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HIGH ORDER TOPOLOGICAL ASYMPTOTICS: RECONCILING LAYER POTENTIALS AND MATCHED ASYMPTOTIC EXPANSIONS

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Abstract. A systematic two-step procedure is proposed for the derivation of full asymptotic expansions of the solution of elliptic partial differential equations set on a domain perforated with a small hole on which a Dirichlet boundary condition is applied. First, an integral representation of the solution is sought, which enables to exploit the explicit dependence with respect to the small parameter to predict the correct form of a two-scale ansatz. Second, the terms of the ansatz are characterized by the method of matched asymptotic expansions as the solutions of a cascade of successive interior and exterior domains. This allows to interpret them as high-order correctors, for which error bounds can be proved using variational estimates. The methodology is illustrated on two different problems: we start by revisiting the perforated Poisson problem with Dirichlet boundary conditions on both the hole and the outer boundary, where we highlight how the method enables to obtain very naturally the correct ansatz in the most delicate two-dimensional setting. Then, we provide original and complete asymptotic expansions for a perforated cell-problem featuring periodic conditions.

Keywords. Topological asymptotics, layer potentials, matched asymptotic expansions, Deny-Lions spaces, exterior Dirichlet problem, periodic cell problem.

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Contents

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1. Introduction

Small inclusion or topological asymptotics are of interest in many applications of mathematical physics, such as topology optimization [23, 34, 36, 4, 25], image processing [12], inverse tomography [10, 11, 14, 26] and imaging [7, 5, 15]. These quantify the sensitivity of the solution to some physical model to the nucleation of a small inclusion in the working domain.

The usual methods to derive topological asymptotics can broadly be classified into two classes: the method of *matched asymptotic expansions*, which compare the solution to the physical model to a tentative *ansatz* thanks to variational estimates [28, 36, 23, 34, 18, 36, 11, 26, 25], and *layer potential methods*, which rely on an integral representation of the solution to characterize explicitly its dependence with respect to the small parameter [10, 7, 10, 14, 17, 31]. If layer potential methods enable one to obtain full asymptotic expansions, the resulting asymptotics involve Neumann series of integral operators, which may be quite difficult to understand physically [9], and rather tedious to compute numerically. On the other hand, the method of matched asymptotic expansions identifies the terms of the guessed ansatz as the solutions to a cascade of well-posed problems which have a clear physical interpretation: higher order terms correct the error accumulated by lower order ones at the boundaries [25]. However, the correct form of the ansatz may be difficult to guess *a priori* (especially in dimension $d = 2$); very often, it is proposed in the literature without justification, and it is only *a posteriori* that one finds that it is amenable to error estimates.

The goal of this paper is to highlight that one can obtain complete small inclusion asymptotics by using a two-step systematic process which benefits from the advantages of both methods:

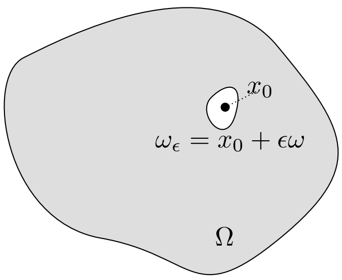
- first, an explicit representation of the physical solution in terms of layer potentials is sought. Owing to a change of variable in these integral operators, the explicit dependence with respect to the small parameter enables to determine the correct form of the ansatz satisfied by the full asymptotic expansion of the solution;
- second, the method of matched asymptotic expansions is applied to characterize the terms of this ansatz as the solutions to a cascade of equations. It is then possible to prove error bounds *a posteriori* with standard variational estimates.

This procedure is systematic and could be applied as well for many other types of perforated Dirichlet problems.

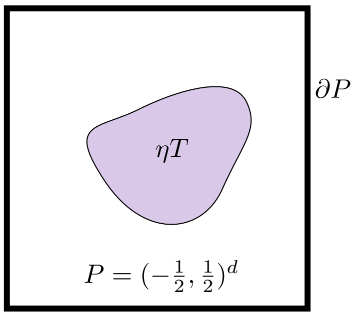
For the purpose of illustrating the method, we derive in this paper full asymptotic expansions of the solutions u_ϵ and $\mathcal{X}_\eta$ as to the following problems:

$$\begin{cases} -\Delta u_\epsilon = f \text{ in } \Omega, \\ u_\epsilon = 0 \text{ on } \partial\omega_\epsilon, \\ u_\epsilon = 0 \text{ on } \partial\Omega, \end{cases} \quad (1.1) \qquad \begin{cases} -\Delta \mathcal{X}_\eta = 1 \text{ in } P\backslash(\eta T), \\ \mathcal{X}_\eta = 0 \text{ on } \partial(\eta T), \\ \mathcal{X}_\eta \text{ is } P-\text{periodic}, \end{cases} \quad (1.2)$$

where we use a different notation for the domains Ω and P, and for the small parameters $\epsilon > 0$ and $\eta > 0$ to emphasize the difference in the physical motivations.



