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Injectivity of sampled Gabor phase retrieval in spaces with general integrability conditions

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Abstract

It was recently shown that functions in $L^4([-B, B])$ can be uniquely recovered up to a global phase factor from the absolute values of their Gabor transform sampled on a rectangular lattice. We prove that this remains true if one replaces $L^4([-B, B])$ by $L^p([-B, B])$ with $p \in [2, \infty]$. To do so, we adapt the original proof by Grohs and Liehr and use sampling results in Bernstein spaces with general integrability parameters. Furthermore, we present some modifications of a result of Müntz–Szász type first proven by Zalik. Finally, we consider the implications of our results for more general function spaces obtained by applying the fractional Fourier transform to $L^p([-B, B])$ and for more general nonuniform sampling sets.

Keywords Phase retrieval, Gabor transform, Sampling theory, Time-frequency analysis, Müntz–Szász type results

Mathematics Subject Classification (2010) 94A12, 94A20

1 Introduction

In this paper, we consider the *Gabor transform* of functions $f \in L^2(\mathbb{R})$ given by

$$\mathcal{G}f(x, \omega) := 2^{1/4} \int_{\mathbb{R}} f(t) \mathrm{e}^{-\pi(t-x)^2} \mathrm{e}^{-2\pi i t \omega} \, \mathrm{d}t, \qquad (x, \omega) \in \mathbb{R}^2,$$

and try to understand if one can recover f from measurements of the absolute value $|\mathcal{G}f|$ on discrete sets $S \subset \mathbb{R}^2$. This so-called sampled Gabor phase retrieval problem has recently been studied extensively [1, 2, 10, 11]. It is an elegant mathematical problem in the sense that it is rather easy to state while, at the same time, being less easy to solve. Moreover, it is connected to certain audio processing applications such as the phase vocoder [6, 14].

A hallmark of all phase retrieval problems is that signals cannot be fully recovered from phaseless measurements. For the sampled Gabor phase retrieval

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problem, we can see that the functions f and $\mathrm{e}^{\mathrm{i}\alpha} f$, where $\alpha \in \mathbb{R}$, generate the same measurements

$$\left|\mathcal{G}(\mathrm{e}^{\mathrm{i}\alpha} f)\right| = \left|\mathrm{e}^{\mathrm{i}\alpha} \mathcal{G} f\right| = |\mathcal{G} f| .$$

Hence, we are not able to distinguish between f and $\mathrm{e}^{\mathrm{i}\alpha} f$ on the basis of their sampled Gabor transform magnitudes. We will therefore consider the equivalence relation $\sim$ on $L^2(\mathbb{R})$ defined by

$$f \sim g :\Longleftrightarrow \exists\, \alpha \in \mathbb{R} : f = \mathrm{e}^{\mathrm{i}\alpha} g. \tag{1}$$

With the help of this relation, we can introduce the phase retrieval operator $\mathcal{A} : X/{\sim} \to [0,\infty)^S$, where X is a subspace of $L^2(\mathbb{R})$, by

$$\mathcal{A}(f)(x,\omega) := |\mathcal{G} f(x,\omega)| , \qquad (x,\omega) \in S,$$

for $f \in X/{\sim}$. The *sampled Gabor phase retrieval problem* is the problem of inverting $\mathcal{A}$ when $S \subset \mathbb{R}^2$ is discrete. We note that it has long been known that one can invert $\mathcal{A}$ for $X = L^2(\mathbb{R})$ and $S = \mathbb{R}^2$:

Lemma 1.1. *The following are equivalent for $f, g \in L^2(\mathbb{R})$:*

1. $f = \mathrm{e}^{\mathrm{i}\alpha} g$ *for some* $\alpha \in \mathbb{R}$,
2. $|\mathcal{G} f| = |\mathcal{G} g|$.

In applications, one does typically not have access to measurements of the Gabor transform magnitude on the entire time-frequency plane, however, and we thus believe that the sampled Gabor phase retrieval problem is a natural first step towards a better understanding of settings encountered in practice. While it is known that one may invert $\mathcal{A}$ for $S = \mathbb{R}^2$, much less was known about the inversion of $\mathcal{A}$ for discrete sets S. Recently, however, a series of breakthroughs was presented in the papers [1, 2, 10, 11]. For the genesis of this paper, the work in [10] was most important. The authors of [10] show that sampled Gabor phase retrieval is unique with $X = L^4([-B,B])$ and $S = \mathbb{Z} \times (4B)^{-1}\mathbb{Z}$.

Lemma 1.2 (Theorem 3.1 on p. 9 of [10])**.** *Let $B > 0$. Then, the following are equivalent for $f, g \in L^4([-B,B])$:*

1. $f = \mathrm{e}^{\mathrm{i}\alpha} g$ *for some* $\alpha \in \mathbb{R}$,
2. $|\mathcal{G} f| = |\mathcal{G} g|$ *on* $\mathbb{Z} \times (4B)^{-1}\mathbb{Z}$.

What is curious about the result above is the use of the space $L^4([-B,B])$. In particular, we find it interesting to ask whether one may extend Lemma 1.2 to spaces with more general integrability conditions and notably to $L^2([-B,B])$. In this paper, we want to answer the prior questions positively by modifying and generalising the original proof of Lemma 1.2. In this way, we obtain the following result.

Theorem 1.3 (Cf. Theorem 5.5)**.** *Let $B > 0$, $b \in (0, \frac{1}{4B})$ and $p \in [2,\infty]$. Then, the following are equivalent for $f, g \in L^p([-B,B])$:*

1. $f = \mathrm{e}^{\mathrm{i}\alpha} g$ *for some* $\alpha \in \mathbb{R}$,

2. $|\mathcal{G}f| = |\mathcal{G}g|$ *on* $\mathbb{N} \times b\mathbb{Z}$.

We observe that the above theorem is almost optimal in view of the results presented in [1]: there, for any lattice $S \subset \mathbb{R}^2$ in the time-frequency plane, explicit examples $f, g \in L^2(\mathbb{R})$ were constructed which do not agree up to global phase but which satisfy that

$$|\mathcal{G}f(x,\omega)| = |\mathcal{G}g(x,\omega)|, \qquad (x,\omega) \in S.$$

In particular, it is necessary to restrict the Gabor phase retrieval problem to a proper subspace X of $L^2(\mathbb{R})$ in order to obtain a uniqueness result from samples.

It may not surprise the reader that one may further generalise Theorem 1.3 in multiple ways to include more general function spaces obtained by taking fractional Fourier transforms of elements in $L^p([-B,B])$ or more general nonuniform sampling sets. Both of these generalisation have already been suggested in [10] and we adapt them here.

Finally, we want to mention that our proof for Theorem 1.3 relies on a Müntz–Szász type result: to be precise, we prove two modifications of a theorem by Zalik [15].

Outline In Section 2, we introduce some basic concepts needed for the further understanding of this paper. Most importantly, we introduce the fractional Fourier transform, the Paley–Wiener spaces and the Bernstein spaces along with some of their most relevant properties.

Thereafter, in Section 3, we reimagine the proof of Lemma 1.2: in particular, we will argue that Lemma 1.2 follows from two core insights. The first of those being that the short-time Fourier transform of a bandlimited function is bandlimited in the first argument (cf. Lemma 3.1) and the second of those being a result of Müntz–Szász type by Zalik. This argument sets the stage for the following sections and the proof of our main result in Section 5 in particular.

In Section 4, we modify one of the Müntz–Szász type results presented in [15]. There, it was shown that certain translates of Gaussians are complete in $L^2([a,b])$, for $a < b$. We extend this result to $L^p([a,b])$, for $p \in [1,\infty)$, by adapting the original proof from [15]. In addition, we show that translates of Gaussians can never be complete in $L^\infty([a,b])$ but that the annihilator of the closed[1] linear hull of certain translates of Gaussians intersects $L^1([a,b])$ trivially.

Finally, in Section 5, we apply the Müntz–Szász type results developed in Section 4 in our reimagination of the proof of Lemma 1.2 presented in Section 3 to generalise the result from [10]. In this way, we obtain Theorem 1.3. Thereafter, we consider certain generalisations with respect to the underlying subspace $X \subset L^2(\mathbb{R})$ and the sampling lattice which are inspired by the original paper [10].

[1] with respect to the L^∞-norm.

Notation Let us denote $\mathbb{N} = \{1, 2, 3, \dots\}$ as well as $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Furthermore, we will denote the canonical inner product on $L^2(\mathbb{R})$ by $(\cdot, \cdot)$ and the open ball of radius $R > 0$ around the origin in $\mathbb{C}$ by

$$B_R := \{z \in \mathbb{C} \mid |z| < R\}.$$

We will make use of the translation operators $\{\mathrm{T}_x\}_{x\in\mathbb{R}}$ given by

$$\mathrm{T}_x f(t) := f(t - x), \qquad t \in \mathbb{R},$$

for $x \in \mathbb{R}$, as well as the modulation operators $\{\mathrm{M}_\omega\}_{\omega\in\mathbb{R}}$ given by

$$\mathrm{M}_\omega f(t) := f(t)\mathrm{e}^{2\pi \mathrm{i} t\omega}, \qquad t \in \mathbb{R},$$

for $\omega \in \mathbb{R}$, repeatedly. Both of these families of operators can be defined for functions $f : \mathbb{R} \to \mathbb{C}$ and are unitary on $L^2(\mathbb{R})$. For sums, we will use notation suggested in [13, 15]. To be precise, we will write

$$\sum{}' r_n^{-1} := \sum_{\substack{n=1 \\ r_n \neq 0}} r_n^{-1},$$

for $(r_n)_{n\in\mathbb{N}} \in [0, \infty)$. Finally, we will often deal with trivial extensions of functions $F : [-B, B] \to \mathbb{C}$, where $B > 0$. To simplify the exposition, we will denote

$$F_0(\xi) := \begin{cases} F(\xi) & \text{if } \xi \in [-B, B], \\ 0 & \text{else}, \end{cases}$$

in this case.

2 Definitions and basic notions

We will use the convention

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(\mathbf{t})\mathrm{e}^{-2\pi \mathrm{i}(\mathbf{t},\boldsymbol{\xi})}\,\mathrm{d}\mathbf{t}, \qquad \boldsymbol{\xi} \in \mathbb{R}^d,$$

for the Fourier transform on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, where $d \geq 1$. It is well-known that the Fourier transform can be extended to $L^2(\mathbb{R}^d)$ by Plancherel's theorem and a density argument. The extension is a unitary map on $L^2(\mathbb{R}^d)$ and therefore its inverse is given by its adjoint

$$\mathcal{F}^{-1}F(\mathbf{t}) = \int_{\mathbb{R}^d} F(\boldsymbol{\xi})\mathrm{e}^{2\pi \mathrm{i}(\boldsymbol{\xi},\mathbf{t})}\,\mathrm{d}\boldsymbol{\xi} = \mathcal{F}F(-\mathbf{t}), \qquad \mathbf{t} \in \mathbb{R}^d,$$

for $F \in L^2(\mathbb{R}^d)$.

A property of the Fourier transform which we will use repeatedly is that it relates complex conjugation to the involution

$$f^{\#}(t) := \overline{f(-t)}, \qquad t \in \mathbb{R}.$$

Indeed, it holds that

$$\mathcal{F}(\overline{f}) = \mathcal{F}(f)^{\#}, \qquad f \in L^1(\mathbb{R}) \cup L^2(\mathbb{R}).$$

We note that $f^{\#}$ is well-defined for $f : \mathbb{R} \to \mathbb{C}$ and one can directly show that $(\cdot)^{\#}$ is an isometry on $L^p(\mathbb{R})$, for $p \in [1, \infty]$.

Let us now consider a function $\phi \in L^2(\mathbb{R})$. We can then define the *short-time Fourier transform with window* ϕ of $f \in L^2(\mathbb{R})$ by

$$\mathcal{V}_\phi f(x, \omega) := \int_{\mathbb{R}} f(t)\overline{\phi(t-x)} \mathrm{e}^{-2\pi \mathrm{i} t\omega} \, \mathrm{d}t, \qquad (x, \omega) \in \mathbb{R}^2.$$

One can show that $\mathcal{V}_\phi f$ is uniformly continuous and that $\mathcal{V}_\phi f \in L^2(\mathbb{R}^2)$ (see Lemma 3.1.1 and Theorem 3.2.1 in [8]). Clearly, the Gabor transform as defined in the introduction corresponds to the short-time Fourier transform with window $\phi = c\mathrm{e}^{-\pi(\cdot)^2}$, where $c = 2^{1/4}$.

It is notable that the short-time Fourier transform at a fixed time $x \in \mathbb{R}$ exactly corresponds to the Fourier transform of a short-time section of the signal f. We will use this insight in some of our proofs so let us be a bit more precise. It holds that

$$\mathcal{V}_\phi f(x, \omega) = \mathcal{F}\left(f \cdot \mathrm{T}_x \overline{\phi}\right)(\omega), \qquad (x, \omega) \in \mathbb{R}^2.$$

Another way of rewriting the short-time Fourier transform which is useful at times is

$$\mathcal{V}_g f(x, \omega) = (f, \mathrm{M}_\omega \mathrm{T}_x g), \qquad x, \omega \in \mathbb{R}.$$

Throughout this paper, we will often refer to the *fundamental identity of time-frequency analysis*, which is the fact that the Fourier transform corresponds to a rotation by 90 degrees of the time-frequency plane (see e.g. Lemma 3.1.1 on p. 39 of [8]):

$$\mathcal{V}_\phi f(x, \omega) = \mathrm{e}^{-2\pi \mathrm{i} x\omega} \mathcal{V}_{\mathcal{F}\phi} \mathcal{F} f(\omega, -x), \qquad (x, \omega) \in \mathbb{R}^2,$$

for $f, \phi \in L^2(\mathbb{R})$.

