

Multilevel Monte Carlo FEM for Elliptic PDEs with Besov Random Tree Priors

Ch. Schwab and A. Stein

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Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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Christoph Schwab ^{*} Andreas Stein ^{*†}

Abstract

We develop a multilevel Monte Carlo (MLMC) FEM algorithm for linear, elliptic diffusion problems in polytopal domain $\mathcal{D} \subset \mathbb{R}^d$, with Besov-tree random coefficients. This is to say that the logarithms of the diffusion coefficients are sampled from so-called Besov-tree priors, which have recently been proposed to model data for fractal phenomena in science and engineering. Numerical analysis of the fully discrete FEM for the elliptic PDE includes quadrature approximation and must account for a) nonuniform pathwise upper and lower coefficient bounds, and for b) low path-regularity of the Besov-tree coefficients.

Admissible non-parametric random coefficients correspond to random functions exhibiting singularities on random fractals with tunable fractal dimension, but involve no a-priori specification of the fractal geometry of singular supports of sample paths. Optimal complexity and convergence rate estimates for quantities of interest and for their second moments are proved. A convergence analysis for MLMC-FEM is performed which yields choices of the algorithmic steering parameters for efficient implementation. A complexity (“error vs work”) analysis of the MLMC-FEM approximations is provided.

1 Introduction

The efficient numerical solution of partial differential equations with uncertain inputs is key in *forward uncertainty quantification*, i.e., the computational quantification of uncertainty of solutions to PDEs with uncertain inputs. It is also crucial in *computational inverse uncertainty quantification*, e.g. via Markov chain Monte Carlo methods, where numerous numerical solves of the forward model subject to realizations of the uncertain input are required. Here, we consider the linear, elliptic diffusion with uncertain coefficient. It models a wide range of phenomena such as diffusion through a medium with uncertain or even unknown permeability, e.g. in subsurface flow, light scattering in dust clouds, to name but a few. Physical modelling of subsurface flow in particular stipulates systems of fractures of uncertain geometry with high permeability along fractures (see, e.g., [7] and the references there). With fracture geometry being only statistically known, it is natural in computational uncertainty quantification (UQ) to specify the geometry in a nonparametric fashion, rather than, for instance, through a Gaussian random field (GRF for short) with a known, parametric two-point correlation to be calibrated from experimental data. This function space perspective has also become topical recently in the context of inverse imaging noisy signals. Modelling with random, fractal geometries also has found applications in biology (roots, lungs [2]). There, Gaussian parametric models have been found computationally efficient due to the availability of padding and circulant embedding based numerics, enabling the use of fast Fourier transform algorithms for sample path generation. However, Gaussian models are perceived as inadequate for the efficient representation of edges and interfaces in imaging. Accordingly, non-parametric representations of inputs with fractal irregularities in sample paths have been proposed

^{*}ETH Zürich, Seminar for Applied Mathematics

[†]andreas.stein@sam.math.ethz.ch

recently, e.g. in [22, 18], and the references there. We also mention the so-called *Besov priors* in Bayesian inverse problems with elliptic PDE constraints (e.g. [23, 21, 4] and the references there).

In the present paper, we investigate so-called *Besov random tree priors* [22], as stochastic log-diffusion coefficient in a linear elliptic PDE. These priors are given by a wavelet series with stochastic coefficients, and certain terms in the expansion vanishing at random, according to the law of so-called *Galton-Watson trees*. Samples of the corresponding random fields involve fractal geometries, hence the Besov random tree prior may be a viable candidate in applications, where models based on GRFs do not allow for sufficiently flexibility. We quantify the pathwise regularity of the random tree prior in terms of Hölder-regularity, and investigate the forward propagation of the uncertainty in the elliptic PDE model in a bounded domain. All results in the present article encompass the "standard" Besov prior from [23] as special case, when no terms in the wavelet series are eliminated. As we point out in our analysis, regularity is inherently low, both with respect to the spatial and stochastic domain of the random field. This is taken into account when developing efficient numerical methods for the elliptic PDE problem at hand.

We develop a multilevel Monte Carlo (MLMC) Finite Element (FE) simulation algorithm and furnish its mathematical analysis for the estimation of quantities of interest (QoI) in the forward PDE model. Multilevel Monte Carlo methods ([13, 14, 5]) are, by now, a well-established methodology in computational UQ, and are effective in regimes with comparably low regularity. In contrast, higher-order methods that consider an equivalent parametric, deterministic PDE problem, such as (multilevel) Quasi-Monte Carlo ([19]), generalized polynomial chaos (gPC) expansions ([20]), or multilevel Smolyak quadrature ([27]) are *not suitable* in the present random tree model: The parametrization of the prior involves discontinuities in the stochastic domain, which strongly violates the regularity requirements of the aforementioned higher-order methods. On the other hand, MLMC techniques merely require square-integrability in the probability space.

1.1 Contributions

For a model linear elliptic diffusion equation, in a polytopal domain $\mathcal{D} \subset \mathbb{R}^d$, we provide the mathematical setting and the numerical analysis of a MLMC-FEM for diffusion in random media with log-fractal Besov random tree structure. In particular, we establish well-posedness of the forward problem including strong measurability of random solutions (a key ingredient in the ensuing MLMC-FE convergence analysis), and pathwise almost sure Besov regularity of weak solutions. Technical results of independent interest include: (i) Bounds on exponential moments of Besov random variables in Hölder norms, generalizing results in [22, 10, 23], (ii) Numerical analysis of elliptic forward problems with fractal coefficient, in particular bounds on the fractal scale truncation error and on the finite element approximation error, as well as the impact of numerical quadrature in view of low (Hölder) path regularity of the random coefficients, (iii) a complete MLMC-FE convergence analysis, for estimating the mean of non-linear functionals of the random solution field.

1.2 Preliminaries and Notation

We denote by \mathcal{V} the topological dual for any vector space \mathcal{V} and by $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ the associated dual pairing. We write $\mathcal{X} \hookrightarrow \mathcal{Y}$ for two metric spaces \mathcal{X}, \mathcal{Y} , if \mathcal{X} is continuously embedded in \mathcal{Y} , i.e., there exists $C > 0$ such that $\|\varphi\|_{\mathcal{Y}} \leq C\|\varphi\|_{\mathcal{X}}$ holds for all $\varphi \in \mathcal{X}$. The Borel σ -algebra of any metric space \mathcal{X} is generated by the open sets in \mathcal{X} and denoted by $\mathcal{B}(\mathcal{X})$. For any σ -finite and complete measure space (E, \mathcal{E}, μ) , a Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, and integrability exponent $p \in [1, \infty]$, we define the Lebesgue-Bochner spaces

$$L^p(E; \mathcal{X}) := \{\varphi : E \rightarrow \mathcal{X} \mid \varphi \text{ is strongly measurable and } \|\varphi\|_{L^p(E; \mathcal{X})} < \infty\},$$

where

$$\|\varphi\|_{L^p(E; \mathcal{X})} := \begin{cases} \left(\int_E \|\varphi(x)\|_{\mathcal{X}}^p \mu(dx) \right)^{1/p}, & p \in [1, \infty) \\ \text{ess sup}_{x \in E} \|\varphi(x)\|_{\mathcal{X}}, & p = \infty. \end{cases}$$

In case that $\mathcal{X} = \mathbb{R}$, we use the shorthand notation $L^p(E) := L^p(E; \mathbb{R})$. If $E \subset \mathbb{R}^d$ is a subset of Euclidean space, we assume $\mathcal{E} = \mathcal{B}(E)$ and μ is the Lebesgue measure, unless stated otherwise. For any bounded and connected spatial domain $\mathcal{D} \subset \mathbb{R}^d$ we denote for $k \in \mathbb{N}$ and $p \in [1, \infty]$ the standard Sobolev space $W^{k,p}(\mathcal{D})$ with k -order weak derivatives in $L^p(\mathcal{D})$. The Sobolev-Slobodeckji space with fractional order $s \geq 0$ is denoted by $W^{s,p}(\mathcal{D})$. Furthermore, $H^s(\mathcal{D}) := W^{s,2}(\mathcal{D})$ for any $s \geq 0$ and we use the identification $H^0(\mathcal{D}) = L^2(\mathcal{D})$. Given that \mathcal{D} is a Lipschitz domain, we define for any $s > 1/2$

$$H_0^s(\mathcal{D}) := \ker(\gamma_0) = \{\varphi \in H^s(\mathcal{D}) \mid \gamma_0(\varphi) = 0 \text{ on } \partial\mathcal{D}\}, \quad (1)$$

Here, $\gamma_0 \in \mathcal{L}(H^s(\mathcal{D}), H^{s-1/2}(\partial\mathcal{D}))$ denotes the trace operator.

Let $C(\overline{\mathcal{D}})$ denote the space of all continuous functions $\varphi : \overline{\mathcal{D}} \rightarrow \mathbb{R}$. For any $\alpha \in \mathbb{N}$, $C^\alpha(\overline{\mathcal{D}})$ is the space of all functions $\varphi \in C(\overline{\mathcal{D}})$ with α continuous partial derivatives. For non-integer $\alpha > 0$, we denote by $C^\alpha(\overline{\mathcal{D}})$ the space of all $\varphi \in C^{\lfloor \alpha \rfloor}(\overline{\mathcal{D}})$ with $\alpha - \lfloor \alpha \rfloor$ -Hölder continuous $\lfloor \alpha \rfloor$ -th partial derivatives. For any positive, real $\alpha > 0$ we further denote by $C^\alpha(\mathcal{D})$ the *Hölder-Zygmund space* of smoothness α . We refer to, e.g., [24, Section 1.2.2] for a Definition. We denote by $S(\mathbb{R}^d)$ the Schwartz space of all smooth, rapidly decaying functions, and with $S'(\mathbb{R}^d)$ its dual, the space of tempered distributions. Moreover, for any open set $O \subseteq \mathbb{R}^d$, $D(O)$ denotes the space of all smooth functions $\varphi \in C^\infty(O)$ with compact support in O .

1.3 Layout of this paper

In Section 2 we introduce the class of random fields taking values in the Besov spaces $B_{p,p}^s$ which we will use in the sequel to model the logarithm of the diffusion coefficient function. Using multiresolution (“wavelet”) bases in $B_{p,p}^s$, in Sections 2.2, 2.3 we construct probability measures on $B_{p,p}^s$ in the spirit of the Gaussian measure on path space for the Wiener process, in Lévy-Cieselski representation. The multilevel structure of the construction will be essential in the ensuing MLMC-FE convergence analysis and its algorithmic realization. In Section 3 we introduce the linear, elliptic divergence-form PDE with Besov-tree coefficients. We recapitulate (mostly known) results on existence, uniqueness and on strong measurability of random solutions. In Section 4 we introduce a conforming Galerkin Finite Element discretization based on continuous, piecewise linear approximations in the physical domain. We account for the error due to finite truncation of the random tree priors, and provide sharp error bounds for the Finite Element discretization errors, under the (generally low) Besov regularity of the coefficient samples. Section 5 then addresses the MLMC-FE error analysis, also for Fréchet-differentiable, possibly nonlinear functionals. Section 6 then illustrates the theory with several numerical experiments, where the impact of the parameter choices in the Besov random tree priors on the overall error convergence of the MLMC-FEM algorithms is studied. Section 7 provides a brief summary of the main results, and indices several generalizations of these and directions of further research. Appendix A collects definitions and key properties of Galton-Watson trees which are used in the main text. Appendix B provides a detailed description of the FE implementation in the experiments reported in Section 6.

2 Random Variables in Besov Spaces

2.1 Wavelet representation of Besov spaces

Let $\mathbb{T}^d := [0, 1]^d$ denote the d -dimensional torus for $d \in \mathbb{N}$. We briefly recall the construction of orthonormal wavelet basis on $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{T}^d)$ and the wavelet representation of the associated Besov spaces. For more detailed accounts we refer to [25, Chapter 1], [26, Chapter 1.2], and to [11, Chapter 5] for orthonormal wavelets in multiresolution analysis (MRA).

2.1.1 Univariate MRA

Let ϕ and ψ be compactly supported scaling and wavelet functions in $C^\alpha(\mathbb{R})$, $\alpha \geq 1$, suitable for multi-resolution analysis in $L^2(\mathbb{R})$. We assume that ψ satisfies *the vanishing moment condition*

$$\int_{\mathbb{R}} \psi(x)x^m dx = 0, \quad m \in \mathbb{N}_0, \quad m < \alpha. \quad (2)$$

One example are Daubechies wavelets with $M := \lfloor \alpha \rfloor \in \mathbb{N}$ vanishing moments also known as $DB(\lfloor \alpha \rfloor)$ -wavelets), that have support $[-M + 1, M]$ and are in $C^1(\mathbb{R})$ for $M \geq 5$ (see, e.g., [11, Section 7.1]). For any $j \in \mathbb{N}_0$ and $k \in \mathbb{Z}$, the MRA is defined by the dilated and translated functions

$$\psi_{j,k,0}(x) := \phi(2^j x - k), \quad \text{and} \quad \psi_{j,k,1}(x) := \psi(2^j x - k), \quad x \in \mathbb{R}. \quad (3)$$

As $\|\phi\|_{L^2(\mathbb{R})} = \|\psi\|_{L^2(\mathbb{R})} = 1$, it follows that $((\psi_{0,k,0}), k \in \mathbb{Z}) \cup ((2^{j/2}\psi_{j,k,1}), (j, k) \in \mathbb{N}_0 \times \mathbb{Z})$ is an orthonormal basis of $L^2(\mathbb{R})$.

2.1.2 Multivariate MRA

A corresponding *isotropic*¹ wavelet basis that is orthonormal in $L^2(\mathbb{R}^d)$, $d \geq 2$ may be constructed by tensorization of univariate MRAs. We define index sets $\mathcal{L}_0 := \{0, 1\}^d$ and $\mathcal{L}_j := \mathcal{L}_0 \setminus \{(0, \dots, 0)\}$ for $j \in \mathbb{N}$. We note that \mathcal{L}_j has cardinality $|\mathcal{L}_j| = 2^d$ if $j = 0$, and $|\mathcal{L}_j| = 2^d - 1$ otherwise. For any $l \in \mathcal{L}_0$, we define furthermore

$$\psi_{j,k,l}(x) := 2^{dj/2} \prod_{i=1}^d \psi_{j,k_i,l(i)}(x_i), \quad j \in \mathbb{N}_0, \quad k \in \mathbb{Z}^d, \quad x \in \mathbb{R}^d, \quad (4)$$

to obtain that $((\psi_{j,k,l}), j \in \mathbb{N}_0, k \in \mathbb{Z}^d, l \in \mathcal{L}_j)$ is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Orthonormal bases consisting of locally supported, periodic functions on the torus \mathbb{T}^d can be introduced by tensorization, as e.g. in [25, Section 1.3]. Given ϕ and ψ , we fix a scaling factor $w \in \mathbb{N}$ such that

$$\text{supp}(\psi_{w,0,l}) \subset \left\{ x \in \mathbb{R}^d \mid \|x\|_2 < \frac{1}{2} \right\}, \quad l \in \mathcal{L}_0.$$

With this choice of w , it follows for $j \in \mathbb{N}_0$ that

$$\text{supp}(\psi_{j+w,0,l}) \subset \{x \in \mathbb{R}^d \mid \|x\|_2 < 2^{-j-1}\}.$$

Now let $K_j := \{k \in \mathbb{Z}^d \mid 0 \leq k_1, \dots, k_d < 2^j\} \subset 2^j \mathbb{T}^d$ and note that $|K_{j+w}| = 2^{d(j+w)}$. Define the index set $\mathcal{I}_w := \{j \in \mathbb{N}_0, k \in K_{j+w}, l \in \mathcal{L}_j\}$, the one-periodic wavelet functions

$$\psi_{j+w,k,l}^{per}(x) := \sum_{n \in \mathbb{Z}^d} \psi_{j+w,k,l}(x - n), \quad (j, k, l) \in \mathcal{I}_w, \quad x \in \mathbb{R}^d,$$

and their restrictions to \mathbb{T}^d by

$$\psi_{j+w,k}^l(x) := \psi_{j+w,k,l}^{per}(x), \quad (j, k, l) \in \mathcal{I}_w, \quad x \in \mathbb{T}^d. \quad (5)$$

We now obtain that

$$\Psi_w := ((\psi_{j+w,k}^l), (j, k, l) \in \mathcal{I}_w) \quad (6)$$

is a $L^2(\mathbb{T}^d)$ -orthonormal basis, see [25, Proposition 1.34]. We next introduce Besov spaces via suitable wavelet-characterization as developed in [25, Chapter 1.3]. We distinguish between spaces of one-periodic functions on \mathbb{R}^d and their restrictions to the torus \mathbb{T}^d :

¹Anisotropic tensorizations leading upon truncation to so-called ‘‘hyperbolic cross approximations’’ may be considered. As such constructions tend to inject preferred directions along the cartesian axes, we do not consider them here.

Definition 2.1.

1. For any $p \in [1, \infty)$ and $s \in (0, \alpha)$ the Besov space $B_{p,p}^{s,per}(\mathbb{R}^d)$ of one-periodic functions on \mathbb{R}^d is given by

$$B_{p,p}^{s,per}(\mathbb{R}^d) := \left\{ \varphi \in S'(\mathbb{R}^d) \left| \sum_{(j,k,l) \in \mathcal{I}_w} 2^{jp(s + \frac{d+w}{2} - \frac{d}{p})} |(\varphi, \psi_{j+w,k,l}^{per})_{L^2(\mathbb{T}^d)}|^p < \infty \right. \right\}. \quad (7)$$

In case that $p = \infty$, one has

$$B_{\infty,\infty}^{s,per}(\mathbb{R}^d) := \left\{ \varphi \in S'(\mathbb{R}^d) \left| \sup_{(j,k,l) \in \mathcal{I}_w} 2^{j(s + \frac{d+w}{2})} |(\varphi, \psi_{j+w,k,l}^{per})_{L^2(\mathbb{T}^d)}| < \infty \right. \right\}. \quad (8)$$

2. For any $p \in [1, \infty)$ and $s \in (0, \alpha)$ the Besov space $B_{p,p}^s(\mathbb{T}^d)$ on \mathbb{T}^d is given by

$$B_{p,p}^s(\mathbb{T}^d) := \left\{ \varphi \in D'(\mathbb{T}^d) \left| \sum_{(j,k,l) \in \mathcal{I}_w} 2^{jp(s + \frac{d+w}{2} - \frac{d}{p})} |(\varphi, \psi_{j+w,k}^l)_{L^2(\mathbb{T}^d)}|^p < \infty \right. \right\}. \quad (9)$$

In case that $p = \infty$, we set

$$B_{\infty,\infty}^s(\mathbb{T}^d) := \left\{ \varphi \in D'(\mathbb{T}^d) \left| \sup_{(j,k,l) \in \mathcal{I}_w} 2^{j(s + \frac{d+w}{2})} |(\varphi, \psi_{j+w,k}^l)_{L^2(\mathbb{T}^d)}| < \infty \right. \right\}. \quad (10)$$

Remark 2.2. Definition 2.1 may be generalized to define the spaces $B_{p,q}^{s,per}(\mathbb{R}^d)$ and $B_{p,q}^s(\mathbb{T}^d)$ with $p, q \in [1, \infty]$ and $p \neq q$, see [25, Chapter 1.3]. The random fields introduced in Subsections 2.2 and 2.3 are naturally $B_{p,p}^s(\mathbb{T}^d)$ -valued by construction, thus we only treat the case $p = q$ for the sake of brevity. By [25, Theorem 1.29] exists an *prolongation isomorphism*

$$\text{prl}^{per} : B_{p,p}^s(\mathbb{T}^d) \rightarrow B_{p,p}^{s,per}(\mathbb{R}^d), \quad (11)$$

that extends $\varphi \in B_{p,p}^s(\mathbb{T}^d)$ to its (unique) one-periodic counterpart in $\text{prl}^{per}(\varphi) \in B_{p,p}^{s,per}(\mathbb{R}^d)$. This in turn allows to identify any $\varphi \in B_{p,p}^s(\mathbb{T}^d)$ as the restriction of a periodic function $\text{prl}^{per}(\varphi) \in B_{p,p}^{s,per}(\mathbb{R}^d)$ to \mathbb{T}^d . We use prl^{per} to define (non-periodic) Besov space-valued random variables on subsets $\mathcal{D} \subset \mathbb{T}^d$ by restriction in Subsection 3.2.

Definition 2.1 is based on an equivalent characterization of the spaces $B_{p,p}^{s,per}(\mathbb{R}^d)$ and $B_{p,p}^s(\mathbb{T}^d)$. They are often (equivalently) defined using a dyadic partition of unity (see e.g. [25, Definitions 1.22 and 1.27]): Using the latter definition for $B_{p,p}^{s,per}(\mathbb{R}^d)$, [25, Theorem 1.36(i)] shows that the spaces (7) resp. (8) are isometrically isomorphic to $B_{p,p}^{s,per}(\mathbb{R}^d)$. As a consequence of the prolongation map prl^{per} in (11), the same holds true for the spaces $B_{p,p}^s(\mathbb{T}^d)$, see [25, Theorem 1.37(i)].

2.1.3 Besov Spaces and MRAs

We define the subspace $V_{w+1} := \text{span}\{\psi_{w,k}^l \mid k \in K_0, l \in \mathcal{L}_0\} \subset L^2(\mathbb{T}^d)$ and observe that $\dim(V_{w+1}) = 2^{d(w+1)}$. By the multiresolution analysis for one-periodic, univariate functions in [11, Chapter 9.3], it follows that $((\psi_{j,k}^l), j \leq w, k \in K_j, l \in \mathcal{L}_j)$ is another orthonormal basis of V_{w+1} . Hence, we may replace the first $2^{d(w+1)}$ basis functions in (5) to obtain the (computationally more convenient) $L^2(\mathbb{T}^d)$ -orthonormal basis

$$\Psi := ((\psi_{j,k}^l), (j, k, l) \in \mathcal{I}_\Psi), \quad \mathcal{I}_\Psi := \{j \in \mathbb{N}_0, k \in K_j, l \in \mathcal{L}_j\}. \quad (12)$$

Based on (12), we define for $s > 0$, $p \in [1, \infty)$, and sufficiently regular $\varphi \in L^2(\mathbb{T}^d)$ the Besov norms

$$\|\varphi\|_{B_{p,p}^s(\mathbb{T}^d)} := \left(\sum_{(j,k,l) \in \mathcal{I}_\Psi} 2^{jp(s + \frac{d}{2} - \frac{d}{p})} |(\varphi, \psi_{j,k}^l)_{L^2(\mathbb{T}^d)}|^p \right)^{1/p}, \quad (13)$$

and

$$\|\varphi\|_{B_{\infty,\infty}^s(\mathbb{T}^d)} := \sup_{(j,k,l) \in \mathcal{I}_{\Psi}} 2^{j(s+\frac{d}{2})} |(\varphi, \psi_{j,k}^l)_{L^2(\mathbb{T}^d)}| < \infty. \quad (14)$$

By Definition 2.1, it follows that $\varphi \in B_{p,p}^s(\mathbb{T}^d)$ if and only if $\|\varphi\|_{B_{p,p}^s(\mathbb{T}^d)} < \infty$.