(A) Schematic for the problem (1.1). Dirichlet boundary conditions are applied on the background domain Ω and on a small inclusion ω_ϵ.



(B) Schematic for the problem (1.2). Periodic boundary conditions are considered on a unit cell P and a Dirichlet boundary condition is applied on the obstacle ηT.

Figure 1. Setting of the two perforated problems considered in this paper.

The problem (1.1) (illustrated on Figure 1a) is known as the perforated Poisson problem; it finds applications e.g. in structural design [23, 25]. Ω is a smooth bounded open subdomain of $\mathbb{R}^d$ with $d \geq 2$, $f \in \mathcal{C}^\infty(\Omega)$ is a smooth right-hand side, and $\omega_\epsilon := x_0 + \epsilon\omega$ is a small inclusion obtained by centering a smooth domain ω around $x_0 \in \Omega$ and rescaling it by a size factor ϵ. In the limit where ϵ converges to zero, it can easily be shown that u_ϵ

converges to the solution u to the same Dirichlet problem with no hole,

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

and the problem of finding the asymptotic of u_ϵ as $\epsilon \to 0$ is to understand how the solution to (1.3) is perturbed by a perforation of the domain Ω with the small hole ω_ϵ.

The problem (1.2) (illustrated on Figure 1b) arises in the context of periodic homogenization of perforated structures [1, 3, 27, 19], where $\mathcal{X}_\eta$ plays the role of a homogenized corrector. $P := (-\frac{1}{2}, \frac{1}{2})^d$ stands for the unit hypercube and ηT is a smooth obstacle T centered around the origin and rescaled by a small size factor $\eta > 0$. The quantity $\int_{P\setminus(\eta T)} \mathcal{X}_\eta \, \mathrm{d}x$ is the analogue of the Darcy porosity tensor in the context of a porous fluid medium filled with solid obstacles of shape ηT; determining its asymptotic as $\eta \to 0$ is of interest to understand the transition from the regime where the effective medium is governed by the Darcy's law to the one when it is governed by the Brinkmann equation [3]. In contrast with (1.1), there is no limit for $\mathcal{X}_\eta$ as $\eta \to 0$ because the "limit problem"

$$\begin{cases} -\Delta \mathcal{X} = 1 \text{ in } P, \\ \quad \mathcal{X} \text{ is } P\text{–periodic}, \end{cases} \tag{1.4}$$

has no solution (see Proposition 5.3 below); as a result, $\mathcal{X}_\eta$ has a singular asymptotic behavior as $\eta \to 0$.

The asymptotic analysis of (1.1) traces back at least to Lions [30, 13]. It can be fully treated by the method of two-scale or matched asymptotic expansions: when the dimension is greater or equal to 3, $d \geq 3$, an ansatz of the form

$$u_\epsilon(x) = \sum_{p=0}^{+\infty} \epsilon^p \left(v_p \left(\frac{x - x_0}{\epsilon} \right) + w_p(x) \right) \tag{1.5}$$

is proposed and the functions $(v_p)_{p\geq 0}$ and $(w_p)_{p\geq 0}$ are identified by inserting (1.5) into (1.1); these are found to be successive correctors of the error committed by the previous orders on the Dirichlet condition on the hole $\partial\omega_\epsilon$ and on the outer boundary $\partial\Omega$. Then one proves using purely variational estimates that the truncation of the formal series (1.5) at rank N yields an approximation of the solution u_ϵ to (1.1) at the order $O(\epsilon^{N+1})$ (see [25] and Proposition 3.7 below). However, this method does not work when the dimension is equal to two ($d = 2$), because terms with logarithmic powers of ϵ occur: for instance, the leading order asymptotic of u_ϵ is given by

$$u_\epsilon(x) = u(x) - \frac{u(x_0)}{\frac{1}{2\pi}\log(\epsilon) - \Phi_\infty + R_\Omega(x_0)} \left[\frac{1}{2\pi} \log \epsilon + R_\Omega(x) + \Phi\left(\frac{x - x_0}{\epsilon} \right) - \Phi^\infty \right] + O(\epsilon), \tag{1.6}$$

for some constant Φ^∞ and auxiliary functions Φ and R_Ω (defined in (2.9) and (4.8) below). Hence a more complicated ansatz needs to be found in this case; a full treatment following this method has been proposed by Kozlov et. al. [28, section 1.4] for a circular hole ω_ϵ.

