$ are adjoints with respect to an $L^2(\Gamma)$-pairing.

5.1. First-kind BIOs for the regularized Hodge-Dirac operator. The two trace operators belonging to Hodge-Dirac operator are $\mathbf{t}$ and $\mathbf{n}$ and the crucial representation formula is (4.30). Applying $\mathbf{t}$ and $\mathbf{n}$ (from either side of Γ, irrelevant due to the jump relations of Proposition 4.8) to the layer potentials $\mathbf{SL}^\kappa[\mathfrak{D}]$ and $\mathbf{DL}^\kappa[\mathfrak{D}]$ from (4.33) we extract two first-kind BIOs for the regularized Hodge–Dirac operator:

$$(5.3a)\qquad \mathsf{V}^\kappa[\mathfrak{D}] := \mathbf{t}\,\mathbf{SL}^\kappa[\mathfrak{D}] = \mathbf{t}\,(\mathfrak{D} - \imath\kappa)\,\mathbf{S}^\lambda : H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}^\Gamma,\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}^\Gamma,\Gamma)\,,$$

$$(5.3b)\qquad \mathsf{W}^\kappa[\mathfrak{D}] := \mathbf{n}\,\mathbf{DL}^\kappa[\mathfrak{D}] = \mathbf{n}\,(\mathfrak{D} - \imath\kappa)\,\mathbf{D}^\lambda : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}^\Gamma,\Gamma) \longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}^\Gamma,\Gamma)\,.$$

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In light of the fact that the trace spaces $H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}^{\Gamma},\Gamma)$ and $H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}^{\Gamma},\Gamma)$ are in duality with respect to the $L^2\Lambda(\Gamma)$ pairing both $\mathsf{V}^{\kappa}[\mathfrak{D}]$ and $\mathsf{W}^{\kappa}[\mathfrak{D}]$ qualify as first-kind boundary integral operators.

Using Corollary 4.4 and (2.9), integration by parts yields

$$\begin{aligned}\langle\!\langle \mathsf{V}^{\kappa}[\mathfrak{D}]\boldsymbol{h},\overline{\boldsymbol{w}}\rangle\!\rangle_{\Gamma} &= \langle\!\langle \mathbf{t}\boldsymbol{\delta}\mathsf{S}^{\lambda}\boldsymbol{h},\overline{\boldsymbol{w}}\rangle\!\rangle_{\Gamma} + \langle\!\langle \mathbf{t}\mathbf{d}\mathsf{S}^{\lambda}\boldsymbol{h},\overline{\boldsymbol{w}}\rangle\!\rangle_{\Gamma} - \imath\kappa\langle\!\langle \mathbf{t}\mathsf{S}^{\lambda}\boldsymbol{h},\overline{\boldsymbol{w}}\rangle\!\rangle_{\Gamma}\\ &= \langle\!\langle \mathbf{t}\mathsf{S}^{\lambda}(\boldsymbol{\delta}^{\Gamma}\boldsymbol{h}),\overline{\boldsymbol{w}}\rangle\!\rangle_{\Gamma} + \langle\!\langle \mathbf{t}\mathsf{S}^{\lambda}\boldsymbol{h},\boldsymbol{\delta}^{\Gamma}\overline{\boldsymbol{w}}\rangle\!\rangle_{\Gamma} - \imath\kappa\langle\!\langle \mathbf{t}\mathsf{S}^{\lambda}\boldsymbol{h},\overline{\boldsymbol{w}}\rangle\!\rangle_{\Gamma}\\ &= (\boldsymbol{\delta}^{\Gamma}\boldsymbol{h},\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} + (\boldsymbol{h},\boldsymbol{\delta}^{\Gamma}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} - \imath\kappa(\boldsymbol{h},\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}}\end{aligned} \tag{5.4}$$

for all $\boldsymbol{h},\boldsymbol{w}\in H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta}^{\Gamma},\Gamma)$. Similarly, we can also compute

