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EXPONENTIAL CONVERGENCE OF DEEP OPERATOR NETWORKS FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We construct and analyze approximation rates of deep operator networks (ONets) between infinite-dimensional spaces that emulate with an exponential rate of convergence the coefficient-to-solution map of elliptic second-order partial differential equations. In particular, we consider problems set in d-dimensional periodic domains, $d = 1, 2, \dots$, and with analytic right-hand sides and coefficients. Our analysis covers linear, elliptic second order divergence-form PDEs as, e.g., diffusion-reaction problems, parametric diffusion equations, and elliptic systems such as linear isotropic elastostatics in heterogeneous materials.

We leverage the exponential convergence of spectral collocation methods for boundary value problems whose solutions are analytic. In the present periodic and analytic setting, this follows from classical elliptic regularity. Within the ONet branch and trunk construction of Chen and Chen [4] and of Lu et al. [18], we show the existence of deep ONets which emulate the coefficient-to-solution map to a desired accuracy in the H^1 norm, uniformly over the coefficient set. We prove that the neural networks in the ONet have size $\mathcal{O}(|\log(\varepsilon)|^\kappa)$, where $\varepsilon > 0$ is the approximation accuracy, for some $\kappa > 0$ depending on the physical space dimension.

Key words. Operator networks, deep neural networks, exponential convergence, elliptic PDEs

AMS subject classifications. 35J15, 65N15, 65N35, 68T07

1. Introduction.

The application of numerical surrogates of solution operators to partial differential equations (PDEs) via algorithms of deep learning has recently received considerable attention. See, e.g., [2, 17, 19, 16] and the references there. Also, expression and approximation rate bounds for such computable operator surrogates have appeared in various settings, see, e.g. [13, 8, 7], and the references there. In the present paper, we construct deep operator network (ONet) emulations of coefficient-to-solution maps for boundary value problems with linear, second order elliptic divergence-form operators. In particular, we consider operator networks with rectified linear unit (ReLU) activation and problems formulated in domains without boundary and with analytic right-hand sides and coefficients. In this setting, we construct operator networks that approximate the (nonlinear) coefficient-to-solution map with exponential accuracy in the corresponding function spaces. We bound—poly-logarithmically with respect to the energy norm of the error—both the size of the approximating network and the number of sampling points where the coefficient is queried.

1.1. Existing Results.

Deep neural networks (DNN) have been employed increasingly in recent years in the numerical solution of differential equations in science and engineering. We refer to the survey [2] for uses and successes of DNN based numerical simulations in computational fluid mechanics, and to [27] for their use in computational finance and computational option pricing. First uses of DNNs in numerical PDE solution in engineering and the sciences focused on leveraging DNNs for "mesh-free" solution approximation and representation (see, e.g., [26, 9]), with good success explained, to some extent, by *approximation properties of DNNs in function spaces* (see, e.g., [24, 22, 23, 20, 28, 10]) in particular overcoming the so-called Curse-of-Dimensionality (CoD) in high-dimensional approximation of PDE solution manifolds [28, 11], of parametric PDEs and of PDEs on high-dimensional state spaces, as arising, e.g., in computational finance (see [27, 1] and the references therein).

Reference [15] addressed the expression rate of ReLU NNs for the solution maps of parametric PDEs. The analysis in that paper proceeds through the DNN emulation of *reduced bases* for the approximation of solutions of the PDEs. The expression rate bounds obtained in [15] are subject to

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strong hypotheses on the DNN expressivity of reduced bases for the PDEs of interest. The parameter sets (i.e., the domains of the solution operator) considered in [15] are finite-dimensional; this paper mostly concerns instead the approximation of solution maps between infinite dimensional spaces. We nonetheless show how expression rates for finitely-parametric PDEs also follow from our main results, see Theorem 5.12 and Remark 5.14.

DNNs have been leveraged in [8, 18, 16] for the *DNN emulation of data-to-solution operators for PDEs*. See also the review [19]. Here, previous investigations have focused on *universality of NNs* for operator approximation. The pioneering work [4] established this for a certain type of NNs with a "branch and trunk" architecture, which will also be used in the present work. While [4] imposed strong compactness assumptions, more recently [16] extended these results to certain settings without the compactness assumptions of [4]. In these papers, focus has been on emulating nonlinear maps, such as domain-to-solution, or coefficient-to-solution maps. For well-posed PDE problems, continuous dependence on the problem data implies that these maps are continuous, in the appropriate topologies on the data and the solution space. We refer to [14, 18] and the references therein. In these references, some theory explaining some of the numerically observed performances of NN emulation of nonlinear operators has been developed (see, e.g., [16, 8, 7]). We also mention the analysis of [13] for Fourier Neural Operators, a different kind of operator networks, introduced in [17].

The *convergence rate estimates* proved in these references indicate that a) DNNs are capable of parsimonious numerical representations of the nonlinear, smooth data-to-solution maps for PDEs, and b) they are not prone to the CoD in connection with the countable number of parameters due, e.g., to series representations of inputs in separable Banach spaces of possibly infinite dimension.

1.2. Contributions. We construct DNN approximations of data-to-solution maps, so-called "Operator Networks" for linear, second order divergence-form elliptic PDEs with non-homogeneous coefficients and source terms. We establish *exponential expression rates* for these coefficient-to-solution operators for elliptic PDEs.

Our argument relies on analytic regularity for elliptic PDEs with analytic coefficients, on the a priori analysis of periodic spectral approximation of PDEs, and on the error analysis of numerical quadrature in fully discrete spectral methods. We consider linear second order divergence-form elliptic boundary value problems with analytic, periodic coefficients, and (uniformly) analytic solutions, whose inputs and solutions admit exponentially convergent spectral collocation approximations from spaces of high-degree, periodically extendable polynomials. Our results show that neural networks can emulate accurately the (nonlinear) data-to-solution operator of Galerkin methods for the elliptic PDEs mentioned above with numerical integration. The operator networks we construct are composed of *encoding*, *approximation*, and *reconstruction* operators. In the encoding step, the input datum is queried on collocation points in the physical domain. The approximation and reconstruction parts of the operator networks are composed of two neural networks, one that approximates a polynomial basis, while the other maps point evaluations of the diffusion coefficient to coefficients over the basis.

Our proof is constructive, based on "NN emulation" of (building blocks of) a spectral method. Our focus is on providing an upper bound on the expression rate of the Operator Network approximation of the coefficient-to-solution maps, rather than to suggest a concrete algorithm to actually construct those networks. Actual applications may be able to perform the numerical Operator Network construction more efficiently.

For the sake of clarity of exposition, we develop this strategy for model, linear second order elliptic PDEs in divergence form, with inhomogeneous coefficients. We then show, using the compositionality of NNs, how to include problems with parametric diffusion, typically arising in computational uncertainty quantification. Finally, we mention the minor modifications required for PDEs with reaction coefficients and discuss in some detail ONet emulation of the coefficient-to-solution map for linear elasticity.

The exponential expression of data-to-solution maps proved in this manuscript is the first result of this kind for operator networks. It is based on exponential compression rates of encoders and decoders which are based on spectral approximations to leverage analyticity of input and output of the data-to-solution maps. Here, analyticity of the solution is a consequence of classical elliptic regularity. The strong compression of spectral encoders and decoders facilitated by analyticity allows to compose ONets from approximate neural network inversion of small, but generally dense spectral Galerkin matrices. ONet constructions for finite regularity input and output pairs with considerably different input encoder and output decoder maps differs substantially from the present construction. They are considered in [12]. The approximation of data-to-solution maps for similar elliptic PDEs has been analyzed with different techniques in [8] under weaker regularity assumptions on the coefficients. These lines of argument yield lower expression rate bounds.

1.3. Structure of this paper. To fix a setting for developing our results, we introduce in Section 2 a scalar, elliptic, isotropic diffusion equation. The *coefficient-to-solution operator* that will be the main target of approximation by neural networks is also introduced in this section. Then, in Section 3, we define feed forward neural networks (with ReLU activation) and operator networks with the branch and trunk architecture of [4, 18], that approximate maps between infinite dimensional spaces. We conclude the section by defining some operations on networks that will then be used for the approximation analysis. In Section 4, we gather (classical) results on the polynomial approximation of solutions to the elliptic problem. The main results of this paper are then proved in Section 5. In Theorem 5.7, we show the exponential convergence of the operator net approximation of the coefficient-to-solution map for the elliptic isotropic diffusion problem. We extend the analysis to parametric diffusion coefficients in Theorem 5.12. Finally, in Section 6 we extend our ONet approximation to further second order problems comprising reaction-diffusion with nonzero reaction coefficients and linear elastostatics.

1.4. Notation. We use standard notation and symbols: $\mathbb{N}$ denotes the set of positive natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We write vectors in lowercase boldface characters and matrices in uppercase boldface characters. We denote by $\|\boldsymbol{a}\|_2$ the ℓ^2-norm of a vector $\boldsymbol{a}$, while for any matrix $\mathbf{A}$, we denote $\|\mathbf{A}\|_2 = \sup_{\|\boldsymbol{x}\|_2=1} \|\mathbf{A}\boldsymbol{x}\|_2$ its operator norm. By $\|\mathbf{A}\|_0$ and $\|\boldsymbol{x}\|_0$ we denote, respectively, the number of nonzero elements of a matrix $\mathbf{A}$ and a vector $\boldsymbol{x}$. The spectrum of a matrix $\mathbf{A}$ is written $\sigma(\mathbf{A})$. For $n \in \mathbb{N}$, $\mathbf{Id}_n$ is the $n \times n$ identity matrix, while $\mathbf{0}_n$ is a vector of zeros of size n. When used between matrices, we denote by $\otimes$ the Kronecker product: given two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{p\times q}$, then $\mathbf{C} = \mathbf{A} \otimes \mathbf{B} \in \mathbb{R}^{mp\times nq}$, such that

$$\mathbf{C} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B} & \cdots & \mathbf{A}_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{m1}\mathbf{B} & \cdots & \mathbf{A}_{mn}\mathbf{B} \end{bmatrix}.$$

Given two functions v_1, v_2, we instead denote by $v_1 \otimes v_2$ the function such that $(v_1 \otimes v_2)(x_1, x_2) = v_1(x_1)v_2(x_2)$. We denote by $[\boldsymbol{a}_1|\dots|\boldsymbol{a}_n]$ the matrix with columns $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$. We indicate by $\mathbf{vec}: \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ and $\mathbf{matr}: \mathbb{R}^{mn} \to \mathbb{R}^{m\times n}$ the vectorization and matricization operators, such that $\mathbf{matr}(\mathbf{vec}(\mathbf{A})) = \mathbf{A}$ for any matrix $\mathbf{A}$. All results are independent of the ordering of the vectorization operation; the dimensions of the matricization operation will be clear from the context. We denote by $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ the space of symmetric matrices of size $n \times n$. Given two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$, we write $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{n} \mathbf{A}_{ij}\mathbf{B}_{ij}$.

Let $d \in \mathbb{N}$. For $k \in \mathbb{N}_0$, $p \in [1, \infty]$, and a domain $D \subset \mathbb{R}^d$, we indicate by $W^{k,p}(D)$ the classical Sobolev spaces. $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d)$ indicates functions that are in $W^{k,p}(D)$ for any bounded subset D of $\mathbb{R}^d$. In the Hilbertian case $p = 2$, we write $H^k(D)$; in addition, $L^p(D) = W^{0,p}(D)$ and $L^2(D) = H^0(D)$.

Given $Q = (0,1)^d$ and $\Omega = (\mathbb{R}/\mathbb{Z})^d$, we denote , for all $k \in \mathbb{N}_0$ and $p \in [1,\infty]$,

$$W^{k,p}(\Omega) = W^{k,p}_{\mathrm{per}}(Q) := \left\{ v|_Q : v \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d) \text{ and } v \text{ is } Q\text{-periodic} \right\},$$

i.e., the restriction to Q of all functions in $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^d)$ that are Q-periodic. We denote by $(\cdot,\cdot)$ the L^2 scalar product in Q.

For $C > 0$, define $\mathrm{Hol}(\Omega; C)$ as the set of functions v that are real analytic in $\mathbb{R}^d$, periodic with period one in all coordinate directions, and such that

$$\|v\|_{W^{k,\infty}(Q)} \leq C^{k+1} k!, \qquad \forall k \in \mathbb{N}_0. \tag{1.1}$$

Define furthermore the set of all real analytic functions in Ω as $\mathrm{Hol}(\Omega) = \bigcup_{C>0} \mathrm{Hol}(\Omega; C)$. By the Arzelà-Ascoli theorem, the set $\mathrm{Hol}(\Omega; C)$ is compact in $L^\infty(\Omega)$.

2. Problem formulation. We introduce the set of admissible diffusion coefficient data $\mathcal{D}$: for each coefficient $a \in \mathcal{D}$ we assume ellipticity in the form that there exist constants $a_{\min}, a_{\max} > 0$ such that

$$\forall \boldsymbol{x} \in Q,\, \forall a \in \mathcal{D} \quad a_{\min} \leq a(\boldsymbol{x}) \leq a_{\max}\,. \tag{2.1}$$

We also assume that all $a \in \mathcal{D}$ are real analytic and Q-periodic, with uniform bounds on the radius of convergence of the Taylor series: there exists a constant $A_{\mathcal{D}} > 0$ such that

$$\mathcal{D} \subset \mathrm{Hol}(\Omega; A_{\mathcal{D}}). \tag{2.2}$$

As it will be useful in the sequel, we define the Poincaré constant $C_{\mathrm{poi}} > 0$ such that

$$\|v - \frac{1}{|Q|}\int_\Omega v\|_{L^2(Q)} \leq C_{\mathrm{poi}} \|\nabla v\|_{L^2(Q)}, \qquad \forall v \in H^1(\Omega) = H^1_{\mathrm{per}}(Q). \tag{2.3}$$

The ellipticity hypotheses (2.1) and the Poincaré inequality (2.3) imply that for every $f \in L^2(\Omega)$ such that $\int_Q f = 0$, and for each $a \in \mathcal{D}$, the elliptic boundary value problem

$$-\nabla \cdot (a \nabla u^a) = f \text{ in } \Omega \tag{2.4}$$

admits a unique solution

$$u^a \in X := \left\{ v \in H^1(\Omega) : \int_Q v = 0 \right\} \simeq H^1_{\mathrm{per}}(Q)/\mathbb{R}.$$

It satisfies the *variational formulation*: given $a \in \mathcal{D}$, find $u \in X$ such that

$$\mathfrak{b}^a(u, v) = (f, v) \quad \forall v \in X. \tag{2.5}$$

Here, for a given $a \in \mathrm{Hol}(\Omega)$, the bilinear form $\mathfrak{b}^a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$\mathfrak{b}^a(w, v) := \int_Q (a \nabla w \cdot \nabla v)\,.$$

In what follows, we assume the fixed source term $f \in \mathrm{Hol}(\Omega) \cap X$ to be given and, for any $a \in \mathcal{D}$, we denote by u^a the unique solution of (2.5) for this choice of f.

We denote (still keeping the source term f in (2.5) fixed) by S the data-to-solution operator $a \mapsto u^a$ in (2.5). We let $\mathcal{U} = S(\mathcal{D})$ the set of solutions of (2.5) corresponding to inputs from $\mathcal{D}$. As shown in Lemma B.1 in Appendix B, for fixed right source term f in (2.5), the data-to-solution map $S : L^\infty(\Omega) \to H^1(\Omega)$ is Lipschitz continuous. Furthermore, standard elliptic regularity (see [21, 5] and Lemma 4.1 below) implies $S(\mathcal{D}) \subset \mathrm{Hol}(\Omega)$.

3. Neural operator networks. Our goal is to derive bounds for the approximation of the solution operator $S : \mathcal{D} \to X \subseteq H^1(\Omega)$, defined in Section 2, by an operator network. To define operator networks, we recall the definition of classical feed forward neural networks with ReLU activation

$$\mathrm{ReLU} : \mathbb{R} \to \mathbb{R},\ x \mapsto \max\{0, x\}.$$

3.1. Feed forward neural network.

Definition 3.1 ([24, Definition 2.1]). *Let $d, L \in \mathbb{N}$. A* neural network Φ with input dimension d and L layers *is a sequence of matrix-vector tuples*

$$\Phi = \big((\mathbf{A}_1, \boldsymbol{b}_1), (\mathbf{A}_2, \boldsymbol{b}_2), \dots, (\mathbf{A}_L, \boldsymbol{b}_L)\big),$$

where $N_0 := d$ and $N_1, \dots, N_L \in \mathbb{N}$, and where $\mathbf{A}_\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$ and $\boldsymbol{b}_\ell \in \mathbb{R}^{N_\ell}$ for $\ell = 1, ..., L$.

For a NN Φ, we define the associated realization of the NN Φ *as*

$$\mathrm{R}(\Phi) : \mathbb{R}^d \to \mathbb{R}^{N_L},\ \boldsymbol{x} \mapsto \boldsymbol{x}_L =: \mathrm{R}(\Phi)(\boldsymbol{x}),$$

where the output $\boldsymbol{x}_L \in \mathbb{R}^{N_L}$ results from

$$\begin{aligned} \boldsymbol{x}_0 &:= \boldsymbol{x}, \\ \boldsymbol{x}_\ell &:= \mathrm{ReLU}(\mathbf{A}_\ell\, \boldsymbol{x}_{\ell-1} + \boldsymbol{b}_\ell), \qquad \textit{for } \ell = 1, \dots, L-1, \\ \boldsymbol{x}_L &:= \mathbf{A}_L\, \boldsymbol{x}_{L-1} + \boldsymbol{b}_L. \end{aligned} \tag{3.1}$$

Here ReLU *is understood to act component-wise on vector-valued inputs, i.e., for $\boldsymbol{y} = (y^1, \dots, y^m) \in \mathbb{R}^m$, $\mathrm{ReLU}(\boldsymbol{y}) := (\mathrm{ReLU}(y^1), \dots, \mathrm{ReLU}(y^m))$. We call $N(\Phi) := d + \sum_{j=1}^{L} N_j$ the* number of neurons of the NN Φ, $\mathrm{L}(\Phi) := L$ *the* number of layers *or* depth, $\mathrm{M}_j(\Phi) := \|A_j\|_0 + \|b_j\|_0$ *the* number of nonzero weights in the j-th layer, *and* $\mathrm{M}(\Phi) := \sum_{j=1}^{L} \mathrm{M}_j(\Phi)$ *the* number of nonzero weights of Φ, *also referred to as its* size. *We refer to N_L as the* dimension of the output layer of Φ.