From our point of view, the matched asymptotic expansion procedure is not completely satisfactory, because the ansatz (1.5) or (1.6) has to be guessed beforehand. On the other hand, the problem (1.1) can also be tackled by the use of layer potential theory. The full asymptotic analysis of (1.1) in the two-dimensional case $d = 2$ by this method is possible and systematic; it has been performed in [16]. However, the authors of [16] consider a double layer potential representation which yields somewhat more complicated representations of the asymptotic expansions (featuring Cauchy products and Leibniz rules), and they do not provide a characterization of the terms of the ansatz as corrector functions.

In this paper, we revisit the derivation of full asymptotic expansions for the problem (1.1), where we show that the correct form of the ansatz, namely some analyticity of the solution with respect to ϵ (and $\log \epsilon$ if $d = 2$), can be conveniently obtained from a *single* layer potential representation. We retrieve the results of [25] and the ansatz (1.5) in the case $d \geq 3$ (Proposition 3.4 below), where we find additionally and very naturally that $w_p = 0$ for $0 \leq p < d - 2$. The correct form of the ansatz for the more delicate two-dimensional case is derived in Proposition 4.5 below and its terms are fully characterized in Proposition 4.6. Our ansatz is different from the one of Kozlov et. al. [28] (valid only for a circular obstacle, see Remark 4.2); its form is fully elucidated when resorting to layer potentials and a variational theory for the solution of exterior Laplace problems.

As for the periodic cell problem (1.2), only the first order expansion of $\mathcal{X}_\eta$ has, to date, been computed: [3] identified the first term of the expansion by using a suitable rescaling and weak convergence arguments, while [27] considered a double layer integral representation of $\mathcal{X}_\eta$ which enables to obtain a first order asymptotic expansion with a quantitative error estimate. In this paper, we show how a suitable single layer potential characterization enables to compute full asymptotic expansions of $\mathcal{X}_\eta$ (in Proposition 5.11 and Proposition 5.13) for the dimensions $d \geq 3$ and $d = 2$ respectively. Quite surprisingly, we find that no logarithmic powers of η arises in the two-dimensional case; this peculiarity is once more fully elucidated thanks to the characterization with integral operators.

The paper outlines as follows. Section 2 to 4 provide a detailed analysis for the Poisson problem (1.1). Section 2 recalls some essential background material on layer potentials and exhibits a simple integral representation of the solution u_ϵ. We then establish the existence of a suitable factorization of the single layer potential in Ω, which elucidates the origin of some analyticity of the solution (in fact, of its normal derivative) with respect to ϵ (and $\log\epsilon$ if $d=2$). A complete asymptotic analysis of the perforated problem (1.1) is then proposed in Sections 2 and 3 for the cases $d\geq 3$, and $d=2$ respectively. We rely very much on the properties of *Deny-Lions* spaces (also called Beppo-Levi or homogeneous Sobolev spaces), which are the appropriate mathematical framework for studying the solutions to the Laplace equation with Dirichlet boundary conditions on some exterior domain. Finally, the same methodology is applied in Section 5 for deriving full asymptotic expansions for the periodic cell problem (1.2).

Through the analysis of (1.1) and (1.2) and its potential generalization to several other physical problems, we hope to bring a pedagogical exposure of the strength of layer potential methods for the asymptotic analysis of PDE problems. In particular, a similar method is applied for the homogenization of non-periodic resonant metamaterials in our work [20].

2. Analytic layer potential representations for the perforated Poisson problem

In this first section, we recall some basic results about single layer potentials in exterior domains in Section 2.1, and about single layer potentials in domains with Dirichlet boundary conditions in Section 2.2. This allows us to introduce an integral representation of the solution u_ϵ to (1.1), which is at the basis of the derivation of a two-scale ansatz for finding its full asymptotic expansion with respect to $\epsilon\to 0$. Then, we introduce in Section 2.3 a rescaling operator $\mathcal{P}_{x_0,\epsilon}$ around the hole, which enables to explicit the analytic dependence of the representation with respect to ϵ when $d\geq 3$, and with respect to $\log\epsilon$ when $d=2$.

