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E. Schulz and R. Hiptmair and S. Kurz

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Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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BOUNDARY INTEGRAL EXTERIOR CALCULUS*

ERICK SCHULZ†, RALF HIPTMAIR† , AND STEFAN KURZ‡

Abstract. We develop first-kind boundary integral equations for Hodge–Dirac and Hodge–Laplace operators associated with de Rham Hilbert complexes on compact Riemannian manifolds and Euclidean space. We show that the first-kind boundary integral operators associated with Hodge–Dirac and Hodge–Laplace boundary value problems posed on submanifolds with Lipschitz boundaries are Hodge–Dirac and Hodge–Laplace operators as well, but associated with a trace de Rham Hilbert complex on the boundary whose spaces are equipped with non-local inner products defined through boundary potentials. The correspondence is to some extent structure-preserving in the sense that zero-order perturbations are also formally reproduced at the level of integral equations.

Key words. Hodge–Dirac operator, Hodge–Laplace operator, Hodge–Yukawa, representation formula, boundary integral operators, compact manifolds, structure-preserving

AMS subject classifications. 31A10, 45A05, 45E05, 45P05, 35F15, 34L40, 35Q61

1. Introduction. Let $\mathcal{M}$ be either a smooth orientable compact Riemannian N-dimensional manifold without boundary or Euclidean space $\mathbb{R}^N$, cf. [1, Chap. 3 and 7] and [19, Chap. 6]. Assume that $\Omega = \Omega^- \subset \mathcal{M}$ is a submanifold of the same dimension with a compatibly oriented Lipschitz boundary $\Gamma := \partial\Omega$, cf. [23, Chap. 3], [25, sect. 2], [28, App. 1], [39, sect. 1] and [41, sect. 1]. Define $\Omega^+ = M\backslash\operatorname{Int}\Omega$ and write $\imath_\Gamma^\mp : \Gamma \to \Omega^\mp$ for the inclusion maps. If $\mathcal{M} = \mathbb{R}^N$, we suppose for simplicity that Ω is bounded to avoid explicitly handling the necessary complications introduced by the need for decay or radiation conditions at infinity, cf. [11, sect. 3.3], [14], [23, Chap. 7] and [33, sect. 4.4].

1.1. Overview. Our goal is to understand the structural properties of *first-kind* boundary integral operators (BIOs) associated with boundary value problems (BVPs) for the Hodge–Dirac and Hodge–Laplace operators

$$\mathfrak{D} = \mathbf{d} + \boldsymbol{\delta} \qquad \text{and} \qquad -\Delta_\ell = \mathrm{d}_{\ell-1}\delta_\ell + \delta_{\ell+1}\mathrm{d}_\ell, \qquad 0 \leq \ell \leq N,$$

cf. [11, 13, 23, 32] and [33]. We will find that the obtained first-kind BIOs are Hodge–Dirac and Hodge–Laplace operators themselves, but associated with trace de Rham complexes whose spaces are equipped with non-local inner products defined through boundary potentials. This discovery promotes the adequacy of the trace de Rham complex to study related BVPs in general.

An unavoidable difficulty arise at the outset of our program. In Euclidean space, both the Hodge–Dirac operator and the Hodge–Laplacian admit two-sided inverses in the sense of distributions when suitable decay conditions are imposed. However, there are topological obstructions on compact manifolds that prevent the existence of fundamental solutions. To recover the crucial feature, we regularize the Hodge–Dirac and Hodge–Laplace operators by adding zero-order terms when they have non-trivial kernels on $\mathcal{M}$. Our main intent being to display the structure-preserving power of first-kind boundary integral equations (BIEs), it will be sufficient motivation to focus on the simplest type of perturbation.

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†Seminar in Applied Mathematics (SAM), Eidgenössische Technische Hochschule Zürich (ETHZ)
‡University of Jyväskylä, Faculty of Information Technology

In that regard, the simplest option is to work with modified Hodge–Dirac and Hodge–Yukawa operators of the form

$$\mathfrak{D} + i\kappa, \qquad \kappa \in \mathbb{R}\backslash\{0\}, \qquad\qquad \text{and} \qquad\qquad -\Delta_\ell + \lambda, \qquad \lambda > 0,$$

which are related by the identity

$$(\mathfrak{D} - i\kappa)\,(\mathfrak{D} + i\kappa) = -\boldsymbol{\Delta} + \kappa^2. \tag{1.1}$$

1.2. Related work. We draw on a previous article by Schulz and Hiptmair in which the correspondence between domain and boundary Hodge–Dirac operators was initially discovered [33]. Inspired by [11] and [12], where first-kind boundary integral equations for Hodge–Helmholtz operators were studied, only three-dimensional Euclidean space $\mathcal{M} = \mathbb{R}^3$ is studied in [33]. The investigation was solely based on classical vector calculus. The idea was to emphasize that although the Hodge–Dirac operator is only first-order, there is a close formal relationship between our arguments and the well-known theory of first-kind boundary integral equations for second-order elliptic operators in Euclidean space. Our goal now is to generalize these results to arbitrary dimensions by translating [11] and [33] into the language of differential forms. In doing so, the theory naturally extends to Riemannian manifolds and hidden structures behind the integral equations are revealed.

We owe to a rich literature on boundary integral equations formulated in the framework of Grassman algebras. Most notably, D. Mitrea, I. Mitrea, M. Mitrea and Taylor extensively studied *second-kind* boundary integral equations related to the Hodge–Laplacian on compact manifolds [26, 28]. Auchmann and Kurz also used exterior algebra to study boundary integral equations for Maxwell-type problems [20].

The important results of D. Mitrea, M. Mitrea, Shaw [27] and Weck [41] on the existence and properties of surjective trace operators for the relevant spaces of differential forms allow the development of boundary integral exterior calculus on boundaries of mere Lipschitz regularity. Abstract trace complexes are also studied in [18], where an alternative proof than that given in [27] is provided for the compactness property of the trace de Rham complex.

1.3. Exterior Calculus. Subspaces of the space of differential forms of order ℓ on $\mathcal{M}$ characterized by coefficient-based regularity properties will be denoted by $L^\infty\Lambda^\ell(\mathcal{M})$, $L^2\Lambda^\ell(\mathcal{M})$, $H^s\Lambda^\ell(\mathcal{M})$ and so forth, cf. [27] and [41]. Similar notation is used for submanifolds. Following [16, Chap. 3], we will shorthand $\mathcal{E}^\ell(\mathcal{M}) = C^\infty\Lambda^\ell(\mathcal{M})$ and $\mathcal{D}^\ell(\mathcal{M}) = C_0^\infty\Lambda^\ell(\mathcal{M})$ for spaces of test functions. Their topological duals will be written $\mathcal{D}'_\ell(\mathcal{M})$ and $\mathcal{E}'_\ell(\mathcal{M})$. Primes always refer to dual spaces or dual maps, e.g. $H^{-1}\Lambda^\ell(\Omega) = (H_0^1\Lambda^\ell(\Omega))'$. We write in a bold font, e.g. $\boldsymbol{U} = (U_\ell)_\ell$, the elements of full Grassman algebras such as $L^2\Lambda(\mathcal{M}) = \oplus_\ell L^2\Lambda^\ell(\mathcal{M})$. For convenience, we let $\ell \in \mathbb{Z}$ run over all integers, but identify forms of rank $\ell < 0$ and $\ell > n$ with zero.

The Hodge star $\star_\ell : L^2\Lambda^\ell(\mathcal{M}) \to L^2\Lambda^{N-\ell}(\mathcal{M})$ is induced by the Riemannian metric on $\mathcal{M}$. The symmetric pairing

$$\langle U_\ell, V_\ell\rangle_\Omega = \int_\Omega U_\ell \wedge \star_\ell V_\ell, \qquad \forall U_\ell, V_\ell \in L^2\Lambda^\ell(\Omega), \tag{1.2}$$

is distinguished from the Hermitian inner product $(U_\ell, V_\ell)_\Omega = \langle U_\ell, \overline{V_\ell}\rangle_\Omega$, where the overline indicates complex conjugation of the coefficients.

The codifferential $\delta_{\ell+1} = (-1)^{\ell+1} \star_\ell^{-1} \mathrm{d}_{N-\ell-1} \star_{\ell+1}$ is formally adjoint to the exterior derivative. We adopt the view that $\mathrm{d}_\ell : L^2\Lambda^\ell(\Omega) \to L^2\Lambda^{\ell+1}(\Omega)$ and $\delta_{\ell+1} : L^2\Lambda^{\ell+1}(\Omega) \to L^2\Lambda^\ell(\Omega)$ are the closed densely defined unbounded linear operators giving rise to the Fredholm Hilbert cochain and chain complexes

$$\text{(1.3a)} \qquad \dots \xrightarrow{\mathrm{d}_{\ell-1}} H\Lambda^\ell(\mathrm{d},\Omega) \xrightarrow{\mathrm{d}_\ell} H\Lambda^{\ell+1}(\mathrm{d},\Omega) \xrightarrow{\mathrm{d}_{\ell+1}} \dots$$

and

$$\text{(1.3b)} \qquad \dots \xleftarrow[\delta_{\ell-1}]{} H\Lambda^{\ell-1}(\delta,\Omega) \xleftarrow[\delta_\ell]{} H\Lambda^\ell(\delta,\Omega) \xleftarrow[\delta_{\ell+1}]{} \dots$$

satisfying the compactness property, cf. [2, Chap. 4 and 6], [4], [25] and [31].

The corresponding diffuse Fredholm–nilpotent operators $\mathbf{d} : L^2\Lambda(\Omega) \to L^2\Lambda(\Omega)$ and $\boldsymbol{\delta} : L^2\Lambda(\Omega) \to L^2\Lambda(\Omega)$ are formally adjoint under the Hermitian inner product $(\boldsymbol{U},\boldsymbol{V})_\Omega = \langle \boldsymbol{U}, \overline{\boldsymbol{V}}\rangle_\Omega$ defined through the symmetric pairing

$$\langle \boldsymbol{U},\boldsymbol{V}\rangle_\Omega = \sum_\ell \langle U_\ell, V_\ell\rangle_\Omega, \qquad \forall \boldsymbol{U},\boldsymbol{V} \in L^2\Lambda(\Omega),$$

cf. [21, sect. 3 and 5] and [5, sect. 3]. As operator matrices acting on vectors of differential forms of the form $\boldsymbol{U} = (U_0,...,U_N)^\top$, the *full* exterior derivative and codifferential read

$$\text{(1.4)} \qquad \mathbf{d} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \mathrm{d}_0 & 0 & 0 & \dots & 0 \\ 0 & \mathrm{d}_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \mathrm{d}_{N-1} & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\delta} = \begin{pmatrix} 0 & \delta_1 & 0 & \dots & 0 \\ 0 & 0 & \delta_2 & \ddots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \delta_N \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Similarly, the full Hodge star is represented by

$$\text{(1.5)} \qquad \star = \begin{pmatrix} & 0 & & \star_0 \\ & & \star_1 & \\ & & \iddots & \\ & \star_{N-1} & & 0 \\ \star_N & & & \end{pmatrix}.$$

We find it convenient to write the duality pairings that extend symmetric L^2-type pairings of the form (1.2) using double angular brackets, e.g. $\langle\!\langle \cdot,\cdot \rangle\!\rangle_\mathcal{M}$.

For later use, we define the Sobolev spaces of ℓ-forms

$$\begin{aligned} H\Lambda^\ell(\mathrm{d}\delta,\Omega) &= \left\{ U_\ell \in H\Lambda^\ell(\delta,\Omega) \;\middle|\; \delta_\ell U_\ell \in H\Lambda^{\ell-1}(\mathrm{d},\Omega) \right\}, \\ H\Lambda^\ell(\delta\mathrm{d},\Omega) &= \left\{ U_\ell \in H\Lambda^\ell(\mathrm{d},\Omega) \;\middle|\; \mathrm{d}_\ell U_\ell \in H\Lambda^{\ell+1}(\delta,\Omega) \right\}, \\ H\Lambda^\ell(\Delta,\Omega) &= H\Lambda^\ell(\mathrm{d}\delta,\Omega) \cap H\Lambda^\ell(\delta\mathrm{d},\Omega), \\ X\Lambda^\ell(\Omega) &= H\Lambda^\ell(\mathrm{d},\Omega) \cap H\Lambda^\ell(\delta,\Omega), \end{aligned}$$

equipped with graph inner products.

Because $\oplus_\ell X\Lambda^\ell(\Omega)$ will be the domain of the Hodge–Dirac operator, we introduce the notation

$$H\Lambda(\mathfrak{D},\Omega) = H\Lambda(\mathbf{d},\Omega)\cap H\Lambda(\boldsymbol{\delta},\Omega)$$

for that space of full forms.

1.4. Trace spaces. We will impose boundary conditions via trace operators. We briefly review their definition and mapping properties, cf. [9, 18, 27, 41]. In accordance with standard practice, we repurpose the notation from Subsection 1.3 for operators on the boundary, but point out that the indices must account for the change in dimension when passing to a submanifold. In particular, notice that the Hodge star associated with the induced metric on the boundary is a continuous mapping $\star_\ell : L^2\Lambda^\ell(\Gamma) \to L^2\Lambda^{N-\ell-1}(\Gamma)$, cf. [27, 41].

1.4.1. Traces of differential forms. Relevant traces for $H_{\text{loc}}\Lambda^\ell(\mathrm{d},\Omega^\mp)$ and $H_{\text{loc}}\Lambda^\ell(\delta,\Omega^\mp)$ are obtained by extending the pullback and "rotated" pullback of differential forms, also called tangential and normal traces. They are defined for all smooth forms $U_\ell \in \mathcal{D}^\ell(\mathcal{M})$ by

$$\mathsf{t}^\mp_\ell U_\ell = \imath^*_\mp U_\ell \qquad \text{and} \qquad \mathsf{n}^\mp_\ell U_\ell = \star^{-1}_{\ell-1}\,\imath^*_\mp\,\star_\ell\, U_\ell. \tag{1.6}$$

Adopting the notation of [20], we define the dual spaces

$$H^{-\frac{1}{2}}_{\parallel}\Lambda^\ell(\Gamma) := (H^{\frac{1}{2}}_{\parallel}\Lambda^\ell(\Gamma))' \qquad \text{and} \qquad H^{-\frac{1}{2}}_{\perp}\Lambda^\ell(\Gamma) := (H^{\frac{1}{2}}_{\perp}\Lambda^\ell(\Gamma))',$$

where the regular trace spaces are given by

$$H^{\frac{1}{2}}_{\parallel}\Lambda^\ell(\Gamma) := \mathsf{t}^\mp_\ell H^1\Lambda^\ell(\Omega^\mp) \qquad \text{and} \qquad H^{\frac{1}{2}}_{\perp}\Lambda^\ell(\Gamma) = \mathsf{n}^\mp_\ell H^1\Lambda^{\ell+1}(\Omega^\mp).$$

They generalize the well-known space of Dirichlet traces $H^{\frac{1}{2}}\Lambda^0(\Gamma)$.

On the boundary, we view the exterior derivative and the codifferential as the closed densely defined unbounded linear operators $\mathrm{d}_\ell : H^{-\frac{1}{2}}_{\perp}\Lambda^\ell(\Gamma) \longrightarrow H^{-\frac{1}{2}}_{\perp}\Lambda^{\ell+1}(\Gamma)$ and $\delta_\ell : H^{-\frac{1}{2}}_{\parallel}\Lambda^\ell(\Gamma) \longrightarrow H^{-\frac{1}{2}}_{\parallel}\Lambda^{\ell-1}(\Gamma)$ giving rise to the Fredholm Hilbert complexes

$$\ldots \xrightarrow{\mathrm{d}_{\ell-1}} H^{-\frac{1}{2}}_{\perp}\Lambda^{\ell-1}(\mathrm{d},\Gamma) \xrightarrow{\mathrm{d}_\ell} H^{-\frac{1}{2}}_{\perp}\Lambda^\ell(\mathrm{d},\Gamma) \xrightarrow{\mathrm{d}_{\ell+1}} \ldots \tag{1.7a}$$

and

$$\ldots \xleftarrow[\delta_{\ell-1}]{} H^{-\frac{1}{2}}_{\parallel}\Lambda^{\ell-1}(\delta,\Gamma) \xleftarrow[\delta_\ell]{} H^{-\frac{1}{2}}_{\parallel}\Lambda^\ell(\delta,\Gamma) \xleftarrow[\delta_{\ell+1}]{} \ldots \tag{1.7b}$$

associated with the domain complexes (1.3a) and (1.3b), cf. [18, 27, 41].

It is the content of the trace theorems studied in [9, 18, 27, 41] that the operators

$$\mathsf{t}^\mp_\ell : H^1\Lambda^\ell_{\text{loc}}(\mathcal{M}) \longrightarrow H^{\frac{1}{2}}_{\parallel}\Lambda^\ell(\Gamma) \quad \text{and} \quad \mathsf{n}^\mp_\ell : H^1\Lambda^\ell_{\text{loc}}(\mathcal{M}) \longrightarrow H^{\frac{1}{2}}_{\perp}\Lambda^{\ell-1}(\Gamma) \tag{1.8}$$

extend to continuous and *surjective* mappings

$$\begin{aligned} \mathsf{t}^\mp_\ell &: H_{\text{loc}}\Lambda^\ell(\mathrm{d},\Omega^\mp) \longrightarrow H^{-\frac{1}{2}}_{\perp}\Lambda^\ell(\mathrm{d},\Gamma) \\ \mathsf{n}^\mp_\ell &: H_{\text{loc}}\Lambda^\ell(\delta,\Omega^\mp) \longrightarrow H^{-\frac{1}{2}}_{\parallel}\Lambda^{\ell-1}(\delta,\Gamma) \end{aligned} \tag{1.9}$$

such that the integration by parts formula

$$\langle \mathrm{d}_\ell\, U_\ell, V_{\ell+1}\rangle_{\Omega^\mp} = \langle U_\ell, \delta_{\ell+1}\, V_{\ell+1}\rangle_{\Omega^\mp} \pm \langle\!\langle \mathsf{t}_\ell^\mp U_\ell, \mathsf{n}_{\ell+1}^\mp V_{\ell+1}\rangle\!\rangle_\Gamma \tag{1.10}$$

holds for all $U_\ell \in H\Lambda^\ell(\mathrm{d},\Omega^\mp)$ and $V_{\ell+1} \in H\Lambda^{\ell+1}(\delta,\Omega^\mp)$.

On the right-hand side of (1.10), the duality pairing on the boundary extends the $L^2\Lambda^\ell(\Gamma)$-pairing. That is, it puts $H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$ in duality with $H_\parallel^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma)$ using $L^2\Lambda^\ell(\Gamma)$ as a pivot space.

In a similar notation to [2, Thm. 6.5],

$$\begin{aligned}
\mathring{H}\Lambda^\ell(\mathrm{d},\Omega) &= \overline{\mathcal{D}(\Omega)}^{H\Lambda^\ell(\mathrm{d},\Omega)} = \ker \mathsf{t}_\ell \cap H\Lambda^\ell(\mathrm{d},\Omega)\\
\mathring{H}\Lambda^\ell(\delta,\Omega) &= \overline{\mathcal{D}(\Omega)}^{H\Lambda^\ell(\delta,\Omega)} = \ker \mathsf{n}_\ell \cap H\Lambda^\ell(\delta,\Omega).
\end{aligned}$$

Despite Γ being merely Lipschitz regular, the usual commutative relations

$$\mathsf{t}_\ell^\mp \circ \mathrm{d}_\ell = \mathrm{d}_\ell \circ \mathsf{t}_\ell^\mp \qquad \text{and} \qquad \mathsf{n}_{\ell-1}^\mp \circ \delta_\ell = -\delta_{\ell-1} \circ \mathsf{n}_\ell^\mp, \tag{1.11}$$

also hold at the level of trace spaces. In particular, the second identity can be obtained from the first:

$$\begin{aligned}
\mathsf{n}_{\ell-1}\delta_\ell &= \star_{\ell-2}^{-1}\imath^* \star_{\ell-1} \left((-1)^\ell \star_{\ell-1}^{-1} \mathrm{d}_{N-\ell}\star_\ell\right)\\
&= -(-1)^{\ell-1} \star_{\ell-2} \mathrm{d}_{N-\ell}\, \imath^*\star_\ell\\
&= -\left((-1)^{\ell-1} \star_{\ell-2} \mathrm{d}_{N-\ell}\star_{\ell-1}\right) \star_{\ell-1}^{-1} \imath^*\star_\ell = -\delta_{\ell-1} \circ \mathsf{n}_\ell^\mp.
\end{aligned}$$

We use a bold font to denote traces acting on the full algebra of forms, i.e.

$$\mathbf{t}^\mp \boldsymbol{U} = \boldsymbol{\imath}_\mp^* \boldsymbol{U} \qquad \text{and} \qquad \mathbf{n}^\mp \boldsymbol{V} = \star^{-1}\, \mathbf{t}_\mp \star \boldsymbol{V}. \tag{1.12}$$

Then, applying the integration by parts formula (1.10) order-wise yields

$$\langle \mathbf{d}\, \boldsymbol{U}, \boldsymbol{V}\rangle_{\Omega^\mp} = \langle \boldsymbol{U}, \boldsymbol{\delta}\, \boldsymbol{V}\rangle_{\Omega^\mp} \pm \langle\!\langle \mathbf{t}^\mp \boldsymbol{U}, \mathbf{n}^\mp \boldsymbol{V}\rangle\!\rangle_\Gamma \tag{1.13}$$

for all $\boldsymbol{U} \in H\Lambda(\mathbf{d},\Omega)$ and $\boldsymbol{V} \in H\Lambda(\boldsymbol{\delta},\Omega)$.

