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Phase retrieval of entire functions and its implications for Gabor phase retrieval

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Abstract

In this paper, we consider the full or partial recovery of entire functions from magnitude measurements on different subsets of the complex plane. This so-called *phase retrieval for entire functions* is inspired by its manifold connections to other phase retrieval problems. A particular connection, illuminated in more detail in this paper, is that to Gabor phase retrieval which can be made by using the Bargmann transform and the Fock space.

By applying well-known techniques from complex analysis — in particular, the famous Hadamard factorisation theorem — we develop many known and numerous new results for the phase retrieval of entire functions. Among other things, we provide a full classification of all (finite order) entire functions whose magnitudes agree on two arbitrary lines in the complex plane as well as a full classification of all entire functions of exponential type whose magnitudes agree on infinitely many equidistant parallel lines.

Our results have interesting implications for Gabor phase retrieval such as giving a full classification of all signals whose Gabor magnitudes agree on two arbitrary lines in the time-frequency plane; or, yielding a machinery with which to generate signals whose Gabor magnitudes agree on infinitely many equidistant lines. The latter has already been harnessed to propose certain counterexamples to sampled Gabor phase retrieval in the literature. We show here that one may also apply this machinery to generate so-called "universal counterexamples": signals which cannot be recovered (up to global phase) from magnitude measurements of their Gabor transforms on infinitely many equidistant parallel lines — no matter how small the distance between those lines.

Keywords Phase retrieval, finite order entire functions, Hadamard factorisation theorem, Gabor transform, time-frequency analysis

Mathematics Subject Classification (2020) 30D20, 94A12

1 Introduction

In this paper, we consider the recovery of entire functions from magnitude-only measurements. This problem has recently been considered in numerous papers because it is connected to many different phase retrieval problems. One particular instance of this is the work by Mallat and Waldspurger on the recovery of analytic signals from Cauchy wavelet transform magnitudes [13]. There, the authors use a factorisation theorem for functions which are analytic on the upper half-plane to show that magnitude information on two scales is sufficient for the recovery of analytic signals from Cauchy wavelet transform magnitudes up to global phase. This is a remarkable result because it stands in stark contrast to the theory of Gabor phase retrieval in which it is not known whether magnitude information in two frequency bins is sufficient for the recovery of signals up to global phase. As the Gabor transform is related to the Fock space of entire functions through its relation with the Bargmann transform [4], we may approach the above gap in knowledge from a complex-analytic perspective. We emphasise that while we will discuss the implications of our theory for Gabor phase retrieval, much of our paper is actually concerned with the phase retrieval of entire functions. In particular, our main results are factorisation or classification theorems for finite order entire functions.

1.1 A short overview of existing research

The main body of our work is an extension of earlier work by Mc Donald on the phase retrieval of entire functions from magnitude information on a single line in the complex plane [3]. Mc Donald realised that such information on a single line is not sufficient for the full recovery of finite order entire functions (up to global phase) and made good use of the Hadamard factorisation theorem to give a full classification of all the sources of non-uniqueness. We postpone a detailed description of the result by Mc Donald to Section 3.

Another paper which is relevant to this work is "Uniqueness results in an extension of Pauli's phase retrieval" by Jaming [10]. Not only does it contain a result on the phase retrieval of certain analytic signals from Cauchy wavelet transform magnitudes which predates the famous results by Mallat and Waldspurger [13], it also includes some interesting results on Gabor phase retrieval and the retrieval of entire functions from phaseless measurements which we re-developed when writing the present paper. In particular, we will encounter the following uniqueness result.

Theorem 1 (Theorem 3.3 on p. 419 of [10]). *Let $\theta_1, \theta_2 \in [0, 2\pi)$, with $\theta_1 - \theta_2 \notin \pi\mathbb{Q}$, and let $f, g \in \mathcal{O}(\mathbb{C})$ be entire functions of finite order such that*

$$\left|f(x\mathrm{e}^{\mathrm{i}\theta_1})\right| = \left|g(x\mathrm{e}^{\mathrm{i}\theta_1})\right| \text{ and } \left|f(x\mathrm{e}^{\mathrm{i}\theta_2})\right| = \left|g(x\mathrm{e}^{\mathrm{i}\theta_2})\right|,$$

for $x \in \mathbb{R}$. Then, there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha} g$.

Hence, finite order entire functions whose magnitudes agree on two lines intersecting at an angle $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$ must agree up to global phase. We will revisit this result in Section 4.1. It is not surprising that the above result has the following noteworthy corollary.

Proposition 2 (Proposition 4.1 on p. 425 of [10]). *Let $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$ and let $f, g \in L^2(\mathbb{R})$ such that*

$$|\mathcal{G}f(x,0)| = |\mathcal{G}g(x,0)| \ \text{ and } \ |\mathcal{G}f(x\cos\theta, x\sin\theta)| = |\mathcal{G}g(x\cos\theta, x\sin\theta)|,$$

for $x \in \mathbb{R}$. Then, there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha} g$.

Thus, two signals whose Gabor magnitudes agree on two lines in the time-frequency plane intersecting at an angle $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$ must agree up to global phase. This is an extension of the classical uniqueness result for Gabor phase retrieval which states that two signals whose Gabor magnitudes agree on the entire time-frequency plane must agree up to global phase.

Let us highlight that the results which we present in this paper have applications in the theory of sampled short-time Fourier transform phase retrieval which has recently seen a stark increase in interest [1, 2, 7, 8, 16]. To be precise, we were able to apply the results in this work to generate the counterexamples presented in an earlier paper [2]. The genesis of the counterexamples presented is not motivated there and, in fact, they are the product of the machinery which we present here. In the meantime, our work in [2] has been generalised by Grohs and Liehr to include window functions different than the Gaussian [8]. We believe that all of these recent advances can be traced back to the ideas which we will present in the following.

Finally, we want to mention the work by Gröchenig on phase retrieval in shift-invariant spaces with Gaussian generator which makes use of techniques similar to the ones we develop here [5].

1.2 Our contributions

As mentioned before, we extend results by Mc Donald on the recovery of entire functions on a single line in the complex plane [3]. To be precise, we consider two arbitrary lines in the complex plane and ask when one may recover a finite order entire function from its magnitudes measured on those two lines. In doing so, we redevelop Theorem 1 by Jaming on the unique recovery of entire functions from magnitude measurements on two lines intersecting at an "irrational angle" $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$. We do moreover consider lines intersecting at a "rational angle" $\theta \in \pi\mathbb{Q}$ as well as parallel lines and show that in both cases the phase retrieval of entire functions does not enjoy uniqueness. More precisely, we are able to characterise all sources of non-uniqueness in these two cases.

For lines intersecting at a rational angle, we obtain the following characterisation.

Theorem 3 (Main theorem — I). *Let $\theta \in \pi\mathbb{Q}$ and let $n \in \mathbb{N}$ be the smallest integer such that $\theta n \in \pi\mathbb{N}$. Let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero entire functions of finite order. The following are equivalent:*

(a) *$|f|$ and $|g|$ agree on two lines intersecting at angle θ at the origin of the complex plane;*

(b) *f and g may be factored as*

$$
\begin{aligned}
f(z) &= r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} c_\ell z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \\
&\qquad \cdot \prod_{a \in \mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta} \frac{z}{a}; p\right)^{m_{\mathcal{Y}}(a)}, \\
g(z) &= r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} c'_\ell z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \\
&\qquad \cdot \prod_{a \in \mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta} \frac{z}{\bar{a}}; p\right)^{m_{\mathcal{Y}}(a)},
\end{aligned}
$$

where $r > 0$, $\phi, \psi \in \mathbb{R}$, and where $(c_\ell)_{\ell=1}^q, (c'_\ell)_{\ell=1}^q \in \mathbb{C}$ are related in some intimate way.

We will explain the precise meaning of the notation used in the statement of the above result in Section 4. For now, we want to point the reader to what we believe to be the most important take-away from this theorem: there are essentially two reasons for which two entire functions f and g may agree on two intersecting lines while not agreeing up to global phase. First, f and g might differ by a factor $\mathrm{e}^{p(z)}$, where p is a special complex polynomial. This is for instance the case when we consider the two lines $\mathbb{R}$ and $\mathrm{i}\mathbb{R}$ along with the functions $f(z) = 1$, $g(z) = \exp(\mathrm{i}z^2)$, for $z \in \mathbb{C}$. Secondly, f and g might have different root sets with rotation symmetry around the point of intersection of the two lines. This is for instance the case when we consider the two lines $\mathbb{R}$ and $\mathrm{i}\mathbb{R}$ along with the functions

$$
f(z) = 1 - \frac{z^2}{(1+\mathrm{i})^2}, \quad g(z) = 1 - \frac{z^2}{(1-\mathrm{i})^2},
$$

for $z \in \mathbb{C}$. Note that f has the root set $\mathcal{R}(f) = \{1+\mathrm{i}, -1-\mathrm{i}\}$ and that g has the root set $\mathcal{R}(g) = \{1-\mathrm{i}, -1+\mathrm{i}\}$. If we rotate $1+\mathrm{i}$ by $2\theta = \pi$ (twice the angle of intersection) repeatedly, we stay within the root set of f. Something similar may be said for g. Our result above states that these two phenomena are the only sources of non-uniqueness for the phase retrieval of entire functions with measurements taken on two intersecting lines.

For parallel lines, we obtain the following characterisation.

Theorem 4 (Main theorem — II). *Let $\tau \in \mathbb{R} \setminus \{0\}$, and let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero entire functions of finite order. Then, the following are equivalent:*

(a) *$|f|$ and $|g|$ agree on two parallel lines with distance τ (parallel to $\mathbb{R}$);*

(b) *f and g may be factored as*

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} c_\ell z^\ell\right) z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};p\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a\in\mathcal{Y}_\mathrm{u}} \prod_{k\in\mathbb{Z}} E\left(\frac{z}{a+2k\mathrm{i}\tau};p\right)^{m_{\mathcal{Y}}(a)},$$

$$g(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} c'_\ell z^\ell\right) z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};p\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a\in\mathcal{Y}_\mathrm{u}} \prod_{k\in\mathbb{Z}} E\left(\frac{z}{\overline{a}+2k\mathrm{i}\tau};p\right)^{m_{\mathcal{Y}}(a)},$$

where $r > 0$, $\phi, \psi \in \mathbb{R}$, and where $(c_\ell)_{\ell=1}^q, (c'_\ell)_{\ell=1}^q \in \mathbb{C}$ are related in some intimate way (which involves the roots of f and g).

We note that for two entire functions f and g, which do not agree up to global phase, to have magnitudes agreeing on two parallel lines in the time-frequency plane, the root sets of f and g must be different. Precisely, the above theorem states that there must be a certain symmetry in these root sets. Indeed, if a is a root of f which is not a root of g, then the same must be true for all elements of $a + 2\mathrm{i}\tau\mathbb{Z}$. In particular, functions f and g as described above must have infinitely many roots. We may consider the parallel lines $\mathbb{R}$ and $\mathbb{R} + \mathrm{i}$ together with the functions

$$f(z) = \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right),$$
$$g(z) = \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right),$$

as an example. The root set of f is given by $\mathcal{R}(f) = \{\mathrm{i}/2 + 2k\mathrm{i} \mid k \in \mathbb{Z}\}$ while the root set of g is given by $\mathcal{R}(g) = \{-\mathrm{i}/2 + 2k\mathrm{i} \mid k \in \mathbb{Z}\}$.

Note that the two theorems discussed above together with Theorem 1 fully characterise all entire functions of finite order whose magnitudes agree on two arbitrary lines in the time-frequency plane. Note also that considering two arbitrary lines in the time-frequency plane always leads to symmetries in the root sets of the entire function under consideration in this way. In particular, intersecting lines correspond to rotation symmetries while parallel lines correspond to translation symmetries. This insight seems to be the most important hallmark of our characterisation and one may use it to consider three or more lines in the complex plane and try to understand how the symmetries imposed by the additional lines change the characterisation. We want to especially focus on infinitely many equidistant parallel lines. As all the lines are equidistant, there are no additional symmetries imposed into the root set of f and g and we may thereby prove the following result.

Theorem 5 (Main theorem — III)**.** *Let $\tau \in \mathbb{R} \setminus \{0\}$ and let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero entire functions of exponential type. Then, the following are equivalent:*

(a) *$|f|$ and $|g|$ agree on infinitely many parallel lines with distance τ (parallel to $\mathbb{R}$);*

(b) *f and g may be factored as*

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{cz}z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}} E\left(\frac{z}{a+2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},$$

$$g(z) = r\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{c'z}z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}} E\left(\frac{z}{\overline{a}+2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},$$

where $r>0$, $\phi,\psi\in\mathbb{R}$, and where $c,c'\in\mathbb{C}$ are intimately related.

We note that we may prove a similar result for general finite order entire functions. However, this general result comes with a very complicated relation for the polynomial coefficients $(c_\ell)_{\ell=1}^q, (c'_\ell)_{\ell=1}^q \in \mathbb{C}$ which seems useless in practice. We note moreover that the above result is the design mechanism according to which we have created the counterexamples presented in [1]. In connection with this, let us remark that the examples

$$f(z) = \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right),$$

$$g(z) = \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right),$$

continue to agree in magnitude on the infinitely many equidistant parallel lines $\mathbb{R}+\mathrm{i}\mathbb{Z}$.

Through the relation of the Gabor transform and the Bargmann transform, the first and second main theorem presented above give rise to a full characterisation of all L^2-functions which do not agree up to global phase but whose Gabor transform magnitudes agree on any two arbitrary lines in the time-frequency plane. Moreover, as hinted at above, the third main theorem may be used to design L^2-functions which do not agree up to global phase but whose Gabor transform magnitudes agree on infinitely many parallel lines in the time-frequency plane. We will explore all of these corollaries and more in greater detail in Section 5.

We finally want to remark that the techniques which we have used in the proofs of the three main theorems may also be applied to generate a "universal counterexample" for Gabor transform phase retrieval. That is a function $f\in L^2(\mathbb{R})$ with the property that for all $n\in\mathbb{N}$, there exists a function $g_n\in L^2(\mathbb{R})$ such that f and g_n do not agree up to global phase while

$$|\mathcal{G}f| = |\mathcal{G}g_n| \text{ on } \mathbb{R}\times\frac{1}{n}\mathbb{Z}.$$

This shows that creating finer and finer sampling sets must not necessarily lead to uniqueness in sampled Gabor phase retrieval even when one of the functions under consideration is fixed.

1.3 Notation

We will denote the space of entire function by $\mathcal{O}(\mathbb{C})$ and, therein (contained as a subspace), the space of polynomials of degree $n \in \mathbb{N}_0$ with complex coefficients by $\mathbb{C}_n[z]$. For an entire function f, we will denote the set of roots by $\mathcal{R}(f)$ and the set of non-zero roots by $\mathcal{R}_*(f)$. Note that by the word "set", we truly mean a set in the mathematically rigorous way. In particular, no repetitions of entries are allowed. We will capture the multiplicity of the roots of non-zero entire functions using a multiset structure over the complex plane. For this purpose, let us denote by $m_f : \mathcal{R}(f) \to \mathbb{N}$ the function mapping each root of f to its multiplicity. We may extend m_f to all of $\mathbb{C}$ by zero and call this extension $M_f : \mathbb{C} \to \mathbb{N}_0$. Remarkably, the function M_f captures all the information about the roots of f: the complement of its null set corresponds to the set of roots of f and its action on the set of roots of f contains all information about the multiplicities. It is in this sense that the following definition should be understood:

Definition 6 (Multiset)**.** *A* multiset over $\mathbb{C}$ *is a function* $M : \mathbb{C} \to \mathbb{N}_0$*.*

We will furthermore denote $\mathbb{R}_* := \mathbb{R} \setminus \{0\}$ as well as $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ following our notation for the roots of entire functions.

1.4 Outline

In Section 2, we provide the necessary background on the theory of entire functions and, in particular, the Hadamard factorisation theorem. Most of the knowledge in this first section is summarised from the excellent book [15].

In Section 3, we present what we consider to be the basics of phase retrieval for entire functions. Those include some simple and elegant statements and proofs about phase retrieval of entire functions with measurements taken on open subsets of $\mathbb{C}$. They additionally include the groundbreaking work by Mc Donald on the phase retrieval of entire functions with measurements taken on a single line in the complex plane [3].

In Section 4, we present the core of our work. In particular, we provide full classifications for entire functions of finite order whose magnitudes agree on two parallel lines or on two intersecting lines in the complex plane. Among other things, we reprove Theorem 1 by Jaming [10]. Additionally, we use the classification of entire functions whose magnitudes agree on two parallel lines in order to give a full classification of all first order entire functions whose magnitudes agree on infinitely many equidistant parallel lines.

In Section 5, we use the relationship between the Gabor transform, the Bargmann transform and the Fock space of entire functions, in order to classify all signals whose Gabor transforms agree on two parallel lines or two intersecting lines in the time-frequency plane. In doing so, we recover Proposition 2 by Jaming [10]. Additionally, we are able to present a machinery which can be used to generate two functions in $L^2(\mathbb{R})$ which do not agree up to global phase but whose Gabor transform magnitudes agree on infinitely many equidistant parallel lines in the time-frequency plane. This is the exact machinery which we used to write our earlier work [1]. Finally, we use Jensen's formula from complex analysis in order to give an upper bound on the number of roots that the Gabor transform can have in circles around the origin.

In Section 6, we prove the existence of so-called "universal counterexamples". In particular, we use the theory on phase retrieval of entire functions with magnitude information on infinitely many parallel lines in order to show that there exists a function $f \in L^2(\mathbb{R})$ such that for all $n \in \mathbb{N}$, there exists a function $g_n \in L^2(\mathbb{R})$ which does not agree with f up to global phase but which satisfies

$$|\mathcal{G}f| = |\mathcal{G}g_n| \text{ on } \mathbb{R} \times \frac{1}{n}\mathbb{Z}.$$

2 Hadamard's factorisation theorem

It is well known that non-zero polynomials $p \in \mathbb{C}_n[z]$ may be factored in terms of their roots $a_1, \ldots, a_n \in \mathbb{C}$. Indeed, suppose that $p(0) \neq 0$ for simplicity. Then, it holds that

$$p(z) = p(0)\left(1 - \frac{z}{a_1}\right) \ldots \left(1 - \frac{z}{a_n}\right), \qquad z \in \mathbb{C}.$$

The same can be said for non-zero entire functions $\mathcal{O}(\mathbb{C})$ but the picture is a bit more complicated for them. The reasons for this are broadly speaking twofold. First, there are many entire functions which do not have any roots but are still distinct (e.g. $\exp(z)$ and $\exp(z^2)$). Secondly, it is, in general, not true that

$$\prod_{a \in \mathcal{R}(f)} \left(1 - \frac{z}{a}\right)^{m_f(a)}$$

converges (think of $\cos(z)$). To deal with the latter problem, one works with the so-called *primary factors*,

$$E(z;0) = 1 - z, \qquad E(z;p) = (1-z)\exp\left(\sum_{\ell=1}^{p} \frac{z^\ell}{\ell}\right), \qquad z \in \mathbb{C},$$

for $p \in \mathbb{N}$. The primary factors can be used to show that any non-zero entire function can be expanded into a product of its roots:

$$f(z) = \mathrm{e}^{g(z)} z^m \prod_{a \in \mathcal{R}_*(f)} E\left(\frac{z}{a}; p(a)\right)^{m_f(a)},$$

where $m = m_f(0)$, $g \in \mathcal{O}(\mathbb{C})$ and $(p(a))_{a \in \mathcal{R}_*(f)} \in \mathbb{N}_0$ is some sequence of integers. For our purposes this result (called *Weierstrass factorisation theorem*) is not precise enough. It only contains little information on the function g and the integers $(p(a))_a$. Luckily enough, we always end up having to work with entire functions of finite order.

Definition 7 (Finite order). *Let $f \in \mathcal{O}(\mathbb{C})$. We say that f is of* finite order *if there exists an $A > 0$ such that*

$$f(z) = \mathcal{O}\left(\exp\left(|z|^A\right)\right), \qquad |z| \to \infty.$$

The infimum over all $A > 0$ such that the above holds is called the order of f *and will be denoted by ρ.*

Finite order functions have the nice property that their root sets are not too dense.

Lemma 8. *Let $f \in \mathcal{O}(\mathbb{C})$ be non-zero and of finite order $\rho \geq 0$. Then, it holds that*

$$\sum_{a \in \mathcal{R}_*(f)} \frac{m_f(a)}{|a|^A}$$

converges for all $A > \rho$.

A beautiful proof of this lemma can be found on the pages 251 and 250 of [15]. It makes use of Jensen's formula which relates the moduli of entire functions to the moduli of their zeroes. Lemma 8 will be used multiple times throughout this section. One of its most immediate uses is in the following corollary, however.

Corollary 9 (Cf. [15]). *Let $f \in \mathcal{O}(\mathbb{C})$ be non-zero and of finite order $\rho \geq 0$. Then, there exists an integer $p \in \mathbb{N}_0$, with $p \leq \rho$, such that*

$$\prod_{a \in \mathcal{R}_*(f)} E\left(\frac{z}{a}; p\right)^{m_f(a)}$$

converges to an entire function.

Interestingly, the above corollary allows for the use of the same integer p in every factor of the product. It remains to refine the information on the function g in the Weierstrass factorisation theorem to obtain the famous Hadamard factorisation theorem.

Theorem 10 (Hadamard factorisation theorem: cf. [15]). *Let $f \in \mathcal{O}(\mathbb{C})$ be non-zero and of finite order $\rho \geq 0$. Then, there exist $r > 0$, $\phi \in \mathbb{R}$, $q \in \mathbb{N}_0$, with $q \leq \rho$, $(c_\ell)_{\ell=1}^q \in \mathbb{C}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$, with $p \leq \rho$, such that*

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} c_\ell z^\ell\right) z^m \prod_{a \in \mathcal{R}_*(f)} E\left(\frac{z}{a}; p\right)^{m_f(a)}, \qquad z \in \mathbb{C}.$$

We want to end this section by stating one more result of which we will make use in the course of this paper. It links the moduli of the roots of certain products to their order.