2.1. **Single layer potential in the exterior domain**

Throughout the paper, we denote by Γ the fundamental solution of the Laplace operator (i.e. $\Delta\Gamma=\delta_0$ in $\mathbb{R}^d$):

$$\Gamma(x):=\begin{cases} \dfrac{1}{2\pi}\log|x| \text{ if } d=2,\\ -\dfrac{1}{(d-2)|\partial B(0,1)|}\dfrac{1}{|x|^{d-2}} \text{ if } d\geq 3,\end{cases} \tag{2.1}$$

where $|\partial B(0,1)|$ is the measure of the unit sphere of $\mathbb{R}^d$. For any Lipschitz open set $D\subset\mathbb{R}^d$, we denote by $\mathcal{S}_D$ the single layer potential on ∂D:

$$\forall x\in\mathbb{R}^d, \forall\phi\in H^{-\frac12}(\partial D),\quad \mathcal{S}_D[\phi](x):=\int_{\partial D}\Gamma(x-y)\phi(y)\,\mathrm{d}\sigma(y). \tag{2.2}$$

The next proposition recalls the main properties of the single layer potential $\mathcal{S}_D$ [21, 6].

Proposition 2.1. *For any $\phi\in H^{-\frac12(\partial D)}$:*

(i) $\mathcal{S}_D[\phi]$ is a harmonic function in $\mathbb{R}^d\backslash\partial D$;

(ii) $\mathcal{S}_D[\phi]$ is continuous on ∂D while its normal derivative has a jump:

$$[\![\mathcal{S}_D[\phi]]\!]=0, \tag{2.3}$$

$$\left[\!\!\left[\frac{\partial\mathcal{S}_D[\phi]}{\partial n}\right]\!\!\right]=\phi, \tag{2.4}$$

where $[\![u]\!]$ denotes the jump of a function u across ∂D, and n is the outward normal:

$$[\![u]\!]:=u|_+-u|_- \text{ with } u|_+(y):=\lim_{\substack{t\to0\\t>0}}u(y+tn),\quad u|_-(y):=\lim_{\substack{t\to0\\t<0}}u(y-tn),\quad y\in\partial D.$$

(iii) $\mathcal{S}_D[\phi]$ has the following asymptotic behavior at infinity:

$$\mathcal{S}_D[\phi](x)=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\nabla^k\Gamma(x)\cdot\int_{\partial D}y^k\phi(y)\,\mathrm{d}\sigma(y), \tag{2.5}$$

where the series converge for any $x\in\mathbb{R}^d\backslash D$ satisfying $|x|>\sup_{y\in D}|y|$.
In (2.5), y^k and $\nabla^k\Gamma$ denote the k-th order tensors

$$y^k=(y_{i_1}y_{i_2}\dots y_{i_k})_{1\leq i_1\dots i_k\leq d},\quad \nabla^k\Gamma=(\partial^k_{i_1\dots i_k}\Gamma)_{1\leq i_1\dots i_k\leq d},$$

and $\nabla^k\Gamma(x)\cdot y^k$ is their contraction:

$$\nabla^k\Gamma(x)\cdot y^k:=\sum_{1\leq i_1\dots i_k\leq d}\partial^k_{i_1\dots i_k}\Gamma(x)y_{i_1}\dots y_{i_k}.$$

Finally, we shall need the following lemma in the proof of Proposition 4.2 and Proposition 5.10 below.

Lemma 2.1. *For any given constant k-th order tensor $\xi^k = (\xi_{i_1 \dots i_k})_{1 \le i_1 \dots i_k \le d} \in \mathbb{R}^{d^k}$ and any smooth open bounded domain $D \subset \mathbb{R}^d$ containing 0,*

$$\int_{\partial D} \frac{\partial \Gamma}{\partial n} \, \mathrm{d}\sigma = 1 \text{ and } \int_{\partial D} \frac{\partial}{\partial n} \nabla^k \Gamma(x) \cdot \xi^k \, \mathrm{d}\sigma(x) = 0 \text{ if } k > 0. \tag{2.6}$$