We introduce the phase retrieval operator $\mathcal{A}_\phi : X/\!\sim\, \to [0, \infty)^S$ by

$$\mathcal{A}_\phi(f)(x, \omega) := \left|\mathcal{V}_\phi f(x, \omega)\right|, \qquad (x, \omega) \in S,$$

for $f \in X/\!\sim$. Here, $\sim$ is the equivalence relation introduced in equation (1), S is a subset of $\mathbb{R}^2$ and X is a subspace of $L^2(\mathbb{R})$. The *short-time Fourier transform phase retrieval problem* then refers to the inversion of $\mathcal{A}$. When S is discrete, we call the corresponding short-time Fourier transform phase retrieval problem *sampled*. Moreover, if $\phi = c\mathrm{e}^{-\pi(\cdot)^2}$, for $c = 2^{1/4}$, we call the short-time Fourier transform phase retrieval problem the *Gabor transform phase retrieval problem.*

Let us now quickly return to the classical uniqueness result for Gabor phase retrieval that we mentioned in the introduction. We may see Lemma 1.1 as an instance of a more general result for short-time Fourier transform phase retrieval.

Lemma 2.1. *Let $\phi \in L^2(\mathbb{R})$ be such that $\mathcal{V}_\phi \phi$ is non-zero almost everywhere. Then, the following are equivalent for $f, g \in L^2(\mathbb{R})$:*

1. *$f = \mathrm{e}^{\mathrm{i}\alpha} g$ for some $\alpha \in \mathbb{R}$,*
2. *$|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$.*

Indeed, if $\phi = c\mathrm{e}^{-\pi(\cdot)^2}$, with $c = 2^{1/4}$, it is well-known that (see Lemma 1.5.2 on p. 18 of [8])

$$\mathcal{V}_\phi \phi(x, \omega) = (\phi, \mathrm{M}_\omega \, \mathrm{T}_x \, \phi) = \mathrm{e}^{-2\pi \mathrm{i} x\omega} \, (\phi, \mathrm{T}_x \, \mathrm{M}_\omega \, \phi) = \mathrm{e}^{-\pi \mathrm{i} x\omega} \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)},$$

for $(x, \omega) \in \mathbb{R}^2$, such that the above result implies Lemma 1.1.

Finally, we want to point out that the proof of Lemma 2.1 — which can, for instance, be found in [9] — can be seen as an application of the following classical result on (radar) ambiguity functions.

Lemma 2.2 (Theorem 2.5 on p. 588 of [4])**.** *Let $f, g \in L^2(\mathbb{R})$ be such that*

$$\mathcal{V}_f f = \mathcal{V}_g g.$$

Then, it holds that there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha} g$.

Notably, we will apply the above lemma in multiple proofs in the present paper.

2.1 The fractional Fourier transform

The fundamental identity of time-frequency analysis which we introduced before can be seen as a special case of a more general principle: the fractional Fourier transform corresponds to a rotation of the time-frequency plane. This principle is tremendously useful when generalising results in time-frequency analysis and we will encounter it multiple times in this paper.

Let us define the *fractional Fourier transform* of a function $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$\mathcal{F}_\theta f(\xi) := c_\theta \mathrm{e}^{\pi \mathrm{i} \xi^2 \cot\theta} \int_{\mathbb{R}} f(t) \mathrm{e}^{\pi \mathrm{i} t^2 \cot\theta} \mathrm{e}^{-2\pi \mathrm{i} \frac{t\xi}{\sin\theta}} \, \mathrm{d}t, \qquad \xi \in \mathbb{R},$$

for $\theta \in \mathbb{R} \setminus \pi\mathbb{Z}$, where $c_\theta \in \mathbb{C}$ is the square root of $1 - \mathrm{i}\cot\theta$ with positive real part, and by $\mathcal{F}_{2k\pi} f := f$ as well as $\mathcal{F}_{(2k+1)\pi} f(\xi) := f(-\xi)$, for $\xi \in \mathbb{R}$, where $k \in \mathbb{Z}$. One can show that the fractional Fourier transform preserves the canonical inner product on $L^2(\mathbb{R})$: to be precise, it holds that for all $\theta \in \mathbb{R}$ and $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have

$$(f, g) = (\mathcal{F}_\theta f, \mathcal{F}_\theta g).$$

It follows that one can extend the fractional Fourier transform to a unitary map on $L^2(\mathbb{R})$.

One important property which the fractional Fourier transform inherits from the classical Fourier transform is that the Gaussian $\phi = c\mathrm{e}^{-\pi(\cdot)^2}$, with $c = 2^{1/4}$, is invariant under its action. More precisely, it holds that

$$\mathcal{F}_\theta \phi = \phi, \qquad \theta \in \mathbb{R}.$$

One can prove this by a direct computation using the classical result which can, for instance, be found on p. 17 of [8]. We have included the calculation in the appendix for the convenience of the reader.

Finally, to state the fundamental principle that the fractional Fourier transform corresponds to a rotation of the time-frequency plane, we will introduce the operator $\mathrm{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\mathrm{R}_\theta(x, \omega) := (x \cos\theta - \omega \sin\theta, x \sin\theta + \omega \cos\theta), \qquad x, \omega \in \mathbb{R}.$$

One can see that R_θ corresponds to a rotation by θ of the time-frequency plane $\mathbb{R}^2$. We can now state the following important identity which we will refer to as the *generalised fundamental identity of time-frequency analysis*.

Lemma 2.3 (Cf. [3, 12])**.** *Let $\theta \in \mathbb{R}$ and $f, g \in L^2(\mathbb{R})$. It holds that*

$$\mathcal{V}_{\mathcal{F}_\theta g} \mathcal{F}_\theta f(x, \omega) = \mathcal{V}_g f(\mathrm{R}_\theta(x, \omega)) \mathrm{e}^{\pi \mathrm{i} \sin\theta \left((x^2 - \omega^2) \cos\theta - 2x\omega \sin\theta \right)},$$

for $x, \omega \in \mathbb{R}$.

Note that the texts [3, 12] do not contain the exact statement of the above lemma but rather results from which the lemma might be deduced. For this reason, we have decided to add a proof of the above result to the appendix of the present paper.

2.2 The Paley–Wiener spaces

In the following, we will mostly work with bandlimited functions. To be precise, we consider the *Paley–Wiener spaces* of bandlimited functions defined via

$$\mathrm{PW}_B^p := \left\{ f : \mathbb{C} \to \mathbb{C} \,\middle|\, \exists\, F \in L^p([-B, B])\, \forall\, z \in \mathbb{C} : f(z) = \int_{-B}^{B} F(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}\xi \right\},$$

for $B > 0$ and $p \in [1, \infty]$. One may see that the Paley–Wiener spaces are nested which is due to the nestedness of L^p-spaces over closed intervals. Since both of these facts will be used heavily in this paper, we state them in the following.

Proposition 2.4. *Let $1 \leq p \leq q \leq \infty$ and $B > 0$. Then, $L^q([-B, B]) \subset L^p([-B, B])$.*

Corollary 2.5. *Let $1 \leq p \leq q \leq \infty$ and $B > 0$. Then, $\mathrm{PW}_B^q \subset \mathrm{PW}_B^p$.*

One of the core properties of the Paley–Wiener spaces is that their elements correspond to entire functions of exponential type. This is, in fact, the message of the famous Paley–Wiener theorem:

Theorem 2.6 (Paley–Wiener theorem)**.** *Let $B > 0$. Then, the following are equivalent:*

1. *$f \in \mathrm{PW}_B^2$,*

2. *f is an entire function such that there exists a constant $A > 0$ for which*

$$|f(z)| \leq A\,\mathrm{e}^{2\pi B|z|}, \qquad z \in \mathbb{C},$$

and

$$\int_{\mathbb{R}} |f(t)|^2 \,\mathrm{d}t < \infty.$$

Another important property of bandlimited functions is that one may recover them from samples on equidistant sets. This classical result is commonly referred to as the Whittaker–Shannon–Kotelnikov (WSK) sampling theorem.

Theorem 2.7 (WSK sampling theorem)**.** *Let $B > 0$ and $f \in \mathrm{PW}_B^2$. Then, we have*

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B}\right) \operatorname{sinc}\left(2Bt - n\right), \qquad t \in \mathbb{R},$$

where the series converges unconditionally in $L^2(\mathbb{R})$.

In the following, we will often refer to the short-time Fourier transform of a function in a Paley–Wiener space. This is a slight abuse of notation since the short-time Fourier transform is not explicitly defined for functions whose domain is $\mathbb{C}$. In this case, the notation $\mathcal{V}_\phi f$ is to be interpreted as $\mathcal{V}_\phi(f|_{\mathbb{R}})$, where $f|_{\mathbb{R}} : \mathbb{R} \to \mathbb{C}$ is understood to be the restriction of $f : \mathbb{C} \to \mathbb{C}$ to the real numbers. Hence, the short-time Fourier transform of a function f in the Paley–Wiener space PW_B^p, with $B > 0$, is well-defined as long as $p \in [2, \infty]$: indeed, we may remember that the Paley–Wiener spaces are nested and that therefore $\mathrm{PW}_B^p \subset \mathrm{PW}_B^2$. It follows that $f \in \mathrm{PW}_B^2$ such that the Paley–Wiener theorem implies that $f|_{\mathbb{R}} \in L^2(\mathbb{R})$. Therefore, the short-time Fourier transform of f is uniformly continuous and an element of the Hilbert space $L^2(\mathbb{R}^2)$.

A final fact about functions $f \in \mathrm{PW}_B^p$ which we will use very often is that their Fourier transforms $\mathcal{F}(f|_{\mathbb{R}})$ are in $L^p(\mathbb{R})$.

Lemma 2.8. *Let $2 \leq p \leq \infty$, $B > 0$ and $f \in \mathrm{PW}_B^p$. Then, we have that $\mathcal{F}(f|_{\mathbb{R}}) \in L^p(\mathbb{R})$ and $\operatorname{supp} \mathcal{F}(f|_{\mathbb{R}}) \subset [-B, B]$.*

Proof. By the definition of the Paley–Wiener spaces, we find that there exists a function $F \in L^p([-B, B]) \subset L^2([-B, B])$ (the inclusion follows from Proposition 2.4 and $p \geq 2$) such that

$$f(z) = \int_{-B}^{B} F(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi z}\,\mathrm{d}\xi, \qquad z \in \mathbb{C}.$$

Using the notation F_0 for the trivial extension of F to $\mathbb{R}$ (as introduced in the paragraph "Notation"), we find that $F_0 \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ and we might write $f|_{\mathbb{R}} = \mathcal{F}^{-1}F_0$. Therefore, we have $\mathcal{F}(f|_{\mathbb{R}}) = F_0$ and the lemma follows. □

2.3 The Bernstein spaces

For our proof of Theorem 1.3, the WSK sampling result is not quite powerful enough. We will instead need to use more general sampling results in the so-called Bernstein spaces which we will introduce in the following.

Let $p \in [1, \infty]$ and $\sigma > 0$. We define the *Bernstein space* B^p_σ to be the space of entire functions f of exponential type $\sigma > 0$, i.e. for every $\epsilon > 0$ there exist constants $A, R > 0$ such that

$$|f(z)| \leq A\mathrm{e}^{(\sigma+\epsilon)|z|}, \qquad z \in \mathbb{C} \setminus B_R,$$

whose restriction to $\mathbb{R}$ is in $L^p(\mathbb{R})$. When $p = 2$, it follows from the classical Paley–Wiener theorem that $\mathrm{PW}^2_B \subset \mathrm{B}^2_{2\pi B}$. We are mostly interested in the following inclusions.

Lemma 2.9. *Let $p \in [1, 2]$ and denote by $q \in [2, \infty]$ its Hölder conjugate. Let moreover $B > 0$. Then, it holds that $\mathrm{PW}^p_B \subset \mathrm{B}^q_{2\pi B}$.*

We include a proof of the above lemma in the appendix. In the Bernstein spaces B^p_σ with $p \in [1, \infty)$, a general sampling theorem holds.

Theorem 2.10 (Cf. Theorem 2.2 on p. 26 of [16])**.** *Let $p \in [1, \infty)$, $\sigma > 0$ and $f \in \mathrm{B}^p_\sigma$. Then, it holds that*

$$f(t) = \sum_{k \in \mathbb{Z}} f\left(\frac{\pi k}{\sigma}\right) \operatorname{sinc}\left(\frac{\sigma t}{\pi} - k\right), \qquad t \in \mathbb{R},$$

where the series converges absolutely and uniformly on every compact subset.

We emphasise that the above result does not continue to hold in the same form for $p = \infty$. This is notable because B^∞_σ is exactly the space which we need to consider when generalising Lemma 1.2 from $L^4([-B, B])$ to $L^2([-B, B])$. Luckily, the following result can be used instead.

Theorem 2.11 (Cf. Theorem 2.3 on p. 29 of [16])**.** *Let $\sigma > 0$ and $f \in \mathrm{B}^\infty_\sigma$. Then, it holds that*

$$f(z) = \sum_{k \in \mathbb{Z}} f\left(\frac{\pi k}{\sigma'}\right) \operatorname{sinc}\left(\frac{\sigma' z}{\pi} - k\right), \qquad z \in \mathbb{C},$$

for $\sigma' > \sigma$, where the series converges uniformly on every compact subset of the complex plane.