$$\begin{aligned}\langle\!\langle \mathsf{W}^{\kappa}[\mathfrak{D}]\boldsymbol{g},\overline{\boldsymbol{v}}\rangle\!\rangle_{\Gamma} &= \langle\!\langle \mathbf{n}\mathbf{d}\mathbf{D}^{\lambda}\boldsymbol{g},\overline{\boldsymbol{v}}\rangle\!\rangle_{\Gamma} + \langle\!\langle \mathbf{n}\boldsymbol{\delta}\mathbf{D}^{\lambda}\boldsymbol{g},\overline{\boldsymbol{v}}\rangle\!\rangle_{\Gamma} - \imath\kappa\langle\!\langle \mathbf{n}\mathbf{D}^{\lambda}\boldsymbol{g},\overline{\boldsymbol{v}}\rangle\!\rangle_{\Gamma}\\ &= -\langle\!\langle \mathbf{n}\mathbf{D}^{\lambda}(\mathbf{d}^{\Gamma}\boldsymbol{g}),\overline{\boldsymbol{v}}\rangle\!\rangle_{\Gamma} - \langle\!\langle \mathbf{n}\mathbf{D}^{\lambda}\boldsymbol{g},\mathbf{d}^{\Gamma}\overline{\boldsymbol{v}}\rangle\!\rangle_{\Gamma} - \imath\kappa\langle\!\langle \mathbf{n}\mathbf{D}^{\lambda}\boldsymbol{g},\overline{\boldsymbol{v}}\rangle\!\rangle_{\Gamma}\\ &= -(\mathbf{d}^{\Gamma}\boldsymbol{g},\boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}} - (\boldsymbol{g},\mathbf{d}^{\Gamma}\boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}} - \imath\kappa(\boldsymbol{g},\boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}}\end{aligned} \tag{5.5}$$

for all $\boldsymbol{g},\boldsymbol{v}\in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d}^{\Gamma},\Gamma)$.

We encourage the reader to compare the bilinear forms in (5.4) and (5.5) with the bilinear forms $\mathcal{A}_{\boldsymbol{\delta}}$ and $\mathcal{A}_{\mathbf{d}}$ appearing in the variational problems (3.4a) and (3.4b) for the regularized Hodge–Dirac operator. Thus, the identities (5.4) and (5.5) can be rephrased as the following main result:

THEOREM 5.2 ($\mathsf{V}^{\kappa}[\mathfrak{D}]$ and $\mathsf{W}^{\kappa}[\mathfrak{D}]$ are Hodge-Dirac operators for the trace De Rham complex). *From a variational point of view, the first-kind BIOs defined in (5.3a) and (5.3b) for the regularized Hodge–Dirac operator $\mathfrak{D}^{\kappa}$ are themselves regularized Hodge–Dirac operators*

$$\mathsf{V}^{\kappa}[\mathfrak{D}] = \boldsymbol{\delta}^{\Gamma} + (\boldsymbol{\delta}^{\Gamma})^{*} - \imath\kappa\mathrm{Id}\,, \tag{5.6a}$$

$$\mathsf{W}^{\kappa}[\mathfrak{D}] = -(\mathbf{d}^{\Gamma} + (\mathbf{d}^{\Gamma})^{*}) - \imath\kappa\mathrm{Id}\,, \tag{5.6b}$$

in the trace de Rham complexes (2.4a) and (2.4b), whose spaces are equipped with the non-local inner products (4.21a) and (4.21b).

5.2. First-kind BIOs for the regularized Hodge-Laplace operators. In Subsection 4.5 we studied the Hodge-Laplacian both in second-order (standard) form and as mixed-order system. Since different trace operators and layer potentials matter for both cases, also the derivation of BIOs will be carried out separately.

5.2.1. First-kind BIOs for second-order form of $\mathfrak{L}_{\ell}^{\lambda}$. Subsection 3.2 introduced the crucial trace operators

$$\mathsf{T}_{\Delta,\ell}^{\mathsf{t},\mp}U_{\ell} := \begin{bmatrix}\mathsf{t}_{\ell-1}^{\mp}\delta_{\ell}U_{\ell}\\ \mathsf{t}_{\ell}^{\mp}U_{\ell}\end{bmatrix} : H\Lambda^{\ell}(\Delta,\Omega^{\mp}) \to H_{\Delta}^{\mathsf{t},\ell}(\Gamma) = H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d}^{\Gamma},\Gamma)\times H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d}^{\Gamma},\Gamma)\,,$$

$$\mathsf{T}_{\Delta,\ell}^{\mathsf{n},\mp}U_{\ell} := \begin{bmatrix}\mathsf{n}_{\ell}^{\mp}U_{\ell}\\ \mathsf{n}_{\ell+1}^{\mp}\mathrm{d}_{\ell}U_{\ell}\end{bmatrix} : H\Lambda^{\ell}(\Delta,\Omega^{\mp}) \to H_{\Delta}^{\mathsf{n},\ell}(\Gamma) = H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta^{\Gamma},\Gamma)\times H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\delta^{\Gamma},\Gamma),$$