Proof. Using the harmonicity of $\nabla^k \Gamma$, we can rewrite the boundary integral on a small sphere $B(0,\epsilon)$ with $\epsilon > 0$:

$$\begin{aligned} \int_{\partial P} \frac{\partial}{\partial n} \nabla^k \Gamma(x) \cdot \xi^k \, \mathrm{d}\sigma(x) &= \int_{\partial P} \nabla^{k+1} \Gamma(x) \cdot (\xi^k \otimes \boldsymbol{n}) \, \mathrm{d}\sigma(x) = \int_{\partial B(0,\epsilon)} \nabla^{k+1} \Gamma(x) \cdot (\xi^k \otimes \boldsymbol{n}) \, \mathrm{d}\sigma(x) \\ &= \epsilon^{d-1} \epsilon^{2-d-k-1} \int_{\partial B(0,1)} \nabla^{k+1} \Gamma(x) \cdot \xi^k \otimes \boldsymbol{n} \, \mathrm{d}\sigma(x) = \epsilon^{-k} \int_{\partial B(0,1)} \nabla^{k+1} \Gamma(x) \cdot \xi^k \otimes \boldsymbol{n} \, \mathrm{d}\sigma(x), \end{aligned} \tag{2.7}$$

where $\otimes$ denotes the usual tensor product. When $k = 0$, the right-hand side can be computed explicitly as

$$\int_{\partial B(0,1)} \nabla \Gamma(x) \cdot \boldsymbol{n} \, \mathrm{d}\sigma(x) = \int_{\partial B(0,1)} \frac{1}{|\partial B(0,1)|} \frac{1}{|x|^{d-1}} \frac{x}{|x|} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 1.$$

When $k > 0$, the left-hand side of (2.7) is a finite quantity, which must be equal to zero, by considering the limit $\epsilon \to 0$. $\square$

2.2. Dirichlet Green function and single layer potential in the interior domain

From the fundamental solution Γ of (2.1), we construct the Laplace Green kernel $G_\Omega(x,y)$ with Dirichlet boundary conditions on Ω, defined as the unique solution to

$$\begin{cases} \Delta_y G_\Omega(x, \cdot) = \delta_x \text{ in } \Omega, \\ \quad G_\Omega(x, \cdot) = 0 \text{ on } \partial\Omega, \end{cases}, \qquad \text{for any } x \in \Omega. \tag{2.8}$$

Classically, the function $G_\Omega(x, \cdot)$ is constructed by using a difference problem [6].

Proposition 2.2. *The Green kernel G_Ω is given by*

$$G_\Omega(x,y) := \Gamma(x-y) + R_\Omega(x,y),$$

where for any $x \in \Omega$, $R_\Omega(x,\cdot)$ is the unique solution to the difference problem

$$\begin{cases} \Delta_y R_\Omega(x, \cdot) = 0 \text{ in } \Omega, \\ \quad R_\Omega(x, \cdot) = -\Gamma(x - \cdot) \text{ on } \partial\Omega. \end{cases} \tag{2.9}$$

The function R_Ω satisfies $R_\Omega(x,y) = R_\Omega(y,x)$ for any $(x,y) \in \Omega \times \Omega$. Furthermore, for any $x \in \Omega$, R_Ω is a smooth function of Ω.

This allows to define the single layer potential $\mathcal{S}_{\Omega,\omega_\epsilon}$ with a Dirichlet boundary condition on $\partial\Omega$ from the formula

$$\forall \phi \in H^{-\frac{1}{2}}(\partial\omega_\epsilon), \quad \forall x \in \Omega, \quad \mathcal{S}_{\Omega,\omega_\epsilon}[\phi](x) := \int_{\partial\omega_\epsilon} G_\Omega(x,y)\phi(y) \, \mathrm{d}\sigma(y). \tag{2.10}$$

We note that the operator $\mathcal{S}_{\Omega,\omega_\epsilon}$ is a compact perturbation of the "classical" single layer potential $\mathcal{S}_{\omega_\epsilon}$:

$$\mathcal{S}_{\Omega,\omega_\epsilon}[\phi] = \mathcal{S}_{\omega_\epsilon}[\phi] + \int_{\partial\omega_\epsilon} R_\Omega(\cdot, y)\phi(y) \, \mathrm{d}\sigma(y). \tag{2.11}$$

The potential $\mathcal{S}_{\Omega,\omega_\epsilon}$ satisfies the following properties.