3 The sampling result from [10] reimagined

One may see Lemma 1.2 (cf. Theorem 3.1 on p. 9 of [10]) as an amalgam of two core insights: the first insight is that the squared magnitude of the short-time Fourier transform of a bandlimited function is bandlimited in its first argument; the second insight is that certain translates of Gaussians are complete in $L^2([a, b])$. We note that the first insight allows for the application of the WSK sampling theorem in the time axis of the time-frequency plane while the second insight can be used to analyse sampling in the frequency axis. It is therefore interesting to think of the proof of Lemma 1.2 as a two step approach: first, time is discretised: secondly, frequency is discretised.

We will start by showing that the square of the magnitude of the short-time Fourier transform of a bandlimited function is bandlimited in its first argument. We note that this first insight holds for general windows $\phi \in L^2(\mathbb{R})$.

Lemma 3.1. *Let $p \in [2, \infty]$ and suppose that $q \in [1, 2]$ is chosen such that*

$$\frac{1}{q} = \frac{1}{2} + \frac{1}{p}.$$

Furthermore, let $B > 0$, $\phi \in L^2(\mathbb{R})$, and $f \in \mathrm{PW}_B^p$. For all $\omega \in \mathbb{R}$ it holds that

1. *$\mathrm{M}_\omega \mathcal{V}_\phi f(\cdot, \omega)$ is the restriction of a function in PW_B^q to $\mathbb{R}$,*
2. *$|\mathcal{V}_\phi f(\cdot, \omega)|^2$ is the restriction of a function in $\mathrm{PW}_{2B}^{p/2}$ to $\mathbb{R}$.*

Proof. We remember that the assumption $p \in [2, \infty]$ ascertains that $f|_\mathbb{R} \in L^2(\mathbb{R})$ and that thereby the short-time Fourier transform of f is a well-defined uniformly continuous function. Let us now fix $\omega \in \mathbb{R}$ arbitrary for this proof.

1. We start by considering the function

$$H_\omega(\xi) := \mathcal{F}(f|_\mathbb{R})(\xi)\overline{\mathcal{F}\phi(\xi - \omega)} = \mathcal{F}(f|_\mathbb{R})(\xi) \cdot \mathrm{T}_\omega \overline{\mathcal{F}\phi}(\xi),$$

for $\xi \in [-B, B]$. Since $f \in \mathrm{PW}_B^p$, it follows from Lemma 2.8 that $\mathcal{F}(f|_\mathbb{R}) \in L^p(\mathbb{R})$. Moreover, the assumption that $\phi \in L^2(\mathbb{R})$ implies by Plancherel's theorem that $\mathcal{F}\phi \in L^2(\mathbb{R})$. Since translations are isometries of $L^2(\mathbb{R})$, we find that $\mathrm{T}_\omega \mathcal{F}\phi \in L^2(\mathbb{R})$. Hence, it follows from Hölder's inequality that

$$\mathcal{F}(f|_\mathbb{R}) \cdot \mathrm{T}_\omega \overline{\mathcal{F}\phi} \in L^q(\mathbb{R})$$

and thus $H_\omega \in L^q([-B, B])$.

We will now define $h_\omega \in \mathrm{PW}_B^q$ by

$$h_\omega(z) := \int_{-B}^{B} H_\omega(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}\xi, \qquad z \in \mathbb{C}.$$

Let $x \in \mathbb{R}$ and note that by definition

$$h_\omega(x) = \int_{-B}^{B} \mathcal{F}(f|_\mathbb{R})(\xi)\overline{\mathcal{F}\phi(\xi - \omega)} \mathrm{e}^{2\pi \mathrm{i} \xi x} \, \mathrm{d}\xi.$$

According to Lemma 2.8, it holds that $\operatorname{supp} \mathcal{F}(f|_\mathbb{R}) \subset [-B, B]$. Therefore,

$$h_\omega(x) = \int_{-B}^{B} \mathcal{F}(f|_\mathbb{R})(\xi)\overline{\mathcal{F}\phi(\xi - \omega)} \mathrm{e}^{2\pi \mathrm{i} \xi x} \, \mathrm{d}\xi$$

$$= \int_{\mathbb{R}} \mathcal{F}(f|_{\mathbb{R}})(\xi)\overline{\mathcal{F}\phi(\xi-\omega)}\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi$$
$$= \mathcal{V}_{\mathcal{F}\phi}\left(\mathcal{F}(f|_{\mathbb{R}})\right)(\omega,-x)$$

holds and we can use the fundamental identity of time-frequency analysis to obtain

$$h_\omega(x) = \mathcal{V}_{\mathcal{F}\phi}\left(\mathcal{F}(f|_{\mathbb{R}})\right)(\omega,-x) = \mathrm{e}^{2\pi\mathrm{i}x\omega}\mathcal{V}_\phi f(x,\omega).$$

It follows that $\mathrm{M}_\omega\,\mathcal{V}_\phi f(\cdot,\omega)$ is the restriction of $h_\omega \in \mathrm{PW}_B^q$ to $\mathbb{R}$.

2. We denote the trivial extension of $H_\omega \in L^q([-B,B])$ to $\mathbb{R}$ by $H_{\omega,0} \in L^q(\mathbb{R})$, as mentioned in the paragraph "Notation". Then, we define the function
$$F_\omega(\xi) := \left(H_{\omega,0} * H_{\omega,0}^{\#}\right)(\xi), \qquad \xi \in [-2B,2B].$$
Notably, Young's convolution inequality implies that
$$H_{\omega,0} * H_{\omega,0}^{\#} \in L^{p/2}(\mathbb{R})$$
as $H_{\omega,0}, H_{\omega,0}^{\#} \in L^q(\mathbb{R})$ and
$$\frac{1}{q} + \frac{1}{q} = 1 + \frac{2}{p}.$$
Therefore, F_ω is a well-defined function in $L^{p/2}([-2B,2B])$.
We furthermore note that
$$\operatorname{supp} H_{\omega,0} * H_{\omega,0}^{\#} \subset [-2B,2B]$$
because $H_{\omega,0}$ and $H_{\omega,0}^{\#}$ are supported in the interval $[-B,B]$. Finally, we remark that $H_\omega \in L^q([-B,B]) \subset L^1([-B,B])$ by Proposition 2.4 and $q \geq 1$. It follows that $H_{\omega,0}$ and $H_{\omega,0}^{\#}$ are in $L^1(\mathbb{R})$.
We may now define $f_\omega \in \mathrm{PW}_{2B}^{p/2}$ via
$$f_\omega(z) := \int_{-2B}^{2B} F_\omega(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi z}\,\mathrm{d}\xi, \qquad z \in \mathbb{C}.$$
As in the proof of item 1, we may consider $x \in \mathbb{R}$ arbitrary but fixed and note that our observation on the support of $H_{\omega,0} * H_{\omega,0}^{\#}$ implies that
$$f_\omega(x) = \int_{-2B}^{2B}\left(H_{\omega,0} * H_{\omega,0}^{\#}\right)(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi = \int_{\mathbb{R}}\left(H_{\omega,0} * H_{\omega,0}^{\#}\right)(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi.$$
We had also noted that $H_{\omega,0}, H_{\omega,0}^{\#} \in L^1(\mathbb{R})$ such that we may apply the Fourier convolution theorem to see
$$f_\omega(x) = \int_{\mathbb{R}}\left(H_{\omega,0} * H_{\omega,0}^{\#}\right)(\xi)\mathrm{e}^{2\pi\mathrm{i}\xi x}\,\mathrm{d}\xi = \mathcal{F}\left(H_{\omega,0} * H_{\omega,0}^{\#}\right)(-x)$$

$$= \mathcal{F}H_{\omega,0}(-x)\mathcal{F}\left(H^{\#}_{\omega,0}\right)(-x) = \mathcal{F}H_{\omega,0}(-x)\overline{\mathcal{F}\left(H_{\omega,0}\right)(-x)}$$
$$= \left|\mathcal{F}H_{\omega,0}(-x)\right|^2.$$

It follows from the considerations in the proof of item 1 that

$$f_\omega(x) = \left|\mathcal{F}H_{\omega,0}(-x)\right|^2 = \left|h_\omega(x)\right|^2 = \left|\mathcal{V}_\phi f(x,\omega)\right|^2.$$

Hence, $\left|\mathcal{V}_\phi f(\cdot,\omega)\right|^2$ is the restriction of $f_\omega \in \mathrm{PW}^{p/2}_{2B}$ to $\mathbb{R}$.

□

Next, we note that certain translates of Gaussians are complete in $L^2([a,b])$. This Müntz–Szász type result was proven in [15] (Theorem 4 on p. 302).

Theorem 3.2 (Zalik's theorem). *Let $-\infty < a < b < \infty$, $c_z \in \mathbb{R}\setminus\{0\}$ and let $(c_n)_{n\in\mathbb{N}} \in \mathbb{R}$ be a sequence of* distinct *numbers. Then,*

$$\left\{ \mathrm{e}^{-c_z^2(\cdot - c_n)^2} \,\middle|\, n \in \mathbb{N} \right\}$$

is complete in $L^2([a,b])$ if and only if

$$\sum{}' \left|c_n\right|^{-1}$$

diverges.

Zalik's theorem together with Lemma 3.1 allows for the proof the following proposition.

Proposition 3.3 (C.f. Proposition 3.4 on p. 11 of [10]). *Let $B > 0$ and $b \in (0, \frac{1}{4B}]$. Then, the following are equivalent for $f, g \in \mathrm{PW}^4_B$:*

1. *$f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$,*
2. *$|\mathcal{G}f| = |\mathcal{G}g|$ on $b\mathbb{Z} \times \mathbb{N}$.*

We may use the fractional Fourier transform to rotate the above result in the time-frequency plane and thus obtain Lemma 1.2 as a corollary. More generally, we can obtain a result for functions in the spaces

$$\mathcal{F}_\theta L^4([-B,B]) := \left\{ f : \mathbb{R} \to \mathbb{C} \,\middle|\, \exists\, F \in L^4([-B,B]) : f = \mathcal{F}_\theta F_0 \right\},$$

for $\theta \in \mathbb{R}$ and $B > 0$, where F_0 is defined as in the paragraph "Notation". We should note that the Gabor transform of elements in $\mathcal{F}_\theta L^4([-B,B])$ is well-defined since $L^4([-B,B]) \subset L^2([-B,B])$ (cf. Proposition 2.4) implies $F_0 \in L^2(\mathbb{R})$. The unitarity of the fractional Fourier transform does therefore imply that $\mathcal{F}_\theta L^4([-B,B]) \subset L^2(\mathbb{R})$.

Proposition 3.4 (C.f. Proposition 3.4 on p. 11 of [10]). *Let $B > 0$, $b \in (0, \frac{1}{4B}]$ and $\theta \in \mathbb{R}$. Then, the following are equivalent for $f, g \in \mathcal{F}_{-\theta} L^4([-B,B])$:*

1. *$f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$,*
2. *$|\mathcal{G}f| = |\mathcal{G}g|$ on $\mathrm{R}_\theta(\mathbb{N} \times b\mathbb{Z})$.*

4 Two modifications of Zalik's theorem

To generalise Proposition 3.3 to PW_B^p with $p < 4$, we need to modify Zalik's theorem. The reason for this is that we will show that

$$h_\xi := \mathcal{F}(f|_\mathbb{R}) \cdot \mathrm{T}_\xi \overline{\mathcal{F}(f|_\mathbb{R})} - \mathcal{F}(g|_\mathbb{R}) \cdot \mathrm{T}_\xi \overline{\mathcal{F}(g|_\mathbb{R})} \in L^{p/2}(\mathbb{R}), \qquad \xi \in \mathbb{R},$$

is orthogonal to a family of translated Gaussians and want to deduce $h_\xi = 0$ from this. If $p/2 < 2$, then generally $h_\xi \notin L^2(\mathbb{R})$ and thus Zalik's theorem is not applicable.

If $p > 2$, we can however prove that the action on h_ξ of certain translated Gaussians — when we view them as elements of the dual of $L^{p/2}([-B,B])$ which is isometrically isomorphic to $L^{p/(p-2)}([-B,B])$ — is trivial. It is thus sufficient to deduce that those translates of Gaussians are complete in $L^{p/(p-2)}([-B,B])$ to conclude that $h_\xi|_{[-B,B]} = 0$. We therefore propose the following extension of Zalik's theorem.

Theorem 4.1. *Let $p \in [1,\infty)$, $-\infty < a < b < \infty$, $c_\mathrm{z} \in \mathbb{R} \setminus \{0\}$ and let $(c_n)_{n\in\mathbb{N}} \in \mathbb{R}$ be a sequence of* distinct *numbers. Then,*

$$\left\{ \mathrm{e}^{-c_\mathrm{z}^2(\cdot - c_n)^2} \,\middle|\, n \in \mathbb{N} \right\}$$

is complete in $L^p([a,b])$ if and only if

$$\sum\nolimits' |c_n|^{-1}$$

diverges.

In the proof of the above result, we will rely on two results from [13]: the first one is the following Müntz–Szász type result.

Theorem 4.2 (Theorem 6.1 on p. 30 of [13])**.** *Let $p \in [1,\infty)$, $0 < a < b < \infty$, and let $(d_n)_{n\in\mathbb{N}} \in \mathbb{C}$ be a sequence of* distinct *numbers such that there exists a $\delta > 0$ and an $N_0 \in \mathbb{N}$ with*

$$|\mathrm{Re}\, d_n| \geq \delta\, |d_n|\,, \qquad n \geq N_0.$$

Then,

$$\left\{ (\cdot)^{d_n} \,\middle|\, n \in \mathbb{N} \right\}$$

is complete in $L^p([a,b])$ and $\mathcal{C}([a,b])$ if and only if

$$\sum\nolimits' |d_n|^{-1}$$

diverges.