Lemma 11 (Cf. [15]). *Let $\{a_j\}_{j=1}^{\infty} \subset \mathbb{C}_*$ be a set of isolated points with corresponding multiplicities $\{m_j\}_{j=1}^{\infty} \subset \mathbb{N}$. Suppose that there exists an $A > 0$ such that*

$$\sum_{j=1}^{\infty} \frac{m_j}{|a_j|^A}$$

converges and denote the infimum over all such $A > 0$ by ρ_1. It holds that

$$f(z) = \prod_{j=1}^{\infty} E\left(\frac{z}{a_j}; p\right)^{m_j}, \qquad z \in \mathbb{C}, \tag{1}$$

is a well-defined entire function of order ρ_1, for $p + 1 > \rho_1$.

Remark 12. *Typically, one calls the non-negative number $\rho_1 \geq 0$ defined in the lemma above the* exponent of convergence of $\{a_j\}_{j=1}^{\infty}$ with multiplicities $\{m_j\}_{j=1}^{\infty}$. *In addition, one calls the smallest integer $p_1 \in \mathbb{N}_0$ such that the product in* (1) *converges the* genus of the canonical product formed with $\{a_j\}_{j=1}^{\infty}$ and multiplicities $\{m_j\}_{j=1}^{\infty}$ *and the corresponding product the* canonical product formed with $\{a_j\}_{j=1}^{\infty}$ and multiplicities $\{m_j\}_{j=1}^{\infty}$. *The attentive reader will have noticed that for an entire function of order $\rho > 0$ with roots $\{a_j\}_{j=1}^{\infty}$ with multiplicities $\{m_j\}_{j=1}^{\infty}$, it holds that*

$$p_1 \leq \rho_1 \leq \rho.$$

The integer p whose existence is postulated by Hadamard's factorisation theorem can actually be chosen in different ways. Often, it is useful to choose p to be the genus of the canonical product formed with $\mathcal{R}_*(f)$. In this way, the factorisation is unique. It is, however, also possible to choose p to be larger to the point where p is even larger than the order of f. In this case, one will need to adapt the factorisation by chosing q just as large, however.

3 Phase retrieval of entire functions: prior arts

In the following two sections, we want to investigate the recoverability of $f \in \mathcal{O}(\mathbb{C})$ from the measurements

$$|f(z)|, \qquad z \in \Omega, \tag{2}$$

where $\Omega \subseteq \mathbb{C}$ is some (not necessarily open) subset of $\mathbb{C}$. We call this the *phase retrieval problem for entire functions*. It is inspired by its connection to the problem of phase retrieval from Gabor and Cauchy wavelet transform measurements [4, 10, 13] which we will partially explain in the Sections 5 and 6 of the present paper. Of course, the choice of Ω dictates what features of f can be recovered from the measurements (2).

Let us start by highlighting the following two fundamental facts related to the phase retrieval problem for entire functions. First, it is obvious that if $\Omega \subseteq \Omega'$ and one can recover a certain feature of f from the measurements (2), then one can also recover that feature from $\{|f(z)| \mid z \in \Omega'\}$. Secondly, it is also clear that no matter how large we make Ω, we can never fully recover f. Indeed, it holds that for all $\alpha \in \mathbb{R}$,

$$\{|f(z)| \mid z \in \mathbb{C}\} = \left\{\left|f(z)\mathrm{e}^{\mathrm{i}\alpha}\right| \mid z \in \mathbb{C}\right\}.$$

We will, however, say that f may be uniquely recovered *up to global phase* if it is possible to recover any element of the set $\{f\mathrm{e}^{\mathrm{i}\alpha} \mid \alpha \in \mathbb{R}\}$ from the measurements (2).

Some recoverability statements for the phase retrieval problem for entire functions come with a proof that is in equal parts easy and elegant. We suspect that these results have been known for a while but could not find a reference for them. The first result is on the recovery of entire functions from absolute value measurements on the entire complex plane.

Theorem 13. *Let $f \in \mathcal{O}(\mathbb{C})$. Then, f may be uniquely recovered up to global phase from the measurements*

$$|f(z)|, \qquad z \in \mathbb{C}.$$

Proof. If f is zero, then the statement is true. Let us therefore consider $f, g \in \mathcal{O}(\mathbb{C})$ non-zero such that

$$|f(z)| = |g(z)|, \qquad z \in \mathbb{C}.$$

We will consider the function $h := f/g$, which is holomorphic on $\mathbb{C} \setminus \mathcal{R}(g)$. It holds that

$$|h(z)| = 1, \qquad z \in \mathbb{C} \setminus \mathcal{R}(g).$$

In particular (as the roots of non-zero entire functions are isolated), we know that for every $a \in \mathcal{R}(g)$, there exists a neighbourhood of a on which h is bounded. By Riemann's theorem on removable singularities, it follows that h is holomorphically extendable over a. Therefore, h extends to an entire function H. In addition,

$$|H(z)| = 1, \qquad z \in \mathbb{C},$$

such that it follows from Liouville's theorem that H is constant. As H has unit absolute value, we know that $H = \mathrm{e}^{\mathrm{i}\alpha}$ for some $\alpha \in \mathbb{R}$. Now, h agrees with H on $\mathbb{C} \setminus \mathcal{R}(g)$ such that, we find

$$f(z) = \mathrm{e}^{\mathrm{i}\alpha} g(z), \qquad z \in \mathbb{C} \setminus \mathcal{R}(g).$$

As both functions in the comparison above are entire (and the roots of non-zero entire functions are isolated), it follows that $f = \mathrm{e}^{\mathrm{i}\alpha} g$. $\square$

A similar result may be proven starting from knowledge of $|f|$ on some open set. To derive this, one only needs to substitute the use of Liouville's theorem with an argument involving the open mapping theorem.

Theorem 14. *Let $\Omega \subset \mathbb{C}$ be open and $f \in \mathcal{O}(\mathbb{C})$. Then, f may be uniquely recovered up to global phase from the measurements*

$$|f(z)|, \qquad z \in \Omega.$$

Proof. If f is zero, then the statement is true. Let us therefore consider $f, g \in \mathcal{O}(\mathbb{C})$ non-zero such that

$$|f(z)| = |g(z)|, \qquad z \in \Omega,$$

We will consider the function $h := f/g$, which is holomorphic on $\mathbb{C} \setminus \mathcal{R}(g)$. It holds that

$$|h(z)| = 1, \qquad z \in \Omega \setminus \mathcal{R}(g),$$

and thus that $h(\Omega \setminus \mathcal{R}(g)) \subset \mathcal{S}^0$ ($\hat{=}$ the complex unit circle). Let us suppose by contradiction that h is non-constant. Then, it follows from the open mapping theorem that $h(\Omega \setminus \mathcal{R}(g))$ is open. This contradicts the fact that $h(\Omega \setminus \mathcal{R}(g)) \subset \mathcal{S}^0$. Therefore, h must be constant. As h has unit absolute value on $\Omega \setminus \mathcal{R}(g)$, we know that $h = \mathrm{e}^{\mathrm{i}\alpha}$, for some $\alpha \in \mathbb{R}$, on $\Omega \setminus \mathcal{R}(g)$. Therefore,

$$f(z) = \mathrm{e}^{\mathrm{i}\alpha} g(z), \qquad z \in \Omega \setminus \mathcal{R}(g).$$

As both functions in the comparison above are entire (and the zeroes of non-zero entire functions are isolated), it follows that $f = \mathrm{e}^{\mathrm{i}\alpha} g$. $\square$

It follows immediately that we can recover f up to global phase from the measurements (2) on any set $\Omega \subset \mathbb{C}$ which contains an open set. To complete our understanding of the phase retrieval problem of entire functions, it is necessary to contemplate what happens in cases in which we only know $|f|$ on sets which do not contain open sets. In the following, we will gradually move towards this goal by considering Ω to be different closed sets. We will start by considering the case in which Ω is a single line in the complex plane.

3.1 Phase retrieval of entire functions from information on a single line

One can quickly convince oneself that knowledge of $|f|$ on a single line is not enough to uniquely recover f up to global phase in general. Indeed, consider $f, g \in \mathcal{O}(\mathbb{C})$ such that

$$f(z) = 1, \qquad g(z) = \mathrm{e}^{\mathrm{i}z},$$

for $z \in \mathbb{C}$. Then, $|f(x)| = |g(x)|$, for $x \in \mathbb{R}$. So $|f|$ and $|g|$ agree on the real number line even though f and g do not agree up to global phase. Similarly, we might consider $f, g \in \mathcal{O}(\mathbb{C})$ such that

$$f(z) = 1 + \mathrm{i}z, \qquad g(z) = 1 - \mathrm{i}z,$$

for $z \in \mathbb{C}$. Again, we find $|f(x)| = |g(x)|$, for $x \in \mathbb{R}$, in spite of f and g not agreeing up to global phase.

Remark 15. *Note that given any single line Ω in the complex plane, there exists a rigid motion $\mathfrak{r} : \mathbb{C} \to \mathbb{C}$ (a rotation followed by a translation) such that $\mathfrak{r}(\Omega) = \mathbb{R}$. Note in addition that rigid motions are entire functions, i.e. $\mathfrak{r} \in \mathcal{O}(\mathbb{C})$. Now, if $f, g \in \mathcal{O}(\mathbb{C})$ are two functions that do not agree up to global phase and satisfy $|f(x)| = |g(x)|$, for $x \in \mathbb{R}$ (we have demonstrated such functions exist). Then, $f \circ \mathfrak{r}, g \circ \mathfrak{r} \in \mathcal{O}(\mathbb{C})$ (where $\circ$ denotes function composition) do not agree up to global phase and satisfy $|f \circ \mathfrak{r}(z)| = |g \circ \mathfrak{r}(z)|$, for $z \in \Omega$. Therefore, we have that* for all lines Ω in the complex plane, there exist entire functions which do not agree up to global phase but whose magnitudes agree on Ω.

In spite of the fact that entire functions are not recoverable from magnitude information on a single line, one can fully characterise all finite order functions $g \in \mathcal{O}(\mathbb{C})$ which satisfy

$$|g(x)| = |f(x)|, \qquad x \in \mathbb{R}, \tag{3}$$

for any given $f \in \mathcal{O}(\mathbb{C})$ of finite order. In this way, one can show that the ambiguities exemplified by the two constructions at the beginning of this subsection really present a complete picture of all possible ambiguities. Of course, this characterisation carries over to general straight lines in $\mathbb{C}$ by the use of rigid motions. The following considerations are due to Mc Donald [3].

For an entire function $f \in \mathcal{O}(\mathbb{C})$, we define another entire function via the involution

$$f^*(z) := \overline{f(\overline{z})}, \qquad z \in \mathbb{C}.$$

If two entire functions satisfy equation (3), then it follows immediately that

$$f^*(x)f(x) = |f(x)|^2 = |g(x)|^2 = g^*(x)g(x), \qquad x \in \mathbb{R}.$$

Since both f^*f and g^*g are entire, we conclude from the above consideration that

$$f^*f = g^*g \tag{4}$$

because the sets of roots of non-zero functions are isolated. Comparing the roots in the above equation yields the following important insight.

Lemma 16 (Root symmetry – I)**.** *Let $f, g \in \mathcal{O}(\mathbb{C})$ be such that*

$$|f(x)| = |g(x)|, \qquad x \in \mathbb{R}.$$

Then, it holds that

$$M_f(z) + M_f(\overline{z}) = M_g(z) + M_g(\overline{z}), \qquad z \in \mathbb{C},$$

where we remind the reader that $M_f, M_g : \mathbb{C} \to \mathbb{N}_0$ denote the multisets of roots of f and g, respectively.

If $a \in \mathbb{C}$ is a root of f, then either a is a root of g or $\overline{a}$ is a root of g. Hence, there is a mirror symmetry in the root sets of f and g along the real number line. We can combine this information with the Hadamard factorisation theorem to obtain a description of all entire functions whose absolute values agree on a single line.

To rigorously describe this, it makes sense to split the roots of f into two sets: i. The roots of f which are roots of g. ii. The roots of f which are not roots of g. We will do this by considering

$$\mathcal{X} := \mathcal{R}_*(f) \cap \mathcal{R}(g), \qquad \mathcal{Y} := \{a \in \mathcal{R}(f) \,|\, m_f(a) > M_g(a)\},$$

together with the map $m_{\mathcal{X}} : \mathcal{X} \to \mathbb{N}$ given by

$$m_{\mathcal{X}}(a) := \min\{m_f(a), m_g(a)\}, \qquad a \in \mathcal{X},$$

as well as the map $m_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{N}$ given by

$$m_{\mathcal{Y}}(a) := m_f(a) - M_g(a), \qquad a \in \mathcal{Y}.$$

We will denote the trivial extensions of these maps to all of $\mathbb{C}$ by $M_{\mathcal{X}}$ and $M_{\mathcal{Y}}$ as usual. It is not hard to see that

$$\begin{aligned} M_f(z) &= \min\{M_f(z), M_g(z)\} + M_f(z) - \min\{M_f(z), M_g(z)\} \\ &= M_{\mathcal{X}}(z) + \max\{0, M_f(z) - M_g(z)\} = M_{\mathcal{X}}(z) + M_{\mathcal{Y}}(z), \end{aligned}$$

for $z \in \mathbb{C}_*$. In addition, it follows from Lemma 16 that

$$\begin{aligned} M_g(z) &= \min\{M_f(z), M_g(z)\} + M_g(z) - \min\{M_f(z), M_g(z)\} \\ &= M_{\mathcal{X}}(z) + \max\{0, M_g(z) - M_f(z)\} \\ &= M_{\mathcal{X}}(z) + \max\{0, M_f(\overline{z}) - M_g(\overline{z})\} = M_{\mathcal{X}}(z) + M_{\mathcal{Y}}(\overline{z}), \end{aligned}$$

for $z \in \mathbb{C}_*$. It is now merely a technical exercise to prove the following theorem:

Theorem 17 (Mc Donald decomposition; [3])**.** *Let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero and of finite order. The following are equivalent:*

(a) *For all $x \in \mathbb{R}$, $|f(x)| = |g(x)|$.*

(b) *There exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $c_\ell, c'_\ell \in \mathbb{R}$, with $\operatorname{Re} c_\ell = \operatorname{Re} c'_\ell$, for $\ell \in \{1, \dots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that*

$$f(z) = r e^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} c_\ell z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{Y}}(a)},$$

$$g(z) = r e^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} c'_\ell z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} E\left(\frac{z}{\overline{a}}; p\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$.

Proof. Let us first show that item (a) implies item (b). According to Hadamard's factorisation theorem and the considerations before, there exist $r, r' > 0$, $\phi, \psi \in \mathbb{R}$, $q, q' \in \mathbb{N}_0$, $a_\ell, b_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $a'_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q'\}$, $m \in \mathbb{N}_0$ and $p, p' \in \mathbb{N}_0$ such that

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell) z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{Y}}(a)},$$

$$g(z) = r'\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q'} (a'_\ell + \mathrm{i}b'_\ell) z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p'\right)^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} E\left(\frac{z}{\overline{a}}; p'\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$. Note that we have used that Lemma 16 directly implies that $m = M_f(0) = M_g(0)$. Further simplifications of the above formulae follow from the considerations in Remark 12 according to which we may assume that $p = p'$ (by setting both of them equal to the maximum of the geni of the canonical products formed with $\mathcal{R}_*(f)$ and with $\mathcal{R}_*(g)$). Note that this implies that we may need to increase the values of q or q'. We will also assume that $q = q'$, for simplicity. This is possible by setting the superfluous polynomial coefficients to zero.

Now, let $x \in \mathbb{R}$ be arbitrary and compute

$$|f(x)| = r \exp\left(\sum_{\ell=1}^{q} a_\ell x^\ell\right) |x|^m \prod_{a \in \mathcal{X}} \left|E\left(\frac{x}{a}; p\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} \left|E\left(\frac{x}{a}; p\right)\right|^{m_{\mathcal{Y}}(a)}$$

as well as

$$|g(x)| = r' \exp\left(\sum_{\ell=1}^{q} a'_\ell x^\ell\right) |x|^m \prod_{a \in \mathcal{X}} \left|E\left(\frac{x}{a}; p\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} \left|E\left(\frac{x}{\overline{a}}; p\right)\right|^{m_{\mathcal{Y}}(a)}.$$

According to

$$|E(z; p)| = |1 - z| \exp\left(\mathrm{Re} \sum_{\ell=1}^{p} \frac{z^\ell}{\ell}\right) = |1 - \overline{z}| \exp\left(\mathrm{Re} \sum_{\ell=1}^{p} \frac{\overline{z}^\ell}{\ell}\right) = |E(\overline{z}; p)|,$$

for $z \in \mathbb{C}$, we have

$$1 = \frac{|f(x)|}{|g(x)|} = \frac{r}{r'} \exp\left(\sum_{\ell=1}^{q} (a_\ell - a'_\ell) x^\ell\right)$$

and thus

$$0 = \log \frac{r}{r'} + \sum_{\ell=1}^{q} (a_\ell - a'_\ell) x^\ell.$$

As $x \in \mathbb{R}$ was arbitrary, we can compare coefficients. In this way, we obtain $r = r'$ as well as $a_\ell = a'_\ell$, for $\ell \in \{1, \ldots, q\}$. Item (b) has been proven.

Next, we show that item (b) implies item (a). Consider $x \in \mathbb{R}$ and note that it follows from item (b) that

$$|f(x)| = r \exp\left(\sum_{\ell=1}^{q} a_\ell x^\ell\right) |x|^m \prod_{a \in \mathcal{X}} \left|E\left(\frac{x}{a}; p\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} \left|E\left(\frac{x}{a}; p\right)\right|^{m_{\mathcal{Y}}(a)}$$

as well as

$$|g(x)| = r \exp\left(\sum_{\ell=1}^{q} a_\ell x^\ell\right) |x|^m \prod_{a \in \mathcal{X}} \left|E\left(\frac{x}{a}; p\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a \in \mathcal{Y}} \left|E\left(\frac{x}{\overline{a}}; p\right)\right|^{m_{\mathcal{Y}}(a)}.$$

Remember that $|E(z; p)| = |E(\overline{z}; p)|$, for $z \in \mathbb{C}$. Therefore, $|f(x)| = |g(x)|$. $\square$

4 Phase retrieval of entire functions: extensions

In the prior section, we had seen that magnitude measurements of entire functions of finite order on any single line in the complex line are not enough to recover the corresponding phases up to a global phase factor. We had also seen that this is due to two ambiguities. First, magnitude information on the line $\mathbb{R}$ is not enough to distinguish between different functions of the form $\exp(iQ)$, where $Q \in \mathbb{R}_n[z]$ is a polynomial with real-coefficients and complex argument. Secondly, and maybe more importantly, magnitude information on a single line is not even enough to determine the roots of the entire function which serves as the signal.

A natural question is whether it is possible to formulate a problem whose solution is unique up to global phase by adding more restrictions on the entire function under consideration. In this section, we want to partially answer this question by determining what happens when we know the absolute value of a finite order entire function on two arbitrary lines in the complex plane. The results we derive can be extended to an arbitrary number of lines in the complex plane in certain cases.

4.1 Intersecting lines

If we look at two arbitrary lines in the complex plane, there are two distinct cases which can come up. In the first case, the lines intersect at some point. In the second case, the lines never intersect and are, in consequence, parallel. In this subsection, we want to focus on intersecting lines. We will assume without loss of generality (remember Remark 15 on rigid motions) that our lines meet at the origin of the complex plane. In this way, we can fully focus on the angle θ in between the intersecting lines. Note that we may assume (again without loss of generality) that one of the two lines corresponds to the line $\mathbb{R}$ and that the other line intersects the reals at an angle $\theta \in (0, \frac{\pi}{2}]$. Indeed, if this should not be the case, we can rotate the entire complex plane about the origin until it is the case. We will furthermore distinguish two cases. First, the case in which $\theta \in (0, \frac{\pi}{2}] \setminus \pi\mathbb{Q}$ (we will call such angles *irrational*). Secondly, the case in which $\theta \in (0, \frac{\pi}{2}] \cap \pi\mathbb{Q}$ (we will call such angles *rational*).

Irrational angles We will start by considering general $\theta \in (0, \frac{\pi}{2}]$ and thus leave considerations concerning the rationality of angles to the side for now. If we have two non-zero functions $f, g \in \mathcal{O}(\mathbb{C})$ such that

$$|f(x)| = |g(x)| \qquad \text{and} \qquad \left|f(e^{i\theta}x)\right| = \left|g(e^{i\theta}x)\right|,$$

for all $x \in \mathbb{R}$, then the symmetries in the root set described by Lemma 16 are accompanied by additional symmmetries stemming from the equality of the absolute values on $e^{i\theta}\mathbb{R}$.