3.2. Operator networks. The operator network approximating the solution operator S can be seen as the composition $\mathcal{R} \circ \mathcal{A} \circ \mathcal{E}$ of three mappings:

- Encoding $\mathcal{E} : \mathcal{D} \to \mathbb{R}^n$, for $n \in \mathbb{N}$,
- Approximation $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$, for $m \in \mathbb{N}$,
- Reconstruction $\mathcal{R} : \mathbb{R}^m \to H^1(Q)$,

see the diagram in Figure 1. We refer the reader to [16, 14] for a broader view on and thorough discussion of operator networks between infinite dimensional spaces. In our analysis, the encoding step will map functions $a \in \mathcal{D}$ to the vector $\boldsymbol{a} \in \mathbb{R}^n$ of their point evaluations, i.e.

$$\boldsymbol{a} = \mathcal{E}_{\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\}}(a) := [a(\boldsymbol{x}_1), \dots, a(\boldsymbol{x}_n)]^\top,$$

for suitable collection of points $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \overline{Q}$. The approximation and reconstruction steps involve feed-forward neural networks that we will, respectively, denote Φ^{branch} and Φ^{trunk}. Specifically, the *approximate solution operator* $\mathcal{A}$ is realized as

$$\mathcal{A}_{\Phi^{\mathrm{branch}}}(\boldsymbol{a}) = \mathrm{R}(\Phi^{\mathrm{branch}})(\boldsymbol{a}).$$

For the reconstruction step $\mathcal{R}$, for all $\boldsymbol{c} \in \mathbb{R}^m$ and $\boldsymbol{x} \in \overline{Q}$, we define

$$\mathcal{R}_{\Phi^{\mathrm{trunk}}}(\boldsymbol{c})(\boldsymbol{x}) = \big(\mathrm{R}(\Phi^{\mathrm{trunk}})(\boldsymbol{x})\big) \cdot \boldsymbol{c}.$$

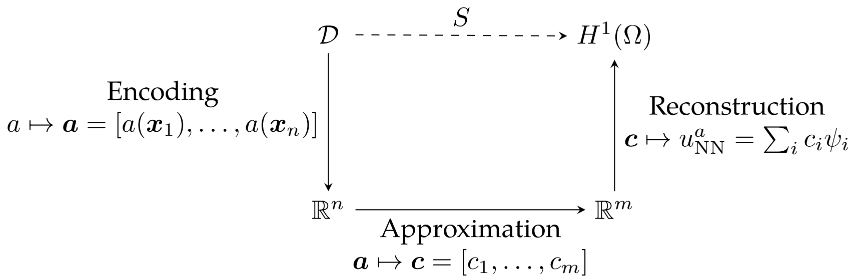

Fig. 1: Diagram of operator network between infinite dimensional spaces

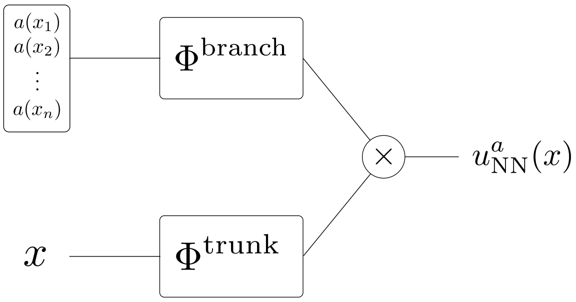

Fig. 2: Structure of the branch and trunk network; $u^a_{\mathrm{NN}}(\boldsymbol{x}) \coloneqq \mathrm{R}(\Phi^{\mathrm{branch}})(\boldsymbol{a}) \cdot \mathrm{R}(\Phi^{\mathrm{trunk}})(\boldsymbol{x})$.

This constructs the operator network mapping from $\mathcal{D}$ to $H^1(Q)$, defined by

$$\mathcal{R}_{\Phi^{\mathrm{trunk}}} \circ \mathcal{A}_{\Phi^{\mathrm{branch}}} \circ \mathcal{E}_{\{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\}} : a \mapsto u^a_{\mathrm{NN}}(\cdot) \coloneqq (\mathrm{R}(\Phi^{\mathrm{trunk}})(\cdot)) \cdot \mathrm{R}(\Phi^{\mathrm{branch}})(\mathcal{E}_{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n}(a))$$

see Figure 2. For the precise definition of the branch and trunk networks used to approximate the solution operator of (2.4) we refer the reader to Sections 5.1 and 5.2.

We aim for operator networks that approximate, for all $a \in \mathcal{D}$, solutions u^a of (2.4) in the $H^1(Q)$-norm, uniformly over the input space $\mathcal{D}$, i.e., such that

$$\sup_{a\in\mathcal{D}} \|u^a - u^a_{\mathrm{NN}}\|_{H^1(Q)} \leq \varepsilon.$$

The main result of this paper consists in proofs for upper bounds on n, m, and on the sizes of Φ^{trunk} and Φ^{branch} as functions of the error ε.

3.3. Operations on neural networks. We introduce and recall some operations on neural networks that will be necessary for the construction of the branch and trunk networks.

3.3.1. Concatenation and sparse concatenation.

Definition 3.2 (NN concatenation, [24, Definition 2.2]). *Let $L_1, L_2 \in \mathbb{N}$ and let*

$$\Phi^1 = ((\mathbf{A}^1_1, \boldsymbol{b}^1_1), \dots, (\mathbf{A}^1_{L_1}, \boldsymbol{b}^1_{L_1})), \qquad \Phi^2 = ((\mathbf{A}^2_1, \boldsymbol{b}^2_1), \dots, (\mathbf{A}^2_{L_2}, \boldsymbol{b}^2_{L_2}))$$

be two neural networks such that the input layer of Φ^1 has the same dimension as the output layer of Φ^2. Then, $\Phi^1 \bullet \Phi^2$ denotes the following $L_1 + L_2 - 1$ layer network:

$$\Phi^1 \bullet \Phi^2 \coloneqq ((\mathbf{A}^2_1, \boldsymbol{b}^2_1), \dots, (\mathbf{A}^2_{L_2-1}, \boldsymbol{b}^2_{L_2-1}), (\mathbf{A}^1_1 \mathbf{A}^2_{L_2}, \mathbf{A}^1_1 \boldsymbol{b}^2_{L_2} + \boldsymbol{b}^1_1), (\mathbf{A}^1_2, \boldsymbol{b}^1_2), \dots, (\mathbf{A}^1_{L_1}, \boldsymbol{b}^1_{L_1})).$$

We call $\Phi^1 \bullet \Phi^2$ *the* concatenation of Φ^1 and Φ^2.

Proposition 3.3 (NN sparse concatenation, [24, Remark 2.6]). *Let* $L_1, L_2 \in \mathbb{N}$, *and let* Φ^1, Φ^2 *be two NNs of respective depths* L_1 *and* L_2 *such that* $N_0^1 = N_{L_2}^2 =: d$, *i.e., the input layer of* Φ^1 *has the same dimension as the output layer of* Φ^2.

Then, there exists a NN $\Phi^1 \odot \Phi^2$, *called the* sparse concatenation of Φ^1 and Φ^2, *such that* $\Phi^1 \odot \Phi^2$ *has* $L_1 + L_2$ *layers,* $\mathrm{R}(\Phi^1 \odot \Phi^2) = \mathrm{R}(\Phi^1) \circ \mathrm{R}(\Phi^2)$ *and* $\mathrm{M}\left(\Phi^1 \odot \Phi^2\right) \le 2\,\mathrm{M}\left(\Phi^1\right) + 2\,\mathrm{M}\left(\Phi^2\right)$.

3.3.2. Emulation of matrix inversion. Dense matrix inversion can be approximated by suitable ReLU NNs. We recall the following result from [15] where, for $Z \in \mathbb{R}_+$ and $N \in \mathbb{N}$,

$$K_N^Z := \{\mathbf{vec}(\mathbf{A}) : \mathbf{A} \in \mathbb{R}^{N\times N}, \|\mathbf{A}\|_2 \le Z\}.$$

Theorem 3.4. [*15, Theorem 3.8*] *For* $\varepsilon, \delta \in (0,1)$ *define*

$$m(\varepsilon,\delta) := \left\lceil \frac{\log(0.5\varepsilon\delta)}{\log(1-\delta)} \right\rceil.$$

There exists a universal constant $C_{\mathrm{inv}} > 0$ *such that for every* $N \in \mathbb{N}$, $\varepsilon \in (0, 1/4)$ *and every* $\delta \in (0,1)$ *there exists a NN* $\Phi^{1-\delta,N}_{\mathrm{inv},\varepsilon}$ *with* N^2*-dimensional input,* N^2*-dimensional output and the following properties:*

1. $\mathrm{L}\left(\Phi^{1-\delta,N}_{\mathrm{inv};\varepsilon}\right) \le C_{\mathrm{inv}}\left(1 + \log\left(m(\varepsilon,\delta)\right)\right)\cdot\left(\log(1/\varepsilon) + \log\left(m(\varepsilon,\delta)\right) + \log(N)\right)$,
2. $\mathrm{M}\left(\Phi^{1-\delta,N}_{\mathrm{inv};\varepsilon}\right) \le C_{\mathrm{inv}} m(\varepsilon,\delta)\left(1 + \log^2(m(\varepsilon,\delta))\right) N^3 \cdot \left(\log(1\varepsilon) + \log\left(m(\varepsilon,\delta)\right) + \log(N)\right)$,
3. $\sup_{\mathbf{vec}(\mathbf{A})\in K_N^{1-\delta}} \left\| (\mathbf{Id}_N - \mathbf{A})^{-1} - \mathbf{matr}\left(\mathrm{R}\left(\Phi^{1-\delta,N}_{\mathrm{inv};\varepsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \le \varepsilon$,
4. *for any* $\mathbf{vec}(\mathbf{A}) \in K_N^{1-\delta}$ *we have*

$$\left\|\mathbf{matr}\left(\mathrm{R}\left(\Phi^{1-\delta,N}_{\mathrm{inv};\varepsilon}\right)(\mathbf{vec}(\mathbf{A}))\right)\right\|_2 \le \varepsilon + \left\|(\mathbf{Id}_N - \mathbf{A})^{-1}\right\|_2 \le \varepsilon + \frac{1}{1 - \|\mathbf{A}\|_2} \le \varepsilon + \frac{1}{\delta}.$$

Remark 3.5. The bounds on the depth and size of the network of Theorem 3.4 are slightly modified compared to those in [15], since some instances of $\log(m(\varepsilon,\delta))$ have been replaced by $1 + \log(m(\varepsilon,\delta))$. Indeed, for all $\delta \in [2/(2+\varepsilon), 1)$, with fixed $\varepsilon > 0$, $m(\varepsilon,\delta) = 1$. In this case, the unmodified estimates would give a degenerate bound on the depth and size of the network. This modification is mathematically inconsequential, the relevant case for the approximation estimates being $\varepsilon \downarrow 0$.

4. Regularity and polynomial approximation. We shall exploit the classical fact that the analyticity of the coefficient a and of the source term f in Ω combined with periodicity implies analyticity of the solution u^a of (2.4). This, in turn, will imply exponential convergence of tensor product polynomial (spectral) approximations of a and u^a, which will be the basis of the NN approximation developed in Section 5 ahead.

4.1. Regularity. The following result follows from [5, Remark 1.6.5 and Theorem 1.7.1].

Lemma 4.1. *There exists* $A_{\mathcal{U}} > 0$ *such that* $S(\mathcal{D}) \subset \mathrm{Hol}(\Omega; A_{\mathcal{U}})$.

Proof. From [5, Theorem 1.7.1], it follows that $S(\mathcal{D}) \subset \mathrm{Hol}(\Omega)$ and, for each $u \in S(\mathcal{D})$, there exists $A_u > 0$ such that

$$\frac{1}{k!}|u|_{H^k(Q)} \le A_u^{k+1}\left(\sum_{j=0}^{k-2}\frac{1}{j!}|f|_{H^j(Q)} + \|u\|_{H^1(Q)}\right), \qquad \forall k \in \mathbb{N}_0.$$

Furthermore, from [5, Remark 1.6.5], inspecting the proof of [5, Theorem 1.7.1], and from (2.2), the proof is completed since it follows that

$$A_{\mathcal{U}} := \sup_{u \in S(\mathcal{D})} A_u < \infty.$$

□

4.2. Polynomial basis and quadrature. Consider the univariate Legendre polynomials L_0, $L_1, \dots$ such that $L_i \in \mathbb{Q}_i((0,1))$, normalized with $L_i(1) = 1$. Define then, for all $i \in \mathbb{N}$,

$$\varphi_0^{1\text{d}} = L_0, \qquad \varphi_{2i-1}^{1\text{d}} = L_{2i}, \qquad \varphi_{2i}^{1\text{d}} = L_{2i+1} - L_1.$$

These functions satisfy, for all $i \in \mathbb{N}_0$, $\varphi_i^{1\text{d}}(0) = \varphi_i^{1\text{d}}(1)$. It follows that, for all $p \in \mathbb{N}$,

$$\operatorname{span}(\varphi_1, \dots, \varphi_p) = \mathbb{Q}_{p+1}((0,1)) \cap \left\{ v \in H^1(\mathbb{R}/\mathbb{Z}) : \int_{(0,1)} v = 0 \right\}.$$

We can then introduce, for all integer $p \geq 2$,

$$\varphi_{i_1 + p i_2 + \cdots + p^{d-1} i_d} = \varphi_{i_1}^{1\text{d}} \otimes \varphi_{i_1}^{1\text{d}} \otimes \cdots \otimes \varphi_{i_d}^{1\text{d}}, \qquad (i_1, \dots, i_d) \in \{0, \dots, p-1\}^d. \tag{4.1}$$

Then, as shown in Lemma D.1 in the appendix, for all integer $p \geq 2$, denoting $n_\text{b} = p^d - 1$,

$$X_{n_\text{b}} := \operatorname{span}(\{\varphi_1, \dots, \varphi_{n_\text{b}}\}) = \left\{ v \in \mathbb{Q}_p(Q) : \int_Q v = 0 \text{ and } v \in H^1(\Omega) \right\} = \mathbb{Q}_p(Q) \cap X. \tag{4.2}$$

The restriction to polynomials of degree $p \geq 2$ is without loss of generality, as the periodicity and vanishing average constraints imply $\mathbb{Q}_1(Q) \cap X = \{0\}$.

For a quadrature order parameter $q \geq 2$, denoting $n_\text{q} = q^d$, we consider the *Gauss-Lobatto quadrature rule* with weights $\{w_k^{(q)}\}_{k=1}^{n_\text{q}}$ and points $\{\boldsymbol{x}_k^{(q)}\}_{k=1}^{n_\text{q}} \subset \overline{Q}$ such that

$$\int_Q g = \sum_{k=1}^{n_\text{q}} w_k^{(q)} g(\boldsymbol{x}_k^{(q)}), \qquad \forall g \in \mathbb{Q}_{2q-3}(Q).$$

There exist constants $c_{\text{quad},1}, c_{\text{quad},2} > 0$ such that

$$c_{\text{quad},1} \|v\|_{L^2(Q)}^2 \leq \sum_{k=1}^{(p+1)^d} w_k^{(p+1)} (v(\boldsymbol{x}_k^{(p+1)}))^2 \leq c_{\text{quad},2} \|v\|_{L^2(Q)}^2, \quad \forall v \in \mathbb{Q}_p(Q), \forall p \in \mathbb{N}, \tag{4.3}$$

see [3, Equation (6.4.52)]. We remark that the constants $c_{\text{quad},1}$ and $c_{\text{quad},2}$ are independent of p, but depend in general exponentially on the dimension d. We may assume, without loss of generality, that $c_{\text{quad},1} \leq 1$ and $c_{\text{quad},2} \geq 1$. We introduce furthermore the bilinear form with quadrature $\mathfrak{b}_{n_\text{q}}^a$

$$\mathfrak{b}_{n_\text{q}}^a(u, v) = \sum_{k=1}^{n_\text{q}} w_k^{(q)} a(\boldsymbol{x}_k^{(q)}) \nabla u(\boldsymbol{x}_k^{(q)}) \cdot \nabla v(\boldsymbol{x}_k^{(q)}), \qquad \forall u, v \in C^1(\overline{Q}).$$

Eventually, here u, v shall be tensor product polynomials in Q.

For each $a \in \mathcal{D}$, we introduce the symmetric matrices

$$\left[\mathbf{A}_{n_\text{b}}^a\right]_{ij} = \mathfrak{b}^a(\varphi_j, \varphi_i), \qquad \left[\mathbf{A}_{n_\text{b}, n_\text{q}}^a\right]_{ij} = \mathfrak{b}_{n_\text{q}}^a(\varphi_j, \varphi_i), \qquad (i,j) \in \{1, \dots, n_\text{b}\}^2.$$

Let $\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}}$ be the matrix obtained with $a \equiv 1$ in Q. Let $q \geq p+1$: for all nonzero $\boldsymbol{x} \in \mathbb{R}^{n_\mathrm{b}}$, there exists $v \in X_{n_\mathrm{b}} \backslash \{0\}$ such that, for all $a \in \mathcal{D}$,

$$\boldsymbol{x}^\top \mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}} \boldsymbol{x} = \mathfrak{b}^a_{n_\mathrm{q}}(v,v) > 0, \tag{4.4}$$

due to the equivalence of norms (4.3) and to the Poincaré inequality (2.3). Hence, the matrices $\mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}}$ and $\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}}$ are invertible. Denote then

$$\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}} = (\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}})^{-1} \mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}}.$$

We also introduce the right-hand side vector $\boldsymbol{c}_{f;n_\mathrm{b}} \in \mathbb{R}^{n_\mathrm{b}}$ such that

$$[\boldsymbol{c}_{f;n_\mathrm{b}}]_i = \int_Q f\varphi_i, \qquad i \in \{1,\dots,n_\mathrm{b}\}\,. \tag{4.5}$$

The Cauchy-Schwarz inequality and

$$\|\varphi_i\|^2_{L^2(Q)} \leq \begin{cases} 1/(2i+3) & \text{if } i \text{ is odd} \\ 1/(2i+3)+1/3 & \text{if } i \text{ is even,} \end{cases} \qquad \forall i \in \mathbb{N},$$

hence $\|\varphi_i\|_{L^2(Q)} \leq 1$, imply that

$$\|\boldsymbol{c}_{f;n_\mathrm{b}}\|_2^2 \leq \|f\|^2_{L^2(Q)} \sum_{i=1}^{n_\mathrm{b}} \|\varphi_i\|^2_{L^2(Q)} \leq n_\mathrm{b} \|f\|^2_{L^2(Q)}. \tag{4.6}$$

Finally, let

$$\widetilde{\boldsymbol{c}}_{f;n_\mathrm{b}} = (\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}})^{-1} \boldsymbol{c}_{f;n_\mathrm{b}}.$$

and

$$\boldsymbol{c}^a_{u;n_\mathrm{b},n_\mathrm{q}} \coloneqq (\mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}})^{-1} \boldsymbol{c}_{f;n_\mathrm{b}}. \tag{4.7}$$

Here and in the sequel, for all $q \in \mathbb{N}$, with $n_\mathrm{q} = q^d$, we will denote $\boldsymbol{a}_{n_\mathrm{q}} \in \mathbb{R}^{n_\mathrm{q}}$ the vector with entries

$$\left[\boldsymbol{a}_{n_\mathrm{q}}\right]_i = a(\boldsymbol{x}^{(q)}_k), \qquad \forall k \in \{1,\dots,n_\mathrm{q}\}.$$

The following two statements concern the norms of the matrices introduced, and will be necessary for later estimates.