1.4.2. Lifting maps. The purpose of this section is twofold. Firstly, it is immediate by surjectivity that the traces in (1.9) admit continuous right inverses into $H\Lambda^\ell(\mathrm{d},\Omega)$ and $H\Lambda^\ell(\delta,\Omega)$, respectively. We want to show in particular that these right inverses can be designed to lift the boundary data into the more regular space $H\Lambda^\ell(\Delta,\Omega)$. Secondly, we also build lifting maps for the continuous traces

$$\begin{aligned}
\mathsf{t}_{\ell-1}^\mp \circ \delta_\ell &: H\Lambda^\ell(\mathrm{d}\delta,\Omega) \to H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma),\\
\mathsf{n}_{\ell+1}^\mp \circ \mathrm{d}_\ell &: H\Lambda^\ell(\delta\mathrm{d},\Omega) \to H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma),
\end{aligned}$$

that will be used to impose boundary conditions for the Hodge–Laplacian.

With the next two lemmas, we generalize to differential forms the results of [11, Sec. 2.5].

Lemma 1.1. *There exist continuous operators* $\mathcal{E}_\ell^{\mathsf{t}} : H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *and* $\mathcal{E}_\ell^{\mathsf{n}} : H_\parallel^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *such that*

$$\mathsf{t}_\ell\, \mathcal{E}_\ell^{\mathsf{t}}\, g_\ell = g_\ell \qquad \textit{and} \qquad \mathsf{n}_\ell\, \mathcal{E}_\ell^{\mathsf{n}}\, h_{\ell-1} = h_{\ell-1}$$

for all $g_\ell \in H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$ *and* $h_{\ell-1} \in H_\parallel^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma)$.

Proof. Given $g_\ell \in H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$, let $\mathcal{E}_\ell^{\mathsf{t}}(g_\ell)$ be the unique element in $H\Lambda(\mathrm{d},\Omega)$ defined by

$$\mathcal{E}_\ell^{\mathsf{t}}(g_\ell) = \underset{\substack{V_\ell \in H\Lambda^\ell(\mathrm{d},\Omega), \\ \mathsf{t}_\ell V_\ell = g_\ell}}{\arg\min} \|V_\ell\|_{H\Lambda^\ell(\mathrm{d},\Omega)} .$$

This minimization problem is equivalent to satisfying the Euler equations

$$\langle \mathrm{d}_\ell \, \mathcal{E}_\ell^{\mathsf{t}}(g_\ell), \mathrm{d}_\ell \, V_\ell \rangle_\Omega + \langle \mathcal{E}_\ell^{\mathsf{t}}(g_\ell), V_\ell \rangle_\Omega = 0$$

for all $V_\ell \in \overset{\circ}{H}\Lambda^\ell(\mathrm{d},\Omega)$. Testing with suitable choices of test functions shows that $\mathcal{E}_\ell^{\mathsf{t}} : H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma) \to H\Lambda^\ell(\mathrm{d},\Omega)$ is a continuous operator satisfying the equations

$$\delta_{\ell+1}\mathrm{d}_\ell \, \mathcal{E}_\ell^{\mathsf{t}} g_\ell + \mathcal{E}_\ell^{\mathsf{t}} g_\ell = 0 \qquad \text{in } \Omega,, \tag{1.14a}$$

$$\mathsf{t}_\ell \mathcal{E}_\ell^{\mathsf{t}} g_\ell = g_\ell \qquad \text{on } \Gamma. \tag{1.14b}$$

In particular, (1.14a) not only reveals that $\mathcal{E}_\ell^{\mathsf{t}} g_\ell \in H\Lambda^\ell(\delta\mathrm{d},\Omega)$, but also that $\delta_\ell \, \mathcal{E}_\ell^{\mathsf{t}} g_\ell = 0$ in Ω. We conclude that $\mathcal{E}_\ell^{\mathsf{t}}$ exhibits the claimed regularity and (1.14b) confirms that the defined map is a right-inverse for the tangential trace.

The map $\mathcal{E}_\ell^{\mathsf{n}}$ can be defined similarly using a minimization problem involving the normal trace. ∎

LEMMA 1.2. *There exist continuous operators* $\mathcal{R}_\ell^{\mathsf{t}} : H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *and* $\mathcal{R}_\ell^{\mathsf{n}} : H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *such that*

$$\mathsf{t}_{\ell-1}\delta_\ell \mathcal{R}_\ell^{\mathsf{t}} \, g_{\ell-1} = g_{\ell-1} \qquad \textit{and} \qquad \mathsf{n}_{\ell+1}\mathrm{d}_\ell \mathcal{R}_\ell^{\mathsf{n}} \, h_\ell = h_\ell$$

for all $g_{\ell-1} \in H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma)$ *and* $h_\ell \in H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma)$.

Proof. Given boundary data $h_\ell \in H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma)$, we define $\mathcal{R}_\ell^{\mathsf{n}}(h_\ell)$ as the unique element of $H\Lambda^\ell(\mathrm{d},\Omega)$ such that

$$\langle \mathrm{d}_\ell \, \mathcal{R}_\ell^{\mathsf{n}}(h_\ell), \mathrm{d}_\ell \, V_\ell \rangle_\Omega + \langle \mathcal{R}_\ell^{\mathsf{t}}(h_\ell), V_\ell \rangle_\Omega = \langle\!\langle h_\ell, \mathsf{t}_\ell V_\ell \rangle\!\rangle_\Gamma$$

for all $V_\ell \in H\Lambda^\ell(\mathrm{d},\Omega)$. Lax-Milgram lemma guarantees that $\mathcal{R}_\ell^{\mathsf{n}}$ is well-defined and continuous as a map $\mathcal{R}_\ell^{\mathsf{n}} : H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma) \to H\Lambda^\ell(\mathrm{d},\Omega)$.

Routine verification using suitable test functions and the integration by parts formula (1.10) shows that it satisfies the equations

$$\delta_{\ell+1}\mathrm{d}_\ell \, \mathcal{R}_\ell^{\mathsf{n}}(h_\ell) + \mathcal{R}_\ell^{\mathsf{n}}(h_\ell) = 0 \qquad \text{in } \Omega, \tag{1.15a}$$

$$\mathsf{n}_{\ell+1}\mathrm{d}_\ell \mathcal{R}_\ell^{\mathsf{n}} \, h_\ell = h_\ell \qquad \text{on } \Gamma. \tag{1.15b}$$

Similarly as in the proof of Lemma 1.1, we obtain from (1.15a) that $\mathcal{R}_\ell^{\mathsf{n}}(h_\ell) \in H\Lambda^\ell(\delta\mathrm{d},\Omega)$ and $\delta_\ell \mathcal{R}_\ell^{\mathsf{t}} h_\ell = 0$ in Ω, i.e. $\mathcal{R}_\ell^{\mathsf{t}} h_\ell \in H\Lambda^\ell(\Delta,\Omega)$. Then, (1.15b) confirms that $\mathcal{R}_\ell^{\mathsf{t}}$ is a right-inverse for the trace $\mathsf{n}_{\ell+1}\mathrm{d}_\ell$.

The analogous result for $\mathsf{t}_{\ell-1}\delta_\ell$ is obtained similarly by defining $\mathcal{R}_\ell^{\mathsf{t}}(h_\ell)$ using the graph inner product on $H\Lambda^\ell(\delta,\Omega)$. ∎

Before moving on, we want to verify that the lifting operators from Lemma 1.1 and Lemma 1.2 can be used to construct right-inverses for the compound traces of the Hodge–Laplacian.

Recalling Subsection 1.4, the traces defined for all $U_\ell \in \mathcal{D}^\ell(\mathcal{M})$ by

$$\mathsf{T}^{\mathsf{t}}_\Delta U_\ell = \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_\ell U_\ell \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \qquad \text{and} \qquad \mathsf{T}^{\mathsf{n}}_\Delta U_\ell = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell \end{pmatrix} \tag{1.16}$$

are continuous as mappings

$$\mathsf{T}^{\mathsf{t}}_\Delta : H\Lambda^\ell(\mathrm{d}\delta,\Omega)\cap H\Lambda^\ell(\mathrm{d},\Omega) \longrightarrow H^{\mathsf{t}}_\Delta(\Gamma),$$
$$\mathsf{T}^{\mathsf{n}}_\Delta : H\Lambda^\ell(\delta,\Omega)\cap H\Lambda^\ell(\delta\mathrm{d},\Omega) \longrightarrow H^{\mathsf{n}}_\Delta(\Gamma),$$

where the product of trace spaces are given by

$$H^{\mathsf{t}}_\Delta(\Gamma) = H_\perp^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma)\times H_\perp^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma),$$
$$H^{\mathsf{n}}_\Delta(\Gamma) = H_\parallel^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma)\times H_\parallel^{-\frac{1}{2}}\Lambda^{\ell}(\delta,\Gamma).$$

We want to show that their restriction to $H\Lambda^\ell(\Delta,\Omega)$ are surjective and admit continuous lifting operators.

LEMMA 1.3. *There exist continuous operators* $\mathcal{L}^{\mathsf{t}}_\ell : H^{\mathsf{t}}_\Delta(\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *and* $\mathcal{L}^{\mathsf{n}}_\ell : H^{\mathsf{n}}_\Delta(\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$ *such that*

$$\mathsf{T}^{\mathsf{t}}_\Delta \mathcal{L}^{\mathsf{t}} \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \qquad \textit{and} \qquad \mathsf{T}^{\mathsf{n}}_\Delta \mathcal{L}^{\mathsf{n}}_\ell \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix}$$

for all $(g_{\ell-1}, g_\ell)^\top \in H^{\mathsf{t}}_\Delta(\Gamma)$ *and* $(h_{\ell-1}, h_\ell)^\top \in H^{\mathsf{n}}_\Delta(\Gamma)$.

Proof. We prove the result for $\mathsf{T}^{\mathsf{n}}_\Delta$. The proof is similar for $\mathsf{T}^{\mathsf{t}}_\Delta$. The trick is to define the lifting for all $(h_{\ell-1}, h_\ell)^\top \in H^{\mathsf{n}}_\Delta(\Gamma)$ by

$$\mathcal{L}^{\mathsf{n}}_\ell \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \mathcal{E}^{\mathsf{n}}_\ell h_{\ell-1} + \mathcal{R}^{\mathsf{n}}_\ell h_\ell - \mathrm{d}_{\ell-1}\mathcal{R}^{\mathsf{n}}_{\ell-1}(\mathsf{n}_\ell \mathcal{R}^{\mathsf{n}}_\ell h_\ell).$$

The first two terms are immediately seen to belong in $H\Lambda^\ell(\Delta,\Omega)$ thanks to the mapping results of Lemma 1.1 and Lemma 1.2. To confirm that the third term also displays the same regularity, we dig deeper into the proof of Lemma 1.2 and simply recall (1.15a).

By construction, $\mathrm{d}_\ell \circ \mathcal{E}^{\mathsf{n}}_\ell = 0$. Indeed, the analogous result for the tangential trace in the proof of Lemma 1.1 was that $\delta_\ell \circ \mathcal{E}^{\mathsf{t}} = 0$. Hence, using the established properties of the lifting operators and the fact that $\mathrm{d}^2 = 0$, we compute

$$\begin{aligned}\mathsf{T}\mathcal{L}^{\mathsf{n}}_\ell \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} &= \begin{pmatrix} \mathsf{n}_\ell\mathcal{E}^{\mathsf{n}}_\ell h_{\ell-1} + \mathsf{n}_\ell\mathcal{R}^{\mathsf{n}}_\ell h_\ell - \mathsf{n}_\ell\mathrm{d}_{\ell-1}\mathcal{R}^{\mathsf{n}}_{\ell-1}(\mathsf{n}_\ell\mathcal{R}^{\mathsf{n}}_\ell h_\ell) \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{E}^{\mathsf{n}}_\ell h_{\ell-1} + \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{R}^{\mathsf{n}}_\ell h_\ell - \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathrm{d}_{\ell-1}\mathcal{R}^{\mathsf{n}}_{\ell-1}(\mathsf{n}_\ell\mathcal{R}^{\mathsf{n}}_\ell h_\ell) \end{pmatrix} \\ &= \begin{pmatrix} h_{\ell-1} + \mathsf{n}_\ell\mathcal{R}^{\mathsf{n}}_\ell h_\ell - \mathsf{n}_\ell\mathcal{R}^{\mathsf{n}}_\ell h_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell\mathcal{R}^{\mathsf{n}}_\ell h_\ell \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix},\end{aligned}$$

which shows that $\mathcal{L}^{\mathsf{n}}_\ell$ is a right-inverse for T^{n}_Δ. ∎

2. Boundary value problems. In this section, we formulate the BVPs of interest in this article. We begin with the Hodge–Dirac operator before moving on to the Hodge–Laplacian. In Subsection 2.1.1, readers might notice that because the Hodge star operators

$$\star_\ell : X\Lambda^\ell(\Omega) \to X\Lambda^{N-\ell}(\Omega) \qquad \text{and} \qquad \star_\ell : H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma) \to H_\parallel^{-\frac{1}{2}}\Lambda^{N-1-\ell}(\delta,\Gamma)$$

are isometric isomorphisms [41, Lem. 5], the two boundary value problems stated for the Hodge–Dirac operator are, at the abstract level of Hilbert complexes, equivalent in terms of solvability. One corresponds to the Hodge–Dirac operator associated with the cochain complex (1.3a), while the other corresponds to the Hodge–Dirac operator associated with the chain complex (1.3b). Each of these problems can be turned into the other, cf. [11, Rmk. 3.3] and [33, Rmk. 5.1]. A similar observation can be made for the two BVPs involving the Hodge–Laplacian that will be presented in Subsection 2.2.1. The reason we insist on formulating each of them explicitly and independently is to highlight the formal difference in the expressions of the self-adjoint operators behind them. It turns out that it is those expressions that we will recognize in the formulas of the associated first-kind BIOs.

2.1. Hodge–Dirac operators. We take $H\Lambda(\mathfrak{D},\Omega)$ to be the domain of the Hodge–Dirac operator

$$\mathfrak{D} = \boldsymbol{\delta} + \mathbf{d} : H\Lambda(\mathfrak{D},\Omega) \to L^2\Lambda(\Omega) \tag{2.1}$$

on which we want to impose boundary conditions.

2.1.1. BVPs for Hodge–Dirac operators. In light of Subsection 1.4, the duality between the trace spaces $H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma)$ and $H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma)$ involved in the integration by parts formula (1.13) points towards two types of boundary conditions. For $\kappa \in \mathbb{R}$, we consider the BVPs

$$\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega) : \begin{cases} (\mathfrak{D} + i\kappa)\,\boldsymbol{U} = \mathbf{0} & \text{in } \Omega \\ \mathbf{t}\,\boldsymbol{U} = \boldsymbol{g} & \text{on } \partial\Omega \end{cases}, \qquad \boldsymbol{g} \in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma), \tag{2.2a}$$

and

$$\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega) : \begin{cases} (\mathfrak{D} + i\kappa)\,\boldsymbol{U} = \mathbf{0} & \text{in } \Omega \\ \mathbf{n}\,\boldsymbol{U} = \boldsymbol{h} & \text{on } \partial\Omega \end{cases}, \qquad \boldsymbol{h} \in H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma). \tag{2.2b}$$

The self-adjoint operators underlying (2.2a) and (2.2b) are

$$\mathfrak{D}_{\mathbf{t}} = \boldsymbol{\delta} + \boldsymbol{\delta}^* : H\Lambda(\boldsymbol{\delta},\Omega) \cap \mathring{H}\Lambda(\mathbf{d},\Omega) \to L^2\Lambda(\Omega), \tag{2.3a}$$

$$\mathfrak{D}_{\mathbf{n}} = \mathbf{d} + \mathbf{d}^* : \mathring{H}\Lambda(\boldsymbol{\delta},\Omega) \cap H\Lambda(\mathbf{d},\Omega) \to L^2\Lambda(\Omega), \tag{2.3b}$$

respectively. These are the Hodge–Dirac operators associated with the nilpotent operators $\boldsymbol{\delta}$ and $\mathbf{d}$ arising from the Hilbert complexes (1.3a) and (1.3b), respectively; cf. [21, Sec. 2], [33, Sec. 3].

We notice that the null-spaces

$$\mathfrak{H}_{\mathbf{t}} = \ker(\mathfrak{D}_{\mathbf{t}}) = \left\{\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega) : \mathbf{d}\boldsymbol{U} = 0,\ \boldsymbol{\delta}\boldsymbol{U} = 0,\ \mathbf{t}\boldsymbol{U} = 0\right\}, \tag{2.4a}$$

$$\mathfrak{H}_{\mathbf{n}} = \ker(\mathfrak{D}_{\mathbf{n}}) = \left\{\boldsymbol{U} \in H\Lambda(\mathfrak{D},\Omega) : \mathbf{d}\boldsymbol{U} = 0,\ \boldsymbol{\delta}\boldsymbol{U} = 0,\ \mathbf{n}\boldsymbol{U} = 0\right\}, \tag{2.4b}$$

are direct sums of harmonic spaces of all orders, cf. [2], [21, sect. 2], [33, sect. 3] and [39, Prop. 5.1]. In particular, when $\mathcal{M} = \mathbb{R}^N$ and $\kappa = 0$, the Hodge–Dirac operators have non-trivial finite dimensional kernels. Nevertheless, the BVPs (2.2a) and (2.2b) are well-posed on the orthogonal complements of $\ker(\mathfrak{D}_{\mathbf{t}})$ and $\ker(\mathfrak{D}_{\mathbf{n}})$ if we impose the following compatibility conditions on the boundary data:

$$\langle \boldsymbol{g}, \mathbf{n}\boldsymbol{V}\rangle_{\Gamma} = 0 \qquad \forall\, \boldsymbol{V} \in \ker(\mathfrak{D}_{\mathbf{n}}), \tag{2.5a}$$

$$\langle \boldsymbol{h}, \mathbf{t}\boldsymbol{V}\rangle_{\Gamma} = 0 \qquad \forall\, \boldsymbol{V} \in \ker(\mathfrak{D}_{\mathbf{t}}), \tag{2.5b}$$

respectively, cf. [21] and [33].

Otherwise, if $\kappa \neq 0$, recall that the inclusion map spawns compact embeddings $\mathrm{dom}(\mathfrak{D}_{\mathbf{t}}) \hookrightarrow L^2\Lambda(\Omega)$ and $\mathrm{dom}(\mathfrak{D}_{\mathbf{n}}) \hookrightarrow L^2\Lambda(\Omega)$, and so $\mathfrak{D}_{\mathbf{t}} + i\kappa$ and $\mathfrak{D}_{\mathbf{n}} + i\kappa$ are Fredholm operators of index zero [18, Lem. 7.2], [31, Lem. 4.1]. Because Lemma 1.1 offers continuous lifting maps from the trace spaces to the domain of the Hodge–Dirac operator $H\Lambda(\mathfrak{D}, \Omega)$, well-posedness of the boundary value problems (2.2a) and (2.2b) thus follow by injectivity, which is evidently guaranteed because the zero-order perturbations are purely imaginary.

2.1.2. Variational formulations for Hodge–Dirac BVPs. As discussed in [33, Sec. 3], a key feature of the Hodge–Dirac operator is that it admits the two distinct fundamental symmetric bilinear forms

$$\mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U}, \boldsymbol{V}) := \langle \boldsymbol{\delta} \boldsymbol{U}, \boldsymbol{V} \rangle_\Omega + \langle \boldsymbol{U}, \boldsymbol{\delta} \boldsymbol{V} \rangle_\Omega, \qquad \forall \boldsymbol{U}, \boldsymbol{V} \in H\Lambda(\boldsymbol{\delta}, \Omega) \tag{2.6a}$$

$$\mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \boldsymbol{V}) := \langle \mathbf{d} \boldsymbol{U}, \boldsymbol{V} \rangle_\Omega + \langle \boldsymbol{U}, \mathbf{d} \boldsymbol{V} \rangle_\Omega, \qquad \forall \boldsymbol{U}, \boldsymbol{V} \in H\Lambda(\mathbf{d}, \Omega), \tag{2.6b}$$

that rest on an equal footing. They arise in first-order analogs of Green's identities

$$\langle \mathfrak{D} \boldsymbol{U}, \boldsymbol{V} \rangle_\Omega = \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U}, \boldsymbol{V}) + \langle\!\langle \mathbf{t} \boldsymbol{U}, \mathbf{n} \boldsymbol{V} \rangle\!\rangle_\Gamma, \tag{2.7a}$$

$$\langle \mathfrak{D} \boldsymbol{U}, \boldsymbol{V} \rangle_\Omega = \mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \boldsymbol{V}) - \langle\!\langle \mathbf{n} \boldsymbol{U}, \mathbf{t} \boldsymbol{V} \rangle\!\rangle_\Gamma, \tag{2.7b}$$

which hold for all $\boldsymbol{U}, \boldsymbol{V} \in H\Lambda(\mathfrak{D}, \Omega)$. They lead to two variational problems associated with (2.2a) and (2.2b), respectively:

$$\boldsymbol{U} \in H\Lambda(\boldsymbol{\delta}, \Omega): \quad \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa \langle \boldsymbol{U}, \boldsymbol{V} \rangle_\Omega = -\langle\!\langle \boldsymbol{g}, \mathbf{n} \boldsymbol{V} \rangle\!\rangle_\Gamma, \quad \forall \boldsymbol{V} \in H\Lambda(\boldsymbol{\delta}, \Omega), \tag{2.8a}$$

$$\boldsymbol{U} \in H\Lambda(\mathbf{d}, \Omega): \quad \mathcal{A}_{\mathrm{d}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa \langle \boldsymbol{U}, \boldsymbol{V} \rangle_\Omega = \langle\!\langle \boldsymbol{h}, \mathbf{t} \boldsymbol{V} \rangle\!\rangle_\Gamma, \quad \forall \boldsymbol{V} \in H\Lambda(\mathbf{d}, \Omega). \tag{2.8b}$$

It is a simple exercise in integration by parts to verify using suitable test functions that the variational problems (2.8a) and (2.8b) are equivalent with the strong formulations (2.2a) and (2.2b), respectively. Nevertheless, the analysis of the Hodge–Dirac operator is not as common as that of the Hodge–Laplacian. Because of the importance of inf-sup inequalities for the analysis of Galerkin discretization, we thus find meaningful to also cover solvability of the variational problems directly without assuming prior knowledge of the Hodge–Dirac operator's properties and take the opportunity to point at important references.