The second one is an interesting construction of an entire function of exponential type which can be seen as the extension of the Fourier transform of a smooth function to $\mathbb{C}$.

Theorem 4.3 (Theorem 5.2 on p. 30 of [13]). *Let $m \in \mathbb{N}_0$, $-\infty < a < b < \infty$, and let $(d_n)_{n\in\mathbb{N}} \in \mathbb{C} \setminus \{0\}$ be an arbitrary sequence of numbers such that*

$$\sum\nolimits' |d_n|^{-1} < \infty.$$

Then, there exists $g \in \mathcal{C}^\infty(\mathbb{R})$ with $\operatorname{supp} g \subset [a, b]$ such that the function

$$G(z) := \int_a^b g(t)\mathrm{e}^{-\mathrm{i}tz}\,\mathrm{d}t, \qquad z \in \mathbb{C},$$

can be factored as

$$G(z) = cz^m \mathrm{e}^{-\mathrm{i}\sigma z} \prod_{n\in\mathbb{N}} \left(1 - \frac{z^2}{d_n^2}\right) \prod_{k\in\mathbb{N}} \cos(\epsilon_k z), \qquad z \in \mathbb{C},$$

where $c \in \mathbb{C} \setminus \{0\}$, the sequence $(\epsilon_k)_{k\in\mathbb{N}} \in (0, \infty)$ is such that

$$\tau = \sum_{k\in\mathbb{N}} \epsilon_k < \infty$$

and $\sigma = a + \tau = b - \tau$.

We may now prove Theorem 4.1 by adapting the proof of Zalik's theorem from [15]. We include this argument in the following for the convenience of the reader.

Proof of Theorem 4.1. Suppose that $\sum'|c_n|^{-1}$ diverges and let $q \in (1, \infty]$ be the Hölder conjugate of p. Then, $L^q([a, b])$ is isometrically isomorphic to the dual of $L^p([a, b])$. We can therefore consider $f \in L^q([a, b])$ such that

$$\int_a^b f(t)\mathrm{e}^{-c_\mathrm{z}^2(t-c_n)^2}\,\mathrm{d}t = 0, \qquad n \in \mathbb{N},$$

and show that $f = 0$ in order to prove that $\{\mathrm{e}^{-c_\mathrm{z}^2(\cdot-c_n)^2} \mid n \in \mathbb{N}\}$ is complete in $L^p([a, b])$. By expanding the square in the exponent of the above integrand, we find that

$$\int_a^b f(t)\mathrm{e}^{-c_\mathrm{z}^2 t^2} \cdot \mathrm{e}^{2c_\mathrm{z}^2 c_n t}\,\mathrm{d}t = 0, \qquad n \in \mathbb{N}. \tag{2}$$

With the notation

$$g(x) := x^{-1/q} f\left(\frac{\log x}{2c_\mathrm{z}^2}\right) \mathrm{e}^{-\frac{\log^2 x}{4c_\mathrm{z}^2}}, \qquad a' := \mathrm{e}^{2c_\mathrm{z}^2 a}, \qquad b' := \mathrm{e}^{2c_\mathrm{z}^2 b},$$

for $x \in [a', b']$, and the substitution $x = \mathrm{e}^{2c_\mathrm{z}^2 t}$, we obtain

$$\begin{aligned}
\int_{a'}^{b'} g(x)x^{c_n-1/p}\,\mathrm{d}x &= \int_{\mathrm{e}^{2c_\mathrm{z}^2 a}}^{\mathrm{e}^{2c_\mathrm{z}^2 b}} x^{-1/q} f\left(\frac{\log x}{2c_\mathrm{z}^2}\right) \mathrm{e}^{-\frac{\log^2 x}{4c_\mathrm{z}^2}} x^{c_n-1/p}\,\mathrm{d}x \\
&= 2c_\mathrm{z}^2 \cdot \int_{\mathrm{e}^{2c_\mathrm{z}^2 a}}^{\mathrm{e}^{2c_\mathrm{z}^2 b}} f\left(\frac{\log x}{2c_\mathrm{z}^2}\right) \mathrm{e}^{-c_\mathrm{z}^2 \frac{\log^2 x}{4c_\mathrm{z}^4}} \cdot x^{c_n} \cdot \frac{1}{2c_\mathrm{z}^2 x}\,\mathrm{d}x \qquad (3) \\
&= 2c_\mathrm{z}^2 \cdot \int_a^b f(t)\mathrm{e}^{-c_\mathrm{z}^2 t^2} \cdot \mathrm{e}^{2c_\mathrm{z}^2 c_n t}\,\mathrm{d}t = 0,
\end{aligned}$$

for $n \in \mathbb{N}$, by equation (2). Moreover, it holds that

$$\begin{aligned}
\|g\|^q_{L^q([a',b'])} &= \int_{a'}^{b'} |g(x)|^q \,\mathrm{d}x = \int_{\mathrm{e}^{2c_\mathrm{z}^2 a}}^{\mathrm{e}^{2c_\mathrm{z}^2 b}} x^{-1} \left| f\left(\frac{\log x}{2c_\mathrm{z}^2}\right)\right|^q \mathrm{e}^{-q\frac{\log^2 x}{4c_\mathrm{z}^2}} \,\mathrm{d}x \\
&= 2c_\mathrm{z}^2 \int_{\mathrm{e}^{2c_\mathrm{z}^2 a}}^{\mathrm{e}^{2c_\mathrm{z}^2 b}} \left| f\left(\frac{\log x}{2c_\mathrm{z}^2}\right)\right|^q \mathrm{e}^{-qc_\mathrm{z}^2\frac{\log^2 x}{4c_\mathrm{z}^4}} \cdot \frac{1}{2c_\mathrm{z}^2 x} \,\mathrm{d}x \\
&= 2c_\mathrm{z}^2 \int_a^b |f(t)|^q \,\mathrm{e}^{-qc_\mathrm{z}^2 t^2} \,\mathrm{d}t \leq 2c_z^2 \,\|f\|^q_{L^q([a,b])} < \infty.
\end{aligned}$$

Since $(c_n)_{n\in\mathbb{N}} \in \mathbb{R}$, it follows that $|\mathrm{Re}(c_n - 1/p)| = |c_n - 1/p|$. It is also true that the numbers $(c_n - 1/p)_{n\in\mathbb{N}}$ are distinct and that $0 < a' < b' < \infty$. Finally, it is readily seen that divergence of

$$\sum\nolimits' |c_n|^{-1} \qquad \text{and} \qquad \sum\nolimits' |c_n - 1/p|^{-1}.$$

are equivalent. It follows from Theorem 4.2 that $\left\{x^{c_n - 1/p} \,\middle|\, n \in \mathbb{N}\right\}$ is complete in $L^p([a',b'])$ and thus equation (3) implies $g = 0$. We conclude that $f = 0$.

Suppose now that $\sum' |c_n|^{-1} < \infty$. If the sequence $(c_n)_{n\in\mathbb{N}}$ contains zero, then we set $m = 1$ and let $(d_n)_{n\in\mathbb{N}} \in \mathbb{C}$ be the sequence obtained from removing zero from $(2\mathrm{i}c_\mathrm{z}^2 c_n)_{n\in\mathbb{N}}$. If $(c_n)_{n\in\mathbb{N}}$ does not contain zero, we set $m = 0$ and let $(d_n)_{n\in\mathbb{N}} = (2\mathrm{i}c_\mathrm{z}^2 c_n)_{n\in\mathbb{N}}$. In any case, we find that

$$\sum\nolimits' |d_n|^{-1} < \infty$$

such that Theorem 4.3 implies that there exists a $g \in \mathcal{C}^\infty(\mathbb{R})$ with $\operatorname{supp} g \subset [a,b]$ and such that

$$G(z) := \int_a^b g(t)\mathrm{e}^{-\mathrm{i}tz} \,\mathrm{d}t, \qquad z \in \mathbb{C},$$

vanishes at the points $(d_n)_{n\in\mathbb{N}}$ (and zero in case the sequence $(c_n)_{n\in\mathbb{N}}$ contains zero). Note also that g is non-trivial: indeed, if g was trivial, then G would be trivial which contradicts its factorisation in Theorem 4.3. So let us define

$$f(t) := g(t)\mathrm{e}^{c_\mathrm{z}^2 t^2}, \qquad t \in [a,b],$$

such that

$$\|f\|_{L^q([a,b])} \leq (b-a)^{1/q} \cdot \sup_{t\in[a,b]} |f(t)| \leq (b-a)^{1/q} \mathrm{e}^{c_\mathrm{z}^2 \max\{|a|,|b|\}^2} \cdot \max_{t\in[a,b]} |g(t)| < \infty,$$

where we used that continuous functions attain their maxima on compact intervals. We therefore have that $f \in L^q([a,b])$ is non-trivial and it holds that

$$\int_a^b f(t)\mathrm{e}^{-c_\mathrm{z}^2 t^2} \cdot \mathrm{e}^{2c_\mathrm{z}^2 c_n t} \,\mathrm{d}t = \int_a^b g(t)\mathrm{e}^{-\mathrm{i}t(2\mathrm{i}c_\mathrm{z}^2 c_n)} \,\mathrm{d}t = G(2\mathrm{i}c_\mathrm{z}^2 c_n) = 0,$$

for $n \in \mathbb{N}$. Multiplying by $\mathrm{e}^{-c_z^2 c_n^2}$, we obtain that

$$\int_a^b f(t)\mathrm{e}^{-c_z^2(t-c_n)^2}\,\mathrm{d}t = 0, \qquad n \in \mathbb{N},$$

and we have thus proven that

$$\left\{\mathrm{e}^{-c_z^2(\cdot-c_n)^2} \,\middle|\, n \in \mathbb{N}\right\}$$

is not complete in $L^p([a,b])$. □

We emphasise that the case $p = \infty$ is excluded from Theorem 4.1. This is unfortunate since in the proof of Theorem 1.3 (for the case $p = 2$), we will show that the action of certain translated Gaussians on

$$h_\xi = \mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_\xi\,\overline{\mathcal{F}(f|_{\mathbb{R}})} - \mathcal{F}(g|_{\mathbb{R}}) \cdot \mathrm{T}_\xi\,\overline{\mathcal{F}(g|_{\mathbb{R}})} \in L^1(\mathbb{R}), \qquad \xi \in \mathbb{R},$$

is trivial. Here, the translated Gaussians are to be seen as elements of the dual of $L^1([a,b])$ which is isometrically isomorphic to $L^\infty([a,b])$. It is however not hard to see that

$$\left\{\mathrm{e}^{-c_z^2(\cdot-c_n)^2} \,\middle|\, n \in \mathbb{N}\right\}$$

cannot be complete in $L^\infty([a,b])$ and that we will thus have to adapt the strategy of our proof at this point: indeed, consider that translated Gaussians are smooth and that uniform limits of continuous functions are continuous. Therefore,

$$\overline{\operatorname{sp}\left\{\mathrm{e}^{-c_z^2(\cdot-c_n)^2} \,\middle|\, n \in \mathbb{N}\right\}}^{L^\infty([a,b])} \subset \mathcal{C}([a,b])$$

and thus translated Gaussians cannot be complete in $L^\infty([a,b])$.

Let us make three remarks on this: first, it might be tempting to show that

$$\left\{\mathrm{e}^{-c_z^2(\cdot-c_n)^2} \,\middle|\, n \in \mathbb{N}\right\}$$

is complete in $\mathcal{C}([a,b])$. While this may be true, it does not seem possible to show it by a simple adaptation of the proof of Zalik's theorem since the dual of $\mathcal{C}([a,b])$ is the space of Radon measures and it is not clear how the steps of the proof would work in this setup.

Secondly, we observe that it is not necessary to show that

$$\left\{\mathrm{e}^{-c_z^2(\cdot-c_n)^2} \,\middle|\, n \in \mathbb{N}\right\}$$

is complete in $\mathcal{C}([a,b])$. In fact, it suffices to show that if the functions $\mathrm{e}^{-c_z^2(\cdot-c_n)^2}$ act trivially on an element $f \in L^1([a,b])$, then $f = 0$. We will express this idea using a standard definition.