Proposition 2.3. *(i) For any $\phi \in H^{-\frac{1}{2}}(\partial\omega_\epsilon)$, $\mathcal{S}_{\Omega,\omega_\epsilon}[\phi]$ satisfies*

$$\begin{cases} -\Delta \mathcal{S}_{\Omega,\omega_\epsilon}[\phi] = 0 \text{ in } \Omega \backslash \partial\omega_\epsilon, \\ \qquad \mathcal{S}_{\Omega,\omega_\epsilon} = 0 \text{ on } \partial\Omega. \end{cases}$$

(ii) $\mathcal{S}_{\Omega,\omega_\epsilon}$ satisfies the jump relations

$$[\![\mathcal{S}_{\Omega,\omega_\epsilon}[\phi]]\!] = 0, \qquad \left[\!\!\left[\frac{\partial \mathcal{S}_{\Omega,\omega_\epsilon}[\phi]}{\partial n}\right]\!\!\right] = \phi. \tag{2.12}$$

(iii) The single layer potential $\mathcal{S}_{\Omega,\omega_\epsilon}$ is invertible when considered as an operator $H^{-\frac{1}{2}}(\partial\omega_\epsilon) \to H^{\frac{1}{2}}(\partial\omega_\epsilon)$.

Proof. The point (i) is obtained from the definition (2.10). The point (ii) follows by using the jump relation on $\mathcal{S}_{\omega_\epsilon}$ and the fact that the perturbation in (2.11) is smoothing. Let us prove the point (iii). Recall that $\mathcal{S}_{\Omega,\omega_\epsilon}$ is a Fredholm operator of index 0 [32], as a compact perturbation of the Fredholm operator $\mathcal{S}_{\omega_\epsilon} : H^{-\frac{1}{2}}(\partial\omega_\epsilon) \to H^{\frac{1}{2}}(\partial\omega_\epsilon)$. Therefore, it is sufficient to show that this operator has a trivial kernel to prove its invertibility.

Let $\phi \in H^{-\frac{1}{2}}(\partial\omega_\epsilon)$ be such that $u := \mathcal{S}_{\Omega,\omega_\epsilon}[\phi] = 0$. The function u satisfies

$$\begin{cases} -\Delta u = 0 \text{ in } \Omega \backslash \partial\omega_\epsilon, \\ \quad u = 0 \text{ on } \partial\omega_\epsilon, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases}$$

which easily implies $u = 0$ on $\Omega\backslash\omega_\epsilon$ and on ω_ϵ. From the jump relation (ii), it holds $\phi = [\![\frac{\partial u}{\partial n}]\!] = 0$ hence $\mathcal{S}_{\Omega,\omega_\epsilon}$ is injective. □

It is now easy to infer the following integral representation for the solution u_ϵ to the perforated problem (1.1).

Proposition 2.4. *The following single layer potential representation holds for the solution u_ϵ to the perforated Poisson problem (1.1):*

$$u_\epsilon(x) = u(x) - \mathcal{S}_{\Omega,\omega_\epsilon}[\mathcal{S}_{\Omega,\omega_\epsilon}^{-1}[u|_{\partial\omega_\epsilon}]], \tag{2.13}$$

where u is the solution to the Poisson problem (1.3) without the hole.

Proof. It is immediate to check from the properties of $\mathcal{S}_{\Omega,\omega_\epsilon}$ that the function $v_\epsilon := -\mathcal{S}_{\Omega,\omega_\epsilon}[\mathcal{S}_{\Omega,\omega_\epsilon}^{-1}[u|_{\partial\omega_\epsilon}]]$ satisfies

$$\begin{cases} -\Delta v_\epsilon = 0 \text{ in } \Omega\backslash\overline{\omega}_\epsilon, \\ \quad v_\epsilon = -u|_{\partial\omega_\epsilon} \text{ on } \partial\omega_\epsilon. \end{cases}$$

Hence, by uniqueness of the solution to (1.1), we find that $u_\epsilon = u + v_\epsilon$. □

2.3. Factorization and analyticity of a single layer potential representation of the perforated solution

In what follows, we consider the mapping $\tau_{x_0,\epsilon}$ which rescales its argument by a factor ϵ around x_0:

$$\tau_{x_0,\epsilon}(t) := x_0 + \epsilon t, \qquad t \in \mathbb{R}^d, \tag{2.14}$$

and we introduce the (pull-back) operator $\mathcal{P}_{x_0,\epsilon} : H^s(\partial\omega_\epsilon) \to H^s(\partial\omega)$ defined for any $s \in \mathbb{R}$ by:

$$\mathcal{P}_{x_0,\epsilon}[\phi] := \phi \circ \tau_{x_0,\epsilon} \text{ for any } \phi \in H^s(\partial\omega_\epsilon). \tag{2.15}$$

The operator $\mathcal{P}_{x_0,\epsilon}$ enables one to factorize $\mathcal{S}_{\Omega,\omega_\epsilon}$ in terms of an operator $\mathcal{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ which is defined on a space independent of ϵ, which is analytic in ϵ when $d \geq 3$, and in $\log\epsilon$ when $d = 2$.