If $\kappa = 0$, solvability of the variational problems (2.8a) and (2.8b) is covered by the theory for the abstract Hodge–Dirac operator provided in [21, Sec. 2]. Specifically, the bilinear forms associated with each of the variational problems

$$\begin{aligned} \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U}, \boldsymbol{V}) + \langle \boldsymbol{P}, \boldsymbol{V} \rangle_\Omega &= -\langle\!\langle \boldsymbol{g}, \mathbf{n} \boldsymbol{V} \rangle\!\rangle_\Gamma, && \forall \boldsymbol{V} \in H\Lambda(\boldsymbol{\delta}, \Omega), \\ \langle \boldsymbol{U}, \boldsymbol{W} \rangle_\Omega &= 0 && \forall \boldsymbol{W} \in \mathfrak{H}_{\mathbf{t}}, \end{aligned} \tag{2.9a}$$

and

$$\begin{aligned} \mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \boldsymbol{V}) + \langle \boldsymbol{Q}, \boldsymbol{V} \rangle_\Omega &= \langle\!\langle \boldsymbol{h}, \mathbf{n} \boldsymbol{V} \rangle\!\rangle_\Gamma, && \forall \boldsymbol{V} \in H\Lambda(\mathbf{d}, \Omega), \\ \langle \boldsymbol{U}, \boldsymbol{W} \rangle_\Omega &= 0 && \forall \boldsymbol{W} \in \mathfrak{H}_{\mathbf{n}}, \end{aligned} \tag{2.9b}$$

satisfy inf-sup inequalities [21, Thm. 6]. The compatibility conditions (2.5b) and (2.5a) ensure compatibility of the right-hand sides and thus well-posedness.

We now show that when $\kappa \neq 0$, generalized Gårding inequalities hold for the bilinear forms associated with the variational problems (2.8a) and (2.8b), cf. [10, Thm. 4], [7, Chap. 11.4].

LEMMA 2.1. *Let $\kappa \neq 0$. The bilinear forms associated with the variational problems* (2.8a) *and* (2.8b) *are T-coercive. In other words, there exist positive constants $C_{\mathsf{t}}, C_{\mathsf{n}} > 0$, isomorphisms $\Xi_{\mathsf{t}} : H\Lambda(\boldsymbol{\delta},\Omega) \to H\Lambda(\boldsymbol{\delta},\Omega)$ and $\Xi_{\mathsf{n}} : H\Lambda(\mathbf{d},\Omega) \to H\Lambda(\mathbf{d},\Omega)$, and compact operators $\mathsf{K}_{\mathsf{t}} : H\Lambda(\boldsymbol{\delta},\Omega) \to H\Lambda(\boldsymbol{\delta},\Omega)$ and $\mathsf{K}_{\mathsf{n}} : H\Lambda(\mathbf{d},\Omega) \to H\Lambda(\mathbf{d},\Omega)$, such that*

$$\|\boldsymbol{U}\|^2_{H\Lambda(\mathrm{d},\Omega)} \leq C_{\mathsf{n}} \left| \mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \overline{\Xi_{\mathsf{n}}\boldsymbol{U}}) + i\kappa\langle \boldsymbol{U}, \overline{\Xi_{\mathsf{n}}\boldsymbol{U}}\rangle_\Omega + \langle \mathsf{K}_{\mathsf{n}}\boldsymbol{U}, \boldsymbol{U}\rangle_\Omega \right| \tag{2.10a}$$

$$\|\boldsymbol{V}\|^2_{H\Lambda(\delta,\Omega)} \leq C_{\mathsf{n}} \left| \mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{V}, \overline{\Xi_{\mathsf{t}}\boldsymbol{V}}) + i\kappa\langle \boldsymbol{U}, \overline{\Xi_{\mathsf{t}}\boldsymbol{V}}\rangle_\Omega + \langle \mathsf{K}_{\mathsf{t}}\boldsymbol{V}, \boldsymbol{V}\rangle_\Omega \right| \tag{2.10b}$$

for all $\boldsymbol{U} \in H\Lambda(\mathbf{d},\Omega)$ and $\boldsymbol{V} \in H\Lambda(\boldsymbol{\delta},\Omega)$.

Proof. By duality, it is sufficient to focus on (2.10a). The isomorphism Ξ_{n} is designed based on the $L^2\Lambda(\Omega)$-orthogonal Hodge decomposition

$$H\Lambda(\mathbf{d},\Omega) = \mathfrak{B} \oplus \mathfrak{H}_{\mathsf{n}} \oplus \mathfrak{Z}^\perp, \tag{2.11}$$

where $\mathfrak{B} = \mathrm{range}(\mathbf{d})$ and $\mathfrak{Z} = \ker(\mathbf{d})$. The intent is to exploit that the identity map spawns compact embeddings $\mathfrak{Z}^\perp \hookrightarrow L^2\Lambda(\Omega)$ and $\mathfrak{H}_{\mathsf{n}} \hookrightarrow L^2\Lambda(\Omega)$, cf. [3], [4], [30, Sec. 2], [31]. According to (2.11), any element $\boldsymbol{U} \in H\Lambda(\mathbf{d},\Omega)$ can be uniquely written as $\boldsymbol{U} = \boldsymbol{U}_{\mathfrak{B}} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \boldsymbol{U}_{\mathfrak{Z}^\perp}$.

Recall that $\mathbf{d}_{\mathfrak{Z}^\perp} = \mathbf{d}\big|_{\mathfrak{Z}^\perp} : \mathfrak{Z}^\perp \to \mathfrak{B}$ is a bounded isomorphism, because $\mathbf{d}$ has closed range (Fredholm property). Therefore, it has a continuous inverse $\mathbf{d}^{-1}_{\mathfrak{Z}^\perp} : \mathfrak{B} \to \mathfrak{Z}^\perp$. We define $\Xi_{\mathsf{n}} : H\Lambda(\mathrm{d},\Omega) \to H\Lambda(\mathrm{d},\Omega)$ by

$$\Xi_{\mathsf{n}}\boldsymbol{U} = \alpha\, \mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp} - i\kappa\alpha\, \boldsymbol{U}_{\mathfrak{B}} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \mathbf{d}^{-1}_{\mathfrak{Z}^\perp}\boldsymbol{U}_{\mathfrak{B}}, \tag{2.12}$$

where $0 < \alpha < 1/\kappa^2$. It is easy to see that Ξ_{n} is bounded.

We claim that Ξ_{n} is injective. Indeed, if we suppose that $\Xi_{\mathsf{n}}\boldsymbol{U} = 0$, then by orthogonality $\|\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}\| = \|\mathbf{d}^{-1}_{\mathfrak{Z}^\perp}\boldsymbol{U}_{\mathfrak{B}}\| = 0$, and thus $\boldsymbol{U}_{\mathfrak{B}} = \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} = 0$. We are left with the identity $0 = \mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp} = \mathbf{d}_{\mathfrak{Z}^\perp}\boldsymbol{U}_{\mathfrak{Z}^\perp}$, from which once again $\boldsymbol{U}_{\mathfrak{Z}^\perp} = 0$.

To see that Ξ_{n} is surjective, we simply verify that

$$\begin{aligned}
&\Xi_{\mathsf{n}}\left(\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \alpha^{-1}\mathbf{d}^{-1}_{\mathfrak{Z}^\perp}\boldsymbol{U}_{\mathfrak{B}} + i\kappa\boldsymbol{U}_{\mathfrak{Z}^\perp}\right) \\
&\qquad = \alpha\,\mathbf{d}\left(\alpha^{-1}\mathbf{d}^{-1}_{\mathfrak{Z}^\perp}\boldsymbol{U}_{\mathfrak{B}} + i\kappa\boldsymbol{U}_{\mathfrak{Z}^\perp}\right) - i\kappa\alpha\,\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \mathbf{d}^{-1}\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp} \\
&\qquad = \mathbf{d}\mathbf{d}^{-1}_{\mathfrak{Z}^\perp}\boldsymbol{U}_{\mathfrak{B}} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \boldsymbol{U}_{\mathfrak{Z}^\perp} = \boldsymbol{U}_{\mathfrak{B}} + \boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}} + \boldsymbol{U}_{\mathfrak{Z}^\perp} = \boldsymbol{U}.
\end{aligned}$$

Now, let us indicate by a hat inequalities and identities that hold *up to compact perturbation*, e.g. $\hat{=}$ and $\hat{\geq}$. Due to orthogonality, we find that

$$\langle \mathbf{d}\boldsymbol{U}, \overline{\Xi_{\mathsf{n}}\boldsymbol{U}}\rangle_\Omega = \alpha\left\|\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp}\right\|^2 + i\kappa\alpha\left(\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp}, \boldsymbol{U}_{\mathfrak{B}}\right)_\Omega, \tag{2.13a}$$

$$\langle \boldsymbol{U}_{\mathfrak{B}}, \mathbf{d}\overline{\Xi_{\mathsf{n}}\boldsymbol{U}}\rangle_\Omega = \|\boldsymbol{U}_{\mathfrak{B}}\|^2, \tag{2.13b}$$

$$i\kappa\langle \boldsymbol{U}, \overline{\Xi_n\boldsymbol{U}}\rangle_\Omega \mathrel{\hat{=}} -\kappa^2\alpha\|\boldsymbol{U}_{\mathfrak{B}}\|^2 + i\kappa\alpha\left(\boldsymbol{U}_{\mathfrak{B}}, \mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp}\right)_\Omega, \tag{2.13c}$$

where compact terms involving $\boldsymbol{U}_{\mathfrak{Z}^\perp}$ and $\boldsymbol{U}_{\mathfrak{H}_{\mathsf{n}}}$ were dropped. Summing the contributions of (2.13a) to (2.13c), we obtain

$$\mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \Xi_{\mathsf{n}}\overline{\boldsymbol{U}}) + i\kappa\langle \boldsymbol{U}, \Xi_{\mathsf{n}}\overline{\boldsymbol{U}}\rangle_\Omega \mathrel{\hat{=}} \alpha\left\|\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp}\right\|^2 + (1-\kappa^2\alpha)\|\boldsymbol{U}_{\mathfrak{B}}\|^2 + i\kappa\alpha(\nu + \overline{\nu}),$$

where $\nu = (\boldsymbol{U}_{\mathfrak{B}}, \mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp})_\Omega$. Since the initial choice of parameter α guarantees that $1-\kappa^2\alpha > 0$ and the last term is purely imaginary, we conclude that

$$\left| \mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \Xi_{\mathsf{n}}\overline{\boldsymbol{U}}) + i\kappa\langle \boldsymbol{U}, \Xi_{\mathsf{n}}\overline{\boldsymbol{U}}\rangle_\Omega \right| \ \hat{\geq}\ C\left(\left\|\mathbf{d}\boldsymbol{U}_{\mathfrak{Z}^\perp}\right\|^2 + \|\boldsymbol{U}_{\mathfrak{B}}\|^2 \right)$$

for $C = \min\{\alpha, 1-\kappa^2\alpha\}$, which concludes the proof. ∎

COROLLARY 2.2. *The variational problems* (2.8a) *and* (2.8b) *are well-posed.*

Proof. Based on Lemma 2.1, the operators associated with the variational problems (2.8a) and (2.8b) are Fredholm of index 0. We thus only need to show that they are injective. We focus on (2.8b).

Suppose that $\boldsymbol{U} \in H\Lambda(\mathrm{d}, \Omega)$ is such that

$$\mathcal{A}_{\mathrm{d}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa\langle \boldsymbol{U}, \boldsymbol{V}\rangle_\Omega = 0$$

for all $\boldsymbol{V} \in H\Lambda(\mathrm{d}, \Omega)$. Testing with $\boldsymbol{V} = \overline{\boldsymbol{U}}$, we find that

$$i\kappa\|\boldsymbol{U}\|^2_\Omega + \omega + \overline{\omega} = 0,$$

where $\omega = (\mathbf{d}\boldsymbol{U}, \boldsymbol{U})_\Omega$. As in the proof of Lemma 2.1, $\omega + \overline{\omega}$ is a real number, so $\|\boldsymbol{U}\|^2_\Omega = 0$, from which we conclude that $\boldsymbol{U} = 0$. ∎

The ability to introduce two distinct bilinear forms associated with (2.2a) and (2.2b) for the Hodge–Dirac operator is crucially rooted in the fact that both the cochain and chain perspective of the de Rham complex can be adopted in formulating BVPs for the Hodge–Dirac operator. Notably, it points to the symmetry between the BVPs (2.2a) and (2.2b) as discussed in the introduction of Subsection 2.1 and emphasizes for $\kappa = 0$ the necessity of imposing the compatibility conditions (2.5a) and (2.5b) on the boundary data. For example, we could alternatively formulate (2.2a) as the variational problem of finding a full form $\boldsymbol{U} \in H\Lambda(\mathbf{d}, \Omega)$ with $\mathbf{t}\boldsymbol{U} = \mathbf{g}$ such that

$$\mathcal{A}_{\mathrm{d}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa\langle \boldsymbol{U}, \boldsymbol{V}\rangle_\Omega = \langle\!\langle \boldsymbol{h}, \mathbf{t}\boldsymbol{V}\rangle\!\rangle_\Gamma, \qquad \forall \boldsymbol{V} \in \overset{\circ}{H}\Lambda^\ell(\mathrm{d}, \Omega). \tag{2.14}$$

Recall that in a formulation such as (2.14), we lift the boundary data and solve

$$\boldsymbol{W} \in \overset{\circ}{H}\Lambda(\mathbf{d}, \Omega): \qquad \mathcal{A}_{\mathrm{d}}(\boldsymbol{W}, \boldsymbol{V}) + i\kappa\langle \boldsymbol{W}, \boldsymbol{V}\rangle_\Omega = \mathsf{F}_{\boldsymbol{g}}(\boldsymbol{V}), \qquad \forall \boldsymbol{V} \in \overset{\circ}{H}\Lambda(\mathbf{d}, \Omega),$$

where $\mathsf{F}_{\boldsymbol{g}}(\boldsymbol{V}) = -\mathcal{A}_{\mathrm{d}}(\boldsymbol{\mathcal{E}}^{\mathsf{t}}\boldsymbol{g}, \boldsymbol{V}) - i\kappa\langle \boldsymbol{\mathcal{E}}^{\mathsf{t}}\boldsymbol{g}, \boldsymbol{V}\rangle_\Omega$. This is the mainstream perspective adopted in the literature of finite element exterior calculus. We depart from this standard because, as opposed to $\mathfrak{D}_{\mathbf{t}}$ and $\mathfrak{D}_{\mathsf{n}}$ in (2.3a) and (2.3b), the self-adjoint operator behind (2.14) is

$$\overset{\circ}{\mathfrak{D}}_{\mathbf{t}} = \overset{\circ}{\mathbf{d}} + \overset{\circ}{\mathbf{d}}{}^*,$$

where

$$\overset{\circ}{\mathrm{d}} : \overset{\circ}{H}\Lambda(\mathbf{d}, \Omega) \to \overset{\circ}{H}\Lambda(\mathbf{d}, \Omega)$$

is obtained by restricting the exterior derivative to the kernel of the tangential trace. For the goal of this article, this approach is inconvinient because it *modifies the exterior derivative* such that it is no longer the one which enters the definition of the Hodge–Dirac operator introduced in (2.1) that leads to the BVPs (2.2a) and (2.2b). It is the *maximal* Hodge–Dirac operator $\mathfrak{D} = \boldsymbol{\delta} + \mathbf{d} : H\Lambda(\mathfrak{D}, \Omega) \to L^2\Lambda(\Omega)$ involving the exterior derivative $\mathbf{d} : H\Lambda(\mathrm{d},\Omega) \to H\Lambda(\mathrm{d},\Omega)$ that appears in the representation formula given in Subsection 4.1.1 from which BIEs are derived. Indeed, the BVPs (2.2a) and (2.2b) lead to four variational problems: to each one of the two BVPs is associated both a variational problem featuring *natural* boundary conditions (such as in (2.8a)) and a variational problem with *essential* boundary conditions imposed on the domain of the operator (such as in (2.14)). As we will see, it is the structure of the variational problems with natural boundary conditions— and accordingly the expressions of the self-adjoint operators (2.3a) and (2.3b)—that is reproduced at the level of the trace de Rham complex in the first-kind BIOs.

2.2. Hodge–Laplace operators. We now turn to the Hodge–Laplacian and zero-order perturbations involving a non-negative constant $\lambda \geq 0$ (non-negative when $\mathcal{M} = \mathbb{R}^N$ and striclty positive when $\mathcal{M}$ is a compact manifold). We will be interested in both *strong* and *mixed* formulations of the operator. While equivalent from the point of view of solvability, the formal distinction in their structure is important in revealing the connection we seek with first-kind BIOs. It is a straightforward exercise in integration by parts to show that all the formulations presented below are indeed equivalent. Since well-posedness of BVPs for the Hodge–Laplacian has been extensively studied and is very well-known, we omit the details and refer to standard references such as [2, Chap. 4].

2.2.1. BVPs for the Hodge–Laplacian. Our starting point is the strong formulation

$$-\Delta_\ell + \lambda : H\Lambda(-\Delta_\ell,\Omega) \to L^2\Lambda^\ell(\Omega). \tag{2.15}$$

Suitable boundary conditions for this operator can be imposed using the surjective compound traces introduced in Subsection 1.4.2. Recall that the traces

$$\mathsf{T}^{\mathsf{t}}_{\Delta}\, U_\ell = \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_\ell U_\ell \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \qquad \text{and} \qquad \mathsf{T}^{\mathsf{n}}_{\Delta}\, U_\ell = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell \end{pmatrix} \tag{2.16}$$

are continuous and surjective as mappings

$$\begin{aligned} \mathsf{T}^{\mathsf{t}}_{\Delta} &: H\Lambda^\ell(\mathrm{d}\delta,\Omega) \cap H\Lambda^\ell(\mathrm{d},\Omega) \longrightarrow H^{\mathsf{t}}_{\Delta}(\Gamma), \\ \mathsf{T}^{\mathsf{n}}_{\Delta} &: H\Lambda^\ell(\delta,\Omega) \cap H\Lambda^\ell(\delta\mathrm{d},\Omega) \longrightarrow H^{\mathsf{n}}_{\Delta}(\Gamma), \end{aligned}$$

where the product of trace spaces are given by

$$\begin{aligned} H^{\mathsf{t}}_{\Delta}(\Gamma) &= H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma) \times H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\mathrm{d},\Gamma), \\ H^{\mathsf{n}}_{\Delta}(\Gamma) &= H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \times H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell}(\delta,\Gamma). \end{aligned}$$

The significance of these traces for the Hodge–Laplacian as long been recognized in related literature. They are covered extensively in [26, Sec.1.1] and [28, Chap. 5]. Imposing boundary conditions using these traces was shown in [36, Sec. 1.6] to render the Hodge–Laplacian elliptic in the sense of Sapiro–Lopatinski. In [11, 12], [15, Sec. 1.c], [17] and [33, 34], these traces are seen to appear naturally in variational problems from identities obtained using integration by parts. In particular, our derivation of a representation formula will use the fact that they give rise to a generalization of Green's second formula to differential forms.

For $0 \leq \ell \leq N$, we consider the BVPs

$$U_\ell \in H\Lambda^\ell(\Delta,\Omega) : \begin{cases} (-\Delta_\ell + \lambda)\, U_\ell = 0 & \text{in } \Omega \\ \mathsf{T}^{\mathsf{t}}_{\Delta}\, U_\ell = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \in H^{\mathsf{t}}_{\Delta}(\Gamma), \tag{2.17a}$$

and

$$U_\ell \in H\Lambda^\ell(\Delta,\Omega) : \begin{cases} (-\Delta_\ell + \lambda)\, U_\ell = 0 & \text{in } \Omega \\ \mathsf{T}^{\mathsf{n}}_{\Delta}\, U_\ell = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} \in H^{\mathsf{n}}_{\Delta}(\Gamma). \tag{2.17b}$$

We will derive BIEs for the BVPs (2.17a) and (2.17b) using the strong formulation of the Hodge–Laplacian in Subsection 4.2.1. However, as mentioned in the closing discussing of Subsection 1.1, if a Hodge–Laplace operator is to appear in the trace de Rham complex, it has to be in mixed form, because the boundary data lies in *product spaces*. With this guiding principle, we now introduce mixed formulations for (2.17a) and (2.17b). We will later recognize their structure in the first-kind BIOs.