Lemma 18 (Root symmetry — II)**.** *Let $\theta \in (0, \frac{\pi}{2}]$ and let $f, g \in \mathcal{O}(\mathbb{C})$ be such that*

$$\left|f(e^{i\theta}x)\right| = \left|g(e^{i\theta}x)\right|, \qquad x \in \mathbb{R}.$$

Then, it holds that

$$M_f(e^{i\theta}z) + M_f(e^{i\theta}\overline{z}) = M_g(e^{i\theta}z) + M_g(e^{i\theta}\overline{z}), \qquad z \in \mathbb{C}.$$

Viewed from a geometrical point of view, the above lemma states that there is a mirror symmetry along the axis $e^{i\theta}\mathbb{R}$ in the root sets of f and g. This symmetry is especially striking when one considers its implications for the multiset described by $M_{\mathcal{Y}}$. If one does consider a root a of f which is not a root of g, then $\overline{a}$ must be a root of g according to the mirror symmetry along the real line. If $\overline{a}$ is not a root of f, then $e^{2i\theta}a$ must be a root of f according to the mirror symmetry along the line $e^{i\theta}\mathbb{R}$. In this way, one observes a rotational symmetry in the roots. To make these considerations precise, we remember the notations

$$\mathcal{X} = \mathcal{R}_*(f) \cap \mathcal{R}(g), \qquad \mathcal{Y} = \{a \in \mathcal{R}_*(f) \,|\, m_f(a) > M_g(a)\},$$

as well as $m_{\mathcal{X}} : \mathcal{X} \to \mathbb{N}$,

$$m_{\mathcal{X}}(a) = \min\{m_f(a), m_g(a)\}, \qquad a \in \mathcal{X},$$

and $m_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{N}$,

$$m_{\mathcal{Y}}(a) = m_f(a) - M_g(a), \qquad a \in \mathcal{Y}.$$

$M_{\mathcal{X}}$ and $M_{\mathcal{Y}}$ were the multisets defined by extending $m_{\mathcal{X}}$ and $m_{\mathcal{Y}}$ to all of $\mathbb{C}$ by zero. One can then convince oneself that the Lemmata 16 and 18 imply that

$$\begin{aligned} M_{\mathcal{Y}}\left(e^{2i\theta}z\right) &= \max\left\{0, M_f\left(e^{2i\theta}z\right) - M_g\left(e^{2i\theta}z\right)\right\} = \max\left\{0, M_g\left(\overline{z}\right) - M_f\left(\overline{z}\right)\right\} \\ &= \max\left\{0, M_f\left(z\right) - M_g\left(z\right)\right\} = M_{\mathcal{Y}}\left(z\right), \end{aligned}$$

for $z \in \mathbb{C}$. It follows immediately that

$$M_{\mathcal{Y}}\left(e^{2ki\theta}z\right) = M_{\mathcal{Y}}\left(z\right), \qquad z \in \mathbb{C}, \tag{5}$$

for $k \in \mathbb{Z}$. The set $\mathcal{Y}$ is rotationally symmetric around the origin with angle of rotation 2θ. Note that rational and irrational angles create fundamentally different situations. For irrational angles, it follows that $\mathcal{Y}$ must contain infinitely many elements whenever it is non-empty: a contradiction for non-zero entire functions. For rational angles, $\mathcal{Y}$ can have finite cardinality even when it is non-empty.

Let us first discuss irrational angles in greater detail. We want to start by stating that the results we present in the following have first been discovered and proven by Philippe Jaming in 2014 [10]. We were not aware of this at the time we started our research on entire functions and do therefore present these results as they occurred to us. In any case, credit for the following result is due to Jaming.

From now on, we will assume that $\theta \in (0, \frac{\pi}{2}] \setminus \pi\mathbb{Q}$. Let us also assume, by contradiction, that $\mathcal{Y}$ is non-empty. It follows immediately from equation (5) that if $a \in \mathcal{Y}$, then

$$C_{a;\theta} := \left\{ \mathrm{e}^{2k\mathrm{i}\theta} a \,\middle|\, k \in \mathbb{Z} \right\} \subset \mathcal{Y}.$$

Note that the countably infinite set $C_{a;\theta}$ is contained in the compact subset

$$\{z \in \mathbb{C} \,|\, |z| = |a|\}$$

of the complex plane. Therefore, there must be a sequence in $C_{a;\theta}$ which converges. This is in contradiction to f having an isolated set of zeroes due to it being a non-zero entire function. We conclude that $\mathcal{Y} = \emptyset$. Therefore, magnitude information on two lines intersecting at an irrational angle is enough to uniquely determine the roots of an entire function. It should come as no surprise that this allows us to fully recover the entire function up to global phase from magnitude information on two lines intersecting at an irrational angle.

Theorem 19 (Cf. Theorem 1)**.** *Let $\theta \in (0, \frac{\pi}{2}] \setminus \pi\mathbb{Q}$, and let $f, g \in \mathcal{O}(\mathbb{C})$ be of finite order. The following are equivalent:*

(a) *For all $x \in \mathbb{R}$, $|f(x)| = |g(x)|$ and $\left|f(\mathrm{e}^{\mathrm{i}\alpha} x)\right| = \left|g(\mathrm{e}^{\mathrm{i}\alpha} x)\right|$.*

(b) *There exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha} g$.*

Proof. We can assume that f and g are non-zero. If one of them happens to be zero, then the other one needs to be zero on the reals. Therefore, it does not have isolated roots such that it is zero itself. We will only show that item (a) implies item (b) and assume that the other direction is clear to the reader. According to the Mc Donald decomposition and the considerations before, there exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \dots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\, z^\ell \right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}, p\right)^{m_{\mathcal{X}}(a)},$$

$$g(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell)\, z^\ell \right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}, p\right)^{m_{\mathcal{X}}(a)},$$

for $z \in \mathbb{C}$.

We may therefore compute

$$1 = \frac{\left|f(e^{i\theta}x)\right|}{|g(e^{i\theta}x)|} = \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\,\mathrm{Im}\left[e^{i\theta\ell}\right]x^\ell\right), \qquad x \in \mathbb{R},$$

such that

$$0 = \sum_{\ell=1}^{q}(b'_\ell - b_\ell)\sin(\theta\ell)x^\ell, \qquad x \in \mathbb{R}.$$

After a comparison of coefficients and realising that $\theta\ell \notin \pi\mathbb{Z}$, for $\ell \in \{1, \ldots, q\}$, we find $c_\ell = b_\ell$, for $\ell \in \{1, \ldots, q\}$. We conclude that f and g differ by a global phase factor ($\alpha = \phi - \psi$). $\square$

Remark 20. *Note that the theorem above deals with the special case where one of the two intersecting lines is* $\mathbb{R}$ *and the other is* $e^{i\theta}\mathbb{R}$*, with* $\theta \in (0, \frac{\pi}{2}]$*. Note also that given any two lines* $L_1, L_2 \subset \mathbb{C}$ *that intersect, we can find a rigid motion* $\mathfrak{r} : \mathbb{C} \to \mathbb{C}$ *such that* $\mathfrak{r}(L_1) = \mathbb{R}$ *and* $\mathfrak{r}(L_2) = e^{i\theta}\mathbb{R}$*, with* $\theta \in (0, \frac{\pi}{2}]$*, (after potentially relabelling* L_1 *and* L_2*). With the help of this rigid motion, we can provide a result for any two intersecting lines which meet at an irrational angle in the complex plane.*

Finally, note that it is actually enough to assume that $|f(x)| = |g(x)|$ *and* $\left|f(e^{i\theta}x)\right| = \left|g(e^{i\theta}x)\right|$*, for* $x \in X \subset \mathbb{R}$*, where* X *is any set with accumulation point. Indeed, it follows from* f *and* g *being entire functions (and the identity theorem), that the above equations must extend to all of* $\mathbb{R}$ *in this case.*

Rational angles We will now turn our attention to rational angles $\theta \in (0, \frac{\pi}{2}] \cap \pi\mathbb{Q}$. It is relatively easy to construct $\Omega \subset \mathbb{C}$ consisting of two lines intersecting at a rational angle ($\theta = \frac{\pi}{2}$) and $f, g \in \mathcal{O}(\mathbb{C})$ such that the magnitudes of f and g agree on Ω yet f and g are not the same up to global phase. Indeed consider $\Omega = \mathbb{R} \cup i\mathbb{R}$ as well as

$$f(z) = 1 - \frac{z^2}{(1+i)^2}, \qquad g(z) = 1 - \frac{z^2}{(1-i)^2},$$

for $z \in \mathbb{C}$. Then, we have

$$|f(x)| = \left|1 - \frac{x^2}{(1+i)^2}\right| = \left|1 - \frac{x^2}{(1-i)^2}\right| = |g(x)|, \qquad x \in \mathbb{R},$$

as well as

$$|f(ix)| = \left|1 + \frac{x^2}{(1+i)^2}\right| = \left|1 + \frac{x^2}{(1-i)^2}\right| = |g(ix)|, \qquad x \in \mathbb{R}.$$

Much like in the Mc Donald decomposition, we can classify all entire functions of finite order whose magnitudes agree on two lines intersecting at a rational angle. Remember from our considerations on irrational angles that two

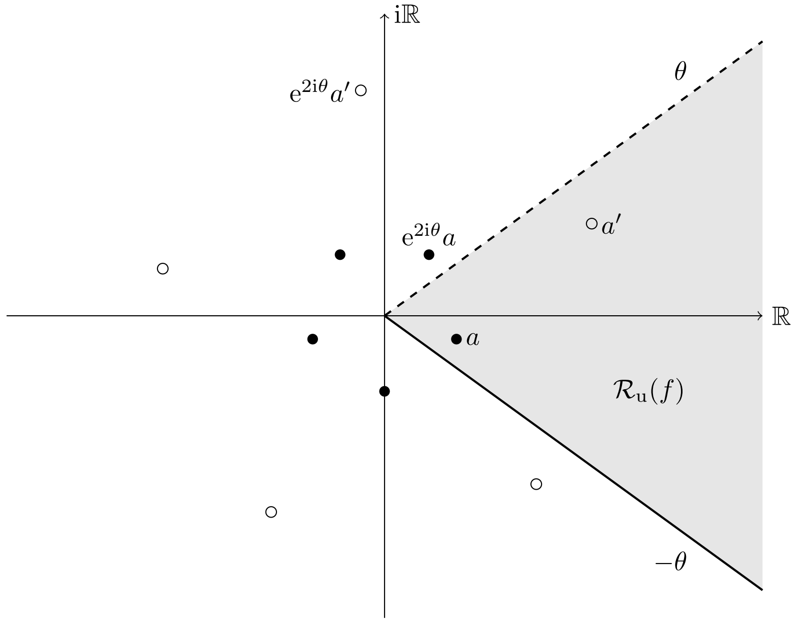

Figure 1: The splitting of $\mathcal{Y}$ according to the rotational symmetry with angle 2θ. The dotted line indicates that no element of $\mathrm{e}^{\mathrm{i}\theta}\mathbb{R}$ is included in $\mathcal{Y}$. Note how for any $a, a' \in \mathcal{Y}$, it holds that $\mathrm{e}^{2k\mathrm{i}\theta}a, \mathrm{e}^{2k\mathrm{i}\theta}a' \in \mathcal{Y}$, for $k = 1, \ldots, n$.

entire functions whose magnitudes agree on two lines which meet at an angle θ will have rotation symmetries around the origin with angle 2θ in their root set. To describe this more precisely, we may consider the smallest integer $n \in \mathbb{N}$ such that $n\theta \in \pi\mathbb{N}$ along with the set

$$\mathcal{Y}_{\mathrm{u}} := \mathcal{Y} \cap \left\{ z \in \mathbb{C} \,\middle|\, \exists\, r > 0,\ \phi \in [-\theta, \theta) : z = r\mathrm{e}^{\mathrm{i}\phi} \right\}$$

which is illustrated in Figure 1. Looking at equation (5), it is not too hard to see that

$$\mathcal{Y} = \mathrm{e}^{2\mathrm{i}\theta(\mathbb{Z}/n\mathbb{Z})}\mathcal{Y}_{\mathrm{u}}.$$

We may now show the following result.

Theorem 21 (Cf. Theorem 3)**.** *Let $\theta \in \pi\mathbb{Q} \cap (0, \frac{\pi}{2}]$, let $n \in \mathbb{N}$ be the smallest integer such that $\theta n \in \pi\mathbb{N}$, and let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero and of finite order. The following are equivalent:*

(a) *For all $x \in \mathbb{R}$, $|f(x)| = |g(x)|$ and $\left|f(\mathrm{e}^{\mathrm{i}\theta}x)\right| = \left|g(\mathrm{e}^{\mathrm{i}\theta}x)\right|$.*

(b) *There exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that*

$$b'_\ell = b_\ell, \qquad \text{for all } \ell \in \{1, \ldots, q\} \setminus n\mathbb{N},$$

as well as

$$\begin{aligned} f(z) &= r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\, z^\ell \right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \\ &\qquad \cdot \prod_{a \in \mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{a}; p\right)^{m_{\mathcal{Y}}(a)}, \\ g(z) &= r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell)\, z^\ell \right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \\ &\qquad \cdot \prod_{a \in \mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{\overline{a}}; p\right)^{m_{\mathcal{Y}}(a)}, \end{aligned}$$

for $z \in \mathbb{C}$.

Proof. We start by showing that item (a) implies item (b). According to the Mc Donald decomposition and the considerations before the statement of Theorem 19, there exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that

$$\begin{aligned} f(z) &= r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\, z^\ell \right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \\ &\qquad \cdot \prod_{a \in \mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{a}; p\right)^{m_{\mathcal{Y}}(a)}, \end{aligned}$$

$$
\begin{aligned}
g(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} \left(a_\ell + \mathrm{i}b'_\ell\right) z^\ell\right) z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};p\right)^{m_{\mathcal{X}}(a)} \\
\cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{\overline{a}};p\right)^{m_{\mathcal{Y}}(a)},
\end{aligned}
$$

for $z \in \mathbb{C}$.

Let $x \in \mathbb{R}$ be arbitrary and compute

$$
\begin{aligned}
\left|f(\mathrm{e}^{\mathrm{i}\theta}x)\right| = r\exp\left(\mathrm{Re}\sum_{\ell=1}^{q}\left(a_\ell + \mathrm{i}b_\ell\right)\mathrm{e}^{\mathrm{i}\theta\ell}x^\ell\right)|x|^m \prod_{a\in\mathcal{X}}\left|E\left(\mathrm{e}^{\mathrm{i}\theta}\frac{x}{a};p\right)\right|^{m_{\mathcal{X}}(a)} \\
\cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k=1}^{n}\left|E\left(\mathrm{e}^{(2k+1)\mathrm{i}\theta}\frac{x}{a};p\right)\right|^{m_{\mathcal{Y}}(a)}
\end{aligned}
$$

as well as

$$
\begin{aligned}
\left|g(\mathrm{e}^{\mathrm{i}\theta}x)\right| = r\exp\left(\mathrm{Re}\sum_{\ell=1}^{q}\left(a_\ell + \mathrm{i}b'_\ell\right)\mathrm{e}^{\mathrm{i}\theta\ell}x^\ell\right)|x|^m \prod_{a\in\mathcal{X}}\left|E\left(\mathrm{e}^{\mathrm{i}\theta}\frac{x}{a};p\right)\right|^{m_{\mathcal{X}}(a)} \\
\cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k=1}^{n}\left|E\left(\mathrm{e}^{(2k+1)\mathrm{i}\theta}\frac{x}{\overline{a}};p\right)\right|^{m_{\mathcal{Y}}(a)}.
\end{aligned}
$$

After a change of index, the latter becomes

$$
\begin{aligned}
\left|g(\mathrm{e}^{\mathrm{i}\theta}x)\right| = r\exp\left(\mathrm{Re}\sum_{\ell=1}^{q}\left(a_\ell + \mathrm{i}b'_\ell\right)\mathrm{e}^{\mathrm{i}\theta\ell}x^\ell\right)|x|^m \prod_{a\in\mathcal{X}}\left|E\left(\mathrm{e}^{\mathrm{i}\theta}\frac{x}{a};p\right)\right|^{m_{\mathcal{X}}(a)} \\
\cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k=1}^{n}\left|E\left(\mathrm{e}^{-(2k+1)\mathrm{i}\theta}\frac{x}{\overline{a}};p\right)\right|^{m_{\mathcal{Y}}(a)}.
\end{aligned}
$$

We can now compute

$$
\begin{aligned}
1 &= \frac{\left|f(\mathrm{e}^{\mathrm{i}\theta}x)\right|}{\left|g(\mathrm{e}^{\mathrm{i}\theta}x)\right|} \\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\sin(\theta\ell)x^\ell\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k=1}^{n}\left|\frac{E\left(\mathrm{e}^{(2k+1)\mathrm{i}\theta}\frac{x}{a};p\right)}{E\left(\mathrm{e}^{-(2k+1)\mathrm{i}\theta}\frac{x}{\overline{a}};p\right)}\right|^{m_{\mathcal{Y}}(a)} \\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\sin(\theta\ell)x^\ell\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k=1}^{n}\left|\frac{\left(1-\mathrm{e}^{(2k+1)\mathrm{i}\theta}\frac{x}{a}\right)}{\left(1-\mathrm{e}^{-(2k+1)\mathrm{i}\theta}\frac{x}{\overline{a}}\right)}\right|^{m_{\mathcal{Y}}(a)} \\
&\qquad \cdot \exp\left(m_{\mathcal{Y}}(a)\,\mathrm{Re}\sum_{\ell=1}^{p}\frac{1}{\ell}\left[\mathrm{e}^{(2k+1)\mathrm{i}\theta\ell}\frac{x^\ell}{a^\ell} - \mathrm{e}^{-(2k+1)\mathrm{i}\theta\ell}\frac{x^\ell}{\overline{a}^\ell}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\sin(\theta\ell)x^\ell\right) \\
&\qquad \cdot \prod_{a\in\mathcal{Y}_\mathrm{u}}\prod_{k=1}^{n}\exp\left(m_\mathcal{Y}(a)\sum_{\ell=1}^{p}\frac{x^\ell}{\ell}\operatorname{Re}\left[\mathrm{e}^{(2k+1)\mathrm{i}\theta\ell}a^{-\ell} - \mathrm{e}^{-(2k+1)\mathrm{i}\theta\ell}\overline{a}^{-\ell}\right]\right) \\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\sin(\theta\ell)x^\ell\right).
\end{aligned}
$$

Therefore,

$$
0 = \sum_{\ell=1}^{q}(b'_\ell - b_\ell)\sin(\theta\ell)x^\ell.
$$

As $x \in \mathbb{R}$ was arbitrary, we can compare coefficients. In this way, we obtain

$$
(b'_\ell - b_\ell)\sin(\theta\ell) = 0, \qquad \ell \in \{1, \ldots, q\}.
$$

For each $\ell \in \{1, \ldots, q\}$ there are two cases: either $\theta\ell \in \pi\mathbb{Z}$ or $\theta\ell \notin \pi\mathbb{Z}$. In the former case, our equation is trivial and $\ell \in \{n, 2n, \ldots\}$. In the latter case, we conclude

$$
b'_\ell = b_\ell.
$$

Finally, we can show that item (b) implies item (a). Note that the assumptions in item (b) are stronger than the assumptions in item (b) of the Mc Donald decomposition theorem. Therefore, it must hold that $|f(x)| = |g(x)|$, for $x \in \mathbb{R}$. In addition, we can follow the computations from earlier in this proof to see that for $x \in \mathbb{R}$,

$$
\frac{\left|f(\mathrm{e}^{\mathrm{i}\theta}x)\right|}{|g(\mathrm{e}^{\mathrm{i}\theta}x)|} = \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\sin(\theta\ell)x^\ell\right) = 1, \qquad x \in \mathbb{R},
$$

since

$$
b'_\ell = b_\ell, \qquad \text{for all } \ell \in \{1, \ldots, q\} \setminus \{n, 2n, \ldots\}.
$$

We conclude that $\left|f(\mathrm{e}^{\mathrm{i}\theta}x)\right| = \left|g(\mathrm{e}^{\mathrm{i}\theta}x)\right|$, for $x \in \mathbb{R}$. □

Remark 22. *As mentioned before, we can use a rigid motion $\mathfrak{r} : \mathbb{C} \to \mathbb{C}$ to transform the above result in such a way that it holds for any two lines in the complex plane which intersect at a rational angle. Additionally, it is enough to assume that $|f(x)| = |g(x)|$ and $\left|f(\mathrm{e}^{\mathrm{i}\alpha}x)\right| = \left|g(\mathrm{e}^{\mathrm{i}\alpha}x)\right|$, for $x \in X \subset \mathbb{R}$, where X is any set with accumulation point.*

4.2 Parallel lines

Having looked at the case of intersecting lines, we want to move on to consider two (and more) parallel lines in the complex plane. In the first part of this section, we will assume without loss of generality (and this is again due to the holomorphy of rigid motions as discussed in Remark 15) that one of those two parallel lines is given by the real numbers and the other of those parallel lines is given by $\mathbb{R} + \mathrm{i}\tau$, for $\tau \in \mathbb{R}_* = \mathbb{R} \setminus \{0\}$. Thereafter, we show that we can use the same considerations to elucidate the case of infinitely many parallel lines $\mathbb{R} + \mathrm{i}\tau\mathbb{Z}$.

Two parallel lines We want to start by noting that it is not enough to know the magnitude of an entire function of finite order on two distinct parallel lines in the complex plane to identify the function up to global phase. Indeed, consider $f, g \in \mathcal{O}(\mathbb{C})$ given by

$$\begin{aligned} f(z) &= \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right), \\ g(z) &= \cosh\left(\frac{\pi z}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi z}{2}\right), \end{aligned}$$

for $z \in \mathbb{C}$. Then, since the complex conjugate of any complex number has the same magnitude as the number itself, it holds that

$$|f(x)| = \left|\cosh\left(\frac{\pi x}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\right| = \left|\cosh\left(\frac{\pi x}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\right| = |g(x)|,$$

for $x \in \mathbb{R}$. Additionally, we can use trigonometric identities to convince ourself of the fact that

$$\begin{aligned} |f(x+\mathrm{i})| &= \left|\cosh\left(\frac{\pi x}{2} + \frac{\pi \mathrm{i}}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi x}{2} + \frac{\pi \mathrm{i}}{2}\right)\right| \\ &= \Big| \cosh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) \\ &\qquad + \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right) \Big| \\ &= \left|\cosh\left(\frac{\pi x}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\right| = \left|\cosh\left(\frac{\pi x}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\right| \\ &= \Big| \cosh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right)\right) \\ &\qquad - \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right) \Big| \\ &= \left|\cosh\left(\frac{\pi x}{2} + \frac{\pi \mathrm{i}}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi x}{2} + \frac{\pi \mathrm{i}}{2}\right)\right| = |g(x+\mathrm{i})|, \end{aligned}$$

for $x \in \mathbb{R}$.