2.1.4 Notation

We fix some notation for Besov, Hölder and Zygmund spaces to be used in the remainder of this paper. As the (periodic) domain \mathbb{T}^d does not vary in the subsequent analysis, we use the abbreviations $B_p^s := B_{p,p}^s(\mathbb{T}^d)$, $C^\alpha := C^\alpha(\mathbb{T}^d)$ and $\mathcal{C}^\alpha := \mathcal{C}^\alpha(\mathbb{T}^d)$ for convenience in the following. Furthermore, we will assume that the basis functions in $\Psi_w, \Psi \subset C^\alpha$, are sufficiently smooth with Hölder index $\alpha > s$ for given $s > 0$, and therefore omit the restriction $s \in (0, \alpha)$ in the following. In this case, it holds that Ψ_w (and thus Ψ) is a basis of B_p^s for $p < \infty$, see [25, Theorem 1.37].

We recall that for any $s > 0$ there holds $\mathcal{C}^s = B_\infty^s$ (see [25, Remark 1.28]), as well as $C^s = \mathcal{C}^s$ for $s \in (0, \infty) \setminus \mathbb{N}$, and $C^s \subsetneq \mathcal{C}^s$ for $s \in \mathbb{N}$ (see [24, Section 1.2.2]). By (13) and (14) we further obtain the continuous embeddings

$$\begin{aligned} B_p^s &\hookrightarrow B_q^t && \text{if } 1 \leq p \leq q < \infty \text{ and } s - \frac{d}{p} \geq t - \frac{d}{q}, \\ B_p^s &\hookrightarrow B_\infty^t = \mathcal{C}^t && \text{for } t \in \left(0, s - \frac{d}{p}\right], \end{aligned} \quad (15)$$

with the embedding constants in (15) bounded by one (cf. [26, Chapter 2.1]).

2.2 Besov priors

To introduce Besov space-valued random variables as in [23], we consider a complete probability space (Ω, \mathcal{A}, P) . Following the constructions in [10, 4, 22], based on the representation in (9), we now define B_p^s -valued random variables by replacing the $L^2(\mathbb{T}^d)$ -orthogonal projection coefficients $(\varphi, \psi_{j,k}^l)_{L^2(\mathbb{T}^d)}$ with suitable random variables. More precisely, consider for any $p \in [1, \infty)$ an independent and identically distributed (i.i.d.) sequence $X = ((X_{j,k}^l), (j, k, l) \in \mathcal{I}_{\Psi})$ of p -exponential random variables. That is, each $X_{j,k}^l : \Omega \rightarrow \mathbb{R}$ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$ -measurable with density

$$\phi_p(x) := \frac{1}{c_p} \exp\left(-\frac{|x|^p}{\kappa}\right), \quad x \in \mathbb{R}, \quad c_p := \int_{\mathbb{R}} \exp\left(-\frac{|x|^p}{\kappa}\right) dx, \quad (16)$$

where $\kappa > 0$ is a fixed scaling parameter. We recover the normal distribution with variance $\frac{\kappa}{2}$ if $p = 2$, and the Laplace distribution with scaling κ for $p = 1$.

Definition 2.3. [23, Definition 9] Let Ψ be the $L^2(\mathbb{T}^d)$ -orthogonal wavelet basis as in (12), let $s > 0$, $p \in [1, \infty)$ and let $X = ((X_{j,k}^l), (j, k, l) \in \mathcal{I}_{\Psi})$ be an i.i.d. sequence of p -exponential random variables with density ϕ_p as in (16). Let the random field $b : \Omega \rightarrow L^2(\mathbb{T}^d)$ be given by

$$b(\omega) := \sum_{(j,k,l) \in \mathcal{I}_{\Psi}} \eta_j X_{j,k}^l(\omega) \psi_{j,k}^l, \quad \omega \in \Omega, \quad \text{where } \eta_j := 2^{-j(s+\frac{d}{2}-\frac{d}{p})}, \quad j \in \mathbb{N}_0. \quad (17)$$

We call b a B_p^s -valued random variable, or simply a B_p^s -random variable.

The random variables b from Definition 2.3 are also referred to as *Besov priors* in the literature on inverse problems. The following regularity results are well-known:

Proposition 2.4.

- [23, Lemma 10] Let b be a B_p^s -random variable for $s > 0$ and $p \in [1, \infty)$. Then, the following conditions are equivalent:

- (i) $\|b\|_{B_p^t} < \infty$ holds P -a.s.;
- (ii) $\mathbb{E} \left(\exp \left(\varepsilon \|b\|_{B_p^t}^p \right) \right) < \infty, \quad \varepsilon \in (0, \frac{1}{\kappa});$
- (iii) $t < s - \frac{d}{p}$.

2. [10, Theorem 2.1] If, in addition, Ψ forms a basis of B_p^t for a $t < s - \frac{d}{p}, t \notin \mathbb{N}$, then it holds

$$\mathbb{E} (\exp (\varepsilon \|b\|_{C^t})) < \infty, \quad \varepsilon \in (0, \bar{\varepsilon}),$$

where $\bar{\varepsilon} > 0$ is a constant depending on p, d, s and t .

Remark 2.5. We derive a considerably stronger version of [10, Theorem 2.1] in Theorem 2.9 below, that implies in particular

$$\mathbb{E} (\exp (\varepsilon \|b\|_{C^t}^p)) < \infty, \quad \varepsilon \in (0, \bar{\varepsilon}),$$

for any $p \geq 1$ and some $\bar{\varepsilon} > 0$. In the Gaussian case with $p = 2$, this estimate would be a consequence of Fernique's theorem, however, we are not aware of a similar result for arbitrary $p \geq 1$ in the literature.

We recall from [25, Theorem 1.37] that Ψ forms an *unconditional basis* of B_p^t (since $p < \infty$), if the scaling and wavelet functions ϕ and ψ satisfy $\phi, \psi \in C^\alpha(\mathbb{R}^d)$ for $\alpha > t > 0$ and the vanishing moment condition (2).

2.3 Besov random tree priors

Taking the cue from [22], we introduce *Besov random tree priors* in this subsection and derive several regularity results for this B_p^s -valued random variable. We investigate all results for periodic functions defined on the torus \mathbb{T}^d in this subsection. For the elliptic problem in Section 3, we will later introduce the corresponding $B_p^s(\mathcal{D})$ -valued random variables on physical domains $\mathcal{D} \subset \mathbb{R}^d$ with $\mathcal{D} \subseteq \mathbb{T}^d$ by their restrictions from \mathbb{T}^d (cf. Definition 3.6). The random tree structure in our prior construction is based on certain set-valued random variables, so-called *Galton-Watson (GW) trees*. For the readers' convenience, definitions of discrete trees, GW trees, along with some other useful results, are listed in Appendix A.

Definition 2.6. [22, Definition 3] Let $\Psi, s > 0, p \in [1, \infty), X = ((X_{j,k}^l), (j, k, l) \in \mathcal{I}_\Psi)$ and $(\eta_j, j \in \mathbb{N}_0)$ be as in Definition 2.3. Let \mathfrak{T} denote the set of all trees with no infinite node (cf. Definition A.1) and let $T : \Omega \rightarrow \mathfrak{T}$ be a GW tree (cf. Definition A.3) with offspring distribution $\mathcal{P} = \text{Bin}(2^d, \beta)$ for $\beta \in [0, 1]$, and independent of X . Furthermore, let \mathcal{J}_T be the set of wavelet indices associated to T from (69). We define the *random tree index set* $\mathcal{I}_T(\omega) := \{(j, k, l) \mid (j, k) \in \mathcal{J}_T(\omega), l \in \mathcal{L}_j\}$ and

$$b_T(\omega) := \sum_{(j,k,l) \in \mathcal{I}_T(\omega)} \eta_j X_{j,k}^l(\omega) \psi_{j,k}^l, \quad \omega \in \Omega. \quad (18)$$

We refer to b_T as a B_p^s -random variable with wavelet density β .

Remark 2.7. Definition 2.6 actually slightly deviates from [22, Definition 3]. By definition of $\mathcal{I}_T(\omega)$, we include the constant function $\psi_{0,0}^{(0,\dots,0)} \equiv 1 \in L^2(\mathbb{T}^d)$ in the series expansion (18). Of course, adding the random constant $X_{0,0}^{(0,\dots,0)}$ does not affect the spatial regularity or integrability of b_T . However, in our definition, series (18) has a natural interpretation as orthogonal expansion of a random function with respect to the (deterministic, fixed) basis Ψ . The tree structure in the "active" (i.e., with index in \mathcal{I}_T) coefficients in the wavelet representation of b_T gives rise to random fractals on \mathbb{T}^d , that occur whenever the tree T in Definition 2.6 does not terminate after a finite number of nodes. It follows by Lemma A.4, that the latter event occurs with positive probability if $\beta \in (2^{-d}, 1]$. In this case the Hausdorff dimension of the fractals is $d + \log_2(\beta) \in (0, d]$, see [22, Section 3] for further details.

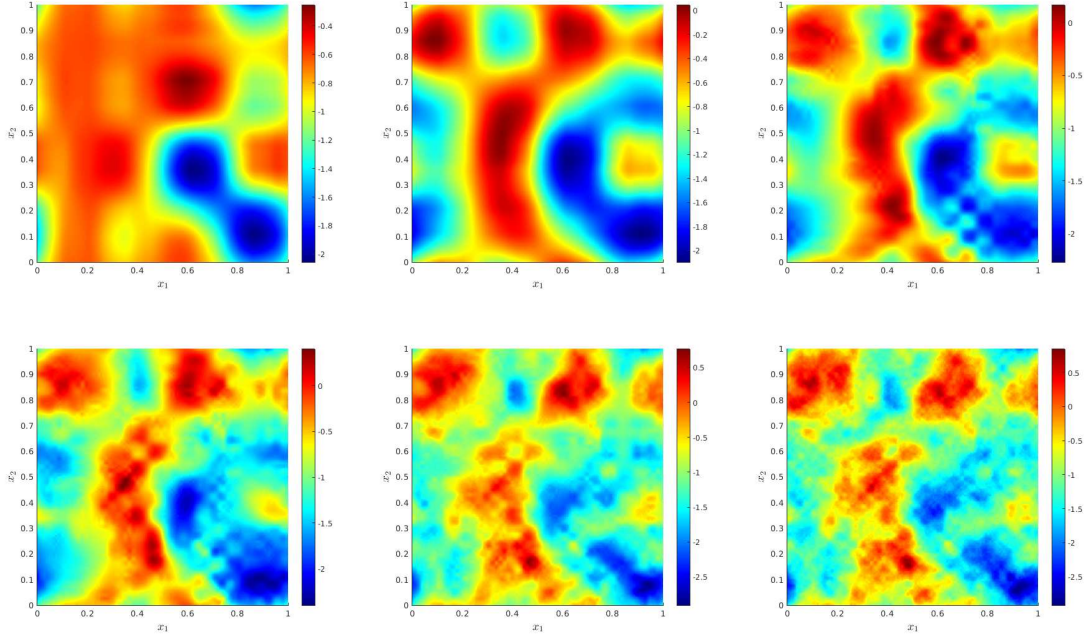


Figure 1: Samples of a B_p^s -valued random variable on $\mathbb{T}^2 = [0, 1]^2$ with $s = p = 2$ and wavelet density $\beta \in \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$ (top row, from left to right) and $\beta \in \{\frac{2}{3}, \frac{3}{4}, 1\}$ (bottom row, from left to right). All samples are based on the same array of random numbers, have been sampled with a spatial resolution of $2^9 \times 2^9$ equidistant grid points, and the expansion in (18) was truncated at $N = 9$ levels of dyadic subdivision (cf. Subsection 4.1). By fixing the array of random numbers, the spatial grid and N , the depicted "evolution" in the panels highlights the effect of an increasing wavelet density β .

Examples of realizations of a B_p^s -random variable on \mathbb{T}^2 with varying wavelet density β are shown in Figure 1.

Proposition 2.8. *Let $s > \frac{d}{p}$, $\beta \in [0, 1]$, and let b_T be a B_p^s -random variable with wavelet density β . Then $b_T : \Omega \rightarrow C(\mathbb{T}^d)$ and b_T is (strongly) $\mathcal{A}/\mathcal{B}(C(\mathbb{T}^d))$ -measurable.*

Proof. Note that $b_T : \Omega \mapsto B_p^s$ holds by (18), and $B_p^s \hookrightarrow C^t$ for $t = s - \frac{d}{p} > 0$ by (15). Thus $b_T : \Omega \mapsto C(\mathbb{T}^d)$ follows. As in Appendix A, we denote by \mathcal{U} the set of finite all sequences in \mathbb{N} and introduce the subset $\mathcal{U}_{\text{Bin}} \subset \mathcal{U}$ with entries in $\{1, \dots, 2^d\}$ as

$$\mathcal{U}_{\text{Bin}} := \{\mathbf{n} \in \mathcal{U} \mid \mathbf{n}_i \in \{1, \dots, 2^d\} \text{ for } i \in \{1, \dots, |\mathbf{n}|\}\}.$$

Note that $T(\omega) \subset \mathcal{U}_{\text{Bin}}$ holds P -a.s., since $\mathcal{P} = \text{Bin}(2^d, \beta)$. Now let $\mathcal{J}_{d,j}$ be as in (68) and recall from Appendix A.2 that $(j, k) \in \mathcal{J}_T(\omega)$ if and only if there is $\mathbf{n} \in T(\omega)$ such that $(j, k) = (|\mathbf{n}|, \mathcal{J}_{d,|\mathbf{n}|}(|\mathbf{n}|))$. Hence, we may rewrite the series expansion (18) as

$$b_T(\omega) = \sum_{(j,k) \in \mathcal{J}_T(\omega)} \eta_j \sum_{l \in \mathcal{L}_j} X_{j,k}^l(\omega) \psi_{j,k}^l = \sum_{\mathbf{n} \in \mathcal{U}_{\text{Bin}}} \mathbb{1}_{\{\mathbf{n} \in T(\omega)\}} \eta_{|\mathbf{n}|} \sum_{l \in \mathcal{L}_{|\mathbf{n}|}} X_{|\mathbf{n}|, \mathcal{J}_{d,|\mathbf{n}|}(|\mathbf{n}|)}^l(\omega) \psi_{|\mathbf{n}|, \mathcal{J}_{d,|\mathbf{n}|}(|\mathbf{n}|)}^l.$$

As $T : \Omega \rightarrow \mathbb{T}$ is $\mathcal{A}/\mathcal{B}(\mathbb{T})$ -measurable, it holds that $\mathbb{1}_{\{\mathbf{n} \in T(\cdot)\}} : \Omega \rightarrow \{0, 1\}$ is measurable for any fixed $\mathbf{n} \in \mathcal{U}_{\text{Bin}}$. Also, the $X_{j,k}^l$ are real-valued random variables and $\psi_{j,k}^l \in C(\mathbb{T}^d)$ by assumption (recall that we assumed $\Psi \subset C^\alpha$ for some $\alpha \geq 1$ in Subsection 2.1). Thus, $b_T : \Omega \rightarrow C(\mathbb{T}^d)$ is measurable, and strongly measurable as $C(\mathbb{T}^d)$ is separable. \square

More insight in the pathwise regularity of Besov random tree priors, in particular with regard to their Hölder regularity, is obtained by the following result.

Theorem 2.9. Let b_T be a B_p^s -random variable with wavelet density $\beta = 2^{\gamma-d}$ as in Definition 2.6 with $\gamma \in (-\infty, d]$.

1.) It holds that $b_T \in L^q(\Omega; B_q^t)$, and hence $b_T \in B_q^t$ P -a.s., for all $t > 0$ and $q \geq 1$ such that $t < s + \frac{d-\gamma}{q} - \frac{d}{p}$.

2.) Let $s - \frac{d}{p} > 0$ and $t \in (0, s - \frac{d}{p})$. Then there is a $\varepsilon_p > 0$ such that

$$\mathbb{E} \left(\exp \left(\varepsilon \|b\|_{\mathcal{C}^t}^p \right) \right) < \infty, \quad \varepsilon \in (0, \varepsilon_p),$$

In particular, it holds $b_T \in L^q(\Omega; \mathcal{C}^t)$ for any $q \geq 1$.

3.) Let $q \geq 1$ and $s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q} > 0$. For any $t \in (0, s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q})$ it holds $b_T \in L^q(\Omega; \mathcal{C}^t)$.

Proof. 1.) For given $q \geq 1$ and $t > 0$ it holds by (13) that

$$\|b_T\|_{B_q^t}^q = \sum_{(j,k,l) \in \mathcal{I}_T(\omega)} 2^{jq(t + \frac{d}{2} - \frac{d}{q})} \eta_j^q |X_{j,k}^l(\omega)|^q = \sum_{(j,k,l) \in \mathcal{I}_T(\omega)} 2^{jq(t - \frac{d}{q} - s + \frac{d}{p})} |X_{j,k}^l(\omega)|^q.$$

For any given $j \in \mathbb{N}$, by Definition 2.6, the number of nodes $v(j)$ on scale j in the random tree T is binomial distributed (conditional on $v(j-1)$) as

$$v(j) := \#\{k \in K_j | (j, k, l) \in \mathcal{I}_T\} \sim \text{Bin}(2^d v(j-1), 2^{\gamma-d}),$$

with initial value $v(0) = 1$. Now let $(X_{j,m}, j \in \mathbb{N}_0, m \in \mathbb{N})$ be an i.i.d. sequence of p -exponential random variables, independent of $v(j)$ for any $j \in \mathbb{N}$. We obtain by Fubini's theorem, Wald's identity, $\mathbb{E}(v(j)) = (2^d \beta)^j = 2^{j\gamma}$, and since $\mathbb{E}(|X_{j,m}|^q) < \infty$ for any $q > 0$ that

$$\begin{aligned} \mathbb{E} \left(\|b_T\|_{B_q^t}^q \right) &= \mathbb{E} \left(\sum_{j=0}^{\infty} 2^{jq(t - \frac{d}{q} - s + \frac{d}{p})} \sum_{m=1}^{(2^d-1)v(j)} |X_{j,m}|^q \right) + \mathbb{E} (|X_{0,2^d}|^q) \\ &= \sum_{j=0}^{\infty} 2^{jq(t - \frac{d}{q} - s + \frac{d}{p})} \mathbb{E} \left((2^d - 1)v(j) \right) \mathbb{E} (|X_{j,m}|^q) + \mathbb{E} (|X_{0,2^d}|^q) \\ &\leq 2^d \mathbb{E} (|X_{1,1}|^q) \sum_{j=0}^{\infty} 2^{jq(t - \frac{d}{q} - s + \frac{d}{p} + \frac{\gamma}{q})}, \end{aligned}$$

with $\mathbb{E} (|X_{1,1}|^q) < \infty$. The series converges if $t < s + \frac{d-\gamma}{q} - \frac{d}{p}$, in which case $b_T \in L^q(\Omega; B_q^t)$, and hence $b_T \in B_q^t$ holds P -a.s.

2.) Now let $q_0 \geq q \geq 1$, $t_0 > \frac{d}{q_0}$ and $t = t_0 - \frac{d}{q_0}$, so that $B_{q_0}^{t_0} \hookrightarrow \mathcal{C}^t$ holds by (15). The embedding follows by a direct comparison of the norms in (13), (14) with $t = t_0 - \frac{d}{q_0}$, and also shows that the corresponding embedding constant $C_0 > 0$ is bounded by $C_0 \leq 1$.

We obtain with Hölder's inequality and analogously to the first part the estimate

$$\begin{aligned} \|b_T\|_{L^q(\Omega; \mathcal{C}^t)}^q &\leq \mathbb{E} \left(\|b_T\|_{\mathcal{C}^t}^{q_0} \right)^{\frac{q}{q_0}} \\ &\leq \mathbb{E} \left(\|b_T\|_{B_{q_0}^{t_0}}^{q_0} \right)^{\frac{q}{q_0}} \\ &\leq 2^{\frac{dq}{q_0}} \mathbb{E} (|X_{1,1}|^{q_0})^{\frac{q}{q_0}} \left(\sum_{j=0}^{\infty} 2^{jq_0(t_0 - \frac{d}{q_0} - s + \frac{d}{p} + \frac{\gamma}{q_0})} \right)^{\frac{q}{q_0}} \\ &= 2^{\frac{dq}{q_0}} \mathbb{E} (|X_{1,1}|^{q_0})^{\frac{q}{q_0}} \left(\sum_{j=0}^{\infty} 2^{jq_0(t - s + \frac{d}{p} + \frac{\gamma}{q_0})} \right)^{\frac{q}{q_0}}. \end{aligned} \tag{19}$$

Now let $t < s - \frac{d}{p}$ be fixed, and let $\gamma \in (0, d]$. We choose $q_0 = \max\left(2\gamma\left(s - \frac{d}{p} - t\right)^{-1}, q\right)$ to obtain for every $q \geq \max(2\gamma\left(s - \frac{d}{p} - t\right)^{-1}, 1)$ that

$$\begin{aligned} \|b_T\|_{L^q(\Omega; \mathcal{C}^t)}^q &\leq 2^d \mathbb{E}(|X_{1,1}|^q) \left(\sum_{j=0}^{\infty} 2^{jq_0(t-s+\frac{d}{p})/2} \right)^{\frac{q}{q_0}} \\ &\leq 2^d \mathbb{E}(|X_{1,1}|^q) \left(\sum_{j=0}^{\infty} 2^{-j\gamma} \right) \\ &\leq C \mathbb{E}(|X_{1,1}|^q), \end{aligned} \tag{20}$$

with a constant $C > 0$ that is independent of q . We now define for given $\varepsilon > 0$, finite $n \in \mathbb{N}$ and $p \in [1, \infty)$ the random variable

$$E_n(\omega) := \sum_{k=0}^n \frac{(\varepsilon \|b_T(\omega)\|_{\mathcal{C}^t}^p)^k}{k!}, \quad \omega \in \Omega.$$

Clearly, $E_n(\omega) \rightarrow \exp(\varepsilon \|b_T(\omega)\|_{\mathcal{C}^t}^p)$ holds P -a.s. as $n \rightarrow \infty$ and Inequality (20) yields, for any $n \in \mathbb{N}$ and $n_\gamma := \lfloor 2\gamma\left(s - \frac{d}{p} - t\right)^{-1} \rfloor$, that

$$\mathbb{E}(E_n) = \sum_{k=0}^n \frac{\varepsilon^k}{k!} \mathbb{E}(\|b_T\|_{\mathcal{C}^t}^{pk}) \leq \tilde{C} + \sum_{k=n_\gamma}^n \frac{\varepsilon^k}{k!} C^{pk} \mathbb{E}(|X_{1,1}|^{pk}) = \tilde{C} + \mathbb{E} \left(\sum_{k=n_\gamma}^n \frac{(\varepsilon C^p |X_{1,1}|^p)^k}{k!} \right),$$

where $\tilde{C} = \tilde{C}(\gamma, s, d, p, t) > 0$. The monotone convergence theorem then shows that for sufficiently small $\varepsilon > 0$ and $t < s - \frac{d}{p}$, it holds

$$\mathbb{E}(\exp(\varepsilon \|b_T\|_{\mathcal{C}^t}^p)) \leq \tilde{C} + \lim_{n \rightarrow \infty} \mathbb{E} \left(\sum_{k=n_\gamma}^n \frac{(\varepsilon C^p |X_{1,1}|^p)^k}{k!} \right) \leq \tilde{C} + \mathbb{E}(\exp(\varepsilon C^p |X_{1,1}|^p)) < \infty.$$

3.) For $\gamma \in (-\infty, 0]$, $q \geq 1$ and $t \in (0, s - \frac{d}{p} - \frac{\gamma}{q})$, we finally use $q_0 = q$ and $t_0 = t + \frac{d}{q}$ in (19) to obtain that

$$\|b_T\|_{L^q(\Omega; \mathcal{C}^t)} \leq C^q \mathbb{E}(|X_{1,1}|^q) \left(\sum_{j=0}^{\infty} 2^{jq(t-s+\frac{d}{p}+\frac{\gamma}{q})} \right) < \infty.$$

□

Remark 2.10. [22, Theorems 4 and 5] state that $b_T \in B_p^t$ holds P -a.s. for all $t \in (0, s - \gamma/p)$, and that $b_T \notin B_p^{s-\gamma/p}$ occurs with probability $1 - p_\beta > 0$ for $\gamma \in (0, d]$, where p_β is the solution to the equation $p_\beta = ((1 - \beta) + \beta p_\beta)^{2^d}$ (cf. Lemma A.4 in the Appendix.) We emphasize that Theorem 2.9 significantly extends these previous results, as we quantify precisely the regularity of b_T in terms of Besov and Hölder-Zygmund norms.