Introducing an auxiliary variable $U_{\ell-1} = \delta_\ell U_\ell \in H\Lambda^{\ell-1}(\mathrm{d},\Omega)$, we obtain the *mixed-order* formulation

$$\begin{aligned} \delta_\ell U_\ell - U_{\ell-1} &= 0, \\ \delta_{\ell+1}\mathrm{d}_\ell U_\ell + \mathrm{d}_{\ell-1} U_{\ell-1} + \lambda U_\ell &= 0. \end{aligned}$$

More succinctly,

$$\mathfrak{M} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the perturbed Hodge–Laplacian in mixed form

$$\mathfrak{M} : \mathrm{dom}(\mathfrak{M}) = H\Lambda^{\ell-1}(\mathrm{d},\Omega) \times \Big(H\Lambda^\ell(\delta\mathrm{d},\Omega) \cap H\Lambda^\ell(\delta,\Omega) \Big) \to L^2\Lambda^{\ell-1}(\Gamma) \times L^2\Lambda^\ell(\Gamma)$$

can be represented by the operator matrix

$$\mathfrak{M} = \begin{pmatrix} -\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \delta_{\ell+1}\mathrm{d}_\ell + \lambda \end{pmatrix}. \tag{2.18}$$

By substituting the auxiliary variable $U_{\ell-1}$ in the traces (2.16) for Hodge–Laplace operators in strong formulation, we obtain the pair of traces

$$\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} \mathsf{t}_{\ell-1} U_{\ell-1} \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \quad \text{and} \quad \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell \end{pmatrix}, \tag{2.19}$$

which are continuous as mappings

$$\begin{aligned} \mathsf{T}^{\mathsf{t}}_{\mathfrak{M}} &: H\Lambda^{\ell-1}(\mathrm{d},\Omega) \times H\Lambda^\ell(\mathrm{d},\Omega) \longrightarrow H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma) = H^{\mathsf{t}}_{\Delta}(\Gamma), \\ \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}} &: L^2\Lambda^{\ell-1}(\Omega) \times \Big(H\Lambda^\ell(\delta,\Omega) \cap H\Lambda^\ell(\delta\mathrm{d},\Omega) \Big) \longrightarrow H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma) = H^{\mathsf{n}}_{\Delta}(\Gamma), \end{aligned}$$

The associated boundary value problems read, respectively:

(2.20a)

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} \in \mathrm{dom}(\mathfrak{M}) : \begin{cases} \mathfrak{M} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{t}}_{\mathfrak{M}} \begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \in H^{\mathsf{t}}_{\Delta}(\Gamma),$$

(2.20b)

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} \in \operatorname{dom}(\mathfrak{M}) : \begin{cases} \mathfrak{M}\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} \in H^{\mathsf{n}}_{\Delta}(\Gamma).$$

Starting from the BVPs (2.17a) and (2.17b) in strong form, we could have alternatively opted for an auxiliary variable $U_{\ell+1} = \mathrm{d}_\ell U_\ell \in H\Lambda^\ell(\delta,\Omega)$ to obtain the equivalent mixed-order formulation

$$\begin{aligned} \delta_{\ell+1}U_{\ell+1} + \mathrm{d}_{\ell-1}\delta_\ell U_\ell + \lambda U_\ell &= 0, \\ \mathrm{d}_\ell U_\ell - U_{\ell+1} &= 0, \end{aligned}$$

or in short,

$$\mathfrak{R}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where this time the perturbed Hodge–Laplacian in mixed form is a continuous map

$$\mathfrak{R} : \operatorname{dom}(\mathfrak{R}) = H\Lambda^{\ell+1}(\delta,\Omega) \times H\Lambda^\ell(\mathrm{d}\delta,\Omega) \cap H\Lambda^\ell(\mathrm{d},\Omega) \to L^2\Lambda^\ell(\Gamma) \times L^2\Lambda^{\ell+1}(\Gamma)$$

whose operator matrix representation reads

$$\mathfrak{R} = \begin{pmatrix} \mathrm{d}_{\ell+1}\delta_\ell + \lambda & \delta_{\ell+1} \\ \mathrm{d}_\ell & -\mathrm{Id} \end{pmatrix}. \tag{2.21}$$

We then reach instead the BVPs

(2.22a)

$$\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} \in \operatorname{dom}(\mathfrak{R}) : \begin{cases} \mathfrak{R}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} \in H^{\mathsf{t}}_{\mathfrak{R}}(\Gamma),$$

(2.22b)

$$\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} \in \operatorname{dom}(\mathfrak{R}) : \begin{cases} \mathfrak{R}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{in } \Omega \\ \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} & \text{on } \partial\Omega \end{cases}, \quad \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} \in H^{\mathsf{n}}_{\mathfrak{R}}(\Gamma),$$

involving the continuous and surjective traces

$$\mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_\ell U_\ell \\ \mathsf{t}_\ell U_\ell \end{pmatrix} \quad \text{and} \quad \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix} U_\ell \\ U_{\ell+1} \end{pmatrix} = \begin{pmatrix} \mathsf{n}_\ell U_\ell \\ \mathsf{n}_{\ell+1} U_{\ell+1} \end{pmatrix}. \tag{2.23}$$

2.3. Variational formulations for Hodge–Laplace BVPs. In line with our goal, it is sufficient for our purposes to present only two variational problems equivalent to the BVPs (2.17a), (2.17b), (2.20a), (2.20b), (2.22a) and (2.22b). We focus on those two because they are the only variational formulations obtained by integration by parts sharing the two following characteristics:

- They are in mixed formulation. Therefore, they involve product spaces of differential forms of order $\ell-1$ and ℓ, or ℓ and $\ell+1$, analogous to the structure of the trace spaces $\mathsf{T}^{\mathsf{t}}_{\Delta}$ and $\mathsf{T}^{\mathsf{n}}_{\Delta}$ for the Hodge–Laplacian.
- The boundary conditions are *natural*, so that no restriction is needed on the domain of the featured exterior derivative or codifferential.

Integrating by parts using (1.10), we find the analog of Green's first identities for Hodge–Laplace operators in mixed formulation:

$$\left(\mathfrak{M}\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right)_\Omega = \mathcal{B}_{\mathrm{d}}\left(\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right) - \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\begin{pmatrix}\overline{V}_{\ell-1}\\\overline{V}_\ell\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma, \tag{2.24a}$$

$$\left(\mathfrak{R}\begin{pmatrix}U_\ell\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_\ell\\V_{\ell+1}\end{pmatrix}\right)_\Omega = \mathcal{B}_{\delta}\left(\begin{pmatrix}U_\ell\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_\ell\\V_{\ell+1}\end{pmatrix}\right) + \left\langle\!\!\left\langle \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}\begin{pmatrix}U_\ell\\U_{\ell+1}\end{pmatrix},\mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix}\overline{V}_\ell\\\overline{V}_{\ell+1}\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma, \tag{2.24b}$$

where the fundamental bilinear forms associated with $\mathfrak{M}$ and $\mathfrak{R}$ are

$$\begin{aligned}\mathcal{B}_{\mathrm{d}}\left(\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right) &= (\mathrm{d}_\ell U_\ell,\mathrm{d}_\ell V_\ell)_\Omega + \lambda\,(U_\ell,V_\ell)_\Omega + (\mathrm{d}_{\ell-1}U_{\ell-1},V_\ell)_\Omega\\ &\quad + (U_\ell,\mathrm{d}_\ell V_{\ell-1})_\Omega - (U_{\ell-1},V_{\ell-1})_\Omega\,,\end{aligned} \tag{2.25}$$

$$\begin{aligned}\mathcal{B}_{\delta}\left(\begin{pmatrix}U_\ell\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_\ell\\V_{\ell+1}\end{pmatrix}\right) &= (\delta_\ell U_\ell,\delta_\ell V_\ell)_\Omega + \lambda\,(U_\ell,V_\ell)_\Omega + (\delta_{\ell+1}U_{\ell+1},V_\ell)_\Omega\\ &\quad + (U_\ell,\delta_{\ell+1}V_{\ell+1})_\Omega - (U_{\ell+1},V_{\ell+1})_\Omega\,.\end{aligned} \tag{2.26}$$

They lead to two variational problems. In the first, we suppose that the boundary data $(h_{\ell-1},h_\ell)^\top \in \mathcal{T}^{\mathsf{n}}_{\mathfrak{M}}(\Gamma)$ is given and we seek $(U_{\ell-1},U_\ell)^\top \in H\Lambda^{\ell-1}(\mathrm{d},\Omega)\times H\Lambda^{\ell}(\mathrm{d},\Omega)$ such that

$$\mathcal{B}_{\mathrm{d}}\left(\begin{pmatrix}U_{\ell-1}\\U_\ell\end{pmatrix},\begin{pmatrix}V_{\ell-1}\\V_\ell\end{pmatrix}\right) = \left\langle\!\!\left\langle \begin{pmatrix}h_{\ell-1}\\h_\ell\end{pmatrix},\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\begin{pmatrix}\overline{V}_{\ell-1}\\\overline{V}_\ell\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \tag{2.27a}$$

for all $(V_{\ell-1},V_\ell)^\top \in H\Lambda^{\ell-1}(\mathrm{d},\Omega)\times H\Lambda^{\ell}(\mathrm{d},\Omega)$. In the second, $(g_\ell,g_{\ell+1})^\top \in H^{\mathsf{t}}_{\mathfrak{R}}(\Gamma)$ is given and we seek $(U_\ell,U_{\ell+1}) \in H\Lambda^{\ell}(\delta,\Omega)\times H\Lambda^{\ell+1}(\delta,\Omega)$ such that

$$\mathcal{B}_{\delta}\left(\begin{pmatrix}U_\ell\\U_{\ell+1}\end{pmatrix},\begin{pmatrix}V_\ell\\V_{\ell+1}\end{pmatrix}\right) = \left\langle\!\!\left\langle \begin{pmatrix}g_{\ell-1}\\g_\ell\end{pmatrix},\mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}\begin{pmatrix}\overline{V}_\ell\\\overline{V}_{\ell+1}\end{pmatrix}\right\rangle\!\!\right\rangle_\Gamma \tag{2.27b}$$

for all $(V_\ell,V_{\ell+1})^\top \in H\Lambda^{\ell}(\delta,\Omega)\times H\Lambda^{\ell+1}(\delta,\Omega)$.

The variational problem (2.27a) is an equivalent reformulation of the problems (2.20b) and (2.17b) where the trace data $\mathsf{T}^{\mathsf{n}}_{\Delta}U_\ell$ is known, while (2.27b) is a variational formulation for (2.22a) and (2.17a) where $\mathsf{T}^{\mathsf{t}}_{\Delta}U_\ell$ is known.

The self-adjoint operators behind these BVPs and associated with the bilinear forms in the analogs of Green's first formulas are

(2.28a)
$$\mathfrak{M}_{\mathsf{n}} = \begin{pmatrix} -\mathrm{Id} & \mathrm{d}_\ell^* \\ \mathrm{d}_{\ell-1} & \mathrm{d}_\ell^*\mathrm{d}_\ell + \lambda \end{pmatrix} : H\Lambda^{\ell-1}(\mathrm{d},\Omega) \times \left(\mathring{H}\Lambda^\ell(\delta,\Omega) \cap \mathring{H}\Lambda^\ell(\delta\mathrm{d},\Omega) \right) \to L^2\Lambda(\Omega),$$

(2.28b)
$$\mathfrak{R}_{\mathsf{t}} = \begin{pmatrix} \delta_\ell^*\delta_\ell + \lambda & \delta_{\ell+1} \\ \delta_{\ell+1}^* & -\mathrm{Id} \end{pmatrix} : \left(\mathring{H}\Lambda^\ell(\mathrm{d},\Omega) \cap \mathring{H}\Lambda^\ell(\mathrm{d}\delta,\Omega) \right) \times H\Lambda^{\ell+1}(\delta,\Omega) \to L^2\Lambda(\Omega),$$

where similarly as for (2.3a) and (2.3b), the traces vanish on

$$\mathring{H}\Lambda^\ell(\delta\mathrm{d},\Omega) = H\Lambda^\ell(\delta\mathrm{d},\Omega) \cap \ker \mathsf{n}_{\ell+1}\mathrm{d}_\ell,$$
$$\mathring{H}\Lambda^\ell(\mathrm{d}\delta,\Omega) = H\Lambda^\ell(\mathrm{d}\delta,\Omega) \cap \ker \mathsf{t}_{\ell-1}\delta_\ell.$$

3. Calculus of boundary potentials. Our main tool in deriving BIEs for Hodge–Dirac and Hodge–Laplace operators is a *calculus of atomic boundary potentials*. We call *atomic* the two boundary potentials defined in this section, because all the other layer potentials in this work are obtained from them by differentiation. We hope to convey that the commutation identities and jump relations that they satisfy involving the exterior derivative and codifferential are valuable instruments that greatly simplify derivations and in that sense viewing these potentials as elementary building blocks unlocks the power of exterior calculus as a framework for calculations. Moreover, these atomic potentials are the crucial components in the definitions of the non-local inner products on the spaces of the trace de Rham complex where the claimed correspondence between the operators entering the BVPs of Section 2 and the first-kind BIOs studied in Section 4 is revealed.

3.1. Newtonian potential. For $\mathcal{M} = \mathbb{R}^N$, a fundamental solution Φ_ℓ^λ for the scalar differential operator $-\Delta_\ell + \lambda$ satisfying suitable decay conditions at infinity exists for all $\lambda \geq 0$, cf.[22, Eq. 4.1], [23, Chap. 8 and 9]. Denote by $\mathcal{I}_\ell$ the identity double form of bi-degree (ℓ,ℓ) on $\mathbb{R}^N \times \mathbb{R}^N$ and for $x \neq y$ let

$$\mathcal{G}_\ell^\lambda(x,y) = \Phi_\lambda\left(|x-y|\right)\, \mathcal{I}_\ell(x,y) \tag{3.1}$$

be the singular kernel of the symmetric integral transformation

$$\mathsf{N}_\ell^\lambda\, U_\ell(x) = \lim_{\epsilon\to 0} \langle\, \mathcal{G}_\ell^\lambda(x,\cdot), U_\ell(\cdot)\,\rangle_{\mathbb{R}^N\setminus B_\epsilon(x)}, \qquad U_\ell \in \mathcal{D}^\ell(\mathbb{R}^N), \tag{3.2}$$

where $B_\epsilon(x)$ is the N-dimensional ball of radius $\epsilon > 0$ centered at x. The extension $\mathsf{N}_\ell^\lambda : \mathcal{E}'_\ell(\mathbb{R}^N) \to \mathcal{D}'_\ell(\mathbb{R}^N)$ to distributions via the dual mapping is a two-sided inverse of $-\Delta_\ell + \lambda$ in the sense of distributions, cf. [16, Chap. 12 and 16], [20, sect. 2.2 and 2.3], [23, Chap. 6], [32, Chap. 3] and [40, sect. 3].

On a compact boundaryless manifold, not only the Hodge–Laplacian is not invertible, we must also be wary of its non-trivial eigenspaces. Thankfully though, $-\Delta + \lambda : H^1\Lambda^\ell(\mathcal{M}) \to H^{-1}\Lambda^\ell(\mathcal{M})$ is invertible for at least all $\lambda > 0$, in which case the Schwartz kernel of its continuous inverse is available, cf. [26, Chap. 3] and [28].

To keep our exposition simple, we thus settle for imposing on $\lambda \geq 0$ the condition that $\lambda > 0$ whenever $\mathcal{M}$ is compact.

Assumption A. *If $\mathcal{M} = \mathbb{R}^N$, we allow $\kappa \in \mathbb{R}$ and $\lambda \geq 0$, but we impose $\kappa > 0$ and $\lambda > 0$ when $\mathcal{M}$ is a compact manifold without boundary.*

Under Assumption A, we can always assume that a Newtonian potential

$$\mathsf{N}_\ell^\lambda : H^{-1}_{\mathrm{comp}}\Lambda^\ell(\mathcal{M}) = H^{-1}\Lambda^\ell(\mathcal{M}) \cap \mathcal{E}'_\ell \to H^1_{\mathrm{loc}}\Lambda^\ell(\mathcal{M}) \cap H_{\mathrm{loc}}\Lambda^\ell(\Delta, \mathcal{M}) \tag{3.3}$$

for the Hodge–Yukawa operator exists whose integrable kernel satisfies

$$\mathrm{d}_{\ell,x}\, \mathcal{G}_\ell^\lambda(x,y) = \delta_{\ell+1,y}\, \mathcal{G}_{\ell+1}^\lambda(x,y) \qquad \text{and} \qquad \delta_{\ell,x}\, \mathcal{G}_\ell^\lambda(x,y) = \mathrm{d}_{\ell-1,y}\, \mathcal{G}_{\ell-1}^\lambda(x,y) \tag{3.4}$$

for $x \neq y$, cf. [20, Lem. 3] and [26, eq. 3.1.44]. Moreover,

$$\star_{\ell,y} \star_{\ell,x} \mathcal{G}_\ell^\lambda = \star_{\ell,x} \star_{\ell,y} \mathcal{G}_\ell^\lambda = \mathcal{G}_{n-\ell}^\lambda, \tag{3.5}$$

cf. [26, 3.1.23] and [20, Lem. 1]. At the level of the full Grassman algebra of differential forms, the identities in (3.4) translate for $\boldsymbol{\mathcal{G}}_\lambda = (\mathcal{G}_\ell^\lambda)_\ell$ to

$$\mathbf{d}_x\, \boldsymbol{\mathcal{G}}_\lambda = \boldsymbol{\delta}_y\, \boldsymbol{\mathcal{G}}_\lambda \qquad \text{and} \qquad \boldsymbol{\delta}_x\, \boldsymbol{\mathcal{G}}_\lambda = \mathbf{d}_y\, \boldsymbol{\mathcal{G}}_\lambda, \tag{3.6}$$

while property (3.5) implies that

$$\bigstar_y \bigstar_x \boldsymbol{\mathcal{G}}_\lambda = \bigstar_x \bigstar_y \boldsymbol{\mathcal{G}}_\lambda = \boldsymbol{\mathcal{G}}_\lambda.$$

3.2. Atomic boundary potentials. Consider the bounded operators

$$\mathsf{t}'_\ell : H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) \to H^{-1}_{\mathrm{comp}}\Lambda^\ell(\mathcal{M}) \qquad \text{and} \qquad \mathsf{n}'_\ell : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \to H^{-1}_{\mathrm{comp}}\Lambda^\ell(\mathcal{M}),$$

dual to the trace mappings in (1.8). As previously stated, the *atomic* boundary potentials

$$\begin{aligned} \mathsf{S}^\ell_\lambda &= \mathsf{N}_\ell^\lambda\, \mathsf{t}'_\ell : H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\Gamma) \longrightarrow H^1_{\mathrm{loc}}\Lambda^\ell(\mathcal{M}), \\ \mathsf{D}^\ell_\lambda &= \mathsf{N}_\ell^\lambda\, \mathsf{n}'_{\ell-1} : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell-1}(\Gamma) \longrightarrow H^1_{\mathrm{loc}}\Lambda^\ell(\mathcal{M}), \end{aligned} \tag{3.7}$$

take center stage throughout this article, cf. [11], [13] and [33].

If $u_\ell \in L^1\Lambda^\ell(\Gamma)$, it follows by symmetry of the fundamental solution that for $x \notin \Gamma$, they admit the integral representations

$$\mathsf{S}_\ell^\lambda u_\ell(x) = \langle u_\ell, \mathsf{t}_\ell\, \mathcal{G}_\ell^\lambda(x,\cdot)\rangle_\Gamma \qquad \text{and} \qquad \mathsf{D}_\ell^\lambda u_{\ell-1}(x) = \langle u_{\ell-1}, \mathsf{n}_\ell \mathcal{G}_\ell^\lambda(x,\cdot)\rangle_\Gamma, \tag{3.8}$$

cf. [23, Thm. 6.10] and [32, Thm. 3.1.6].

Since $\star_\ell$ is an isometry, we observe using (3.5) that for $x \notin \Gamma$,

$$\begin{aligned} \star_{\ell+1}^{-1}\langle \star_\ell\, u_\ell(\cdot), \mathsf{t}_{N-\ell-1}\, \mathcal{G}_{N-\ell-1}^\lambda(x,\cdot)\rangle_\Gamma &= \langle\, u_\ell(\cdot), \star_\ell^{-1} \mathsf{t}_{N-\ell-1} \underset{(\cdot)}{\star_{\ell+1}}\, \mathcal{G}_{\ell+1}^\lambda(x,\cdot)\rangle_\Gamma \\ &= \langle\, u_\ell(\cdot), \mathsf{n}_{\ell+1}\mathcal{G}_{\ell+1}^\lambda(x,\cdot)\rangle_\Gamma. \end{aligned}$$

Therefore, a density argument eventually shows that

$$\star_\ell^{-1}\, \mathsf{S}_{N-\ell}^\lambda \star_{\ell-1} = \mathsf{D}_\ell^\lambda \qquad \text{and} \qquad \bigstar^{-1}\, \mathbf{S}_\lambda \bigstar = \mathbf{D}_\lambda, \tag{3.9}$$

where a bold font denote the boundary potentials $\mathbf{S}_\lambda = (\mathsf{S}_\ell^\lambda)_\ell$ and $\mathbf{D}_\lambda = (\mathsf{D}_\ell^\lambda)_\ell$ acting on the full algebra of differential forms.

Lemma 3.1. *For all $v_\ell \in H_{\|}^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma)$ and $u_\ell \in H_{\perp}^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$,*

$$\delta_\ell\, \mathsf{S}_\ell^\lambda(v_\ell) = \mathsf{S}_{\ell-1}^\lambda(\delta_\ell\, v_\ell) \qquad \text{and} \qquad \mathrm{d}_{\ell+1}\, \mathsf{D}_{\ell+1}^\lambda(u_\ell) = -\mathsf{D}_{\ell+2}^\lambda(\mathrm{d}_\ell\, u_\ell).$$

Proof. We refer to [20, Lem. 3] and [26, Eq. 3.2.41] for the first identity. The second then follows as a consequence of (3.9), but can also be verified directly using (3.4) and integration by parts as follows.