Proposition 2.5. *The following factorizations holds:*

$$\mathcal{S}_{\Omega,\omega_\epsilon} = \epsilon\mathcal{P}_{x_0,\epsilon}^{-1}\mathcal{S}_\omega(\epsilon)\mathcal{P}_{x_0,\epsilon},$$

where $\mathcal{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ is the operator defined by

$$\mathcal{S}_\omega(\epsilon)[\phi](t) = \frac{1}{2\pi}\log\epsilon\int_{\partial\omega}\phi\,\mathrm{d}\sigma\,\delta_{d=2} + \mathcal{S}_\omega[\phi](t) + \epsilon^{d-2}\int_{\partial\omega}R_\Omega(x_0+\epsilon t, x_0+\epsilon t')\phi(t')\,\mathrm{d}\sigma(t'), \quad t \in \partial\omega. \tag{2.16}$$

Proof. For $\phi \in H^{-\frac{1}{2}}(\partial\omega)$ and $t \in \partial\omega$, we compute

$$\begin{aligned} \mathcal{S}_\omega(\epsilon)[\phi](t) &:= \mathcal{P}_{x_0,\epsilon}\mathcal{S}_{\Omega,\omega_\epsilon}\mathcal{P}_{x_0,\epsilon}^{-1}[\phi](t) = \mathcal{S}_{\Omega,\omega_\epsilon}[\phi\circ\tau_{x_0,\epsilon}^{-1}]\circ\tau_{x_0,\epsilon}(t) \\ &= \int_{\partial\omega_\epsilon} G_\Omega(x_0+\epsilon t, y)\phi\circ\tau_{x_0,\epsilon}^{-1}(y)\,\mathrm{d}\sigma(y) = \epsilon^{d-1}\int_{\partial\omega} G_\Omega(x_0+\epsilon t, x_0+\epsilon t')\phi(t')\,\mathrm{d}\sigma(t') \\ &= \epsilon^{d-1}\int_{\partial\omega}\Gamma(\epsilon(t-t'))\phi(t')\,\mathrm{d}\sigma(t') + \epsilon^{d-1}\int_{\partial\omega}R_\Omega(x_0+\epsilon t, x_0+\epsilon t')\phi(t')\,\mathrm{d}\sigma(t'). \end{aligned}$$

The identity (2.16) is obtained by using $\Gamma(\epsilon(t-t')) = \frac{1}{2\pi}\log\epsilon\delta_{d=2} + \epsilon^{2-d}\Gamma(t-t')$. □

In order to obtain full asymptotic expansions of the solution u_ϵ to (1.1), we rewrite (2.13) in terms of $\mathcal{S}_\omega(\epsilon)$:

$$u_\epsilon(x) = u(x) - \mathcal{S}_\omega(\epsilon)\Big[\mathcal{S}_\omega(\epsilon)^{-1}[u|_{\partial\omega_\epsilon}\circ\tau_{x_0,\epsilon}]\Big]\left(\frac{x-x_0}{\epsilon}\right). \tag{2.17}$$

In the next sections, we compute asymptotic expansions for the inverse of the operator $\mathcal{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\omega) \to H^{\frac{1}{2}}(\omega)$ which allow to infer full asymptotic expansions of u_ϵ based on this representation. Since the mathematical treatment when the dimension is greater than three, $d \geq 3$, is substantially different from the one when $d = 2$ (because of the logarithmic term in (2.16)), we present the analysis in two dedicated Sections 3 and 4.

3. Full asymptotic analysis of the perforated Poisson problem in dimension $d \geq 3$

This section is devoted to the full asymptotic analysis of the perforated Dirichlet problem (1.1) in the case where the dimension is at least three-dimensional: $d \geq 3$. Section 3.1 recalls some background material on Deny-Lions spaces and on the exterior Dirichlet problem in $\mathbb{R}^d \backslash \overline{\omega}$ in this context. The two-scale ansatz for the solution u_ϵ is then derived in Section 3.2 based on the analytic expression (2.16) of the single layer potential. The terms of the ansatz are identified as solutions of a fully determined cascade of interior and exterior problems in Section 3.3. Finally, an error estimate for the truncation of the ansatz at a finite order is proved in Section 3.4.