Definition 4.4 (Annihilator)**.** *Let V be a normed space with (continuous) dual space V' and let $W \subset V$ be a closed linear subspace. The* annihilator *of W is given by*

$$W^\perp := \{\phi \in V' \,|\, W \subset \ker\phi\}.$$

Note that we want to consider the normed space $L^\infty([a,b])$ in which the underlying measure space is given by $([a,b],\mathcal{B}([a,b]),\lambda|_{\mathcal{B}([a,b])})$, where $\mathcal{B}([a,b])$ denotes the Borel σ-algebra on $[a,b]$ and λ denotes the Lebesgue measure. In this setup, the dual of $L^\infty([a,b])$ can be identified with the space $\mathrm{ba}([a,b],\mathcal{L},\lambda|_{\mathcal{L}})$ (see Theorem 16 on p. 196 of [5]), where

$$\mathcal{L} = \{A \subset [a,b] \,|\, \exists\, B_0, B_1 \in \mathcal{B}([a,b]) : B_0 \subset A \subset B_1 \text{ and } \lambda(B_1 \setminus B_0) = 0\}$$

is the set of Lebesgue measurable subsets of $[a,b]$. Here, $\mathrm{ba}([a,b],\mathcal{L},\lambda|_{\mathcal{L}})$ denotes the space of all bounded, finitely additive signed measures on $\mathcal{L}$ which are absolutely continuous with respect to λ equipped with the total variation norm. In this setting, the space $L^1([a,b])$ can be identified as a subspace of $\mathrm{ba}([a,b],\mathcal{L},\lambda|_{\mathcal{L}})$ through the definition

$$\mu(A) := \int_A f \,\mathrm{d}\lambda, \qquad A \in \mathcal{L}.$$

Using the annihilator notation, we may thus see that realising our second remark amounts to proving that

$$\left(\overline{\mathrm{sp}\left\{\mathrm{e}^{-c_{\mathrm{z}}^2(\cdot - c_n)^2} \,\middle|\, n \in \mathbb{N}\right\}}^{L^\infty([a,b])}\right)^{\perp} \cap L^1([a,b]) = \{0\}. \tag{4}$$

Thirdly, approximating continuous functions with linear combinations of elements of

$$\{(\cdot)^{c_n} \,|\, n \in \mathbb{N}\}$$

is sufficient for proving equation (4) because the Fourier characters $\chi_\omega = \mathrm{e}^{2\pi\mathrm{i}\omega\cdot}$ are continuous.

Theorem 4.5. *Let $-\infty < a < b < \infty$, $c_{\mathrm{z}} \in \mathbb{R} \setminus \{0\}$ and let $(c_n)_{n\in\mathbb{N}} \in \mathbb{R}$ be a sequence of* distinct *numbers. Then, it holds that*

$$\left(\overline{\mathrm{sp}\left\{\mathrm{e}^{-c_{\mathrm{z}}^2(\cdot - c_n)^2} \,\middle|\, n \in \mathbb{N}\right\}}^{L^\infty([a,b])}\right)^{\perp} \cap L^1([a,b]) = \{0\}$$

if and only if

$$\sum{}' |c_n|^{-1}$$

diverges.

Proof. Suppose that $\sum{}'|c_n|^{-1}$ diverges and let us consider $f \in L^1([a,b])$ such that

$$\int_a^b f(t)\mathrm{e}^{-c_{\mathrm{z}}^2(t-c_n)^2}\,\mathrm{d}t = 0, \qquad n \in \mathbb{N}.$$

Our goal is to show that $f = 0$. As in the proof of Theorem 4.1, the notation

$$g(x) := x^{-1} f\left(\frac{\log x}{2c_{\mathrm{z}}^2}\right)\mathrm{e}^{-\frac{\log^2 x}{4c_{\mathrm{z}}^2}}, \qquad a' := \mathrm{e}^{2c_{\mathrm{z}}^2 a}, \qquad b' := \mathrm{e}^{2c_{\mathrm{z}}^2 b},$$

for $x \in [a', b']$, and the substitution $x = \mathrm{e}^{2c_z^2 t}$ allows us to compute that

$$\int_{a'}^{b'} g(x) x^{c_n} \,\mathrm{d}x = 0, \qquad n \in \mathbb{N}, \tag{5}$$

and that $g \in L^1([a', b'])$. Moreover, Theorem 4.2 implies that $\{(\cdot)^{c_n} \,|\, n \in \mathbb{N}\}$ is complete in $\mathcal{C}([a', b'])$. So, let us consider $\xi \in \mathbb{R}$ arbitrary but fixed and note that there exists a sequence $s_k \in \mathcal{C}([a', b'])$ of the form

$$s_k(x) = \sum_{n=1}^{N(k)} \lambda_n(k) x^{c_n}, \qquad x \in [a', b'],$$

where $(N(k))_{k \in \mathbb{N}} \in \mathbb{N}$ and $(\lambda_n(k))_{n,k \in \mathbb{N}} \in \mathbb{C}$, such that

$$\sup_{x \in [a', b']} |\chi_\xi(x) - s_k(x)| < \frac{1}{k \cdot \|g\|_{L^1([a', b'])}},$$

for $k \in \mathbb{N}$. It follows from the linearity of the integral and equation (5) that

$$\int_{a'}^{b'} g(x) s_k(x) \,\mathrm{d}x = \sum_{n=1}^{N(k)} \lambda_n(k) \int_{a'}^{b'} g(x) x^{c_n} \,\mathrm{d}x = 0, \qquad k \in \mathbb{N}.$$

Hence, we may estimate

$$\begin{aligned} \left| \int_{a'}^{b'} g(x) \chi_\xi(x) \,\mathrm{d}x \right| &= \left| \int_{a'}^{b'} g(x) \left(\chi_\xi(x) - s_k(x) + s_k(x) \right) \mathrm{d}x \right| \\ &\leq \left| \int_{a'}^{b'} g(x) \left(\chi_\xi(x) - s_k(x) \right) \mathrm{d}x \right| \\ &\leq \|g\|_{L^1([a,b])} \cdot \sup_{x \in [a', b']} |\chi_\xi(x) - s_k(x)| \leq k^{-1}, \end{aligned}$$

for $k \in \mathbb{N}$. Therefore,

$$\mathcal{F} g_0(-\xi) = \int_{a'}^{b'} g(x) \mathrm{e}^{2\pi \mathrm{i} x \xi} \,\mathrm{d}x = \int_{a'}^{b'} g(x) \chi_\xi(x) \,\mathrm{d}x = 0$$

and, since $\xi \in \mathbb{R}$ was arbitrary, we conclude that $\mathcal{F} g_0 = 0$. This implies that $g_0 = 0$ and thus that $g = 0$ which shows that $f = 0$.

Suppose now that $\sum' |c_n|^{-1} < \infty$. As in the proof of Theorem 4.1, we may find a non-trivial function $f \in L^1([a, b])$ such that

$$\int_a^b f(t) \mathrm{e}^{-c_z^2 (t - c_n)^2} \,\mathrm{d}t = 0, \qquad n \in \mathbb{N}.$$

Therefore,

$$\left(\overline{\mathrm{sp}\left\{ \mathrm{e}^{-c_z^2 (\cdot - c_n)^2} \,\middle|\, n \in \mathbb{N} \right\}}^{L^\infty([a,b])} \right)^\perp \cap L^1([a, b]) \neq \{0\}.$$

□

5 Generalisation of the sampling result: step by step

As mentioned in the introduction, it is remarkable that the sampling result in [10] which we have reimagined in Section 3 (Proposition 3.4) is only stated and proven in $\mathcal{F}_\theta L^4([-B,B])$. This immediately raises the question whether a similar result continues to hold if we replace $\mathcal{F}_\theta L^4([-B,B])$ by the more general spaces

$$\mathcal{F}_\theta L^p([-B,B]) := \{f : \mathbb{R} \to \mathbb{C} \,|\, \exists\, F \in L^p([-B,B]) : f = \mathcal{F}_\theta F_0\},$$

for $\theta \in \mathbb{R}$, $p \in [2,\infty]$, and $B > 0$. In particular, the case $p = 2$ seems interesting as Gabor phase retrieval is usually stated with respect to the Hilbert space $L^2(\mathbb{R})$. We should note that, just as the L^p-spaces on closed intervals and the Paley–Wiener spaces, the spaces $\mathcal{F}_\theta L^p([-B,B])$ are nested.

Proposition 5.1. *Let $1 \leq p \leq q \leq \infty$, $B > 0$, and $\theta \in \mathbb{R}$. Then, we have $\mathcal{F}_\theta L^q([-B,B]) \subset \mathcal{F}_\theta L^p([-B,B])$.*

Proof. Let $f \in \mathcal{F}_\theta L^q([-B,B])$ be arbitrary. Then, by definition, there exists an $F \in L^q([-B,B])$ such that $f = \mathcal{F}_\theta F_0$. By Proposition 2.4, we find that $F \in L^p([-B,B])$ and thus that $f \in \mathcal{F}_\theta L^p([-B,B])$. □

It follows that the most general case which we will be considering is $f, g \in \mathcal{F}_\theta L^2([-B,B])$. Interestingly, the difficulty of generalising Proposition 3.4 to $\mathcal{F}_\theta L^2([-B,B])$ can already be understood from considering Lemma 3.1. Indeed, the case $\mathcal{F}_\theta L^4([-B,B])$ is in some sense particularly easy to deal with since $f \in \mathrm{PW}_B^4$ implies that $|\mathcal{G}f|^2 \in \mathrm{PW}_{2B}^2$ which is exactly the space for which we can apply the WSK sampling theorem. Additionally,

$$h_\xi = \mathcal{F}(f|_\mathbb{R}) \cdot \mathrm{T}_\xi\, \overline{\mathcal{F}(f|_\mathbb{R})} - \mathcal{F}(g|_\mathbb{R}) \cdot \mathrm{T}_\xi\, \overline{\mathcal{F}(g|_\mathbb{R})} \in L^2(\mathbb{R}), \qquad \xi \in \mathbb{R},$$

allows for the application of Zalik's theorem. If $f \in \mathrm{PW}_B^2$, then $|\mathcal{G}f|^2 \in \mathrm{PW}_{2B}^1$, however, and we need to replace the use of the WSK sampling theorem by the use of a sampling theorem in a Bernstein space. In addition, as we have discussed in the prior section, we cannot apply Zalik's theorem and will instead need to make use of Theorem 4.5.

As advertised in the title of the present section, we will generalise Proposition 3.4 step by step. To be precise, we will prove that Proposition 3.4 will continue to hold if we replace $\mathcal{F}_\theta L^4([-B,B])$ by $\mathcal{F}_\theta L^p([-B,B])$, for general $p \in [2,\infty]$. We will do this in three steps which are naturally ordered by difficulty: first, we consider $p \geq 4$, then we consider $p \in (2,4)$ and finally we consider $p = 2$.

5.1 $p \geq 4$

Let us start with the case $p \geq 4$. In this case, we obtain the following result as a direct corollary to Proposition 3.3.

Corollary 5.2. *Let $p \in [4, \infty]$, $B > 0$ and $b \in (0, \frac{1}{4B}]$. Then, the following are equivalent for $f, g \in \mathrm{PW}_B^p$:*

1. *$f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$,*
2. *$|\mathcal{G}f| = |\mathcal{G}g|$ on $b\mathbb{Z} \times \mathbb{N}$.*

Proof. According to Corollary 2.5, it holds that $\mathrm{PW}_B^p \subset \mathrm{PW}_B^4$, for $p \in [4, \infty]$. Therefore, $f, g \in \mathrm{PW}_B^p$ satisfy that $f, g \in \mathrm{PW}_B^4$ and thus the equivalence of item 1 and item 2 follows immediately from Proposition 3.3. □

5.2 $p \in (2, 4)$

Next, we can consider $p \in (2, 4)$. In this case, we will need to make use of a generalised version of the WSK sampling theorem. To be precise, we may apply Lemma 2.9 to see that $\mathrm{PW}_B^p \subset B_{2\pi B}^q$, where $q \in (2, \infty)$ is the Hölder conjugate of p, and then utilise Theorem 2.10. In addition, we will apply Theorem 4.1.

Proposition 5.3. *Let $p \in (2, 4)$, $B > 0$ and $b \in (0, \frac{1}{4B}]$. Then, the following are equivalent for $f, g \in \mathrm{PW}_B^p$:*

1. *$f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$,*
2. *$|\mathcal{G}f| = |\mathcal{G}g|$ on $b\mathbb{Z} \times \mathbb{N}$.*

Proof. First, note that if $f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$, then it follows immediately that $|\mathcal{G}f| = |\mathcal{G}g|$. Secondly, suppose that $|\mathcal{G}f| = |\mathcal{G}g|$ on $b\mathbb{Z} \times \mathbb{N}$. If $k \in \mathbb{N}$ is arbitrary but fixed, it follows directly from the Lemmata 2.9 and 3.1 that $|\mathcal{G}f(\cdot, k)|^2$ and $|\mathcal{G}g(\cdot, k)|^2$ are restrictions of functions in

$$\mathrm{PW}_{2B}^{p/2} \subset \mathrm{B}_{4\pi B}^{p/(p-2)} \subset \mathrm{B}_{\pi/b}^{p/(p-2)}$$

to $\mathbb{R}$. Therefore, Theorem 2.10 implies

$$|\mathcal{G}f(x, k)|^2 = |\mathcal{G}g(x, k)|^2, \qquad x \in \mathbb{R}. \tag{6}$$

To apply Theorem 4.1, we need to reformulate the equation above. For this purpose, we remember that $f \in \mathrm{PW}_B^p \subset \mathrm{PW}_B^2$ by $p > 2$ (Corollary 2.5) and that the Paley–Wiener theorem does therefore imply that $f|_{\mathbb{R}} \in L^2(\mathbb{R})$. Using that the Gaussian is invariant under the Fourier transform as well as the fundamental identity of time-frequency analysis, we can compute that

$$|\mathcal{G}f(x, k)|^2 = |\mathcal{G}\left(\mathcal{F}(f|_{\mathbb{R}})\right)(k, -x)|^2 = \mathcal{G}\left(\mathcal{F}(f|_{\mathbb{R}})\right)(k, -x)\overline{\mathcal{G}\left(\mathcal{F}(f|_{\mathbb{R}})\right)(k, -x)}.$$

As the short-time Fourier transform corresponds to the Fourier transform of the short-time sections of the underlying function, we find that

$$\begin{aligned} |\mathcal{G}f(x, k)|^2 &= \mathcal{G}\left(\mathcal{F}(f|_{\mathbb{R}})\right)(k, -x)\overline{\mathcal{G}\left(\mathcal{F}(f|_{\mathbb{R}})\right)(k, -x)} \\ &= \mathcal{F}\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)(-x)\overline{\mathcal{F}\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)(-x)} \end{aligned}$$

$$= \mathcal{F}\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)(-x) \cdot \mathcal{F}\left(\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)^{\#}\right)(-x).$$