Let $u_\ell \in L^\infty\Lambda^\ell(\Gamma) \cap H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$ be the tangential trace of a smooth ℓ-form on $\mathcal{M}$. Then, for $x \notin \Gamma$, we can evaluate directly using the integral representation of the boundary potential that

$$\begin{aligned}
\mathrm{d}_{\ell+1}\mathsf{D}^\lambda_{\ell+1}u_\ell(x) &= \int_\Gamma u_\ell(y) \wedge_y \imath_y^* \star_{\ell+1,y} \mathrm{d}_{\ell+1,x}\, \mathcal{G}^\lambda_{\ell+1,\ell+1}(x,y)\,\mathrm{d}y \\
&= \int_\Gamma u_\ell(y) \wedge_y \imath_y^* \star_{\ell+1,y} \delta_{\ell+2,y}\, \mathcal{G}^\lambda_{\ell+2,\ell+2}(x,y)\,\mathrm{d}y \qquad (3.10)\\
&= (-1)^{\ell+2}\int_\Gamma u_\ell(y) \wedge_y \mathrm{d}_{n-\ell-2,y}\, \imath_y^* \star_{\ell+2,y} \mathcal{G}^\lambda_{\ell+2,\ell+2}(x,y)\,\mathrm{d}y \qquad (3.11)\\
&= -(-1)^{\ell}(-1)^{\ell+2}\int_\Gamma \mathrm{d}_{\ell,y}\, u_\ell(y) \wedge_y \imath_y^* \star_{\ell+2,y} G^\lambda_{\ell+2,\ell+2}(x,y)\,\mathrm{d}y \qquad (3.12)\\
&= -\langle \mathrm{d}_\ell\, u_\ell, \mathsf{n}_{\ell+2}\mathcal{G}^\lambda_{\ell+2,\ell+2}\rangle_\Gamma,
\end{aligned}$$

where (3.10) is obtained using (3.4), (3.11) holds because the exterior derivative commutes with pullbacks, and (3.12) follows by integration by parts. ∎

Corollary 3.2. *For all* $\boldsymbol{v} \in H_\parallel^{-\frac{1}{2}}\Lambda(\delta,\Gamma)$ *and* $\boldsymbol{u} \in H_\perp^{-\frac{1}{2}}\Lambda(\mathrm{d},\Gamma)$*,*

$$\boldsymbol{\delta}\mathbf{S}_\lambda(\boldsymbol{v}) = \mathbf{S}_\lambda(\boldsymbol{\delta v}) \qquad \text{and} \qquad \mathbf{dD}_\lambda(\boldsymbol{u}) = -\mathbf{D}_\lambda(\mathbf{d}\boldsymbol{u}).$$

Lemma 3.3. *The boundary potentials restrict to continuous mappings*

$$\begin{aligned}
\mathsf{S}^\ell_\lambda &: H_\parallel^{-\frac{1}{2}}\Lambda^\ell(\delta,\Gamma) \longrightarrow H^1_{\mathrm{loc}}\Lambda^\ell(\mathcal{M}) \cap H\Lambda^\ell(-\Delta,\Omega),\\
\mathsf{D}^{\ell+1}_\lambda &: H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma) \longrightarrow H^1_{\mathrm{loc}}\Lambda^{\ell+1}(\mathcal{M}) \cap H\Lambda^{\ell+1}(-\Delta,\Omega),
\end{aligned}$$

satisfying, in the sense of distributions,

$$(-\Delta_\ell + \lambda)\,\mathsf{S}^\lambda_\ell(u_\ell) = 0, \qquad \textit{and} \qquad (-\Delta_{\ell+1} + \lambda)\,\mathsf{D}^\lambda_{\ell+1}(w_\ell) = 0, \tag{3.13}$$

for all $u_\ell \in H_\parallel^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma)$ *and* $w_\ell \in H_\perp^{-\frac{1}{2}}\Lambda^\ell(\mathrm{d},\Gamma)$*.*

Proof. For the first identity in (3.13), we refer to [26, Eq. 3.2.5] and [20, Lem. 3 (ii)]. The second is obtained as a corollary using (3.9), because the Hodge star commutes with the Hodge–Laplacian [26, Lem. 2.8]. ∎

Denote the jump of a trace across Γ by $[\![\bullet]\!] = \bullet^+ - \bullet^-$, where $\bullet = \mathsf{t}$ or n.

Lemma 3.4. *We have the jump relations*

$$\begin{aligned}
&[\![\mathsf{t}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, && [\![\mathsf{t}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, && [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, && (3.14a)\\
&[\![\mathsf{n}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, && [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{S}^\lambda_\ell = -\mathrm{Id}, && [\![\mathsf{n}_{\ell-1}\delta_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, && (3.14b)\\
&[\![\mathsf{t}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0, && [\![\mathsf{t}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0, && [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\,\mathsf{D}^\lambda_\ell = \mathrm{Id}, && (3.14c)\\
&[\![\mathsf{n}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0, && [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0, && [\![\mathsf{n}_{\ell-1}\delta_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0. && (3.14d)
\end{aligned}$$

Proof. We appeal to continuity and [20, Lem. 10], which already gives us

$$[\![\mathsf{t}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, \qquad [\![\mathsf{t}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0,$$
$$[\![\mathsf{n}_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{S}^\lambda_\ell = -\mathrm{Id}, \qquad [\![\mathsf{n}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0.$$

Based on these jump identities, the commutative relations for the trace operators in (1.11) and for the boundary potentials given in Lemma 3.1 immediately yields

$$[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, \qquad [\![\mathsf{t}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0, \qquad [\![\mathsf{n}_{\ell-1}\delta_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0,$$
$$[\![\mathsf{n}_{\ell-1}\delta_\ell]\!]\,\mathsf{S}^\lambda_\ell = 0, \qquad [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\,\mathsf{D}^\lambda_\ell = 0.$$

It only remains to verify that $[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\,\mathsf{D}^\lambda_\ell = \mathrm{Id}$. Using (3.9), we expand the definition of the codifferential to get

$$\begin{aligned}\mathsf{t}_{\ell-1}\delta_\ell\,\mathsf{D}^\lambda_\ell &= (-1)^\ell \mathsf{t}_{\ell-1} \star^{-1}_{\ell-1} \mathrm{d}_{N-\ell} \star_\ell \star^{-1}_\ell \mathsf{S}^\lambda_{N-\ell} \star_{\ell-1} \\ &= (-1)^\ell \mathsf{t}_{\ell-1} \star^{-1}_{\ell-1} \mathrm{d}_{N-\ell}\,\mathsf{S}^\lambda_{N-\ell} \star_{\ell-1}.\end{aligned}$$

It is a tedious but straightforward calculation to invert the domain and boundary Hodge star operators to further conclude that

$$\mathsf{t}_{\ell-1}\delta_\ell\,\mathsf{D}^\lambda_\ell = (-1)^\ell(-1)^{\ell+1}\mathsf{t}_{\ell-1} \star_{N-\ell+1} \mathrm{d}_{N-\ell}\,\mathsf{S}^\lambda_{N-\ell} \star^{-1}_{N-\ell} = \mathsf{t}_{\ell-1}\Phi,$$

where we have recognized the double layer potential

$$\Phi = -\star_{N-\ell+1} \mathrm{d}_{N-\ell}\,\mathsf{S}^\lambda_{N-\ell} \star^{-1}_{N-\ell}$$

studied in [20] and for which we know from [20, Lem. 10] that $[\![\mathsf{t}_{\ell-1}]\!]\,\Phi = \mathrm{Id}$. ∎

3.3. Trace de Rham complex with non-local inner products. The trace de Rham complexes introduced in (1.7a) and (1.7b) are the canvas on which the theory in this article is drawn. Ultimately, it is by formulating the first-kind boundary integral equations for the Hodge–Dirac and Hodge–Laplace operators as variational problems in the trace de Rham complex that their structure is revealed with the most clarity.

We generalize to differential forms of any degree and to arbitrary dimension the theory presented in [33, Sec. 8]. It should be compared with [18] and [27].

The key observation is that the continuous bilinear forms

$$(u_\ell, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}} = \langle\!\langle u_\ell, \mathsf{t}_\ell \mathsf{S}^\lambda_\ell(\overline{v}_\ell)\rangle\!\rangle_\Gamma, \qquad u_\ell, v_\ell \in H^{-\frac{1}{2}}_\parallel \Lambda^\ell(\Gamma), \tag{3.15a}$$

$$(w_\ell, z_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} = \langle\!\langle w_\ell, \mathsf{n}_{\ell+1} \mathsf{D}^\lambda_\ell(\overline{z}_\ell)\rangle\!\rangle_\Gamma, \qquad w_\ell, z_\ell \in H^{-\frac{1}{2}}_\perp \Lambda^\ell(\Gamma), \tag{3.15b}$$

define non-local inner products on the spaces $H^{-\frac{1}{2}}_\parallel \Lambda^\ell(\Gamma)$ and $H^{-\frac{1}{2}}_\perp \Lambda^\ell(\Gamma)$.

In the following results, it is a convenient notation to write

$$\|U_\ell\|^2_{\lambda, X\Lambda^\ell(\mathcal{M})} = \lambda\,\|U_\ell\|^2_{\mathcal{M}} + \|\mathrm{d}_\ell U_\ell\|^2_{\mathcal{M}} + \|\delta_\ell U_\ell\|^2_{\mathcal{M}}, \qquad \forall U_\ell \in X\Lambda^\ell(\mathcal{M}),$$

where we allow $\lambda = 0$. Evidently, $\mathcal{M}$ can be replaced by $\Omega^\mp$.

Lemma 3.5. *For all $h_\ell \in H^{-\frac{1}{2}}_\parallel \Lambda^\ell(\Gamma)$ and $g_\ell \in H^{-\frac{1}{2}}_\perp \Lambda^\ell(\Gamma)$, we have*

$$\|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}} = \langle\!\langle \overline{h}_\ell, \mathsf{t}_\ell \mathsf{S}^\lambda_\ell(v_\ell)\rangle\!\rangle_\Gamma = \left\|\mathsf{S}^\lambda_\ell h_\ell\right\|^2_{\lambda, X\Lambda^\ell(\mathcal{M})}, \tag{3.16a}$$

$$\|g_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{n}} = \langle\!\langle \overline{u}_\ell, \mathsf{n}_\ell \mathsf{S}^\lambda_\ell(g_\ell)\rangle\!\rangle_\Gamma = \left\|\mathsf{D}^\lambda_\ell g_\ell\right\|^2_{\lambda, X\Lambda^\ell(\mathcal{M})}. \tag{3.16b}$$

Proof. For convenience, let us shorthand $\Psi = \mathsf{S}^\lambda_\ell h_\ell$. Integrating by parts the first term on the right-hand side of

$$\|\Psi\|^2_{\lambda, X\Lambda^\ell(\mathcal{M})} = \|\Psi\|^2_{\lambda, X\Lambda^\ell(\lambda,\Omega^-)} + \|\Psi\|^2_{\lambda, X\Lambda^\ell(\lambda,\Omega^+)},$$

we obtain

$$\begin{aligned}\|\Psi\|^2_{X\Lambda^\ell(\lambda,\Omega^-)} &= (\mathrm{d}_\ell\Psi, \mathrm{d}_\ell\Psi)_{\Omega^-} + (\delta_\ell\Psi, \delta_\ell\Psi)_{\Omega^-} + \lambda\|\Psi\|^2_{\Omega^-} \\ &= (-\Delta_\ell\Psi, \Psi)_{\Omega^-} + \langle\!\langle \mathsf{n}^-_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma - \langle\!\langle \mathsf{t}^-_{\ell-1}\delta_\ell\Psi, \mathsf{n}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma + \lambda\|\Psi\|^2_{\Omega^-} \\ &= \langle\!\langle \mathsf{n}^-_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma - \langle\!\langle \mathsf{t}^-_{\ell-1}\delta_\ell\Psi, \mathsf{n}^-_\ell\overline{\Psi}\rangle\!\rangle_\Gamma,\end{aligned}$$

where we have used the fact that Ψ satisfies the equation $-\Delta\Psi = -\lambda\Psi$ in Ω^-, i.e. $(-\Delta_\ell\Psi, \Psi)_{\Omega^-} = \lambda(\Psi, \Psi)_{\Omega^-} = \|\Psi\|^2_{\Omega^-}$. We find similarly in Ω^+ that

$$\|\Psi\|^2_{\lambda, X\Lambda^\ell(\Omega^+)} = -\langle\!\langle \mathsf{n}^+_{\ell+1}\mathrm{d}_\ell\Psi, \mathsf{t}^+_\ell\overline{\Psi}\rangle\!\rangle_\Gamma + \langle\!\langle \mathsf{t}^+_{\ell-1}\delta_\ell\Psi, \mathsf{n}^+_\ell\overline{\Psi}\rangle\!\rangle_\Gamma$$

Summing these contributions and using the jump relations from Lemma 3.4 yields

$$\|\Psi\|^2_{\lambda, X\Lambda^\ell(\mathcal{M})} = \langle\!\langle -[\![\, \mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\Psi, \mathsf{t}\overline{\Psi}\rangle\!\rangle_\Gamma = \langle\!\langle \overline{h}_\ell, \mathsf{t}\mathsf{S}^\lambda_\ell u_\ell\rangle\!\rangle_\Gamma = \|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}} \qquad \square$$

The next result generalizes [8, Thm. 4] to arbitrary dimensions. We indicate inequalities that hold up to a positive constant multiple depending on Ω using $\lesssim$.

THEOREM 3.6. *Under Assumption A, we have*

$$\begin{aligned}\|h_\ell\|^2_{H_\|^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} &\lesssim (h_\ell, h_\ell)_{-\frac{1}{2},\lambda,\mathsf{t}}, && \forall h_\ell \in H_\|^{-\frac{1}{2}}\Lambda^\ell(\Gamma), \\ \|g_\ell\|^2_{H_\perp^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} &\lesssim (g_\ell, g_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}, && \forall w_\ell \in H_\perp^{-\frac{1}{2}}\Lambda^\ell(\Gamma).\end{aligned}$$

Proof. We focus on the first inequality. The second can be obtained using analogous arguments.

In the first step, we use two ingredients:

- Recall from Subsection 1.4 that the tangential trace $\mathsf{t}_\ell : H^1\Lambda^\ell(\Omega) \to H_\|^{\frac{1}{2}}\Lambda^\ell(\Gamma)$ is a *surjective* operator admitting a bounded right-inverse

$$\mathsf{t}^\dagger : H_\|^{\frac{1}{2}}\Lambda^\ell(\Gamma) \to H^1\Lambda^\ell(\Omega),$$

  i.e $\left\|\mathsf{t}^\dagger g_\ell\right\|_{H^1\Lambda^\ell(\Omega)} \lesssim \|g_\ell\|_{H_\|^{\frac{1}{2}}\Lambda^\ell}(\Gamma)$ and $\mathsf{t}_\ell \circ \mathsf{t}^\dagger g_\ell = g_\ell$ for all $g_\ell \in H_\|^{\frac{1}{2}}\Lambda^\ell(\Gamma)$.
- According to [27, prop. 3.1], there exists a continuous *extension* operator

$$\mathsf{E} : H^1\Lambda^\ell(\Omega) \to H^1\Lambda^\ell(\mathcal{M})$$

  such that $(\mathsf{E}\, U_\ell)\big|_\Omega = U_\ell$ for all $U_\ell \in H^1\Lambda(\Omega)$.

Given $h_\ell \in H_\|^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$, we can introduce these operators in the definition of

$H_{\parallel}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)$ to obtain the estimate

$$\begin{aligned}
\|h_\ell\|^2_{H_{\parallel}^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} &= \sup_{g_\ell\in H_{\parallel}^{\frac{1}{2}}\Lambda^\ell(\Gamma)} \frac{\left|\langle\!\langle h_\ell, g_\ell\rangle\!\rangle_\Gamma\right|}{\|g_\ell\|_{H_{\parallel}^{\frac{1}{2}}\Lambda^\ell(\Gamma)}} \lesssim \sup_{g_\ell\in H_{\parallel}^{\frac{1}{2}}\Lambda^\ell(\Gamma)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell\mathsf{t}^\dagger g_\ell\rangle\!\rangle_\Gamma\right|}{\left\|\mathsf{t}^\dagger g_\ell\right\|_{H^1\Lambda^\ell(\Omega)}} \\
&= \sup_{W_\ell\in H^1\Lambda^\ell(\Omega)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell W_\ell\rangle\!\rangle_\Gamma\right|}{\|W_\ell\|_{H^1\Lambda^\ell(\Omega)}} \lesssim \sup_{W_\ell\in H^1\Lambda^\ell(\Omega)} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell \mathsf{E} W_\ell\rangle\!\rangle_\Gamma\right|}{\|\mathsf{E}W_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}} \\
&\lesssim \sup_{W_\ell\in H^1\Lambda^\ell(\mathcal{M})} \frac{\left|\langle\!\langle h_\ell, \mathsf{t}_\ell W_\ell\rangle\!\rangle_\Gamma\right|}{\|W_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}}.
\end{aligned} \tag{3.17}$$

In the second step, we recognize in the numerator the definition of the atomic boundary potential. Recall that $\mathsf{S}_\ell^\lambda = \mathsf{N}_\ell^\lambda \circ \mathsf{t}'_\ell = (-\Delta_\ell + \lambda \mathrm{Id})^{-1} \circ \mathsf{t}'_\ell$. In other words, $\mathsf{S}_\ell^\lambda h_\ell$ satisfies the variational equation

$$\left(\mathrm{d}_\ell \mathsf{S}_\ell^\lambda g_\ell, \mathrm{d}_\ell V_\ell\right)_\Omega + \left(\delta_\ell \mathsf{S}_\ell^\lambda g_\ell, \delta_\ell V_\ell\right)_\Omega + \lambda\left(\mathsf{S}_\ell^\lambda g_\ell, V_\ell\right)_\Omega = \langle\!\langle \mathsf{t}'_\ell h_\ell, V_\ell\rangle\!\rangle_\Gamma = \langle\!\langle h_\ell, \mathsf{t}_\ell V_\ell\rangle\!\rangle_\Gamma \tag{3.18}$$

for all $V_\ell \in H^1\Lambda^\ell(\Omega)$. Hence, we arrive at the identify

$$\left|\langle\!\langle h_\ell, \mathsf{t}_\ell V_\ell\rangle\!\rangle_\Gamma\right| = \left|\left(\mathsf{S}_\ell^\ell h_\ell, V_\ell\right)_{\lambda, X\Lambda^\ell(\mathcal{M})}\right| \tag{3.19}$$

and plug it into (3.17) to obtain

$$\|h_\ell\|^2_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \sup_{V_\ell\in H^1\Lambda^\ell(\mathcal{M})} \frac{\left|\left(\mathsf{S}_\ell^\ell h_\ell, V_\ell\right)_{\lambda, X\Lambda^\ell(\mathcal{M})}\right|}{\|V_\ell\|_{H^1\Lambda^\ell(\mathcal{M})}}. \tag{3.20}$$

In the third step, we simply appeal a Gaffney inequality (the easy direction), which states that

$$\|V_\ell\|_{H^1\Lambda^\ell(\mathcal{M})} \sim \|V_\ell\|_{\lambda, X\Lambda^\ell(\mathcal{M})} \tag{3.21}$$

for all $V_\ell \in H^1\Lambda^\ell(\Omega)$, cf. [37], [29, Thm. 7.2.6], [6]. Going back to (3.20) and applying the Cauchy-Schwartz inequality yields

$$\|h_\ell\|^2_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \sup_{V_\ell\in X\Lambda^\ell(\mathcal{M})} \frac{\left|\left(\mathsf{S}_\ell^\ell h_\ell, V_\ell\right)_{\lambda, X\Lambda^\ell(\mathcal{M})}\right|}{\|V_\ell\|_{X\Lambda^\ell(\lambda,\mathcal{M})}} = \left\|\mathsf{S}_\ell^\lambda h_\ell\right\|^2_{\lambda, X\Lambda^\ell(\mathcal{M})}. \tag{3.22}$$

□

Finally, using Lemma 3.5, we arrive at

$$\|h_\ell\|^2_{H^{-\frac{1}{2}}\Lambda^\ell(\Gamma)} \lesssim \|h_\ell\|^2_{-\frac{1}{2},\lambda,\mathsf{t}}.$$

4. Boundary integral equations. In this section, we exploit the results of Section 3 to derive boundary integral equations for the BVPs of Section 2. We follow the same recipe for every operator. The approach has a long history. Standard references for scalar-valued BVPs in Euclidean space are [24, 32] and [38]. We also particularly recommend [13]. We refer to [10] for classical electromagnetism, where the perspective is also prominently adopted. As mentioned in the introduction—and directly relevant to this work—the abstract procedure was used to derive BIEs for the

Hodge–Dirac and Hodge–Helmholtz operators using vector calculus in [12] and [33] under the hypothesis that $\mathcal{M} = \mathbb{R}^3$. An overview similar to what follows is given in [34].

Let $\mathfrak{L}$ stand for any one of the operators introduced in Section 2: $\mathfrak{D}+i\kappa$, $-\Delta_\ell+\lambda$, $\mathfrak{M}$ or $\mathfrak{R}$.