Remark 23. *Note that given any two distinct parallel lines $L_1, L_2 \subset \mathbb{C}$, we may find a holomorphic map $\mathfrak{h} : \mathbb{C} \to \mathbb{C}$ such that $\mathfrak{h}(L_1) = \mathbb{R}$ and $\mathfrak{h}(L_2) = \mathbb{R} + \mathrm{i}$. Therefore, the example given above naturally translates to an example for any given two distinct parallel lines (when we consider $f \circ \mathfrak{h}$ as well as $g \circ \mathfrak{h}$) and we conclude that* given two parallel lines L_1 and L_2 in the complex plane, there exist entire functions which do not agree up to global phase and whose magnitudes agree on $L_1 \cup L_2$.

As we have done twice at this point, we will continue by characterising the non-uniqueness of phase retrieval in terms of the Hadamard factorisation of finite order entire functions. Let us consider $\tau \in \mathbb{R}_*$ and suppose that we have two non-zero functions $f, g \in \mathcal{O}(\mathbb{C})$ such that

$$|f(x)| = |g(x)| \qquad \text{and} \qquad |f(x + \mathrm{i}\tau)| = |g(x + \mathrm{i}\tau)|,$$

for all $x \in \mathbb{R}$. Then, the symmetries in the root set described by Lemma 16 are accompanied by additional symmmetries stemming from the equality of the absolute values on $\mathbb{R} + \mathrm{i}\tau$.

Lemma 24 (Root symmetry — III)**.** *Let $\tau \in \mathbb{R}_*$ and let $f, g \in \mathcal{O}(\mathbb{C})$ be such that*

$$|f(x + \mathrm{i}\tau)| = |g(x + \mathrm{i}\tau)|, \qquad x \in \mathbb{R}.$$

Then, it holds that

$$M_f(z + \mathrm{i}\tau) + M_f(\overline{z} + \mathrm{i}\tau) = M_g(z + \mathrm{i}\tau) + M_g(\overline{z} + \mathrm{i}\tau), \qquad z \in \mathbb{C}.$$

Viewed from a geometrical point of view, the above lemma states that there is a mirror symmetry along the axis $\mathbb{R}+\mathrm{i}\tau$ in the root sets of f and g. Again this symmetry becomes especially interesting when one considers its implications for the multiset described by $M_{\mathcal{Y}}$. If one does consider a root a of f which is not a root of g, then $\overline{a}$ must be a root of g according to the mirror symmetry along the real line. If $\overline{a}$ is not a root of f, then $a+2\mathrm{i}\tau$ must be a root of f according to the mirror symmetry along the line $\mathbb{R}+\mathrm{i}\tau$. In this way, one observes a translational symmetry in the roots.

More precisely, one can convince oneself that the Lemmata 16 and 24 imply that

$$\begin{aligned} M_{\mathcal{Y}}(z + 2\mathrm{i}\tau) &= \max\{0, M_f(z + 2\mathrm{i}\tau) - M_g(z + 2\mathrm{i}\tau)\} \\ &= \max\{0, M_g(\overline{z}) - M_f(\overline{z})\} = \max\{0, M_f(z) - M_g(z)\} \\ &= M_{\mathcal{Y}}(z), \end{aligned}$$

for $z \in \mathbb{C}$. It follows immediately that

$$M_{\mathcal{Y}}(z + 2k\mathrm{i}\tau) = M_{\mathcal{Y}}(z), \qquad z \in \mathbb{C},$$

for $k \in \mathbb{Z}$, and the set $\mathcal{Y}$ is symmetric with respect to translation by $2\mathrm{i}\tau$. Note that it follows that $\mathcal{Y}$ must contain infinitely many elements when it is non-empty. Unlike in the case of irrational angles, this does not lead to a contradiction, however, since those elements do not need to have an accumulation point. It is interesting to note that $\mathcal{Y} \neq \emptyset$ implies that f and g must at least be of order one by Lemma 11.

We might introduce the set

$$\mathcal{Y}_{\mathrm{u}} := \mathcal{Y} \cap (\mathbb{R} + \mathrm{i}\,[-\tau, \tau))$$

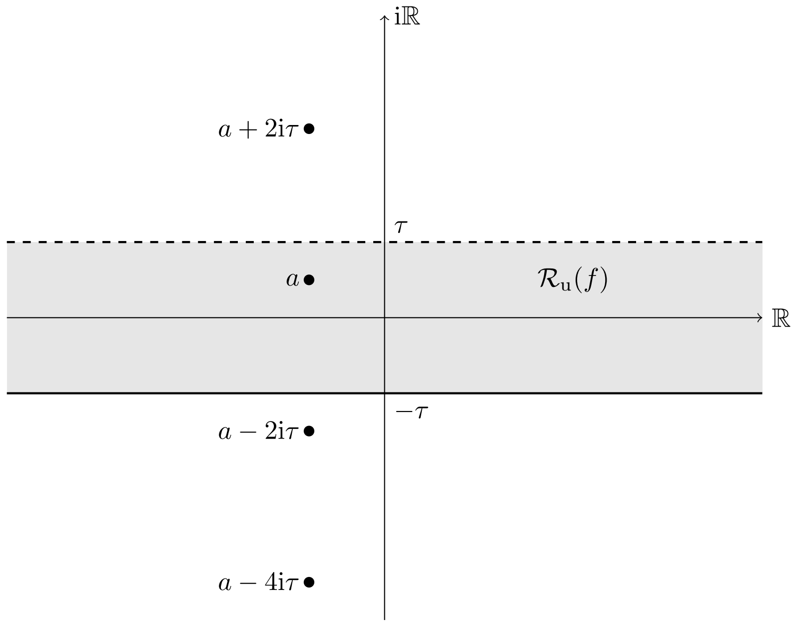

Figure 2: Depiction of $\mathcal{Y}_{\mathrm{u}}$. Note that the dashed line indicates that $\mathbb{R}+\mathrm{i}\tau$ is not contained in $\mathcal{Y}_{\mathrm{u}}$.

visualised in Figure 2 to see that

$$\mathcal{Y} = \mathcal{Y}_{\mathrm{u}} + 2\mathrm{i}\tau\mathbb{Z}.$$

We may now prove the following result.

Theorem 25 (Cf. Theorem 4)**.** *Let $\tau \in \mathbb{R}_*$ and let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero and of finite order. Then, the following are equivalent:*

(a) *For all $x \in \mathbb{R}$, $|f(x)| = |g(x)|$ and $|f(x+\mathrm{i}\tau)| = |g(x+\mathrm{i}\tau)|$.*

(b) *There exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that*

$$\sum_{\ell=1}^{q} (b_\ell - b'_\ell) \operatorname{Im}\left[(x+\mathrm{i}\tau)^\ell\right] = \sum_{a \in \mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k \in \mathbb{Z}} \Bigg(\log \left| \frac{a + 2(k-1)\mathrm{i}\tau}{a + 2k\mathrm{i}\tau} \right| + \sum_{\ell=1}^{p} \frac{1}{\ell} \operatorname{Re}\left[\left(\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau} \right)^\ell - \left(\frac{x+\mathrm{i}\tau}{\overline{a} - 2(k-1)\mathrm{i}\tau} \right)^\ell \right] \Bigg), \quad (6)$$

for all $x \in \mathbb{R}$*, as well as*

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\, z^\ell\right) z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k\in\mathbb{Z}} E\left(\frac{z}{a + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$

$$g(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell)\, z^\ell\right) z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k\in\mathbb{Z}} E\left(\frac{z}{\overline{a} + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$*.*

Proof. Let us first show that item (a) implies item (b). According to the Mc Donald decomposition and the prior considerations there exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\, z^\ell\right) z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k\in\mathbb{Z}} E\left(\frac{z}{a + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$

$$g(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell)\, z^\ell\right) z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k\in\mathbb{Z}} E\left(\frac{z}{\overline{a} + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$.

Let $x \in \mathbb{R}$ be arbitrary and compute

$$|f(x + \mathrm{i}\tau)| = r \exp\left(\mathrm{Re} \sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\,(x + \mathrm{i}\tau)^\ell\right) |x + \mathrm{i}\tau|^m \cdot \prod_{a\in\mathcal{X}} \left| E\left(\frac{x + \mathrm{i}\tau}{a}; p\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k\in\mathbb{Z}} \left| E\left(\frac{x + \mathrm{i}\tau}{a + 2k\mathrm{i}\tau}; p\right)\right|^{m_{\mathcal{Y}}(a)},$$

as well as

$$|g(x + \mathrm{i}\tau)| = r \exp\left(\mathrm{Re} \sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell)\,(x + \mathrm{i}\tau)^\ell\right) |x + \mathrm{i}\tau|^m$$

$$\cdot \prod_{a\in\mathcal{X}} \left| E\left(\frac{x+\mathrm{i}\tau}{a};p\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}} \left| E\left(\frac{x+\mathrm{i}\tau}{\overline{a}+2k\mathrm{i}\tau};p\right)\right|^{m_{\mathcal{Y}}(a)}.$$

After a change of index, the latter becomes

$$|g(x+\mathrm{i}\tau)| = r\exp\left(\mathrm{Re}\sum_{\ell=1}^{q}(a_\ell+\mathrm{i}b'_\ell)\,(x+\mathrm{i}\tau)^\ell\right)|x+\mathrm{i}\tau|^m$$
$$\cdot \prod_{a\in\mathcal{X}} \left| E\left(\frac{x+\mathrm{i}\tau}{a};p\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}} \left| E\left(\frac{x+\mathrm{i}\tau}{\overline{a}+2(k-1)\mathrm{i}\tau};p\right)\right|^{m_{\mathcal{Y}}(a)}.$$

We can now compute

$$\begin{aligned}
1 &= \frac{|f(x+\mathrm{i}\tau)|}{|g(x+\mathrm{i}\tau)|}\\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell-b_\ell)\,\mathrm{Im}\left[(x+\mathrm{i}\tau)^\ell\right]\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|\frac{E\left(\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau};p\right)}{E\left(\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau};p\right)}\right|^{m_{\mathcal{Y}}(a)}\\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell-b_\ell)\,\mathrm{Im}\left[(x+\mathrm{i}\tau)^\ell\right]\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|\frac{1-\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}}{1-\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}}\right|^{m_{\mathcal{Y}}(a)}\\
&\qquad\cdot\exp\left(m_{\mathcal{Y}}(a)\sum_{\ell=1}^{p}\frac{1}{\ell}\,\mathrm{Re}\left[\left(\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^\ell-\left(\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}\right)^\ell\right]\right)\\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell-b_\ell)\,\mathrm{Im}\left[(x+\mathrm{i}\tau)^\ell\right]\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right|^{m_{\mathcal{Y}}(a)}\\
&\qquad\cdot\exp\left(m_{\mathcal{Y}}(a)\sum_{\ell=1}^{p}\frac{1}{\ell}\,\mathrm{Re}\left[\left(\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^\ell-\left(\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}\right)^\ell\right]\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{\ell=1}^{q}(b_\ell-b'_\ell)\,\mathrm{Im}\left[(x+\mathrm{i}\tau)^\ell\right] = \sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}&\left(\log\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right|\right.\\
&\left.+\sum_{\ell=1}^{p}\frac{1}{\ell}\,\mathrm{Re}\left[\left(\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^\ell-\left(\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}\right)^\ell\right]\right).
\end{aligned}$$

Finally, we show that item (b) implies item (a). Note that the decompositions in item (b) satisfy the assumptions under item (b) of the Mc Donald decomposition theorem. Therefore, it follows that $|f(x)| = |g(x)|$, for $x\in\mathbb{R}$. We can now follow the computations in the first part of this proof to find that for $x\in\mathbb{R}$,

$$
\begin{aligned}
\frac{|f(x+\mathrm{i}\tau)|}{|g(x+\mathrm{i}\tau)|} &= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\operatorname{Im}\left[(x+\mathrm{i}\tau)^\ell\right]\right) \\
&\qquad \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right|^{m_{\mathcal{Y}}(a)} \\
&\quad \cdot \exp\left(m_{\mathcal{Y}}(a)\sum_{\ell=1}^{p}\frac{1}{\ell}\operatorname{Re}\left[\left(\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^\ell - \left(\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}\right)^\ell\right]\right).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\log\frac{|f(x+\mathrm{i}\tau)|}{|g(x+\mathrm{i}\tau)|} &= \sum_{\ell=1}^{q}(b'_\ell - b_\ell)\operatorname{Im}\left[(x+\mathrm{i}\tau)^\ell\right] \\
&\qquad + \sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\left(\log\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right|\right. \\
&\qquad\left. + \sum_{\ell=1}^{p}\frac{1}{\ell}\operatorname{Re}\left[\left(\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^\ell - \left(\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}\right)^\ell\right]\right) = 0.
\end{aligned}
$$

We conclude that $|f(x+\mathrm{i}\tau)| = |g(x+\mathrm{i}\tau)|$. □

Remark 26. *Note that the theorem above deals with the special case where one of the two parallel lines is $\mathbb{R}$ and the other is $\mathbb{R}+\mathrm{i}\tau$, with $\tau \in \mathbb{R}_*$. Note also that given any two parallel lines $L_1, L_2 \subset \mathbb{C}$, we can find a rigid motion $\mathfrak{r} : \mathbb{C} \to \mathbb{C}$ such that $\mathfrak{r}(L_1) = \mathbb{R}$ and $\mathfrak{r}(L_2) = \mathbb{R}+\mathrm{i}\tau$, with $\tau \in \mathbb{R}_*$. With the help of this rigid motion, we can provide results for any two parallel lines in the complex plane.*

Finally, as before, it is enough to assume that $|f(x)| = |g(x)|$ as well as $|f(x+\mathrm{i}\tau)| = |g(x+\mathrm{i}\tau)|$, for $x \in X \subset \mathbb{R}$, where X is any set with accumulation point.

The condition on the numbers $b_\ell, b'_\ell \in \mathbb{R}$, for $\ell = 1, \ldots, q$, given in equation (6) seems of limited use in this generality. The main difficulty in simplifying the relation between $\{b_\ell\}_{\ell=1}^q$ and $\{b'_\ell\}_{\ell=1}^q$ lies in the fact that generally the sums involved in the relation are infinite and therefore a straight-forward comparison of coefficients like the ones performed in the proofs of Theorems 19 and 21 can only work when all the sums one needs to reorder actually converge unconditionally. In the following pages, we will illuminate one special case in which this turns out to be true and thus the condition on the numbers b_ℓ, b'_ℓ simplifies significantly.

Infinitely many parallel lines It is remarkable that the functions $f, g \in \mathcal{O}(\mathbb{C})$ given by

$$
\begin{aligned}
f(z) &= \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right), \\
g(z) &= \cosh\left(\frac{\pi z}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi z}{2}\right),
\end{aligned}
$$

for $z \in \mathbb{C}$, do have magnitudes which not only agree on $\mathbb{R}$ and $\mathbb{R} + \mathrm{i}$ but which agree on $\mathbb{R} + \mathrm{i}\mathbb{Z}$. In other words, the magnitudes of f and g agree on infinitely many parallel lines in the complex plane. To see this, one can recycle the computation from before and convince oneself (by noting that the terms involving a cosine must vanish when n is odd and, likewise, the terms involving sine must vanish when n is even in the third step) that

$$\begin{aligned}
|f(x+n\mathrm{i})| &= \left|\cosh\left(\frac{\pi x}{2} + \frac{\pi n \mathrm{i}}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi x}{2} + \frac{\pi n \mathrm{i}}{2}\right)\right| \\
&= \Big| \cosh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi n}{2}\right) - \sin\left(\frac{\pi n}{2}\right)\right) \\
&\qquad + \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi n}{2}\right) + \sin\left(\frac{\pi n}{2}\right)\right) \Big| \\
&= \Big| \cosh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi n}{2}\right) + \sin\left(\frac{\pi n}{2}\right)\right) \\
&\qquad - \mathrm{i}\sinh\left(\frac{\pi x}{2}\right)\left(\cos\left(\frac{\pi n}{2}\right) - \sin\left(\frac{\pi n}{2}\right)\right) \Big| \\
&= \left|\cosh\left(\frac{\pi x}{2} + \frac{\pi n \mathrm{i}}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi x}{2} + \frac{\pi n \mathrm{i}}{2}\right)\right| = |g(x+n\mathrm{i})|,
\end{aligned}$$

for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. This suggests that Theorem 25 might be adaptable to the case in which we consider infinitely many parallel lines. This hunch is further supported by the fact that additional equispaced parallel lines will lead to symmetries with respect to translations which are already implied by the symmetry stemming from the equations on $\mathbb{R}$ and $\mathbb{R} + \mathrm{i}\tau$. In other words, the magnitude of an entire function on infinitely many equidistant parallel lines does not contain more information on the roots of the function in question than the magnitude on two parallel lines[1]. At least for functions of order one, we can show that Theorem 25 is directly adaptable to the case in which infinitely many lines are considered:

Theorem 27 (Cf. Theorem 5)**.** *Let $\tau \in \mathbb{R}_*$ and let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero and of order one. Then, the following are equivalent:*

(a) *For all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, $|f(x+n\mathrm{i}\tau)| = |g(x+n\mathrm{i}\tau)|$.*

(b) *There exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $a', b, b' \in \mathbb{R}$ as well as $m \in \mathbb{N}_0$ such that*

$$b = b' + \frac{1}{\tau}\sum_{a \in \mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k \in \mathbb{Z}} \left(\log\left|\frac{a + 2(k-1)\mathrm{i}\tau}{a + 2k\mathrm{i}\tau}\right| - 2\tau\,\mathrm{Im}\,\frac{1}{a + 2k\mathrm{i}\tau}\right).$$

as well as

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{(a'+\mathrm{i}b)z} z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a \in \mathcal{Y}_{\mathrm{u}}}\prod_{k \in \mathbb{Z}} E\left(\frac{z}{a + 2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},$$

[1]Note that these two parallel lines have to be carefully chosen to be two lines which have smallest distance between each other.

$$g(z) = r\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{(a'+\mathrm{i}b')z}z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)} \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}} E\left(\frac{z}{\overline{a}+2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$.

Proof. Let us start by collecting three facts about the convergence of certain series along with their proofs.

Fact i. Let $\ell \in \mathbb{Z}$ be arbitrary but fixed. Then,

$$\sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k\in\mathbb{Z}} \left|\log\left|\frac{a+2(k-\ell-1)\mathrm{i}\tau}{a+2(k-\ell)\mathrm{i}\tau}\right| - 2\tau\operatorname{Im}\frac{1}{a+2(k-\ell)\mathrm{i}\tau}\right| < \infty.$$

Proof of Fact i. Let us start by noting that

$$\begin{aligned}
\sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k\in\mathbb{Z}} &\left|\log\left|\frac{a+2(k-\ell-1)\mathrm{i}\tau}{a+2(k-\ell)\mathrm{i}\tau}\right| - 2\tau\operatorname{Im}\frac{1}{a+2(k-\ell)\mathrm{i}\tau}\right| \\
&= \sum_{a\in\mathcal{Y}} m_{\mathcal{Y}}(a)\left|\log\left|\frac{a-2\mathrm{i}\tau}{a}\right| - 2\tau\operatorname{Im}\frac{1}{a}\right| \\
&= \sum_{a\in\mathcal{Y}} m_{\mathcal{Y}}(a)\left|\log\left|1-\frac{2\mathrm{i}\tau}{a}\right| + \operatorname{Re}\frac{2\mathrm{i}\tau}{a}\right| \\
&= \sum_{a\in\mathcal{Y}} m_{\mathcal{Y}}(a)\left|\operatorname{Re}\left[\log\left(1-\frac{2\mathrm{i}\tau}{a}\right)+\frac{2\mathrm{i}\tau}{a}\right]\right| \\
&\leq \sum_{a\in\mathcal{Y}} m_{\mathcal{Y}}(a)\left|\log\left(1-\frac{2\mathrm{i}\tau}{a}\right)+\frac{2\mathrm{i}\tau}{a}\right|.
\end{aligned}$$

Since f is an entire function, it follows that $\mathcal{Y} \subset \mathcal{R}(f)$ is a set of isolated points. Therefore, we have

$$\sum_{\substack{a\in\mathcal{Y}\\ |a|\leq 4|\tau|}} m_{\mathcal{Y}}(a)\left|\log\left(1-\frac{2\mathrm{i}\tau}{a}\right)+\frac{2\mathrm{i}\tau}{a}\right| < \infty$$

due to the sum being taken over a finite set. We may now use the Mercator series of the logarithm and the fact that f is of order one (see Lemma 8) to see that

$$\begin{aligned}
\sum_{\substack{a\in\mathcal{Y}\\ |a|>4|\tau|}} m_{\mathcal{Y}}(a)\left|\log\left(1-\frac{2\mathrm{i}\tau}{a}\right)+\frac{2\mathrm{i}\tau}{a}\right| &= \sum_{\substack{a\in\mathcal{Y}\\ |a|>4|\tau|}} m_{\mathcal{Y}}(a)\left|\sum_{j=2}^{\infty}\frac{1}{j}\left(\frac{2\mathrm{i}\tau}{a}\right)^j\right| \\
&\leq \frac{1}{2}\sum_{\substack{a\in\mathcal{Y}\\ |a|>4|\tau|}} m_{\mathcal{Y}}(a)\sum_{j=2}^{\infty}\left|\frac{2\mathrm{i}\tau}{a}\right|^j
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{\substack{a\in\mathcal{Y}\\ |a|>4|\tau|}} m_{\mathcal{Y}}(a) \left|\frac{2\mathrm{i}\tau}{a}\right|^2 \sum_{j=0}^{\infty} \left|\frac{2\mathrm{i}\tau}{a}\right|^j \\
&\leq 2\tau^2 \sum_{\substack{a\in\mathcal{Y}\\ |a|>4|\tau|}} \frac{m_{\mathcal{Y}}(a)}{|a|^2} \sum_{j=0}^{\infty} 2^{-j} \\
&= 4\tau^2 \sum_{\substack{a\in\mathcal{Y}\\ |a|>4|\tau|}} \frac{m_{\mathcal{Y}}(a)}{|a|^2} < \infty.
\end{aligned}
$$