Lemma 4.2. *There exist $C_\mathrm{coer}, C_\mathrm{cont} > 0$ such that for all $a \in \mathcal{D}$, all $p \in \mathbb{N}$, and all integer $q \geq p+1$,*

$$\sigma(\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}}) \subset [C_\mathrm{coer}, C_\mathrm{cont}], \qquad \sigma((\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}})^{-1}) \subset [1/C_\mathrm{cont}, 1/C_\mathrm{coer}]\,, \tag{4.8}$$

with $n_\mathrm{b} = p^d - 1$ and $n_\mathrm{q} = q^d$.

Proof. For all $\boldsymbol{x} \in \mathbb{R}^{n_\mathrm{b}}$,

$$a_\mathrm{min} \boldsymbol{x}^\top \mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}} \boldsymbol{x} \leq \boldsymbol{x}^\top \mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}} \boldsymbol{x} \leq a_\mathrm{max} \boldsymbol{x}^\top \mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}} \boldsymbol{x}.$$

Since the matrices $\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}}$ and $\mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}}$ are symmetric and positive definite, see (4.4), this implies, by Lemma C.1 in the Appendix,

$$\sigma((\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}})^{-1} \mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}}) \subset [a_\mathrm{min}, a_\mathrm{max}].$$

The assertion follows with $C_\mathrm{coer} = a_\mathrm{min}$ and $C_\mathrm{cont} = a_\mathrm{max}$. ∎

We assume, for ease of notation, that $C_{\mathrm{coer}} \leq 1$ and $C_{\mathrm{cont}} \geq 1$.

LEMMA 4.3. *There exists a constant $C_{\mathrm{A}} > 0$ such that, for all $p \in \mathbb{N}$, and for all integer $q \geq p+1$,*

$$\|(\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}})^{-1}\|_2 \leq C_{\mathrm{A}} n_{\mathrm{b}},$$

with $n_{\mathrm{b}} = p^d - 1$ and $n_{\mathrm{q}} = q^d$.

Proof. From (4.3), (2.3), the symmetry of the bilinear form, and Lemma D.2, it follows that

$$\lambda_{\min}(\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}}) \coloneqq \min_{\lambda \in \sigma(\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}})} \lambda = \inf_{\boldsymbol{x} \in \mathbb{R}^{n_{\mathrm{b}}}} \frac{\boldsymbol{x}^\top \mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}} \boldsymbol{x}}{\|\boldsymbol{x}\|_2^2} \overset{(D.1)}{\geq} \inf_{v \in X_{n_{\mathrm{b}}}} \frac{\mathfrak{b}^1_{n_{\mathrm{q}}}(v,v)}{C_{L^2}^2 n_{\mathrm{b}} \|v\|^2_{L^2(Q)}}$$

$$\overset{(4.3)}{\geq} \frac{c^2_{\mathrm{quad},1}}{C^2_{L^2} n_{\mathrm{b}}} \inf_{v \in X_{n_{\mathrm{b}}}} \frac{\|\nabla v\|^2_{L^2(Q)}}{\|v\|^2_{L^2(Q)}} \overset{(2.3)}{\geq} \frac{c^2_{\mathrm{quad},1}}{C^2_{\mathrm{poi}} C^2_{L^2} n_{\mathrm{b}}}.$$

This concludes the proof, since $\|(\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}})^{-1}\|_2 = 1/\lambda_{\min}(\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}})$. ∎

5. NN approximation. We detail the structure of the branch and trunk networks (see Fig. 2) and state and prove their convergence rate bounds.

5.1. Branch network.

5.1.1. Input layer. For all $k \in \{1,\dots,n_{\mathrm{q}}\}$, let $\widehat{\mathbf{D}}_{n_{\mathrm{b}}}(\boldsymbol{x}_k^{(q)})$ denote the matrix with entries

$$\left[\widehat{\mathbf{D}}_{n_{\mathrm{b}}}(\boldsymbol{x}_k^{(q)})\right]_{ij} = w_k^{(q)} \nabla\varphi_i(\boldsymbol{x}_k^{(q)}) \cdot \nabla\varphi_j(\boldsymbol{x}_k^{(q)}), \qquad (i,j) \in \{1,\dots,n_{\mathrm{b}}\}^2.$$

The following statement follows from this definition.

LEMMA 5.1. *For all $\alpha \in \mathbb{R}$, the one-layer NN*

$$\Phi^{A,\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}} \coloneqq \left(\left(-\alpha\left[\mathbf{vec}(\widehat{\mathbf{D}}_{n_{\mathrm{b}}}(\boldsymbol{x}_1^{(q)}))|\dots|\,\mathbf{vec}(\widehat{\mathbf{D}}_{n_{\mathrm{b}}}(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}))\right], \mathbf{0}_{n_{\mathrm{b}}^2}\right)\right)$$

satisfies $\mathrm{M}(\Phi^{A,\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}}) \leq n_{\mathrm{b}}^2 n_{\mathrm{q}}$ and is such that

$$\mathbf{matr}\left(\mathrm{R}(\Phi^{A,\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}})(\boldsymbol{a}_{n_{\mathrm{q}}})\right) = -\alpha \mathbf{A}^a_{n_{\mathrm{b}},n_{\mathrm{q}}}. \tag{5.1}$$

Proof. We have

$$\left[\mathrm{R}\left(\Phi^{A,\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}}\right)(\boldsymbol{a}_{n_{\mathrm{q}}})\right]_i = -\alpha \sum_{k=1}^{n_{\mathrm{q}}} \left[\mathbf{vec}(\widehat{\mathbf{D}}(\boldsymbol{x}_k^{(q)}))\right]_i a(\boldsymbol{x}_k^{(q)}),$$

hence the equality after matricization. The size bound follows from the fact that

$$\|\widehat{\mathbf{D}}_{n_{\mathrm{b}}}(x_k^{(q)})\|_0 \leq n_{\mathrm{b}}^2, \quad k \in \{1,\dots,n_{\mathrm{q}}\}.$$

∎

LEMMA 5.2. *For all $\alpha \in \mathbb{R}$, the two-layer NN*

$$\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}} \coloneqq \left(\left(\mathbf{Id}_{n_{\mathrm{b}}} \otimes (\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}})^{-1}, \mathbf{vec}(\mathbf{Id}_{n_{\mathrm{b}}})\right)\right) \odot \Phi^{A,\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}} \tag{5.2}$$

is such that

$$\mathbf{matr}\left(\mathrm{R}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}})(\boldsymbol{a}_{n_{\mathrm{q}}})\right) = -\alpha \widetilde{\mathbf{A}}^a_{n_{\mathrm{b}},n_{\mathrm{q}}} + \mathbf{Id}_{n_{\mathrm{b}}},$$

and $\mathrm{M}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}}) \leq 2n_{\mathrm{b}}(1 + n_{\mathrm{b}}n_{\mathrm{q}} + n_{\mathrm{b}}^2)$.

Proof. For all $m, n, l \in \mathbb{N}$ and all $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{X} \in \mathbb{R}^{n\times l}$, $\mathbf{vec}(\mathbf{AX}) = (\mathbf{Id}_l \otimes \mathbf{A})\,\mathbf{vec}(\mathbf{X})$. Hence,

$$\begin{aligned}
\mathrm{R}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_\mathrm{q}}) &\overset{(5.2)}{=} (\mathbf{Id}_{n_\mathrm{b}} \otimes (\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}})^{-1})\,\mathrm{R}(\Phi^{A,\alpha}_{n_\mathrm{b},n_\mathrm{q}}) + \mathbf{vec}(\mathbf{Id}_{n_\mathrm{b}}) \\
&\overset{(5.1)}{=} -\alpha(\mathbf{Id}_{n_\mathrm{b}} \otimes (\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}})^{-1})\,\mathbf{vec}(\mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}}) + \mathbf{vec}(\mathbf{Id}_{n_\mathrm{b}}) \\
&= -\alpha\,\mathbf{vec}((\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}})^{-1}\mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}}) + \mathbf{vec}(\mathbf{Id}_{n_\mathrm{b}}).
\end{aligned}$$

Finally, by Proposition 3.3,

$$\mathrm{M}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_\mathrm{q}}) \le 2\,\mathrm{M}(\Phi^{A,\alpha}_{n_\mathrm{b},n_\mathrm{q}}) + 2\left(\|\mathbf{Id}_{n_\mathrm{b}} \otimes (\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}})^{-1}\|_0 + \|\mathbf{Id}_{n_\mathrm{b}}\|_0\right) \le 2\,\mathrm{M}(\Phi^{A,\alpha}_{n_\mathrm{b},n_\mathrm{q}}) + 2(n_\mathrm{b}^3 + n_\mathrm{b}). \qquad \square$$

The following statement is a consequence of Lemmas 4.2 and 5.2.

LEMMA 5.3. *Let $C_\mathrm{cont}, C_\mathrm{coer}$ be the constants introduced in Lemma 4.2. For all $a \in \mathcal{D}$, for all $p \in \mathbb{N}$, for all integer $q \ge p+1$, and for all $\alpha \in (0, 1/C_\mathrm{cont})$,*

$$\|\,\mathbf{matr}\left(\mathrm{R}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_\mathrm{q}})(\boldsymbol{a}_{n_\mathrm{q}})\right)\|_2 \le 1 - \alpha C_\mathrm{coer} =: 1-\delta,$$

with $n_\mathrm{b} = p^d - 1$ and $n_\mathrm{q} = q^d$.

Proof. By Lemma 5.2, $\mathbf{matr}\left(\mathrm{R}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_\mathrm{q}})(\boldsymbol{a}_{n_\mathrm{q}})\right) = \mathbf{Id}_{n_\mathrm{b}} - \alpha\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}}$. Due to Lemma 4.2 and since $\alpha \le 1/C_\mathrm{cont}$, this matrix is symmetric positive definite and

$$\|\mathbf{Id}_{n_\mathrm{b}} - \alpha\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}}\|_2 = \sup_{\substack{\boldsymbol{x}\in\mathbb{R}^{n_\mathrm{b}}\\ \|\boldsymbol{x}\|_2=1}} \boldsymbol{x}^\top(\mathbf{Id}_{n_\mathrm{b}} - \alpha\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}})\boldsymbol{x} = 1 - \alpha \inf_{\substack{\boldsymbol{x}\in\mathbb{R}^{n_\mathrm{b}}\\ \|\boldsymbol{x}\|_2=1}} \boldsymbol{x}^\top\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}}\boldsymbol{x} \le 1 - \alpha C_\mathrm{coer},$$

where the last inequality follows from Lemma 4.2. $\square$

Thanks to Theorem 3.4 we can construct the network that approximates the inversion of the "preconditioned" Galerkin-Numerical Integration matrix $\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}}$ (more precisely, the network that emulates the map $\boldsymbol{a}_{n_\mathrm{q}} \mapsto (\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}})^{-1}$).

PROPOSITION 5.4. *Let $C_\mathrm{coer}, C_\mathrm{cont}$ be defined as in Lemma 4.2. There exists a constant $C_{\mathrm{inv},A} > 0$ such that for all $n_\mathrm{b} \in \mathbb{N}$ and for all $\varepsilon_\mathrm{inv} \in (0,1)$, writing $\alpha = 1/(C_\mathrm{coer} + C_\mathrm{cont})$, $\delta = \alpha C_\mathrm{coer}$, $n_\mathrm{q} = n_\mathrm{b}+1$, and denoting*

$$\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_\mathrm{inv},n_\mathrm{b}} := ((\alpha\mathbf{Id}_{n_\mathrm{b}}, \mathbf{0}_{n_\mathrm{b}}))\bullet\Phi^{1-\delta,n_\mathrm{b}}_{\mathrm{inv};\frac{\varepsilon_\mathrm{inv}}{\alpha}} \odot \Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_\mathrm{q}}, \tag{5.3}$$

we have

$$\sup_{a\in\mathcal{D}} \|(\mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_\mathrm{inv},n_\mathrm{b}}))(\boldsymbol{a}_{n_\mathrm{q}})\|_2 \le \varepsilon_\mathrm{inv},$$

and

$$\begin{aligned}
\mathrm{L}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_\mathrm{inv},n_\mathrm{b}}) &\le C_{\mathrm{inv},A}\left[1 + \log(1+|\log\varepsilon_\mathrm{inv}|) + \log(n_\mathrm{b})\right] \\
&\qquad \times \left[1 + |\log\varepsilon_\mathrm{inv}| + \log(n_\mathrm{b}) + \log(1+|\log\varepsilon_\mathrm{inv}|)\right] \\
\mathrm{M}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_\mathrm{inv},n_\mathrm{b}}) &\le C_{\mathrm{inv},A} n_\mathrm{b}^3\left[1 + |\log\varepsilon_\mathrm{inv}|\right]\left[1 + \log(1+|\log\varepsilon_\mathrm{inv}|) + \log(n_\mathrm{b})\right]^2 \\
&\qquad \times \left[1 + |\log\varepsilon_\mathrm{inv}| + \log(n_\mathrm{b}) + \log(1+|\log\varepsilon_\mathrm{inv}|)\right].
\end{aligned}$$

Proof. We start by estimating the approximation error. By Lemma 5.3,

$$\|\,\mathbf{matr}(\mathrm{R}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_\mathrm{q}}(\boldsymbol{a}_{n_\mathrm{q}})))\|_2 \le 1-\delta.$$

We temporarily restrict $\varepsilon_\mathrm{inv} \in (0,\alpha/4)$. Then, we have, for all $a \in \mathcal{D}$,

$$\begin{aligned}
&\|(\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_\mathrm{inv},n_\mathrm{b}}))(\boldsymbol{a}_{n_\mathrm{q}})\|_2\\
&\overset{(5.3)}{=} \alpha\|(\alpha\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi^{1-\delta,n_\mathrm{b}}_{\mathrm{inv};\frac{\varepsilon_\mathrm{inv}}{\alpha}} \odot \Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_\mathrm{q}}))(\boldsymbol{a}_{n_\mathrm{q}})\|_2\\
&\overset{\mathrm{L.}\,5.2}{=} \alpha\|(\alpha\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}})^{-1} - \mathbf{matr}(\mathrm{R}(\Phi^{1-\delta,n_\mathrm{b}}_{\mathrm{inv};\frac{\varepsilon_\mathrm{inv}}{\alpha}})(-\alpha\,\mathbf{vec}(\widetilde{\mathbf{A}}^a_{n_\mathrm{b},n_\mathrm{q}}) + \mathbf{vec}(\mathbf{Id}_{n_\mathrm{b}}))\|_2\\
&\overset{\mathrm{T.}\,3.4}{\le} \alpha\frac{\varepsilon_\mathrm{inv}}{\alpha} = \varepsilon_\mathrm{inv}.
\end{aligned}$$

We now have to bound the depth and size of $\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_\mathrm{inv},n_\mathrm{b}}$. First, we remark that

$$m(\varepsilon_\mathrm{inv}/\alpha,\delta) = \left\lceil \frac{\log\left(C_\mathrm{coer}\varepsilon_\mathrm{inv}/2\right)}{\log(1-\delta)}\right\rceil,$$

where $m(\cdot,\cdot)$ is defined in Theorem 3.4. Now, we use the fact that there exists $C_1 > 1$ such that, for all $\varepsilon_\mathrm{inv} \in (0,1)$,

$$|\log(C_\mathrm{coer}\varepsilon_\mathrm{inv}/2)| \le C_1(1+|\log\varepsilon_\mathrm{inv}|).$$

Furthermore, there exists $C_2 > 1$ such that for all $n_\mathrm{b} \in \mathbb{N}$, $\delta \ge C_2^{-1}$. Remark then that $|\log(1-y)| \ge y$ for all $y \in (0,1)$, hence $|\log(1-\delta)|^{-1} \le C_2$. We infer that for all $\varepsilon_\mathrm{inv} \in (0,1)$ and for all $n_\mathrm{b} \in \mathbb{N}$,

$$m(\varepsilon_\mathrm{inv}/\alpha,\delta) \le C_1C_2(2+|\log\varepsilon_\mathrm{inv}|)\ .$$

Therefore, from Theorem 3.4 we obtain that there exist constants $C_4, C_5 > 0$ dependent only on C_coer, C_cont, and d such that

$$\mathrm{L}\left(\Phi^{1-\delta,n_\mathrm{b}}_{\mathrm{inv};\frac{\varepsilon_\mathrm{inv}}{\alpha}}\right) \le C_4\left(1+\log(1+|\log\varepsilon_\mathrm{inv}|)+\log(n_\mathrm{b})\right)\cdot\left(1+|\log\varepsilon_\mathrm{inv}|+\log(n_\mathrm{b})+\log(1+|\log\varepsilon_\mathrm{inv}|)\right)$$

and

$$\begin{aligned}
\mathrm{M}\left(\Phi^{1-\delta,n_\mathrm{b}}_{\mathrm{inv};\frac{\varepsilon_\mathrm{inv}}{\alpha}}\right) \le C_5(1+|\log\varepsilon_\mathrm{inv}|)n_\mathrm{b}^3\left[1+\log(1+|\log\varepsilon_\mathrm{inv}|)+\log(n_\mathrm{b})\right]^2&\\
\times\left[1+|\log\varepsilon_\mathrm{inv}|+\log(n_\mathrm{b})+\log(1+|\log\varepsilon_\mathrm{inv}|)\right]&.
\end{aligned}$$