1. We confirm that the operator satisfies an identity of the form

$$\langle \mathfrak{L}U, V\rangle_{\Omega^\mp} = \langle U, \mathfrak{L}V\rangle_{\Omega^\mp} \pm \langle\langle \mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}U, \mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}V\rangle\rangle_\Gamma \mp \langle\langle \mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}V, \mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}U\rangle\rangle_\Gamma \tag{4.1}$$

   resembling Green's second formula and identify a "Newton potential operator" $\mathsf{N}[\mathfrak{N}]$, i.e. an inverse of $\mathfrak{L}$ in the sense of distributions. Together, these two ingredients enable us to find a representation formula of the form

$$U = \mathsf{N}[\mathfrak{L}]U - \mathsf{SL}[\mathfrak{L}]\left([\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}U]\!]\right) + \mathsf{DL}[\mathfrak{L}]\left([\![\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}U]\!]\right), \tag{4.2}$$

   where $\mathsf{SL}[\mathfrak{L}]$ and $\mathsf{DL}[\mathfrak{L}]$ are potential operators playing roles analogous to the single and double layer potentials in the classical theory of BIEs for scalar Laplace problems or electromagnetic scattering.
2. We apply average traces $\{\mathsf{T}^{\bullet}_{\mathfrak{L}}\} = (\mathsf{T}^{\bullet,-}_{\mathfrak{L}} + \mathsf{T}^{\bullet,+}_{\mathfrak{L}})/2$ to the obtained boundary potentials $\mathsf{SL}[\mathfrak{L}]$ and $\mathsf{DL}[\mathfrak{L}]$ to define four BIOs:

$$\begin{aligned}
\mathsf{V}[\mathfrak{L}] &= \{\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}\}\mathsf{SL}[\mathfrak{L}] : H^{\mathsf{n}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathsf{t}}_{\mathfrak{L}}(\Gamma),\\
\mathsf{K}[\mathfrak{L}] &= \{\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}\}\mathsf{DL}[\mathfrak{L}] : H^{\mathsf{t}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathsf{t}}_{\mathfrak{L}}(\Gamma),\\
\mathsf{A}[\mathfrak{L}] &= \{\mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}\}\mathsf{SL}[\mathfrak{L}] : H^{\mathsf{n}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathsf{n}}_{\mathfrak{L}_\ell}(\Gamma),\\
\mathsf{W}[\mathfrak{L}] &= \{\mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}\}\mathsf{DL}[\mathfrak{L}] : H^{\mathsf{t}}_{\mathfrak{L}}(\Gamma) \longrightarrow H^{\mathsf{n}}_{\mathfrak{L}}(\Gamma).
\end{aligned}$$

3. We verify that the jump relations

$$\begin{aligned}
&[\![\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}]\!]\mathsf{SL}[\mathfrak{L}] = 0, && [\![\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}]\!]\mathsf{DL}[\mathfrak{L}] = \mathrm{Id},\\
&[\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}]\!]\mathsf{SL}[\mathfrak{L}] = -\mathrm{Id}, && [\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}]\!]\mathsf{DL}[\mathfrak{L}] = 0,
\end{aligned}$$

   hold in the trace spaces $T^{\mathsf{t}}_{\mathfrak{L}}$ and $T^{\mathsf{n}}_{\mathfrak{L}}$. Applying average traces on both sides of the representation formula and appealing to these jump relations lead to a Calderón operator

$$\mathsf{C}[\mathfrak{L}] = \begin{pmatrix} \frac{1}{2}\mathrm{Id} + \mathsf{K}[\mathfrak{L}] & -\mathsf{V}[\mathfrak{L}] \\ -\mathsf{W}[\mathfrak{L}] & \frac{1}{2}\mathrm{Id} - \mathsf{A}[\mathfrak{L}] \end{pmatrix} \tag{4.3}$$

   whose kernel fully characterizes the space of valid Cauchy data. In other words, boundary data $(g,h)^\top \in T^{\mathsf{t}}_{\mathfrak{L}}(\Gamma) \times T^{\mathsf{n}}_{\mathfrak{L}}(\Gamma)$ satisfies $\mathsf{C}[\mathfrak{L}](g,h)^\top = 0$ if and only if there exists $U \in \mathrm{dom}(\mathfrak{L})$ such that $\mathfrak{L}U = 0$ with $\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}U = g$ and $\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}U = h$.
4. We extract two first-kind BIEs:

$$h \in T^{\mathsf{n}}_{\mathfrak{L}}(\Gamma): \qquad \mathsf{V}[\mathfrak{L}]h = (\frac{1}{2}\mathrm{Id} + \mathsf{K}[\mathfrak{L}])g, \qquad g \in T^{\mathsf{t}}_{\mathfrak{L}}(\Gamma), \tag{4.4a}$$

$$g \in H^{\mathsf{t}}_{\mathfrak{L}}(\Gamma): \qquad \mathsf{W}[\mathfrak{L}]g = (\frac{1}{2}\mathrm{Id} - \mathsf{A}[\mathfrak{L}])h, \qquad h \in H^{\mathsf{n}}_{\mathfrak{L}}(\Gamma). \tag{4.4b}$$

   It suffices to take duality pairing on both sides of these equations to obtain the equivalent variational problems

$$\langle\langle \mathsf{V}[\mathfrak{L}]h, \overline{w}\rangle\rangle_\Gamma = \langle\langle (\frac{1}{2}\mathrm{Id} + \mathsf{K}[\mathfrak{L}])g, \overline{w}\rangle\rangle_\Gamma, \qquad \forall w \in T^{\mathsf{n}}_{\mathfrak{L}}(\Gamma), \tag{4.5a}$$

$$\langle\langle \mathsf{W}[\mathfrak{L}]g, \overline{v}\rangle\rangle_\Gamma = \langle\langle (\frac{1}{2}\mathrm{Id} - \mathsf{A}[\mathfrak{L}])h, \overline{v}\rangle\rangle_\Gamma, \qquad \forall v \in T^{\mathsf{t}}_{\mathfrak{L}}(\Gamma). \tag{4.5b}$$

We will show that the first-kind BIOs operators $\mathsf{V}[\mathfrak{L}]$ and $\mathsf{W}[\mathfrak{L}]$ associated with the bilinear forms on the left-hand side of these variational problems are Hodge–Dirac and Hodge–Laplace operators in variational form in the trace de Rham complex with non-local inner products presented in Subsection 3.3.

Remark 4.1. The signs in (4.3), and thus accordingly in (4.4a) and (4.4b), were chosen as per convention to mimic well-known theory for the scalar Laplacian. This choice is somewhat arbitrary and the equations can be altered to avoid some sign flips that occur in the next sections. We restrain ourselves from doing so as we do not believe there is much to gain so far by departing from classical sign conventions.

4.1. BIEs for Hodge–Dirac BVPs. In this section, the abstract theory is instantiated according to the following table.

$\mathfrak{L}$	$\mathrm{dom}(\mathfrak{L})$	$\mathsf{T}^{\mathsf{t}}_{\mathfrak{L}}$	$\mathsf{T}^{\mathsf{n}}_{\mathfrak{L}}$	$H^{\mathsf{t}}_{\mathfrak{L}}(\Gamma)$	$H^{\mathsf{n}}_{\mathfrak{L}}(\Gamma)$
$\mathfrak{D} + i\kappa$	$H\Lambda(\mathfrak{D}, \Omega)$	$\mathbf{t}$	$\mathbf{n}$	$H_{\perp}^{-\frac{1}{2}}\Lambda(\mathrm{d}, \Gamma)$	$H_{\parallel}^{-\frac{1}{2}}\Lambda(\delta, \Gamma)$

Integration by parts reveals that the Hodge–Dirac operator satisfies an identity such as (4.1):

$$\langle \mathfrak{D}\boldsymbol{U}, \boldsymbol{V} \rangle_{\Omega^{\mp}} = \langle \boldsymbol{U}, \mathfrak{D}\boldsymbol{V} \rangle_{\Omega^{\mp}} \pm \langle\!\langle \mathbf{t}\,\boldsymbol{U}, \mathbf{n}\,\boldsymbol{V} \rangle\!\rangle_{\Gamma} \mp \langle\!\langle \mathbf{t}\,\boldsymbol{V}, \mathbf{n}\,\boldsymbol{U} \rangle\!\rangle_{\Gamma} \tag{4.6}$$

for all $\boldsymbol{U}, \boldsymbol{V} \in H\Lambda(\mathfrak{D}, \Omega)$

Remark 4.2. It is remarkable that an identity resembling Green's second formula is available *despite* the operator being only first-order. Evidently, this is due to its symmetric structure. The Hodge–Dirac operator is a sum of two operators that are formally adjoint to each other. Thanks to that, a representation by boundary potentials can be derived using the approach promoted by Costabel for second order elliptic operators, cf. [11, sect. 4.2], [13], [33, sect. 4.4] and [34, Sec. 2.4].

4.1.1. Representation formula for Hodge–Dirac operators. This section generalizes [33, Sec. 4]. It follows immediately from (3.6) that

$$\mathfrak{D}_x \boldsymbol{\mathcal{G}}_\lambda = \mathfrak{D}_y \boldsymbol{\mathcal{G}}_\lambda.$$

Integrating by parts after using the commutative relations (3.6) eventually verifies that

$$\mathbf{N}_\lambda\,\mathfrak{D} = \mathfrak{D}\,\mathbf{N}_\lambda \tag{4.7}$$

in the sense of distributions. Going back to (1.1) with $\lambda = \kappa^2$, we find that

$$(\mathfrak{D} - i\kappa)\,\mathbf{N}_\lambda\,(\mathfrak{D} + i\kappa) = (-\boldsymbol{\Delta} + \kappa^2)\mathbf{N}_\lambda = \mathrm{Id}.$$

In other words,

$$\mathsf{N}[\mathfrak{D}] = (\mathfrak{D} - i\kappa)\,\mathbf{N}_\lambda = \mathbf{N}_\lambda\,(\mathfrak{D} - i\kappa)$$

is a fundamental solution for the perturbed Hodge–Dirac operator $(\mathfrak{D} + i\kappa)$.

PROPOSITION 4.3. *If $\boldsymbol{U} \in L^2\Lambda(\mathcal{M})$ is compactly supported and there exists $\boldsymbol{F} \in L^2\Lambda(\mathcal{M})$ such that $\boldsymbol{F}|_{\Omega} = (\mathfrak{D} + i\kappa)\boldsymbol{U}|_{\Omega}$ and $\boldsymbol{F}|_{\Omega^+} = (\mathfrak{D} + i\kappa)\boldsymbol{U}|_{\Omega^+}$, then*

$$\boldsymbol{U} = (\mathfrak{D} - i\kappa)\left(\mathbf{N}_\lambda \boldsymbol{F} - \mathbf{S}_\lambda[\![\mathsf{n}\boldsymbol{U}]\!] + \mathbf{D}_\lambda[\![\mathsf{t}\boldsymbol{U}]\!]\right). \tag{4.8}$$

Proof. According to (4.6), we have

$$\begin{aligned}\langle\!\langle (\mathfrak{D}+i\kappa)\,\boldsymbol{U}, \boldsymbol{V}\rangle\!\rangle_{\mathcal{M}} &= \langle \boldsymbol{U}, (\mathfrak{D}+i\kappa)\boldsymbol{V}\rangle_{\Omega} + \langle \boldsymbol{U}, (\mathfrak{D}+i\kappa)\boldsymbol{V}\rangle_{\Omega^+}\\ &= \langle \boldsymbol{F}, \boldsymbol{V}\rangle_{\Omega} + \langle\!\langle \mathbf{t}^-\boldsymbol{V}, \mathbf{n}^-\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} - \langle\!\langle \mathbf{t}^-\,\boldsymbol{U}, \mathbf{n}^-\boldsymbol{V}\rangle\!\rangle_{\Gamma}\\ &\qquad + \langle \boldsymbol{F}, \boldsymbol{V}\rangle_{\Omega^+} - \langle\!\langle \mathbf{t}^+\boldsymbol{V}, \mathbf{n}^+\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} + \langle\!\langle \mathbf{t}^+\,\boldsymbol{U}, \mathbf{n}^+\boldsymbol{V}\rangle\!\rangle_{\Gamma}\\ &= \langle \boldsymbol{F}, \boldsymbol{V}\rangle_{\mathcal{M}} - \langle\!\langle \mathbf{t}\,\boldsymbol{V}, [\![\mathbf{n}]\!]\,\boldsymbol{U}\rangle\!\rangle_{\Gamma} + \langle\!\langle [\![\mathbf{t}]\!]\boldsymbol{U}, \mathbf{n}\boldsymbol{V}\rangle\!\rangle_{\Gamma}\end{aligned}$$

for all $\boldsymbol{V} \in \mathcal{D}(\mathcal{M})$. The regularity assumption on $\boldsymbol{U}$ guarantees that the traces are well-defined. We have used the fact that $\boldsymbol{V}$ is smooth across the boundary to obtain the last equality, because smoothness guarantees that $\mathbf{t}^+\boldsymbol{V} = \mathbf{t}^-\boldsymbol{V}$ and $\mathbf{n}^+\boldsymbol{V} = \mathbf{n}^-\boldsymbol{V}$, i.e. the jumps vanish on Γ. Hence, in the sense of distributions,

$$(\mathfrak{D}+i\kappa)\,\boldsymbol{U} = \boldsymbol{F} - \mathbf{t}'\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathbf{n}'\,[\![\mathbf{t}\,\boldsymbol{U}]\!].$$

Since $\boldsymbol{U}$ has compact support, it can be interpreted as a continuous linear functional on $\boldsymbol{\mathcal{E}}(\mathcal{M})$. With the definitions of the atomic boundary potentials from (3.7) at hand, applying the Newton potential operator $\mathbf{N}_{\boldsymbol{\lambda}}$ on both sides of this equation yields

$$\begin{aligned}\mathbf{N}_{\boldsymbol{\lambda}}(\mathfrak{D}+i\kappa)\,\boldsymbol{U} &= \mathbf{N}_{\boldsymbol{\lambda}}\boldsymbol{F} - \mathbf{N}_{\boldsymbol{\lambda}}\mathbf{t}'\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathbf{N}_{\boldsymbol{\lambda}}\mathbf{n}'\,[\![\mathbf{t}\,\boldsymbol{U}]\!]\\ &= \mathbf{N}_{\boldsymbol{\lambda}}\boldsymbol{F} - \mathbf{S}_{\lambda}\,[\![\mathbf{n}\,\boldsymbol{U}]\!] + \mathbf{D}_{\lambda}\,[\![\mathbf{t}\,\boldsymbol{U}]\!].\end{aligned} \tag{4.9}$$

Since $(\mathfrak{D}+i\kappa)\boldsymbol{U}$ is square-integrable, the mapping properties of the Newton potential provided in (3.3) ensures that the left-hand side of this identity lies in the domain of the Hodge–Dirac operator, since it is in fact component-wise weakly differentiable. Moreover, that the images of the atomic boundary potentials belong to $\boldsymbol{H}^1_{\mathrm{loc}}\Lambda(\mathcal{M})$ was the result of Lemma 3.3. Therefore, we can apply $\mathfrak{D} - i\kappa$ on both sides of (4.9) and use the commutation relation (4.7) to reach (4.8). ∎

We are tempted to call *single* and *double* layer potentials for the Hodge–Dirac operator the boundary potentials

$$\mathsf{SL}[\mathfrak{D}] = (\mathfrak{D}-i\kappa)\,\mathbf{S}_{\lambda} : H_{\parallel}^{-\frac{1}{2}}\Lambda^{\ell-1}(\delta,\Gamma) \longrightarrow H\Lambda(\mathfrak{D},\Omega), \tag{4.10a}$$

$$\mathsf{DL}[\mathfrak{D}] = (\mathfrak{D}-i\kappa)\,\mathbf{D}_{\lambda} : H_{\perp}^{-\frac{1}{2}}\Lambda^{\ell}(\mathrm{d},\Gamma) \longrightarrow H\Lambda(\mathfrak{D},\Omega), \tag{4.10b}$$

respectively. However, while this nomenclature is a convenient way to highlight the similarities between our development and the classical theory of boundary integral equations for second-order elliptic operators, we stress that it may also be misleading. Both traces in (1.6) rest on an equal footing in that none involves a differential operator. We saw in (3.9) that the two boundary potentials are not only isometrically isomorphic, but also symmetric in the sense of Hodge duality.

It follows immediately from Lemma 3.4 that these boundary potentials satisfy the abstract jump relations stated above. For example,

$$[\![\mathsf{T}^{\mathsf{n}}_{\mathfrak{D}}]\!]\mathsf{SL}[\mathfrak{D}] = [\![\mathbf{n}]\!]\,(\mathfrak{D}-i\kappa)\,\mathbf{S}_{\lambda} = [\![\mathbf{nd}]\!]\mathbf{S}_{\lambda} + [\![\mathbf{n}\boldsymbol{\delta}]\!]\mathbf{S}_{\lambda} - i\kappa[\![\mathbf{n}]\!]\mathbf{S}_{\lambda} = -\mathrm{Id}.$$

The other relations are computed similarly.

4.1.2. BIOs for Hodge–Dirac operators. Since the boundary potentials may jump across Γ, we resort as per convention to the average traces $\{\bullet\} = (\bullet^- + \bullet^+)/2$, where $\bullet = \mathsf{t}$ or n. In particular, we let

$$\begin{aligned}
\mathsf{V}[\mathfrak{D}] &= \{\mathsf{t}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}_\lambda : H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma),\\
\mathsf{K}[\mathfrak{D}] &= \{\mathsf{t}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{D}_\lambda : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma),\\
\mathsf{A}[\mathfrak{D}] &= \{\mathsf{n}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}_\lambda : H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma) \longrightarrow H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma),\\
\mathsf{W}[\mathfrak{D}] &= \{\mathsf{n}\}\,(\mathfrak{D} - i\kappa)\,\mathbf{D}_\lambda : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma) \longrightarrow H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma).
\end{aligned}$$

As a consequence of the jump relations, these boundary integral operators enter a Calderón operator $\mathsf{C}[\mathfrak{D}]$ such as (4.3) whose kernel fully characterize the space of valid Cauchy data. This last property is a consequence of three ingredients: the jump relations, the representation formula and the lifting maps from Subsection 1.4.2.

From the jump relations, $\mathsf{V}[\mathfrak{D}] = \mathsf{t}\,(\mathfrak{D} - i\kappa)\,\mathbf{S}_\lambda$, i.e. the average of the traces is equal to taking a single-sided trace. Using Corollary 3.2, it follows by integration by parts and (1.11) that

$$\begin{aligned}
\langle\!\langle\, \mathsf{V}[\mathfrak{D}]\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma &= \langle\!\langle\, \mathsf{t}\boldsymbol{\delta}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma + \langle\!\langle\, \mathsf{t}\mathbf{d}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{t}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\rangle\!\rangle_\Gamma\\
&= \langle\!\langle\, \mathsf{t}\mathbf{S}_\lambda\boldsymbol{\delta}\boldsymbol{h}, \overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma + \langle\!\langle\, \mathsf{t}\mathbf{S}_\lambda\boldsymbol{h}, \boldsymbol{\delta}\overline{\boldsymbol{w}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{t}\mathbf{S}_\lambda\boldsymbol{h}, \overline{\boldsymbol{w}}\rangle\!\rangle_\Gamma\\
&= (\boldsymbol{\delta}\boldsymbol{h}, \boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} + (\boldsymbol{h}, \boldsymbol{\delta}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} - i\kappa(\boldsymbol{h}, \boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}}.
\end{aligned} \tag{4.11}$$

for all $\boldsymbol{h}, \boldsymbol{w} \in H_{\parallel}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma)$.

Similarly, we can also compute

$$\begin{aligned}
\langle\!\langle\, \mathsf{W}[\mathfrak{D}]\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma &= \langle\!\langle\, \mathsf{n}\mathbf{d}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma + \langle\!\langle\, \mathsf{n}\boldsymbol{\delta}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{n}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\rangle\!\rangle_\Gamma\\
&= -\langle\!\langle\, \mathsf{n}\mathbf{D}_\lambda\mathbf{d}\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma - \langle\!\langle\, \mathsf{n}\mathbf{D}_\lambda\boldsymbol{g}, \mathbf{d}\overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma - i\kappa\langle\!\langle \mathsf{n}\mathbf{D}_\lambda\boldsymbol{g}, \overline{\boldsymbol{v}}\rangle\!\rangle_\Gamma\\
&= -(\mathbf{d}\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}} - (\boldsymbol{g}, \mathbf{d}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{n}} - i\kappa(\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}},
\end{aligned} \tag{4.12}$$

for all $\boldsymbol{g}, \boldsymbol{v} \in H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma)$.

We urge the reader to compare these bilinear forms on the boundary with the bilinear forms $\mathcal{A}_{\boldsymbol{\delta}}$ and $\mathcal{A}_{\mathbf{d}}$ that appear in the variational problems (2.8a) and (2.8b) for the Hodge–Dirac operator in the domain Ω.