We note that $\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi$ is in $L^1(\mathbb{R})$ because $f|_{\mathbb{R}}, \phi \in L^2(\mathbb{R})$. As the involution $(\cdot)^{\#}$ is an isometry of $L^1(\mathbb{R})$, we may apply the Fourier convolution theorem to the above equation. In this way, we obtain that

$$\begin{aligned} |\mathcal{G}f(x,k)|^2 &= \mathcal{F}\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)(-x) \cdot \mathcal{F}\left(\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)^{\#}\right)(-x) \\ &= \mathcal{F}\left(\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right) * \left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)^{\#}\right)(-x). \end{aligned}$$

We may also note that $\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi$ is in $L^{4/3}(\mathbb{R})$. To see this, we can use Hölder's inequality together with the facts that $\mathrm{T}_k\, \phi \in L^4(\mathbb{R})$ and $\mathcal{F}(f|_{\mathbb{R}}) \in L^2(\mathbb{R})$. The prior follows from translations being isometries of $L^4(\mathbb{R})$ and $\phi \in L^4(\mathbb{R})$. Since the involution $(\cdot)^{\#}$ is an isometry of $L^{4/3}(\mathbb{R})$, it follows from Young's convolution inequality that

$$\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right) * \left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)^{\#} \in L^2(\mathbb{R}).$$

Therefore, we find that for almost every $\xi \in \mathbb{R}$, it holds that

$$\begin{aligned} \mathcal{F}\left(|\mathcal{G}f(\cdot,k)|^2\right)(\xi) &= \left(\left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right) * \left(\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_k\, \phi\right)^{\#}\right)(\xi) \\ &= \int_{\mathbb{R}} \mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} \cdot \phi(\omega-k)\phi(\omega-\xi-k)\,\mathrm{d}\omega. \end{aligned}$$

We may now use that $\phi = c\mathrm{e}^{-\pi(\cdot)^2}$, with $c = 2^{1/4}$, and compute

$$\begin{aligned} \mathcal{F}\left(|\mathcal{G}f(\cdot,k)|^2\right)(\xi) &= \int_{\mathbb{R}} \mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} \cdot \phi(\omega-k)\phi(\omega-\xi-k)\,\mathrm{d}\omega \\ &= \sqrt{2}\int_{\mathbb{R}} \mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} \cdot \mathrm{e}^{-\pi(\omega-k)^2-\pi(\omega-\xi-k)^2}\,\mathrm{d}\omega \\ &= \sqrt{2}\int_{\mathbb{R}} \mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} \cdot \mathrm{e}^{-2\pi\left(\omega-k-\frac{\xi}{2}\right)^2-\frac{\pi\xi^2}{2}}\,\mathrm{d}\omega \end{aligned}$$

by completing the square in the exponent. By Lemma 2.8, we know that $\operatorname{supp}\mathcal{F}(f|_{\mathbb{R}}) \subset [-B,B]$ and thus

$$\begin{aligned} \mathcal{F}\left(|\mathcal{G}f(\cdot,k)|^2\right)(\xi) &= \sqrt{2}\mathrm{e}^{-\frac{\pi\xi^2}{2}}\int_{\mathbb{R}} \mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} \cdot \mathrm{e}^{-2\pi\left(\omega-k-\frac{\xi}{2}\right)^2}\,\mathrm{d}\omega \\ &= \sqrt{2}\mathrm{e}^{-\frac{\pi\xi^2}{2}}\int_{-B}^{B} \mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} \cdot \mathrm{e}^{-2\pi\left(\omega-k-\frac{\xi}{2}\right)^2}\,\mathrm{d}\omega. \end{aligned}$$

Of course, the exact same can be shown for $g \in \mathrm{PW}_B^p$ and therefore it follows from equation (6) that

$$\sqrt{2}\mathrm{e}^{-\frac{\pi\xi^2}{2}}\int_{-B}^{B} \mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} \cdot \mathrm{e}^{-2\pi\left(\omega-k-\frac{\xi}{2}\right)^2}\,\mathrm{d}\omega$$

$$= \sqrt{2}\mathrm{e}^{-\frac{\pi\xi^2}{2}} \int_{-B}^{B} \mathcal{F}(g|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(g|_{\mathbb{R}})(\omega-\xi)} \cdot \mathrm{e}^{-2\pi\left(\omega-k-\frac{\xi}{2}\right)^2}\,\mathrm{d}\omega$$

holds, for almost every $\xi \in \mathbb{R}$. Hence, we have

$$\int_{-B}^{B} \Big(\mathcal{F}(f|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(f|_{\mathbb{R}})(\omega-\xi)} - \mathcal{F}(g|_{\mathbb{R}})(\omega)\overline{\mathcal{F}(g|_{\mathbb{R}})(\omega-\xi)}\Big) \cdot \mathrm{e}^{-2\pi\left(\omega-k-\frac{\xi}{2}\right)^2}\,\mathrm{d}\omega = 0, \quad (7)$$

for almost every $\xi \in \mathbb{R}$. By Lemma 2.8, $\mathcal{F}(f|_{\mathbb{R}}), \mathcal{F}(g|_{\mathbb{R}}) \in L^p(\mathbb{R})$ and thus

$$h_\xi = \mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_\xi\, \overline{\mathcal{F}(f|_{\mathbb{R}})} - \mathcal{F}(g|_{\mathbb{R}}) \cdot \mathrm{T}_\xi\, \overline{\mathcal{F}(g|_{\mathbb{R}})} \in L^{p/2}(\mathbb{R}), \qquad \xi \in \mathbb{R}.$$

The dual of $L^{p/2}(\mathbb{R})$ is isometrically isomorphic to $L^{p/(p-2)}(\mathbb{R})$ and since $p/(p-2) \in [2,\infty)$, we may apply Theorem 4.1.

Let us fix $\xi \in \mathbb{R}$ arbitrary in a set of full measure in which equation (7) holds and set $a = -B$, $b = B$, $c_{\mathrm{z}} = \sqrt{2\pi}$ as well as $c_n = n + \xi/2$, for $n \in \mathbb{N}$. As

$$\sum{}' \, |n + \xi/2|'$$

diverges, Theorem 4.1 implies that

$$\left\{ \mathrm{e}^{-2\pi\left(\cdot-n-\frac{\xi}{2}\right)^2} \,\middle|\, n \in \mathbb{N} \right\}$$

is complete in $L^{p/(p-2)}([-B,B])$. Since $k \in \mathbb{N}$ was arbitrary in our computations above, this together with equation (7) implies that $h_\xi|_{[-B,B]} = 0$ and thus that

$$\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_\xi\, \overline{\mathcal{F}(f|_{\mathbb{R}})} = \mathcal{F}(g|_{\mathbb{R}}) \cdot \mathrm{T}_\xi\, \overline{\mathcal{F}(g|_{\mathbb{R}})} \quad (8)$$

as functions in $L^{p/2}([-B,B])$. By the support properties of $\mathcal{F}(f|_{\mathbb{R}})$ and $\mathcal{F}(g|_{\mathbb{R}})$ this equation extends to $L^{p/2}(\mathbb{R})$.

Since ξ was chosen arbitrarily in a set of full measure, we may take the Fourier transform of equation (8) to obtain that

$$\mathcal{V}_{\mathcal{F}(f|_{\mathbb{R}})}\mathcal{F}(f|_{\mathbb{R}}) = \mathcal{V}_{\mathcal{F}(g|_{\mathbb{R}})}\mathcal{F}(g|_{\mathbb{R}}).$$

Hence, by the fundamental identity of time-frequency analysis,

$$\mathcal{V}_{f|_{\mathbb{R}}} f = \mathcal{V}_{g|_{\mathbb{R}}} g.$$

Finally, Lemma 2.2 implies that there exists an $\alpha \in \mathbb{R}$ such that $f|_{\mathbb{R}} = \mathrm{e}^{\mathrm{i}\alpha} g|_{\mathbb{R}}$. As both f and g are entire, this equality extends to $f = \mathrm{e}^{\mathrm{i}\alpha} g$. □

5.3 $p = 2$

Finally, we may consider the most general case $p = 2$. As before, we can see that it follows from Lemma 2.9 that $\mathrm{PW}_B^2 \subset B_{2\pi B}^\infty$ and that we can therefore use Theorem 2.11 to take care of the sampling in time. We observe here that Theorem 2.11 does not guarantee unique recovery from samples at the critical rate in contrast to the WSK sampling theorem and Theorem 2.10. Additionally, we make use of Theorem 4.5 to take care of the sampling in frequency.

Proposition 5.4. *Let $B > 0$ and $b \in (0, \frac{1}{4B})$. Then, the following are equivalent for $f, g \in \mathrm{PW}_B^2$:*

1. $f = \mathrm{e}^{\mathrm{i}\alpha} g$, *for some* $\alpha \in \mathbb{R}$,
2. $|\mathcal{G}f| = |\mathcal{G}g|$ *on* $b\mathbb{Z} \times \mathbb{N}$.

Proof. First, note that if $f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$, then it follows immediately that $|\mathcal{G}f| = |\mathcal{G}g|$. Secondly, suppose that $|\mathcal{G}f| = |\mathcal{G}g|$ on $b\mathbb{Z} \times \mathbb{N}$. If $k \in \mathbb{N}$ is arbitrary but fixed, it follows directly from the Lemmata 2.9 and 3.1 that $|\mathcal{G}f(\cdot, k)|^2$ and $|\mathcal{G}g(\cdot, k)|^2$ are restrictions of functions in

$$\mathrm{PW}_{2B}^1 \subset \mathrm{B}_{4\pi B}^\infty$$

to $\mathbb{R}$. Therefore, Theorem 2.11 implies

$$|\mathcal{G}f(x,k)|^2 = |\mathcal{G}g(x,k)|^2, \qquad x \in \mathbb{R}. \tag{9}$$

We may now exactly follow the calculations in the proof of Proposition 5.3 to see that

$$\int_{-B}^{B} \Big(\mathcal{F}(f|_{\mathbb{R}})(\omega) \overline{\mathcal{F}(f|_{\mathbb{R}})(\omega - \xi)} - \mathcal{F}(g|_{\mathbb{R}})(\omega) \overline{\mathcal{F}(g|_{\mathbb{R}})(\omega - \xi)} \Big) \cdot \mathrm{e}^{-2\pi \left(\omega - k - \frac{\xi}{2}\right)^2} \, \mathrm{d}\omega = 0 \tag{10}$$

holds, for almost every $\xi \in \mathbb{R}$. By Lemma 2.8, $\mathcal{F}(f|_{\mathbb{R}}), \mathcal{F}(g|_{\mathbb{R}}) \in L^2(\mathbb{R})$ and thus

$$h_\xi = \mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_\xi \, \overline{\mathcal{F}(f|_{\mathbb{R}})} - \mathcal{F}(g|_{\mathbb{R}}) \cdot \mathrm{T}_\xi \, \overline{\mathcal{F}(g|_{\mathbb{R}})} \in L^1(\mathbb{R}), \qquad \xi \in \mathbb{R}.$$

The dual of $L^1(\mathbb{R})$ is isometrically isomorphic to $L^\infty(\mathbb{R})$ such that we may apply Theorem 4.5.

Let us fix $\xi \in \mathbb{R}$ arbitrary in a set of full measure in which equation (7) holds and set $a = -B$, $b = B$, $c_{\mathrm{z}} = \sqrt{2\pi}$ as well as $c_n = n + \xi/2$, for $n \in \mathbb{N}$. As

$$\sum{}' \, |n + \xi/2|'$$

diverges, Theorem 4.5 implies that

$$\left(\overline{\mathrm{sp}\left\{ \mathrm{e}^{-c_{\mathrm{z}}^2 (\cdot - c_n)^2} \;\middle|\; n \in \mathbb{N} \right\}}^{L^\infty([a,b])} \right)^{\perp} \cap L^1([a,b]) = \{0\}.$$

Since $k \in \mathbb{N}$ was arbitrary in our computations above, this together with equation (10) implies $h_\xi|_{[-B,B]} = 0$ and thus

$$\mathcal{F}(f|_{\mathbb{R}}) \cdot \mathrm{T}_\xi \, \overline{\mathcal{F}(f|_{\mathbb{R}})} = \mathcal{F}(g|_{\mathbb{R}}) \cdot \mathrm{T}_\xi \, \overline{\mathcal{F}(g|_{\mathbb{R}})}$$

as functions in $L^1([-B, B])$. By the support properties of $\mathcal{F}(f|_{\mathbb{R}})$ and $\mathcal{F}(g|_{\mathbb{R}})$ this equation extends to $L^1(\mathbb{R})$. As in the proof of Proposition 5.3, we may now deduce that there exists an $\alpha \in \mathbb{R}$ such that $f = \mathrm{e}^{\mathrm{i}\alpha} g$. □

5.4 Main results

We can now use the fractional Fourier transform to rotate our results in the time-frequency plane. In this way, we might unifyingly state the following theorem.