We can now drop the restriction $\varepsilon_\mathrm{inv} < \alpha/4$, adjusting the constants C_4 and C_5. Since, in addition,

$$\mathrm{L}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_q}) = 2, \qquad \mathrm{M}(\Phi^{\widetilde{A},\mathrm{Id},\alpha}_{n_\mathrm{b},n_q}) \le C_6 n_\mathrm{b}^3,$$

for $C_6 > 0$ independent of n_b, we obtain the bounds on the depth and size of $\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_\mathrm{inv},n_\mathrm{b}}$. ∎

5.1.2. Computation of the coefficients.

Proposition 5.5. *There exists a constant $C_{c_u} > 0$ such that for all $n_\mathrm{b} \in \mathbb{N}$ and for all $\varepsilon_u \in (0,1)$, writing $n_\mathrm{q} = n_\mathrm{b}+1$ and*

$$\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}} := \left((\widetilde{\boldsymbol{c}}^\top_{f;n_\mathrm{b}} \otimes \mathbf{Id}_{n_\mathrm{b}}, \mathbf{0}_{n_\mathrm{b}})\right) \odot \Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}n_\mathrm{b}^{3/2}C_\mathrm{A}),n_\mathrm{b}},$$

where C_{A} is the constant from Lemma 4.3, we have

$$\sup_{a\in\mathcal{D}} \|\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}} - \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}})(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2 \leq \varepsilon_u$$

and

$$\begin{aligned}
\mathrm{L}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}}) &\leq C_{c_u}\left[1+\log(1+|\log\varepsilon_u|+\log n_{\mathrm{b}})+\log(n_{\mathrm{b}})\right] \\
&\qquad\qquad \times \left[1+|\log\varepsilon_u|+\log(n_{\mathrm{b}})+\log(1+|\log\varepsilon_u|+\log n_{\mathrm{b}})\right] \\
\mathrm{M}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}}) &\leq C_{c_u} n_{\mathrm{b}}^3\left[1+|\log\varepsilon_u|\,\right]\left[1+\log(1+|\log\varepsilon_u|+\log n_{\mathrm{b}})+\log(n_{\mathrm{b}})\right]^2 \\
&\qquad\qquad \times \left[1+|\log\varepsilon_u|+\log(n_{\mathrm{b}})+\log(1+|\log\varepsilon_u|)\right].
\end{aligned}$$

Proof. For all $m,n,l\in\mathbb{N}$, let $\mathbf{B}\in\mathbb{R}^{m\times n}$ and $\mathbf{C}\in\mathbb{R}^{n\times l}$. Then

$$\mathbf{vec}(\mathbf{BC}) = (\mathbf{C}^\top\otimes\mathbf{Id}_m)\,\mathbf{vec}(\mathbf{B}).$$

This identity implies

$$(5.4)\qquad \left(\widetilde{\boldsymbol{c}}^\top_{f;n_{\mathrm{b}}}\otimes\mathbf{Id}_{n_{\mathrm{b}}}\right)\mathrm{R}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}),n_{\mathrm{b}}})(\boldsymbol{a}_{n_{\mathrm{q}}}) = \mathbf{matr}\left(\mathrm{R}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}),n_{\mathrm{b}}})(\boldsymbol{a}_{n_{\mathrm{q}}})\right)\widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}.$$

We assume that $n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}\|f\|_{L^2(Q)}\geq 1$; if this does not hold, it is sufficient to temporarily restrict $\varepsilon_u < n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}\|f\|_{L^2(Q)}$ and drop this restriction at the end of the proof by adjusting the constants. Therefore, for all $a\in\mathcal{D}$,

$$\begin{aligned}
&\|\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}} - \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_{\mathrm{b}}})(\boldsymbol{a}_{n_{\mathrm{q}}})\|_2 \\
&\overset{(4.7)}{=} \left\|\left((\widetilde{\mathbf{A}}^a_{n_{\mathrm{b}},n_{\mathrm{q}}})^{-1} - \mathbf{matr}\left(\mathrm{R}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}),n_{\mathrm{b}}})(\boldsymbol{a}_{n_{\mathrm{q}}})\right)\right)\widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}\right\|_2 \\
&\leq \left\|(\widetilde{\mathbf{A}}^a_{n_{\mathrm{b}},n_{\mathrm{q}}})^{-1} - \mathbf{matr}\left(\mathrm{R}(\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}),n_{\mathrm{b}}})(\boldsymbol{a}_{n_{\mathrm{q}}})\right)\right\|_2\|\widetilde{\boldsymbol{c}}_{f;n_{\mathrm{b}}}\|_2 \\
&\overset{\text{P. }5.4}{\leq} \frac{\varepsilon_u}{n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}\|f\|_{L^2(Q)}}\|(\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}})^{-1}\|_2\|\boldsymbol{c}_{f;n_{\mathrm{b}}}\|_2 \\
&\overset{(4.6)}{\leq} \frac{\varepsilon_u}{n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}\|f\|_{L^2(Q)}}\|(\mathbf{A}^1_{n_{\mathrm{b}},n_{\mathrm{q}}})^{-1}\|_2\sqrt{n_{\mathrm{b}}}\|f\|_{L^2(Q)} \\
&\overset{\text{L. }4.3}{\leq} \frac{\varepsilon_u}{C_{\mathrm{A}}n_{\mathrm{b}}\|f\|_{L^2(Q)}}C_{\mathrm{A}}n_{\mathrm{b}}\|f\|_{L^2(Q)} = \varepsilon_u,
\end{aligned}$$

where in the last three steps we have used Proposition 5.4, bound (4.6), and Lemma 4.3. To derive the bounds on the size and depth of $\Phi^{\widetilde{A}}_{\mathrm{inv};\varepsilon_u/(\|f\|_{L^2(\Omega)}n_{\mathrm{b}}^{3/2}C_{\mathrm{A}}),n_{\mathrm{b}}}$, first remark that

$$\|\widetilde{\boldsymbol{c}}^\top_{f;n_{\mathrm{b}}}\otimes\mathbf{Id}_{n_{\mathrm{b}}}\|_0 \leq n_{\mathrm{b}}^2.$$

Then, defining $\varepsilon_{\mathrm{inv}} \coloneqq \varepsilon_u/(n_{\mathrm{b}}^{3/2}\|f\|_{L^2(Q)}C_{\mathrm{A}})$, there exists $C_1>0$ such that for all $n_{\mathrm{b}}\in\mathbb{N}$ and for all $\varepsilon_u\in(0,1)$,

$$|\log\varepsilon_{\mathrm{inv}}| \leq C_1\left(1+|\log\varepsilon_u|+\log n_{\mathrm{b}}\right).$$

Inserting this bound in Proposition 5.4 and applying Proposition 5.4 concludes the proof. ∎

5.2. Trunk network. The following emulation rates for the approximation of the polynomial basis are a direct consequence of [23, Proposition 2.13].

Proposition 5.6. *There exists $C_{\mathrm{b}} > 0$ such that, for all $\varepsilon_{\mathrm{b}} \in (0,1)$ and all $n_{\mathrm{b}} \in \mathbb{N}$, there exists a NN $\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}$ such that $\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}) : \mathbb{R}^d \to \mathbb{R}^{n_{\mathrm{b}}}$,*

$$\max_{i\in\{1,\dots,n_{\mathrm{b}}\}} \|\varphi_i - \left[\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}})\right]_i \|_{H^1(Q)} \leq \varepsilon_{\mathrm{b}},$$

and

$$\begin{aligned} \mathrm{L}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}) &\leq C_{\mathrm{b}} \left(1 + |\log\varepsilon_{\mathrm{b}}| + n_{\mathrm{b}}^{1/d}\right)(1+\log n_{\mathrm{b}}) \\ \mathrm{M}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}},n_{\mathrm{b}}}) &\leq C_{\mathrm{b}} \left(n_{\mathrm{b}}^{2/d} + n_{\mathrm{b}}^{1/d}|\log\varepsilon_{\mathrm{b}}| + n_{\mathrm{b}}(1+\log n_{\mathrm{b}} + |\log\varepsilon_{\mathrm{b}}|)\right). \end{aligned}$$

5.3. Operator network expression rates. Combining the results from Sections 5.1 and 5.2, we obtain the main result on the operator network approximation of (2.4). The structure of the operator network is schematically represented in Figures 1 and 2.

Theorem 5.7. *There exists $C > 0$ such that, for all $\varepsilon \in (0,1)$, for all $a \in \mathcal{D}$ with $u^a = S(a)$ and solution operator S as defined in Section 2, there exist*

(*a*) $n_{\mathrm{b}}, n_{\mathrm{q}} \in \mathbb{N}$,
(*b*) *a set of points* $\boldsymbol{x}_{\mathrm{enc}} := \{\boldsymbol{x}_1, \dots \boldsymbol{x}_{n_{\mathrm{q}}}\} \subset \overline{Q}$,
(*c*) *two NNs* $\Phi^{\mathrm{br}}_\varepsilon$ *and* $\Phi^{\mathrm{tr}}_\varepsilon$ *with* $\mathrm{R}(\Phi^{\mathrm{br}}_\varepsilon) : \mathbb{R}^{n_{\mathrm{q}}} \to \mathbb{R}^{n_{\mathrm{b}}}$ *and* $\mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon) : Q \to \mathbb{R}^{n_{\mathrm{b}}}$,

such that

(*i*) $n_{\mathrm{b}}, n_{\mathrm{q}} \leq C(1+|\log\varepsilon|^d)$,
(*ii*) *the following error bound holds:*

$$\sup_{a\in\mathcal{D}} \|u^a - (\mathcal{R}_{\Phi^{\mathrm{tr}}_\varepsilon} \circ \mathcal{A}_{\Phi^{\mathrm{br}}_\varepsilon} \circ \mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}})(a)\|_{H^1(Q)} \leq \varepsilon,$$

(*iii*) *as* $\varepsilon \downarrow 0$,

$$\mathrm{L}(\Phi^{\mathrm{br}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|\,(\log|\log\varepsilon|)\right), \qquad \mathrm{M}(\Phi^{\mathrm{br}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|^{3d+2}(\log|\log\varepsilon|)^2\right),$$

and

$$\mathrm{L}(\Phi^{\mathrm{tr}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|\,(\log|\log\varepsilon|)\right), \qquad \mathrm{M}(\Phi^{\mathrm{tr}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|^{d+1}\right).$$

Proof. Due to Lemma A.2, there exist constants $C_{\mathrm{G}}, b_{\mathrm{G}}, C_{\mathrm{q}} > 0$ such that for all $n_{\mathrm{b}} \in \mathbb{N}$, there exists $n_{\mathrm{q}} \leq C_{\mathrm{q}} n_{\mathrm{b}}$ such that

$$\sup_{a\in\mathcal{D}} \|u^a - u^a_{n_{\mathrm{b}},n_{\mathrm{q}}}\|_{H^1(Q)} \leq C_{\mathrm{G}} \exp(-b_{\mathrm{G}} n_{\mathrm{b}}^{1/d}),$$

where $u^a_{n_{\mathrm{b}},n_{\mathrm{q}}} = \sum_{i=1}^{n_{\mathrm{b}}} \left[\boldsymbol{c}^a_{u;n_{\mathrm{b}},n_{\mathrm{q}}}\right]_i \varphi_i \in X_{n_{\mathrm{b}}}$ is the fully discrete Galerkin projection of u^a onto $X_{n_{\mathrm{b}}}$,

$$u^a_{n_{\mathrm{b}},n_{\mathrm{q}}} \in X_{n_{\mathrm{b}}} : \quad \mathfrak{b}^a_{n_{\mathrm{q}}}(u^a_{n_{\mathrm{b}},n_{\mathrm{q}}}, v) = (f, v), \qquad \forall v \in X_{n_{\mathrm{b}}}.$$

We assume, without loss of generality and for ease of notation, that $C_{\mathrm{G}} \geq 1$. Fix now

$$p(\varepsilon) = \left\lceil \frac{|\log(\varepsilon/3)| + \log C_{\mathrm{G}}}{b_{\mathrm{G}}} \right\rceil + 1, \qquad n_{\mathrm{b}}(\varepsilon) = p(\varepsilon)^d - 1, \qquad n_{\mathrm{q}}(\varepsilon) = n_{\mathrm{b}}(\varepsilon) + 1. \tag{5.5}$$

Then, we observe that (5.5) implies the existence of a constant $C_{\mathrm{b}} > 0$ such that, for all $\varepsilon \in (0,1)$, we have $n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon) \leq C_{\mathrm{b}}(1+|\log\varepsilon|^d)$, i.e., item (i) of the statement of the theorem.

We also define $C_{\mathrm{pol}} > 0$ as a constant such that, for all $p \in \mathbb{N}$,

$$\|\nabla q\|_{L^2(Q)} \leq C_{\mathrm{pol}} p^2 \|q\|_{L^2(Q)}, \qquad \forall q \in \mathbb{Q}_p(Q). \tag{5.6}$$

This inverse inequality follows straightforwardly from the classical Markov inequality in $(0,1)$, with a tensorization argument (which yields that $C_{\mathrm{pol}} \sim \sqrt{d}$). With $n_{\mathrm{b}}(\varepsilon)$ as in (5.5) define

$$\varepsilon_{\mathrm{b}} := \frac{\varepsilon}{3n_{\mathrm{b}}(\varepsilon)(2+\sup_{a\in\mathcal{D}}\|u^a\|_{L^2(Q)})}, \quad \varepsilon_u := \frac{\varepsilon}{3(1+C_{\mathrm{pol}}^2 n_{\mathrm{b}}(\varepsilon)^{4/d})^{1/2} n_{\mathrm{b}}(\varepsilon)^{1/2}}, \quad \varepsilon_{\mathrm{G}} := \frac{\varepsilon}{3}. \tag{5.7}$$

We assume that $\varepsilon \in (0,1)$ implies $\varepsilon_{\mathrm{b}}, \varepsilon_u \in (0,1)$. If this is not the case, it is sufficient to restrict ε to values such that $\varepsilon_{\mathrm{b}}, \varepsilon_u \in (0,1)$; the restriction can be dropped at the end of the proof. Due to (5.5),

$$\sup_{a\in\mathcal{D}} \|u^a - u^a_{n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon)}\|_{H^1(Q)} \leq \varepsilon_{\mathrm{G}}. \tag{5.8}$$

Define then

$$\Phi^{\mathrm{br}}_\varepsilon = \Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)} \qquad \text{and} \qquad \Phi^{\mathrm{tr}}_\varepsilon = \Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)},$$

where the NNs $\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)}$ and $\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}$ are defined in Propositions 5.5 and 5.6, respectively.

Error estimate. For all $a \in \mathcal{D}$,

$$\begin{aligned}
&\|u^a - \left(\mathrm{R}(\Phi^{\mathrm{br}}_\varepsilon)(\boldsymbol{a}_{n_{\mathrm{q}}})\right) \cdot \mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon)\|_{H^1(Q)} \\
&\quad \leq \|u^a - u^a_{n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon)}\|_{H^1(Q)} + \|u^a_{n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon)} - \left(\mathrm{R}(\Phi^{\mathrm{br}}_\varepsilon)(\boldsymbol{a}_{n_{\mathrm{q}}})\right) \cdot \mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon)\|_{H^1(Q)} =: (I) + (II).
\end{aligned}$$

We have already established that $(I) \leq \varepsilon_{\mathrm{G}} = \varepsilon/3$.