Recall that we may replace the Hölder-Zygmund spaces \mathcal{C}^t in Theorem 2.9 by the "usual" Hölder spaces C^t if $t \notin \mathbb{N}$ (which is not necessarily true for integer t). Theorem 2.9 shows that a wavelet density $\beta = 2^{\gamma-d} < 1$ improves smoothness in B_q^t , as the upper bound $t < s + \frac{d-\gamma}{q} - \frac{d}{p}$ is decreasing in $\gamma \in (-\infty, d]$. However, given that $\gamma > 0$ we may not expect to gain any (pathwise) Hölder regularity beyond $t < s - \frac{d}{p}$. This is not surprising with regard to Remark 2.7: b_T admits an infinite series expansion on random fractals in \mathcal{D} for $\beta > 2^{-d}$ with positive probability. Hence, the local Hölder-regularity of b_T on such fractals corresponds to a B_p^s -random variable b as in Definition 2.3 (with full wavelet density $\beta = 1$). In case that $\gamma \leq 0$, the series expansion of b_T

terminates almost surely after a finite number of terms. This may also increase the smoothness exponent t for $b_T \in L^q(\Omega; \mathcal{C}^t)$ to the admissible range $t < s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q}$. For large q , we see that essentially the restriction $t < s - \frac{d}{p}$ applies as for $\gamma > 0$. This in turn indicates that the bound

$$\mathbb{E}(\exp(\varepsilon \|b\|_{\mathcal{C}^t}^p)) < \infty, \quad \varepsilon \in (0, \varepsilon_p),$$

from part 2.) of Theorem 2.9 can not be improved to Hölder indices $t \geq s - \frac{d}{p}$, even if $\gamma \leq 0$.

3 Elliptic PDEs with Besov Coefficients

In this section, we first recall well-posedness and regularity results for linear, second order elliptic diffusion problems with random coefficient. Thereafter, we transfer the results to a setting with Besov tree random diffusion coefficient by exploiting the results from Section 2.

3.1 Well-posedness and regularity

Let $\mathcal{D} \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$ be a convex polygonal domain, with the boundary $\partial\mathcal{D}$ consisting of a finite number of line or plane segments. We consider the random elliptic problem to find $u(\omega) : \mathcal{D} \rightarrow \mathbb{R}$ for given $\omega \in \Omega$ such that

$$-\nabla \cdot (a(\omega) \nabla u(\omega)) = f \quad \text{in } \mathcal{D}, \quad u(\omega) = 0 \quad \text{on } \partial\mathcal{D}. \quad (21)$$

The diffusion coefficient a in Problem (21) admits positive paths on \mathcal{D} , i.e., $a(\omega) : \mathcal{D} \rightarrow \mathbb{R}_{>0}$. Moreover, a is a random variable $a : \Omega \rightarrow \mathcal{X}$, taking values in a suitable Banach space $\mathcal{X} \subset L^\infty(\mathcal{D})$. The source term $f : \mathcal{D} \rightarrow \mathbb{R}$ is assumed to be a deterministic function for the sake of simplicity, but may as well be modeled by a random function $f : \mathcal{D} \times \Omega \rightarrow \mathbb{R}$. For the variational formulation of Problem (21) we define $H := L^2(\mathcal{D})$, $V := H_0^1(\mathcal{D})$ and recall that $\|\cdot\|_V : V \rightarrow \mathbb{R}_{\geq 0}$, $v \mapsto \|\nabla v\|_H$ defines a norm on V by Poincaré's inequality. The weak formulation of Problem (21) for fixed $\omega \in \Omega$ is to find $u(\omega) \in V$ such that for any $v \in V$ it holds

$$\int_{\mathcal{D}} a(\omega) \nabla u(\omega) \cdot \nabla v dx = {}_{V'} \langle f, v \rangle_V. \quad (22)$$

Definition 3.1. The map $\omega \mapsto u(\omega) \in V$ with $u(\omega)$ the solution of (22) is the *pathwise weak solution*.

Existence and uniqueness of pathwise weak solutions are ensured by the following theorem.

Theorem 3.2. Let $a : \Omega \rightarrow L^\infty(\mathcal{D})$ be strongly $\mathcal{A}/\mathcal{B}(L^\infty(\mathcal{D}))$ -measurable such that

$$a_-(\omega) := \operatorname{ess\,inf}_{x \in \mathcal{D}} a(x, \omega) > 0, \quad P\text{-a.s.}, \quad (23)$$

and let $f \in V'$. Then, there exists for all $\omega \in \Omega$ a unique weak solution $u(\omega) \in V$ to Problem (21). The map $u : \Omega \rightarrow V$ is strongly $\mathcal{A}/\mathcal{B}(V)$ -measurable.

Proof. By the completeness of (Ω, \mathcal{A}, P) , we may assume without loss of generality that $a_-(\omega) > 0$ and $a(\omega) \in L^\infty(\mathcal{D})$ holds for all $\omega \in \Omega^2$.

Existence and uniqueness of a pathwise solution $u(\omega)$ now follows for all $\omega \in \Omega$ by the Lax-Milgram Lemma. To show measurability of u , consider two diffusion coefficients $a_1, a_2 : \Omega \rightarrow$

²If this holds only P -a.s., we may modify $a : \Omega \rightarrow L^\infty(\mathcal{D})$ on a P -nullset to obtain a strongly measurable modification $\tilde{a} : \Omega \rightarrow L^\infty(\mathcal{D})$ of a , so that $\operatorname{ess\,inf}_{x \in \mathcal{D}} \tilde{a}(x, \omega) > 0$ and $\tilde{a} \in L^\infty(\mathcal{D})$ holds for all $\omega \in \Omega$. In fact, let

$$A_0 := \{\omega \in \Omega \mid a_-(\omega) \leq 0 \text{ or } a(\omega) \notin L^\infty(\mathcal{D})\}.$$

Then $P(A_0) = 0$ by assumption, and hence $A_0 \in \mathcal{A}$ by completeness of (Ω, \mathcal{A}, P) . Thus, we may consider, for instance, the modification $\tilde{a}(\omega) := a(\omega) \mathbb{1}_{\{\omega \notin A_0\}} + \mathbb{1}_{\{\omega \in A_0\}}$. It is readily verified that \tilde{a} is strongly $\mathcal{A}/\mathcal{B}(L^\infty(\mathcal{D}))$ -measurable, and for all $\omega \in \Omega$ it holds $\operatorname{ess\,inf}_{x \in \mathcal{D}} \tilde{a}(x, \omega) > 0$ and $\tilde{a} \in L^\infty(\mathcal{D})$.

$L^\infty(\mathcal{D})$ that satisfy the assumption of the theorem with lower bounds $a_{1,-}, a_{2,-} > 0$ as in (23) and denote by $u_1, u_2 : \Omega \rightarrow V$ the associated unique weak solutions. Equation (21) together with $\|v\|_V^2 = \|\nabla v\|_H^2$ and Hölder's inequality yields for any fixed coefficients $a_1, a_2 \in L^\infty(\mathcal{D})$ such that $a_{i,-} := \text{ess inf}_{x \in \mathcal{D}} a_i(x) > 0$ that

$$\|u_1 - u_2\|_V \leq \frac{\|u_2\|_V}{a_{1,-}} \|a_1 - a_2\|_{L^\infty(\mathcal{D})} \leq \frac{\|f\|_{V'}}{a_{1,-} a_{2,-}} \|a_1 - a_2\|_{L^\infty(\mathcal{D})}. \quad (24)$$

Therefore, the data-to-solution map $U : S \rightarrow V$, $a \mapsto u$ is (Lipschitz) continuous on the set $S := \{a \in L^\infty(\mathcal{D}) \mid \text{ess inf}_{x \in \mathcal{D}} a(x) > 0\}$. Since the pathwise weak solution $u : \Omega \rightarrow V$ of (21) may be written as $u = U \circ a$, the claim follows with the strong $\mathcal{A}/\mathcal{B}(L^\infty(\mathcal{D}))$ -measurability of $a : \Omega \rightarrow L^\infty(\mathcal{D})$. \square

Lipschitz continuity (24) of the data-to-solution map will be essential in deriving error estimates in Section 4 ahead, and also implies strong measurability of random solutions.

Proposition 3.3. *Let $a_1, a_2 : \Omega \rightarrow L^\infty(\mathcal{D})$ be strongly $\mathcal{A}/\mathcal{B}(L^\infty(\mathcal{D}))$ -measurable such that*

$$a_{i,-}(\omega) := \text{ess inf}_{x \in \mathcal{D}} a_i(x, \omega) > 0, \quad P\text{-a.s. for } i \in \{1, 2\}. \quad (25)$$

Then, for every $f \in V'$ exists for $i \in \{1, 2\}$ and for all $\omega \in \Omega$ a unique weak solution $u_i(\omega) \in V$ to Problem (21) (with a in place of a_i). There holds the continuous-dependence estimate

$$\|u_1 - u_2\|_V \leq \frac{\|f\|_{V'}}{a_{1,-} a_{2,-}} \|a_1 - a_2\|_{L^\infty(\mathcal{D})}.$$

Proof. This follows immediately with Theorem 3.2 and (24). \square

From the regularity analysis of deterministic linear elliptic problems it is well known that $H^s(\mathcal{D})$ -regularity of u may be derived for certain $s > 1$, provided that a is Hölder continuous. The corresponding estimates usually do not reveal the explicit dependence of constants on $a(\omega)$ or bounds on the Hölder norm $\|a(\omega)\|_{C^\iota}$. For the stochastic problem and the ensuing numerical analysis in Sections 4 and 5, however, we need the explicit dependence for given ω to ensure that all pathwise estimates also hold in $L^q(\Omega; H^s(\mathcal{D}))$ for suitable $q \geq 1$. To obtain explicit estimates, we follow the approach from [12, Chapter 3.3] for parametric elliptic PDEs, where regularity estimates are derived via the K-method of function space interpolation.³ One obtains in particular Hölder spaces $C^r(\overline{\mathcal{D}})$ by interpolation ([3, Lemma 7.36]):

$$C^r(\overline{\mathcal{D}}) = [L^\infty(\mathcal{D}), W^{1,\infty}(\mathcal{D})]_{r,\infty}, \quad r \in (0, 1).$$

To investigate spatial regularity of solutions to (21), we introduce the normed space

$$W := \{v \in V \mid \Delta v \in H\}, \quad \|v\|_W := \|\Delta v\|_H. \quad (26)$$

³Recall the K-method of interpolation of two Banach spaces $(A_0, \|\cdot\|_{A_0})$ and $(A_1, \|\cdot\|_{A_1})$ with continuous embedding $A_1 \hookrightarrow A_0$: their K-functional is defined by

$$K(a, z; A_0, A_1) := \inf_{a_1 \in A_1} \{\|a - a_1\|_{A_0} + z\|a_1\|_{A_1}\}, \quad a \in A_0, z > 0.$$

For any $r \in (0, 1)$ and $q \in [1, \infty]$ the interpolation space of order r with fine index q is

$$[A_0, A_1]_{r,q} = \left\{ a \in A_0 \mid \|a\|_{[A_0, A_1]_{r,q}} < \infty \right\},$$

where

$$\|a\|_{[A_0, A_1]_{r,q}} := \begin{cases} \left(\int_0^\infty z^{-rq} K(a, z; A_0, A_1)^q \frac{1}{z} dz \right)^{\frac{1}{q}}, & q \in [1, \infty), \\ \sup_{z>0} z^{-r} K(a, z; A_0, A_1), & q = \infty. \end{cases}$$

The set $[A_0, A_1]_{r,q}$ is a (generally non-separable) Banach space.

Note that $v = 0 \Leftrightarrow \|v\|_W = 0$ follows by the maximum principle, since $v \in V = H_0^1(\mathcal{D})$ has vanishing trace. We formulate regularity results in terms of the interpolation space

$$W^r := [V, W]_{r, \infty}, \quad r \in (0, 1). \quad (27)$$

For a concise notation, we further set $W^1 := W$ in the following.

Lemma 3.4. [12, Propositions 3.2 and 3.5] *Let $a : \Omega \rightarrow C^r(\overline{\mathcal{D}}) \subset L^\infty(\mathcal{D})$ be strongly measurable for some $r \in (0, 1]$ such that $a_-(\omega) > 0$ holds P -a.s. and let $f \in H$. Then, there is a constant $C = C(r, \mathcal{D})$, such that it holds*

$$\|u(\omega)\|_{W^r} \leq \frac{C}{a_-(\omega)} \left(1 + \left(\frac{\|a(\omega)\|_{C^r(\overline{\mathcal{D}})}}{a_-(\omega)} \right)^{1/r} \right) \|f\|_H. \quad (28)$$

All results from this subsection so far hold under the considerably weaker assumption that $\mathcal{D} \subset \mathbb{R}^d$ is a bounded Lipschitz domain. However, since \mathcal{D} is assumed convex, we are able to embed W^r in (fractional) Sobolev spaces. This is made precise in the following Lemma, which is in required for the finite element error analysis in Section 4.2.

Lemma 3.5. *Let \mathcal{D} be convex, $W^r := [V, W]_{r, \infty}$ for $r \in (0, 1)$ and let $W^1 := W$. Then, it holds that $W = W^1 \hookrightarrow H^2(\mathcal{D})$. Moreover, $W^r \hookrightarrow H^{1+r_0}(\mathcal{D})$ for any $r_0 \in (0, r)$.*

Proof. By convexity of \mathcal{D} , we have that $\|v\|_{H^2(\mathcal{D})} \leq C_{\mathcal{D}} \|v\|_W$ holds for all $v \in W$, where $C_{\mathcal{D}}$ only depends on the diameter of \mathcal{D} , see, e.g., [15, Theorem 3.2.1.2]. Thus, $W \hookrightarrow H^2(\mathcal{D}) \cap V$ follows.

For the case $r \in (0, 1)$, we recall that there is $C_V > 0$, such that $\|v\|_{H^1(\mathcal{D})} \leq C_V \|v\|_V$ holds for all $v \in V$ by Poincaré's inequality. Moreover, we have $\|w\|_{H^2(\mathcal{D})} \leq C_{\mathcal{D}} \|w\|_W$ for any $w \in W$, and hence $W \subset H^2 \cap V$ from the first part. For $v \in V \subset H^1$ this yields

$$\begin{aligned} \|v\|_{[H^1(\mathcal{D}), H^2(\mathcal{D})]_{r, \infty}} &= \sup_{z > 0} z^{-r} \inf_{w \in H^2} \{ \|v - w\|_{H^1(\mathcal{D})} + \|w\|_{H^2(\mathcal{D})} \} \\ &\leq \sup_{z > 0} z^{-r} \inf_{w \in W} \{ \|v - w\|_{H^1(\mathcal{D})} + \|w\|_{H^2(\mathcal{D})} \} \\ &\leq \sup_{z > 0} z^{-r} \inf_{w \in W} \{ C_V \|v - w\|_V + C_{\mathcal{D}} \|w\|_W \} \\ &\leq \max(C_V, C_{\mathcal{D}}) \|v\|_{W^r}. \end{aligned}$$

Hence, $W^r \hookrightarrow [H^1(\mathcal{D}), H^2(\mathcal{D})]_{r, \infty}$. The claim now follows since for any $\varepsilon \in (0, 1 + r)$ there holds

$$[H^1(\mathcal{D}), H^2(\mathcal{D})]_{r, \infty} = [H^{1+r-\varepsilon}(\mathcal{D}), H^{1+r+\varepsilon}(\mathcal{D})]_{\frac{1}{2}, \infty} \hookrightarrow H^{1+r-\varepsilon}(\mathcal{D}),$$

see [3, Section 7.32]. □

3.2 Besov random tree priors as log-diffusion coefficient

To formulate Problem (21) with a Besov random tree coefficient, we assume that $\mathcal{D} \subseteq \mathbb{T}^d$. We follow [25, Section 2] and define, for given $\omega \in \Omega$, the random element $b_T(\omega) : \mathcal{D} \rightarrow \mathbb{R}$ as the restriction of a periodic function in $B_{p,p}^{s,per}(\mathbb{T}^d)$ to the domain \mathcal{D} . The restriction $\varphi|_{\mathcal{D}}$ of any $\varphi \in S'(\mathbb{R}^d)$ to \mathcal{D} is in turn given by the element $\varphi|_{\mathcal{D}} \in D'(\mathcal{D})$ such that

$$D'(\mathcal{D}) \langle \varphi|_{\mathcal{D}}, v \rangle_{D(\mathcal{D})} = S'(\mathbb{R}^d) \langle \varphi, v_0 \rangle_{S(\mathbb{R}^d)}, \quad v \in D(\mathcal{D}),$$

where $v_0 \in D(\mathbb{R}^d) \subset S(\mathbb{R}^d)$ denotes the zero-extension of any $v \in D(\mathcal{D})$.

Definition 3.6. Let $\mathcal{D} \subseteq \mathbb{T}^d$ be a bounded, connected domain. Let b_T be given in Definition 2.6 for $p \in [1, \infty)$, $s > 0$ and $\beta = 2^{\gamma-d} \in [0, 1]$, and let $\text{prl}^{per} : B_{p,p}^s(\mathbb{T}^d) \rightarrow B_{p,p}^{s,per}(\mathbb{R}^d)$ denote the isomorphic extension operator from (11). Then we define for any $\omega \in \Omega$

$$b_{T,\mathcal{D}}(\omega) := (\text{prl}^{per} b_T(\omega))|_{\mathcal{D}},$$

and call $b_{T,\mathcal{D}}$ a $B_p^s(\mathcal{D})$ -valued random variable.

Remark 3.7. In case that $\mathcal{D} = \mathbb{T}^d$, we may readily use the identification $b_{T,\mathcal{D}} = b_T$. Note that $b_{T,\mathcal{D}}$ is periodic in this case, in the sense that there exists an extension $\text{prl}^{\text{per}} b_{T,\mathcal{D}} \in B_{p,p}^{s,\text{per}}(\mathbb{R}^d)$. If $\mathcal{D} \subsetneq \mathbb{T}^d$, however, $b_{T,\mathcal{D}}$ is not (necessarily) periodic, but *merely the restriction of a periodic function* from the torus \mathbb{T}^d .

Remark 3.8. The same procedure could be applied for general bounded domains $\mathcal{D} \subsetneq \mathbb{T}^d$, by extending Definition (2.6) from the torus \mathbb{T}^d to a sufficiently large (periodic) domain $[-L, L]^d$ for $L > 1$. This would increase the index-set K_j of wavelet coefficients by at most a constant factor on each dyadic scale j . However, all regularity proofs from Section 2 are carried out similar in this setting, with minor changes to absolute constants. For instance, the admissible range of ε in Proposition 2.4 may become smaller if $L > 1$, but the smoothness parameter $t \in (0, s - \frac{d}{p})$ is unaffected. Therefore, assuming $\mathcal{D} \subseteq \mathbb{T}^d$ for the sake of brevity does not have any substantial impact on the following results.

We consider Problem (21), resp. its weak formulation (22), with $a(\omega) := \exp(b_T(\omega))$, where b_T is a B_p^s -random variable with wavelet density β . That is, we model the log-diffusion by a Besov random tree prior to incorporate fractal structures. With this preparation, we are able to derive well-posedness and regularity of the corresponding pathwise weak solution.

Theorem 3.9. *Let $a := \exp(b_{T,\mathcal{D}})$ with $b_{T,\mathcal{D}}$ given in Definition 3.6 for $p \in [1, \infty)$, $s > 0$ and $\beta = 2^{\gamma-d} \in [0, 1]$, so that $sp > d$. Furthermore, let $f \in V'$.*

- 1.) *Then, there exists almost surely a unique weak solution $u(\omega) \in V$ to (21) and $u : \Omega \rightarrow V$ is strongly measurable.*
- 2.) *For sufficiently small $\kappa > 0$ in (16), there are constants $\bar{q} \in (1, \infty)$ and $C > 0$ such that*

$$\|u\|_{L^q(\Omega; V)} \leq C \|f\|_{V'} < \infty \quad \begin{cases} \text{for } q \in [1, \bar{q}) \text{ if } p = 1, \text{ and} \\ \text{for any } q \in [1, \infty) \text{ if } p > 1. \end{cases}$$

- 3.) *Let $r \in (0, s - \frac{d}{p}) \cap (0, 1]$, $f \in H$ and W^r as in (27). For sufficiently small $\kappa > 0$ in (16), there are constants $\bar{q} \in (1, \infty)$ and $C > 0$ such that*

$$\|u\|_{L^q(\Omega; W^r)} \leq C \|f\|_H < \infty \quad \begin{cases} \text{for } q \in [1, \bar{q}) \text{ if } p = 1 \text{ and} \\ \text{for any } q \in [1, \infty) \text{ if } p > 1. \end{cases}$$

Proof. 1.) As $sp > d$, Theorem 2.9 shows that $b_T \in C(\mathbb{T}^d)$ holds P -a.s. Moreover, $b_T : \Omega \rightarrow C(\mathbb{T}^d)$ is strongly measurable by Proposition 2.8, and thus in particular strongly $\mathcal{A}/\mathcal{B}(L^\infty(\mathbb{T}^d))$ -measurable, since $\mathcal{B}(C(\mathbb{T}^d)) \subset \mathcal{B}(L^\infty(\mathbb{T}^d))$. As $b_{T,\mathcal{D}}$ in Definition 3.6 is the restriction of $\text{ext}^{\text{per}} b_T$ to $\mathcal{D} \subset \mathbb{T}^d$, and $a = \exp \circ b_{T,\mathcal{D}}$, it follows that, $a : \Omega \rightarrow C(\bar{\mathcal{D}})$, and a is strongly $\mathcal{A}/\mathcal{B}(L^\infty(\mathcal{D}))$ -measurable such that $a_- > 0$ holds P -a.s. Theorem 3.2 then guarantees the P -a.s. existence of a unique pathwise weak solution $u(\omega)$. Moreover, $u : \Omega \rightarrow V$ is strongly measurable and Equation (22) shows that

$$\|u(\omega)\|_V \leq \frac{\|f\|_{V'}}{a_-(\omega)}.$$

2.) To show the second part, we fix $t \in (0, s - \frac{d}{p})$ and $q \geq 1$ to see that

$$\begin{aligned}
\|u\|_{L^q(\Omega; V)}^q &\leq \mathbb{E}(a_-^{-q}) \|f\|_V^q, \\
&= \mathbb{E}\left(\left(\operatorname{ess\,inf}_{x \in \mathcal{D}} \exp(-b_{T, \mathcal{D}}(x))\right)^q\right) \|f\|_V^q, \\
&= \mathbb{E}\left(\exp\left(\operatorname{ess\,inf}_{x \in \mathcal{D}} -qb_{T, \mathcal{D}}(x)\right)\right) \|f\|_V^q, \\
&\leq \mathbb{E}\left(\exp\left(\operatorname{ess\,sup}_{x \in \mathbb{T}^d} qb_T(x)\right)\right) \|f\|_V^q, \\
&= \mathbb{E}\left(\exp(q\|b_T\|_{L^\infty(\mathbb{T}^d)})\right) \|f\|_V^q, \\
&\leq \mathbb{E}\left(\exp(q\|b_T\|_{\mathcal{C}^t})\right) \|f\|_V^q.
\end{aligned}$$

We have used that $\exp(\cdot)$ is strictly increasing for the second equality, and that $b_T(x)$ is a centered random variable such that b_T and $-b_T$ are equal in distribution. The last estimate is due to $\|b_T\|_{L^\infty(\mathbb{T}^d)} \leq \|b_T\|_{\mathcal{C}^t}$ for any $t > 0$. For $p = 1$, we note that ε_p in the second part of Theorem 2.9 may be chosen as $\varepsilon_p = (\kappa C)^{-1}$, where $C > 0$ is the constant in (20). Therefore, for sufficiently small $\kappa > 0$, we may set $\bar{q} := \varepsilon_p > 1$ in the claim. In case that $p > 1$, Young's inequality shows that for any $q \geq 1$ there is an arbitrary small $\varepsilon > 0$ and a constant $C_\varepsilon \in (0, \infty)$ such that

$$q\|b_T\|_{\mathcal{C}^t} \leq \varepsilon\|b_T\|_{\mathcal{C}^t}^p + C_\varepsilon.$$

Thus, we have no restrictions on $q \in [1, \infty)$, which proves the second part of the claim.