Theorem 5.5 (Main theorem)**.** *Let* $p \in [2, \infty]$*,* $B > 0$ *and* $\theta \in \mathbb{R}$*. Let* $b \in (0, \frac{1}{4B})$*, if* $p = 2$*, and* $b \in (0, \frac{1}{4B}]$*, if* $p \in (2, \infty]$*. Then, the following are equivalent for* $f, g \in \mathcal{F}_{-\theta} L^p([-B, B])$*:*

1. $f = \mathrm{e}^{\mathrm{i}\alpha} g$*, for some* $\alpha \in \mathbb{R}$*,*
2. $|\mathcal{G}f| = |\mathcal{G}g|$ *on* $\mathrm{R}_\theta(\mathbb{N} \times b\mathbb{Z})$*.*

Proof. It is obvious that item 1 implies item 2. Let us therefore assume that $|\mathcal{G}f| = |\mathcal{G}g|$ on $\mathrm{R}_\theta(\mathbb{N} \times b\mathbb{Z})$, i.e.

$$|\mathcal{G}f(\mathrm{R}_\theta(k, bn))| = |\mathcal{G}g(\mathrm{R}_\theta(k, bn))|, \qquad k \in \mathbb{N},\ n \in \mathbb{Z}.$$

Now, note that by definition there exist $F, G \in L^p([-B, B])$ such that $f = \mathcal{F}_{-\theta} F_0$ and $g = \mathcal{F}_{-\theta} G_0$, respectively. According to the generalised fundamental identity of time-frequency analysis (Lemma 2.3), the fact that the Gaussian is invariant under the fractional Fourier transform, and $\mathrm{R}_{-\theta} \mathrm{R}_\theta = \mathrm{id}$, we thus find that

$$|\mathcal{G}F_0(k, bn)| = |\mathcal{G}G_0(k, bn)|, \qquad k \in \mathbb{N},\ n \in \mathbb{Z}. \tag{11}$$

Let us next define the functions

$$h_f(z) := \int_{-B}^{B} F(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}\xi, \qquad h_g(z) := \int_{-B}^{B} G(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}\xi,$$

for $z \in \mathbb{C}$. Since $F, G \in L^p([-B, B])$, it follows that $h_f, h_g \in \mathrm{PW}_B^p$. By the definition of F_0 and G_0, we find that $h_f|_\mathbb{R} = \mathcal{F}^{-1} F_0$ as well as $h_g|_\mathbb{R} = \mathcal{F}^{-1} G_0$ and thus equation (11) implies

$$|\mathcal{G}\mathcal{F}(h_f|_\mathbb{R})(k, bn)| = |\mathcal{G}\mathcal{F}(h_g|_\mathbb{R})(k, bn)|, \qquad k \in \mathbb{N},\ n \in \mathbb{Z}.$$

According to the fundamental identity of time-frequency analysis, we find that

$$|\mathcal{G}h_f(-bn, k)| = |\mathcal{G}h_g(-bn, k)|, \qquad k \in \mathbb{N},\ n \in \mathbb{Z}.$$

Therefore, it follows from Corollary 5.2 (if $p \geq 4$), Proposition 5.4 (if $p \in (2, 4)$) and Proposition 5.3 (if $p = 2$) that there exists an $\alpha \in \mathbb{R}$ such that $h_f = \mathrm{e}^{\mathrm{i}\alpha} h_g$ which immediately implies that $f = \mathrm{e}^{\mathrm{i}\alpha} g$ by the relations $h_f|_\mathbb{R} = \mathcal{F}^{-1} F_0 = \mathcal{F}^{-1} \mathcal{F}_\theta f$ and $h_g|_\mathbb{R} = \mathcal{F}^{-1} G_0 = \mathcal{F}^{-1} \mathcal{F}_\theta g$. □

It is clear from the proofs presented in this paper that the main theorem continues to hold for more general nonuniform sampling lattices. In particular, $\mathbb{N}$ may be replaced by any sequence $(c_n)_{n \in \mathbb{N}}$ of distinct real numbers such that

$$\sum{}' |c_n|^{-1}$$

diverges, and $b\mathbb{Z}$ may be replaced by any sequence $(t_n)_{n\in\mathbb{Z}}$ of real numbers which satisfies that $f(t_n) = 0$ implies $f = 0$, for all $f \in \mathrm{B}^{\infty}_{4\pi B}$. According to Theorem 3.2 on p. 44 of [16], the condition

$$\sup_{n\in\mathbb{Z}} |t_n - bn| < \frac{b}{4}, \tag{12}$$

with $0 < b < \frac{1}{4B}$, is sufficient to guarantee this and therefore the following holds.

Theorem 5.6. *Let $p \in [2,\infty]$, $B > 0$ and $\theta \in \mathbb{R}$. Let $(c_n)_{n\in\mathbb{N}} \in \mathbb{R}$ be a sequence of distinct numbers such that*

$$\sum\nolimits' |c_n|^{-1}$$

diverges and let $(t_n)_{n\in\mathbb{Z}} \in \mathbb{R}$ be a sequence which satisfies condition (12). Then, the following are equivalent for $f, g \in \mathcal{F}_{-\theta}L^p([-B,B])$:

1. *$f = \mathrm{e}^{\mathrm{i}\alpha} g$, for some $\alpha \in \mathbb{R}$,*
2. *$|\mathcal{G}f| = |\mathcal{G}g|$ on $\mathrm{R}_\theta(\{c_n\}_{n\in\mathbb{N}} \times \{t_n\}_{n\in\mathbb{Z}})$.*

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A Properties of the fractional Fourier transform

Lemma A.1. *Let $\theta \in \mathbb{R}$ and $\phi = c\mathrm{e}^{-\pi(\cdot)^2}$, with $c = 2^{1/4}$. Then, it holds that*

$$\mathcal{F}_\theta \phi = \phi.$$

Proof. Let us start by considering that

$$\mathcal{F}_{2k\pi}\phi = \phi, \qquad \mathcal{F}_{(2k+1)\pi}\phi(\xi) = \phi(-\xi) = \phi(\xi),$$

for $\xi \in \mathbb{R}$, where we have used that ϕ is even. We consider $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ next and compute

$$\begin{aligned}\mathcal{F}_\theta\phi(\xi) &= cc_\theta \mathrm{e}^{\pi\mathrm{i}\xi^2\cot\theta} \int_{\mathbb{R}} \mathrm{e}^{-\pi t^2} \mathrm{e}^{\pi\mathrm{i}t^2\cot\theta} \mathrm{e}^{-2\pi\mathrm{i}\frac{t\xi}{\sin\theta}}\,\mathrm{d}t \\ &= cc_\theta \mathrm{e}^{\pi\mathrm{i}\xi^2\cot\theta} \int_{\mathbb{R}} \mathrm{e}^{-\pi(1-\mathrm{i}\cot\theta)t^2} \mathrm{e}^{-2\pi\mathrm{i}t\frac{\xi}{\sin\theta}}\,\mathrm{d}t,\end{aligned}$$

for $\xi \in \mathbb{R}$ arbitrary but fixed. The above expression involves the classical Fourier transform of the Gaussian

$$\varphi(t) := \mathrm{e}^{-\pi(1-\mathrm{i}\cot\theta)t^2}, \qquad t \in \mathbb{R},$$

which according to Lemma 1.5.1 on p. 17 of [8] and the paragraph thereafter is given by

$$\mathcal{F}\varphi(\xi) = c_\theta^{-1} \mathrm{e}^{-\frac{\pi}{1-\mathrm{i}\cot\theta}\xi^2}.$$

It follows that

$$\begin{aligned}\mathcal{F}_\theta\phi(\xi) &= cc_\theta \mathrm{e}^{\pi\mathrm{i}\xi^2\cot\theta} \int_{\mathbb{R}} \mathrm{e}^{-\pi(1-\mathrm{i}\cot\theta)t^2} \mathrm{e}^{-2\pi\mathrm{i}t\frac{\xi}{\sin\theta}}\,\mathrm{d}t = cc_\theta \mathrm{e}^{\pi\mathrm{i}\xi^2\cot\theta}\mathcal{F}\varphi\left(\frac{\xi}{\sin\theta}\right) \\ &= c\mathrm{e}^{\pi\mathrm{i}\xi^2\cot\theta}\mathrm{e}^{-\frac{\pi}{1-\mathrm{i}\cot\theta}\left(\frac{\xi}{\sin\theta}\right)^2} = c\mathrm{e}^{\pi\left(\mathrm{i}\cot\theta - \frac{1}{(1-\mathrm{i}\cot\theta)\sin^2\theta}\right)\xi^2}.\end{aligned}$$

Finally, we may compute

$$\begin{aligned}\mathrm{i}\cot\theta - \frac{1}{(1-\mathrm{i}\cot\theta)\sin^2\theta} &= \frac{\mathrm{i}\cos\theta}{\sin\theta} - \frac{1}{(\sin\theta - \mathrm{i}\cos\theta)\sin\theta} \\ &= \frac{\mathrm{i}\cos\theta(\sin\theta - \mathrm{i}\cos\theta) - 1}{(\sin\theta - \mathrm{i}\cos\theta)\sin\theta} \\ &= \frac{\mathrm{i}\cos\theta\sin\theta + \cos^2\theta - 1}{(\sin\theta - \mathrm{i}\cos\theta)\sin\theta} \\ &= \frac{\mathrm{i}\cos\theta\sin\theta - \sin^2\theta}{(\sin\theta - \mathrm{i}\cos\theta)\sin\theta} = -1\end{aligned}$$

such that

$$\mathcal{F}_\theta\phi(\xi) = c\mathrm{e}^{\pi\left(\mathrm{i}\cot\theta - \frac{1}{(1-\mathrm{i}\cot\theta)\sin^2\theta}\right)\xi^2} = c\mathrm{e}^{-\pi\xi^2} = \phi(\xi).$$

□

Lemma A.2 (Cf. [3, 12])**.** *Let $\theta \in \mathbb{R}$ and $f, g \in L^2(\mathbb{R})$. It holds that*

$$\mathcal{V}_{\mathcal{F}_\theta g}\mathcal{F}_\theta f(x,\omega) = \mathcal{V}_g f(\mathrm{R}_\theta(x,\omega))\mathrm{e}^{\pi\mathrm{i}\sin\theta\left(\left(x^2-\omega^2\right)\cos\theta - 2x\omega\sin\theta\right)},$$

for $x, \omega \in \mathbb{R}$.

Proof. Let $x, \omega \in \mathbb{R}$ be arbitrary but fixed and consider

$$\mathcal{V}_{\mathcal{F}_{2k\pi}g}\mathcal{F}_{2k\pi}f(x,\omega) = \mathcal{V}_g f(x,\omega)$$

as well as

$$\begin{aligned}\mathcal{V}_{\mathcal{F}_{(2k+1)\pi}g}\mathcal{F}_{(2k+1)\pi}f(x,\omega) &= \int_{\mathbb{R}} \mathcal{F}_{(2k+1)\pi}f(t)\overline{\mathcal{F}_{(2k+1)\pi}g(t-x)}\mathrm{e}^{-2\pi\mathrm{i}t\omega}\,\mathrm{d}t \\ &= \int_{\mathbb{R}} f(-t)\overline{g(x-t)}\mathrm{e}^{-2\pi\mathrm{i}t\omega}\,\mathrm{d}t \\ &= \int_{\mathbb{R}} f(s)\overline{g(x+s)}\mathrm{e}^{2\pi\mathrm{i}s\omega}\,\mathrm{d}s \\ &= \mathcal{V}_g f(-x,-\omega).\end{aligned}$$

We may therefore focus on $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ from here on out: we consider

$$\mathcal{V}_{\mathcal{F}_\theta g}\mathcal{F}_\theta f(x,\omega) = (\mathcal{F}_\theta f, \mathrm{M}_\omega \mathrm{T}_x \mathcal{F}_\theta g).$$

To progress, we need to understand how modulations and translations act on the fractional Fourier transform. Let us start by considering the action of translations through the following calculation:

$$\begin{aligned}
\mathrm{T}_x \mathcal{F}_\theta g(\tau) &= \mathcal{F}_\theta g(\tau - x) = c_\theta \mathrm{e}^{\pi\mathrm{i}(\tau-x)^2\cot\theta} \int_{\mathbb{R}} g(t)\mathrm{e}^{\pi\mathrm{i}t^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{t(\tau-x)}{\sin\theta}}\,\mathrm{d}t \\
&= c_\theta \mathrm{e}^{-2\pi\mathrm{i}\tau x\cot\theta}\mathrm{e}^{\pi\mathrm{i}\left(\tau^2+x^2\right)\cot\theta} \int_{\mathbb{R}} g(t)\mathrm{e}^{2\pi\mathrm{i}\frac{tx}{\sin\theta}} \cdot \mathrm{e}^{\pi\mathrm{i}t^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{t\tau}{\sin\theta}}\,\mathrm{d}t \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\left(\tau^2+x^2\right)\cot\theta} \int_{\mathbb{R}} g(t)\mathrm{e}^{2\pi\mathrm{i}\frac{tx}{\sin\theta}} \cdot \mathrm{e}^{\pi\mathrm{i}t^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{t}{\sin\theta}+x\cot\theta\right)\tau}\,\mathrm{d}t \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\left(\tau^2+x^2\right)\cot\theta} \int_{\mathbb{R}} g(t)\mathrm{e}^{2\pi\mathrm{i}\frac{tx}{\sin\theta}} \cdot \mathrm{e}^{\pi\mathrm{i}t^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{(t+x\cos\theta)\tau}{\sin\theta}}\,\mathrm{d}t \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\left(\tau^2+x^2\right)\cot\theta} \\
&\qquad \cdot \int_{\mathbb{R}} g(s-x\cos\theta)\mathrm{e}^{2\pi\mathrm{i}\frac{(s-x\cos\theta)x}{\sin\theta}} \cdot \mathrm{e}^{\pi\mathrm{i}(s-x\cos\theta)^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{s\tau}{\sin\theta}}\,\mathrm{d}s \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\left(\tau^2-x^2\right)\cot\theta} \\
&\qquad \cdot \int_{\mathbb{R}} g(s-x\cos\theta)\mathrm{e}^{2\pi\mathrm{i}\frac{sx}{\sin\theta}} \cdot \mathrm{e}^{\pi\mathrm{i}(s-x\cos\theta)^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{s\tau}{\sin\theta}}\,\mathrm{d}s \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\tau^2\cot\theta}\mathrm{e}^{\pi\mathrm{i}x^2\left(\cos^2\theta-1\right)\cot\theta} \\
&\qquad \cdot \int_{\mathbb{R}} g(s-x\cos\theta)\mathrm{e}^{2\pi\mathrm{i}sx\left(\frac{1}{\sin\theta}-\cos\theta\cot\theta\right)} \cdot \mathrm{e}^{\pi\mathrm{i}s^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{s\tau}{\sin\theta}}\,\mathrm{d}s \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\tau^2\cot\theta}\mathrm{e}^{-\pi\mathrm{i}x^2\sin\theta\cos\theta} \\
&\qquad \cdot \int_{\mathbb{R}} g(s-x\cos\theta)\mathrm{e}^{2\pi\mathrm{i}sx\sin\theta} \cdot \mathrm{e}^{\pi\mathrm{i}s^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{s\tau}{\sin\theta}}\,\mathrm{d}s \\
&= \mathrm{e}^{-\pi\mathrm{i}x^2\sin\theta\cos\theta} \cdot \mathcal{F}_\theta \mathrm{M}_{x\sin\theta} \mathrm{T}_{x\cos\theta}\, g(\tau).
\end{aligned}$$