and Proposition 4.9 provided the relevant representation formula, from which we extracted the layer potentials $\mathsf{SL}_\ell^\lambda[\Delta] : H_\Delta^{\mathsf{n},\ell}(\Gamma) \to H\Lambda^\ell(\Delta, \mathcal{M} \setminus \Gamma)$ and $\mathsf{DL}_\ell^\lambda[\Delta] : H_\Delta^{\mathsf{t},\ell}(\Gamma) \to H\Lambda^\ell(\Delta, \mathcal{M} \setminus \Gamma)$ defined in (4.38). Remember that the trace spaces $H_\Delta^{\mathsf{t},\ell}(\Gamma)$ and $H_\Delta^{\mathsf{n},\ell}(\Gamma)$ are in duality with respect to the $L^2\Lambda^{\ell-1}(\Gamma) \times L^2\Lambda^\ell(\Gamma)$ inner product, which ensures that the BIOs

$$\mathsf{V}_\ell^\lambda[\Delta] := \mathsf{T}_{\Delta,\ell}^{\mathsf{t}} \mathsf{SL}_\ell^\lambda[\Delta] : H_\Delta^{\mathsf{n},\ell}(\Gamma) \longrightarrow H_\Delta^{\mathsf{t},\ell}(\Gamma) \,, \tag{5.7a}$$

$$\mathsf{W}_\ell^\lambda[\Delta] := \mathsf{T}_{\Delta,\ell}^{\mathsf{n}} \mathsf{DL}_\ell^\lambda[\Delta] : H_\Delta^{\mathsf{t},\ell}(\Gamma) \longrightarrow H_\Delta^{\mathsf{n},\ell}(\Gamma) \tag{5.7b}$$

are first-kind. Again, thanks to the jump relations (4.41) it does not matter from which side of Γ we apply the trace operator and, therefore, we omit the superscripts $\pm$ here and in the sequel.

Starting with $\mathsf{V}_\ell^\lambda[\Delta]$, we evaluate using Lemma 4.3 (terms in green and blue), Lemma 4.1 (terms in blue) and (2.9) (terms in red), that

$$\begin{aligned}\mathsf{T}_{\Delta,\ell}^{\mathsf{t}} \mathsf{SL}_\ell^\lambda[\Delta] \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix} &= \begin{bmatrix} \mathsf{t}_{\ell-1}\delta_\ell \mathrm{d}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda h_{\ell-1} + \mathsf{t}_{\ell-1}\delta_\ell \mathsf{S}_\ell^\lambda h_\ell \\ \mathsf{t}_\ell \mathrm{d}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda h_{\ell-1} + \mathsf{t}_\ell \mathsf{S}_\ell^\lambda h_\ell \end{bmatrix} \\ &= \begin{bmatrix} -\mathrm{d}_{\ell-2}^\Gamma \mathsf{t}_{\ell-2} \mathsf{S}_{\ell-2}^\lambda (\delta_{\ell-1}^\Gamma h_{\ell-1}) - \lambda \mathsf{t}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda h_{\ell-1} + \mathsf{t}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda (\delta_\ell^\Gamma h_\ell) \\ \mathrm{d}_\ell^\Gamma \mathsf{t}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda h_{\ell-1} + \mathsf{t}_\ell \mathsf{S}_\ell^\lambda h_\ell \end{bmatrix} ,\end{aligned}$$