3.) For any fixed $r \in (0, s - \frac{d}{p}) \cap (0, 1]$ and $t \in (0, s - \frac{d}{p})$ Lemma 3.4 shows that

$$\begin{aligned}
\|u\|_{L^q(\Omega; W^r)}^q &\leq C^q \mathbb{E}\left[\exp\left(\operatorname{ess\,inf}_{x \in \mathcal{D}} -qb_T(x)\right) \left(1 + \|b_{T, \mathcal{D}}\|_{\mathcal{C}^r(\overline{\mathcal{D}})}^{\frac{q}{r}} \exp\left(\operatorname{ess\,inf}_{x \in \mathcal{D}} -\frac{q}{r}b_{T, \mathcal{D}}(x)\right)\right)\right] \|f\|_H^q \\
&\leq C^q \mathbb{E}\left(\exp(qq_1\|b_{T, \mathcal{D}}\|_{L^\infty(\mathcal{D})})\right)^{\frac{1}{q_1}} \\
&\quad \cdot \left[1 + \mathbb{E}\left(\exp\left(\frac{qq_2}{r}\|b_{T, \mathcal{D}}\|_{L^\infty(\mathcal{D})}\right)\right)^{\frac{1}{q_2}} \mathbb{E}\left(\exp\left(\frac{qq_3}{r}\|b_{T, \mathcal{D}}\|_{\mathcal{C}^r(\overline{\mathcal{D}})}\right)\right)^{\frac{1}{q_3}}\right] \|f\|_H^q \\
&\leq C^q \left[\mathbb{E}\left(\exp(qq_1\|b_T\|_{\mathcal{C}^t})\right)^{\frac{1}{q_1}} + \mathbb{E}\left(\exp\left(q\max(q_1, \frac{q_2}{r}, \frac{q_3}{r})\|b_T\|_{\mathcal{C}^r}\right)\right)\right] \|f\|_H^q,
\end{aligned} \tag{29}$$

where we have used Hölder's inequality with $q_1, q_2, q_3 > 1$ such that $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$, together with $\mathbb{E}(a_-^{-q}) = \mathbb{E}(\exp(q\|b_T\|_{L^\infty(\mathcal{D})}))$ in the second step. In the last estimate, we used $\|b_{T, \mathcal{D}}\|_{L^\infty(\mathcal{D})} \leq \|b_{T, \mathcal{D}}\|_{\mathcal{C}^t}$ and $\|b_T\|_{\mathcal{C}^r(\overline{\mathcal{D}})} \leq \|b_T\|_{\mathcal{C}^r}$ for any $r > 0$, as well as $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} = 1$.

To bound the Hölder-norm $\|b_T\|_{\mathcal{C}^r}$ in (29), we first consider the case $r < 1$. Then, we recall from Subsection 2.1 that $\mathcal{C}^r = \mathcal{C}^r$ with equivalent norms, thus $\|b_T\|_{\mathcal{C}^r} \leq C\|b_T\|_{\mathcal{C}^r}$. If $r = 1$, then $s - \frac{d}{p} > 1$, and we use the same argument to derive the bound $\|b_T\|_{\mathcal{C}^r} \leq \|b_T\|_{\mathcal{C}^{r+\varepsilon}} \leq C\|b_T\|_{\mathcal{C}^{r+\varepsilon}}$ for any $\varepsilon \in (0, s - \frac{d}{p} - 1)$.

For $p = 1$, Theorem 2.9 now shows again that for sufficiently small $\kappa > 0$, there are admissible choices $q, q_1, q_2, q_3 \in [1, \infty)$, dependent on r , such that the right hand side in (29) is finite. The proof is concluded by noting that $q \in [1, \infty)$ may again be arbitrary large in (29) if $p > 1$, independent of r . \square

4 Pathwise Finite Element Approximation

4.1 Dimension truncation

To obtain a tractable approximation of b_T in (18), we truncate the wavelet series expansion after $N \in \mathbb{N}$ scales to obtain the *truncated random tree Besov prior*

$$b_{T,N}(\omega) := \sum_{\substack{(j,k,l) \in \mathcal{I}_T(\omega) \\ j \leq N}} \eta_j X_{j,k}^l(\omega) \psi_{j,k}^l, \quad \omega \in \Omega. \quad (30)$$

The corresponding diffusion problem in weak form with truncated coefficient for fixed $\omega \in \Omega$ is to find $u_N(\omega) \in V$ such that for all $v \in V$

$$\int_{\mathcal{D}} a_N(\omega) \nabla u_N(\omega) \cdot \nabla v dx =_{V'} \langle f, v \rangle_V, \quad (31)$$

where

$$a_N : \Omega \rightarrow L^\infty(\mathcal{D}), \quad \omega \mapsto \exp(b_{T,N}(\omega)|_{\mathcal{D}}). \quad (32)$$

Existence, uniqueness, and regularity of u_N follows analogously as for u in the previous section.

Corollary 4.1. *Let $N \in \mathbb{N}$, $a_N = \exp(b_{T,N}|_{\mathcal{D}})$ with $b_{T,N}$ be given as in (31) for $p \in [1, \infty)$, $s > 0$ and $\beta = 2^{\gamma-d} \in [0, 1]$, so that $sp > d$. Furthermore, let $f \in V'$. Then the following holds.*

- 1.) *There exists almost surely a unique weak solution $u_N(\omega) \in V$ to the truncated Problem (31) and $u_N : \Omega \rightarrow V$ is strongly measurable.*
- 2.) *For sufficiently small $\kappa > 0$ in (16), there are constants $\bar{q} \in (1, \infty)$ and $C > 0$ such that for any $N \in \mathbb{N}$*

$$\|u_N\|_{L^q(\Omega; V)} \leq C \|f\|_{V'} < \infty \quad \begin{cases} \text{for } q \in [1, \bar{q}] \text{ if } p = 1, \text{ and} \\ \text{for any } q \in [1, \infty) \text{ if } p > 1. \end{cases}$$

- 3.) *Let $r \in (0, s - \frac{d}{p}) \cap (0, 1]$ and $f \in H$. There are constants $\bar{q} \in (1, \infty)$ and $C > 0$ such that for any $N \in \mathbb{N}$*

$$\|u_N\|_{L^q(\Omega; W^r)} \leq C \|f\|_H < \infty \quad \begin{cases} \text{for } q \in [1, \bar{q}] \text{ if } p = 1 \text{ and} \\ \text{for any } q \in [1, \infty) \text{ if } p > 1. \end{cases}$$

Proof. The result follows analogously to Theorems 2.9 and 3.9, upon observing that $\|b_{T,N}(\omega)\|_{B_q^t} \leq \|b_T(\omega)\|_{B_q^t}$ holds P -a.s. for any $t > 0$, $q \in [1, \infty]$, and $N \in \mathbb{N}$. \square

The important observation from Corollary 4.1 is that the bounds are independent of N , which is crucial when estimating the finite element discretization error of u_N in the next subsection. We bound the truncation errors $a - a_N$ and $u - u_N$ in the remainder of this section.

Proposition 4.2. *Let $a := \exp(b_{T,\mathcal{D}})$ with $b_{T,\mathcal{D}}$ as given in Definition 3.6 with $p \in (1, \infty)$, $s > 0$, $\beta = 2^{\gamma-d} \in [0, 1]$ and such that $sp > d + \min(\gamma, 0)$. Let $b_{T,N}$ and a_N be the approximations of b_T and a for given $N \in \mathbb{N}$ as in (31) and (32), respectively.*

- 1.) *For any $q \geq 1$ and $t \in (0, s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q})$ there is a constant $C > 0$ such that for every $N \in \mathbb{N}$ it holds*

$$\|b_{T,\mathcal{D}} - b_{T,N}|_{\mathcal{D}}\|_{L^q(\Omega; C^t(\bar{\mathcal{D}}))} \leq C 2^{N(t - s + \frac{d}{p} + \frac{\min(\gamma, 0)}{q})}.$$

2.) Moreover, for any $q \geq 1$, $\varepsilon > 0$ and $t \in (0, s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q})$ there is a $C > 0$ such that for every $N \in \mathbb{N}$ it holds

$$\|a - a_N\|_{L^q(\Omega; \mathcal{C}^t(\overline{\mathcal{D}}))} \leq C 2^{N(t-s+\frac{d}{p}+\frac{\min(\gamma+\varepsilon, 0)}{q})}.$$

Proof. 1.) Let $q_0 \geq q$, $t_0 > \frac{d}{q_0}$ and $t = t_0 - \frac{d}{q_0}$, so that $B_{q_0}^{t_0} \hookrightarrow \mathcal{C}^t$. For any fixed $N \in \mathbb{N}$, we obtain with Hölder's inequality analogously to the proof of Theorem 2.9 the estimate

$$\begin{aligned} \|b_{T, \mathcal{D}} - b_{T, N}|_{\mathcal{D}}\|_{L^q(\Omega; \mathcal{C}^t(\overline{\mathcal{D}}))} &\leq \|b_T - b_{T, N}\|_{L^q(\Omega; \mathcal{C}^t)} \\ &\leq \mathbb{E} \left(\|b_T - b_{T, N}\|_{\mathcal{C}^t}^{q_0} \right)^{\frac{1}{q_0}} \\ &\leq \mathbb{E} \left(\|b_T - b_{T, N}\|_{B_{q_0}^{t_0}}^{q_0} \right)^{\frac{1}{q_0}} \\ &\leq \left(\sum_{j=N+1}^{\infty} 2^{jq_0(t_0 - \frac{d}{q_0} - s + \frac{d}{p} + \frac{\gamma}{q_0})} \right)^{\frac{1}{q_0}} \\ &= 2^{N(t-s+\frac{d}{p}+\frac{\gamma}{q_0})} \left(\sum_{j=1}^{\infty} 2^{jq_0(t-s+\frac{d}{p}+\frac{\gamma}{q_0})} \right)^{\frac{1}{q_0}}. \end{aligned} \quad (33)$$

Now let $t < s - \frac{d}{p}$ and $\gamma \in (0, d]$ in (33), and choose $q_0 = \max\left(\gamma N, 2\gamma(s - \frac{d}{p} - t)^{-1}, q\right)$ (for sufficiently large, given N) to obtain that

$$\begin{aligned} \|b_{T, \mathcal{D}} - b_{T, N}|_{\mathcal{D}}\|_{L^q(\Omega; \mathcal{C}^t(\overline{\mathcal{D}}))} &\leq 2^{N(t-s+\frac{d}{p})+1} \left(\sum_{j=1}^{\infty} 2^{jq_0(t-s+\frac{d}{p})\frac{1}{2}} \right)^{\frac{1}{q_0}} \\ &\leq 2^{N(t-s+\frac{d}{p})} 2 \left(\sum_{j=1}^{\infty} 2^{j(t-s+\frac{d}{p})\frac{1}{2}} \right). \end{aligned} \quad (34)$$

The final bound in (34) is independent of $q_0 = q_0(N)$, which shows the first part of the claim for $\gamma \in (0, d]$. For $\gamma \in (-\infty, 0]$, we use $q_0 = q$ in (33) to obtain for any $t \in (0, s - \frac{d}{p} - \frac{\gamma}{q})$ that

$$\|b_{T, \mathcal{D}} - b_{T, N}|_{\mathcal{D}}\|_{L^q(\Omega; \mathcal{C}^t(\overline{\mathcal{D}}))} \leq 2^{N(t-s+\frac{d}{p}+\frac{\gamma}{q})} \left(\sum_{j=1}^{\infty} 2^{jq(t-s+\frac{d}{p}+\frac{\gamma}{q})} \right)^{\frac{1}{q}} \leq 2^{N(t-s+\frac{d}{p}+\frac{\gamma}{q})}. \quad (35)$$

2.) To prove the second part, we take $t \in (0, s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q})$ and use Taylor's expansion together with Hölder's inequality for exponents $p_1, p_2 > 1$ such that $\frac{1}{p_1} + \frac{1}{p_2} = 1$ to obtain

$$\begin{aligned} \|a - a_N\|_{L^q(\Omega; \mathcal{C}^t(\overline{\mathcal{D}}))} &\leq \mathbb{E} \left(\exp \left(q(\|b_{T, \mathcal{D}}\|_{L^\infty(\mathcal{D})} + \|b_{T, N}|_{\mathcal{D}}\|_{\mathcal{C}^t(\overline{\mathcal{D}})}) \right) \|b_{T, \mathcal{D}} - b_{T, N}|_{\mathcal{D}}\|_{\mathcal{C}^t(\overline{\mathcal{D}})}^q \right)^{\frac{1}{q}} \\ &\leq \mathbb{E} \left(\exp \left(p_1 q(\|b_{T, \mathcal{D}}\|_{\mathcal{C}^t(\overline{\mathcal{D}})} + \|b_{T, N}|_{\mathcal{D}}\|_{\mathcal{C}^t(\overline{\mathcal{D}})}) \right) \right)^{(qp_1)^{-1}} \\ &\quad \cdot \mathbb{E} \left(\|b_{T, \mathcal{D}} - b_{T, N}|_{\mathcal{D}}\|_{\mathcal{C}^t(\overline{\mathcal{D}})}^{p_2 q} \right)^{(qp_2)^{-1}} \\ &\leq \mathbb{E} \left(\exp(2p_1 q \|b_T\|_{\mathcal{C}^t}) \right)^{(qp_1)^{-1}} \|b_{T, \mathcal{D}} - b_{T, N}|_{\mathcal{D}}\|_{L^{p_2 q}(\Omega; \mathcal{C}^t(\overline{\mathcal{D}}))} \\ &\leq C \mathbb{E} \left(\exp(2p_1 q \|b_T\|_{\mathcal{C}^t}) \right)^{(qp_1)^{-1}} 2^{N(t-s+\frac{d}{p}+\frac{\min(\gamma, 0)}{p_2 q})} \end{aligned}$$

We have used $\|b_{T, N}|_{\mathcal{D}}\|_{\mathcal{C}^t(\overline{\mathcal{D}})} \leq \|b_{T, \mathcal{D}}\|_{\mathcal{C}^t(\overline{\mathcal{D}})} \leq \|b_T\|_{\mathcal{C}^t}$ for the third estimate, and combined (34) and (35) to obtain the last bound. As $p > 1$, Theorem 2.9 yields that $\mathbb{E}(\exp(2p_1 q \|b_T\|_{\mathcal{C}^t})) < \infty$ for any $p_1, q \in [1, \infty)$. The claim follows for any $\varepsilon > 0$ by choosing $p_2 > 1$ so small that $\min(\gamma, 0) \leq p_2 \min(\gamma + \varepsilon, 0)$. \square

Remark 4.3. We emphasize that all estimates in Proposition 4.2 are independent of $\mathcal{D} \subset \mathbb{T}^d$, as all uniform error bounds are derived with respect to \mathbb{T}^d . Proposition 4.2 shows in particular that for any $q \geq 1$ and $t \in (0, s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q})$ there is a $C > 0$ such that for any $N \in \mathbb{N}$ it holds

$$\|a - a_N\|_{L^q(\Omega; L^\infty(\mathcal{D}))} \leq C2^{-Nt}.$$

This estimate is essential to bound the truncation error $u - u_N$ of the approximated elliptic problem in (31), see Theorem 4.4 below. In the borderline case $p = 1$ with sufficiently small $\kappa > 0$ and $sp > d$, we still recover the slightly weaker estimates

$$\|a - a_N\|_{L^q(\Omega; C^t(\overline{\mathcal{D}}))} \leq C2^{N(t-s+\frac{d}{p})}, \quad \|a - a_N\|_{L^q(\Omega; L^\infty(\mathcal{D}))} \leq C2^{-tN} \quad (36)$$

for *sufficiently small* $q \geq 1$ (depending on κ) and $t \in (0, s - \frac{d}{p})$, *independently* of γ . This may be seen from by letting $p_1 \rightarrow 1$ and $p_2 \rightarrow \infty$ in the last part of the proof for Proposition 4.2.

Theorem 4.4. *Let u be as in (21) with $a = \exp(b_{T, \mathcal{D}})$ and let u_N be as in (21) with $a_N = \exp(b_{T, N} |_{\mathcal{D}})$ given by (32). Furthermore, let $b_{T, \mathcal{D}}$ be such that $p \in (1, \infty)$, $s > 0$, $\beta = 2^{\gamma-d} \in [0, 1]$, and $sp > d \geq d + \min(\gamma, 0)$. Then, for any $q \geq 1$ and $t \in (0, s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q})$ there is a $C > 0$ such that for every $N \in \mathbb{N}$ and it holds*

$$\|u - u_N\|_{L^q(\Omega; V)} \leq C2^{-Nt}.$$

Proof. For fixed $\omega \in \Omega$ and $N \in \mathbb{N}$, we obtain by Proposition 3.3

$$\|u(\omega) - u_N(\omega)\|_V \leq \frac{\|f\|_{V'}}{a_-(\omega)a_{N,-}(\omega)} \|a(\omega) - a_N(\omega)\|_{L^\infty(\mathcal{D})},$$

where $a_{N,-}(\omega) := \text{ess inf}_{x \in \mathcal{D}} a_N(\omega, x)$. Taking expectations yields with Hölder's inequality

$$\|u - u_N\|_{L^q(\Omega; V)} \leq \|f\|_{V'} \|a_-^{-1}\|_{L^{q_1}(\Omega)} \|a_{N,-}^{-1}\|_{L^{q_2}(\Omega)} \|a - a_N\|_{L^{q_3}(\Omega; L^\infty(\mathcal{D}))}, \quad (37)$$

where $q_1, q_2, q_3 > 1$ are such that $\frac{1}{q} = \sum_{i=1}^3 \frac{1}{q_i}$ and $\|f\|_{V'} < \infty$. As in the proof of part 2.) in Theorem 3.2, we conclude for any $q_1 \in [1, \infty)$ and $t \in (0, s - \frac{d}{p})$ with Theorem 2.9 that

$$\|a_-^{-1}\|_{L^{q_1}(\Omega)} \leq \|\exp(\|b_{T, \mathcal{D}}\|_{L^\infty(\mathcal{D})})\|_{L^{q_1}(\Omega)} \leq \|\exp(\|b_T\|_{C^t})\|_{L^{q_1}(\Omega)} < \infty.$$

Similarly, it follows for all $q_2 \in [1, \infty)$ that

$$\|a_{N,-}^{-1}\|_{L^{q_2}(\Omega)} \leq \|\exp(\|b_{T, N}\|_{C^t})\|_{L^{q_2}(\Omega)} \leq \|\exp(\|b_T\|_{C^t})\|_{L^{q_2}(\Omega)} < \infty,$$

where we emphasize that the last bound is uniform with respect to N . Proposition 4.2 and Remark 4.3 show for $q_3 \in [1, \infty)$ and $t \in (0, s - \frac{d}{p} - \frac{\min(\gamma, 0)}{q_3})$ that

$$\|a - a_N\|_{L^{q_3}(\Omega; L^\infty(\mathcal{D}))} \leq C2^{-Nt}.$$

This, together with (37), shows the claim, as $q_3 > q$ may be chosen arbitrary close to q , and

$$\|a_-^{-1}\|_{L^{q_1}(\Omega)} + \|a_{N,-}^{-1}\|_{L^{q_2}(\Omega)} \leq C < \infty$$

holds for all $q_1, q_2 \in [1, \infty)$ with $C = C(q_1, q_2) > 0$, and uniform with respect to N . \square

Remark 4.5. In view of Remark 4.3, we note that for $p = 1$ with sufficiently small $\kappa > 0$ and $sp > d$ there holds the slightly weaker estimate

$$\|u - u_N\|_{L^q(\Omega; V)} \leq C2^{-Nt}.$$

for *sufficiently small* $q \geq 1$ (depending on κ) and $t \in (0, s - \frac{d}{p})$, *independently* of γ . This may also be seen by letting $q_1, q_2 \rightarrow \frac{1}{2q}$ and $q_3 \rightarrow \infty$ in the proof of Theorem 4.4.

4.2 Finite element discretization

The solution $u_N : \Omega \rightarrow V$ to Problem (31) with truncated diffusion coefficient is still not fully tractable, as it takes values in the infinite-dimensional Hilbert space V . Thus, we consider Galerkin-finite element approximations of u_N in a finite-dimensional subspace of V . Corollary 4.1 provides the necessary regularity of u_N , independent of the truncation index N , therefore we fix $N \in \mathbb{N}$ for the remainder of this section.

We partition the convex, polytopal domain $\mathcal{D} \subset \mathbb{T}^d$, $d \in \{1, 2, 3\}$ by a sequence of simplices (intervals/triangles/tetrahedra) or parallelotopes (intervals/parallelograms/parallelepipeds), denoted by $(\mathcal{K}_h)_{h \in \mathfrak{H}}$. The refinement parameter $h > 0$ takes values in a countable index set $\mathfrak{H} \subset (0, \infty)$ and corresponds to the longest edge of a simplex/parallelotope $K \in \mathcal{K}_h$. We impose the following assumptions on $(\mathcal{K}_h)_{h \in \mathfrak{H}}$ to obtain a sequence of "well-behaved" triangulations.