We conclude from (4.11) and (4.12) that the first-kind boundary integral operators $\mathsf{V}[\mathfrak{D}]$ and $\mathsf{W}[\mathfrak{D}]$ associated with the direct first-kind boundary integral equations (4.4a) and (4.4b) are zero-order perturbations of Hodge–Dirac operators in the trace de Rham complexes of Subsection 3.3. More precisely,

$$\mathsf{V}[\mathfrak{D}] = \boldsymbol{\delta} + \boldsymbol{\delta}^* - i\kappa, \tag{4.13a}$$

$$\mathsf{W}[\mathfrak{D}] = -(\mathbf{d} + \mathbf{d}^*) - i\kappa, \tag{4.13b}$$

where the closed densely defined unbounded operators

$$\begin{aligned}
\boldsymbol{\delta}^* &: H_{\parallel}^{-\frac{1}{2}}\Lambda(\Gamma) \longrightarrow H_{\parallel}^{-\frac{1}{2}}\Lambda(\Gamma),\\
\mathbf{d}^* &: H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma) \longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma),
\end{aligned}$$

are the Hilbert space adjoint of the closed densely defined unbounded operators

$$\begin{aligned}\boldsymbol{\delta} : H_{\|}^{-\frac{1}{2}}\Lambda(\boldsymbol{\delta},\Gamma) \subset H_{\|}^{-\frac{1}{2}}\Lambda(\Gamma) &\longrightarrow H_{\|}^{-\frac{1}{2}}\Lambda(\Gamma),\\ \mathbf{d} : H_{\perp}^{-\frac{1}{2}}\Lambda(\mathbf{d},\Gamma) \subset H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma) &\longrightarrow H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma),\end{aligned}$$

introduced in Subsection 1.4, but where the spaces $H_{\|}^{-\frac{1}{2}}\Lambda(\Gamma)$ and $H_{\perp}^{-\frac{1}{2}}\Lambda(\Gamma)$ are equipped with the non-local inner products defined in Subsection 3.3.

As in Subsection 2.1, it follows immediately from the abstract theory for the Hodge–Dirac operator in Hilbert complexes that $\mathsf{V}[\mathfrak{D}]$ and $\mathsf{W}[\mathfrak{D}]$ are invertible for $\kappa \neq 0$. They are Fredholm operators of index zero when $\kappa = 0$, in which case the dimension of their finite dimensional kernel is the sum of the Betti numbers of the boundary Γ.

The expressions (4.13a) and (4.13b) should be compared with the self-adjoint operators (2.3a) and (2.3b).

unknown	$\mathbf{n}\boldsymbol{U}$
boundary data	$\mathbf{t}\boldsymbol{U}$
self-adjoint op. in Ω	$\mathfrak{D}_{\mathbf{t}} + i\kappa = \boldsymbol{\delta} + \boldsymbol{\delta}^* + i\kappa$
first-kind BIO	$\mathsf{V}[\mathfrak{D}] = \boldsymbol{\delta} + \boldsymbol{\delta}^* - i\kappa$
bilinear form on Γ	$\langle\!\langle \mathsf{V}[\mathfrak{D}]\boldsymbol{h}, \overline{\boldsymbol{w}} \rangle\!\rangle_\Gamma = (\boldsymbol{\delta}\boldsymbol{h},\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}} + (\boldsymbol{h},\boldsymbol{\delta}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}}$ $-i\kappa(\boldsymbol{h},\boldsymbol{\delta}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{t}}$
bilinear form in Ω	$\mathcal{A}_{\boldsymbol{\delta}}(\boldsymbol{U},\boldsymbol{V}) + i\kappa(\boldsymbol{U},\boldsymbol{V})_\Omega = (\boldsymbol{\delta}\boldsymbol{U},\boldsymbol{V})_\Omega + (\boldsymbol{U},\boldsymbol{\delta}\boldsymbol{V})_\Omega$ $+i\kappa\,(\boldsymbol{U},\boldsymbol{V})_\Omega$

Fig. 1. *Table of relations for the BVPs* (2.2a) *and* (2.8a).

4.2. BIEs for Hodge–Laplace BVPs. In Subsection 2.2, the notation was kept consistent with the abstract overview given at the beginning of Section 4, so we can jump straight into calculations. It is routine to verify that Green's second formulas such as (4.1) hold for the Hodge–Laplacian in both strong and mixed formulations, cf. [11, 12, 34, 35]. For the mixed formulations, this can be seen directly from the fact the bilinear forms in Subsection 2.3 are symmetric.

Remark 4.4. It is worth noting that the strong form of the Hodge–Laplace operator *fails to admit* an identity akin to Green's *first* formula:

$$\langle -\Delta_\ell U_\ell, V_\ell\rangle_\Omega \neq \mathcal{B}_\Delta(U_\ell,V_\ell) \pm \langle\!\langle \mathsf{T}^{\mathsf{n}}_\Delta U_\ell, \mathsf{T}^{\mathsf{t}}_\Delta U_\ell\rangle\!\rangle_\Gamma,$$

where $\mathcal{B}_\Delta(U_\ell,V_\ell) = \langle \mathrm{d}_\ell U_\ell, \mathrm{d}_\ell V_\ell\rangle_\Omega + \langle \delta_\ell U_\ell, \delta_\ell V_\ell\rangle_\Omega$ is the fundamental bilinear form associated with $-\Delta_\ell$. When it comes to the use of BIEs in scattering and transmission

unknown	$\mathbf{t}\boldsymbol{U}$
boundary data	$\mathbf{n}\boldsymbol{U}$
self-adjoint op. in Ω	$\mathfrak{D}_{\mathsf{n}} = \mathbf{d} + \mathbf{d}^* + i\kappa$
first-kind BIO	$\mathsf{W}[\mathfrak{D}] = -\mathbf{d} - \mathbf{d}^* - i\kappa$
bilinear form on Γ	$\langle\!\langle\, \mathsf{W}[\mathfrak{D}]\boldsymbol{g}, \overline{\boldsymbol{v}}\,\rangle\!\rangle_\Gamma = -(\mathbf{d}\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}} - (\boldsymbol{g}, \mathbf{d}\boldsymbol{w})_{-\frac{1}{2},\lambda,\mathsf{n}}$ $-i\kappa(\boldsymbol{g}, \boldsymbol{v})_{-\frac{1}{2},\lambda,\mathsf{n}}$
bilinear form in Ω	$\mathcal{A}_{\mathbf{d}}(\boldsymbol{U}, \boldsymbol{V}) + i\kappa(\boldsymbol{U}, \boldsymbol{V})_\Omega = (\mathbf{d}\boldsymbol{U}, \boldsymbol{V})_\Omega + (\boldsymbol{U}, \mathbf{d}\boldsymbol{V})_\Omega$ $+i\kappa\,(\boldsymbol{U}, \boldsymbol{V})_\Omega$

FIG. 2. *Table of relations for the BVPs* (2.2b) *and* (2.8b).

problems, this can be a serious drawback. The perfect match between the boundary term that arises in domain variational problems for the mixed Hodge–Laplacian and the product space of traces on which first-kind BIEs are defined was crucial in [35] to establish variational formulations which coupled the two.

4.2.1. Representation formula for the strong Hodge–Laplacian. We have already seen in Subsection 3.1 that a Newton operator $\mathsf{N}[\Delta] = \mathsf{N}_\ell^\lambda$ is available for the Hodge–Laplacian in strong second-order formulation.

PROPOSITION 4.5. *If* $U_\ell \in L^2\Lambda^\ell(\mathcal{M})$ *is compactly supported and there exists* $F_\ell \in L^2\Lambda^\ell(\mathcal{M})$ *such that* $F_\ell|_\Omega = (-\Delta_\ell + \lambda)U_\ell|_\Omega$ *and* $F_\ell|_{\Omega^+} = (-\Delta_\ell + \lambda)U_\ell|_{\Omega^+}$*, then*

$$U_\ell = \mathsf{N}_\ell^\lambda F_\ell - \begin{pmatrix} \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{S}_{\ell-1}^\lambda [\![\mathsf{n}_\ell U_\ell]\!] \\ \mathsf{S}_\ell^\lambda [\![\mathsf{n}_{\ell+1} \mathrm{d}_\ell U_\ell]\!] \end{pmatrix} + \begin{pmatrix} \mathrm{Id} & \delta_{\ell+1} \end{pmatrix} \begin{pmatrix} \mathsf{D}_\ell^\lambda [\![\mathsf{t}_{\ell-1}\delta_\ell U_\ell]\!] \\ \mathsf{D}_{\ell+1}^\lambda [\![\mathsf{t}_\ell U_\ell]\!] \end{pmatrix}.$$

Proof. Details of the argument are similar to those in the proof of (4.3), so we proceed faster through the derivation. From Green's second formula,

$$\begin{aligned} \langle\!\langle(-\Delta_\ell + \lambda\mathrm{Id})U_\ell, V_\ell\rangle\!\rangle_\Omega &= \langle U_\ell, (-\Delta_\ell + \lambda\mathrm{Id})V_\ell\rangle_{\Omega^-} + \langle U_\ell, (-\Delta_\ell + \lambda\mathrm{Id})V_\ell\rangle_{\Omega^+} \\ &= \langle F_\ell, V_\ell\rangle_{\Omega^-} - \langle\!\langle \mathsf{T}_{\Delta_\ell}^{\mathsf{t},-} U_\ell, \mathsf{T}_{\Delta_\ell}^{\mathsf{n},-} V_\ell\rangle\!\rangle_\Gamma + \langle\!\langle \mathsf{T}_{\Delta_\ell}^{\mathsf{n},-} U_\ell, \mathsf{T}_{\Delta_\ell}^{\mathsf{t},-} V_\ell\rangle\!\rangle_\Gamma \\ &\quad + \langle F_\ell, V_\ell\rangle_{\Omega^+} + \langle\!\langle \mathsf{T}_{\Delta_\ell}^{\mathsf{t},+} U_\ell, \mathsf{T}_{\Delta_\ell}^{\mathsf{n},+} V_\ell\rangle\!\rangle_\Gamma - \langle\!\langle \mathsf{T}_{\Delta_\ell}^{\mathsf{n},+} U_\ell, \mathsf{T}_{\Delta_\ell}^{\mathsf{t},+} V_\ell\rangle\!\rangle_\Gamma \\ &= \langle F_\ell, V_\ell\rangle_{\mathcal{M}} + \langle\!\langle [\![\mathsf{T}_{\Delta_\ell}^{\mathsf{t}} U_\ell]\!], \mathsf{T}_{\Delta_\ell}^{\mathsf{n}} V_\ell\rangle\!\rangle_\Gamma - \langle\!\langle [\![\mathsf{T}_{\Delta_\ell}^{\mathsf{n}} U_\ell]\!], \mathsf{T}_{\Delta_\ell}^{\mathsf{t}} V_\ell\rangle\!\rangle_\Gamma \end{aligned}$$

for all $V_\ell \in \mathcal{D}^\ell(\mathcal{M})$. Hence, in the sense of distributions, we have

$$U_\ell = \mathsf{N}_\ell^\lambda(-\Delta_\ell + \lambda\mathrm{Id})U_\ell = \mathsf{N}_\ell^\lambda F_\ell + \mathsf{N}_\ell^\lambda \left(\mathsf{T}_{\Delta_\ell}^{\mathsf{n}}\right)' [\![\mathsf{T}_{\Delta_\ell}^{\mathsf{t}} U_\ell]\!] - \mathsf{N}_\ell^\lambda \left(\mathsf{T}_{\Delta_\ell}^{\mathsf{t}}\right)' [\![\mathsf{T}_{\Delta_\ell}^{\mathsf{n}} U_\ell]\!].$$

Explicitly, we appeal to the integral representations provided in (3.8) to evaluate

$$\begin{aligned}\mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{t}}_{\Delta_\ell}\right)' (h_\ell,\ h_{\ell-1})^\top &= \langle\!\langle h_{\ell-1}(y), \mathsf{t}_{\ell-1,y}\delta_{\ell,y}\,\mathcal{G}_\ell^\lambda(x,y)\rangle\!\rangle_\Gamma + \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma \\ &= \mathrm{d}_{\ell-1,x}\langle\!\langle h_{\ell-1}, \mathsf{t}_{\ell-1}\mathcal{G}_{\ell-1}^\lambda\rangle\!\rangle_\Gamma + \langle\!\langle h_\ell, \mathsf{t}_\ell\mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma \\ &= \mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + \mathsf{S}_\ell^\lambda(h_\ell),\end{aligned}$$

and

$$\begin{aligned}\mathsf{N}_\ell^\lambda \left(\mathsf{T}^{\mathsf{n}}_{\Delta_\ell}\right)' (g_\ell,\ g_{\ell-1})^\top &= \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell\mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma + \langle\!\langle g_\ell(y), \mathsf{n}_{\ell+1,y}\mathrm{d}_{\ell,y}\mathcal{G}_\ell^\lambda(x,y)\rangle\!\rangle_\Gamma \\ &= \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell\mathcal{G}_\ell^\lambda\rangle\!\rangle_\Gamma + \delta_{\ell+1,x}\langle\!\langle g_\ell, \mathsf{n}_{\ell+1}\mathcal{G}_{\ell+1}^\lambda\rangle\!\rangle_\Gamma \\ &= \mathsf{D}_\ell^\lambda(g_{\ell-1}) + \delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda(g_\ell),\end{aligned}$$

where we have used the identities stated in (3.4) to proceed.

We have arrived at the representation formula

$$U_\ell = \mathsf{N}_\ell^\lambda F_\ell - \mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda[\![\mathsf{n}_\ell U_\ell]\!] - \mathsf{S}_\ell^\lambda[\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell U_\ell]\!] + \mathsf{D}_\ell^\lambda[\![\mathsf{t}_{\ell-1}\delta_\ell U_\ell]\!] + \delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda[\![\mathsf{t}_\ell U_\ell]\!]. \quad \square$$

In the representation formula of Proposition 4.5, the boundary potentials

$$\mathsf{SL}[\Delta] : T^{\mathsf{n}}_\Delta(\Gamma) \to H\Lambda^\ell(\Delta,\Omega) \qquad \text{and} \qquad \mathsf{DL}[\Delta] : T^{\mathsf{t}}_\Delta(\Gamma) \to H\Lambda^\ell(\Delta,\Omega)$$

defined by

$$\begin{aligned}\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top &= \mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + \mathsf{S}_\ell^\lambda(h_\ell), \\ \mathsf{DL}[\Delta](g_{\ell-1},\ g_\ell)^\top &= \mathsf{D}_\ell^\lambda(g_{\ell-1}) + \delta_{\ell+1}\mathsf{D}_{\ell+1}^\lambda(g_\ell),\end{aligned}$$

play the roles of single and double layer for the Hodge–Laplacian in strong form.

Once again, the jump relations for these potentials are obtained from those of the atomic potentials stated in Lemma 3.4. However, unlike for the Hodge–Dirac operator, for which the calculations were direct, we now need to appeal to Lemma 3.3. For example,

$$[\![\mathsf{T}^{\mathsf{n}}_\Delta]\!]\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top = \begin{pmatrix} [\![\mathsf{n}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{n}_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell) \\ \\ [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{n}_{\ell+1}\mathrm{d}_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell)\end{pmatrix} = \begin{pmatrix} h_{\ell-1} \\ \\ h_\ell \end{pmatrix}$$

simply follows from Lemma 3.4 because $\mathrm{d}^2 = 0$, but we must evaluate in

$$[\![\mathsf{T}^{\mathsf{t}}_\Delta]\!]\mathsf{SL}[\Delta](h_{\ell-1},\ h_\ell)^\top = \begin{pmatrix} [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell) \\ \\ [\![\mathsf{t}_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda(h_{\ell-1}) + [\![\mathsf{t}_\ell]\!]\mathsf{S}_\ell^\lambda(h_\ell)\end{pmatrix}$$

the jump $[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda$, which we haven't encountered before. To show that it vanishes, we use the fact that the atomic potential satisfies the equation in the interior and exterior domains to compute

$$[\![\mathsf{t}_{\ell-1}\delta_\ell]\!]\mathrm{d}_{\ell-1}\mathsf{S}_{\ell-1}^\lambda = -\mathrm{d}_{\ell-2}[\![\mathsf{t}_{\ell-1}\delta_{\ell-1}]\!]\mathsf{S}_{\ell-1}^\lambda - \lambda[\![\mathsf{t}_{\ell-1}]\!]\mathsf{S}_{\ell-1}^\lambda = 0.$$

The other jump relations are obtained similarly.

4.2.2. BIOs for the strong formulation of the Hodge–Laplacian. We want to find explicit expressions for the first-kind BIOs

$$\begin{aligned} \mathsf{V}[\Delta] &= \{\mathsf{T}^{\mathsf{t}}_{\Delta}\}\mathsf{SL}[\Delta] : H^{\mathsf{n}}_{\Delta}(\Gamma) \longrightarrow H^{\mathsf{t}}_{\Delta}(\Gamma), \\ \mathsf{W}[\Delta] &= \{\mathsf{T}^{\mathsf{n}}_{\Delta}\}\mathsf{DL}[\Delta] : H^{\mathsf{t}}_{\Delta}(\Gamma) \longrightarrow H^{\mathsf{n}}_{\Delta}(\Gamma). \end{aligned}$$

Once again, we work under the duality pairings on the left of the variational problems (4.5a) and (4.5b), which allows us to combine the "integration by parts trick" with the commutative relations of Lemma 3.1. Starting with $\mathsf{V}[\Delta]$, we evaluate using Lemma 3.1 (terms in green and blue), Lemma 3.3 (terms in blue) and (1.11) (terms in red) that

$$\begin{aligned} &\mathsf{T}^{\mathsf{t}}_{\Delta}\mathsf{SL}[\Delta](h_{\ell-1},\ h_{\ell})^{\top} \\ &\quad= \begin{pmatrix} \mathsf{t}_{\ell-1}\delta_{\ell}\mathrm{d}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell-1}\delta_{\ell}\mathsf{S}^{\lambda}_{\ell}h_{\ell} \\ \mathsf{t}_{\ell}\mathrm{d}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell}\mathsf{S}^{\lambda}_{\ell}h_{\ell} \end{pmatrix} \\ &\quad= \begin{pmatrix} -\mathrm{d}_{\ell-2}\mathsf{t}_{\ell-2}\mathsf{S}^{\lambda}_{\ell-2}\delta_{\ell-1}h_{\ell-1} - \lambda\mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}\delta_{\ell}h_{\ell} \\ \mathrm{d}_{\ell}\mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell}\mathsf{S}^{\lambda}_{\ell}h_{\ell} \end{pmatrix}, \end{aligned}$$

from which we can further obtain

$$\begin{aligned} &\langle\!\langle \mathsf{T}^{\mathsf{t}}_{\Delta}\mathsf{SL}[\Delta](h_{\ell-1},\ h_{\ell})^{\top}, (\overline{w}_{\ell-1},\ \overline{w}_{\ell})^{\top}\rangle\!\rangle_{\Gamma} \\ &\quad= -\langle\!\langle \mathsf{t}_{\ell-2}\mathsf{S}^{\lambda}_{\ell-2}\delta_{\ell-1}h_{\ell-1}, \delta_{\ell-1}\overline{w}_{\ell-1}\rangle\!\rangle_{\Gamma} - \lambda\langle\!\langle \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1}, \overline{w}_{\ell-1}\rangle\!\rangle_{\Gamma} \\ &\qquad+\langle\!\langle \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}\delta_{\ell}h_{\ell}, \overline{w}_{\ell-1}\rangle\!\rangle_{\Gamma} + \langle\!\langle \mathsf{t}_{\ell-1}\mathsf{S}^{\lambda}_{\ell-1}h_{\ell-1}, \delta_{\ell}\overline{w}_{\ell}\rangle\!\rangle_{\Gamma} \\ &\qquad+\langle\!\langle \mathsf{t}_{\ell}\mathsf{S}^{\lambda}_{\ell}h_{\ell}, \overline{w}_{\ell}\rangle\!\rangle_{\Gamma} \\ &\quad= -\left(\delta_{\ell-1}h_{\ell-1}, \delta_{\ell-1}\overline{w}_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} - \lambda\left(h_{\ell-1}, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} \\ &\qquad+ \left(\delta_{\ell}h_{\ell}, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + \left(h_{\ell-1}, \delta_{\ell}w_{\ell}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} \\ &\qquad+ \left(h_{\ell}, w_{\ell}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} \end{aligned} \tag{4.14}$$

using integration by parts.