from which we obtain, using integration by parts on Γ,

$$\begin{aligned} &\left\langle\!\!\left\langle \mathsf{T}_{\Delta,\ell}^{\mathsf{t}} \mathsf{SL}_\ell^\lambda[\Delta] \begin{bmatrix} h_{\ell-1} \\ h_\ell \end{bmatrix}, \begin{bmatrix} \overline{w}_{\ell-1} \\ \overline{w}_\ell \end{bmatrix} \right\rangle\!\!\right\rangle_\Gamma \\ &\quad = -\langle\!\langle \mathsf{t}_{\ell-2} \mathsf{S}_{\ell-2}^\lambda (\delta_{\ell-1}^\Gamma h_{\ell-1}), \delta_{\ell-1}^\Gamma \overline{w}_{\ell-1} \rangle\!\rangle_\Gamma - \lambda \langle\!\langle \mathsf{t}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda h_{\ell-1}, \overline{w}_{\ell-1} \rangle\!\rangle_\Gamma \\ &\qquad + \langle\!\langle \mathsf{t}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda (\delta_\ell^\Gamma h_\ell), \overline{w}_{\ell-1} \rangle\!\rangle_\Gamma + \langle\!\langle \mathsf{t}_{\ell-1} \mathsf{S}_{\ell-1}^\lambda h_{\ell-1}, \delta_\ell^\Gamma \overline{w}_\ell \rangle\!\rangle_\Gamma \\ &\qquad + \langle\!\langle \mathsf{t}_\ell \mathsf{S}_\ell^\lambda h_\ell, \overline{w}_\ell \rangle\!\rangle_\Gamma \\ &\qquad\qquad = -\left(\delta_{\ell-1}^\Gamma h_{\ell-1}, \delta_{\ell-1}^\Gamma w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} - \lambda \left(h_{\ell-1}, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + \\ &\qquad\qquad\quad \left(\delta_\ell^\Gamma h_\ell, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + \left(h_{\ell-1}, \delta_\ell^\Gamma w_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + \\ &\qquad\qquad\quad (h_\ell, w_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}} \,. \end{aligned} \tag{5.8}$$

Similarly for $\mathsf{W}_\ell^\lambda[\Delta]$, evaluating

$$\begin{aligned} &\mathsf{T}_{\Delta,\ell}^{\mathsf{n}} \mathsf{DL}_\ell^\lambda[\Delta] \begin{bmatrix} g_{\ell-1} \\ g_\ell \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathsf{n}_\ell \mathsf{D}_\ell^\lambda g_{\ell-1} + \mathsf{n}_\ell \delta_{\ell+1} \mathsf{D}_{\ell+1}^\lambda g_\ell \\ \mathsf{n}_{\ell+1} \mathrm{d}_\ell \mathsf{D}_\ell^\lambda g_{\ell-1} + \mathsf{n}_{\ell+1} \mathrm{d}_\ell \delta_{\ell+1} \mathsf{D}_{\ell+1}^\lambda g_\ell \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathsf{n}_\ell \mathsf{D}_\ell^\lambda g_{\ell-1} - \delta_\ell^\Gamma \mathsf{n}_{\ell+1} \mathsf{D}_{\ell+1}^\lambda g_\ell \\ -\mathsf{n}_{\ell+1} \mathsf{D}_{\ell+1}^\lambda (\mathrm{d}_{\ell-1}^\Gamma g_{\ell-1}) - \mathsf{n}_{\ell+1} \delta_{\ell+2} \mathrm{d}_{\ell+1} \mathsf{D}_{\ell+1}^\lambda g_\ell - \lambda \mathsf{n}_{\ell+1} \mathsf{D}_{\ell+1}^\lambda g_\ell \end{bmatrix} \\ &\quad = \begin{bmatrix} \mathsf{n}_\ell \mathsf{D}_\ell^\lambda g_{\ell-1} - \delta_\ell^\Gamma \mathsf{n}_{\ell+1} \mathsf{D}_{\ell+1}^\lambda g_\ell \\ -\mathsf{n}_{\ell+1} \mathsf{D}_{\ell+1}^\lambda (\mathrm{d}_{\ell-1}^\Gamma g_{\ell-1}) - \delta_{\ell+1}^\Gamma \mathsf{n}_{\ell+2} \mathsf{D}_{\ell+2}^\lambda (\mathrm{d}_\ell^\Gamma g_\ell) - \lambda \mathsf{n}_{\ell+1} \mathsf{D}_{\ell+1}^\lambda g_\ell \end{bmatrix} \end{aligned}$$

eventually leads to

$$\left\langle\!\!\left\langle \mathsf{T}^{\mathsf{n}}_{\Delta,\ell}\mathsf{DL}^{\lambda}_{\ell}[\Delta]\begin{bmatrix} g_{\ell-1}\\ g_{\ell}\end{bmatrix}, \begin{bmatrix}\overline{v}_{\ell-1}\\ \overline{v}_{\ell}\end{bmatrix}\right\rangle\!\!\right\rangle_{\Gamma} = (g_{\ell-1}, v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(g_{\ell}, \mathrm{d}^{\Gamma}_{\ell-1}v_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(\mathrm{d}^{\Gamma}_{\ell-1}g_{\ell-1}, v_{\ell}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(\mathrm{d}^{\Gamma}_{\ell}g_{\ell}, \mathrm{d}^{\Gamma}_{\ell}v_{\ell}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \lambda\,(g_{\ell}, v_{\ell})_{-\frac{1}{2},\lambda,\mathsf{n}}\,. \tag{5.9}$$

Again, we ask the reader to compare the sesqui-linear forms (5.8) and (5.9) with the bilinear forms (3.11) and (3.16). They match apart from the underlying inner products and this observation amounts to our second main result.