Consider term (II). We have

$$\begin{aligned}
(II) &= \| \sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \left(\left[\boldsymbol{c}^a_{u; n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon)}\right]_i \varphi_i - \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\right]_i \left[\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)})\right]_i \right) \|_{H^1(Q)} \\
&\leq \| \sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \left(\left[\boldsymbol{c}^a_{u; n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon)}\right]_i - \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\right]_i \right) \varphi_i \|_{H^1(Q)} \\
&\qquad\qquad + \| \sum_{i=1}^{n_{\mathrm{b}}(\varepsilon)} \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)})\right]_i \left(\varphi_i - \left[\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)})\right]_i \right) \|_{H^1(Q)} \\
&=: (IIa) + (IIb).
\end{aligned}$$

Denote, for all $i \in \{1, \ldots, n_{\mathrm{b}}(\varepsilon)\}$,

$$\eta_i := \left[\boldsymbol{c}^a_{u; n_{\mathrm{b}}(\varepsilon), n_{\mathrm{q}}(\varepsilon)}\right]_i - \left[\mathrm{R}(\Phi^{c_u}_{\varepsilon_u, n_{\mathrm{b}}(\varepsilon)})(\boldsymbol{a}_{n_{\mathrm{q}}})\right]_i.$$

Using the Cauchy-Schwarz inequality, the bound $\|\varphi_i\|_{L^2(Q)} \leq 1$, the polynomial inverse inequality

(5.6) and Proposition 5.5, we obtain

$$
\begin{aligned}
(IIa)^2 &\leq \| \sum_{i=1}^{n_\mathrm{b}(\varepsilon)} \eta_i \varphi_i \|^2_{H^1(Q)} = \int_Q \left(\sum_{i=1}^{n_\mathrm{b}(\varepsilon)} \eta_i \varphi_i\right)^2 + \int_Q \left(\sum_{i=1}^{n_\mathrm{b}(\varepsilon)} \eta_i \nabla\varphi_i\right)^2 \\
&\overset{\text{C-S}}{\leq} \left(\sum_{i=1}^{n_\mathrm{b}(\varepsilon)} \eta_i^2\right) \sum_{i=1}^{n_\mathrm{b}(\varepsilon)} \left(\|\varphi_i\|^2_{L^2(Q)} + \|\nabla\varphi_i\|^2_{L^2(Q)}\right) \\
&\overset{(5.6)}{\leq} \|\boldsymbol{c}^a_{u;n_\mathrm{b}(\varepsilon),n_\mathrm{q}(\varepsilon)} - \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}(\varepsilon)})(\boldsymbol{a}_{n_\mathrm{q}})\|_2^2 \left(1 + C^2_{\mathrm{pol}} n_\mathrm{b}(\varepsilon)^{4/d}\right) n_\mathrm{b}(\varepsilon) \\
&\overset{\text{P. 5.5}}{\leq} \varepsilon_u^2 \left(1 + C^2_{\mathrm{pol}} n_\mathrm{b}(\varepsilon)^{4/d}\right) n_\mathrm{b}(\varepsilon) \\
&\overset{(5.7)}{\leq} \left(\frac{\varepsilon}{3}\right)^2.
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
\| \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}(\varepsilon)})(\boldsymbol{a}_{n_\mathrm{q}})\|_2 &\leq \| \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}(\varepsilon)})(\boldsymbol{a}_{n_\mathrm{q}}) - \boldsymbol{c}^a_{u;n_\mathrm{b}(\varepsilon),n_\mathrm{q}(\varepsilon)}\|_2 + \|\boldsymbol{c}^a_{u;n_\mathrm{b}(\varepsilon),n_\mathrm{q}(\varepsilon)}\|_2 \\
&\overset{\text{P. 5.5,(D.1)}}{\leq} 1 + C_{L^2} n_\mathrm{b}(\varepsilon)^{1/2} \|u^a_{n_\mathrm{b}(\varepsilon),n_\mathrm{q}(\varepsilon)}\|_{L^2(Q)} \\
&\leq 1 + C_{L^2} n_\mathrm{b}(\varepsilon)^{1/2} \|u^a_{n_\mathrm{b}(\varepsilon),n_\mathrm{q}(\varepsilon)} - u^a\|_{L^2(Q)} + C_{L^2} n_\mathrm{b}(\varepsilon)^{1/2}\|u^a\|_{L^2(Q)} \\
&\overset{(5.8)}{\leq} \left(2 + \|u^a\|_{L^2(Q)}\right) C_{L^2} n_\mathrm{b}(\varepsilon)^{1/2}.
\end{aligned} \tag{5.9}
$$

Then,

$$
\begin{aligned}
(IIb)^2 &\overset{\text{C-S}}{\leq} \| \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}(\varepsilon)})(\boldsymbol{a}_{n_\mathrm{q}})\|_2^2 \sum_{i=1}^{n_\mathrm{b}(\varepsilon)} \|\varphi_i - \left[\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_\mathrm{b},n_\mathrm{b}(\varepsilon)})\right]_i \|^2_{H^1(Q)} \\
&\leq n_\mathrm{b}(\varepsilon) \| \mathrm{R}(\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}(\varepsilon)})(\boldsymbol{a}_{n_\mathrm{q}})\|_2^2 \max_{i\in\{1,\dots,n_\mathrm{b}(\varepsilon)\}} \|\varphi_i - \left[\mathrm{R}(\Phi^{\mathrm{b}}_{\varepsilon_\mathrm{b},n_\mathrm{b}(\varepsilon)})\right]_i \|^2_{H^1(Q)} \\
&\overset{\text{P. 5.6,(5.9)}}{\leq} n_\mathrm{b}(\varepsilon)(2 + \|u_a\|_{L^2(Q)})^2 C^2_{L^2} n_\mathrm{b}(\varepsilon) \varepsilon_\mathrm{b}^2 \\
&\overset{(5.7)}{\leq} \left(\frac{\varepsilon}{3}\right)^2.
\end{aligned}
$$

We can conclude that

$$
\|u^a - \left(\mathrm{R}(\Phi^{\mathrm{br}}_\varepsilon)(\boldsymbol{a}_{n_\mathrm{q}})\right) \cdot \mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon)\|_{H^1(Q)} \leq (I) + (IIa) + (IIb) \leq \varepsilon.
$$

Depth and size bounds. Using (5.5) and the definitions (5.7), we obtain that there exists a constant $C_1 > 0$ such that, for all $\varepsilon \in (0, 1)$,

$$
1 + \max\left(|\log\varepsilon_\mathrm{b}|, |\log\varepsilon_\mathrm{G}|, |\log\varepsilon_u|\right) \leq C_1(1 + |\log\varepsilon|).
$$

We infer then, from Proposition 5.5, that there exists $C_2 > 0$ such that, for all $\varepsilon \in (0, 1)$,

$$
\mathrm{L}(\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}(\varepsilon)}) \leq C_2 \left(1 + \log(1 + |\log\varepsilon|^d)\right)(1 + |\log\varepsilon|)
$$

and

$$
\mathrm{M}(\Phi^{c_u}_{\varepsilon_u,n_\mathrm{b}(\varepsilon)}) \leq C_2 \left(1 + |\log\varepsilon|^d\right)^3 (1 + |\log\varepsilon|)^2 \left(1 + \log(1 + |\log\varepsilon|^d)\right)^2.
$$

Furthermore, from Proposition 5.6, we have that there exists $C_3 > 0$ such that for all $\varepsilon \in (0,1)$

$$\mathrm{L}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}) \leq C_3(1 + |\log \varepsilon|)\left(1 + \log(1 + |\log \varepsilon|^d)\right)$$

and

$$\mathrm{M}(\Phi^{\mathrm{b}}_{\varepsilon_{\mathrm{b}}, n_{\mathrm{b}}(\varepsilon)}) \leq C_3(1 + |\log \varepsilon|^{d+1}).$$

Using the definition of $\Phi^{\mathrm{tr}}_{\varepsilon}$ and $\Phi^{\mathrm{br}}_{\varepsilon}$ gives Item (iii) and concludes the proof. ∎

Remark 5.8. The implicit constants in the size bounds of the operator networks in Theorem 5.7 and in the theorems of the upcoming sections depend, in general, exponentially on the dimension d.

Remark 5.9. In Theorem 5.7, we have considered the data-to-solution map $a \mapsto u^a$ for a fixed, given right-hand side f. The present analysis may be extended to the operator $S : (a, c, f) \mapsto u$ where u is the solution to the reaction-diffusion equation

$$-\nabla \cdot (a \nabla u) + cu = f \text{ in } \Omega$$

with (analytic in $\overline{Q}$ and Q-periodic) positive diffusion coefficient function a, nonnegative reaction coefficient function c and source-term f. This requires a straightforward modification of the branch network so that

1. it takes the point evaluations of f at $\boldsymbol{x}_{\mathrm{enc}}$ as input and outputs an exponentially consistent numerical quadrature approximation of $\boldsymbol{c}_{f;n_{\mathrm{b}}}$ in (4.5);
2. the approximation of $\boldsymbol{c}_{f;n_{\mathrm{b}}}$ is then passed to a network approximating matrix-vector multiplication (see [15, Proposition 3.7]) with the output of the network of Proposition 5.4.

The construction of the remaining parts of the operator network follows along the same lines.

5.4. Parametric diffusion coefficient. In many applications, for example in uncertainty quantification, one is interested in the case where the diffusion coefficient in (2.4) is parametric. This is naturally accommodated for by composition with solution operator networks and we briefly detail this here. Specifically, suppose that there exists $d_{\mathrm{p}} \in \mathbb{N}$ and a compact parameter set $\mathcal{P} \subset \mathbb{R}^{d_{\mathrm{p}}}$ such that $\mathfrak{a} : \mathcal{P} \to \mathrm{Hol}(\Omega)$ and that there exist constants $a_{\min}, C_{\mathrm{p}}, b_{\mathrm{p}}, \alpha_{\mathrm{p}}, A_{\mathrm{p}}, A_{\psi} > 0$, and functions $\psi_i : Q \to \mathbb{R}$ and $a_i : \mathcal{P} \to \mathbb{R}$, $i \in \mathbb{N}$, such that

$$\inf_{\boldsymbol{y}\in\mathcal{P}} \inf_{\boldsymbol{x}\in Q} \mathfrak{a}(\boldsymbol{y})(\boldsymbol{x}) \geq a_{\min}, \tag{5.10}$$

that

$$\forall n_{\mathrm{p}} \in \mathbb{N}, \qquad \sup_{\boldsymbol{y}\in\mathcal{P}} \|\mathfrak{a}(\boldsymbol{y}) - \sum_{i=1}^{n_{\mathrm{p}}} a_i(\boldsymbol{y})\psi_i\|_{L^\infty(Q)} \leq C_{\mathrm{p}} \exp(-b_{\mathrm{p}} n_{\mathrm{p}}^{\alpha_{\mathrm{p}}}), \tag{5.11}$$

with

$$\forall i \in \mathbb{N}, \qquad \psi_i \in \mathrm{Hol}(\Omega; A_\psi), \qquad a_i \in \mathrm{Hol}(\mathcal{P}; A_{\mathrm{p}}), \tag{5.12}$$

and that

$$\sup_{\boldsymbol{y}\in\mathcal{P}} \sum_{i=1}^{\infty} |a_i(\boldsymbol{y})| \leq A_{\mathrm{p}}. \tag{5.13}$$

Here, we use the same constant A_{p} in the second hypothesis in (5.12) and in (5.13) only to simplify notation. For all $\boldsymbol{y} \in \mathcal{P}$, we denote $u_{\boldsymbol{y}} \in X$ the solution to

$$-\nabla \cdot (\mathfrak{a}(\boldsymbol{y}) \nabla u_{\boldsymbol{y}}) = f, \qquad \text{in } \Omega. \tag{5.14}$$

Remark 5.10. Diffusion coefficient functions that can be written in Fourier series as

$$\mathfrak{a}(\boldsymbol{y})(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (a_{\boldsymbol{k}}(\boldsymbol{y}) + ib_{\boldsymbol{k}}(\boldsymbol{y}))e^{i2\pi\boldsymbol{k}\cdot\boldsymbol{x}},$$

where $a_{\boldsymbol{k}}, b_{\boldsymbol{k}} \in \mathrm{Hol}(\mathcal{P}, A_{\mathrm{p}})$ are chosen so that $\mathfrak{a}$ is a real function for all $\boldsymbol{y}$, with exponential decrease of $\sup_{\boldsymbol{y}\in\mathcal{P}}(|a_{\boldsymbol{k}}(\boldsymbol{y})| + |b_{\boldsymbol{k}}(\boldsymbol{y})|)$ with respect to $|\boldsymbol{k}|$, and such that $\mathfrak{a}$ is uniformly bounded from below by a positive constant in the sense that (5.10) holds for its real part, fulfill conditions (5.11), (5.12) and (5.13).

LEMMA 5.11. *There exists $C > 0$ such that for all $n_{\mathrm{p}} \in \mathbb{N}$ and for all $\varepsilon \in (0,1)$, there exists a NN $\Phi^{a,\mathrm{coef}}_{\varepsilon,n_{\mathrm{p}}}$ with input dimension d_{p} and output dimension n_{p} such that*

$$\max_{i=1,\dots,n_{\mathrm{p}}} \|a_i - \left[\mathrm{R}(\Phi^{a,\mathrm{coef}}_{\varepsilon,n_{\mathrm{p}}})\right]_i \|_{L^\infty(\mathcal{P})} \leq \varepsilon \tag{5.15}$$

and that $\mathrm{L}(\Phi^{a,\mathrm{coef}}_{\varepsilon,n_{\mathrm{p}}}) \leq C(1+|\log\varepsilon|)(1+\log|\log\varepsilon|)$ *and* $\mathrm{M}(\Phi^{a,\mathrm{coef}}_{\varepsilon,n_{\mathrm{p}}}) \leq C(1+|\log\varepsilon|^{d_{\mathrm{p}}+1})n_{\mathrm{p}}$.

Proof. The statement follows from a parallelization of the network of [23, Theorem 3.6]. ∎

THEOREM 5.12. *Let $d_{\mathrm{p}} \in \mathbb{N}$ and let $\mathfrak{a}$ and $u_{\boldsymbol{y}}$ be defined as above. There exists $C > 0$ such that, for all $\varepsilon \in (0,1)$, there exist*

(*a*) $n_{\mathrm{b}} \in \mathbb{N}$,

(*b*) *two NNs $\Phi^{\mathrm{br}}_\varepsilon$ and $\Phi^{\mathrm{tr}}_\varepsilon$ with $\mathrm{R}(\Phi^{\mathrm{br}}_\varepsilon) : \mathbb{R}^{d_{\mathrm{p}}} \to \mathbb{R}^{n_{\mathrm{b}}}$ and $\mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon) : Q \to \mathbb{R}^{n_{\mathrm{b}}}$,*

such that

(*i*) $n_{\mathrm{b}} \leq C(1+|\log\varepsilon|^d)$,

(*ii*) *the following error estimate holds:*

$$\sup_{\boldsymbol{y}\in\mathcal{P}} \|u_{\boldsymbol{y}} - \left(\mathrm{R}(\Phi^{\mathrm{br}}_\varepsilon)(\boldsymbol{y})\right) \cdot \mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon)\|_{H^1(Q)} \leq \varepsilon,$$

(*iii*) *as $\varepsilon \downarrow 0$,*

$$\mathrm{L}(\Phi^{\mathrm{br}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|\left(\log|\log\varepsilon|\right)\right),$$
$$\mathrm{M}(\Phi^{\mathrm{br}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|^{3d+2}\left(\log|\log\varepsilon|\right)^2 + |\log\varepsilon|^{1+d_{\mathrm{p}}+1/\alpha_{\mathrm{p}}}\right),$$

and

$$\mathrm{L}(\Phi^{\mathrm{tr}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|\left(\log|\log\varepsilon|\right)\right), \qquad \mathrm{M}(\Phi^{\mathrm{tr}}_\varepsilon) = \mathcal{O}\left(|\log\varepsilon|^{d+1}\right).$$

Proof. The proof proceeds in several steps. We first prove a consistency bound, then detail the construction of the ONet, and conclude with verification of the asserted bounds on the depth and size of the ONet.

Let $C_L > 0$ be the constant such that, given $\mathfrak{a}_1, \mathfrak{a}_2 \in L^\infty(\Omega)$ such that

$$0 < \frac{a_{\min}}{4} \leq \mathfrak{a}_i \leq \max(a_{\max}, (1+A_{\mathrm{p}})A_\psi), \qquad \text{a. e. in } Q \text{ and for } i = 1,2, \tag{5.16}$$

and $u_i = S(\mathfrak{a}_i)$, $i = 1,2$, then

$$\|u_1 - u_2\|_{H^1(Q)} \leq C_L \|\mathfrak{a}_1 - \mathfrak{a}_2\|_{L^\infty(Q)},$$

see Lemma B.1. We suppose, without loss of generality and for ease of notation, that $A_\psi \geq 1$ and $C_L \geq 1$. Let now n_{p} be the smallest integer such that

$$C_L C_{\mathrm{p}} \exp(-b_{\mathrm{p}} n_{\mathrm{p}}^{\alpha_{\mathrm{p}}}) \leq \min\left(\frac{\varepsilon}{3}, \frac{a_{\min}}{2}\right). \tag{5.17}$$

This implies that there exists a constant $C_1 > 0$ (depending only on $C_L, C_\mathrm{p}, b_\mathrm{p}, a_\mathrm{min}$) such that

$$n_\mathrm{p} \le C_1(1 + |\log\varepsilon|^{1/\alpha_\mathrm{p}})$$

and that, due to (5.10), (5.11), (5.12), and (5.13),

$$\inf_{\boldsymbol{y}\in\mathcal{P}}\inf_{x\in Q}\sum_{i=1}^{n_\mathrm{p}} a_i(\boldsymbol{y})\psi_i(x) \ge \frac{a_\mathrm{min}}{2}, \qquad \sup_{\boldsymbol{y}\in\mathcal{P}}\|\sum_{i=1}^{n_\mathrm{p}} a_i(\boldsymbol{y})\psi_i\|_{L^\infty(Q)} \le A_\mathrm{p}A_\psi. \tag{5.18}$$

Let also

$$\varepsilon_\mathrm{p} \coloneqq \frac{1}{n_\mathrm{p}A_\psi}\min\left(\frac{\varepsilon}{3C_L}, \frac{a_\mathrm{min}}{4}\right), \qquad \widetilde{\mathfrak{a}} \coloneqq \sum_{i=1}^{n_\mathrm{p}}\left[\mathrm{R}(\Phi^{a,\mathrm{coef}}_{\varepsilon_\mathrm{p},n_\mathrm{p}})\right]_i\psi_i, \tag{5.19}$$

where the network $\Phi^{a,\mathrm{coef}}_{\varepsilon_\mathrm{p},n_\mathrm{p}}$ is defined in Lemma 5.11. We now show that $\widetilde{\mathfrak{a}}$ fulfills conditions like (2.1) and (2.2) (with updated values of the constants a_min, a_max, $A_\mathcal{D}$), uniformly with respect to n_p and ε_p. From (5.15) and (5.18), it follows that, for all $k \in \mathbb{N}_0$,

$$\begin{aligned}\sup_{\boldsymbol{y}\in\mathcal{P}}\|\widetilde{\mathfrak{a}}(\boldsymbol{y})\|_{W^{k,\infty}(Q)} &\le \sup_{\boldsymbol{y}\in\mathcal{P}}\sum_{i=1}^{n_\mathrm{p}}\Big(|a_i(\boldsymbol{y}) - \left[\mathrm{R}(\Phi^{a,\mathrm{coef}}_{\varepsilon_\mathrm{p},n_\mathrm{p}})(\boldsymbol{y})\right]_i| + |a_i(\boldsymbol{y})|\Big)\|\psi_i\|_{W^{k,\infty}(Q)}\\ &\le (n_\mathrm{p}\varepsilon_\mathrm{p} + A_\mathrm{p})\,A_\psi^{k+1}k! \le (1 + A_\mathrm{p})\,A_\psi^{k+1}k!.\end{aligned} \tag{5.20}$$

Furthermore, for all $\boldsymbol{y} \in \mathcal{P}$ and all $x \in Q$,

$$\begin{aligned}\widetilde{\mathfrak{a}}(\boldsymbol{y})(x) &\ge \sum_{i=1}^{n_\mathrm{p}}\Big(\left[\mathrm{R}(\Phi^{a,\mathrm{coef}}_{\varepsilon_\mathrm{p},n_\mathrm{p}})(\boldsymbol{y})\right]_i - a_i(\boldsymbol{y})\Big)\psi_i(x) + \sum_{i=1}^{n_\mathrm{p}} a_i(\boldsymbol{y})\psi_i(x)\\ &\ge \frac{a_\mathrm{min}}{2} - n_\mathrm{p}\varepsilon_\mathrm{p}A_\psi,\\ &\ge \frac{a_\mathrm{min}}{4}.\end{aligned} \tag{5.21}$$

Here we have used (5.13), (5.15), (5.18), and the definition of ε_p in (5.19).