Similarly for $\mathsf{W}[\Delta]$, evaluating

$$\begin{aligned} &\mathsf{T}^{\mathsf{n}}_{\Delta_{\ell}}\mathsf{DL}^{\lambda}_{\ell}[\Delta](g_{\ell-1},\ g_{\ell})^{\top} \\ &\quad= \begin{pmatrix} \mathsf{n}_{\ell}\mathsf{D}^{\lambda}_{\ell}g_{\ell-1} + \mathsf{n}_{\ell}\delta_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_{\ell} \\ \mathsf{n}_{\ell+1}\mathrm{d}_{\ell}\mathsf{D}^{\lambda}_{\ell}g_{\ell-1} + \mathsf{n}_{\ell+1}\mathrm{d}_{\ell}\delta_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_{\ell} \end{pmatrix} \\ &\quad= \begin{pmatrix} \mathsf{n}_{\ell}\mathsf{D}^{\lambda}_{\ell}g_{\ell-1} - \delta_{\ell}\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_{\ell} \\ -\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}\mathrm{d}_{\ell-1}g_{\ell-1} - \mathsf{n}_{\ell+1}\delta_{\ell+2}\mathrm{d}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_{\ell} - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_{\ell} \end{pmatrix} \\ &\quad= \begin{pmatrix} \mathsf{n}_{\ell}\mathsf{D}^{\lambda}_{\ell}g_{\ell-1} - \delta_{\ell}\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_{\ell} \\ -\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}\mathrm{d}_{\ell-1}g_{\ell-1} - \delta_{\ell+1}\mathsf{n}_{\ell+2}\mathsf{D}^{\lambda}_{\ell+2}\mathrm{d}_{\ell}g_{\ell} - \lambda\mathsf{n}_{\ell+1}\mathsf{D}^{\lambda}_{\ell+1}g_{\ell} \end{pmatrix} \end{aligned}$$

eventually leads to

$$
\begin{aligned}
&\langle\!\langle \mathsf{T}^{\mathsf{n}}_{\Delta_\ell}\mathsf{DL}^{\lambda}_{\ell}[\Delta](h_{\ell-1},\ h_\ell)^\top, (\overline{v}_{\ell-1},\ \overline{v}_\ell)^\top \rangle\!\rangle_\Gamma \\
&\qquad = (g_{\ell-1}, v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} - (g_\ell, \mathrm{d}_{\ell-1}v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} \\
&\qquad\quad - (\mathrm{d}_{\ell-1}g_{\ell-1}, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} - (\mathrm{d}_\ell g_\ell, \mathrm{d}_\ell v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} \\
&\qquad\quad - \lambda\,(g_\ell, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}\,.
\end{aligned} \tag{4.15}
$$

We conclude from (4.14) and (4.15) that the first-kind boundary integral operators $\mathsf{V}[\Delta]$ and $\mathsf{W}[\Delta]$ associated with the direct first-kind boundary integral equations (4.4a) and (4.4b) are zero-order perturbations of Hodge–Laplace operators in the trace de Rham complexes equipped with the non-local inner products introduced in Subsection 3.3. More precisely, in the same sense as (4.13a) and (4.13b), we have

$$
\mathsf{V}[\Delta] = \begin{pmatrix} -\delta^*_{\ell-1}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \delta^*_\ell & \mathrm{Id} \end{pmatrix}, \tag{4.16a}
$$

$$
\mathsf{W}[\Delta] = \begin{pmatrix} \mathrm{Id} & -\mathrm{d}^*_\ell \\ -\mathrm{d}_{\ell-1} & -\mathrm{d}^*_\ell\mathrm{d}_\ell - \lambda\mathrm{Id} \end{pmatrix}. \tag{4.16b}
$$

The results of the abstract theory of Hodge–Laplace operators in Hilbert complexes is therefore available to analyze the BIOS. When $\lambda > 0$, $\mathsf{V}[\Delta]$ and $\mathsf{W}[\Delta]$ are invertible. They are Fredholm operators of index zero when $\lambda = 0$, in which case the dimension of their finite dimensional kernel is the same as the Betti number of corresponding order on the boundary.

The expressions (4.16a) and (4.16b) should be compared with the self-adjoint operators (2.28a) and (2.28b), while the bilinear forms (4.14) and (4.15) should be compared with the bilinear forms (2.25) and (2.26).

4.2.3. Representation formula for the mixed-order Hodge–Laplacian. Similarly as for the Hodge–Dirac operator, we can build a fundamental solution for the mixed-order Hodge–Laplacian using the one available for the Hodge–Laplacian in strong formulation. Notice that

$$
\begin{aligned}
\begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix}\mathfrak{M} &= \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix}\begin{pmatrix} -\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \delta_{\ell+1}\mathrm{d}_\ell + \lambda\,\mathrm{Id} \end{pmatrix} \\
&= \begin{pmatrix} -\Delta_\ell + \lambda\mathrm{Id} & 0 \\ 0 & -\Delta_\ell + \lambda\mathrm{Id} \end{pmatrix}.
\end{aligned}
$$

Moreover, integrating by parts after using the commutative relations (3.4) verifies that the commutation property

$$\begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} = \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix}$$

holds in the sense of distributions. We conclude that

$$\mathsf{N}[\mathfrak{M}] = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix}$$

is a fundamental solution for the Hodge–Laplacian $\mathfrak{M}$ in mixed formulation.

A similar fundamental solution can be designed for $\mathfrak{R}$, but since the following development is mirrored for the mixed formulation involving $\mathfrak{R}$, we will focus our attention on $\mathfrak{M}$.

PROPOSITION 4.6. *If* $(U_{\ell-1}, U_\ell)^\top \in L^2\Lambda^{\ell-1}(\mathcal{M}) \times L^2\Lambda^\ell(\mathcal{M})$ *is compactly supported and there exists* $(F_{\ell-1}, F_\ell)^\top \in L^2\Lambda^{\ell-1}(\mathcal{M}) \times L^2\Lambda^\ell(\mathcal{M})$ *such that* $(F_{\ell-1}, F_\ell)^\top|_\Omega = \mathfrak{M}(U_{\ell-1}, U_\ell)^\top|_\Omega$ *and* $(F_{\ell-1}, F_\ell)^\top|_{\Omega^+} = \mathfrak{M}(U_{\ell-1}, U_\ell)^\top|_{\Omega^+}$*, then*

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \Bigg(\begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} F_{\ell-1} \\ \mathsf{N}^\lambda_\ell F_\ell \end{pmatrix} - \begin{pmatrix} 0 \\ \mathsf{D}^\lambda_\ell \left[\!\left[\mathsf{t}_{\ell-1} U_{\ell-1} \right]\!\right] + \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} \left[\!\left[\mathsf{t}_\ell U_\ell \right]\!\right] \end{pmatrix} + \begin{pmatrix} \mathsf{S}^\lambda_{\ell-1} \left[\!\left[\mathsf{n}_\ell U_\ell \right]\!\right] \\ \mathsf{S}^\lambda_\ell \left[\!\left[\mathsf{n}_{\ell+1} \mathrm{d}_\ell U_\ell \right]\!\right] \end{pmatrix} \Bigg)$$

Proof. As before, it follows from Green's second formula that

$$\begin{pmatrix} U_{\ell-1} \\ U_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \Bigg(\begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} F_{\ell-1} \\ \mathsf{N}^\lambda_\ell F_\ell \end{pmatrix} - \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} (\mathsf{T}^\mathsf{n}_\mathfrak{M})' \left[\!\left[\mathsf{T}^\mathsf{t}_\mathfrak{M} (U_{\ell-1}, U_\ell)^\top \right]\!\right] + \begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} \left(\mathsf{T}^\mathsf{t}_\mathfrak{M}\right)' \left[\!\left[\mathsf{T}^\mathsf{n}_\mathfrak{M} (U_{\ell-1}, U_\ell)^\top \right]\!\right] \Bigg).$$

Explicitly, we evaluate

$$\begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} (\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}})' \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} = \begin{pmatrix} 0 \\ \langle\!\langle g_{\ell-1}, \mathsf{n}_\ell \mathcal{G}^\lambda_\ell \rangle\!\rangle_\Gamma + \langle\!\langle g_\ell, \mathsf{n}_{\ell+1} \mathrm{d}_\ell \mathcal{G}^\lambda_\ell \rangle\!\rangle_\Gamma \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ \mathsf{D}^\lambda_\ell g_{\ell-1} + \delta_{\ell+1} \mathsf{D}^\lambda_{\ell+1} g_\ell \end{pmatrix}$$

and

$$\begin{pmatrix} \mathsf{N}^\lambda_{\ell-1} & 0 \\ 0 & \mathsf{N}^\lambda_\ell \end{pmatrix} (\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}})' \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \begin{pmatrix} \langle\!\langle h_{\ell-1}, \mathsf{t}_{\ell-1} \mathcal{G}^\lambda_{\ell-1} \rangle\!\rangle_\Gamma \\ \langle\!\langle h_\ell, \mathsf{t}_\ell \mathcal{G}^\lambda_\ell \rangle\!\rangle_\Gamma \end{pmatrix} = \begin{pmatrix} \mathsf{S}^\lambda_{\ell-1} h_{\ell-1} \\ \mathsf{S}^\lambda_\ell h_\ell \end{pmatrix}. \qquad \square$$

In the representation formula of Proposition 4.6, the potentials

$$\mathsf{SL}[\mathfrak{M}] : H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma) \to \mathrm{dom}(\mathfrak{M}) \qquad \text{and} \qquad \mathsf{DL}[\mathfrak{M}] : H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma) \to \mathrm{dom}(\mathfrak{M})$$

defined by

$$\mathsf{SL}[\mathfrak{M}] \begin{pmatrix} h_{\ell-1} \\ h_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathsf{S}^\lambda_{\ell-1} h_{\ell-1} \\ \mathsf{S}^\lambda_\ell h_\ell \end{pmatrix},$$
$$\mathsf{DL}^\lambda_\ell[\mathfrak{M}] \begin{pmatrix} g_{\ell-1} \\ g_\ell \end{pmatrix} = \begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} 0 \\ \mathsf{D}^\lambda_\ell g_{\ell-1} + \delta_{\ell+1} \mathsf{D}^\lambda_{\ell+1} g_\ell \end{pmatrix}$$
$$= \begin{pmatrix} \delta_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} \\ \mathsf{D}^\lambda_\ell g_{\ell-1} + \delta_{\ell+1} \mathsf{D}^\lambda_{\ell+1} g_\ell \end{pmatrix},$$

play the roles of single and double layer potentials for the Hodge–Laplace operator in mixed form $\mathfrak{M}$.

Jump relations for these boundary potentials are obtained using the same techniques as in the previous sections.

4.2.4. Boundary integral operators for the mixed Hodge–Laplacian. We now derive explicit expressions for the first-kind BIOs

$$\mathsf{V}[\mathfrak{M}] = \{\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\}\mathsf{SL}[\mathfrak{M}] : H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma) \longrightarrow H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma),$$
$$\mathsf{W}[\mathfrak{M}] = \{\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}\}\mathsf{DL}[\mathfrak{M}] : H^{\mathsf{t}}_{\mathfrak{M}}(\Gamma) \longrightarrow H^{\mathsf{n}}_{\mathfrak{M}}(\Gamma).$$

After evaluating

$$\{\mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}\}\mathsf{SL}[\mathfrak{M}](h_{\ell-1}, h_\ell)^\top$$
$$= \begin{pmatrix} -\mathrm{d}_{\ell-2}\mathsf{t}_{\ell-2}\mathsf{S}_{\ell-2}\delta_{\ell-1}h_{\ell-1} - \lambda \mathsf{t}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + \mathsf{t}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}\delta_\ell h_\ell \\ \mathrm{d}_{\ell-1}\mathsf{t}_{\ell-1}\mathsf{S}^\lambda_{\ell-1}h_{\ell-1} + \mathsf{t}_\ell \mathsf{S}^\lambda_\ell h_\ell \end{pmatrix},$$

we find that

$$
\begin{aligned}
&\langle\!\langle \mathsf{V}[\mathfrak{M}](h_{\ell-1}, h_\ell)^\top, (\overline{w}_{\ell-1}, \overline{w}_\ell)^\top \rangle\!\rangle_\Gamma \\
&\qquad = -\left(\delta_{\ell-1} h_{\ell-1}, \delta_{\ell-1}\overline{w}_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} - \lambda \left(h_{\ell-1}, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} \\
&\qquad\quad + \left(\delta_\ell h_\ell, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + \left(h_{\ell-1}, \delta_\ell w_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{t}} \\
&\qquad\quad + \left(h_\ell, w_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{t}} .
\end{aligned} \tag{4.17}
$$

Similarly, evaluating

$$
\begin{aligned}
\mathsf{T}^{\mathsf{n}}_{\mathfrak{M}} \mathsf{DL}^\lambda_\ell[\mathfrak{M}](g_{\ell-1}, g_\ell)^\top &= \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}} \begin{pmatrix} \delta_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} \\ \mathsf{D}^\lambda_\ell g_{\ell-1} + \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \end{pmatrix} \\
&= \begin{pmatrix} \mathsf{n}_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} + \mathsf{n}_\ell \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \\ \mathsf{n}_{\ell+1} \mathrm{d}_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} + \mathsf{n}_{\ell+1}\mathrm{d}_\ell \delta_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \end{pmatrix} \\
&= \begin{pmatrix} \mathsf{n}_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} - \delta_\ell \mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \\ -\mathsf{n}_{\ell+1} \mathsf{D}^\lambda_{\ell+1} \mathrm{d}_{\ell-1} g_{\ell-1} + \mathsf{n}_{\ell+1}\delta_{\ell+2}\mathsf{D}^\lambda_{\ell+2} \mathrm{d}_\ell g_\ell - \lambda \mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \end{pmatrix} \\
&= \begin{pmatrix} \mathsf{n}_\ell \mathsf{D}^\lambda_\ell g_{\ell-1} - \delta_\ell \mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \\ -\mathsf{n}_{\ell+1} \mathsf{D}^\lambda_{\ell+1} \mathrm{d}_{\ell-1} g_{\ell-1} - \delta_{\ell+1}\mathsf{n}_{\ell+2}\mathsf{D}^\lambda_{\ell+2} \mathrm{d}_\ell g_\ell - \lambda \mathsf{n}_{\ell+1}\mathsf{D}^\lambda_{\ell+1} g_\ell \end{pmatrix}
\end{aligned}
$$

leads to

$$
\begin{aligned}
&\langle\!\langle \mathsf{W}[\mathfrak{M}](g_{\ell-1}, g_\ell)^\top, (\overline{v}_{\ell-1}, \overline{v}_\ell)^\top \rangle\!\rangle_\Gamma \\
&\qquad = \left(g_{\ell-1}, v_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(g_\ell, \mathrm{d}_{\ell-1} v_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{n}} \\
&\qquad\quad - \left(\mathrm{d}_{\ell-1} g_{\ell-1}, v_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{n}} - \left(\mathrm{d}_\ell g_\ell, \mathrm{d}_\ell v_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{n}} \\
&\qquad\quad - \lambda \left(g_\ell, v_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{n}} .
\end{aligned} \tag{4.18}
$$

We conclude from (4.17) and (4.18) that the first-kind boundary integral operators $\mathsf{V}[\mathfrak{M}]$ and $\mathsf{W}[\mathfrak{M}]$ associated with the direct first-kind boundary integral equations (4.4a) and (4.4b) are zero-order perturbations of Hodge–Laplace operators in the trace de Rham complexes equipped with the non-local inner products introduced in Subsection 3.3. More precisely, in the same sense as

(4.13a), (4.13b), (4.16a) and (4.16b) we have

$$\mathsf{V}[\mathfrak{M}] = \begin{pmatrix} -\delta^*_{\ell-1}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \delta^*_\ell & \mathrm{Id} \end{pmatrix}, \tag{4.19a}$$

$$\mathsf{W}[\mathfrak{M}] = \begin{pmatrix} \mathrm{Id} & -\mathrm{d}^*_\ell \\ -\mathrm{d}_{\ell-1} & -\mathrm{d}^*_\ell\mathrm{d}_\ell - \lambda\mathrm{Id} \end{pmatrix}. \tag{4.19b}$$

The results of the abstract theory of Hodge–Laplace operators in Hilbert complexes is therefore available to analyze the BIOS. When $\lambda > 0$, $\mathsf{V}[\Delta]$ and $\mathsf{W}[\Delta]$ are invertible. They are Fredholm operators of index zero when $\lambda = 0$, in which case the dimension of their finite dimensional kernel is the same as the Betti number of corresponding order on the boundary.

Importantly, we have unveiled that

$$\mathsf{V}[\Delta] = \mathsf{V}[\mathfrak{W}],$$
$$\mathsf{W}[\Delta] = \mathsf{W}[\mathfrak{W}].$$

Notice that in the mixed formulation, the tangential trace was relieved of differential operators, but these were account for in the factor

$$\begin{pmatrix} -\mathrm{d}_{\ell-2}\delta_{\ell-1} - \lambda\mathrm{Id} & \delta_\ell \\ \mathrm{d}_{\ell-1} & \mathrm{Id} \end{pmatrix}$$

appearing in the fundamental solution, which should be compared with (4.19a) and the mixed Hodge–Laplacian $\mathfrak{R}$.

5. Conclusion. We have seen in Subsection 4.2 that while the correspondence between the BVPs for the Hodge–Dirac operator in Ω and its first-kind BIOs on Γ displays a striking simplicity and elegance, the correspondence claimed in the introduction for the Hodge–Laplacian hid that the first-kind BIOs on Γ turn out to be Hodge–Laplace operators in *mixed* formulation. However, far from undermining the relevance of the connections revealed by boundary integral exterior calculus, this interesting complication sheds new light on the structure of the so-called "compound" traces for the Hodge–Laplacian, which appear naturally from integration by parts. In mixed formulation, the Hodge–Laplacian in Ω can be represented by an operator matrix acting on a product space. The associated BIOs are then also Hodge–Laplace operators in mixed formulation acting on products of trace spaces. But it is clear that such a correspondence cannot materialize for the second-order strong formulation of the Hodge–Laplacian: we cannot expect to obtain an Hodge–Laplacian on the boundary in strong formulation, because such an operator only acts on forms of a given order. In fact, if the BIOs associated with the strong formulation are to be Hodge–Laplace operators at all, then they *must* be in mixed formulation, because they operate on boundary data that lives in product spaces. It turns out that first-kind

unknown	$\mathsf{T}^{\mathsf{n}}_{\Delta}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}$
boundary data	$\mathsf{T}^{\mathsf{t}}_{\Delta}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}$
self-adjoint operator in Ω	$\mathfrak{R}_{\mathsf{t}} = \begin{pmatrix} \delta_\ell^* \delta_\ell + \lambda & \delta_{\ell+1} \\ \delta_{\ell+1}^* & -\mathrm{Id} \end{pmatrix}$
first-kind BIOs	$\mathsf{V}[\mathfrak{M}] = \mathsf{V}[\Delta] = \begin{pmatrix} -\delta_{\ell-1}^* \delta_{\ell-1} - \lambda \mathrm{Id} & \delta_\ell \\ \delta_\ell^* & \mathrm{Id} \end{pmatrix}$
bilinear form on Γ	$\langle\!\langle \mathsf{V}[\mathfrak{M}](h_{\ell-1}, h_\ell)^\top, (\overline{w}_{\ell-1}, \overline{w}_\ell)^\top \rangle\!\rangle$ $= -\left(\delta_{\ell-1} h_{\ell-1}, \delta_{\ell-1} \overline{w}_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} - \lambda \left(h_{\ell-1}, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}}$ $+ \left(\delta_\ell h_\ell, w_{\ell-1}\right)_{-\frac{1}{2},\lambda,\mathsf{t}} + \left(h_{\ell-1}, \delta_\ell w_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{t}}$ $+ \left(h_\ell, w_\ell\right)_{-\frac{1}{2},\lambda,\mathsf{t}}$
bilinear form in Ω	$\mathcal{B}_\delta \left((U_\ell, U_{\ell+1})^\top, (V_\ell, V_{\ell+1})^\top \right)$ $= \left(\delta_\ell U_\ell, \delta_\ell V_\ell\right)_\Omega + \lambda \left(U_\ell, V_\ell\right)_\Omega + \left(\delta_{\ell+1} U_{\ell+1}, V_\ell\right)_\Omega$ $+ \left(U_\ell, \delta_{\ell+1} V_{\ell+1}\right)_\Omega - \left(U_{\ell+1}, V_{\ell+1}\right)_\Omega$

FIG. 3. *Table of relations for the BVPs* (2.17a)*,* (2.20a)*,* (2.22a) *and* (2.27b)*.*

BIOs for the Hodge–Laplacian in strong form and the first-kind BIOs for the Hodge–Laplacian in mixed form *are the same*! The difference in meaning of the solutions of the BIEs is accounted for on the right hand sides.

As a by product of our study, an exterior calculus of boundary potentials was described that eases calculations. Recognizing the structure of the BIOs as operators in trace de Rham complexes also enables us to harness a rich and powerful literature on Hilbert complexes for their analysis.

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unknown	$\mathsf{T}^{\mathsf{t}}_{\Delta}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{t}}_{\mathfrak{R}}$
boundary data	$\mathsf{T}^{\mathsf{n}}_{\Delta}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{M}}, \mathsf{T}^{\mathsf{n}}_{\mathfrak{R}}$
self-adjoint operator in Ω	$\mathfrak{M}_{\mathsf{n}} = \begin{pmatrix} -\mathrm{Id} & \mathrm{d}_\ell^* \\ \mathrm{d}_{\ell-1} & \mathrm{d}_\ell^*\mathrm{d}_\ell + \lambda \end{pmatrix}$
first-kind BIO	$\mathsf{W}[\mathfrak{M}] = \mathsf{W}[\Delta] = \begin{pmatrix} \mathrm{Id} & -\mathrm{d}_\ell^* \\ -\mathrm{d}_{\ell-1} & -\mathrm{d}_\ell^*\mathrm{d}_\ell - \lambda\mathrm{Id} \end{pmatrix}$
bilinear form on Γ	$\langle\!\langle \mathsf{W}[\mathfrak{M}](g_{\ell-1}, g_\ell)^\top, (\overline{v}_{\ell-1}, \overline{v}_\ell)^\top \rangle\!\rangle_\Gamma$ $= (g_{\ell-1}, v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}} - (g_\ell, \mathrm{d}_{\ell-1}v_{\ell-1})_{-\frac{1}{2},\lambda,\mathsf{n}}$ $- (\mathrm{d}_{\ell-1}g_{\ell-1}, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}} - (\mathrm{d}_\ell g_\ell, \mathrm{d}_\ell v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}$ $-\lambda\,(g_\ell, v_\ell)_{-\frac{1}{2},\lambda,\mathsf{n}}$
bilinear form in Ω	$\mathcal{B}_{\mathrm{d}}\left((U_{\ell-1}, U_\ell)^\top, (V_{\ell-1}, V_\ell)^\top\right)$ $= (\mathrm{d}_\ell U_\ell, \mathrm{d}_\ell V_\ell)_\Omega + \lambda\,(U_\ell, V_\ell)_\Omega + (\mathrm{d}_{\ell-1}U_{\ell-1}, V_\ell)_\Omega$ $+ (U_\ell, \mathrm{d}_\ell V_{\ell-1})_\Omega - (U_{\ell-1}, V_{\ell-1})_\Omega$

FIG. 4. *Table of relations for the BVPs* (2.17b)*,* (2.20b)*,* (2.22b) *and* (2.27a)*.*

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