Assumption 4.6. The sequence $(\mathcal{K}_h)_{h \in \mathfrak{H}}$ satisfies:

1. *Admissibility:* For each $h \in \mathfrak{H}$, \mathcal{K}_h consists of open, non-empty simplices/parallelotopes K such that
 - $\overline{\mathcal{D}} = \bigcup_{K \in \mathcal{K}_h} \overline{K}$,
 - $K_1 \cap K_2 = \emptyset$ for any two $K_1, K_2 \in \mathcal{K}_h$ such that $K_1 \neq K_2$, and
 - the intersection $\overline{K}_1 \cap \overline{K}_2$ for $K_1 \neq K_2$ is either empty, a common edge, a common vertex, or (in space dimension $d = 3$) a common face of K_1 and K_2 .
2. *Shape-regularity:* Let $\rho_{K,in}$ and $\rho_{K,out}$ denote the radius of the largest in- and circumscribed circle, respectively, for a given $K \in \mathcal{K}_h$. There is a constant $\rho > 0$ such that

$$\rho := \sup_{h \in \mathfrak{H}} \sup_{K \in \mathcal{K}_h} \frac{\rho_{K,out}}{\rho_{K,in}} < \infty.$$

Based on a given tessellation \mathcal{K}_h , we define the space of piecewise (multi-)linear finite elements

$$V_h := \begin{cases} \{v \in V \mid v|_T \text{ is linear for all } K \in \mathcal{K}_h\}, & \text{if } \mathcal{K}_h \text{ consists of simplices,} \\ \{v \in V \mid v|_T \text{ is } d\text{-linear for all } K \in \mathcal{K}_h\}, & \text{if } \mathcal{K}_h \text{ consists of parallelotopes.} \end{cases}$$

Clearly, $V_h \subset V$ is a finite-dimensional space and we define $n_h := \dim(V_h) \in \mathbb{N}$. This yields for fixed $\omega \in \Omega$ the *fully discrete problem* to find $u_{N,h}(\omega) \in V_h$ such that for all $v_h \in V_h$

$$\int_{\mathcal{D}} a_N(\omega) \nabla u_{N,h}(\omega) \cdot \nabla v_h dx =_{V'} \langle f, v_h \rangle_V. \quad (38)$$

Theorem 4.7. Let $(\mathcal{K}_h)_{h \in \mathfrak{H}}$ be a sequence of triangulations satisfying Assumption 4.6, and let u_N and $u_{N,h}$ be the pathwise weak solutions to (31) and (38). Furthermore, let $N \in \mathbb{N}$, a_N be given as in (32) for $p \in [1, \infty)$ and $s > 0$, such that $sp > d$, and with $\beta = 2^{\gamma-d} \in [0, 1]$.

For any $f \in H$, sufficiently small $\kappa > 0$ in (16) and any $r \in (0, s - \frac{d}{p}) \cap (0, 1]$, there are constants $\bar{q} \in (1, \infty)$ and $C > 0$ such that for any $N \in \mathbb{N}$ and $h \in \mathfrak{H}$ there holds

$$\|u_N - u_{N,h}\|_{L^q(\Omega; V)} \leq Ch^r \begin{cases} \text{for } q \in [1, \bar{q}] \text{ if } p = 1, \text{ and} \\ \text{for any } q \in [1, \infty) \text{ if } p > 1. \end{cases}$$

Proof. We recall that $a_{N,-}(\omega) := \text{ess inf}_{x \in \mathcal{D}} a_{N,-}(\omega) > 0$ and obtain by Cea's Lemma

$$\|u_N(\omega) - u_{h,N}(\omega)\|_V \leq \frac{\|a_N(\omega)\|_{L^\infty(\mathcal{D})}}{a_{N,-}(\omega)} \|f\|_{V'} \inf_{v_h \in V_h} \|u_N(\omega) - v_h\|_V. \quad (39)$$

Now first suppose that $p > 1$. Since $f \in H$, it holds by Corollary 4.1 for any $q \geq 1$ that $u_N \in L^q(\Omega; W^r)$ for $r \in (0, s - \frac{d}{p}) \cap (0, 1]$. For $0 < s - \frac{d}{p} \leq 1$, we have $r \in (0, s - \frac{d}{p})$, and Lemma 3.5, shows $u_N \in L^q(\Omega; H^{1+r_0}(\mathcal{D}))$ for any $r_0 \in (0, r)$. It hence follows for $r_0 \in (0, r)$ that

$$\inf_{v_h \in V_h} \|u_N(\omega) - v_h\|_V \leq C \|u_N(\omega)\|_{H^{1+r_0}(\mathcal{D})} h^{r_0}. \quad (40)$$

This is a standard result for first order Lagrangian FEM, see, e.g., [16, Theorems 8.62/8.69] or [8, Theorem 4.4.20]. The constant $C > 0$ in (40) depends on the shape-regularity parameter ρ and on \mathcal{D} , but is independent of u_N and h . Combining (39) and (40) shows with Hölder's inequality

$$\begin{aligned} \|u_N(\omega) - u_{h,N}(\omega)\|_{L^q(\Omega;V)} &\leq C \|f\|_{V'} \|a_N\|_{L^{3q}(\Omega;L^\infty(\mathcal{D}))} \|a_{N,-}^{-1}\|_{L^{3q}(\Omega)} \|u_N\|_{L^{3q}(\Omega;H^{1+r_0}(\mathcal{D}))} h^{r_0} \\ &\leq C \|a_N\|_{L^{3q}(\Omega;L^\infty(\mathcal{D}))}^2 \|u_N\|_{L^{3q}(\Omega;H^{1+r_0}(\mathcal{D}))} h^{r_0} \\ &\leq C \|a_N\|_{L^{3q}(\Omega;L^\infty(\mathcal{D}))}^2 \|u_N\|_{L^{3q}(\Omega;W^r)} h^{r_0} \\ &\leq Ch^{r_0}. \end{aligned} \tag{41}$$

We have used that $a_{N,-}$ and $\|a_N\|_{L^\infty(\mathcal{D})}$ are equal in distribution for the second estimate, and Lemma 3.5 in the third line. The last step follows for any $q \in [1, \infty)$ by Corollary 4.1 and Proposition 4.2 since $p > 1$. Moreover, as a further consequence of Corollary 4.1 and Proposition 4.2, the constant $C > 0$ in the final estimate in (41) bounded independently of N and h . Since $0 < s - \frac{d}{p} \leq 1$, we may choose $r_0 < r < s - \frac{d}{p}$ arbitrary close to $s - \frac{d}{p}$.

On the other hand, if $s - \frac{d}{p} > 1$ and $r = 1$, Lemma 3.5 implies that $u_N \in L^q(\Omega;H^2(\mathcal{D}))$. Estimates (40) and (41) then hold for $r_0 = r = 1$, which proves the claim in case that $p > 1$.

For $p = 1$ and given $q \geq 1$, we need to assume in addition that $\kappa > 0$ be sufficiently small such that Corollary 4.1 and (36) in Remark 4.3 hold with q replaced $3q$. In this case, the claim for $p = 1$ follows analogously as for $p > 1$. \square

Theorem 4.8. *Let the assumptions of Theorem 4.7 hold. For any $f \in H$, sufficiently small $\kappa > 0$ in (16) and for any $r \in (0, s - \frac{d}{p}) \cap (0, 1]$, there are constants $\bar{q} \in (1, \infty)$ and $C > 0$ such that for any $N \in \mathbb{N}$ and $h \in \mathfrak{H}$ there holds*

$$\|u_N - u_{N,h}\|_{L^q(\Omega;H)} \leq Ch^{2r} \begin{cases} \text{for } q \in [1, \bar{q}) \text{ if } p = 1, \text{ and} \\ \text{for any } q \in [1, \infty) \text{ if } p > 1. \end{cases}$$

Proof. The proof uses the well-known *Aubin-Nitsche duality* argument. Let $e_{N,h} := u_N - u_{N,h}$ and consider for fixed $\omega \in \Omega$ the dual problem to find $\varphi(\omega) \in V$ such that for all $v \in V$ it holds

$$\int_{\mathcal{D}} a_N(\omega) \nabla \varphi(\omega) \cdot \nabla v dx = {}_{V'} \langle e_{N,h}(\omega), v \rangle_V. \tag{42}$$

We need to investigate the regularity and integrability of φ as a first step. Lemma 3.4 shows that

$$\|\varphi(\omega)\|_{W^r} \leq \frac{C}{a_-(\omega)} \left(1 + \left(\frac{\|a(\omega)\|_{C^r(\overline{\mathcal{D}})}}{a_-(\omega)} \right)^{1/r} \right) \|e_{N,h}(\omega)\|_H. \tag{43}$$

Let $t \in (0, s - \frac{d}{p})$ be fixed. We integrate both sides of (43) and use Hölder's inequality as in the third part of Theorem 3.9 to obtain for $q_0 \geq 1$ and $q_1, \dots, q_4 \in [1, \infty)$ such that $1 = \sum_{i=1}^4 \frac{1}{q_i}$

$$\begin{aligned} \|\varphi\|_{L^{q_0}(\Omega;W^r)} &\leq C^q \left[\mathbb{E} \left(\exp(q_0 q_1 \|b_T\|_{C^t}) \right)^{\frac{1}{q_1}} + \mathbb{E} \left(\exp \left(q_0 \max(q_1, \frac{q_2}{r}, \frac{q_3}{r}) \|b_T\|_{C^r} \right) \right)^{\frac{1}{q_3} + \frac{1}{q_2} + \frac{1}{q_3}} \right] \\ &\quad \cdot \mathbb{E} (\|e_{N,h}\|_H^{q_0 q_4})^{1/q_4}, \end{aligned} \tag{44}$$

By Theorems 2.9, 4.7 and Proposition 4.2, we now conclude that the right hand side in (44) is finite and bounded uniformly in N for any $q_0 \geq 1$ if $p > 1$, as the Hölder conjugates $q_1, \dots, q_4 \in [1, \infty)$ may be arbitrary large. For $p = 1$, we further need that $\kappa > 0$ in (16) is sufficiently small, so that $\varepsilon_p > q_0 \max(q_1, \frac{q_2}{r}, \frac{q_3}{r})$ in Theorem 2.9 and that $\bar{q} \geq q_0 q_4$ in Theorem 4.7. Given that $\kappa > 0$ is sufficiently small, there is for any $p \geq 1$ a $q_0 \geq 1$ such that $\varphi \in L^{q_0}(\Omega;W^r)$.

For the next step, we combine Equations (31) and (38) to show the Galerkin orthogonality

$$\int_{\mathcal{D}} a_N(\omega) \nabla e_{N,h}(\omega) \cdot \nabla v_h dx = 0, \quad v_h \in V_h. \quad (45)$$

Let $P_h : V \rightarrow V_h$ denote the V -orthogonal projection onto V_h . Testing with $v = e_{N,h}(\omega) \in V$ in (42) then shows together with $v_h = P_h \varphi(\omega)$ in (45) that

$$\begin{aligned} \|e_{N,h}(\omega)\|_H^2 &= \int_{\mathcal{D}} a_N(\omega) \nabla \varphi(\omega) \cdot \nabla e_{N,h}(\omega) dx \\ &\leq \|a_N(\omega)\|_{L^\infty(\mathcal{D})} \|e_{N,h}(\omega)\|_V \|(I - P_h)\varphi(\omega)\|_V. \end{aligned} \quad (46)$$

Estimate (46) then yields for $q \in [1, \infty)$ with Hölder's inequality

$$\|e_{N,h}\|_{L^q(\Omega;H)} \leq \|a_N\|_{L^{\frac{3q}{2}}(\Omega;L^\infty(\mathcal{D}))} \|e_{N,h}\|_{L^{\frac{3q}{2}}(\Omega;V)} \|(I - P_h)\varphi\|_{L^{\frac{3q}{2}}(\Omega;V)}.$$

First, suppose again that $p > 1$, where $\varphi \in L^{q_0}(\Omega;W^r)$ holds for any $q_0 \geq 1$. Proposition 4.2 and Theorem 4.7 yield

$$\|e_{N,h}\|_{L^q(\Omega;H)} \leq Ch^r \|(I - P_h)\varphi\|_{L^{\frac{3q}{2}}(\Omega;V)},$$

where $C > 0$ is independent of N and h . Lemma 3.5, $\varphi \in L^{q_0}(\Omega;W^r)$, and (40) further show that

$$\|e_{N,h}\|_{L^q(\Omega;H)} \leq Ch^{r+r_0}, \quad r_0 \in (0, r) \cup \{r\}. \quad (47)$$

The claim follows as in Theorem 4.7, since $r = r_0 = 1$ if $s - \frac{d}{p} > 1$, and $r_0 < r < s - \frac{d}{p}$ may be arbitrary close to $s - \frac{d}{p}$ otherwise.

For $p = 1$ and given $q \geq 1$ on the other hand, we need to assume that $\kappa > 0$ is sufficiently small so that $\varphi \in L^{q_0}(\Omega;W^r(\mathcal{D}))$ for $q_0 = \frac{3q}{2}$, and that (36) and Theorem 4.7 hold with q replaced by $\frac{3q}{2}$. The claim then follows as for $p > 1$ from (4.2). \square

Bounds on the overall approximation errors with respect to V and H now follow as an immediate consequence of Theorems 4.4, 4.7, 4.8 and Remark 4.5.

Corollary 4.9. *Let the assumptions of Theorem 4.7 hold, let $f \in H$, let $t \in (0, s - \frac{d}{p})$ and assume given $r \in (0, s - \frac{d}{p}) \cap (0, 1]$. Then there holds:*

- 1.) *For $p = 1$ and sufficiently small $\kappa > 0$ in (16), there are constants $\bar{q} = \bar{q}(\kappa) \in (1, \infty)$ and $C > 0$ such that for any $q \in [1, \bar{q})$, $N \in \mathbb{N}$ and $h \in \mathfrak{H}$ there holds*

$$\begin{aligned} \|u - u_{N,h}\|_{L^q(\Omega;V)} &\leq C(2^{-tN} + h^r), \\ \|u - u_{N,h}\|_{L^q(\Omega;H)} &\leq C(2^{-tN} + h^{2r}). \end{aligned}$$

- 2.) *For $p \in (1, \infty)$ and any $q \in [1, \infty)$ there is a constant $C > 0$ such that for any $N \in \mathbb{N}$ and $h \in \mathfrak{H}$ there holds*

$$\begin{aligned} \|u - u_{N,h}\|_{L^q(\Omega;V)} &\leq C(2^{N(-t + \frac{\min(\gamma, 0)}{q})} + h^r), \\ \|u - u_{N,h}\|_{L^q(\Omega;H)} &\leq C(2^{N(-t + \frac{\min(\gamma, 0)}{q})} + h^{2r}). \end{aligned}$$

5 Multilevel Monte Carlo Estimation

We consider Monte Carlo estimation of $\mathbb{E}(\Psi(u))$ for a given functional Ψ and u as solution to (22) with Besov random tree coefficient a . We replace u by a tractable approximation $u_{N,h}$ to evaluate $\Psi(u_{N,h}) \approx \Psi(u)$ and bound the overall error consisting of the pathwise discretization from Section 4 and the statistical error of the Monte Carlo approximation.

Assumption 5.1.

1.) Let $\theta \in [0, 1]$, let $\Psi : H^\theta(\mathcal{D}) \rightarrow \mathbb{R}$ be Fréchet-differentiable on $H^\theta(\mathcal{D})$ and denote by

$$\Psi' : H^\theta(\mathcal{D}) \rightarrow \mathcal{L}(H^\theta(\mathcal{D}); \mathbb{R}) = (H^\theta(\mathcal{D}))'$$

the Fréchet-derivative of Ψ . There are constants $C > 0$, $\rho_1, \rho_2 \geq 0$ such that for all $v \in H^\theta(\mathcal{D})$

$$|\Psi(v)| \leq C(1 + \|v\|_{H^\theta(\mathcal{D})}^{\rho_1}), \quad \|\Psi'(v)\|_{\mathcal{L}(H^\theta(\mathcal{D}); \mathbb{R})} \leq C(1 + \|v\|_{H^\theta(\mathcal{D})}^{\rho_2}). \quad (48)$$

2.) For $q := 2 \max(\rho_1, \rho_2 + 1)$, there holds $u \in L^q(\Omega; V)$.

3.) $(\mathcal{K}_h)_{h \in \mathfrak{H}}$ is a collection of triangulations satisfying Assumption 4.6.

4.) There are constants $t > 0$, $r \in (0, 1]$ and $C > 0$ such that for $q = 2 \max(\rho_1, \rho_2 + 1)$ and any $N \in \mathbb{N}$ and $h \in \mathfrak{H}$ it holds

$$\|u - u_{N,h}\|_{L^q(\Omega; V)} \leq C(2^{-tN} + h^r), \quad \|u - u_{N,h}\|_{L^q(\Omega; H)} \leq C(2^{-tN} + h^{2r}).$$

Remark 5.2. Assumption 5.1 is natural, and includes in particular bounded linear functions Ψ , where $\rho_1 = 1$ and $\rho_2 = 0$. Item 2 follows by Theorem 3.9 and Item 4 by Corollary 4.9, with no further restrictions whenever $p > 1$. Only in case that $p = 1$, $\kappa > 0$ needs to be sufficiently small to ensure that all bounds hold for $q = 2 \max(\rho_1, \rho_2 + 1) \geq 2$.

5.1 Singlelevel Monte Carlo

We use Monte Carlo (MC) methods to approximate $\mathbb{E}(\Psi(u))$ for a given functional Ψ . To this end, we first consider the standard MC estimator for (general) real-valued random variables.

Definition 5.3. Let $Y : \Omega \rightarrow \mathbb{R}$ be a random variable and let $(Y^{(i)}, i \in \mathbb{N})$ be a sequence of i.i.d. copies of Y . For $M \in \mathbb{N}$ we define *Monte Carlo estimator* $E_M(Y) : \Omega \rightarrow \mathbb{R}$ as

$$E_M(Y) := \frac{1}{M} \sum_{i=1}^M Y^{(i)}. \quad (49)$$

As we are not able to sample directly from the distribution of u , we rely on i.i.d. copies $(u_{N,h}^{(i)}, i \in \mathbb{N})$ of the pathwise approximation $u_{N,h}$ from Section 4. Thereby, in addition to the statistical MC error of order $\mathcal{O}(M^{-1/2})$, we introduce a sampling bias that depends on N and h .

Theorem 5.4. *Let $M \in \mathbb{N}$, let $E_M(\Psi(u_{N,h}))$ be the MC estimator as in (49), and let Assumption 5.1 hold. Then, there is a constant $C > 0$, such that for any $M, N \in \mathbb{N}$ and $h \in \mathfrak{H}$ it holds*

$$\|\mathbb{E}(\Psi(u)) - E_M(\Psi(u_{N,h}))\|_{L^2(\Omega)} \leq C \left(M^{-1/2} + 2^{-tN} + h^{(2-\theta)r} \right)$$

Proof. We split the overall error via

$$\begin{aligned} \|\mathbb{E}(\Psi(u)) - E_M(\Psi(u_{N,h}))\|_{L^2(\Omega)} &\leq \|\mathbb{E}(\Psi(u)) - E_M(\Psi(u))\|_{L^2(\Omega)} \\ &\quad + \|E_M(\Psi(u)) - E_M(\Psi(u_{N,h}))\|_{L^2(\Omega)} \\ &:= I + II. \end{aligned}$$

To bound I , we use independence of $\Psi(u)^{(i)}$ and $\Psi(u)^{(j)}$ for $i \neq j$ to see that

$$\begin{aligned} I^2 &= \mathbb{E} \left(\left(\mathbb{E}(\Psi(u)) - \frac{1}{M} \sum_{i=1}^M \Psi(u)^{(i)} \right)^2 \right) \\ &= \mathbb{E}(\Psi(u))^2 - \frac{2}{M} \sum_{i=1}^M \mathbb{E}(\Psi(u))^2 + \frac{1}{M^2} \sum_{i,j=1}^M \mathbb{E} \left(\Psi(u)^{(i)} \Psi(u)^{(j)} \right) \\ &= \frac{\text{Var}(\Psi(u))}{M}. \end{aligned}$$

Assumption 5.1 further shows that

$$I \leq \frac{\|\Psi(u)\|_{L^2(\Omega)}}{M^{1/2}} \leq C \frac{1 + \|u\|_{L^{2\rho_1}(\Omega; H^\theta(\mathcal{D}))}}{M^{1/2}} \leq C \frac{1 + \|u\|_{L^{2\rho_1}(\Omega; V)}}{M^{1/2}} \leq CM^{-1/2},$$

where we have used that $\theta \leq 1$ and $u \in L^{2\rho_1}(\Omega; V)$.

To bound II , we use Equation (48) and derive the pathwise estimate

$$\begin{aligned} |\Psi(u) - \Psi(u_{N,h})| &= \left| \int_0^1 \Psi'(u + z(u_{N,h} - u))(u - u_{N,h}) dz \right| \\ &\leq \int_0^1 \|\Psi'(u + z(u_{N,h} - u))\|_{\mathcal{L}(H^\theta(\mathcal{D}); \mathbb{R})} \|u - u_{N,h}\|_{H^\theta(\mathcal{D})} dz \\ &\leq C \left(1 + \|u\|_{H^\theta(\mathcal{D})}^{\rho_2} + \|u - u_{N,h}\|_{H^\theta(\mathcal{D})}^{\rho_2} \right) \|u - u_{N,h}\|_{H^\theta(\mathcal{D})} \\ &\leq C \left(1 + \|u\|_{H^\theta(\mathcal{D})}^{\rho_2} + \|u_{N,h}\|_{H^\theta(\mathcal{D})}^{\rho_2} \right) \|u - u_{N,h}\|_{H^\theta(\mathcal{D})}. \end{aligned} \quad (50)$$

By Assumption 5.1, there is a $C > 0$ such that for every N and every $0 < h \leq 1$ it holds

$$\|u_{N,h}\|_{L^{2(\rho_2+1)}(\Omega; H^\theta(\mathcal{D}))} \leq C \|u\|_{L^{2(\rho_2+1)}(\Omega; H^\theta(\mathcal{D}))} \leq C \|u\|_{L^{2(\rho_2+1)}(\Omega; V)} < \infty. \quad (51)$$

Furthermore, as $\theta \in [0, 1]$, we have by the Gagliardo-Nirenberg interpolation inequality

$$\|u - u_{N,h}\|_{L^{2(\rho_2+1)}(\Omega; H^\theta(\mathcal{D}))} \leq \|u - u_{N,h}\|_{L^{2(\rho_2+1)}(\Omega; H)}^{1-\theta} \|u - u_{N,h}\|_{L^{2(\rho_2+1)}(\Omega; V)}^\theta \leq C(2^{-tN} + h^{(2-\theta)r}). \quad (52)$$

Thus, Hölder's inequality with conjugate exponents $q_1 = \frac{\rho_2+1}{\rho_2}$, $q_2 = \rho_2 + 1$ (and $q_1 = \infty$, $q_2 = 1$ for $\rho_2 = 0$) shows that

$$\begin{aligned} II &\leq \|\Psi(u) - \Psi(u_{N,h})\|_{L^2(\Omega)} \\ &\stackrel{(50)}{\leq} C(1 + \|u\|_{L^{q_1 2\rho_2}(\Omega; H^\theta(\mathcal{D}))} + \|u_{N,h}\|_{L^{q_1 2\rho_2}(\Omega; H^\theta(\mathcal{D}))}) \|u - u_{N,h}\|_{L^{q_2 2}(\Omega; H^\theta(\mathcal{D}))} \\ &\stackrel{(51)}{\leq} C(1 + \|u\|_{L^{2(\rho_2+1)}(\Omega; V)}) \|u - u_{N,h}\|_{L^{2(\rho_2+1)}(\Omega; H^\theta(\mathcal{D}))} \\ &\stackrel{(52)}{\leq} C(2^{-tN} + h^{(2-\theta)r}). \end{aligned} \quad (53)$$

□

The error contributions in Theorem 5.4 are balanced by choosing

$$M \approx 2^{2tN} \approx h^{-2(2-\theta)r}. \quad (54)$$

With this choice, achieving the target accuracy $\|\mathbb{E}(\Psi(u)) - E_M(\Psi(u_{N,h}))\|_{L^2(\Omega)} = \mathcal{O}(\varepsilon)$ requires sampling $M = \mathcal{O}(\varepsilon^{-2})$ high-fidelity approximations $u_{N,h}$ with $N = \mathcal{O}(\frac{\log(\varepsilon)}{t})$ scales and mesh refinement $h = \mathcal{O}(\varepsilon^{\frac{1}{(2-\theta)r}})$. This is computationally challenging in dimension $d \geq 2$ and for low-regularity problems, i.e., when $t, r > 0$ are small. To alleviate the computational burden, we propose a multilevel Monte Carlo extension of the estimator E_M in the next subsection.