Construction of the operator network and error estimate. For $q \in \mathbb{N}$, $n_\mathrm{q} = q^d$, we introduce the matrix $\mathbf{V}_{n_\mathrm{q},n_\mathrm{p}} \in \mathbb{R}^{n_\mathrm{q}\times n_\mathrm{p}}$ with entries

$$\left[\mathbf{V}_{n_\mathrm{q},n_\mathrm{p}}\right]_{ij} = \psi_j(\boldsymbol{x}_i^{(q)}), \qquad i = 1,\dots,n_\mathrm{q},\ j = 1,\dots,n_\mathrm{p}, \tag{5.22}$$

where $\boldsymbol{x}_1^{(q)}, \dots, \boldsymbol{x}_{n_\mathrm{q}}^{(q)}$ are the quadrature nodes introduced in Section 4.2. Then the NN

$$\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_\mathrm{p},n_\mathrm{p},n_\mathrm{q}} = \big((\mathbf{V}_{n_\mathrm{q},n_\mathrm{p}}, \mathbf{0}_{n_\mathrm{q}})\big) \odot \Phi^{a,\mathrm{coef}}_{\varepsilon_\mathrm{p},n_\mathrm{p}} \tag{5.23}$$

has realization such that

$$\mathrm{R}(\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_\mathrm{p},n_\mathrm{p},n_\mathrm{q}})(\boldsymbol{y}) = \begin{pmatrix}\widetilde{\mathfrak{a}}(\boldsymbol{y})(\boldsymbol{x}_1^{(q)})\\ \vdots\\ \widetilde{\mathfrak{a}}(\boldsymbol{y})(\boldsymbol{x}_{n_\mathrm{q}}^{(q)})\end{pmatrix}.$$

Let $\widetilde{u}_{\boldsymbol{y}} \in X$ denote, for each $\boldsymbol{y} \in \mathcal{P}$, the solution to

$$-\nabla\cdot(\widetilde{\mathfrak{a}}(\boldsymbol{y})\nabla\widetilde{u}_{\boldsymbol{y}}) = f \quad \text{in } \Omega.$$

Thanks to (5.20), (5.21), and to Theorem 5.7, there exists a constant C_2 independent of ε, $n_{\mathrm{q}} \in \mathbb{N}$ such that $n_{\mathrm{q}} \leq C_2(1+|\log\varepsilon|)$, and networks $\widetilde{\Phi}^{\mathrm{br}}_{\varepsilon/3}$ and $\widetilde{\Phi}^{\mathrm{tr}}_{\varepsilon/3}$ such that

$$\forall \boldsymbol{y} \in \mathcal{P}: \quad \|\widetilde{u}_{\boldsymbol{y}} - \left(\mathrm{R}(\widetilde{\Phi}^{\mathrm{br}}_{\varepsilon/3}) \circ \mathrm{R}(\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}},n_{\mathrm{q}}})\right)(\boldsymbol{y}) \cdot \mathrm{R}(\widetilde{\Phi}^{\mathrm{tr}}_{\varepsilon/3})\|_{H^1(Q)} \leq \frac{\varepsilon}{3}.$$

Furthermore, for all $\boldsymbol{y} \in \mathcal{P}$, $\mathfrak{a}(\boldsymbol{y})$ and $\widetilde{\mathfrak{a}}(\boldsymbol{y})$ satisfy the conditions in (5.16), hence for all $\boldsymbol{y} \in \mathcal{P}$

$$\begin{aligned}\|u_{\boldsymbol{y}} - \widetilde{u}_{\boldsymbol{y}}\|_{H^1(Q)} &\leq C_L \|\mathfrak{a}(\boldsymbol{y}) - \widetilde{\mathfrak{a}}(\boldsymbol{y})\|_{L^\infty(Q)} \\ &\leq C_L \left(\|\mathfrak{a}(\boldsymbol{y}) - \sum_{i=1}^{n_{\mathrm{p}}} a_i(\boldsymbol{y})\psi_i\|_{L^\infty(Q)} + \|\sum_{i=1}^{n_{\mathrm{p}}} \left(a_i(\boldsymbol{y}) - \left[\mathrm{R}(\Phi^{a,\mathrm{coef}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}}})(\boldsymbol{y})\right]_i\right)\psi_i\|_{L^\infty(Q)}\right) \\ &\leq \frac{\varepsilon}{3} + C_L n_{\mathrm{p}} \varepsilon_{\mathrm{p}} A_\psi \\ &\leq \frac{2}{3}\varepsilon,\end{aligned}$$

where we have used (5.11), (5.15), (5.17), and (5.19) in the third inequality and (5.19) in the last one. We deduce that

$$\sup_{\boldsymbol{y}\in\mathcal{P}} \|u_{\boldsymbol{y}} - \left(\mathrm{R}(\widetilde{\Phi}^{\mathrm{br}}_{\varepsilon/3}) \circ \mathrm{R}(\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}},n_{\mathrm{q}}})\right)(\boldsymbol{y}) \cdot \mathrm{R}(\widetilde{\Phi}^{\mathrm{tr}}_{\varepsilon/3})\|_{H^1(Q)} \leq \varepsilon,$$

which is Item (ii), with

$$\Phi^{\mathrm{br}}_{\varepsilon} := \widetilde{\Phi}^{\mathrm{br}}_{\varepsilon/3} \odot \Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}},n_{\mathrm{q}}}, \qquad \Phi^{\mathrm{tr}}_{\varepsilon} := \widetilde{\Phi}^{\mathrm{tr}}_{\varepsilon/3}.$$

Depth and size bounds. The bounds on the depth and size of $\Phi^{\mathrm{tr}}_{\varepsilon}$ can be inferred directly from Theorem 5.7. To compute bounds on the size and depth of $\Phi^{\mathrm{br}}_{\varepsilon}$, note that, by Lemma 5.11, there exist C_3, C_4, C_5, C_6 independent of ε such that

$$\mathrm{L}(\Phi^{a,\mathrm{coef}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}}}) \leq C_3(1+|\log\varepsilon_{\mathrm{p}}|)(1+\log|\log\varepsilon_{\mathrm{p}}|) \leq C_4(1+|\log\varepsilon|)(1+\log|\log\varepsilon|) \tag{5.24}$$

and

$$\mathrm{M}(\Phi^{a,\mathrm{coef}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}}}) \leq C_5(1+|\log\varepsilon_{\mathrm{p}}|^{d_{\mathrm{p}}+1})n_{\mathrm{p}} \leq C_6(1+|\log\varepsilon|^{d_{\mathrm{p}}+1+1/\alpha_{\mathrm{p}}}). \tag{5.25}$$

Furthermore, there exists C_7 independent of ε such that

$$\|\mathbf{V}_{n_{\mathrm{q}},n_{\mathrm{p}}}\|_0 \leq n_{\mathrm{p}} n_{\mathrm{q}} \leq C_7(1+|\log\varepsilon|^{1+1/\alpha_{\mathrm{p}}}). \tag{5.26}$$

From (5.25) and (5.26) it follows that

$$\mathrm{L}(\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}},n_{\mathrm{q}}}) \leq C_8(1+|\log\varepsilon|)(1+\log|\log\varepsilon|), \qquad \mathrm{M}(\Phi^{\widetilde{\mathfrak{a}}}_{\varepsilon_{\mathrm{p}},n_{\mathrm{p}},n_{\mathrm{q}}}) \leq C_9(1+|\log\varepsilon|^{d_{\mathrm{p}}+1+1/\alpha_{\mathrm{p}}}), \tag{5.27}$$

for constants C_8, C_9 independent of ε. Combining the bounds in (5.27) with the bounds on the depth and size of $\widetilde{\Phi}^{\mathrm{br}}_{\varepsilon/3}$ coming from Theorem 5.7 concludes the proof. □

Remark 5.13. If each function a_i does not depend on all the parameters but only on a subset of them, the size bound of Theorem 5.12 results in an overestimation. Specifically, for all $i \in \mathbb{N}$, let $\mathcal{P}_i$ be the domain of a_i and denote $d_{\mathrm{p},i} := \dim(\mathcal{P}_i)$. Then, modifying Lemma 5.11 so that the subnetworks approximating each of the a_i only take a $d_{\mathrm{p},i}$-dimensional input, we obtain in Theorem 5.12 that there exists a constant $c > 0$ independent of ε such that for $\varepsilon \downarrow 0$,

$$\mathrm{M}(\Phi^{\mathrm{br}}_{\varepsilon}) = \mathcal{O}\left(|\log\varepsilon|^{3d+2}(\log|\log\varepsilon|)^2 + \sum_{i=1}^{c|\log\varepsilon|^{1/\alpha_{\mathrm{p}}}} |\log\varepsilon|^{1+d_{\mathrm{p},i}}\right).$$

Clearly, setting $d_{\mathrm{p},i} = d_{\mathrm{p}}$ for all i in the equation above gives the estimate in Theorem 5.12.

Remark 5.14. Similar results to Theorem 5.12 can be obtained through the technique in [15], by using the exponential convergence of polynomial approximations to the functions in the solution manifold $\mathcal{M} = \{u(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{P}\}$ to derive an upper bound on the n-width of $\mathcal{M}$.

6. Generalizations. All steps of the analysis of ONet emulation rates for the coefficient-to-solution map of (2.4) directly generalize to other, structurally similar, linear divergence-form elliptic PDEs. We illustrate the extension of the preceding result by two of these: anisotropic diffusion-reaction equations and linear elastostatics.

6.1. Linear anisotropic diffusion-reaction equations.

6.1.1. Definition of the problem. We consider again the torus $\Omega = (\mathbb{R}/\mathbb{Z})^d$. For a constant $A_{\mathcal{D}^{\mathrm{rd}}} > 0$, introduce the set of admissible data

$$\mathcal{D}^{\mathrm{rd}} \subset \mathrm{Hol}(\Omega; A_{\mathcal{D}^{\mathrm{rd}}})^{d\times d} \times \mathrm{Hol}(\Omega; A_{\mathcal{D}^{\mathrm{rd}}})$$

of pairs $(\mathbf{A}, c)$ and suppose there exist $Q_0 \subset Q$, $a_{\min}, c_{\min} > 0$ such that for all $(\mathbf{A}, c) \in \mathcal{D}^{\mathrm{rd}}$,

- $\mathbf{A}$ is symmetric and is uniformly positive definite, i.e., $\mathbf{A}_{ij} = \mathbf{A}_{ji}$ and

$$\forall \boldsymbol{x} \in Q,\ \forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad \boldsymbol{\xi}^\top \mathbf{A}(\boldsymbol{x})\boldsymbol{\xi} \geq a_{\min}|\boldsymbol{\xi}|^2 ,$$

- $c(\boldsymbol{x}) \geq c_{\min}$ for all $\boldsymbol{x} \in Q_0$.

For all $(\mathbf{A}, c) \in \mathcal{D}^{\mathrm{rd}}$, the bilinear form $\mathfrak{b}^{(\mathbf{A},c)}(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) = H^1_{\mathrm{per}}(Q) \times H^1_{\mathrm{per}}(Q) \to \mathbb{R}$ given by

$$\mathfrak{b}^{(\mathbf{A},c)}(w, v) := \int_Q ((\mathbf{A}\nabla w) \cdot \nabla v + cwv)$$

is coercive, i.e., there exists a constant $\alpha_0 > 0$ independent of $(\mathbf{A}, c)$ such that

$$\forall v \in H^1(\Omega), \quad \mathfrak{b}^{(\mathbf{A},c)}(v, v) \geq \alpha_0 \|v\|^2_{H^1(Q)} .$$

The continuity of the form $\mathfrak{b}^{(\mathbf{A},c)}(\cdot,\cdot)$ on $H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ being evident, the Lax-Milgram Lemma implies that for every $f \in \mathrm{Hol}(\Omega)$ there exists a unique solution

$$u \in H^1(\Omega) : \quad \mathfrak{b}^{(\mathbf{A},c)}(u, v) = (f, v) \quad \forall v \in H^1(\Omega) = H^1_{\mathrm{per}}(Q).$$

For given, fixed $f \in \mathrm{Hol}(\Omega)$, the coefficient-to-solution map

$$S^{\mathrm{rd}} : (\mathbf{A}, c) \mapsto u$$

is analytic. Furthermore, there exists $A_{\mathcal{U}^{\mathrm{rd}}} > 0$ such that

$$S^{\mathrm{rd}}(\mathcal{D}^{\mathrm{rd}}) \subset \mathrm{Hol}(\Omega; A_{\mathcal{U}^{\mathrm{rd}}}),$$

which can be proven as in Lemma 4.1.

6.1.2. Operator network approximation. We introduce, for all $n_{\mathrm{q}} \in \mathbb{N}$ such that $q := n_{\mathrm{q}}^{1/d} \in \mathbb{N}$, the encoding operator $\mathcal{E}^{\mathrm{rd}}_{\boldsymbol{x}_{\mathrm{enc}}} : C(\Omega)^{d\times d} \times C(\Omega) \to \mathbb{R}^{d^2 n_{\mathrm{q}} + n_{\mathrm{q}}}$ such that

$$\mathcal{E}^{\mathrm{rd}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathbf{A}, c) = \begin{pmatrix} \mathbf{vec}(\mathbf{A}(\boldsymbol{x}_1^{(q)})) \\ \vdots \\ \mathbf{vec}(\mathbf{A}(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)})) \\ c(\boldsymbol{x}_1^{(q)}) \\ \vdots \\ c(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}) \end{pmatrix},$$

where $\boldsymbol{x}_{\mathrm{enc}} = \boldsymbol{x}_1^{(q)}, \dots, \boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}$ are the points from Section 4.2. Theorem 5.7 can then be extended to this class of reaction-diffusion equations.

THEOREM 6.1. *Theorem 5.7 holds with $a \in \mathcal{D}$ replaced by $(\mathbf{A}, c) \in \mathcal{D}^{\mathrm{rd}}$, $S(a)$ replaced by $S^{\mathrm{rd}}(\mathbf{A}, c)$, and $\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}(a)$ replaced by $\mathcal{E}^{\mathrm{rd}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathbf{A}, c)$.*

For the proof, for all $p \in \mathbb{N}$, writing $\widetilde{n}_b = p^d$, consider the basis functions $\{\widetilde{\varphi}_i\}_{i=1}^{\widetilde{n}_b}$ of $\mathbb{Q}_p(Q)$ such that

$$\{\widetilde{\varphi}_i\}_{i=1}^{\widetilde{n}_b} = \{\varphi_i\}_{i=1}^{n_{\mathrm{b}}} \cup \{L_0 \otimes \cdots \otimes L_0\},$$

where $L_0 \equiv 1$ is the constant, unit function in $(0, 1)$. Define further

$$\widetilde{X}_{n_{\mathrm{b}}} := \mathrm{span}(\{\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_{\widetilde{n}_b}\}) \subset \widetilde{X} := H^1(\Omega) = H^1_{\mathrm{per}}(Q).$$

In order to prove Theorem 6.1, we have to replace the input layer network introduced in Lemma 5.1 with an input layer adapted for anisotropic diffusion-reaction problems, as introduced in Lemma 6.2 below. For $k \in \{1, \dots, n_{\mathrm{q}}\}$, we introduce $\widetilde{\mathbf{D}}(\boldsymbol{x}_k^{(q)})$ such that

$$\widetilde{\mathbf{D}}^{ij}_{mn}(\boldsymbol{x}_k^{(q)}) = w_k^{(q)} (\partial_{x_n} \widetilde{\varphi}_j)(\boldsymbol{x}_k^{(q)}) (\partial_{x_m} \widetilde{\varphi}_i)(\boldsymbol{x}_k^{(q)}), \qquad (i, j) \in \{1, \dots, \widetilde{n}_b\}^2,\ (m, n) \in \{1, \dots, d\}^2.$$

Furthermore, let $\mathfrak{v} : \mathbb{N}^2 \to \mathbb{N}$ be the reordering such that for any matrix $\mathbf{A}$,

$$\mathbf{vec}(\mathbf{A})_{\mathfrak{v}(i,j)} = \mathbf{A}_{ij}. \tag{6.1}$$

We introduce the operation $\widetilde{\mathbf{vec}} : \mathbb{R}^{\widetilde{n}_b \times \widetilde{n}_b \times d \times d} \to \mathbb{R}^{\widetilde{n}_b^2 \times d^2}$

$$\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}_k^{(q)}))_{\mathfrak{v}(i,j)\mathfrak{v}(m,n)} = \widetilde{\mathbf{D}}^{ij}_{mn}(\boldsymbol{x}_k^{(q)}).$$

Finally, define

$$\widehat{\mathbf{M}}_{ij}(\boldsymbol{x}_k^{(q)}) = w_k^{(q)} \widetilde{\varphi}_i(\boldsymbol{x}_k^{(q)}) \widetilde{\varphi}_j(\boldsymbol{x}_k^{(q)}).$$

LEMMA 6.2. *For all $\alpha \in \mathbb{R}$ the one-layer NN*

$$\Phi^{(\mathbf{A},c),\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}} := \left(\left(-\alpha \left[\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}_1^{(q)})) | \dots | \widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)})) | \,\mathbf{vec}(\widehat{\mathbf{M}}(\boldsymbol{x}_1^{(q)})) | \dots | \,\mathbf{vec}(\widehat{\mathbf{M}}(\boldsymbol{x}_{n_{\mathrm{q}}}^{(q)}))\right], \mathbf{0}_{\widetilde{n}_b^2}\right)\right)$$

is such that

$$\mathbf{matr}\left(\mathrm{R}(\Phi^{(\mathbf{A},c),\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}})(\mathcal{E}^{\mathrm{rd}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathbf{A}, c))\right)_{ij} = -\alpha \mathfrak{b}^{(\mathbf{A},c)}(\widetilde{\varphi}_j, \widetilde{\varphi}_i)$$

and $\mathrm{M}(\Phi^{(\mathbf{A},c),\alpha}_{n_\mathrm{b},n_\mathrm{q}}) \leq (d^2+1)\widetilde{n}_b^2 n_\mathrm{q}$.