Fact ii. Let $\ell', \ell \in \mathbb{N}_0$, such that $\ell' \leq \ell$, be arbitrary but fixed. Then,

$$\sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k\in\mathbb{Z}} \left| \frac{1}{a+2(k-\ell')\mathrm{i}\tau} - \frac{1}{a+2(k-\ell)\mathrm{i}\tau} \right| < \infty.$$

Proof of Fact ii. We can estimate

$$
\begin{aligned}
&\sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k\in\mathbb{Z}} \left| \frac{1}{a+2(k-\ell')\mathrm{i}\tau} - \frac{1}{a+2(k-\ell)\mathrm{i}\tau} \right| \\
&\qquad = \sum_{a\in\mathcal{Y}} m_{\mathcal{Y}}(a) \left| \frac{1}{a-2\ell'\mathrm{i}\tau} - \frac{1}{a-2\ell\mathrm{i}\tau} \right| \\
&\qquad = \sum_{a\in\mathcal{Y}} m_{\mathcal{Y}}(a) \left| \frac{2(\ell'-\ell)\mathrm{i}\tau}{(a-2\ell'\mathrm{i}\tau)(a-2\ell\mathrm{i}\tau)} \right| \\
&\qquad\qquad = 2(\ell-\ell')\,|\tau| \sum_{a\in\mathcal{Y}} \frac{m_{\mathcal{Y}}(a)}{|a-2\ell'\mathrm{i}\tau|\,|a-2\ell\mathrm{i}\tau|}.
\end{aligned}
\tag{7}
$$

We might again see that

$$\sum_{\substack{a\in\mathcal{Y}\\ |a|\leq 4\ell|\tau|}} \frac{m_{\mathcal{Y}}(a)}{|a-2\ell'\mathrm{i}\tau|\,|a-2\ell\mathrm{i}\tau|} < \infty$$

since f being entire implies that its roots are isolated and thus the above sum is taken over a finite set. Additionally, we have (since f is of order one)

$$
\begin{aligned}
\sum_{\substack{a\in\mathcal{Y}\\ |a|>4\ell|\tau|}} \frac{m_{\mathcal{Y}}(a)}{|a-2\ell'\mathrm{i}\tau|\,|a-2\ell\mathrm{i}\tau|} &\leq \sum_{\substack{a\in\mathcal{Y}\\ |a|>4\ell|\tau|}} \frac{m_{\mathcal{Y}}(a)}{(|a|-2\ell'\,|\tau|)(|a|-2\ell\,|\tau|)} \\
&\leq 4 \sum_{\substack{a\in\mathcal{Y}\\ |a|>4\ell|\tau|}} \frac{m_{\mathcal{Y}}(a)}{|a|^2} < \infty.
\end{aligned}
$$

Fact iii. Let $\ell', \ell \in \mathbb{N}_0$, such that $\ell' \leq \ell$, be arbitrary but fixed. Then,

$$\sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k\in\mathbb{Z}} \left(\frac{1}{a+2(k-\ell')\mathrm{i}\tau} - \frac{1}{a+2(k-\ell)\mathrm{i}\tau} \right) = 0.$$

Proof of Fact iii. Let us start by noting that

$$\frac{1}{a+2(k-\ell')\mathrm{i}\tau} - \frac{1}{a+2(k-\ell)\mathrm{i}\tau} = \sum_{j=\ell'}^{\ell-1} \left(\frac{1}{a+2(k-j)\mathrm{i}\tau} - \frac{1}{a+2(k-j-1)\mathrm{i}\tau} \right),$$

for $a \in \mathcal{Y}_{\mathrm{u}}$ and $k \in \mathbb{Z}$. By Fact ii (which guarantees the unconditional convergence of the sums involved), we may write

$$\sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k\in\mathbb{Z}} \left(\frac{1}{a+2(k-\ell')\mathrm{i}\tau} - \frac{1}{a+2(k-\ell)\mathrm{i}\tau} \right) = \sum_{j=\ell'}^{\ell-1} \sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a) \sum_{k\in\mathbb{Z}} \left(\frac{1}{a+2(k-j)\mathrm{i}\tau} - \frac{1}{a+2(k-j-1)\mathrm{i}\tau} \right).$$

We may now consider

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \left(\frac{1}{a+2(k-j)\mathrm{i}\tau} - \frac{1}{a+2(k-j-1)\mathrm{i}\tau} \right) &= \lim_{N\to\infty} \sum_{k=-N}^{N} \left(\frac{1}{a+2(k-j)\mathrm{i}\tau} - \frac{1}{a+2(k-j-1)\mathrm{i}\tau} \right) \\ &= \lim_{N\to\infty} \left(\frac{1}{a+2(N-j)\mathrm{i}\tau} - \frac{1}{a-2(N+j+1)\mathrm{i}\tau} \right) = 0. \end{aligned}$$

We can now turn to the proof of the Theorem. Let us first show that item (b) implies item (a). If we assume that item (b) holds, then we know by the Mc Donald decomposition theorem that $|f| = |g|$ on the real numbers. We may now consider $x \in \mathbb{R}$ as well as $n \in \mathbb{Z}_*$ and compute

$$|f(x+n\mathrm{i}\tau)| = r\mathrm{e}^{\mathrm{Re}\left((a'+\mathrm{i}b)(x+n\mathrm{i}\tau)\right)} |x+n\mathrm{i}\tau|^m \prod_{a\in\mathcal{X}} \left| E\left(\frac{x+n\mathrm{i}\tau}{a};1\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k\in\mathbb{Z}} \left| E\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau};1\right)\right|^{m_{\mathcal{Y}}(a)}.$$

as well as (by multiplying over $n-k$)

$$|g(x+n\mathrm{i}\tau)| = r\mathrm{e}^{\mathrm{Re}\left((a'+\mathrm{i}b')(x+n\mathrm{i}\tau)\right)} |x+n\mathrm{i}\tau|^m \prod_{a\in\mathcal{X}} \left| E\left(\frac{x+n\mathrm{i}\tau}{a};1\right)\right|^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k\in\mathbb{Z}} \left| E\left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau};1\right)\right|^{m_{\mathcal{Y}}(a)}.$$

Therefore, we find that

$$\begin{aligned}
\frac{|f(x+n\mathrm{i}\tau)|}{|g(x+n\mathrm{i}\tau)|} &= \mathrm{e}^{\operatorname{Re}\left(\mathrm{i}(b-b')(x+n\mathrm{i}\tau)\right)} \prod_{a\in\mathcal{Y}_\mathrm{u}} \prod_{k\in\mathbb{Z}} \left| \frac{E\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau};1\right)}{E\left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau};1\right)} \right|^{m_\mathcal{Y}(a)} \\
&= \mathrm{e}^{(b'-b)\operatorname{Im}(x+n\mathrm{i}\tau)} \prod_{a\in\mathcal{Y}_\mathrm{u}} \prod_{k\in\mathbb{Z}} \left| \frac{1-\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau}}{1-\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau}} \right|^{m_\mathcal{Y}(a)} \\
&\qquad \cdot \exp\left(m_\mathcal{Y}(a) \operatorname{Re}\left[\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau} - \frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau} \right] \right) \\
&= \mathrm{e}^{n\tau(b'-b)} \prod_{a\in\mathcal{Y}_\mathrm{u}} \prod_{k\in\mathbb{Z}} \left| \frac{a+2(k-n)\mathrm{i}\tau}{a+2k\mathrm{i}\tau} \right|^{m_\mathcal{Y}(a)} \\
&\qquad \cdot \exp\left(m_\mathcal{Y}(a) \operatorname{Re}\left[\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau} - \frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau} \right] \right),
\end{aligned}$$

whenever this fraction is well-defined. From this computation, it is clear that $|f(x+n\mathrm{i}\tau)| = |g(x+n\mathrm{i}\tau)|$ if

$$\begin{aligned}
b-b' = \frac{1}{n\tau} \sum_{a\in\mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k\in\mathbb{Z}} \Bigg(&\log\left| \frac{a+2(k-n)\mathrm{i}\tau}{a+2k\mathrm{i}\tau} \right| \\
&+ \operatorname{Re}\left[\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau} - \frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau} \right] \Bigg).
\end{aligned}$$

It will turn out that this condition on the coefficients $b, b' \in \mathbb{R}$ is equivalent to the condition stated in the theorem. Showing this is a lengthy technical exercise which must in particular explain why the condition above is independent of $x \in \mathbb{R}$ and $n \in \mathbb{Z}_*$.

Let us consider $n \in \mathbb{N}$ (the case in which $-n \in \mathbb{N}$ can be dealt with analogously) as well as $x \in \mathbb{R}$. It holds that

$$\log\left| \frac{a+2(k-n)\mathrm{i}\tau}{a+2k\mathrm{i}\tau} \right| = \sum_{\ell=1}^{n} \log\left| \frac{a+2(k-\ell)\mathrm{i}\tau}{a+2(k-\ell+1)\mathrm{i}\tau} \right|,$$

for $a \in \mathcal{Y}_\mathrm{u}$ and $k \in \mathbb{Z}$, as well as

$$\begin{aligned}
\operatorname{Re}\left[\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau} - \frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau} \right] = x \operatorname{Re}&\left[\frac{1}{a+2k\mathrm{i}\tau} - \frac{1}{a+2(k-n)\mathrm{i}\tau} \right] \\
&- n\tau \operatorname{Im}\left[\frac{1}{a+2k\mathrm{i}\tau} + \frac{1}{a+2(k-n)\mathrm{i}\tau} \right].
\end{aligned}$$

We may now rewrite the coefficients of the sum relating b and b':

$$\log\left| \frac{a+2(k-n)\mathrm{i}\tau}{a+2k\mathrm{i}\tau} \right| + \operatorname{Re}\left[\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau} - \frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau} \right]$$

$$= \sum_{\ell=1}^{n} \log \left| \frac{a + 2(k-\ell)\mathrm{i}\tau}{a + 2(k-\ell+1)\mathrm{i}\tau} \right| - n\tau \operatorname{Im} \left[\frac{1}{a + 2k\mathrm{i}\tau} + \frac{1}{a + 2(k-n)\mathrm{i}\tau} \right] + x \operatorname{Re} \left[\frac{1}{a + 2k\mathrm{i}\tau} - \frac{1}{a + 2(k-n)\mathrm{i}\tau} \right].$$

We can elegantly add zero to the above expression and obtain

$$\begin{aligned} \log \left| \frac{a + 2(k-n)\mathrm{i}\tau}{a + 2k\mathrm{i}\tau} \right| &+ \operatorname{Re} \left[\frac{x + n\mathrm{i}\tau}{a + 2k\mathrm{i}\tau} - \frac{x + n\mathrm{i}\tau}{\overline{a} + 2(n-k)\mathrm{i}\tau} \right] \\ &= \sum_{\ell=1}^{n} \left(\log \left| \frac{a + 2(k-\ell)\mathrm{i}\tau}{a + 2(k-\ell+1)\mathrm{i}\tau} \right| - 2\tau \operatorname{Im} \frac{1}{a + 2(k-\ell+1)\mathrm{i}\tau} \right) \\ &\quad - \tau \sum_{\ell=1}^{n} \operatorname{Im} \left[\frac{1}{a + 2k\mathrm{i}\tau} - \frac{1}{a + 2(k-\ell+1)\mathrm{i}\tau} \right] \\ &\quad + \tau \sum_{\ell=1}^{n} \operatorname{Im} \left[\frac{1}{a + 2(k-\ell+1)\mathrm{i}\tau} - \frac{1}{a + 2(k-n)\mathrm{i}\tau} \right] \\ &\quad + x \operatorname{Re} \left[\frac{1}{a + 2k\mathrm{i}\tau} - \frac{1}{a + 2(k-n)\mathrm{i}\tau} \right]. \end{aligned}$$

According to the Facts i and ii, it follows that the sum relating b and b' splits into $3n + 1$ parts:

$$\begin{aligned} \frac{1}{n\tau} \sum_{a \in \mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k \in \mathbb{Z}} \Bigg(\log &\left| \frac{a + 2(k-n)\mathrm{i}\tau}{a + 2k\mathrm{i}\tau} \right| + \operatorname{Re} \left[\frac{x + n\mathrm{i}\tau}{a + 2k\mathrm{i}\tau} - \frac{x + n\mathrm{i}\tau}{\overline{a} + 2(n-k)\mathrm{i}\tau} \right] \Bigg) \\ &= \frac{1}{n\tau} \sum_{\ell=1}^{n} \sum_{a \in \mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k \in \mathbb{Z}} \Bigg(\log \left| \frac{a + 2(k-\ell)\mathrm{i}\tau}{a + 2(k-\ell+1)\mathrm{i}\tau} \right| - 2\tau \operatorname{Im} \frac{1}{a + 2(k-\ell+1)\mathrm{i}\tau} \Bigg) \\ &\quad - \frac{1}{n} \sum_{\ell=1}^{n} \sum_{a \in \mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k \in \mathbb{Z}} \operatorname{Im} \left[\frac{1}{a + 2k\mathrm{i}\tau} - \frac{1}{a + 2(k-\ell+1)\mathrm{i}\tau} \right] \\ &\quad + \frac{1}{n} \sum_{\ell=1}^{n} \sum_{a \in \mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k \in \mathbb{Z}} \operatorname{Im} \left[\frac{1}{a + 2(k-\ell+1)\mathrm{i}\tau} - \frac{1}{a + 2(k-n)\mathrm{i}\tau} \right] \\ &\quad + \frac{x}{n\tau} \sum_{a \in \mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k \in \mathbb{Z}} \operatorname{Re} \left[\frac{1}{a + 2k\mathrm{i}\tau} - \frac{1}{a + 2(k-n)\mathrm{i}\tau} \right]. \end{aligned}$$

According to Fact iii, we find that $2n+1$ sums of these $3n+1$ sums vanish. We conclude that

$$
\begin{aligned}
&\frac{1}{n\tau}\sum_{a\in\mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\Bigg(\log\left|\frac{a+2(k-n)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right| \\
&\qquad\qquad\qquad + \operatorname{Re}\left[\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau}-\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau}\right]\Bigg) \\
&\qquad = \frac{1}{n\tau}\sum_{\ell=1}^{n}\sum_{a\in\mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\Bigg(\log\left|\frac{a+2(k-\ell)\mathrm{i}\tau}{a+2(k-\ell+1)\mathrm{i}\tau}\right| \\
&\qquad\qquad\qquad - 2\tau\operatorname{Im}\frac{1}{a+2(k-\ell+1)\mathrm{i}\tau}\Bigg) \\
&= \frac{1}{n\tau}\sum_{\ell=1}^{n}\sum_{a\in\mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\left(\log\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right| - 2\tau\operatorname{Im}\frac{1}{a+2k\mathrm{i}\tau}\right) \\
&\qquad = \frac{1}{\tau}\sum_{a\in\mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\left(\log\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right| - 2\tau\operatorname{Im}\frac{1}{a+2k\mathrm{i}\tau}\right),
\end{aligned}
$$

where we changed the index of summation in the second step. It follows (under the assumptions in item (b)) that $|f(x+n\mathrm{i}\tau)| = |g(x+n\mathrm{i}\tau)|$.

All that is left to do is to show that item (a) implies item (b). Since we can recycle some of our considerations from above this is now relatively simple. Indeed, it follows from Theorem 25 that there must exist $r>0$, $\phi,\psi\in\mathbb{R}$, $a',b,b'\in\mathbb{R}$, and $m\in\mathbb{N}_0$ such that

$$
\begin{aligned}
b-b' = \frac{1}{\tau}\sum_{a\in\mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\Bigg(&\log\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right| \\
&+\operatorname{Re}\left[\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}-\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}\right]\Bigg),
\end{aligned}
$$

for all $x\in\mathbb{R}$, as well as

$$
f(z) = r\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{(a'+\mathrm{i}b)z}z^m\prod_{a\in\mathcal{X}}E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)}\prod_{a\in\mathcal{Y}_\mathrm{u}}\prod_{k\in\mathbb{Z}}E\left(\frac{z}{a+2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},
$$

$$
g(z) = r\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{(a'+\mathrm{i}b')z}z^m\prod_{a\in\mathcal{X}}E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)}\prod_{a\in\mathcal{Y}_\mathrm{u}}\prod_{k\in\mathbb{Z}}E\left(\frac{z}{\overline{a}+2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},
$$

for $z\in\mathbb{C}$. We have just shown that (when we showed that item (b) implies item (a))

$$
\begin{aligned}
&\frac{1}{\tau}\sum_{a\in\mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\left(\log\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right| + \operatorname{Re}\left[\frac{x+\mathrm{i}\tau}{a+2k\mathrm{i}\tau}-\frac{x+\mathrm{i}\tau}{\overline{a}-2(k-1)\mathrm{i}\tau}\right]\right) \\
&\qquad = \frac{1}{\tau}\sum_{a\in\mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\left(\log\left|\frac{a+2(k-1)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right| - 2\tau\operatorname{Im}\frac{1}{a+2k\mathrm{i}\tau}\right).
\end{aligned}
$$

Therefore, item (b) follows. □

We want to finish this section by noting that we can also prove a result for entire functions of finite order $\rho > 1$.

Lemma 28. *Let $\tau \in \mathbb{R}_*$ and let $f, g \in \mathcal{O}(\mathbb{C})$ be non-zero and of finite order. Then, the following are equivalent:*

(a) *For all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$, $|f(x + n\mathrm{i}\tau)| = |g(x + n\mathrm{i}\tau)|$.*

(b) *There exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that*

$$\sum_{\ell=1}^{q} (b_\ell - b'_\ell) \operatorname{Im}\left[(x + n\mathrm{i}\tau)^\ell\right] = \sum_{a \in \mathcal{Y}_\mathrm{u}} m_{\mathcal{Y}}(a) \sum_{k \in \mathbb{Z}} \left(\log\left|\frac{a + 2(k-n)\mathrm{i}\tau}{a + 2k\mathrm{i}\tau}\right| \right.$$
$$\left. + \sum_{\ell=1}^{p} \frac{1}{\ell} \operatorname{Re}\left[\left(\frac{x + n\mathrm{i}\tau}{a + 2k\mathrm{i}\tau}\right)^\ell - \left(\frac{x + n\mathrm{i}\tau}{\overline{a} + 2(n-k)\mathrm{i}\tau}\right)^\ell\right] \right),$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$, as well as

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\, z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)}$$
$$\cdot \prod_{a \in \mathcal{Y}_\mathrm{u}} \prod_{k \in \mathbb{Z}} E\left(\frac{z}{a + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$
$$g(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell)\, z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)}$$
$$\cdot \prod_{a \in \mathcal{Y}_\mathrm{u}} \prod_{k \in \mathbb{Z}} E\left(\frac{z}{\overline{a} + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$.