5.2 Multilevel Monte Carlo

The multilevel Monte Carlo (MLMC) algorithm was invented by Heinrich [17] to compute parametric integrals, then rediscovered and popularized by Giles [13, 14], and has since then found various applications in uncertainty quantification and beyond.

To apply this methodology to our model problem we fix a maximum refinement level $L \in \mathbb{N}$ and consider a *sequence of approximated solutions* u_{N_ℓ, h_ℓ} with $(N_\ell, h_\ell) \in \mathbb{N} \times \mathfrak{H}$ for $\ell \in \{1, \dots, L\}$. We assume that $N_1 < \dots < N_L$ and $h_1 > \dots > h_L$, so that the error $u - u_{N_\ell, h_\ell}$ decreases with

respect to the level ℓ . For notational convenience, we define $\Psi_\ell := \Psi(u_{N_\ell, h_\ell})$ as the approximation of the quantity of interest $\Psi(u)$ on level ℓ , and set $\Psi_0 := 0$. The basic idea of the MLMC method for estimating $\mathbb{E}(\Psi(u))$ is to exploit the telescopic expansion

$$\mathbb{E}(\Psi(u)) \approx \mathbb{E}(\Psi_L) = \mathbb{E}(\Psi_L) - \mathbb{E}(\Psi_0) = \sum_{\ell=1}^L \mathbb{E}(\Psi_\ell - \Psi_{\ell-1}) \quad (55)$$

of the high-fidelity approximation Ψ_L . On each level ℓ , the correction $\mathbb{E}(\Psi_\ell - \Psi_{\ell-1})$ is estimated by (standard) MC estimator with M_ℓ samples. This yields the *multilevel Monte Carlo estimator*

$$E^L(\Psi_L) := \sum_{\ell=1}^L E_{M_\ell}(\Psi_\ell - \Psi_{\ell-1}), \quad (56)$$

with level-dependent numbers of samples $M_1, \dots, M_L \in \mathbb{N}$. We assume that the estimators $E_{M_\ell}(\Psi_\ell - \Psi_{\ell-1})$ are *jointly independent across the levels* $\ell = 1, \dots, L$.

Provided that $\text{Var}(\Psi_\ell - \Psi_{\ell-1})$ decays sufficiently fast in ℓ , we choose $M_1 > \dots > M_L$ such that the majority of samples are generated cheaply on low levels ℓ , while only a few expensive samples for large ℓ are necessary. This entails massive computational savings compared to a *singlelevel Monte Carlo* (SLMC) estimator as in (49), that requires a large number of expensive samples on level L , and does not exploit the level hierarchy whatsoever. The computational gain of the MLMC method is precisely quantified under certain assumptions in Giles' complexity theorem ([13, Theorem 3.1]). Given some $\varepsilon > 0$, Giles derives the optimal number of refinement levels L and associated numbers of samples M_1, \dots, M_L that guarantee $\|\mathbb{E}(\Psi(u)) - E^L(\Psi(u))\|_{L^2(\Omega)} \leq \varepsilon$. The latter requires exact knowledge of all constants in Assumption 5.1, and, furthermore, exact knowledge of the cost for sampling one instance of Ψ_ℓ . As this is not feasible a-priori, we choose a slightly different approach to determine the MLMC parameters. We retain the optimal order of complexity as in [13, Theorem 3.1].

Assumption 5.5. Let $(\mathcal{K}_h)_{h \in \mathfrak{H}}$ be a sequence of triangulations satisfying Assumption 4.6, and assume that $h_\ell \in \mathfrak{H}$ for any $\ell \in \mathbb{N}$. Furthermore, in view of the multilevel convergence analysis, we assume that there are $0 < \underline{c}_\mathcal{K} \leq \bar{c}_\mathcal{K} < 1$ and $h_0 > 0$ such that

$$\underline{c}_\mathcal{K}^\ell h_0 \leq h_\ell \leq \bar{c}_\mathcal{K}^\ell h_0, \quad \ell \in \mathbb{N}. \quad (57)$$

One sample of $\Psi_\ell = \Psi(u_{N_\ell, h_\ell})$ with $u_{N_\ell, h_\ell} \in V_{h_\ell}$ and $n_\ell := \dim(V_{h_\ell}) = \mathcal{O}(h_\ell^{-d})$ is realized in $\mathcal{O}(n_\ell)$ work and memory.

Remark 5.6. Assumption 5.5 is natural and holds, for instance, with $\underline{c}_\mathcal{K}, \bar{c}_\mathcal{K} \approx \frac{1}{2}$ if the mesh \mathcal{K}_{h_ℓ} is obtained from $\mathcal{K}_{h_{\ell-1}}$ by bisection of the longest edge of each $K \in \mathcal{K}_{h_{\ell-1}}$. We remark that in general, it may be hard to achieve $\underline{c}_\mathcal{K} = \bar{c}_\mathcal{K}$, which is why we imposed an upper and lower bound in (57). Simulating Ψ_ℓ requires $\mathcal{O}(n_\ell) = \mathcal{O}(h_\ell^{-d})$ floating point operations per sample when using multilevel solvers for continuous piecewise linear or multi-linear elements. We also refer to Lemma B.2 in Appendix B.3, where we show that the expected cost of sampling $b_{N,T}$ on the associated grid is of order $\mathcal{O}(h_\ell^{-d})$ if $\beta < 1$.

Theorem 5.7. Let Assumptions 5.1 and 5.5 hold, and let $\varepsilon > 0$. For t, r and θ as in Assumption 5.1, select the MLMC parameters in (56) for $\ell \in \{1, \dots, L\}$ as

$$L := \left\lceil \frac{\log(\varepsilon)}{(2-\theta)r \log(\underline{c}_\mathcal{K})} - \frac{\log(h_0)}{\log(\underline{c}_\mathcal{K})} \right\rceil, \quad M_\ell := \left\lceil \left(\frac{h_\ell}{h_L} \right)^{2(2-\theta)r} w_\ell \right\rceil, \quad N_\ell := \left\lceil -\frac{\log(h_\ell)(2-\theta)r}{\log(2)t} \right\rceil. \quad (58)$$

For given $L \geq 1$, choose the weights $w_\ell > 0$ to determine M_ℓ such that $\sum_{\ell=1}^L w_\ell^{-1} \leq C_w < \infty$, for sufficiently large, fixed $C_w > 0$ independent of L .

Then, there is a $C > 0$, such that for any $\varepsilon > 0$ it holds

$$\|\mathbb{E}(\Psi(u)) - E^L(\Psi_L)\|_{L^2(\Omega)} \leq C\varepsilon.$$

Proof. We use the error splitting

$$\|\mathbb{E}(\Psi(u)) - E^L(\Psi_L)\|_{L^2(\Omega)} \leq \|\mathbb{E}(\Psi(u)) - \mathbb{E}(\Psi_L)\|_{L^2(\Omega)} + \|\mathbb{E}(\Psi_L) - E^L(\Psi_L)\|_{L^2(\Omega)}.$$

We obtain in the same fashion as for the term II in the proof of Theorem 5.4 that

$$\|\mathbb{E}(\Psi(u)) - \mathbb{E}(\Psi_L)\|_{L^2(\Omega)} \leq \|\Psi(u) - \Psi(u_{N_L, h_L})\|_{L^2(\Omega)} \leq C(2^{-tN_L} + h_L^{(2-\theta)r}) \leq Ch_L^{(2-\theta)r},$$

where we have used $2^{-tN_L} \leq h_L^{(2-\theta)r}$ by (58) in the last step. To bound the second term, we expand $\mathbb{E}(\Psi_L)$ in a telescopic sum to obtain with (56)

$$\begin{aligned} \|\mathbb{E}(\Psi_L) - E^L(\Psi_L)\|_{L^2(\Omega)}^2 &= \sum_{\ell=1}^L \|\mathbb{E}(\Psi_\ell - \Psi_{\ell-1}) - E_{M_\ell}(\Psi_\ell - \Psi_{\ell-1})\|_{L^2(\Omega)}^2 \\ &= \sum_{\ell=1}^L M_\ell^{-1} \|\Psi_\ell - \Psi_{\ell-1}\|_{L^2(\Omega)}^2. \end{aligned}$$

The first equality holds since the MC estimators $E_{M_1}(\Psi_1), \dots, E_{M_L}(\Psi_L - \Psi_{L-1})$ are jointly independent and unbiased in the sense that $\mathbb{E}(\Psi_\ell - \Psi_{\ell-1}) = \mathbb{E}(E_{M_\ell}(\Psi_\ell - \Psi_{\ell-1}))$. The triangle inequality and the estimate (53) (from the proof of Theorem 5.4) then further yield

$$\begin{aligned} \|\mathbb{E}(\Psi_L) - E^L(\Psi_L)\|_{L^2(\Omega)}^2 &\leq 2 \sum_{\ell=1}^L M_\ell^{-1} \left[\|\Psi_\ell - \Psi(u)\|_{L^2(\Omega)}^2 + \|\Psi(u) - \Psi_{\ell-1}\|_{L^2(\Omega)}^2 \right] \\ &\leq C \sum_{\ell=1}^L M_\ell^{-1} \left[2^{-2tN_\ell} + h_\ell^{2(2-\theta)r} + 2^{-2tN_{\ell-1}} + h_{\ell-1}^{2(2-\theta)r} \right] \\ &\leq C \sum_{\ell=1}^L w_\ell^{-1} h_L^{2(2-\theta)r}, \end{aligned}$$

where we have used Assumption 5.5 and the choices for M_ℓ and N_ℓ in (58) in the last step. As $\sum_{\ell=1}^L w_\ell^{-1} \leq C_w < \infty$ is bounded independently of L , we conclude with (57), L as in (58), and $0 < \underline{c}_K \leq \bar{c}_K < 1$ that

$$\|\mathbb{E}(\Psi(u)) - E^L(\Psi_L)\|_{L^2(\Omega)} \leq Ch_L^{(2-\theta)r} \leq C(\bar{c}_K h_0)^{(2-\theta)r} \leq C\varepsilon^{\frac{\log(\bar{c}_K)}{\log(\underline{c}_K)}} h_0^{\left(1 - \frac{\log(\bar{c}_K)}{\log(\underline{c}_K)}\right)(2-\theta)r} \leq C\varepsilon. \quad \square$$

The computational advantages of the MLMC method are precisely quantified in the next statement. Therein, the choice of w_ℓ plays a key role and depends on the relation of variance decay and cost of sampling on each level.

Theorem 5.8. *Let Assumptions 5.1 and 5.5 hold, and let $\varepsilon > 0$. Given t, r and θ and $\varepsilon > 0$, set L, M_ℓ and N_ℓ as in Theorem 5.7 and choose the weight functions*

$$w_\ell = \begin{cases} \ell^{1+\iota} & \text{if } 2(2-\theta)r > d \\ L & \text{if } 2(2-\theta)r = d, \quad \ell \in \{1, \dots, L\}, \\ \underline{c}_K^{(2(2-\theta)r-d)\iota\ell} & \text{if } 2(2-\theta)r < d \end{cases}$$

where $\iota \in (0, \frac{1}{L}]$ is an arbitrary small constant. Then, the MLMC estimator satisfies

$$\|\mathbb{E}(\Psi(u)) - E^L(\Psi_L)\|_{L^2(\Omega)} \leq C\varepsilon,$$

with computational cost \mathcal{C}_{MLMC} for $\varepsilon \rightarrow 0$ of order

$$\mathcal{C}_{MLMC} = \begin{cases} \mathcal{O}(\varepsilon^{-2}) & \text{if } 2(2-\theta)r > d \\ \mathcal{O}(\varepsilon^{-2} \log(\varepsilon)^2) & \text{if } 2(2-\theta)r = d \\ \mathcal{O}(\varepsilon^{-2 - \frac{d-2(2-\theta)r}{(2-\theta)r}}) & \text{if } 2(2-\theta)r < d. \end{cases}$$

Proof. Since $\underline{c}_{\mathcal{K}} \in (0, 1)$, it holds in each scenario $\sum_{\ell=1}^L w_{\ell}^{-1} \leq C$ for a constant $C > 0$, and uniform with respect to L . Therefore, we conclude by Theorem 5.7 that

$$\|\mathbb{E}(\Psi(u)) - E^L(\Psi_L)\|_{L^2(\Omega)} \leq C\varepsilon,$$

and it remains to derive the complexity in terms of ε .

Assumption 5.5 implies that $h_L \leq h_{\ell} \underline{c}_{\mathcal{K}}^{(L-\ell)}$. We obtain with (58) that

$$M_{\ell} = \left\lceil \underline{c}_{\mathcal{K}}^{(\ell-L)2(2-\theta)r} w_{\ell} \right\rceil. \quad (59)$$

Let \mathcal{C}_{ℓ} denote the work required to generate on sample of Ψ_{ℓ} . As $h_{\ell} \geq \underline{c}_{\mathcal{K}}^{\ell} h_0$, it holds that

$$\mathcal{C}_{\ell} = \mathcal{O}(\dim(V_{h_{\ell}})) = \mathcal{O}(h_{\ell}^{-d}) \leq C \underline{c}_{\mathcal{K}}^{-d\ell} \quad (60)$$

Since we generate M_{ℓ} independent $\Psi_{\ell} - \Psi_{\ell-1}$ on each level (and also generate independent samples across all levels), the overall cost of the MLMC estimator is with (59) and (60) bounded by

$$\begin{aligned} \mathcal{C}_{MLMC} &:= \sum_{\ell=1}^L M_{\ell} (\mathcal{C}_{\ell} + \mathcal{C}_{\ell-1}) \\ &\leq C \sum_{\ell=1}^L \underline{c}_{\mathcal{K}}^{(\ell-L)2(2-\theta)r} w_{\ell} (\underline{c}_{\mathcal{K}}^{-d\ell} + \underline{c}_{\mathcal{K}}^{-d(\ell-1)}) \\ &\leq C \underline{c}_{\mathcal{K}}^{-L2(2-\theta)r} \sum_{\ell=1}^L \left(\underline{c}_{\mathcal{K}}^{2(2-\theta)r-d} \right)^{\ell} w_{\ell}. \end{aligned}$$

Now first suppose that $2(2-\theta)r - d > 0$. Since $\underline{c}_{\mathcal{K}} \in (0, 1)$, the ratio test for sum convergence shows that for any $\iota > 0$ we obtain the uniform bound (with respect $L \in \mathbb{N}$)

$$\sum_{\ell=1}^L \left(\underline{c}_{\mathcal{K}}^{2(2-\theta)r-d} \right)^{\ell} w_{\ell} \leq \sum_{\ell \in \mathbb{N}} \left(\underline{c}_{\mathcal{K}}^{2(2-\theta)r-d} \right)^{\ell} \ell^{1+\iota} < \infty.$$

On the other, hand if $2(2-\theta)r - d = 0$, there holds with $w_{\ell} = L$

$$\sum_{\ell=1}^L \left(\underline{c}_{\mathcal{K}}^{2(2-\theta)r-d} \right)^{\ell} w_{\ell} = L^2.$$

Finally, for $2(2-\theta)r - d < 0$, it follows with $\iota \in (0, \frac{1}{L}]$ that there is a $C > 0$ such that

$$\sum_{\ell=1}^L \left(\underline{c}_{\mathcal{K}}^{2(2-\theta)r-d} \right)^{\ell} w_{\ell} = \sum_{\ell=1}^L \underline{c}_{\mathcal{K}}^{\ell(1+\iota)(2(2-\theta)r-d)} \leq C \underline{c}_{\mathcal{K}}^{L(2(2-\theta)r-d)}.$$

Altogether, we obtain that there exists a constant $C > 0$ independent of L such that

$$\mathcal{C}_{MLMC} \leq C \begin{cases} \underline{c}_{\mathcal{K}}^{-L2(2-\theta)r} & \text{if } 2(2-\theta)r > d, \\ \underline{c}_{\mathcal{K}}^{-L2(2-\theta)r} L^2 & \text{if } 2(2-\theta)r = d, \\ \underline{c}_{\mathcal{K}}^{-L2(2-\theta)r} \underline{c}_{\mathcal{K}}^{L(2(2-\theta)r-d)} & \text{if } 2(2-\theta)r < d. \end{cases}$$

With L from (58) it follows that $\underline{c}_{\mathcal{K}}^L = \mathcal{O}(\varepsilon^{\frac{1}{(2-\theta)r}} h_0^{-1})$ for $\varepsilon \rightarrow 0$. This shows the following asymptotics for the ε -cost bounds as $\varepsilon \rightarrow 0$

$$\mathcal{C}_{MLMC}(\varepsilon) = \begin{cases} \mathcal{O}(\varepsilon^{-2}) & \text{if } 2(2-\theta)r > d, \\ \mathcal{O}(\varepsilon^{-2} \log(\varepsilon)^2) & \text{if } 2(2-\theta)r = d, \\ \mathcal{O}(\varepsilon^{-2 - \frac{d-2(2-\theta)r}{(2-\theta)r}}) & \text{if } 2(2-\theta)r < d. \end{cases}$$

□

Remark 5.9. The asymptotic complexity bounds for $\varepsilon \rightarrow 0$ are of the same magnitude as in [13, Theorem 3.1] and [9, Theorem 1], but only require knowledge of the parameters r, t and θ , but not on further absolute constants. From Theorem 5.4, the SLMC estimator requires for given $\varepsilon > 0$ a total of $M \approx \varepsilon^{-2}$ samples with refinement parameters satisfying $2^{-tN} \approx h^{(2-\theta)r} \approx \varepsilon$. Hence, $h = \mathcal{O}(\varepsilon^{\frac{1}{(2-\theta)r}})$ and, assuming availability of a linear complexity solver such as multigrid, the computational cost per sample is bounded asymptotically as $\mathcal{O}(\dim(V_h)) = \mathcal{O}(h^{-d}) = \mathcal{O}(\varepsilon^{-\frac{d}{(2-\theta)r}})$. The total cost of the SLMC estimator to achieve ε -accuracy is therefore

$$\mathcal{C}_{SLMC}(\varepsilon) = \mathcal{O}(\varepsilon^{-2 - \frac{d}{(2-\theta)r}}) > \mathcal{C}_{MLMC}(\varepsilon), \quad \text{as } \varepsilon \rightarrow 0. \quad (61)$$

Consequently, under the stated assumptions, MLMC-FEM achieves a considerable reduction in (asymptotic) ε -complexity, even in low-regularity regimes with $2(2-\theta)r < d$.

In case that $2(2-\theta)r > d$, the assumption that $E_{M_1}(\Psi_1), \dots, E_{M_L}(\Psi_L - \Psi_{L-1})$ are independent MC estimators is not required to derive the optimal complexity $\mathcal{C}_{MLMC} = \mathcal{O}(\varepsilon^{-2})$. Instead, setting $w_\ell = \ell^{2(1+\iota)}$ is sufficient to compensate for the dependence across discretization levels. This could be exploited in a simulation to "recycle" samples from coarser discretization levels in order to further increase efficiency. We refer, e.g., to the discussion in [6, Section 5.2].

6 Numerical Experiments

We consider numerical experiments in the rectangular domain $\mathcal{D} := \mathbb{T}^2 = (0, 1)^2$ and use the constant source function $f \equiv 1$. For the spatial discretization we use bilinear finite elements that may be efficiently computed by exploiting their tensor product-structure, see Appendix B.2 for details. The initial mesh width is given by $h_0 = \frac{1}{2}$ and we use dyadic refinements with factor $c_{\mathcal{K}} = \bar{c}_{\mathcal{K}} = \frac{1}{2}$ to obtain a sequence of tessellations $(\mathcal{K}_h, h = 2^{-\ell}h_0, \ell \in \mathbb{N})$ that satisfies Assumption 4.6 for the MLMC algorithm. We further use midpoint quadrature to assemble the stiffness matrix for each realization of the diffusion coefficient. The resulting quadrature error does not dominate the FE error convergence from Theorems 4.7 and 4.8, as we show in Lemma B.1 in the Appendix. For given N and a rectangular mesh \mathcal{K}_h , we evaluate $b_{T,N}$ at the midpoint of each $K \in \mathcal{K}_h$ as described in Appendix B.3.

We investigate different parameter regimes of varying smoothness for the diffusion coefficient, the values and resulting pathwise approximation rates r and t as in Corollary 4.9 are collected in Table 1. In all experiments, we build the random field b_T resp. $b_{T,N}$ based on Daubechies wavelets with five vanishing moments ("DB(5)-wavelets"), with smoothness $\phi, \psi \in C^{1.177}(\mathbb{R})$ (see [11, Section 7.1.2]). We consider the $L^2(\mathcal{D})$ -norm of the gradient as quantity of interest (QoI), with associated functional given by

$$\Psi : H^1(\mathcal{D}) \rightarrow [0, \infty), \quad u \mapsto \left(\int_{\mathcal{D}} |\nabla u|^2 dx \right)^{1/2},$$

so that Assumption 5.1 holds with $\theta = 1$.

Parameter values	s	p	κ	β	t	r	MLMC complexity for $\theta = 1$
"smooth Gaussian"	2	2	1	$\frac{1}{2}$	1	1	$\mathcal{O}(\varepsilon^{-2} \log(\varepsilon) ^2)$
"rough Gaussian"	$\frac{3}{2}$	2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\mathcal{O}(\varepsilon^{-4})$
" p -exponential"	2	$\frac{8}{5}$	1	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\mathcal{O}(\varepsilon^{-\frac{8}{3}})$

Table 1: Parameters values for the random tree Besov priors in the numerical experiments.