Proof. We have

$$\left[\mathrm{R}\left(\Phi^{(\mathbf{A},c),\alpha}_{n_\mathrm{b},n_\mathrm{q}}\right)(\mathcal{E}^{\mathrm{rd}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathbf{A},c))\right]_{\mathfrak{v}(i,j)}$$
$$= -\alpha \sum_{k=1}^{n_\mathrm{q}} w_k^{(q)} \left(\sum_{m,n=1}^{d} \left[\mathbf{A}_{mn}(\partial_{x_n}\widetilde{\varphi}_j)(\partial_{x_m}\widetilde{\varphi}_i)\right](\boldsymbol{x}_k^{(q)}) + \left[c\widetilde{\varphi}_j\widetilde{\varphi}_i\right](\boldsymbol{x}_k^{(q)}) \right)$$
$$= -\alpha \sum_{k=1}^{n_\mathrm{q}} w_k^{(q)} \left(\left[(\mathbf{A}\nabla\widetilde{\varphi}_i)\cdot(\nabla\widetilde{\varphi}_j)\right](\boldsymbol{x}_k^{(q)}) + \left[c\widetilde{\varphi}_j\widetilde{\varphi}_i\right](\boldsymbol{x}_k^{(q)}) \right),$$

hence the equality after matricization. The size bound follows from the fact that

$$\|\widetilde{\mathbf{D}}(\boldsymbol{x}_k^{(q)})\|_0 \leq d^2\widetilde{n}_b^2, \qquad \|\widehat{\mathbf{M}}(\boldsymbol{x}_k^{(q)})\|_0 \leq \widetilde{n}_b^2,$$

for all $k \in \{1,\ldots,n_\mathrm{q}\}$. □

We can now prove Theorem 6.1.

Proof of Theorem 6.1. The proof follows along the same lines as the proof of Theorem 5.7. In particular, in the construction of $\Phi^{\mathrm{br}}_\varepsilon$, the input network $\Phi^{A,\alpha}_{n_\mathrm{b},n_\mathrm{q}}$ and Lemma 5.1 are replaced by the network $\Phi^{(\mathbf{A},c),\alpha}_{n_\mathrm{b},n_\mathrm{q}}$ and Lemma 6.2. Then, the spaces X and X_{n_b} are replaced by $\widetilde{X}$ and $\widetilde{X}_{\widetilde{n}_\mathrm{b}}$. The basis $\{\widetilde{\varphi}_1,\ldots,\widetilde{\varphi}_{\widetilde{n}_\mathrm{b}}\}$ is equal to $\{\varphi_1,\ldots,\varphi_{n_\mathrm{b}}\}$ with the addition of a constant function, which can be emulated exactly by deep ReLU neural networks. Hence, Proposition 5.6 can be extended to this case. Finally, the matrices $\mathbf{A}^a_{n_\mathrm{b},n_\mathrm{q}}$ and $\mathbf{A}^1_{n_\mathrm{b},n_\mathrm{q}}$ used in the proof of Theorem 5.7 are replaced, respectively, by the matrices with entries

$$\mathfrak{b}^{(\mathbf{A},c)}_{n_\mathrm{q}}(\widetilde{\varphi}_j,\widetilde{\varphi}_i) \quad \text{and} \quad \mathfrak{b}^{(\mathbf{Id}_d,1)}_{n_\mathrm{q}}(\widetilde{\varphi}_j,\widetilde{\varphi}_i), \qquad (i,j)\in\{1,\ldots,\widetilde{n}_b\}^2,$$

where

$$\mathfrak{b}^{(\mathbf{A},c)}_{n_\mathrm{q}}(u,v) := \sum_{k=1}^{n_\mathrm{q}} w_k^{(q)} \left(\mathbf{A}(\boldsymbol{x}_k^{(q)})\nabla u(\boldsymbol{x}_k^{(q)})\right)\cdot\nabla v(\boldsymbol{x}_k^{(q)}) + \sum_{k=1}^{n_\mathrm{q}} w_k^{(q)} c(\boldsymbol{x}_k^{(q)})u(\boldsymbol{x}_k^{(q)})v(\boldsymbol{x}_k^{(q)}),$$

for all $u,v\in C^1(\Omega)$. Since the bilinear form $\mathfrak{b}^{(\mathbf{A},c)}$ is coercive and continuous on $H^1(\Omega)$, results equivalent to Lemmas 4.2 and 4.3 with the new matrices can be proven directly. The rest of the proof is the same as the proof of Theorem 5.7. □

6.2. Linear Elastostatics.

6.2.1. Definition of the problem. We assume $d = 2,3$. Small, linear elastic deformation of a body occupying $Q = (0,1)^d$ with periodic boundary conditions and subject to a prescribed, periodic body force $\boldsymbol{f} : \Omega = \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d$ can be described by the displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ which satisfies the equilibrium of stress

$$\operatorname{div}\boldsymbol{\sigma}[\boldsymbol{u}] + \boldsymbol{f} = 0 \quad \text{in } \Omega. \tag{6.2}$$

Here $\boldsymbol{\sigma}[\boldsymbol{u}] : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ is symmetric matrix function, the so-called *stress tensor*. It depends on the displacement field $\boldsymbol{u}$ via the (linearized) *strain tensor* $\boldsymbol{\varepsilon}[\boldsymbol{u}] : \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ which is given by

$$\boldsymbol{\varepsilon}[\boldsymbol{u}] := \frac{1}{2}\left(\operatorname{grad}\boldsymbol{u} + (\operatorname{grad}\boldsymbol{u})^\top\right), \quad (\boldsymbol{\varepsilon}[\boldsymbol{u}])_{ij} := \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad i,j = 1,...,d\,. \tag{6.3}$$

In the linearized theory, the tensors $\boldsymbol{\sigma}$ and ε in (6.2), (6.3) are related by the linear constitutive stress-strain relation ("Hooke's law")

$$\boldsymbol{\sigma} = \mathtt{A}\boldsymbol{\varepsilon}. \tag{6.4}$$

In (6.4), $\mathtt{A}$ is a fourth order tensor field, i.e. $\mathtt{A} = \{\mathtt{A}_{ijkl} : i, j, k, l = 1, ..., d\}$, with certain symmetries: the d^4 component functions $\mathtt{A}_{ijkl}(x)$ are assumed analytic in $[0, 1]^d$ and 1-periodic with respect to each coordinate, and satisfy for every $x \in \Omega$,

$$\forall \boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ \mathtt{A}(x)\boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \quad \text{and} \quad \forall \boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ (\mathtt{A}(x)\boldsymbol{\tau}) : \boldsymbol{\sigma} = (\mathtt{A}(x)\boldsymbol{\sigma}) : \boldsymbol{\tau}\ . \tag{6.5}$$

Key assumption on $\mathtt{A}$ is *coercivity*: there exists a constant $a_{\min} > 0$ such that

$$\forall x \in \Omega,\ \forall \boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \quad (\mathtt{A}(x)\boldsymbol{\tau}) : \boldsymbol{\tau} \geq a_{\min}\|\boldsymbol{\tau}\|_2^2\ . \tag{6.6}$$

see, e.g., [30] for details. Inserting (6.4) into (6.2), integrating by parts and noting the periodic boundary conditions, the so-called "primal variational formulation" of (6.2) reads: find $\boldsymbol{u}^{\mathtt{A}} \in [H^1(\Omega)/\mathbb{R}]^d$ such that

$$\mathfrak{b}^{\mathtt{A}}(\boldsymbol{u}^{\mathtt{A}}, \boldsymbol{v}) := \int_\Omega \boldsymbol{\varepsilon}[\boldsymbol{v}] : (\mathtt{A}\boldsymbol{\varepsilon}[\boldsymbol{u}^{\mathtt{A}}]) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad \forall \boldsymbol{v} \in [H^1(\Omega)/\mathbb{R}]^d\ . \tag{6.7}$$

Unique solvability of (6.7) is implied by the Lax-Milgram Lemma with (6.6) and Korn's inequality upon noticing that the space $X^d = [H^1(\Omega)/\mathbb{R}]^d$ does not contain rigid body motions: rigid body rotations are eliminated due to the periodicity of the present setting, and rigid body translations with the factoring of constants in each component. The Korn inequality and the Poincaré inequality (2.3) imply existence of a positive constant c such that

$$\forall \boldsymbol{v} \in X^d : \quad \mathfrak{b}^{\mathtt{A}}(\boldsymbol{v}, \boldsymbol{v}) \geq c a_{\min}\|\boldsymbol{v}\|^2_{H^1(Q)}\ .$$

For given, fixed, Q-periodic $\boldsymbol{f} \in [\mathrm{Hol}(\Omega)/\mathbb{R}]^d$, there exists a unique solution of (6.7). Furthermore, the coefficient-to-solution map $S^{\mathrm{el}} : \mathtt{A} \mapsto \boldsymbol{u}^{\mathtt{A}}$ is analytic from the set $\mathcal{D}^{\mathrm{el}} = \{\mathtt{A} \in \mathrm{Hol}(\Omega, A_{\mathcal{D}^{\mathrm{el}}})^{d^4} :$ (6.6) and (6.5) hold$\}$ to $\mathcal{U}^{\mathrm{el}} = S^{\mathrm{el}}(\mathcal{D}^{\mathrm{el}}) \subset X^d \cap \mathrm{Hol}(\Omega, A_{\mathcal{U}^{\mathrm{el}}})^d$, for positive constants $A_{\mathcal{D}^{\mathrm{el}}}, A_{\mathcal{U}^{\mathrm{el}}}$.

6.2.2. Operator network approximation. For the operator network approximation of the map S^{el}, we introduce modified encoding and reconstruction operators. To construct the encoding operator, we extend the definition of the vectorization operation to fourth order tensors so that, for all $\mathtt{B} \in R^{n_1\times\cdots\times n_4}$, $\mathbf{vec}(\mathtt{B}) \in \mathbb{R}^{n_1\cdots n_4}$. We consequently extend the definition of the reordering function introduced in Section 6.1 to $\mathfrak{v} : \mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that

$$\mathbf{vec}(\mathtt{B})_{\mathfrak{v}(m,n,p,q)} = \mathtt{B}_{mnpq}. \tag{6.8}$$

The modified encoding operator $\mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}} : [C([0, 1]^d)]^{d\times d\times d\times d} \to \mathbb{R}^{d^4 n_{\mathrm{q}}}$ is then given by

$$\mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathtt{A}) := \begin{pmatrix} \mathbf{vec}(\mathtt{A}(\boldsymbol{x}^{(q)}_1)) \\ \vdots \\ \mathbf{vec}(\mathtt{A}(\boldsymbol{x}^{(q)}_{n_{\mathrm{q}}})) \end{pmatrix}, \tag{6.9}$$

where $\boldsymbol{x}_{\mathrm{enc}} = \boldsymbol{x}^{(q)}_1, \dots, \boldsymbol{x}^{(q)}_{n_{\mathrm{q}}}$ are the usual quadrature points. For all $m \in \mathbb{N}$, the modified reconstruction operator $\mathcal{R}^{\mathrm{el}} : \mathbb{R}^{dm} \to H^1(\Omega)^d$ is instead defined, given a neural network Φ^{branch} such that $\mathrm{R}(\Phi^{\mathrm{branch}}) : \overline{Q} \to \mathbb{R}^m$, as

$$\mathcal{R}^{\mathrm{el}}_{\Phi^{\mathrm{branch}}}(\boldsymbol{c})(x) = \left(\mathbf{Id}_d \otimes \mathrm{R}(\Phi^{\mathrm{branch}})(x)\right)^\top \boldsymbol{c}, \qquad \forall x \in \overline{Q},\ \forall \boldsymbol{c} \in \mathbb{R}^{dm}. \tag{6.10}$$

We can now state the operator network approximation result for problem (6.2).

THEOREM 6.3. *Theorem 5.7 holds with* $a \in \mathcal{D}$ *replaced by* $\mathtt{A} \in \mathcal{D}^{\mathrm{el}}$, $S(a)$ *replaced by* $S^{\mathrm{el}}(\mathtt{A})$, $\mathcal{E}_{\boldsymbol{x}_{\mathrm{enc}}}(a)$ *replaced by* $\mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathtt{A})$*, and* $\mathcal{R}_{\Phi^{\mathrm{br}}_\varepsilon}$ *replaced by* $\mathcal{R}^{\mathrm{el}}_{\Phi^{\mathrm{br}}_\varepsilon}$.

Proof. We construct a basis of the dn_{b}-dimensional discrete space $X^d_{n_{\mathrm{b}}}$ approximating X^d as

$$\boldsymbol{\psi}_1 = \begin{pmatrix} \varphi_1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \boldsymbol{\psi}_{n_{\mathrm{b}}} = \begin{pmatrix} \varphi_{n_{\mathrm{b}}} \\ \vdots \\ 0 \end{pmatrix}, \dots, \boldsymbol{\psi}_{(d-1)n_{\mathrm{b}}+1} = \begin{pmatrix} 0 \\ \vdots \\ \varphi_1 \end{pmatrix}, \boldsymbol{\psi}_{dn_{\mathrm{b}}} = \begin{pmatrix} 0 \\ \vdots \\ \varphi_{n_{\mathrm{b}}} \end{pmatrix},$$

where $\varphi_1, \dots, \varphi_{n_{\mathrm{b}}}$ are the basis functions defined in Section 4.2. The trunk network $\Phi^{\mathrm{tr}}_\varepsilon$ is then constructed as in the proof of Theorem 5.7: it follows that the jth column of

$$\left(\mathbf{Id}_d \otimes \mathrm{R}(\Phi^{\mathrm{tr}}_\varepsilon)\right)^\top$$

contains an approximation of $\boldsymbol{\psi}_j$, for each $j \in \{1, \dots, dn_{\mathrm{b}}\}$.

To construct the branch network $\Phi^{\mathrm{br}}_\varepsilon$, we replace the input layer used in the proof of Theorem 5.7, in a similar way as we did in Lemma 6.2. Define, for all $i, j \in \{1, \dots, dn_{\mathrm{b}}\}$ and $m, n, p, q \in \{1, \dots, d\}$,

$$\widetilde{\mathbf{D}}^{ij}_{mnpq}(\boldsymbol{x}^{(q)}_k) = w^{(q)}_k \left(\boldsymbol{\varepsilon}[\boldsymbol{\psi}_i](\boldsymbol{x}^{(q)}_k)\right)_{mn} \left(\boldsymbol{\varepsilon}[\boldsymbol{\psi}_j](\boldsymbol{x}^{(q)}_k)\right)_{pq}$$

and let $\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_k)) \in \mathbb{R}^{d^2 n_{\mathrm{b}}^2 \times d^4}$ such that

$$\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_k))_{\mathfrak{v}(i,j)\mathfrak{v}(m,n,p,q)} = \widetilde{\mathbf{D}}^{ij}_{mnpq}(\boldsymbol{x}^{(q)}_k),$$

with $\mathfrak{v}$ defined in (6.1) and (6.8) for two and four arguments, respectively. Then,

$$\Phi^{\mathtt{A},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}} := \left(\left(-\alpha \left[\widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_1)) | \dots | \widetilde{\mathbf{vec}}(\widetilde{\mathbf{D}}(\boldsymbol{x}^{(q)}_{n_{\mathrm{q}}}))\right], \mathbf{0}_{d^2 n_{\mathrm{b}}^2}\right)\right)$$

is such that

$$\mathbf{matr}\left(\mathrm{R}(\Phi^{\mathtt{A},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}})(\mathcal{E}^{\mathrm{el}}_{\boldsymbol{x}_{\mathrm{enc}}}(\mathtt{A}))\right)_{ij} = -\alpha \mathfrak{b}^{\mathtt{A}}(\boldsymbol{\psi}_j, \boldsymbol{\psi}_i), \qquad \forall (i,j) \in \{1, \dots, dn_{\mathrm{b}}\}^2.$$

We can then construct $\Phi^{\mathrm{tr}}_\varepsilon$ as in the proof of Theorem 5.7, with $\Phi^{\mathtt{A},\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}}$ replacing $\Phi^{A,\alpha}_{n_{\mathrm{b}},n_{\mathrm{q}}}$. The rest of the proof follows the same argument as the proof of Theorem 5.7. ∎

7. Conclusions. We proved, in the periodic setting on $\Omega = \mathbb{R}^d/\mathbb{Z}^d$, the exponential convergence of deep operator network emulation of the coefficient-to-solution map of some linear elliptic equations, under the assumption of analytic coefficients a and right-hand sides f. The proof used the analytic regularity of solutions u^a of (2.4) implied by classical elliptic regularity results and the consequential exponential convergence of polynomial approximations of a and u^a and of fully discrete spectral-Galerkin numerical schemes. The expression rate bounds were not explicit in the physical domain dimension d which is moderate in engineering applications. We have developed the analysis for isotropic diffusion equations and extended it to problems with parametric diffusion, with anisotropic diffusion and reaction, and to linear elastostatics. We also obtained corresponding expression rate bounds for PDEs with parametric inputs. Here, leveraging NN composition, dependence of NN expression rates for the parametric inputs on the parameter dimension d_{p} is inherited by the solution expression rate bounds.

Further directions comprise the derivation of NN expression rate bounds of data-to-solution maps for *smooth, nonlinear forward problems* and for *inverse problems*. In the analysis of such problems, the presently established NN expression rate bounds constitute an essential building block in the construction of parameter-sparse NN approximations. Our analysis extends to ONet approximations of operators mapping both coefficients and right-hand sides to the solutions, see Remark 5.9.

Appendix A. Exponential convergence of fully discrete Spectral-Galerkin Solution. We present here the exponential convergence of fully discrete Spectral-Galerkin solutions of the problems considered in this paper. The following classical approximation result will be useful.