Next, we may consider the action of modulations. For this purpose, we consider $h \in L^2(\mathbb{R})$ and compute

$$\begin{aligned}
\mathrm{M}_\omega \mathcal{F}_\theta h(\tau) &= \mathcal{F}_\theta h(\tau)\mathrm{e}^{2\pi\mathrm{i}\tau\omega} = c_\theta \mathrm{e}^{2\pi\mathrm{i}\tau\omega}\mathrm{e}^{\pi\mathrm{i}\tau^2\cot\theta} \int_{\mathbb{R}} h(t)\mathrm{e}^{\pi\mathrm{i}t^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{t\tau}{\sin\theta}}\,\mathrm{d}t \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\tau^2\cot\theta} \int_{\mathbb{R}} h(t)\mathrm{e}^{\pi\mathrm{i}t^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\left(\frac{t}{\sin\theta}-\omega\right)\tau}\,\mathrm{d}t \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\tau^2\cot\theta} \int_{\mathbb{R}} h(t)\mathrm{e}^{\pi\mathrm{i}t^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{(t-\omega\sin\theta)\tau}{\sin\theta}}\,\mathrm{d}t \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\tau^2\cot\theta} \int_{\mathbb{R}} h(s+\omega\sin\theta)\mathrm{e}^{\pi\mathrm{i}(s+\omega\sin\theta)^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{s\tau}{\sin\theta}}\,\mathrm{d}s \\
&= c_\theta \mathrm{e}^{\pi\mathrm{i}\tau^2\cot\theta}\mathrm{e}^{\pi\mathrm{i}\omega^2\sin\theta\cos\theta} \\
&\qquad \cdot \int_{\mathbb{R}} h(s+\omega\sin\theta)\mathrm{e}^{2\pi\mathrm{i}s\omega\cos\theta} \cdot \mathrm{e}^{\pi\mathrm{i}s^2\cot\theta}\mathrm{e}^{-2\pi\mathrm{i}\frac{s\tau}{\sin\theta}}\,\mathrm{d}s
\end{aligned}$$

$$= \mathrm{e}^{\pi \mathrm{i} \omega^2 \sin\theta\cos\theta} \cdot \mathcal{F}_\theta \, \mathrm{M}_{\omega\cos\theta} \, \mathrm{T}_{-\omega\sin\theta} \, h(\tau).$$

The action of a translation followed by a modulation is therefore given by

$$\begin{aligned}
\mathrm{M}_\omega \, \mathrm{T}_x \, \mathcal{F}_\theta g(\tau) &= \mathrm{e}^{-\pi \mathrm{i} x^2 \sin\theta\cos\theta} \cdot \mathcal{F}_\theta \, \mathrm{M}_{x\sin\theta} \, \mathrm{T}_{x\cos\theta} \, g(\tau) \mathrm{e}^{2\pi \mathrm{i} \tau\omega} \\
&= \mathrm{e}^{\pi \mathrm{i} \left(\omega^2 - x^2\right) \sin\theta\cos\theta} \cdot \mathcal{F}_\theta \, \mathrm{M}_{\omega\cos\theta} \, \mathrm{T}_{-\omega\sin\theta} \, \mathrm{M}_{x\sin\theta} \, \mathrm{T}_{x\cos\theta} \, g(\tau) \\
&= \mathrm{e}^{\pi \mathrm{i} \left(\omega^2 - x^2\right) \sin\theta\cos\theta} \mathrm{e}^{2\pi \mathrm{i} x\omega \sin^2\theta} \\
&\qquad \cdot \mathcal{F}_\theta \, \mathrm{M}_{x\sin\theta + \omega\cos\theta} \, \mathrm{T}_{x\cos\theta - \omega\sin\theta} \, g(\tau) \\
&= \mathrm{e}^{\pi \mathrm{i} \sin\theta \left(\left(\omega^2 - x^2\right) \cos\theta + 2x\omega\sin\theta \right)} \\
&\qquad \cdot \mathcal{F}_\theta \, \mathrm{M}_{x\sin\theta + \omega\cos\theta} \, \mathrm{T}_{x\cos\theta - \omega\sin\theta} \, g(\tau),
\end{aligned}$$

where we have used that

$$\mathrm{T}_t \, \mathrm{M}_\xi = \mathrm{e}^{-2\pi \mathrm{i} t\xi} \, \mathrm{M}_\xi \, \mathrm{T}_t, \qquad t, \xi \in \mathbb{R}.$$

It does therefore follow from the unitarity of the fractional Fourier transform that

$$\begin{aligned}
\mathcal{V}_{\mathcal{F}_\theta g} \mathcal{F}_\theta f(x,\omega) &= \left(\mathcal{F}_\theta f, \mathrm{M}_\omega \, \mathrm{T}_x \, \mathcal{F}_\theta g\right) \\
&= \mathrm{e}^{-\pi \mathrm{i} \sin\theta \left(\left(\omega^2 - x^2\right) \cos\theta + 2x\omega\sin\theta \right)} \\
&\qquad \cdot \left(\mathcal{F}_\theta f, \mathcal{F}_\theta \, \mathrm{M}_{x\sin\theta + \omega\cos\theta} \, \mathrm{T}_{x\cos\theta - \omega\sin\theta} \, g\right) \\
&= \mathrm{e}^{-\pi \mathrm{i} \sin\theta \left(\left(\omega^2 - x^2\right) \cos\theta + 2x\omega\sin\theta \right)} \\
&\qquad \cdot \left(f, \mathrm{M}_{x\sin\theta + \omega\cos\theta} \, \mathrm{T}_{x\cos\theta - \omega\sin\theta} \, g\right) \\
&= \mathrm{e}^{-\pi \mathrm{i} \sin\theta \left(\left(\omega^2 - x^2\right) \cos\theta + 2x\omega\sin\theta \right)} \\
&\qquad \cdot \mathcal{V}_g f(x\cos\theta - \omega\sin\theta, x\sin\theta + \omega\cos\theta).
\end{aligned}$$

□

B Paley–Wiener spaces and Bernstein spaces

Lemma B.1. *Let $p \in [1,2]$ and denote by $q \in [2,\infty]$ its Hölder conjugate. Let moreover $B > 0$. Then, it holds that $\mathrm{PW}_B^p \subset \mathrm{B}_{2\pi B}^q$.*

Proof. Let us denote $\chi_z(\xi) := \mathrm{e}^{2\pi \mathrm{i} \xi z}$, for $\xi \in \mathbb{R}$ and $z \in \mathbb{C}$, throughout this proof. Moreover, we will drop the interval $[-B,B]$ in the notation of the L^r-norms $\|\cdot\|_r = \|\cdot\|_{L^r([-B,B])}$, for $r \in [1,\infty]$, to shorten notation.

Let $f \in \mathrm{PW}_B^p$. By definition, there exists an $F \in L^p([-B,B])$ such that

$$f(z) = \int_{-B}^{B} F(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}\xi, \qquad z \in \mathbb{C}.$$

Based on the above formula, we may show that $f : \mathbb{C} \to \mathbb{C}$ is continuous: indeed, consider $z_0, z \in \mathbb{C}$ arbitrary and apply the triangle inequality as well as Hölder's inequality multiple times to obtain

$$\begin{aligned}
|f(z) - f(z_0)| &= \left| \int_{-B}^{B} F(\xi) \left(\mathrm{e}^{2\pi \mathrm{i} \xi z} - \mathrm{e}^{2\pi \mathrm{i} \xi z_0} \right) \mathrm{d}\xi \right| \\
&\leq \int_{-B}^{B} |F(\xi) \left(\chi_z(\xi) - \chi_{z_0}(\xi) \right)| \, \mathrm{d}\xi \\
&\leq \|F\|_p \|\chi_z - \chi_{z_0}\|_q \leq (2B)^{\frac{1}{q}} \|F\|_p \|\chi_z - \chi_{z_0}\|_\infty \\
&= (2B)^{\frac{1}{q}} \|F\|_p \|\chi_{z_0} \left(\chi_{z-z_0} - 1 \right)\|_\infty \\
&\leq (2B)^{\frac{1}{q}} \|F\|_p \|\chi_{z_0}\|_\infty \|\chi_{z-z_0} - 1\|_\infty \\
&\leq (2B)^{\frac{1}{q}} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \|\chi_{z-z_0} - 1\|_\infty \\
&= (2B)^{\frac{1}{q}} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \cdot \sup_{\xi \in \mathbb{R}} \left| \mathrm{e}^{2\pi \mathrm{i} \xi (z - z_0)} - 1 \right|,
\end{aligned}$$

where we have used that $\chi_{z+z'} = \chi_z \chi_{z'}$, for $z, z' \in \mathbb{C}$. We may now assume that $|z - z_0| < (4\pi B)^{-1}$, expand the exponential function and estimate

$$\begin{aligned}
|f(z) - f(z_0)| &\leq (2B)^{\frac{1}{q}} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \cdot \sup_{\xi \in \mathbb{R}} \left| \mathrm{e}^{2\pi \mathrm{i} \xi (z - z_0)} - 1 \right| \\
&= (2B)^{\frac{1}{q}} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \cdot \sup_{\xi \in \mathbb{R}} \left| \sum_{k=1}^{\infty} \frac{(2\pi \mathrm{i} \xi (z - z_0))^k}{k!} \right| \\
&\leq (2B)^{\frac{1}{q}} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \cdot \sum_{k=1}^{\infty} \frac{(2\pi B |z - z_0|)^k}{k!} \\
&\leq (2B)^{\frac{1}{q}} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \cdot \sum_{k=1}^{\infty} (2\pi B |z - z_0|)^k \\
&= (2B)^{\frac{1}{q}} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \cdot \frac{2\pi B |z - z_0|}{1 - 2\pi B |z - z_0|} \\
&\leq 2^{2+1/q} \pi B^{1+1/q} \mathrm{e}^{2\pi B |z_0|} \|F\|_p \cdot |z - z_0|,
\end{aligned}$$

where we have used the convergence of the geometric series which holds due to $2\pi B |z - z_0| < 1/2 < 1$. It follows that $f : \mathbb{C} \to \mathbb{C}$ is continuous.

Next, we can apply Morera's theorem to see that f is entire. Indeed, we might consider a closed piecewise $\mathcal{C}^1$ curve γ in $\mathbb{C}$ and compute

$$\int_\gamma f(z) \, \mathrm{d}z = \int_\gamma \int_{-B}^{B} F(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}\xi \, \mathrm{d}z = \int_{-B}^{B} F(\xi) \int_\gamma \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}z \, \mathrm{d}\xi = 0,$$

where we used Fubini's theorem to exchange integration and applied Morera's theorem to see that the contour integral of $z \mapsto \mathrm{e}^{2\pi \mathrm{i} \xi z}$ vanishes, for all $\xi \in \mathbb{R}$ — which works because $z \mapsto \mathrm{e}^{2\pi \mathrm{i} \xi z}$ is entire, for all $\xi \in \mathbb{R}$. It therefore follows from Morera's theorem that f is entire.

We can finally estimate

$$|f(z)| = \left| \int_{-B}^{B} F(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi z} \, \mathrm{d}\xi \right| \leq \int_{-B}^{B} |F(\xi) \chi_z(\xi)| \, \mathrm{d}\xi \leq \|F\|_p \, \|\chi_z\|_q$$
$$\leq (2B)^{1/q} \, \|F\|_p \, \|\chi_z\|_\infty \leq (2B)^{1/q} \, \|F\|_p \, \mathrm{e}^{2\pi B |z|},$$

for $z \in \mathbb{C}$. It follows that f is of exponential type $2\pi B$. Additionally, it follows from the classical Hausdorff–Young inequality (see for instance Proposition 2.2.16 on p. 114 of [7]) that $f|_{\mathbb{R}} \in L^q(\mathbb{R})$, where q is the Hölder conjugate of p. □

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