Given θ, r and t , we prescribes target accuracies $\varepsilon = 2^{-r\xi}$, $\xi \in \{5, \dots, 9\}$ and select, for given $\varepsilon > 0$, the MLMC parameters as in Theorem 5.7. The maximum refinement level is denoted by L_ε and the corresponding estimators by $E^{L_\varepsilon}(\Psi_{L_\varepsilon})$. We sample $n_{ML} = 2^8$ realizations of $E^{L_\varepsilon}(\Psi_{L_\varepsilon})$

for every ε . As reference solution, we use $n_{ref} = 2^4$ realizations of $E^{L_{ref}}(\Psi_{L_{ref}})$ with parameters adjusted to achieve $\varepsilon_{ref} := 2^{-r_{11}}$. We report for prescribed ε the *realized empirical RMSE*

$$\text{RMSE}(\varepsilon) = \left(\frac{1}{n_{ML}} \sum_{i=1}^{n_{ML}} \left(E^{L_\varepsilon}(\Psi_{L_\varepsilon})(\omega_i) - \left(\frac{1}{n_{ref}} \sum_{j=1}^{n_{ref}} E^{L_{ref}}(\Psi_{L_{ref}})(\omega_j) \right) \right)^2 \right)^{\frac{1}{2}}.$$

All computations are realized using MATLAB on a workstation with two Intel Xeon E5-2697 CPUs with 2.7 GHz, a total of 12 cores, and 256 Gigabyte RAM.

We start with the "smooth Gaussian" case from Table 1. A sample of the diffusion coefficient and the associated bilinear FE approximation of u is given in Figure 2, where we also plot the (average) CPU times against the realized RMSE and the prescribed accuracies ε . As we see, the realized error is very close to the prescribed accuracy ε , which corresponds to the error estimate from Theorem 5.7. Moreover, the empirical results are in line with the work estimates from Theorem 5.8, as is seen in the right plot of Figure 2, since the computational complexity is (asymptotically) of order $\mathcal{O}(\varepsilon^{-2} |\log(\varepsilon)|^2)$.

Next, we decrease the parameter s to obtain the "rough Gaussian" scenario from Table 1. A sample of the diffusion coefficient and the associated bilinear FE approximation of u is given in Figure 3. Compared to Figure 2, we now see more detailed, sharp features in the diffusion coefficient, due to the slower decay factor of the wavelet basis. Average CPU times vs. realized RMSE are given in Figure 3. Again, the realized error is of order $\mathcal{O}(\varepsilon)$, and the computational times are asymptotically of order $\mathcal{O}(\varepsilon^{-4})$, as expected from Theorem 5.8.

Finally, we investigate the " p -exponential" scenario from Table 1, where we use a heavier-tailed distribution of $X_{j,k}^l$ and increase the wavelet density to $\beta = \frac{3}{4}$. We use a standard Acceptance-Rejection algorithm to sample from the p -exponential density for $p = 1.6$. A sample of the diffusion coefficient and the associated bilinear FE approximation of u is given in Figure 4. We observe that the variance of coefficient and solution is increased, compared to the previous two examples. This is indicated by the larger bars of the confidence intervals in the right plot of Figure 4. The a-priori accuracy has been scaled by a factor of three in this plot, for a better visual comparison of realized and prescribed RMSE. Although absolute magnitude and variance of the realized RMSE have increased, we still recover in line with Theorem 5.8 the asymptotic error decay of order $\mathcal{O}(\varepsilon)$, together with CPU times of order $\mathcal{O}(\varepsilon^{-\frac{8}{3}})$.

7 Conclusions

We have developed a computational framework for the efficient discretization of linear, elliptic PDEs with log-Besov random field coefficients which are modelled by a multiresolution in the physical domain whose coefficients are standard normal with random choices of active coefficients according to GW-trees. The corresponding pathwise diffusion coefficients generally admit only rather low path regularity, thereby mandating low order Finite Element discretizations in the physical domain. We established strong pathwise solution regularity, and FE error bounds for the corresponding single-level Monte Carlo-FEM algorithm. The corresponding error vs. work bounds for the multi-level Monte Carlo algorithm follow then in the standard way. We emphasize again that higher order sampling methods seem to be obstructed by the GW-tree structure which has recently been identified in [22] as well-suited for modelling diffuse media such as clouds, fog and aerosols. The presently proposed MLMC-FE error analysis for Elliptic PDEs with (log-) Besov random tree coefficients will imply corresponding complexity bounds in multilevel Markov Chain Monte Carlo sampling strategies for Bayesian Inverse Problems on log-Besov random tree priors, as considered for example in imaging applications in [23, 10, 4, 18]. Details on their analysis and computation will be developed elsewhere.

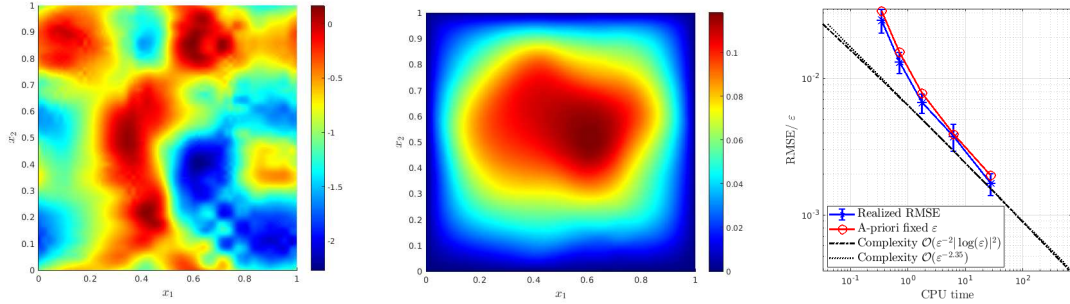


Figure 2: Sample of a Besov random tree prior/log-diffusion coefficient b_T (left) and the corresponding finite element approximation of u (middle) for the “smooth Gaussian” case in Table 1. Coefficient and solution in the figures have been sampled on a grid with $2^9 \times 2^9$ equidistant points, and wavelet series truncation was after $N = 9$ scales. Fractal structures in the log-diffusion caused by the wavelet density $\beta = \frac{1}{2}$ are clearly visible in the left plot. The realized RMSE (blue) resp. predicted RMSE (red) vs. computational complexity is depicted in the right plot. Both curves exhibit the predicted asymptotic behavior of $\mathcal{O}(\varepsilon^{-2}|\log(\varepsilon)|^2)$, as indicated by the dashed line. The support line $\mathcal{O}(\varepsilon^{-2.35})$ has been added to show that the complexity is indeed non-linear in the log-scale. The bars indicate 95%-confidence intervals of the realized RMSE.

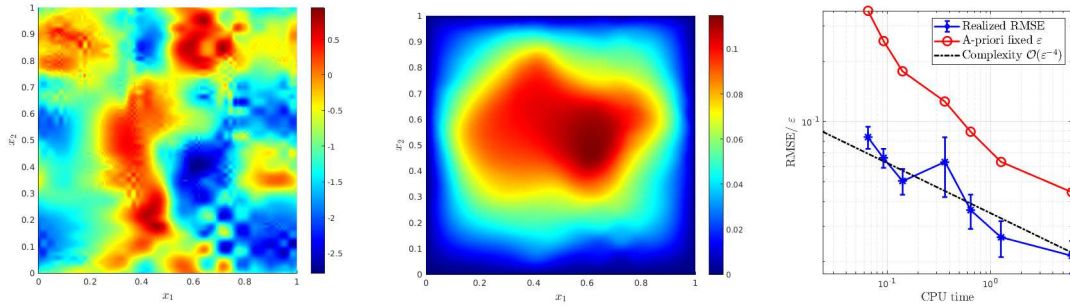


Figure 3: Sample of a Besov random tree prior/log-diffusion coefficient b_T (left) and the corresponding finite element approximation of u (middle) for the “rough Gaussian” case in Table 1. Coefficient and solution in the figures have been sampled on a grid with $2^9 \times 2^9$ equidistant points, and wavelet series truncation was after $N = 9$ scales. The diffusion coefficient exhibits sharper features, as compared to the smooth case Figure 2. The realized RMSE (blue) resp. predicted RMSE (red) vs. computational complexity is depicted in the right plot, both curves show the predicted asymptotic behavior of $\mathcal{O}(\varepsilon^{-4})$. The bars indicate 95%-confidence intervals of the realized RMSE.

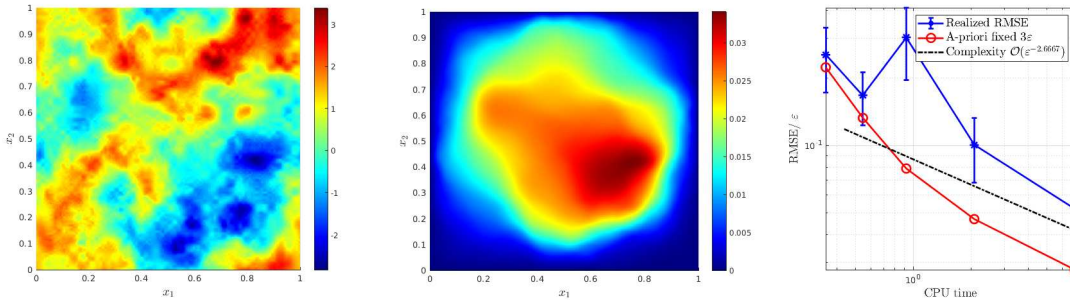


Figure 4: Sample of a Besov random tree prior/log-diffusion coefficient b_T (left) and the corresponding finite element approximation of u (middle) for the “p-exponential” case in Table 1. Coefficient and solution in the figures have been sampled on a grid with $2^9 \times 2^9$ equidistant points, and wavelet series truncation after $N = 9$ scales. The variance is significantly increased compared to the previous examples, as indicated by the bars 95%-confidence intervals of the realized RMSE (right plot). The a-priori fixed ε (red curve) has been scaled by a factor of three in this plot. We still recover the predicted asymptotic error of order $\mathcal{O}(\varepsilon)$ with computational work of order $\mathcal{O}(\varepsilon^{-3/2})$.

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A Galton-Watson trees

We provide some basic concepts of discrete trees, give a formal definition of *Galton-Watson trees*, and record a result on their extinction probabilities. The presentation follows [1, Section 2], with modified notation where necessary.

A.1 Notation and basic definitions

Let $\mathcal{U} := \bigcup_{n \geq 0} \mathbb{N}^n$ be the set of all finite sequences of positive integers, where $\varrho := ()$ denotes the empty sequence, and we use the convention $\mathbb{N}^0 = \{\varrho\}$. For any $\mathbf{n} \in \mathcal{U}$, let $|\mathbf{n}|$ denote the length of \mathbf{n} , with $|\varrho| := 0$. For $\mathbf{n}, \mathbf{m} \in \mathcal{U}$, we denote by \mathbf{nm} the concatenation of two sequences, with the convention $\mathbf{n}\varrho = \varrho\mathbf{n} = \mathbf{n}$ for all $\mathbf{n} \in \mathcal{U}$. There exists a partial order, called the *genealogical order*, on \mathcal{U} : we say that $\mathbf{m} \preceq \mathbf{n}$, whenever there is a $\mathbf{n}_0 \in \mathcal{U}$ such that $\mathbf{m}\mathbf{n}_0 = \mathbf{n}$. We say that \mathbf{m} is an ancestor of \mathbf{n} and write $\mathbf{m} \prec \mathbf{n}$ if $\mathbf{m} \preceq \mathbf{n}$ and $\mathbf{m} \neq \mathbf{n}$. The *set of all ancestors of \mathbf{n}* is denoted by $A_{\mathbf{n}} := \{\mathbf{m} \in \mathcal{U} \mid \mathbf{m} \prec \mathbf{n}\} \subset \mathcal{U}$.

Definition A.1. [1, Section 2.1] A tree \mathbf{t} is a subset $\mathbf{t} \subset \mathcal{U}$ that satisfies

- $\varrho \in \mathbf{t}$,
- If $\mathbf{n} \in \mathbf{t}$, then $A_{\mathbf{n}} \subset \mathbf{t}$,
- For any $\mathbf{n} \in \mathbf{t}$, there exists $\mathfrak{K}_{\mathbf{n}}(\mathbf{t}) \in \mathbb{N} \cup \{\infty\}$, such that for every $n \in \mathbb{N}$, $\mathbf{n}n \in \mathbf{t}$ if, and only if, $1 \leq n \leq \mathfrak{K}_{\mathbf{n}}(\mathbf{t})$.

We denote the *set of all trees* by \mathfrak{T}_{∞} . Let $|\mathbf{t}| \in \mathbb{N} \cup \{\infty\}$ be the cardinality of the tree $\mathbf{t} \in \mathfrak{T}_{\infty}$, and introduce the *set of all finite trees* by $\mathfrak{T}_0 := \{\mathbf{t} \in \mathfrak{T} \mid |\mathbf{t}| < \infty\}$. The set \mathfrak{T}_0 is countable. The integer $\mathfrak{K}_{\mathbf{n}}(\mathbf{t})$ represents the number of offsprings in \mathbf{t} at the node \mathbf{n} . The *set of all trees without infinite nodes* is a subset \mathfrak{T}_{∞} and denoted by

$$\mathfrak{T} := \{\mathbf{t} \in \mathfrak{T}_{\infty} \mid \mathfrak{K}_{\mathbf{n}}(\mathbf{t}) < \infty \text{ for all } \mathbf{n} \in \mathbf{t}\} \quad (62)$$

For $\mathbf{n} \in \mathbf{t}$, the *sub-tree $\mathfrak{S}_{\mathbf{n}}(\mathbf{t})$ of \mathbf{t} above node \mathbf{n}* is defined as:

$$\mathfrak{S}_{\mathbf{n}}(\mathbf{t}) := \{\mathbf{m} \in \mathcal{U} \mid \mathbf{nm} \in \mathbf{t}\}. \quad (63)$$

We also need the *restriction functions* $\mathfrak{r}_n : \mathfrak{T} \rightarrow \mathfrak{T}$, $n \in \mathbb{N}_0$ which are given by

$$\mathfrak{r}_n(\mathbf{t}) := \{\mathbf{n} \in \mathbf{t} \mid |\mathbf{n}| \leq n\}. \quad (64)$$

With these preparations, we are in position to define metric and associated Borel σ -algebra on \mathfrak{T} .

Definition A.2. [1, Section 2.1] Let

$$\delta : \mathfrak{T} \times \mathfrak{T} \rightarrow [0, 1], \quad (\mathbf{t}, \mathbf{t}') \mapsto 2^{-\sup\{n \in \mathbb{N}_0 \mid \mathfrak{r}_n(\mathbf{t}) = \mathfrak{r}_n(\mathbf{t}')\}}. \quad (65)$$

Furthermore, define the σ -algebra

$$\mathcal{B}(\mathfrak{T}) := \sigma \left(\bigcup_{t \in \mathfrak{T}} \bigcup_{n \in \mathbb{N}} \mathfrak{r}_n^{-1}(\mathfrak{r}_n(\mathbf{t})) \right) = \sigma \left(\bigcup_{t \in \mathfrak{T}} \bigcup_{n \in \mathbb{N}} \{\mathbf{t}' \mid \delta(\mathbf{t}, \mathbf{t}') \leq 2^{-n}\} \right). \quad (66)$$

By [1, Lemma 2.1], (\mathfrak{T}, δ) is a complete and separable metric space. The countable set of all finite trees \mathfrak{T}_0 is dense in \mathfrak{T} : for all $\mathbf{t} \in \mathfrak{T}$, we have $\mathfrak{r}_n(\mathbf{t}) \rightarrow \mathbf{t}$ as $n \rightarrow \infty$ in (\mathfrak{T}, δ) . Further, δ is an ultra-metric (see [1, Section 2.1]), hence $\mathfrak{r}_n^{-1}(\mathfrak{r}_n(\mathbf{t}))$ is the set of open (and closed) balls with center \mathbf{t} and radius n [1, Section 2.1] with respect to δ . By separability, $\mathcal{B}(\mathfrak{T})$ coincides with the Borel σ -algebra on \mathfrak{T} , that is generated by all open sets on (\mathfrak{T}, δ) . Given a probability space (Ω, \mathcal{A}, P) , we then call any $\mathcal{A}/\mathcal{B}(\mathfrak{T})$ -measurable mapping $\tau : \Omega \rightarrow \mathfrak{T}$ a \mathfrak{T} -valued random variable. This allows us to formalize *Galton-Watson trees*:

Definition A.3. [Galton-Watson(GW) tree with offspring distribution \mathcal{P}] A \mathfrak{T} -valued random variable T has the *branching property* if, for any $n \in \mathbb{N}$, conditionally on $\{k_\varrho(\tau) = n\}$, the subtrees $\mathfrak{S}_1(T), \dots, \mathfrak{S}_n(T)$ are independent and distributed as the original tree τ . Now let \mathcal{P} be a probability distribution on \mathbb{N}_0 , i.e., a probability measure on $(\mathbb{N}_0, \mathcal{B}(\mathbb{N}_0))$. A \mathfrak{T} -valued random variable T is called a *Galton-Watson(GW) tree with offspring distribution \mathcal{P}* if T has the branching property and if $\mathfrak{K}_\varrho(T) \sim \mathcal{P}$.

According to [1, Equation (12)], the *distribution \mathbb{P}_T of a GW tree T , restricted to the set of finite trees*, is given by

$$\mathbb{P}_T(T = \mathbf{t}) = \prod_{\mathbf{n} \in \mathbf{t}} \mathcal{P}(\mathfrak{K}_{\mathbf{n}}(\mathbf{t})), \quad \mathbf{t} \in \mathfrak{T}_0, \quad (67)$$

where the product on the right hand side is finite.

A.2 Random wavelet trees and extinction probabilities

Now we consider again the d -dimensional torus \mathbb{T}^d with wavelet basis Ψ as in (12). To relate the nodes of a GW tree to the wavelet indices in $\mathcal{I}_\Psi = \{j \in \mathbb{N}_0, k \in K_j, l \in \mathcal{L}_j\}$, we assume that T is a GW tree with offspring distribution $\mathcal{P} = \text{Bin}(2^d, \beta)$ for some $\beta \in [0, 1]$. For a given realization $T(\omega)$ and node $\mathbf{n} \in T(\omega)$, we identify the length of \mathbf{n} with the corresponding wavelet scale via $j := |\mathbf{n}| \in \mathbb{N}_0$. Further, since \mathcal{P} is binomial, there are at most 2^{dj} nodes \mathbf{n} of length j in $T(\omega)$. Each of this nodes has j entries in $\{1, \dots, 2^d\}$. We assign an integer to all $\mathbf{n} \in T(\omega)$ with $|\mathbf{n}| = j$ via the bijection

$$\mathfrak{J}_{d,j}^1 : \{1, \dots, 2^d\}^j \rightarrow \{1, \dots, 2^{dj}\}, \quad \mathbf{n} \mapsto 1 + \sum_{i=1}^j 2^{d(j-i)}(\mathbf{n}_i - 1).$$

On the other hand, we assign for any $\{1, \dots, 2^{jd}\}$ an index in $K_j = \{0, \dots, 2^j - 1\}^d$ via

$$\mathfrak{J}_{d,j}^2 : \{1, \dots, 2^{jd}\} \rightarrow K_j, \quad n \mapsto \left(\max(n - 2^{j(i-1)}, 0) \bmod 2^j \right)_{i=1, \dots, d},$$

which yields a one-to-one mapping

$$\mathfrak{J}_{d,j} : \{1, \dots, 2^d\}^j \rightarrow K_j, \quad \mathbf{n} \mapsto (\mathfrak{J}_{d,j}^2 \circ \mathfrak{J}_{d,j}^1)(\mathbf{n}). \quad (68)$$

Thus, each $\mathbf{n} \in T(\omega)$ corresponds to a unique pair of indices (j, k) via $\mathbf{n} \mapsto (|\mathbf{n}|, \mathfrak{J}_{d,|\mathbf{n}|}^2 \circ \mathfrak{J}_{d,|\mathbf{n}|}^1(\mathbf{n}))$. Collecting the pairs (j, k) for all nodes in T yields the *random active index set*

$$\mathfrak{J}_T(\omega) := \bigcup_{\mathbf{n} \in T(\omega)} \left(|\mathbf{n}|, \mathfrak{J}_{d,|\mathbf{n}|}^2 \circ \mathfrak{J}_{d,|\mathbf{n}|}^1(\mathbf{n}) \right) \subset \{j \in \mathbb{N}_0, k \in K_j\}. \quad (69)$$

Then, only wavelets with indices $\mathcal{I}_T(\omega) := \{(j, k, l) \mid (j, k) \in \mathfrak{J}_T(\omega), l \in \mathcal{L}_j\} \subset \mathcal{I}_{\Psi}$ are “activated” in the series expansion of a sample of b_T in Definition 2.6.

It is then of course of interest whether the GW tree T terminates after a finite number of nodes, in which case \mathfrak{J}_T is finite, or if T has infinitely many nodes. In the latter case, b_T exhibits fractal structures on \mathbb{T}^d , in areas where the wavelet expansion has infinitely many terms. Let the *extinction event of a GW tree T* be denoted by $\mathcal{E}(T) := \{T \in \mathfrak{T}_0\}$. The *extinction probability of GW trees* are quantified in the following result:

Lemma A.4. [1, Corollary 2.5/Lemma 2.6] *Let $T : \Omega \rightarrow \mathfrak{T}$ be GW tree with offspring distribution \mathcal{P} and let $\zeta \sim \mathcal{P}$.*

1. *If $\mathcal{P}(0) = 0$, then $\mathbb{P}_T(\mathcal{E}(T)) = 0$,*
2. *If $\mathcal{P}(0) \in (0, 1)$ and $\mathcal{P}(0) + \mathcal{P}(1) = 1$, then $\mathbb{P}_T(\mathcal{E}(T)) = 1$,*
3. *If $\mathcal{P}(0) \in (0, 1)$, $\mathcal{P}(0) + \mathcal{P}(1) < 1$, and $\mathbb{E}(\zeta) \leq 1$, then $\mathbb{P}_T(\mathcal{E}(T)) = 1$,*
4. *If $\mathcal{P}(0) \in (0, 1)$, $\mathcal{P}(0) + \mathcal{P}(1) < 1$, and $\mathbb{E}(\zeta) > 1$, then $\mathbb{P}_T(\mathcal{E}(T)) = q \in (0, 1)$. Here q is the smallest root in $[0, 1]$ of the equation $\mathbb{E}(q^\zeta) = q$.*

Lemma A.4 does not require a Binomial offspring distribution, but remains true for arbitrary distributions \mathcal{P} on \mathbb{N}_0 . Moreover, we conclude that a GW tree T with offspring distribution $\mathcal{P} = \text{Bin}(2^d, \beta)$ generates finite wavelet expansions via \mathfrak{J}_T P -a.s. if and only if $\beta \in [0, 2^{-d}]$.

B Finite element approximation

This appendix collects details on the (standard) implementation of the pathwise FE discretization from Section 4. In particular, we analyze the quadrature error arising during matrix assembly, describe an assembly routine based on tensorization for bilinear finite elements on \mathbb{T}^2 , and comment on the efficient evaluation and sampling cost of a_N .