LEMMA A.1. *Let $A > 0$. Let $X = \{v \in H^1(\Omega) : \int_Q v = 0\}$ or $X = H^1(\Omega)$. Then, there exist constants $C, b > 0$ such that for all $p \in \mathbb{N}_0$ and for all $v \in \mathrm{Hol}(\Omega; A)$,*

$$\inf_{w\in\mathbb{Q}_p(Q)\cap X} \|v-w\|_{H^1(Q)} \leq C\exp(-bp), \qquad \inf_{w\in\mathbb{Q}_p(Q)\cap X} \|v-w\|_{L^\infty(Q)} \leq C\exp(-bp).$$

Proof. The error bound in the $H^1(Q)$-norm follows from tensorization of a univariate interpolation operator that is exact at the endpoints, see, e.g., [6, Theorem A.2]. This implies

$$\inf_{w\in\mathbb{Q}_p(Q)\cap H^1(\Omega)} \|v-w\|_{H^1(Q)} \leq C_1\exp(-bp).$$

The bound in the $L^\infty(Q)$-norm is a consequence of, e.g., [23, Remark 3.1 and Theorem 3.5], which implies

$$\inf_{w\in\mathbb{Q}_p(Q)} \|v-w\|_{L^\infty(Q)} \leq C_1\exp(-bp).$$

To ensure conformity in $H^1(\Omega) = H^1_{\mathrm{per}}(Q)$, i.e., to impose continuity across matching hyperfaces, it is sufficient to lift the difference at vertices, followed (if $d \geq 2$) by edges, and iteratively up to $d-1$ dimensional hyperfaces. The norm of the lifting is bounded by a constant (exponential in d) multiplying the $L^\infty(Q)$-norm of $v - w$.

If X includes the vanishing average constraint, it is sufficient to remark that for all $w \in L^2(Q)$ and all $v \in X$, there holds $\left|\int_Q w\right| \leq \|v-w\|_{L^2(Q)}$. ◻

The following lemma, then, concerns the convergence of fully discrete Spectral-Galerkin solutions for problems in Ω, with analytic right-hand sides and coefficients.

LEMMA A.2. *Let $\Theta \in \{\mathcal{D}, \mathcal{D}^{\mathrm{rd}}, \mathcal{D}^{\mathrm{el}}\}$ and $\mathfrak{d} = d$ for linear elasticity, $\mathfrak{d} = 1$ otherwise. Let X be the space of solutions of the problem considered and denote $X_p = \mathbb{Q}_p(Q)^{\mathfrak{d}} \cap X$. Let $f \in \mathrm{Hol}(\Omega)^{\mathfrak{d}}$ and, for coefficients $\theta \in \Theta$, let $\mathfrak{b}^\theta(\cdot,\cdot)$ be one of the bilinear forms defined in Sections 2, 6.1.1, or 6.2.1. There exists $C_1, C_2 > 0$ such that, for all $p \in \mathbb{N}$ and for all integer $q \geq p+1$,*

$$\sup_{\theta\in\Theta} \|u^\theta - u^\theta_{n_\mathrm{b},n_\mathrm{q}}\|_{H^1(Q)} \leq C_1\exp(-C_2 p),$$

where $u^\theta_{n_\mathrm{b},n_\mathrm{q}} \in X_p$ is such that $\mathfrak{b}^\theta_{q^d}(u^\theta_{n_\mathrm{b},n_\mathrm{q}}, v) = (f, v)$ for all $v \in X_p$.

Proof. Strang's lemma [25, Lemma 10.1] implies that there exists $C > 0$ independent of $\theta \in \Theta$, p, and q, such that

$$\|u^\theta - u^\theta_{n_\mathrm{b},n_\mathrm{q}}\|_{H^1(Q)} \leq C \inf_{v\in X_p}\left(\|u^\theta - v\|_{H^1(Q)} + \sup_{w\in X_p\setminus\{0\}} \frac{|\mathfrak{b}^\theta(v,w) - \mathfrak{b}^\theta_{n_\mathrm{q}}(v,w)|}{\|v\|_{H^1(Q)}\|w\|_{H^1(Q)}} \right).$$

By [3, Section 6.4.3], then, denoting $\widetilde{p} = \lfloor p/2 \rfloor$, there exists $\widetilde{C} > 0$ independent of $\theta \in \Theta$, p, and q

such that

$$\|u^\theta - u^\theta_{n_\mathrm{b},n_\mathrm{q}}\|_{H^1(Q)} \leq \widetilde{C}\left(\inf_{v\in X_p}\|u^\theta - v\|_{H^1(Q)} + \inf_{v\in Y_{\widetilde{p}}}\|\theta - v\|_{L^\infty(Q)}\right),$$

where the space $Y_{\widetilde{p}}$ depends on the problem under consideration:

$$Y_{\widetilde{p}} = \begin{cases} X_{\widetilde{p}} & \text{if } \Theta = \mathcal{D}, \\ \left\{\mathbf{A} \in X_{\widetilde{p}}^{d\times d} : \mathbf{A}_{ij} = \mathbf{A}_{ji}\right\} \times X_{\widetilde{p}} & \text{if } \Theta = \mathcal{D}^{\mathrm{rd}}, \\ \left\{\mathtt{A} \in X_{\widetilde{p}}^{d\times d\times d\times d} : (6.5) \text{ holds}\right\} & \text{if } \Theta = \mathcal{D}^{\mathrm{el}}. \end{cases}$$

Since functions in Θ and in $S(\mathcal{D})$, $S^{\mathrm{rd}}(\mathcal{D}^{\mathrm{rd}})$, or $S^{\mathrm{el}}(\mathcal{D}^{\mathrm{el}})$ are analytic with uniform bounds on the norms at all orders, using Lemma A.1 concludes the proof. □

Appendix B. Lipschitz continuity of the data-to-solution map. For the readers' convenience, we provide a proof of the (known) Lipschitz dependence of the solution of the PDEs considered in this paper on the coefficients.

Lemma B.1. *Let X be a Hilbert space, let Y be a Banach space, and let $\Theta \subset Y$. Let furthermore $\mathfrak{b}^\theta : X \times X \to \mathbb{R}$ be a bilinear form that is also linear with respect to the coefficient θ. Suppose that there exists $C_{\mathrm{cont}} > 0$ such that*

$$\forall\theta \in Y: \qquad \mathfrak{b}^\theta(u,v) \leq C_{\mathrm{cont}}\|\theta\|_Y\|u\|_X\|v\|_X, \qquad \forall u,v\in X\,. \tag{B.1}$$

Furthermore, suppose there exists $\theta_{\min} > 0$ such that

$$\mathfrak{b}^\theta(u,u) \geq \theta_{\min}\|u\|_X^2, \qquad \forall u \in X,\ \forall\theta\in\Theta. \tag{B.2}$$

For fixed $f \in X'$ and for each $\theta \in \Theta$, define $u^\theta \in X$ as the function such that

$$\mathfrak{b}^\theta(u^\theta, v) = \langle f, v\rangle, \qquad \forall v \in X. \tag{B.3}$$

Then, there exists $C_L > 0$ (depending only on C_{cont}, $\theta_{\min}$, and f) such that

$$\|u^{\theta_1} - u^{\theta_2}\|_X \leq C_L\|\theta_1 - \theta_2\|_Y, \qquad \forall\theta_1,\theta_2\in\Theta.$$

Proof. Denote $u_i = u^{\theta_i}$, $i = 1, 2$. Using (B.2), (B.3), the continuity of the bilinear form with respect to the coefficient, and (B.1)

$$\begin{aligned}\|u_1 - u_2\|_X^2 &\leq \frac{1}{\theta_{\min}}\mathfrak{b}^{\theta_1}(u_1 - u_2, u_1 - u_2)\\ &\leq \frac{1}{\theta_{\min}}\left(\mathfrak{b}^{\theta_2}(u_2, u_1 - u_2) - \mathfrak{b}^{\theta_1}(u_2, u_1-u_2)\right)\\ &\leq \frac{C_{\mathrm{cont}}}{\theta_{\min}}\|\theta_2 - \theta_1\|_Y\|u_2\|_X\|u_1 - u_2\|_X.\end{aligned}$$

The Lax-Milgram bound $\|u_2\|_X \leq \frac{1}{\theta_{\min}}\|f\|_{X'}$ concludes the proof. □

Appendix C. Generalized eigenvalue problems. We recall and a result on the relationship between Rayleigh quotients and generalized eigenvalue problems (see, e.g., [29, Section I.10]).

Lemma C.1. *Let $n \in \mathbb{N}$ and let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ be symmetric, positive definite matrices. Suppose that there exist constants $c, C > 0$ such that*

$$\forall \boldsymbol{x} \in \mathbb{R}^n: \qquad c \leq \frac{\boldsymbol{x}^\top\mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^\top\mathbf{B}\boldsymbol{x}} \leq C.$$

Then, the spectrum of $\mathbf{B}^{-1}\mathbf{A}$ is contained in $[c, C]$.

Proof. Denote by $\mathbf{B}^{1/2}$ the symmetric positive definite matrix such that $\mathbf{B}^{1/2}\mathbf{B}^{1/2} = \mathbf{B}$ and by $\mathbf{B}^{-1/2}$ its inverse. For all $\boldsymbol{y} \in \mathbb{R}^n$, we can write $\boldsymbol{x} = \mathbf{B}^{-1/2}\boldsymbol{y}$ and by hypothesis

$$\frac{\boldsymbol{y}^\top \mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}\boldsymbol{y}}{\boldsymbol{y}^\top \boldsymbol{y}} = \frac{\boldsymbol{x}^\top \mathbf{A}\boldsymbol{x}}{\boldsymbol{x}^\top \mathbf{B}\boldsymbol{x}} \in [c, C].$$

Now, $\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}$ is a symmetric matrix, and we have shown that its Rayleigh quotient is contained in $[c, C]$. Therefore,

$$\sigma(\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}) \subset [c, C].$$

We conclude by remarking that the matrices $\mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2}$ and $\mathbf{B}^{-1}\mathbf{A}$ are similar, hence have the same eigenvalues [29, p. 38] ◻

Appendix D. Polynomial basis in $H^1_{\mathrm{per}}(Q)$. Let X_{n_b} be defined as in (4.2); we adopt the notation of Section 4.2. In the next lemma, we show that X_{n_b} is a basis for polynomials that is conforming in $H^1(\Omega)$.

LEMMA D.1. *Let $X = \{v \in H^1(\Omega) = H^1_{\mathrm{per}}(Q) : \int_Q v = 0\}$. Let $p \in \mathbb{N}$ and $n_\mathrm{b} = p^d - 1$. Then,*

$$X_{n_\mathrm{b}} = \mathbb{Q}_p(Q) \cap X.$$

Proof. The inclusion $X_{n_\mathrm{b}} \subset \mathbb{Q}_p(Q) \cap X$ is a consequence of the definition of $\varphi_1, \dots, \varphi_{n_\mathrm{b}}$. We now prove that the dimensions of X_{n_b} and $\mathbb{Q}_p(Q) \cap X$ are equal. Set $\zeta^{\mathrm{1d}}_0 = L_0$, $\zeta^{\mathrm{1d}}_1 = L_1$, and

$$\begin{aligned} \zeta^{\mathrm{1d}}_{2i} &= L_{2i}, \qquad i = 1, \dots, \lfloor p/2 \rfloor, \\ \zeta^{\mathrm{1d}}_{2i+1} &= L_{2i+1} - L_1, \qquad i = 1, \dots, \lfloor (p-1)/2 \rfloor. \end{aligned}$$

Since $\{L_0, \dots, L_p\}$ is a basis for $\mathbb{Q}_p((0,1))$, then $\{\zeta^{\mathrm{1d}}_0, \dots, \zeta^{\mathrm{1d}}_p\}$ is also a basis for $\mathbb{Q}_p((0,1))$. Define

$$\zeta_{i_1,\dots,i_d} = \zeta^{\mathrm{1d}}_{i_1} \otimes \cdots \otimes \zeta^{\mathrm{1d}}_{i_d}, \qquad (i_1, \dots, i_d) \in \mathbb{N}^d_0.$$

It follows that $\{\zeta_{i_1,\dots,i_d} : (i_1, \dots, i_d) \in \{0, \dots, p\}^d\}$ is a basis for $\mathbb{Q}_p(Q)$. Now, given any function

$$v = \sum_{i_1,\dots,i_d \in \{0,\dots,p\}^d} v_{i_1,\dots,i_d} \zeta_{i_1,\dots,i_d},$$

requiring that $v \in H^1_{\mathrm{per}}(Q)$ is equivalent to imposing that

$$v_{i_1,\dots,i_d} = 0, \qquad \forall (i_1, \dots, i_d) \in \{0, \dots, p\}^d : \exists j : i_j = 1.$$

Since $\dim(\mathbb{Q}_p(Q)) = (p+1)^d$ and

$$\operatorname{card}\left(\{(i_1, \dots, i_d) \in \{0, \dots, p\}^d : \exists j : i_j = 1\}\right) = \sum_{n=1}^{d} \binom{d}{n} p^{d-n},$$

we infer that

$$\dim(\mathbb{Q}_p(Q) \cap H^1_{\mathrm{per}}(Q)) = (p+1)^d - \sum_{n=1}^{d} \binom{d}{n} p^{d-n} = p^d.$$

Imposing vanishing average removes another degree of freedom, which implies $\dim(\mathbb{Q}_p(Q) \cap X) = n_\mathrm{b}$ and concludes the proof. ◻

In the next lemma we bound the ℓ^2 norm of the coefficients (in the basis introduced in Section 4.2) of a function in X_{n_b} by its L^2-norm.

LEMMA D.2. *Let φ_i, $i \in \mathbb{N}_0$, be the functions defined in (4.1). There exists $C_{L^2} \geq 1$ such that, for all $p \in \mathbb{N}$ with $p \geq 2$ and for all $v \in X_{n_\mathrm{b}}$ with $n_\mathrm{b} = p^d - 1$ such that $v = \sum_{i=1}^{n_\mathrm{b}} w_i \varphi_i$,*

$$\|\boldsymbol{w}\|_2 \leq C_{L^2} n_\mathrm{b}^{1/2} \|v\|_{L^2(Q)} \tag{D.1}$$

where $\boldsymbol{w} = (w_1, \dots, w_{n_\mathrm{b}})$.

Proof. For ease of notation, we introduce the functions

$$\nu^{1\mathrm{d}}_{2i} = L_{2i}, \qquad \nu^{1\mathrm{d}}_{2i+1} = L_{2i+1} - L_1, \qquad i \in \mathbb{N}_0,$$

and

$$\nu_{i_1,\dots,i_d} = \nu^{1\mathrm{d}}_{i_1} \otimes \cdots \otimes \nu^{1\mathrm{d}}_{i_d}, \qquad (i_1, \dots, i_d) \in \mathbb{N}_0^d.$$

Remark that $\nu^{1\mathrm{d}}_1 = 0$. We then rewrite v as

$$v = \sum_{(i_1,\dots,i_d)\in\{0,\dots,p\}^d} x_{i_1,\dots,i_d} \nu_{i_1,\dots,i_d}.$$

Note that, since $v \in X_{n_\mathrm{b}}$, we have $x_{0,\dots,0} = 0$. Furthermore, we assume that $x_{i_1,\dots,i_d} = 0$ if there exists at least one $j \in \{1, \dots, d\}$ such that $i_j = 1$. We have

$$\{x_{i_1,\dots,i_d} : (i_1, \dots, i_d) \neq (0, \dots, 0) \text{ and } \forall j \in \{1, \dots, d\}, i_j \neq 1\} = \{w_1, \dots, w_{n_\mathrm{b}}\}, \tag{D.2}$$

with one-to-one correspondence between the sets. We now introduce the $L^2((0,1))$-orthonormal basis

$$\chi^{1\mathrm{d}}_i = \frac{L_i}{\|L_i\|_{L^2((0,1))}}, \qquad i \in \mathbb{N}_0,$$

and the $L^2(Q)$-orthonormal basis

$$\chi_{i_1,\dots,i_d} = \chi^{1\mathrm{d}}_{i_1} \otimes \cdots \otimes \chi^{1\mathrm{d}}_{i_d}, \qquad (i_1, \dots, i_d) \in \mathbb{N}_0^d.$$

It follows that

$$\begin{aligned}
\|v\|^2_{L^2(Q)} &= \sum_{(i_1,\dots,i_d)\in\{0,\dots,p\}^d} (v, \chi_{i_1,\dots,i_d})^2_{L^2(Q)} \\
&= \sum_{(i_1,\dots,i_d)\in\{0,\dots,p\}^d} \left(\sum_{(j_1,\dots,j_d)\in\{0,\dots,p\}^d} x_{j_1,\dots,j_d} \prod_{k=1}^d (\nu^{1\mathrm{d}}_{j_k}, \chi^{1\mathrm{d}}_{i_k})_{L^2((0,1))} \right)^2.
\end{aligned} \tag{D.3}$$

Now, for $i, j \in \{0, \dots, p\}$, we have

$$(\nu^{1\mathrm{d}}_j, \chi^{1\mathrm{d}}_i)_{L^2((0,1))} = \begin{cases} \delta_{ij} \|L_i\|_{L^2((0,1))} & \text{if } i \neq 1 \\ -\|L_1\|_{L^2((0,1))} & \text{if } i = 1 \text{ and } j \text{ is odd, } j \neq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where δ_{ij} is the Kronecker delta. Hence,

$$\|v\|^2_{L^2(Q)} = \sum_{(i_1,\dots i_d): i_j\neq 1, \forall j} x^2_{i_1,\dots,i_d} \prod_{k=1}^d \|L_{i_k}\|^2_{L^2((0,1))} + \sum_{(i_1,\dots i_d): \exists j \text{ s.t. } i_j=1} \left(\sum_{(j_1,\dots,j_d)\in\{0,\dots,p\}^d} x_{j_1,\dots,j_d} \prod_{k=1}^d (\nu^{1\mathrm{d}}_{j_k}, \chi^{1\mathrm{d}}_{i_k})_{L^2((0,1))} \right)^2,$$

where in the sums we still require also $(i_1,\dots,i_d) \in \{0,\dots,p\}^d$. Since $\|L_i\|^2_{L^2((0,1))} = 1/(2i+1)$, it follows from the equation above that

$$\|v\|^2_{L^2(Q)} \geq \frac{1}{(2p+1)^d} \sum_{(i_1,\dots i_d): i_j\neq 1, \forall j} x^2_{i_1,\dots,i_d}.$$

From the correspondence between coefficients (D.2) and since $n_{\mathrm{b}} = p^d - 1$, we obtain the bound in (D.1). We remark that the constant C_{L^2} depends exponentially on d. ∎

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