Proof. We will only show that item (a) implies item (b). The other direction is much simpler and its proof is overwhelmingly similar to what has already been presented in the proof of Theorem 25. Let us assume that item (a) holds. Then, it follows from Theorem 25 that there must exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $q \in \mathbb{N}_0$, $a_\ell, b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \ldots, q\}$, $m \in \mathbb{N}_0$ and $p \in \mathbb{N}_0$ such that

$$f(z) = r\mathrm{e}^{\mathrm{i}\phi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b_\ell)\, z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)}$$
$$\cdot \prod_{a \in \mathcal{Y}_\mathrm{u}} \prod_{k \in \mathbb{Z}} E\left(\frac{z}{a + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$
$$g(z) = r\mathrm{e}^{\mathrm{i}\psi} \exp\left(\sum_{\ell=1}^{q} (a_\ell + \mathrm{i}b'_\ell)\, z^\ell\right) z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_{\mathcal{X}}(a)}$$
$$\cdot \prod_{a \in \mathcal{Y}_\mathrm{u}} \prod_{k \in \mathbb{Z}} E\left(\frac{z}{\overline{a} + 2k\mathrm{i}\tau}; p\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$. We may now compute

$$|f(x+ni\tau)| = r \exp\left(\operatorname{Re}\sum_{\ell=1}^{q}(a_\ell + \mathrm{i}b_\ell)\,(x+n\mathrm{i}\tau)^\ell\right)|x+n\mathrm{i}\tau|^m \cdot \prod_{a\in\mathcal{X}}\left|E\left(\frac{x+n\mathrm{i}\tau}{a};p\right)\right|^{m_{\mathcal{X}}(a)}\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|E\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau};p\right)\right|^{m_{\mathcal{Y}}(a)}.$$

By multiplying over $n-k$ instead of k in the rightmost product of g, one can obtain

$$|g(x+n\mathrm{i}\tau)| = r \exp\left(\operatorname{Re}\sum_{\ell=1}^{q}(a_\ell + \mathrm{i}b'_\ell)\,(x+n\mathrm{i}\tau)^\ell\right)|x+n\mathrm{i}\tau|^m \cdot \prod_{a\in\mathcal{X}}\left|E\left(\frac{x+n\mathrm{i}\tau}{a};p\right)\right|^{m_{\mathcal{X}}(a)}\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|E\left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau};p\right)\right|^{m_{\mathcal{Y}}(a)}.$$

We may now compute the fraction (whenever it is well-defined)

$$\begin{aligned}
&\frac{|f(x+n\mathrm{i}\tau)|}{|g(x+n\mathrm{i}\tau)|}\\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\operatorname{Im}\left[(x+n\mathrm{i}\tau)^\ell\right]\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|\frac{E\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau};p\right)}{E\left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau};p\right)}\right|^{m_{\mathcal{Y}}(a)}\\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\operatorname{Im}\left[(x+n\mathrm{i}\tau)^\ell\right]\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|\frac{1-\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau}}{1-\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau}}\right|^{m_{\mathcal{Y}}(a)}\\
&\qquad\cdot\exp\left(m_{\mathcal{Y}}(a)\sum_{\ell=1}^{p}\frac{1}{\ell}\operatorname{Re}\left[\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^\ell - \left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau}\right)^\ell\right]\right)\\
&= \exp\left(\sum_{\ell=1}^{q}(b'_\ell - b_\ell)\operatorname{Im}\left[(x+n\mathrm{i}\tau)^\ell\right]\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}\left|\frac{a+2(k-n)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right|^{m_{\mathcal{Y}}(a)}\\
&\qquad\cdot\exp\left(m_{\mathcal{Y}}(a)\sum_{\ell=1}^{p}\frac{1}{\ell}\operatorname{Re}\left[\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^\ell - \left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau}\right)^\ell\right]\right).
\end{aligned}$$

The relation between the coefficients b_ℓ and b'_ℓ does follow from considering the logarithm of the above equation. □

Unfortunately, the condition on the coefficients $b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1,\dots,q\}$, i.e. the relation

$$\sum_{\ell=1}^{q}(b_\ell - b'_\ell)\operatorname{Im}\left[(x+n\mathrm{i}\tau)^\ell\right] = \sum_{a\in\mathcal{Y}_{\mathrm{u}}} m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\left(\log\left|\frac{a+2(k-n)\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right|\right.$$

$$+\sum_{\ell=1}^{p}\frac{1}{\ell}\operatorname{Re}\left[\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^{\ell}-\left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau}\right)^{\ell}\right]\Bigg),$$

for all $x\in\mathbb{R}$ and all $n\in\mathbb{Z}$, is not very clear unless $\rho=1$. Specifically, the following problem arises. Suppose that we want to construct two entire functions of finite order such that their magnitudes agree on the infinite set of parallel lines $\mathbb{R}+\mathrm{i}\tau\mathbb{Z}$. We can choose a set of isolated points $\mathcal{Y}_{\mathrm{u}}\subset\mathbb{R}+\mathrm{i}[-\tau,\tau)$ along with some multiplicities $m_{\mathcal{Y}}:\mathcal{Y}_{\mathrm{u}}\to\mathbb{N}$, and an integer $p\in\mathbb{N}$ such that

$$\sum_{a\in\mathcal{Y}_{\mathrm{u}}}\sum_{k\in\mathbb{Z}}\frac{m_{\mathcal{Y}}(a)}{|a+2k\mathrm{i}\tau|^{p+2}}<\infty$$

and define

$$f(z):=\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}E\left(\frac{z}{a+2k\mathrm{i}\tau};p\right)^{m_{\mathcal{Y}}(a)}.$$

We might as well choose another integer $q\in\mathbb{N}$ and some coefficients b_ℓ, for $\ell\in\{1,\dots,q\}$, and define

$$g(z):=\exp\left(\mathrm{i}\sum_{\ell=1}^{q}b_\ell z^\ell\right)\prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}}E\left(\frac{z}{\overline{a}+2k\mathrm{i}\tau};p\right)^{m_{\mathcal{Y}}(a)}.$$

It is however not clear if we can choose the coefficients such that

$$\sum_{\ell=1}^{q}b_\ell\operatorname{Im}\left[(x+n\mathrm{i}\tau)^\ell\right]=\sum_{a\in\mathcal{Y}_{\mathrm{u}}}m_{\mathcal{Y}}(a)\sum_{k\in\mathbb{Z}}\Bigg(\log\left|\frac{a+2k\mathrm{i}\tau}{a+2(k-n)\mathrm{i}\tau}\right|$$
$$+\sum_{\ell=1}^{p}\frac{1}{\ell}\operatorname{Re}\left[\left(\frac{x+n\mathrm{i}\tau}{\overline{a}+2(n-k)\mathrm{i}\tau}\right)^{\ell}-\left(\frac{x+n\mathrm{i}\tau}{a+2k\mathrm{i}\tau}\right)^{\ell}\right]\Bigg),$$

for all $x\in\mathbb{R}$ and all $n\in\mathbb{Z}$. The above is a necessary condition for f and g to agree in magnitude on the lines $\mathbb{R}+\mathrm{i}\tau\mathbb{Z}$.

Of course, if we design the functions f and g such that their order is one, then we can use the Theorem 27 in which the condition on the coefficients $b,b'\in\mathbb{R}$ is actually tractable. Let us consider the following simple example: we let $\tau=1$,

$$\mathcal{Y}_{\mathrm{u}}=\left\{\frac{\mathrm{i}}{2}\right\},$$

$m_{\mathcal{Y}}(\mathrm{i}/2)=1$, as well as $b=\pi/2$ and define

$$f(z)=\mathrm{e}^{\frac{\pi\mathrm{i}}{2}z}\prod_{k\in\mathbb{Z}}E\left(\frac{2z}{(4k+1)\mathrm{i}};1\right).$$

We can now compute

$$b'=\frac{\pi}{2}-\sum_{k\in\mathbb{Z}}\left(\log\left|\frac{4k-3}{4k+1}\right|+\frac{4}{4k+1}\right)=-\frac{\pi}{2}$$

and define

$$g(z) = \mathrm{e}^{-\frac{\pi \mathrm{i}}{2} z} \prod_{k \in \mathbb{Z}} E\left(\frac{2z}{(4k+3)\mathrm{i}}; 1\right).$$

It follows from Theorem 27 that

$$|f(x+n\mathrm{i})| = |g(x+n\mathrm{i})|, \qquad x \in \mathbb{R},\ n \in \mathbb{Z}.$$

Finally, one can actually show that

$$\begin{aligned} f(z) &= \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right), \\ g(z) &= \cosh\left(\frac{\pi z}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi z}{2}\right). \end{aligned}$$

Therefore, we recover the examples from the beginning of this paragraph in this way.

Of course, we can now find two entire functions f and g of general order ρ whose magnitudes agree on the lines $\mathbb{R} + \mathrm{i}\mathbb{Z}$ but which do not agree up to global phase by picking $h \in \mathcal{O}(\mathbb{C})$ of order ρ and considering

$$\begin{aligned} f(z) &= h(z)\left(\cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right)\right), \\ g(z) &= h(z)\left(\cosh\left(\frac{\pi z}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi z}{2}\right)\right). \end{aligned}$$

But if we suppose that we have an arbitrary but fixed function $f \in \mathcal{O}(\mathbb{C})$ of order $\rho > 1$, then it is not clear whether we can harness the lemma above to specifiy all $g \in \mathcal{O}(\mathbb{C})$ such that $|f|$ and $|g|$ agree on $\mathbb{R} + \mathrm{i}\tau\mathbb{Z}$. Simplifying and clarifying the relation on the coefficients $b_\ell, b'_\ell \in \mathbb{R}$, for $\ell \in \{1, \dots, q\}$, might thus be a tempting topic for future research.

5 Phase retrieval from Gabor measurements

In this section, we will show how our work for entire functions relates to Gabor phase retrieval through the intimate relationship of the Gabor transform, the Bargmann transform and the Fock space of second order entire functions. We do so in keeping with the ordering chosen in section 4 of presenting results on intersecting lines in the time-frequency plane in the first subsection and presenting results on parallel lines in the time-frequency plane in the second subsection. In a third subsection, we will discuss a restriction on the root set of the Gabor transform imposed by Jensen's formula from complex analysis.

As stated above, we want to make use of the relation between the Gabor transform, the Bargmann transform and the Fock space. To do this, we start by introducing the relevant notation in line with the excellent exposition in [4]. First, let us consider the Gaussian $\phi(t) := \mathrm{e}^{-\pi t^2}$, for $t \in \mathbb{R}$. We may define the Gabor transform of a signal $f \in L^2(\mathbb{R})$ as

$$\mathcal{G}f(x, \omega) := \int_{\mathbb{R}} f(t)\phi(t-x)\mathrm{e}^{-2\pi \mathrm{i} t\omega}\, \mathrm{d}t, \qquad x, \omega \in \mathbb{R}.$$

Similarly, we might define the Bargmann transform of $f \in L^2(\mathbb{R})$ as

$$\mathcal{B}f(z) := \int_{\mathbb{R}} f(t)\mathrm{e}^{2\pi tz - \pi t^2 - \frac{\pi}{2}z^2}\,\mathrm{d}t, \qquad z \in \mathbb{C}.$$

Thirdly, we introduce the Fock space $\mathcal{F}^2(\mathbb{C})$ as the Hilbert space of entire functions F such that

$$\|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |F(z)|\,\mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z < \infty,$$

equipped with the inner product

$$(F, G)_{\mathcal{F}} := \int_{\mathbb{C}} F(z)\overline{G(z)}\mathrm{e}^{-\pi|z|^2}\,\mathrm{d}z,$$

for $F, G \in \mathcal{F}^2(\mathbb{C})$. One can show that, after rescaling by the factor $2^{1/4}$, the Bargmann transform is a unitary operator from $L^2(\mathbb{R})$ onto $\mathcal{F}^2(\mathbb{C})$ (see Theorem 3.4.3 on p. 56 of [4]). This implies that the Bargmann transform is bijective with inverse

$$\mathcal{B}^{-1}F(t) = \sqrt{2}\int_{\mathbb{C}} F(z)\mathrm{e}^{2\pi t\bar{z} - \pi t^2 - \frac{\pi}{2}\bar{z}^2 - \pi|z|^2}\,\mathrm{d}z, \qquad t \in \mathbb{R},$$

where $F \in \mathcal{F}^2(\mathbb{C})$. Moreover, and this should already seem natural in light of the definitions above, the Gabor and the Bargmann transform are intimately related via the formula (see Proposition 3.4.1 in [4])

$$\mathcal{G}f(x, -\omega) = \mathrm{e}^{\pi\mathrm{i}x\omega}\mathcal{B}f(z)\mathrm{e}^{-\pi|z|^2/2}, \qquad z = x + \mathrm{i}\omega \in \mathbb{C}. \tag{8}$$

Finally, it is useful to know that any function in the Fock space $F \in \mathcal{F}^2(\mathbb{C})$ satisfies the reproducing kernel Hilbert space property

$$|F(z)| \leq \|F\|_{\mathcal{F}}\,\mathrm{e}^{\pi|z|^2/2}, \qquad z \in \mathbb{C}.$$

Most importantly, we conclude from this that any $f \in L^2(\mathbb{R})$ gives rise to an entire function of order two with upper bound of type $\pi/2$ via

$$F(z) = \mathcal{B}f(z) = \mathrm{e}^{-\pi\mathrm{i}x\omega}\mathcal{G}f(x, -\omega)\mathrm{e}^{\pi|z|^2/2}, \qquad z = x + \mathrm{i}\omega \in \mathbb{C},$$

and that vice versa any entire function F of order at most two with sufficiently small type (to be more specific any function in the Fock space $\mathcal{F}^2(\mathbb{C})$) gives rise to an L^2-function via $f = \mathcal{B}^{-1}F$. We can now apply our results for entire functions directly to the Gabor transform.

5.1 Intersecting lines

First, we want to consider what happens in the case in which we have two intersecting lines in the time-frequency plane which meet at an irrational angle. We want to emphasise that the two following results have first been proven by Philippe Jaming in 2014 [10].

Remember from Theorem 19 that two finite order entire functions $F, G \in \mathcal{O}(\mathbb{C})$ satisfying $|F| = |G|$ on $\mathbb{R} \cup \mathrm{e}^{\mathrm{i}\theta}\mathbb{R}$, for $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$, agree up to global phase, i.e. $F \equiv G$. We can easily transfer this uniqueness to the Gabor transform.

Theorem 29 (Cf. Proposition 2)**.** *Let $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$ and $f, g \in L^2(\mathbb{R})$ such that*

$$|\mathcal{G}f(x,0)| = |\mathcal{G}g(x,0)| \ \text{ and } \ |\mathcal{G}f(x\cos\theta, -x\sin\theta)| = |\mathcal{G}g(x\cos\theta, -x\sin\theta)|,$$

for all $x \in \mathbb{R}$. Then, there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha} g$.

Proof. Let us define the entire functions $F = \mathcal{B}f$ along with $G = \mathcal{B}g$. Then, both F and G are in the Fock space and thus of order two. Additionally, by the relation of the Bargmann and the Gabor transform:

$$|\mathcal{B}f(x)| = |\mathcal{G}f(x,0)|\, \mathrm{e}^{\pi x^2/2} = |\mathcal{G}g(x,0)|\, \mathrm{e}^{\pi x^2/2} = |\mathcal{B}g(x)|, \qquad x \in \mathbb{R},$$

as well as

$$\begin{aligned} \left|\mathcal{B}f(\mathrm{e}^{\mathrm{i}\theta}x)\right| &= |\mathcal{G}f(x\cos\theta, -x\sin\theta)|\, \mathrm{e}^{\pi x^2/2} = |\mathcal{G}g(x\cos\theta, -x\sin\theta)|\, \mathrm{e}^{\pi x^2/2} \\ &= \left|\mathcal{B}g(\mathrm{e}^{\mathrm{i}\theta}x)\right|, \end{aligned}$$

for $x \in \mathbb{R}$. It follows from Theorem 19 that there exists an $\alpha \in \mathbb{R}$ such that $F = \mathrm{e}^{\mathrm{i}\alpha} G$. Therefore, we find that

$$f = \mathcal{B}^{-1}F = \mathrm{e}^{\mathrm{i}\alpha}\mathcal{B}^{-1}G = \mathrm{e}^{\mathrm{i}\alpha} g.$$

□

Remembering the remark after Theorem 19, we see that two functions f and g whose Gabor transform magnitudes agree on any two lines in the time-frequency plane intersecting at an irrational angle must agree up to global phase. This is a generalisation of the well-known uniqueness result for phase retrieval from Gabor measurements (which guarantees the unique recovery of signals from measurements taken on the entire time-frequency plane). We can go one step further, however, if we remember that for the uniqueness result for entire functions (Theorem 19) it was sufficient to assume that magnitudes agree on $X \cup \mathrm{e}^{\mathrm{i}\theta} X$, where $X \subset \mathbb{R}$ is any set with an accumulation point.

Theorem 30. *Let $\theta_1, \theta_2 \in \mathbb{R}$ such that $\theta_1 - \theta_2 \in \mathbb{R} \setminus \pi\mathbb{Q}$, let $(x_k)_{k\geq 1} \in \mathbb{R}$ be a sequence with accumulation point and let $f, g \in L^2(\mathbb{R})$. If*

$$|\mathcal{G}f(x_k\cos\theta_j, -x_k\sin\theta_j)| = |\mathcal{G}g(x_k\cos\theta_j, -x_k\sin\theta_j)|, \qquad j = 1,2, \ k \geq 1,$$

then there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha} g$.

Next, we will consider lines in the time-frequency plane which intersect at rational angles, i.e. $\theta \in \pi\mathbb{Q}$. In this case, we will not obtain a uniqueness results as can be seen from Theorem 21 and can instead precisely classify the non-uniqueness in the following way.

Lemma 31. *Let $\theta \in \pi\mathbb{Q} \cap (0, \frac{\pi}{2}]$, let $n \in \mathbb{N}$ be the smallest integer such that $\theta n \in \pi\mathbb{N}$, and let $f, g \in L^2(\mathbb{R})$ be non-zero. We can define*

$$\mathcal{X} = \mathcal{R}_*(\mathcal{B}f) \cap \mathcal{R}(\mathcal{B}g), \qquad \mathcal{Y} = \{a \in \mathcal{R}_*(\mathcal{B}f) \,|\, m_{\mathcal{B}f}(a) > M_{\mathcal{B}g}(a)\},$$

and

$$\mathcal{Y}_{\mathrm{u}} := \mathcal{Y} \cap \left\{ z \in \mathbb{C} \,\middle|\, \exists\, r > 0,\ \phi \in [-\theta, \theta) : z = r\mathrm{e}^{\mathrm{i}\phi} \right\},$$

as well as $m_{\mathcal{X}} : \mathcal{X} \to \mathbb{N}$,

$$m_{\mathcal{X}}(a) = \min\{m_{\mathcal{B}f}(a), m_{\mathcal{B}g}(a)\}, \qquad a \in \mathcal{X},$$

and $m_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{N}$,

$$m_{\mathcal{Y}}(a) = m_{\mathcal{B}f}(a) - M_{\mathcal{B}g}(a), \qquad a \in \mathcal{Y}.$$

Then,

$$|\mathcal{G}f(x,0)| = |\mathcal{G}g(x,0)| \ \text{ and } \ |\mathcal{G}f(x\cos\theta, -x\sin\theta)| = |\mathcal{G}g(x\cos\theta, -x\sin\theta)|,$$

for $x \in \mathbb{R}$, *is equivalent to the existence of* $r > 0$, $\phi, \psi \in \mathbb{R}$, $a_1, a_2, b_1, b_2, b_2' \in \mathbb{R}$, $m \in \mathbb{N}_0$ *and* $p \in \{0,1,2\}$ *such that* $b_2' = b_2$ *if* $n \neq 2$ *and*

$$\begin{aligned}
\mathcal{B}f(z) &= r\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{(a_1+\mathrm{i}b_1)z+(a_2+\mathrm{i}b_2)z^2} z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};p\right)^{m_{\mathcal{X}}(a)} \\
&\qquad\qquad \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{a};p\right)^{m_{\mathcal{Y}}(a)}, \\
\mathcal{B}g(z) &= r\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{(a_1+\mathrm{i}b_1)z+(a_2+\mathrm{i}b_2')z^2} z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};p\right)^{m_{\mathcal{X}}(a)} \\
&\qquad\qquad \cdot \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{\overline{a}};p\right)^{m_{\mathcal{Y}}(a)},
\end{aligned}$$

for $z \in \mathbb{C}$.

We could alternatively state this result in terms of the Gabor transforms of f and g via the equation (8) but decided against it to keep the mathematical formulae as short as possible. In any case, it is clear from the result above that signals are not uniquely determined from Gabor magnitude measurements on two lines which intersect at a rational angle. More precisely, the result above allows for an exact description of all $g \in L^2(\mathbb{R})$ for which the Gabor transform of g agrees with the Gabor transform of a fixed $f \in L^2(\mathbb{R})$ on two such lines. Moreoever, it simplifies the construction of signals whose Gabor transform magnidues do agree on two intersecting lines massively. We might namely take any two sets of isolated points $\mathcal{X} \subset \mathbb{C}_*$ and $\mathcal{Y}_{\mathrm{u}} \subset \left\{ z \in \mathbb{C} \,\middle|\, \exists\, r > 0,\ \phi \in [-\theta, \theta) : z = r\mathrm{e}^{\mathrm{i}\phi} \right\}$ together with multiplicity functions $m_{\mathcal{X}} : \mathcal{X} \to \mathbb{N}$ and $m_{\mathcal{Y}} : \mathcal{Y}_{\mathrm{u}} \to \mathbb{N}$ such that

$$\sum_{a\in\mathcal{X}} \frac{m_{\mathcal{X}}(a)}{|a|^A} + \sum_{a\in\mathcal{Y}_{\mathrm{u}}} \frac{m_{\mathcal{Y}}(a)}{|a|^A} < \infty,$$

for some $A \in (0,2)$. It follows from Lemma 11 that the functions

$$F(z) = r\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{(a_1+\mathrm{i}b_1)z} z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};p\right)^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}} \prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{a};p\right)^{m_{\mathcal{Y}}(a)},$$

$$G(z) = r\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{(a_1+\mathrm{i}b_1)z}z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};p\right)^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_\mathrm{u}}\prod_{k=1}^{n} E\left(\mathrm{e}^{2k\mathrm{i}\theta}\frac{z}{\overline{a}};p\right)^{m_{\mathcal{Y}}(a)},$$

for $r > 0$, $\phi, \psi \in \mathbb{R}$, $a_1, b_1 \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $p > A - 1$, are of order at most A and must thus lie in the Fock space. Therefore, $f = \mathcal{B}^{-1}F$ and $g = \mathcal{B}^{-1}G$ are L^2-functions and by the prior lemma

$$|\mathcal{G}f(x,0)| = |\mathcal{G}g(x,0)| \text{ and } |\mathcal{G}f(x\cos\theta, -x\sin\theta)| = |\mathcal{G}g(x\cos\theta, -x\sin\theta)|,$$

for $x \in \mathbb{R}$. If $\mathcal{Y}_\mathrm{u}$ is chosen such that $\mathcal{Y} = \mathrm{e}^{2\mathrm{i}\theta(\mathbb{Z}/n\mathbb{Z})}\mathcal{Y}_\mathrm{u}$ satisfies $\overline{\mathcal{Y}} \neq \mathcal{Y}$, then f and g do not agree up to global phase. An example of such a choice is given by $\theta = \frac{\pi}{2}$, $\mathcal{X} = \emptyset$, $\mathcal{Y}_\mathrm{u} = \{\sqrt{2}\mathrm{e}^{\pi\mathrm{i}/4}\}$ and $m_{\mathcal{X}} = m_{\mathcal{Y}} = 1$. Taking $r = \phi = \psi = a_1 = b_1 = m = p = 0$ yields

$$F(z) = \left(1 - \frac{z}{1+\mathrm{i}}\right)\left(1 + \frac{z}{1+\mathrm{i}}\right) = 1 - \frac{z^2}{(1+\mathrm{i})^2}, \qquad G(z) = 1 - \frac{z^2}{(1-\mathrm{i})^2},$$

for $z \in \mathbb{C}$, which coincides with the examples we had given at the beginning of the paragraph "Rational angles" in Section 4. We can now compute the corresponding L^2-functions by using the inverse Bargmann transform. We obtain

$$f(t) = \mathcal{B}^{-1}F(t) = \frac{2\pi + (4\pi t^2 - 1)\mathrm{i}}{\sqrt{2}\pi}\mathrm{e}^{-\pi t^2},$$
$$g(t) = \mathcal{B}^{-1}G(t) = \frac{2\pi - (4\pi t^2 - 1)\mathrm{i}}{\sqrt{2}\pi}\mathrm{e}^{-\pi t^2},$$

for $t \in \mathbb{R}$.