B.1 Finite element quadrature error

Let $h > 0$ be the FE meshwidth and let $\{v_1, \dots, v_{n_h}\}$ be a suitable basis of V_h . As $u_{h,N} = \sum_{i=1}^{n_h} \underline{u}_i v_i$ for a coefficient vector \underline{u} , problem (38) is equivalent to the linear system of equations

$$\mathbf{A} \underline{u}_h = \underline{F}. \quad (70)$$

For any $i, j \in \{1, \dots, n_h\}$, the entries of \mathbf{A} and \underline{F} are given by

$$\mathbf{A}_{i,j} := \int_{\mathcal{D}} a_N(\omega) \nabla v_i \cdot \nabla v_j dx, \quad \text{and} \quad \underline{F}_i := {}_{V'} \langle f, v_i \rangle_V, \quad (71)$$

and have in general to be evaluated by numerical quadrature. Thereby, we commit a variational crime in the assembling of \mathbf{A} and \underline{F} . As a_n resp. a is of low regularity, we have to make sure to choose an appropriate quadrature method, that does not spoil the convergence rate of the finite element approximation. It turns out that the midpoint rule is sufficient for (d -)linear elements, as we show in the remainder of this subsection. We restrict ourselves to the quadrature error analysis for the stiffness matrix \mathbf{A} for brevity, the corresponding analysis for the load vector \underline{F} is carried out analogously. We denote for any simplex/parallelotope $K \in \mathcal{K}_h$ its midpoint or barycenter by $x_K^m \in K$. Furthermore, we define the piecewise constant approximation \bar{a}_N of a_n given by

$$\bar{a}_N(\omega, x) := a_N(\omega, x_K^m), \quad x \in K, \quad K \in \mathcal{K}_h.$$

As we consider piecewise (d -)linear finite elements, approximating \mathbf{A}_{ij} in (71) by midpoint quadrature on each K is equivalent to solving the following discrete problem: Find $\bar{u}_{N,h}(\omega) \in V_h$ such that for all $v \in V$ it holds

$$\int_{\mathcal{D}} \bar{a}_N(\omega) \nabla \bar{u}_{N,h}(\omega) \cdot \nabla v dx = {}_{V'} \langle f, v \rangle_V.$$

There exists a.s. a unique solution $\bar{u}_{N,h}(\omega)$ and the quadrature error is bounded in the following.

Lemma B.1. *Let the assumptions of Theorem 4.7 hold. For any $f \in H$, sufficiently small $\kappa > 0$ in (16) and any $r \in (0, s - \frac{d}{p}) \cap (0, 1]$, there are constants $\bar{q} \in (1, \infty)$ and $C > 0$ such that for any $N \in \mathbb{N}$ and $h \in \mathfrak{H}$ there holds*

$$\|u_{N,h} - \bar{u}_{N,h}\|_{L^q(\Omega;V)} + h^{-r} \|u_{N,h} - \bar{u}_{N,h}\|_{L^q(\Omega;H)} \leq Ch^r \begin{cases} \text{for } q \in [1, \bar{q}) \text{ if } p = 1, \text{ and} \\ \text{for any } q \in [1, \infty) \text{ if } p > 1. \end{cases}$$

Furthermore, if $s - \frac{d}{p} > 1$, the statement holds for $t = 1$.

Proof. There exists a.s. a unique solution $\bar{u}_{N,h}(\omega)$ and the distance to $u_{N,h}(\omega)$ is readily bounded with Proposition 3.3 by

$$\|u_{N,h}(\omega) - \bar{u}_{N,h}(\omega)\|_V \leq \frac{\|f\|_{V'}}{a_N(\omega)\bar{a}_N(\omega)} \|a_N(\omega) - \bar{a}_N(\omega)\|_{L^\infty(\mathcal{D})}. \quad (72)$$

Theorems 2.9 and Proposition 4.2 show that $a_N \in L^{3q}(\Omega; \mathcal{C}^t(\mathcal{D}))$ for all $t \in (0, s - \frac{d}{p})$ and $q \geq \frac{1}{3}$. For $p > 1$, we may again choose any $q \in [\frac{1}{3}, \infty)$, for $p = 1$ we have $q \in [\frac{1}{3}, \bar{q})$, where $\bar{q} > 1$ for sufficiently small $\kappa > 0$. This yields for $q \in [1, \bar{q})$ with Hölder's inequality

$$\begin{aligned} \|u_{N,h} - \bar{u}_{N,h}\|_{L^q(\Omega;V)} &\leq C \|a_{N,-}^2\|_{L^{3q/2}(\Omega)} \|a_N - \bar{a}_N\|_{L^{3q}(\Omega;L^\infty(\mathcal{D}))} \\ &\leq C \|a_{N,-}\|_{L^{3q}(\Omega)}^2 \|a_N\|_{L^{3q}(\Omega; \mathcal{C}^t(\mathcal{D}))} h^{\min(t,1)} \\ &\leq Ch^r. \end{aligned}$$

To prove the error with respect to H , we recall the duality argument from Theorem 4.8: Let $\bar{e}_{N,h} := u_{N,h} - \bar{u}_{N,h}$ and consider for fixed $\omega \in \Omega$ the dual problem to find $\bar{\varphi}(\omega) \in V$ such that for all $v \in V$ it holds

$$\int_{\mathcal{D}} a_N(\omega) \nabla \bar{\varphi}(\omega) \cdot \nabla v dx = {}_{V'} \langle \bar{e}_{N,h}(\omega), v \rangle_V,$$

Analogously to the proof of Theorem 4.8, we derive the pathwise estimate

$$\|\bar{e}_{N,h}(\omega)\|_H^2 \leq \|a_N(\omega)\|_{L^\infty(\mathcal{D})} \|\bar{e}_{N,h}(\omega)\|_V \|(I - P_h)\bar{\varphi}(\omega)\|_V.$$

Provided sufficiently integrability if $p = 1$, we find that $\|u_{N,h} - \bar{u}_{N,h}\|_{L^q(\Omega;H)} \leq Ch^{2r}$. \square

Note that r in Lemma B.1 is identical to r in Theorems 4.7 and 4.8. Hence the quadrature error does not dominate the finite element convergence rate. We further emphasize that Lemma B.1 holds for arbitrary piecewise (multi-)linear elements, regardless if we use simplices or parallelotopes to discretize \mathcal{D} .

B.2 Bilinear finite element discretization

We focus on the rectangular domain $\mathcal{D} = \mathbb{T}^2 = [0, 1]^2$ in this subsection. It is convenient to use a spatial discretization based on bilinear finite elements. Let therefore $h = 1/n$ for a $n \in \mathbb{N}$ and consider the nodes $\bar{x}_i := ih$, $i \in \{0, \dots, n\}$. Then $\Xi_h := \{\bar{x}_0, \dots, \bar{x}_n\} \subset [0, 1]$ defines an equidistant mesh of \mathbb{T}^1 . A rectangular tessellation of \mathcal{K}_h of \mathbb{T}^2 is then given by the $(n+1)^2$ grid points $\Xi_h^2 := \{(\bar{x}_{i_1}, \bar{x}_{i_2}) \mid i_1, i_2 \in \{0, \dots, n\}\} \subset \mathbb{T}^2$. Let

$$\phi_i(x) := \max \left\{ 0, 1 - \frac{|\bar{x}_i - h|}{h} \right\}, \quad x \in \mathbb{R}, \quad i \in \{0, \dots, n\}$$

be the one-dimensional hat function basis at the nodes in Ξ_h . Then, the space of bilinear finite elements corresponding to Ξ_h^2 resp. \mathcal{K}_h is given by

$$V_h := \text{span}_{\mathbb{R}} \{\phi_{i_1} \otimes \phi_{i_2}, \quad i_1, i_2 \in \{1, \dots, n-1\}\}.$$

The dyads in the tensor product basis coincide with the pointwise products

$$\phi_{i_1} \otimes \phi_{i_2}(x) := \phi_{i_1}(x_1)\phi_{i_2}(x_2), \quad x \in \mathbb{R}^2.$$

Note that $\dim(V_h) = (n-1)^2$ due to the homogeneous Dirichlet boundary conditions. Now let $i, j \in \{1, \dots, (n-1)^2\}$ be indices such that $v_i = \phi_{i_1} \otimes \phi_{i_2} \in V_h$ and $v_j = \phi_{j_1} \otimes \phi_{j_2} \in V_h$. The entries of the associated stiffness matrix $\mathbf{A} \in \mathbb{R}^{(n-1)^2 \times (n-1)^2}$ are given by

$$\mathbf{A}_{i,j} := \int_{\mathbb{T}^2} a_N(\omega, x) \nabla(\phi_{i_1} \otimes \phi_{i_2})(x) \cdot \nabla(\phi_{j_1} \otimes \phi_{j_2})(x) dx. \quad (73)$$

We approximate the entries of \mathbf{A} by midpoint quadrature on each square in \mathcal{K}_h , which may be realized by replacing a in (73) by a suitable piecewise constant interpolation at the midpoints as in Appendix B.1. Let the midpoint of each square $K = [\bar{x}_{i_1}, \bar{x}_{i_1+1}] \times [\bar{x}_{i_2}, \bar{x}_{i_2+1}] \in \mathcal{K}_h$ be given by $x_K^m = x_{i_1, i_2}^m := (\bar{x}_{i_1} + \frac{h}{2}, \bar{x}_{i_2} + \frac{h}{2})$ for $i_1, i_2 \in \{0, \dots, n-1\}$ and define

$$\bar{a}_N(\omega, x) := a_N(\omega, x_{i_1, i_2}^m), \quad x \in [\bar{x}_{i_1}, \bar{x}_{i_1+1}] \times [\bar{x}_{i_2}, \bar{x}_{i_2+1}].$$

With indices i, j as above this yields

$$\begin{aligned} \mathbf{A}_{i,j} &= \int_{\mathbb{T}^2} a_N(\omega) \nabla(\phi_{i_1} \otimes \phi_{i_2})(x) \cdot \nabla(\phi_{j_1} \otimes \phi_{j_2})(x) dx \\ &\approx \int_{\mathbb{T}^2} \bar{a}_N(\omega) \nabla(\phi_{i_1} \otimes \phi_{i_2})(x) \cdot \nabla(\phi_{j_1} \otimes \phi_{j_2})(x) dx \\ &= \int_{\bar{x}_{i_1-1}}^{\bar{x}_{i_1+1}} \int_{\bar{x}_{i_2-1}}^{\bar{x}_{i_2+1}} \bar{a}_N(\omega) \nabla(\phi_{i_1} \otimes \phi_{i_2})(x) \cdot \nabla(\phi_{j_1} \otimes \phi_{j_2})(x) dx \\ &= a_N(\omega, x_{i_1-1, i_2-1}^m) \left[\int_{\bar{x}_{i_1-1}}^{\bar{x}_{i_1}} \phi'_{i_1} \phi'_{j_1} dx_1 \int_{\bar{x}_{i_2-1}}^{\bar{x}_{i_2}} \phi_{i_2} \phi_{j_2} dx_2 + \int_{\bar{x}_{i_1-1}}^{\bar{x}_{i_1}} \phi_{i_1} \phi_{j_1} dx_1 \int_{\bar{x}_{i_2-1}}^{\bar{x}_{i_2}} \phi'_{i_2} \phi'_{j_2} dx_2 \right] \\ &\quad + a_N(\omega, x_{i_1, i_2-1}^m) \left[\int_{\bar{x}_{i_1}}^{\bar{x}_{i_1+1}} \phi'_{i_1} \phi'_{j_1} dx_1 \int_{\bar{x}_{i_2-1}}^{\bar{x}_{i_2}} \phi_{i_2} \phi_{j_2} dx_2 + \int_{\bar{x}_{i_1}}^{\bar{x}_{i_1+1}} \phi_{i_1} \phi_{j_1} dx_1 \int_{\bar{x}_{i_2-1}}^{\bar{x}_{i_2}} \phi'_{i_2} \phi'_{j_2} dx_2 \right] \\ &\quad + a_N(\omega, x_{i_1-1, i_2}^m) \left[\int_{\bar{x}_{i_1-1}}^{\bar{x}_{i_1}} \phi'_{i_1} \phi'_{j_1} dx_1 \int_{\bar{x}_{i_2}}^{\bar{x}_{i_2+1}} \phi_{i_2} \phi_{j_2} dx_2 + \int_{\bar{x}_{i_1-1}}^{\bar{x}_{i_1}} \phi_{i_1} \phi_{j_1} dx_1 \int_{\bar{x}_{i_2}}^{\bar{x}_{i_2+1}} \phi'_{i_2} \phi'_{j_2} dx_2 \right] \\ &\quad + a_N(\omega, x_{i_1, i_2}^m) \left[\int_{\bar{x}_{i_1}}^{\bar{x}_{i_1+1}} \phi'_{i_1} \phi'_{j_1} dx_1 \int_{\bar{x}_{i_2}}^{\bar{x}_{i_2+1}} \phi_{i_2} \phi_{j_2} dx_2 + \int_{\bar{x}_{i_1}}^{\bar{x}_{i_1+1}} \phi_{i_1} \phi_{j_1} dx_1 \int_{\bar{x}_{i_2}}^{\bar{x}_{i_2+1}} \phi'_{i_2} \phi'_{j_2} dx_2 \right]. \end{aligned}$$

We define the matrices \mathbf{S} and \mathbf{M} via

$$\begin{aligned} \mathbf{S}_{i_1, j_1} &:= \int_{\bar{x}_{i_1-1}}^{\bar{x}_{i_1}} \phi'_{i_1} \phi'_{j_1} dx_1 = \begin{cases} \frac{1}{h}, & i_1 = j_1, \\ -\frac{1}{h}, & i_1 = j_1 + 1, \\ 0, & \text{else,} \end{cases} \\ \mathbf{M}_{i_1, j_1} &:= \int_{\bar{x}_{i_1-1}}^{\bar{x}_{i_1}} \phi_{i_1} \phi_{j_1} dx_1 = \begin{cases} \frac{h}{3}, & i_1 = j_1, \\ \frac{h}{6}, & i_1 = j_1 + 1, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

for $i_1, j_1 \in \{1, \dots, n-1\}$, and observe that

$$\mathbf{S}_{i_1, j_1}^\top = \mathbf{S}_{j_1, i_1} = \int_{\bar{x}_{i_1}}^{\bar{x}_{i_1+1}} \phi'_{i_1} \phi'_{j_1} dx_1, \quad \mathbf{M}_{i_1, j_1}^\top = \mathbf{M}_{j_1, i_1} = \int_{\bar{x}_{i_1}}^{\bar{x}_{i_1+1}} \phi_{i_1} \phi_{j_1} dx_1.$$

This yields

$$\begin{aligned} \mathbf{A}_{i,j} &= a_N(\omega, x_{i_1-1, i_2-1}^m) (\mathbf{S}_{i_1, j_1} \mathbf{M}_{i_2, j_2} + \mathbf{M}_{i_1, j_1} \mathbf{S}_{i_2, j_2}) \\ &\quad + a_N(\omega, x_{i_1, i_2-1}^m) (\mathbf{S}_{i_1, j_1}^\top \mathbf{M}_{i_2, j_2} + \mathbf{M}_{i_1, j_1}^\top \mathbf{S}_{i_2, j_2}) \\ &\quad + a_N(\omega, x_{i_1-1, i_2}^m) (\mathbf{S}_{i_1, j_1} \mathbf{M}_{i_2, j_2}^\top + \mathbf{M}_{i_1, j_1} \mathbf{S}_{i_2, j_2}^\top) \\ &\quad + a_N(\omega, x_{i_1, i_2}^m) (\mathbf{S}_{i_1, j_1}^\top \mathbf{M}_{i_2, j_2}^\top + \mathbf{M}_{i_1, j_1}^\top \mathbf{S}_{i_2, j_2}^\top), \end{aligned} \quad (74)$$

and we only obtain a contribution to $\mathbf{A}_{i,j}$ if $|i_1 - j_1|, |i_2 - j_2| \leq 1$. Hence, the representation in (74) may be used for an efficiently assembly of \mathbf{A} .

B.3 Evaluation of a_N

To assemble \mathbf{A} via (74) it still remains to evaluate $a_N(\omega) = \exp(b_N(\omega))$ at the quadrature points in Ξ_h^2 , or, more generally, at the d -dimensional grid $\Xi_h^d := \otimes_{i=1}^d \Xi_h \subset [0, 1]^d$. We recall that

$$b_{T,N}(\omega) = \sum_{\substack{(j,k,l) \in \mathcal{I}_T(\omega) \\ j \leq N}} \eta_j X_{j,k}^l(\omega) \psi_{j,k}^l, \quad \omega \in \Omega.$$

The tensor-product representation of $\psi_{j,k}^l \in L^2(\mathbb{T}^2)$ given in (5) and (4) then shows

$$\begin{aligned} b_{T,N}(\omega, x) &= \sum_{\substack{(j,k,l) \in \mathcal{I}_T(\omega) \\ j \leq N}} \eta_j X_{j,k}^l(\omega) 2^{\frac{dj}{2}} \bigotimes_{i=1}^d \psi_{j+w, k_i, l(i)}(x) \\ &= \sum_{\substack{(j,k,l) \in \mathcal{I}_T(\omega) \\ j \leq N}} \eta_j X_{j,k}^l(\omega) 2^{\frac{dj}{2}} \prod_{i=1}^d \psi_{j+w, k_i, l(i)}(x_i), \quad x \in \mathbb{T}^d. \end{aligned}$$

We define the vectors $\underline{\psi}_{j,k_i, l(i)} := 2^{\frac{j}{2}} \psi_{j+w, k_i, l(i)}(\bar{x})|_{\bar{x} \in \Xi} \in \mathbb{R}^n$ and finally obtain

$$b_{T,N}(\omega, x)|_{x \in \Xi_h^2} = \sum_{\substack{(j,k,l) \in \mathcal{I}_T(\omega) \\ j \leq N}} \eta_j X_{j,k}^l(\omega) \prod_{i=1}^d \underline{\psi}_{j,k_i, l(i)} \in \mathbb{R}^{dn}.$$

Therefore, it is sufficient to evaluate the scaled and shifted functions $\psi_{j,k,l(i)}$ on the one-dimensional grid Ξ_h , the values of b_N , resp. a_N , at the d -dimensional grid Ξ_h^d are then obtained by tensorization. Evaluating $\psi_{j,k,l(i)}$ eventually requires to approximate the fractal functions ϕ and ψ at a discrete set of points. This is feasible to arbitrary precision with the iterative *Cascade algorithm*, see, e.g., [11, Chapter 6.5]. Using $J \in \mathbb{N}$ iterations in the Cascade algorithm yields approximate values of ϕ, ψ at 2^J dyadic grid points, which are then interpolated to obtain piece-wise linear or constant approximation of continuous ϕ and ψ interpolation. The resulting error is of order $\mathcal{O}(2^{-J\alpha})$ if $\phi, \psi \in C^\alpha(\mathbb{R})$ with $\alpha \in (0, 1]$. Consequently, we use $J_\ell := \lceil \frac{N_\ell t}{\alpha} \rceil$ on each level in the MLMC algorithm to match the midpoint quadrature error in Lemma B.1. The cost of sampling $b_{T,N}$ on a uniform, dyadic grid is quantified in the following.

Lemma B.2. *Let $h_\ell = 2^{-(\ell+1)}$ for $\ell \in \mathbb{N}_0$, let $\Xi_{h_\ell}^d := \otimes_{i=1}^d \Xi_{h_\ell} \subset [0, 1]^d$ for $d \in \mathbb{N}$, and let $\mathcal{C}_{\text{sample}}$ denote the random cost (in terms of work and memory required) of sampling $b_{T,N}$ with respect to the grid $\Xi_{h_\ell}^d$. Then, there is a constant $C > 0$, independent of h_ℓ and N , such that*

$$\mathbb{E}(\mathcal{C}_{\text{sample}}) \leq C \begin{cases} h_\ell^{-d}(N+1) & \text{if } \beta = 1, \text{ and} \\ h_\ell^{-d} & \text{if } \beta \in (0, 1). \end{cases}$$

Proof. Given that ψ and ϕ are evaluated at the $2^{\ell+1} \in \mathbb{N}$ grid points in Ξ_{h_ℓ} , we need to calculate $2^d - 1$ tensor products of scaled and translated vectors $\underline{\psi}_{j,k_i, l(i)} \in \mathbb{R}^{2^{\ell+1}}$. Recall from the MRA in Subsection 2.1.2 that tensorization yields 2^j one-periodic wavelet functions $\psi_{j,k}^l$ on each scale $j \in \mathbb{N}_0$. Moreover, the support of $\psi_{j+w, k}^l$ has diameter bounded by $2^{-d(j+1-w)}$ in \mathbb{T}^d for fixed index $(j, k, l) \in \mathcal{I}_\Psi$, where $w \in \mathbb{N}$ is a scaling factor that only depends on the choice of ϕ and ψ . Hence, the number of grid points lying in the support of $\psi_{j,k}^l$ is given by

$$|\text{supp}(\psi_{j+w, k}^l) \cap \Xi_{h_\ell}^d| \leq 2^{-d(j+1-w)} 2^{d(\ell+1)} = 2^{d(\ell-j+w)}. \quad (75)$$

Now we also fix a realization $(T(\omega), X(\omega))$ of the random tree T and the coefficients X . If $(j, k, l) \in \mathcal{I}_T(\omega)$, we multiply the corresponding $2^{d(l-j+w)}$ grid points with the coefficient $X_{j,k}^l(\omega)$. Otherwise, if $(j, k, l) \notin \mathcal{I}_T(\omega)$, there is no contribution to $b_{T,N}(\omega)$ from this index. Summing over all non-zero contributions and grid points thus requires computational cost of

$$C_{\text{sample}} \leq 2 \sum_{\substack{(j,k,l) \in \mathcal{I}_T(\omega) \\ j \leq N}} 2^{d(l-j+w)} = 2 \sum_{\substack{(j,k,l) \in \mathcal{I}_\Psi \\ j \leq N}} \mathbb{1}_{\{(j,k,l) \in \mathcal{I}_T(\omega)\}} 2^{d(l-j+w)}.$$

Since $\mathcal{P} = \text{Bin}(2^d, \beta)$, it readily follows that $P((j, k, l) \in \mathcal{I}_T(\omega)) = \beta^j$, which in turn shows

$$\mathbb{E}(C_{\text{sample}}) = 2 \sum_{j=0}^N 2^{dj} (2^d - 1) \beta^j 2^{d(l-j+w)} \leq \begin{cases} 2^{dw+1} (2^d - 1) h_\ell^{-d} (N + 1) & \text{if } \beta = 1, \text{ and} \\ \frac{2^{dw+1} (2^d - 1)}{1 - \beta} h_\ell^{-d} & \text{if } \beta \in (0, 1). \end{cases}$$

□

Remark B.3. Lemma B.2 shows that for given $\beta \in (0, 1)$, the expected cost of sampling $b_{N,T}$ is bounded by Ch_ℓ^{-d} uniformly with respect to $N \in \mathbb{N}$. Thus, the condition that a sample of u_{N_ℓ, h_ℓ} may be realized with (expected) work $\mathcal{O}(h_\ell^{-d})$ from Assumption 5.5 is indeed justified. On the other hand, we note that $C = C(d, \beta) \rightarrow \infty$ as $\beta \rightarrow 1$, resulting in a possibly large hidden constant within the asymptotic costs in Theorem 5.8. However, if we choose the error balancing $N \propto |\log(h_\ell)|$ according to (54), we still recover the only slightly worse complexity bound of $\mathcal{O}(h_\ell^{-d} |\log(h_\ell)|)$ per sample in the limit $\beta = 1$.