THEOREM 5.3 ($\mathsf{V}^{\lambda}_{\ell}[\Delta]$ and $\mathsf{W}^{\lambda}_{\ell}[\Delta]$ are mixed-order regularized Hodge Laplace operators the trace De Rham complex). *From a variational point of view, the* first-kind *BIOs defined in* (5.7a) *and* (5.7b) *for the regularized Hodge–Laplace operator $\mathfrak{L}^{\lambda}_{\ell}$ are mixed-order regularized Hodge–Dirac operators*

$$\mathsf{V}^{\lambda}_{\ell}[\Delta] = \begin{bmatrix} -(\delta^{\Gamma}_{\ell-1})^{*}\delta^{\Gamma}_{\ell-1} - \lambda\mathrm{Id} & \delta^{\Gamma}_{\ell}\\ (\delta^{\Gamma}_{\ell})^{*} & \mathrm{Id}\end{bmatrix}, \tag{5.10a}$$

$$\mathsf{W}^{\lambda}_{\ell}[\Delta] = \begin{bmatrix} \mathrm{Id} & -(\mathrm{d}^{\Gamma}_{\ell-1})^{*}\\ -\mathrm{d}^{\Gamma}_{\ell-1} & -(\mathrm{d}^{\Gamma}_{\ell})^{*}\mathrm{d}^{\Gamma}_{\ell} - \lambda\mathrm{Id}\end{bmatrix} \tag{5.10b}$$

in the trace de Rham complexes (2.4a) *and* (2.4b)*, which are based on the non-local inner products* (4.21a) *and* (4.21b)*.*

5.2.2. BIOs for mixed-order form of $\mathfrak{L}^{\lambda}_{\ell}$. We focus on the mixed-order regularized Hodge-Laplace operator $\mathfrak{M}^{\lambda}_{\ell}$ defined in (3.9) and recall from Subsection 3.1, (3.12) the associated complementary trace operators $\mathsf{T}^{\mathsf{t},\mp}_{\mathfrak{M},\ell}$ and $\mathsf{T}^{\mathsf{n},\mp}_{\mathfrak{M},\ell}$, whose co-domains are in duality, cf. (3.13). We apply those trace operators to the layer potentials $\mathsf{SL}^{\lambda}_{\ell}[\mathfrak{M}]$ and $\mathsf{DL}^{\lambda}_{\ell}[\mathfrak{M}]$ that we extracted from the representation formula of Proposition 4.11. Thus we obtain the first-kind BIOs

$$\mathsf{V}^{\lambda}_{\ell}[\mathfrak{M}] := \mathsf{T}^{\mathsf{t}}_{\mathfrak{M},\ell}\mathsf{SL}^{\lambda}_{\ell}[\mathfrak{M}] : H^{\mathsf{n},\ell}_{\Delta}(\Gamma) \longrightarrow H^{\mathsf{t},\ell}_{\Delta}(\Gamma)\,, \tag{5.11a}$$

$$\mathsf{W}^{\lambda}_{\ell}[\mathfrak{M}] := \mathsf{T}^{\mathsf{n}}_{\mathfrak{M},\ell}\mathsf{DL}^{\lambda}_{\ell}[\mathfrak{M}] : H^{\mathsf{t},\ell}_{\Delta}(\Gamma) \longrightarrow H^{\mathsf{n},\ell}_{\Delta}(\Gamma)\,, \tag{5.11b}$$

where thanks to Proposition 4.10 the side of Γ from which we take the traces does not matter. After evaluating