5.2 Parallel lines

Let us now turn to the analysis of parallel lines in the time-frequency plane. The following result follows immediately from Theorem 25 and shows that Gabor magnitude measurements on two parallel lines are, in general, not sufficient to recover a signal up to global phase.

Lemma 32. *Let $\tau \in \mathbb{R}_*$ and let $f, g \in L^2(\mathbb{R})$ be non-zero. We can define*

$$\mathcal{X} = \mathcal{R}_*(\mathcal{B}f) \cap \mathcal{R}(\mathcal{B}g), \qquad \mathcal{Y} = \{a \in \mathcal{R}_*(\mathcal{B}f) \,|\, m_{\mathcal{B}f}(a) > M_{\mathcal{B}g}(a)\},$$

and

$$\mathcal{Y}_\mathrm{u} := \mathcal{Y} \cap (\mathbb{R} + \mathrm{i}\,[-\tau, \tau)),$$

as well as $m_{\mathcal{X}} : \mathcal{X} \to \mathbb{N}$,

$$m_{\mathcal{X}}(a) = \min\{m_{\mathcal{B}f}(a), m_{\mathcal{B}g}(a)\}, \qquad a \in \mathcal{X},$$

and $m_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{N}$,

$$m_{\mathcal{Y}}(a) = m_{\mathcal{B}f}(a) - M_{\mathcal{B}g}(a), \qquad a \in \mathcal{Y}.$$

Then,

$$|\mathcal{G}f(x,0)| = |\mathcal{G}g(x,0)| \text{ and } |\mathcal{G}f(x,-\tau)| = |\mathcal{G}g(x,-\tau)|,$$

for $x \in \mathbb{R}$, is equivalent to the existence of $r > 0$, $\phi, \psi \in \mathbb{R}$, $a_1, a_2, b_1, b_1', b_2, b_2' \in \mathbb{R}$, $m \in \mathbb{N}_0$, and $p \in \{0,1,2\}$ such that

$$\begin{aligned} 2x(b_2 - b_2') + (b_1 - b_1') = \frac{1}{\tau} \sum_{a \in \mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k \in \mathbb{Z}} \Bigg(\log \left| \frac{a + 2(k-1)\mathrm{i}\tau}{a + 2k\mathrm{i}\tau} \right| \\ + \sum_{\ell=1}^{p} \frac{1}{\ell} \operatorname{Re} \left[\left(\frac{x + \mathrm{i}\tau}{a + 2k\mathrm{i}\tau} \right)^{\ell} - \left(\frac{x + \mathrm{i}\tau}{\overline{a} - 2(k-1)\mathrm{i}\tau} \right)^{\ell} \right] \Bigg), \end{aligned}$$

for all $x \in \mathbb{R}$, as well as

$$\begin{aligned} \mathcal{B}f(z) &= r\mathrm{e}^{\mathrm{i}\phi} \mathrm{e}^{(a_1 + \mathrm{i}b_1)z + (a_2 + \mathrm{i}b_2)z^2} z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_\mathcal{X}(a)} \\ &\quad \cdot \prod_{a \in \mathcal{Y}_\mathrm{u}} \prod_{k \in \mathbb{Z}} E\left(\frac{z}{a + 2k\mathrm{i}\tau}; p\right)^{m_\mathcal{Y}(a)}, \\ \mathcal{B}g(z) &= r\mathrm{e}^{\mathrm{i}\psi} \mathrm{e}^{(a_1 + \mathrm{i}b_1')z + (a_2 + \mathrm{i}b_2')z^2} z^m \prod_{a \in \mathcal{X}} E\left(\frac{z}{a}; p\right)^{m_\mathcal{X}(a)} \\ &\quad \cdot \prod_{a \in \mathcal{Y}_\mathrm{u}} \prod_{k \in \mathbb{Z}} E\left(\frac{z}{\overline{a} + 2k\mathrm{i}\tau}; p\right)^{m_\mathcal{Y}(a)}, \end{aligned}$$

for $z \in \mathbb{C}$.

As before in the case of intersecting lines meeting at a rational angle, this result allows for a full classification of all $g \in L^2(\mathbb{R})$ whose Gabor transform magnitudes agree with those of any fixed signal $f \in L^2(\mathbb{R})$. In addition, we might use the lemma to find signals whose Gabor magnitudes agree on parallel lines.

Finally, we can consider what we believe to be the most interesting non-uniqueness result for the Gabor transform in this section: the result on infinitely many parallel lines. The following is an immediate corollary of Theorem 27.

Theorem 33. *Let $\tau \in \mathbb{R}_*$ and consider two sets of isolated points $\mathcal{X} \subset \mathbb{C}_*$ and $\mathcal{Y}_\mathrm{u} \subset (\mathbb{R} + \mathrm{i}[-\tau, \tau))$ as well as multiplicity functions $m_\mathcal{X} : \mathcal{X} \to \mathbb{N}$ and $m_\mathcal{X} : \mathcal{Y}_\mathrm{u} \to \mathbb{N}$ such that*

$$\sum_{a \in \mathcal{X}} \frac{m_\mathcal{X}(a)}{|a|^{1+\epsilon}} + \sum_{a \in \mathcal{Y}_\mathrm{u}} \sum_{k \in \mathbb{Z}} \frac{m_\mathcal{Y}(a)}{|a + 2k\mathrm{i}\tau|^{1+\epsilon}} < \infty,$$

for all $\epsilon > 0$. Additionally, let $r > 0$, $\phi, \psi \in \mathbb{R}$, $a', b, b' \in \mathbb{R}$ and $m \in \mathbb{N}_0$ such that

$$b = b' + \frac{1}{\tau} \sum_{a \in \mathcal{Y}_\mathrm{u}} m_\mathcal{Y}(a) \sum_{k \in \mathbb{Z}} \left(\log \left| \frac{a + 2(k-1)\mathrm{i}\tau}{a + 2k\mathrm{i}\tau} \right| - 2\tau \operatorname{Im} \frac{1}{a + 2k\mathrm{i}\tau} \right),$$

as well as $f,g \in L^2(\mathbb{R})$ defined via

$$\mathcal{B}f(z) = r\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{(a'+\mathrm{i}b)z}z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}} E\left(\frac{z}{a+2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},$$

$$\mathcal{B}g(z) = r\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{(a'+\mathrm{i}b')z}z^m \prod_{a\in\mathcal{X}} E\left(\frac{z}{a};1\right)^{m_{\mathcal{X}}(a)} \prod_{a\in\mathcal{Y}_{\mathrm{u}}}\prod_{k\in\mathbb{Z}} E\left(\frac{z}{\overline{a}+2k\mathrm{i}\tau};1\right)^{m_{\mathcal{Y}}(a)},$$

for $z \in \mathbb{C}$. Then, it holds that

$$|\mathcal{G}f(x,n\tau)| = |\mathcal{G}g(x,n\tau)|, \qquad x\in\mathbb{R},\ n\in\mathbb{Z}.$$

Proof. By assumption, it holds that

$$\sum_{a\in\mathcal{X}} \frac{m_{\mathcal{X}}(a)}{|a|^{1+\epsilon}} + \sum_{a\in\mathcal{Y}_{\mathrm{u}}}\sum_{k\in\mathbb{Z}} \frac{m_{\mathcal{Y}}(a)}{|a+2k\mathrm{i}\tau|^{1+\epsilon}} < \infty,$$

for all $\epsilon > 0$. Therefore, the functions $\mathcal{B}f$ and $\mathcal{B}g$ are entire functions of order one according to Lemma 11. It follows that $\mathcal{B}f$ and $\mathcal{B}g$ are indeed in the Fock space and must thus be well-defined Bargmann transforms of two signals f and g respectively. The theorem follows by applying Theorem 27 and remembering the relation in equation (8). □

Of course, we could also have stated the above theorem in terms of an equivalence just like we did for the cases of two lines intersecting at a rational angle and two parallel lines. We chose this formulation, however, since an important interest of ours is to construct $f,g \in L^2(\mathbb{R})$ which do not agree up to global phase but whose Gabor transform magnitudes agree on different sets in the time-frequency plane. Following this interest, we may consider $\tau = 1$, $\mathcal{X} = \emptyset$, $\mathcal{Y}_{\mathrm{u}} = \{\mathrm{i}/2\}$ and $m_{\mathcal{X}} = m_{\mathcal{Y}} = 1$. With these choices, we note that

$$\sum_{k\in\mathbb{Z}} \frac{1}{|\mathrm{i}/2+2k\mathrm{i}\tau|^{1+\epsilon}} < \infty,$$

for all $\epsilon > 0$, such that the assumptions of the above theorem are satisfied. We may additionally consider $r = 1$, $\phi = \psi = a' = m = 0$ as well as $b = \pi/2$ and

$$\begin{aligned} b' &= \frac{\pi}{2} - \sum_{k\in\mathbb{Z}} \left(\log\left| \frac{\mathrm{i}/2 + 2(k-1)\mathrm{i}}{\mathrm{i}/2+2k\mathrm{i}} \right| - 2\,\mathrm{Im}\,\frac{1}{\mathrm{i}/2+2k\mathrm{i}} \right) \\ &= \frac{\pi}{2} - \sum_{k\in\mathbb{Z}} \left(\log\left| \frac{4k-3}{4k+1} \right| + \frac{4}{4k+1} \right) = -\frac{\pi}{2} \end{aligned}$$

to obtain the same examples as considered in Section 4.2,

$$\begin{aligned} F(z) &= \cosh\left(\frac{\pi z}{2}\right) + \mathrm{i}\sinh\left(\frac{\pi z}{2}\right), \\ G(z) &= \cosh\left(\frac{\pi z}{2}\right) - \mathrm{i}\sinh\left(\frac{\pi z}{2}\right), \end{aligned}$$

for $z \in \mathbb{C}$. These entire functions correspond (after taking the inverse Bargmann transform) to the signals

$$f(t) = \sqrt{2}\mathrm{e}^{-\pi\left(\frac{1}{8}+t^2\right)} (\cosh \pi t + \mathrm{i} \sinh \pi t),$$
$$g(t) = \sqrt{2}\mathrm{e}^{-\pi\left(\frac{1}{8}+t^2\right)} (\cosh \pi t - \mathrm{i} \sinh \pi t),$$

for $t \in \mathbb{R}$, which originally inspired our work in [1]. We note that the signals f and g do not agree up to global phase but that their Gabor transform magnitudes agree on $\mathbb{R} \times \mathbb{Z}$.

5.3 On the number of roots in disks around the origin of functions in the Fock space

In this final subsection, we want to discuss an important detail in the construction of functions in the Fock space through the Hadamard factorisation theorem. To make this discussion as palpable as possible, we want to start by presenting an interesting example. Suppose we are given $a, b > 0$ and are asked to construct a non-zero function $f \in L^2(\mathbb{R})$ such that $\mathcal{G}f = 0$ on $a\mathbb{Z} \times b\mathbb{Z}$. A natural way of approaching this problem is to consider that

$$\begin{aligned}
\sum_{j,k \geq 2} \frac{1}{(a^2 j^2 + b^2 k^2)^{1+\epsilon}} &= \sum_{j,k \geq 2} \int_{j-1}^{j} \int_{k-1}^{k} \frac{1}{(a^2 j^2 + b^2 k^2)^{1+\epsilon}} \,\mathrm{d}x \,\mathrm{d}y \\
&\leq \sum_{j,k \geq 2} \int_{j-1}^{j} \int_{k-1}^{k} \frac{1}{(a^2 x^2 + b^2 y^2)^{1+\epsilon}} \,\mathrm{d}x \,\mathrm{d}y \\
&= \int_{1}^{\infty} \int_{1}^{\infty} \frac{1}{(a^2 x^2 + b^2 y^2)^{1+\epsilon}} \,\mathrm{d}x \,\mathrm{d}y < \infty,
\end{aligned}$$

for $\epsilon > 0$. Therefore, it follows that

$$\sum_{(j,k) \in \mathbb{Z}_*^2} \frac{1}{(a^2 j^2 + b^2 k^2)^{1+\epsilon}} < \infty,$$

for $\epsilon > 0$. According to Lemma 11, we have that the entire function

$$F(z) = z \prod_{(j,k) \in \mathbb{Z}_*^2} E\left(\frac{z}{aj + \mathrm{i}bk}; 2\right), \qquad z \in \mathbb{C},$$

is of order two. It follows that if we can give an upper bound on the type of F, i.e. if we could say that

$$\sigma(F) := \limsup_{r \to \infty} \frac{\log \sup_{|z| \leq r} |F(z)|}{r^2} < \frac{\pi}{2},$$

then $F \in \mathcal{F}^2(\mathbb{C})$ and thus we could write $f = \mathcal{B}^{-1} F \in L^2(\mathbb{R})$. So, the important question is: can we upper bound the type of F?

Should we not be able to do this, a natural follow up question would be, whether we can upper bound the number of roots a general function in the Fock space can have. If we consider disks centered around the origin, we may use the well-known Jensen formula from complex analysis to investigate this question.

Lemma 34. *Let $f \in L^2(\mathbb{R})$ be non-zero, let $n : (0, \infty) \to \mathbb{N}_0$ be such that for all $r > 0$, $n(r)$ denotes the number of roots of $\mathcal{B}f$ (counted with multiplicity) in the disc of radius r centered at the origin, let $k \in \mathbb{N}_0$ denote the multiplicity of the root at zero of $\mathcal{B}f$ (with the convention that $k = 0$ should $\mathcal{B}f$ not have a root at the origin), and let $c \in \mathbb{R}$ be such that*

$$c = \begin{cases} \log\left(\frac{1}{\sqrt{2}} \|f\|_2^2\right) - 2\log|\mathcal{B}f(0)| & \text{if } k = 0, \\ \log\left(\frac{1}{\sqrt{2}\mathrm{e}^k} \|f\|_2^2\right) - 2\log\left|\lim_{z\to 0} \frac{\mathcal{B}f(z)}{z^k}\right| + k & \text{if } k > 0. \end{cases}$$

Then,

$$n(r) \leq \pi \mathrm{e} r^2 - 2k\log r + c, \qquad r > 0.$$

Proof. Let $R \geq r > 0$ be such that $R^2 = \mathrm{e}r^2$, let

$$F(z) := \frac{\mathcal{B}f(z)}{z^k}, \qquad z \in \mathbb{C},$$

and denote by $n_F(t)$ the number of roots of F (counted with multiplicity) in the disc of radius $t > 0$ centered at the origin. Note that F is an entire function with removable singularity at $z = 0$. Therefore, we have

$$\int_0^R \frac{n_F(t)}{t}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \log\left|F(R\mathrm{e}^{\mathrm{i}\theta})\right|\,\mathrm{d}\theta - \log\left|\lim_{z\to 0} F(z)\right|,$$

according to Jensen's formula. We can use the reproducing property of the Fock space along with the fact that $2^{1/4}\mathcal{B}$ is an isometry from L^2 into the Fock space to see that

$$\left|F(R\mathrm{e}^{\mathrm{i}\theta})\right| = \frac{\left|\mathcal{B}f(R\mathrm{e}^{\mathrm{i}\theta})\right|}{R^k} \leq \frac{\|\mathcal{B}f\|_{\mathcal{F}}}{R^k}\mathrm{e}^{\frac{\pi}{2}R^2} = 2^{-1/4}\mathrm{e}^{-k/2}\frac{\|f\|_2}{r^k}\mathrm{e}^{\frac{\pi\mathrm{e}}{2}r^2}.$$

In addition, we can use that $n_F(t)$ is increasing in t to see that

$$\int_r^R \frac{n_F(t)}{t}\,\mathrm{d}t \geq n_F(r)\int_r^R \frac{1}{t}\,\mathrm{d}t = \frac{n_F(r)}{2}$$

and thus

$$n_F(r) \leq 2\int_r^R \frac{n_F(t)}{t}\,\mathrm{d}t \leq 2\int_0^R \frac{n_F(t)}{t}\,\mathrm{d}t.$$

We conclude that

$$n_F(r) \leq \pi\mathrm{e}r^2 - 2k\log r + \log\left(\frac{1}{\sqrt{2}\mathrm{e}^k}\|f\|_2^2\right) - 2\log\left|\lim_{z\to 0} F(z)\right|.$$

□

Let us try to apply this knowledge to our example. We can estimate

$$n(r) = |(a\mathbb{Z} + \mathrm{i}b\mathbb{Z}) \cap \{z \in \mathbb{C} \,|\, |z| \leq r\}| = \frac{\pi r^2}{ab} + \mathcal{O}(r^{2/3}), \qquad r \to \infty,$$

where we used a result of Levitan [11]. Comparing this calculation to the result in our lemma, we find that $ab \geq \mathrm{e}^{-1}$ is a necessary condition for F being in the Fock space. In words, if we try to design a signal $f \in L^2(\mathbb{R})$ such that the Gabor transform vanishes on a grid that is sufficiently fine, then we will fail.

Corollary 35 (Sampling for the Gabor transform)**.** *Let $a, b > 0$ such that $ab < \mathrm{e}^{-1}$ and let $f, g \in L^2(\mathbb{R})$. If*

$$\mathcal{G}f = \mathcal{G}g \qquad \text{on } a\mathbb{Z} \times b\mathbb{Z},$$

then $f = g$.

Proof. Suppose that $\mathcal{G}f = \mathcal{G}g$ on $a\mathbb{Z} \times b\mathbb{Z}$. Then, $h = f - g \in L^2(\mathbb{R})$ satisfies that $\mathcal{G}h = 0$ on $a\mathbb{Z} \times b\mathbb{Z}$ and thus $H = \mathcal{B}h \in \mathcal{F}^2(\mathbb{C})$ satisfies that $H = 0$ on $a\mathbb{Z} \times b\mathbb{Z}$. If assume by contradiction that h is non-zero, then it follows from the prior lemma and the considerations before this corollary that $ab \geq \mathrm{e}^{-1}$. The latter contradicts our assumptions and therefore, the corollary is proven. □

We should remark that the assumption $ab < \mathrm{e}^{-1}$ is far from optimal. In fact, the corollary continues to hold if we assume $ab \leq 1$ according to the celebrated results by Lyubarskiĭ [12] and Seip [14] from 1992.

We can also directly estimate the type of the F constructed in the beginning of this subsection. Indeed, F is a special case of the Weierstrass σ-function and one may use its quasi-periodicity to show that its type satisfies [6, 9]

$$\sigma(F) \leq \frac{\pi}{2ab}$$

and therefore $F \in \mathcal{F}^2(\mathbb{C})$ if $ab > 1$. It does thus follow that we can construct a non-zero function $f \in L^2(\mathbb{R})$ such that $\mathcal{G}f = 0$ on $a\mathbb{Z} \times b\mathbb{Z}$ if $ab > 1$.

Finally, we still believe that Lemma 34 is interesting because it can be used when constructing signals $f \in L^2(\mathbb{R})$ through their Bargmann transform. In particular, Lemma 34 can be used to get a rough idea about how dense our choice of the roots of f may be.

6 Universal counterexamples in Gabor phase retrieval

In the prior sections (cf. also [1]), we have shown how, given infinitely many equidistant parallel lines Ω in the time-frequency plane, one may construct functions $f, g \in L^2(\mathbb{R})$ (depending on Ω) such that f and g do not agree up to global phase but

$$|\mathcal{G}f| = |\mathcal{G}g| \ \text{ on } \Omega.$$

This raises the question whether f (and g) can be chosen independently on Ω as well. To partially answer this question, we want to consider the simplified setup in which $\Omega = \mathbb{R} \times a\mathbb{Z}$, for some $a > 0$.

Let us introduce the equivalence relation $\sim$ on $L^2(\mathbb{R})$ via

$$f \sim g :\Leftrightarrow \exists\, \alpha \in \mathbb{R} : f = \mathrm{e}^{\mathrm{i}\alpha} g,$$

for $f, g \in L^2(\mathbb{R})$. We are interested in the following question.

Question 36 (Existence of universal counterexamples — I)**.** *Does there exist a function $f \in L^2(\mathbb{R})$ such that for all $a > 0$, there exists a $g = g_a \in L^2(\mathbb{R})$ such that $f \not\sim g$ yet*

$$|\mathcal{G} f| = |\mathcal{G} g| \qquad \text{on } \mathbb{R} \times a\mathbb{Z}?$$

In the following, we will show that the answer to this question is negative because the condition $a > 0$ allows for uncountably many different choices of a.

Lemma 37. *For all $f \in L^2(\mathbb{R})$, there exists an $a > 0$ such that for all $g \in L^2(\mathbb{R})$, it holds that*

$$(|\mathcal{G} f| = |\mathcal{G} g| \;\text{ on } \mathbb{R} \times a\mathbb{Z}) \implies f \sim g.$$

Proof. Suppose, by contradiction, that there exists an $f \in L^2(\mathbb{R})$ such that for all $a > 0$, there exists a $g_a \in L^2(\mathbb{R})$ satisfying $f \not\sim g_a$ yet

$$|\mathcal{G} f| = |\mathcal{G} g_a| \quad \text{on } \mathbb{R} \times a\mathbb{Z}.$$

Let $a > 0$ be arbitrary but fixed for now and consider that our assumption implies

$$|\mathcal{G} f| = |\mathcal{G} g_a| \quad \text{on } \mathbb{R} \times \{0, -a\},$$

and thereby (remember equation (8))

$$|\mathcal{B} f| = |\mathcal{B} g_a| \qquad \text{on } \mathbb{R} + \{0, \mathrm{i}a\}.$$

At the same time, $f \not\sim g_a$ implies that $\mathcal{B} f \not\sim \mathcal{B} g_a$ which together with the equation above tells us that $\mathcal{B} f$ is non-trivial.

Next, let us remember the notations

$$\begin{aligned} \mathcal{X} = \mathcal{X}_a &= \mathcal{R}_*(\mathcal{B} f) \cap \mathcal{R}(\mathcal{B} g_a), \\ \mathcal{Y} = \mathcal{Y}_a &= \left\{ w \in \mathcal{R}_*(\mathcal{B} f) \,|\, m_{\mathcal{B} f}(w) > M_{\mathcal{B} g_a}(w) \right\}, \end{aligned}$$

as well as $m_{\mathcal{X}} : \mathcal{X} \to \mathbb{N}$,

$$m_{\mathcal{X}}(w) = \min\{m_{\mathcal{B} f}(w), m_{\mathcal{B} g_a}(w)\}, \qquad w \in \mathcal{X},$$

and $m_{\mathcal{Y}} : \mathcal{Y} \to \mathbb{N}$,

$$m_{\mathcal{Y}}(w) = m_{\mathcal{B} f}(w) - M_{\mathcal{B} g_a}(w), \qquad w \in \mathcal{Y}.$$

$M_{\mathcal{X}}$ and $M_{\mathcal{Y}}$ will continue to denote the extensions of $m_{\mathcal{X}}$ and $m_{\mathcal{Y}}$ to $\mathbb{C}$ by zero, respectively. If we suppose, by contradiction, that $\mathcal{Y}_a = \emptyset$, then it follows from $|\mathcal{B}f| = |\mathcal{B}g_a|$ and the Mc Donald decomposition that there exist $r > 0$, $\phi, \psi \in \mathbb{R}$, $a_1, b_1, b_1', a_2, b_2, b_2' \in \mathbb{R}$ and $m \in \mathbb{N}_0$ such that

$$\mathcal{B}f(z) = r\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{(a_1+\mathrm{i}b_1)z+(a_2+\mathrm{i}b_2)z^2} z^m \prod_{w\in\mathcal{X}} E\left(\frac{z}{w};2\right)^{m_{\mathcal{X}}(w)},$$

$$\mathcal{B}g_a(z) = r\mathrm{e}^{\mathrm{i}\psi}\mathrm{e}^{(a_1+\mathrm{i}b_1')z+(a_2+\mathrm{i}b_2')z^2} z^m \prod_{w\in\mathcal{X}} E\left(\frac{z}{w};2\right)^{m_{\mathcal{X}}(w)},$$

for $z \in \mathbb{C}$. By

$$1 = \frac{|\mathcal{B}f(x+\mathrm{i}a)|}{|\mathcal{B}g_a(x+\mathrm{i}a)|} = \left|\mathrm{e}^{\mathrm{i}(b_1-b_1')(x+\mathrm{i}a)+\mathrm{i}(b_2-b_2')(x+\mathrm{i}a)^2}\right| = \mathrm{e}^{(b_1'-b_1)a+2(b_2'-b_2)xa},$$

for $x \in \mathbb{R}$, we conclude that $b_1 = b_1'$ and $b_2 = b_2'$ and thus we find that $\mathcal{B}f \sim \mathcal{B}g_a$ which is a contradiction. Hence, it holds that $\mathcal{Y}_a \neq \emptyset$. We can thus find $y_a \in \mathcal{Y}_a \subset \mathcal{R}_*(\mathcal{B}f) \subset \mathbb{C}_*$. By the Lemmata 16 and 24, we can show that (see Subsection 4.2)

$$M_{\mathcal{Y}}(z + 2a\mathrm{i}k) = M_{\mathcal{Y}}(z), \qquad z \in \mathbb{C}.$$

Therefore, it follows that $y_a + 2a\mathrm{i}\mathbb{Z} \subset \mathcal{Y}_a \subset \mathcal{R}_*(\mathcal{B}f)$.

As $a > 0$ was arbitrary, we have shown that for all $a > 0$, there exists a $y_a \in \mathbb{C}_*$ such that $y_a + 2a\mathrm{i}\mathbb{Z} \subset \mathcal{R}_*(\mathcal{B}f)$. Therefore, $\mathcal{R}_*(\mathcal{B}f)$ is uncountable and must thus contain an accumulation point. As non-zero entire functions have the property that their roots do not have any accumulation point (except infinity), we have reached a contradiction. ☐

Interestingly, the proof of the prior lemma relies on the uncountability of $(0, \infty)$ rather than on the fact that there are arbitrarily small numbers in $(0, \infty)$. In particular, we can also prove the following result.

Lemma 38. *Let $T \subset (0, \infty)$ be uncountable. For all $f \in L^2(\mathbb{R})$, there exists an $a \in T$ such that for all $g \in L^2(\mathbb{R})$, it holds that*

$$(|\mathcal{G}f| = |\mathcal{G}g| \ \text{ on } \mathbb{R} \times a\mathbb{Z}) \implies f \sim g.$$

This raises the question whether the same remains true if we replace T by a countable set containing arbitrarily small numbers such as $T = \mathbb{N}^{-1}$, for instance.

Question 39 (Existence of universal counterexamples — II)**.** *Does there exist a function $f \in L^2(\mathbb{R})$ such that for all $n \in \mathbb{N}$, there exists a function $g_n \in L^2(\mathbb{R})$ such that $f \not\sim g_n$ yet*

$$|\mathcal{G}f| = |\mathcal{G}g_n| \qquad \text{on } \mathbb{R} \times \frac{1}{n}\mathbb{Z}?$$

To answer Question 39, we consider a direct construction that is based on the work which we have presented thus far in this paper. In particular, we introduce the function

$$F(z) := \prod_{n\in\mathbb{N}} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right) \exp\left(\frac{z}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right),$$

for $z \in \mathbb{C}$. Through the theory surrounding the growth of entire functions and the Hadamard factorisation theorem, we will show that F is a well-defined entire function of first order.

To be precise, we will apply Lemma 11. We will therefore consider the roots of F which are given by

$$a_{n,k} := \mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}, \qquad n \in \mathbb{N},\ k \in \mathbb{Z},$$

and consider the convergence of sums of the form

$$\begin{aligned}\sum_{n,k} \frac{1}{|a_{n,k}|^\lambda} &= \sum_{n,k} \frac{1}{\left(\mathrm{e}^{2n} + \left(\frac{2k}{n} + \frac{1}{2n}\right)^2\right)^{\frac{\lambda}{2}}}\\ &= 2^\lambda \sum_{n\in\mathbb{N}} n^\lambda \sum_{k\in\mathbb{Z}} \frac{1}{(4n^2\mathrm{e}^{2n} + (4k+1)^2)^{\frac{\lambda}{2}}}.\end{aligned}$$

Let us distinguish between two different cases: first, if $k \in \mathbb{N}_0$, then $(4k+1)^2 \geq 16k^2$ and thus

$$\begin{aligned}\frac{2^\lambda n^\lambda}{(4n^2\mathrm{e}^{2n} + (4k+1)^2)^{\frac{\lambda}{2}}} &\leq \frac{2^\lambda n^\lambda}{(4n^2\mathrm{e}^{2n} + 16k^2)^{\frac{\lambda}{2}}} = \frac{n^\lambda}{(n^2\mathrm{e}^{2n} + 4k^2)^{\frac{\lambda}{2}}}\\ &= \frac{1}{\left(\mathrm{e}^{2n} + 4\frac{k^2}{n^2}\right)^{\frac{\lambda}{2}}},\end{aligned}$$

for $n \in \mathbb{N}$. Secondly, if $k \in \mathbb{N}$, then $(4(k-1)+3)^2 \geq 16(k-1)^2$ such that

$$\begin{aligned}\frac{2^\lambda n^\lambda}{(4n^2\mathrm{e}^{2n} + (-4k+1)^2)^{\frac{\lambda}{2}}} &= \frac{2^\lambda n^\lambda}{(4n^2\mathrm{e}^{2n} + (4(k-1)+3)^2)^{\frac{\lambda}{2}}}\\ &\leq \frac{2^\lambda n^\lambda}{(4n^2\mathrm{e}^{2n} + 16(k-1)^2)^{\frac{\lambda}{2}}}\\ &= \frac{1}{\left(\mathrm{e}^{2n} + 4\frac{(k-1)^2}{n^2}\right)^{\frac{\lambda}{2}}},\end{aligned}$$

for $n \in \mathbb{N}$. Therefore, we have

$$\sum_{n,k} \frac{1}{|a_{n,k}|^\lambda} = 2^\lambda \sum_{n\in\mathbb{N}} n^\lambda \sum_{k\in\mathbb{Z}} \frac{1}{(4n^2\mathrm{e}^{2n} + (4k+1)^2)^{\frac{\lambda}{2}}}$$

$$
\begin{aligned}
&\le 2\sum_{n\in\mathbb{N}} \mathrm{e}^{-\lambda n} + 2\sum_{n,k\in\mathbb{N}} \frac{1}{\left(\mathrm{e}^{2n} + 4\frac{k^2}{n^2}\right)^{\frac{\lambda}{2}}} \\
&= \frac{2\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}} + 2\sum_{n,k\in\mathbb{N}} \frac{1}{\left(\mathrm{e}^{2n} + 4\frac{k^2}{n^2}\right)^{\frac{\lambda}{2}}},
\end{aligned}
$$

when $\lambda > 0$.

It remains to understand when the rightmost sum converges. The easiest way to do so seems to be to establish the convergence by some integral test. We will therefore consider the function

$$
f(t) := \sum_{n\in\mathbb{N}} \frac{1}{\left(\mathrm{e}^{2n} + \frac{4}{n^2}t^2\right)^{\frac{\lambda}{2}}}, \qquad t \in [1,\infty),
$$

which is well-defined since for every fixed $t \in [1,\infty)$, we have

$$
\sum_{n\in\mathbb{N}} \frac{1}{\left(\mathrm{e}^{2n} + \frac{4}{n^2}t^2\right)^{\frac{\lambda}{2}}} \le \sum_{n\in\mathbb{N}} \mathrm{e}^{-\lambda n} = \frac{\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}.
$$

By the integral test, the convergence of

$$
\sum_{k\in\mathbb{N}} f(k)
$$

is guaranteed by the convergence of

$$
\int_1^\infty f(t)\,\mathrm{d}t = \int_1^\infty \sum_{n\in\mathbb{N}} \frac{1}{\left(\mathrm{e}^{2n} + \frac{4}{n^2}t^2\right)^{\frac{\lambda}{2}}}\,\mathrm{d}t.
$$

By Tonelli's theorem, we can exchange the summation and the integration to obtain

$$
\int_1^\infty f(t)\,\mathrm{d}t = \int_1^\infty \sum_{n\in\mathbb{N}} \frac{1}{\left(\mathrm{e}^{2n} + \frac{4}{n^2}t^2\right)^{\frac{\lambda}{2}}}\,\mathrm{d}t = \sum_{n\in\mathbb{N}} \int_1^\infty \frac{1}{\left(\mathrm{e}^{2n} + \frac{4}{n^2}t^2\right)^{\frac{\lambda}{2}}}\,\mathrm{d}t.
$$

We might now rewrite the integral a little bit by using the substitution $t = \frac{n\exp(n)}{2}s$ and find

$$
\begin{aligned}
\int_1^\infty f(t)\,\mathrm{d}t &= \sum_{n\in\mathbb{N}} \int_1^\infty \frac{1}{\left(\mathrm{e}^{2n} + \frac{4}{n^2}t^2\right)^{\frac{\lambda}{2}}}\,\mathrm{d}t = \sum_{n\in\mathbb{N}} \mathrm{e}^{-\lambda n} \int_1^\infty \frac{1}{\left(1 + \frac{4}{n^2\mathrm{e}^{2n}}t^2\right)^{\frac{\lambda}{2}}}\,\mathrm{d}t \\
&= \frac{1}{2}\sum_{n\in\mathbb{N}} n\mathrm{e}^{-(\lambda-1)n} \int_{\frac{2}{n\exp(n)}}^\infty \frac{1}{(1+s^2)^{\frac{\lambda}{2}}}\,\mathrm{d}t \\
&\le \frac{1}{2}\sum_{n\in\mathbb{N}} n\mathrm{e}^{-(\lambda-1)n} \int_0^\infty \frac{1}{(1+s^2)^{\frac{\lambda}{2}}}\,\mathrm{d}t.
\end{aligned}
$$

One may show that

$$I(\lambda) := \int_0^\infty \frac{1}{(1+s^2)^{\frac{\lambda}{2}}}\,\mathrm{d}t < \infty,$$

when $\lambda > 1$. One way of doing this is demonstrated in the appendix where we prove that

$$I(\lambda) = \frac{\sqrt{\pi}\Gamma\left(\frac{\lambda-1}{2}\right)}{2\Gamma\left(\frac{\lambda}{2}\right)}, \qquad \lambda > 1,$$

where

$$\Gamma(z) := \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad z \in \mathbb{C},\ \operatorname{Re} z > 0,$$

denotes the Euler gamma function. Finally, we may bound

$$\int_1^\infty f(t)\,\mathrm{d}t \leq \frac{I(\lambda)}{2}\sum_{n\in\mathbb{N}} n\mathrm{e}^{-(\lambda-1)n} < \infty$$

by using the ratio test when $\lambda > 1$.

We may now conclude from the integral test that

$$\sum_{n,k\in\mathbb{N}} \frac{1}{\left(\mathrm{e}^{2n} + 4\frac{k^2}{n^2}\right)^{\frac{\lambda}{2}}} = \sum_{k\in\mathbb{N}} f(k) < \infty$$

and thus that

$$\sum_{n,k} \frac{1}{|a_{n,k}|^\lambda} < \infty,$$

as long as $\lambda > 1$. Finally, it follows from Lemma 11 that

$$F(z) = \prod_{n\in\mathbb{N}}\prod_{k\in\mathbb{Z}}\left(1 - \frac{z}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right)\exp\left(\frac{z}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right),$$

is a well-defined entire function of order at most 1.

Since F is of first order, it is, in particular, in the Fock space $\mathcal{F}^2(\mathbb{C})$. Therefore, we may define $f := \mathcal{B}^{-1}F \in L^2(\mathbb{R})$. In the following, we show that f is a universal counterexample.

Theorem 40 (Existence of universal counterexamples — version I)**.** *The function*

$$f = \mathcal{B}^{-1}\prod_{n\in\mathbb{N}}\prod_{k\in\mathbb{Z}}\left(1 - \frac{\cdot}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right)\exp\left(\frac{\cdot}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right)$$

is a well-defined function in $L^2(\mathbb{R})$. Additionally, the functions

$$g_m := \mathcal{B}^{-1}\mathrm{e}^{-\mathrm{i}b_m\cdot}\prod_{k\in\mathbb{Z}}\left(1 - \frac{\cdot}{\mathrm{e}^m - \frac{\mathrm{i}}{2m} + \frac{2\mathrm{i}k}{m}}\right)\exp\left(\frac{\cdot}{\mathrm{e}^m - \frac{\mathrm{i}}{2m} + \frac{2\mathrm{i}k}{m}}\right)$$

$$\cdot \prod_{n\in\mathbb{N}\setminus\{m\}} \prod_{k\in\mathbb{Z}} \left(1 - \frac{\cdot}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right) \exp\left(\frac{\cdot}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right),$$

where

$$b_m = m \sum_{k\in\mathbb{Z}} \left(\log \left| \frac{\mathrm{e}^m + \frac{\mathrm{i}}{2m} + \frac{2(k-1)\mathrm{i}}{m}}{\mathrm{e}^m + \frac{\mathrm{i}}{2m} + \frac{2k\mathrm{i}}{m}} \right| - \frac{2}{m} \operatorname{Im} \frac{1}{\mathrm{e}^m + \frac{\mathrm{i}}{2m} + \frac{2k\mathrm{i}}{m}} \right) \in \mathbb{R},$$

for $m \in \mathbb{N}$, are well-defined functions in $L^2(\mathbb{R})$. Finally, it holds that $f \not\sim g_m$ as well as

$$|\mathcal{G}f| = |\mathcal{G}g_m| \quad \textit{on } \mathbb{R} \times \frac{1}{m}\mathbb{Z}.$$

Proof. We have already shown that $f \in L^2(\mathbb{R})$ is well-defined. Let us now consider $m \in \mathbb{N}$ arbitrary but fixed. We remark that the considerations before the statement of the theorem imply that

$$\sum_{k\in\mathbb{Z}} \frac{1}{\left|\mathrm{e}^m - \frac{\mathrm{i}}{2m} + \frac{2\mathrm{i}k}{m}\right|^\lambda} + \sum_{n\in\mathbb{N}\setminus\{m\}} \sum_{k\in\mathbb{Z}} \frac{1}{|a_{n,k}|^\lambda} < \infty.$$

Therefore, it follows from Lemma 11 that

$$G_m(z) := \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\mathrm{e}^m - \frac{\mathrm{i}}{2m} + \frac{2\mathrm{i}k}{m}}\right) \exp\left(\frac{z}{\mathrm{e}^m - \frac{\mathrm{i}}{2m} + \frac{2\mathrm{i}k}{m}}\right)$$
$$\cdot \prod_{n\in\mathbb{N}\setminus\{m\}} \prod_{k\in\mathbb{Z}} \left(1 - \frac{z}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right) \exp\left(\frac{z}{\mathrm{e}^n + \frac{\mathrm{i}}{2n} + \frac{2\mathrm{i}k}{n}}\right),$$

for $z \in \mathbb{C}$, is a well-defined entire function of first order. At the same time, the convergence of the sums mentioned above also implies that the sum defining b_m is convergent. It follows that $G_m \mathrm{e}^{-\mathrm{i}b_m \cdot} \in \mathcal{F}^2(\mathbb{C})$ and thus $g_m = \mathcal{B}^{-1}(G_m \mathrm{e}^{-\mathrm{i}b_m \cdot}) \in L^2(\mathbb{R})$ is well-defined.

Next, we may see that $f \not\sim g_m$ by definition. Indeed, $\mathrm{e}^m + \frac{\mathrm{i}}{2m}$ is a root of $\mathcal{B}f$ but not a root of $\mathcal{B}g_m$. Finally, we may apply Theorem 27 with $\tau = \frac{1}{m}$, $r = 1$, $\phi = \psi = a' = b = 0$, $b' = -b_m$, $m = 0$,

$$\mathcal{Y}_{\mathrm{u}} = \left\{ \mathrm{e}^m + \tfrac{\mathrm{i}}{2m} \right\},$$

$m_{\mathcal{Y}} = 1$,

$$\mathcal{X} = \left\{ \mathrm{e}^n + \tfrac{\mathrm{i}}{2n} + \tfrac{2\mathrm{i}k}{n} \,\middle|\, n \in \mathbb{N} \setminus \{m\},\ k \in \mathbb{Z} \right\},$$

and $m_{\mathcal{X}} = 1$. In this way, we see that

$$|\mathcal{B}f| = |\mathcal{B}g_m| \quad \text{on } \mathbb{R} + \mathrm{i}\frac{1}{m}\mathbb{Z}.$$

According to the relation of the Gabor transform and the Bargmann transform (see equation (8)), we conclude

$$|\mathcal{G}f| = |\mathcal{G}g_m| \quad \text{on } \mathbb{R} \times \frac{1}{m}\mathbb{Z}.$$

□

The following statement about existence of universal counterexamples follows directly from the prior theorem.

Corollary 41 (Existence of universal counterexamples — version II)**.** *There exists an $f \in L^2(\mathbb{R})$ such that for all $n \in \mathbb{N}$, there exists a $g_n \in L^2(\mathbb{R})$ such that $f \not\sim g_n$ and yet*

$$|\mathcal{G}f| = |\mathcal{G}g_n| \quad \text{on } \mathbb{R} \times \frac{1}{n}\mathbb{Z}.$$

A Relating the convergence of a certain integral to the Euler gamma function

The following lemma might be well-known. We found it by comparing it to the well-known relation of the Euler gamma function and the beta function.

Lemma 42. *It holds that*

$$\int_0^\infty \frac{1}{(1+t^2)^{\lambda/2}}\,\mathrm{d}t = \frac{\sqrt{\pi}\Gamma\left(\frac{\lambda-1}{2}\right)}{2\Gamma\left(\frac{\lambda}{2}\right)}, \qquad \lambda > 1.$$

Proof. Let us consider

$$\Gamma\left(\frac{\lambda}{2}\right) \cdot \int_0^\infty \frac{1}{(1+t^2)^{\lambda/2}}\,\mathrm{d}t = \int_0^\infty \int_0^\infty \frac{u^{\lambda/2-1}\mathrm{e}^{-u}}{(1+t^2)^{\lambda/2}}\,\mathrm{d}t\,\mathrm{d}u.$$

We may now use the substitution $u = x + y$, $t = \sqrt{x/y}$ with Jacobian

$$J = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2\sqrt{xy}} & -\frac{\sqrt{x}}{2y^{3/2}} \end{pmatrix}.$$

We compute

$$|\det J| = \frac{1}{2\sqrt{xy}} + \frac{\sqrt{x}}{2y^{3/2}} = \frac{x+y}{2x^{1/2}y^{3/2}}$$

and note that

$$x = \frac{ut^2}{1+t^2}, \qquad y = \frac{u}{1+t^2},$$

which implies that the domain of integration remains unchanged. Therefore,

$$\begin{aligned}
\Gamma\left(\frac{\lambda}{2}\right) \cdot \int_0^\infty \frac{1}{(1+t^2)^{\lambda/2}}\,\mathrm{d}t &= \int_0^\infty \int_0^\infty \frac{u^{\lambda/2-1}\mathrm{e}^{-u}}{(1+t^2)^{\lambda/2}}\,\mathrm{d}t\,\mathrm{d}u \\
&= \int_0^\infty \int_0^\infty \left(\frac{u}{1+t^2}\right)^{\lambda/2} \mathrm{e}^{-u}u^{-1}\,\mathrm{d}t\,\mathrm{d}u \\
&= \int_0^\infty \int_0^\infty y^{\lambda/2}\mathrm{e}^{-x-y}(x+y)^{-1}\frac{x+y}{2x^{1/2}y^{3/2}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int_0^\infty \int_0^\infty y^{\lambda/2-3/2}x^{-1/2}\mathrm{e}^{-x-y}\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\lambda-1}{2}\right).$$

The lemma follows from the well-known facts that $\Gamma(1/2) = \sqrt{\pi}$ and that the Gamma function has no zeros. $\square$

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