$$\mathsf{T}^{\mathsf{t}}_{\mathfrak{M},\ell}\mathsf{SL}^{\lambda}_{\ell}[\mathfrak{M}]\begin{bmatrix} h_{\ell-1}\\ h_{\ell}\end{bmatrix} = \begin{bmatrix} -\mathrm{d}^{\Gamma}_{\ell-2}\mathsf{t}_{\ell-2}\mathsf{S}^{\lambda}_{\ell-2}(\delta^{\Gamma}_{\ell-1}h_{\ell-1}) - \lambda\mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}(\delta^{\Gamma}_{\ell}h_{\ell})\\ \mathrm{d}^{\Gamma}_{\ell-1}\mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell}\mathsf{S}^{\lambda}_{\ell}h_{\ell}\end{bmatrix},$$

we find that

$$\left\langle\!\!\left\langle \mathsf{V}^{\lambda}_{\ell}[\mathfrak{M}] \begin{bmatrix} h_{\ell-1} \\ h_{\ell} \end{bmatrix}, \begin{bmatrix} \overline{w}_{\ell-1} \\ \overline{w}_{\ell} \end{bmatrix} \right\rangle\!\!\right\rangle_{\Gamma} \tag{5.12}$$

$$= -\left(\delta^{\Gamma}_{\ell-1} h_{\ell-1}, \delta^{\Gamma}_{\ell-1} w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} - \lambda\,(h_{\ell-1}, w_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{t}} + \left(\delta^{\Gamma}_{\ell} h_{\ell}, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + \left(h_{\ell-1}, \delta^{\Gamma}_{\ell} w_{\ell}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + (h_{\ell}, w_{\ell})_{-\frac{1}{2},\lambda,\mathsf{t}}\ .$$

Similarly, by using the definitions

$$\begin{aligned}
\mathsf{T}^{\mathsf{n}}_{\mathfrak{M},\ell}\mathsf{DL}^{\lambda}_{\ell}[\mathfrak{M}] \begin{bmatrix} g_{\ell-1} \\ g_{\ell} \end{bmatrix} &= \mathsf{T}^{\mathsf{n}}_{\mathfrak{M},\ell} \begin{bmatrix} \delta_{\ell}\mathsf{D}^{\lambda}_{\ell} g_{\ell-1} \\ \mathsf{D}^{\lambda}_{\ell} g_{\ell-1} + \delta_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1} g_{\ell} \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{n}_{\ell}\mathsf{D}^{\lambda}_{\ell} g_{\ell-1} + \mathsf{n}_{\ell}\delta_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1} g_{\ell} \\ \mathsf{n}_{\ell+1}\mathrm{d}_{\ell}\mathsf{D}^{\lambda}_{\ell} g_{\ell-1} + \mathsf{n}_{\ell+1}\mathrm{d}_{\ell}\delta_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1} g_{\ell} \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{n}_{\ell}\mathsf{D}^{\lambda}_{\ell} g_{\ell-1} - \delta^{\Gamma}_{\ell}\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1} g_{\ell} \\ -\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}(\mathrm{d}^{\Gamma}_{\ell-1} g_{\ell-1}) + \mathsf{n}_{\ell+1}\delta_{\ell+2}\mathsf{D}^{\lambda}_{\ell+2}(\mathrm{d}^{\Gamma}_{\ell} g_{\ell}) - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1} g_{\ell} \end{bmatrix} \\
&= \begin{bmatrix} \mathsf{n}_{\ell}\mathsf{D}^{\lambda}_{\ell} g_{\ell-1} - \delta^{\Gamma}_{\ell}\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1} g_{\ell} \\ -\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}(\mathrm{d}^{\Gamma}_{\ell-1} g_{\ell-1}) - \delta^{\Gamma}_{\ell+1}\mathsf{n}_{\ell+2}\mathsf{D}^{\lambda}_{\ell+2}(\mathrm{d}^{\Gamma}_{\ell} g_{\ell}) - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1} g_{\ell} \end{bmatrix}
\end{aligned}$$

leads to

$$\left\langle\!\!\left\langle \mathsf{W}^{\lambda}_{\ell}[\mathfrak{M}] \begin{bmatrix} g_{\ell-1} \\ g_{\ell} \end{bmatrix}, \begin{bmatrix} \overline{v}_{\ell-1} \\ \overline{v}_{\ell} \end{bmatrix} \right\rangle\!\!\right\rangle_{\Gamma} = (g_{\ell-1}, v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(g_{\ell}, \mathrm{d}^{\Gamma}_{\ell-1} v_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(\mathrm{d}^{\Gamma}_{\ell-1} g_{\ell-1}, v_{\ell}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(\mathrm{d}^{\Gamma}_{\ell} g_{\ell}, \mathrm{d}^{\Gamma}_{\ell} v_{\ell}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \lambda\,(g_{\ell}, v_{\ell})_{-\frac{1}{2},\lambda,\mathsf{n}}\ . \tag{5.13}$$

Again, there is a striking correspondence between the sesqui-linear forms induced by $\mathsf{V}^{\lambda}_{\ell}[\mathfrak{M}]$ and $\mathsf{W}^{\lambda}_{\ell}[\mathfrak{M}]$ and the bilinear forms (3.16) and (3.11) providing a weak incarnation of the mixed-order Hodge Laplacians for the de Rham complex. As before, our observations can be summarized as follows.

THEOREM 5.4 ($\mathsf{V}_\ell^\lambda[\mathfrak{M}]$ and $\mathsf{W}_\ell^\lambda[\mathfrak{M}]$ are weak mixed-order regularized Hodge-Laplace operators in the trace De Rham complex). *From a variational point of view, the* first-kind *BIOs defined in* (5.11a) *and* (5.11b) *for the regularized mixed-order Hodge–Laplace operator* $\mathfrak{M}_\ell^\lambda$ *are mixed-order regularized Hodge–Laplace operators*

$$\mathsf{V}_\ell^\lambda[\mathfrak{M}] = \begin{bmatrix} -(\delta_{\ell-1}^\Gamma)^*\delta_{\ell-1}^\Gamma - \lambda\mathrm{Id} & \delta_\ell^\Gamma \\ (\delta_\ell^\Gamma)^* & \mathrm{Id} \end{bmatrix}, \tag{5.14a}$$

$$\mathsf{W}_\ell^\lambda[\mathfrak{M}] = \begin{bmatrix} \mathrm{Id} & -(\mathrm{d}_{\ell-1}^\Gamma)^* \\ -\mathrm{d}_{\ell-1}^\Gamma & -(\mathrm{d}_\ell^\Gamma)^*\mathrm{d}_\ell^\Gamma - \lambda\mathrm{Id} \end{bmatrix}. \tag{5.14b}$$

in the trace de Rham complexes (2.4a) *and* (2.4b) *whose spaces are equipped with the non-local inner products* (4.21a) *and* (4.21b).

By combining Theorem 5.3 and Theorem 5.4 we find that in terms of first-kind BIOs it does not matter whether we consider the second-order or mixed-order form of the Hodge-Laplacians.

COROLLARY 5.5. *The first-kind BIOs spawned by the Hodge-Laplacians in second-order form and those in mixed-order form agree*

$$\mathsf{V}_\ell^\lambda[\Delta] = \mathsf{V}_\ell^\lambda[\mathfrak{W}] \quad \textit{and} \quad \mathsf{W}_\ell^\lambda[\Delta] = \mathsf{W}_\ell^\lambda[\mathfrak{W}]\,.$$

6. Conclusion. The message of Theorem 5.2, Theorem 5.3 and Theorem 5.4 is that from a variational perspective the reduction to the boundary of boundary value problems for (regularized) Hodge-Dirac and Hodge-Laplace operators in the De Rham complex yields "boundary value problems" of the same type now set in the trace De Rham complex. In fact, these results unify several "integration by parts" formulas for first kind BIOs in variational form scattered across literature, e.g., [20, Thm. 3.3.22], [6, Equs. (64) and (71)], [12, Lemma 11, (83d)].

Variational problems linked to boundary value problems for (regularized) Hodge-Dirac and Hodge-Laplace operators in the De Rham complex lend themselves to analysis solely drawing on the fact that the de Rham complex is a Hilbert complex enjoying the Fredholm property, see [2, Section 2] and [13, Section 2.2]. This Fredholm property also holds true for the trace de Rahm Hilbert complex. Therefore, from the results of this article we can immediately conclude existence, uniqueness, and stability of solution of first-kind boundary integral equations induced by the (regularized) Hodge-Dirac and Hodge-Laplace operators.

Moreover, also the techniques used in the numerical analysis of the Galerkin finite-element discretization of (regularized) Hodge-Dirac and Hodge-Laplace operators by means of discrete differential forms [2, Section 7], [13, Section 3] remain relevant for understanding boundary element Galerkin methods for the corresponding first-kind boundary integral equations. Thus, the results of [7] just demonstrate a special case and the techniques can be generalized.

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