3.1. Preliminaries: Deny-Lions spaces and exterior Dirichlet problem in dimension $d \geq 3$

Our analysis relies on several fundamental results about the Deny-Lions spaces (also called Beppo-Levi, or also homogeneous Sobolev spaces) which are recalled in this section. For any $d \geq 2$, the Deny-Lions space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is defined as the completion of the space of compactly supported functions with respect to the L^2 norm of the gradient (see [29, 33]):

$$\mathcal{D}^{1,2}(\mathbb{R}^d) := \overline{\mathcal{C}_c^\infty(\mathbb{R}^d)}^{||\nabla\cdot||_{L^2(\mathbb{R}^d)}}. \tag{3.1}$$

When the space is at least three-dimensional, we have the following characterization of $\mathcal{D}^{1,2}(\mathbb{R}^d)$ (see [2]).

Proposition 3.1. *Assume $d \geq 3$. The following Poincaré inequality holds in the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$:*

$$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^d) \qquad \left|\left| \frac{v}{1+|x|} \right|\right|_{L^2(\mathbb{R}^d)} \leq C ||\nabla v||_{L^2(\mathbb{R}^d)}, \tag{3.2}$$

for some independent constant $C > 0$. Reciprocally, the space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ has the following characterization:

$$\mathcal{D}^{1,2}(\mathbb{R}^d) = \left\{ v \,\middle|\, \frac{v}{1+|x|} \in L^2(\mathbb{R}^d) \text{ and } \nabla v \in L^2(\mathbb{R}^d) \right\}, \tag{3.3}$$

and $||v||_{\mathcal{D}^{1,2}(\mathbb{R}^d)} := ||\nabla v||_{L^2(\mathbb{R}^d)}$ defines a norm on $\mathcal{D}^{1,2}(\mathbb{R}^d)$.

Remark 3.1. The condition $v/(1+|x|) \in L^2(\mathbb{R}^d)$ can be interpreted formally as $v(x) = o(|x|^{1-d/2})$ at infinity. In particular, constant functions do not belong to $\mathcal{D}^{1,2}(\mathbb{R}^d)$ for $d \geq 3$.

Similarly, we define $\mathcal{D}^{1,2}(\mathbb{R}^d \backslash \overline{\omega})$ for a bounded domain ω, and $\mathcal{D}_0^{1,2}(\mathbb{R}^d \backslash \partial\omega)$ the subspace of functions of $\mathcal{D}^{1,2}(\mathbb{R}^d)$ vanishing on $\partial\omega$:

$$\mathcal{D}_0^{1,2}(\mathbb{R}^d \backslash \partial\omega) := \{ v \in \mathcal{D}^{1,2}(\mathbb{R}^d) \,|\, v = 0 \text{ on } \partial\omega \},$$

where the trace makes sense due to the inclusion $\mathcal{D}^{1,2}(\mathbb{R}^d) \subset H^1_{\text{loc}}(\mathbb{R}^d)$. The space $\mathcal{D}^{1,2}(\mathbb{R}^d)$ is the natural space for the solutions to the exterior Laplace problem.

Proposition 3.2. *Let $g \in H^{\frac{1}{2}}(\partial D)$.*

(i) There exists a unique solution to the problem

$$\begin{cases} -\Delta v = 0 \text{ in } \mathbb{R}^d \backslash \partial\omega, \\ \quad v = g \text{ on } \partial\omega, \\ \quad v \in \mathcal{D}^{1,2}(\mathbb{R}^d). \end{cases} \tag{3.4}$$

(ii) The solution v can be represented as a single layer potential:

$$v = \mathcal{S}_\omega \left[\left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \right] \text{ in } \mathbb{R}^d, \tag{3.5}$$

where $[\![\frac{\partial v}{\partial n}]\!] = \frac{\partial v}{\partial n}\big|_+ - \frac{\partial v}{\partial n}\big|_-$ is the jump of the normal derivative of v across $\partial\omega$. Consequently, v has the following asymptotic expansion at infinity:

$$v(x) = \left(\int_{\partial\omega} \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] \mathrm{d}\sigma \right) \Gamma(x) + O\left(\frac{1}{|x|^{d-1}} \right) \text{ as } |x| \to +\infty. \tag{3.6}$$

More precisely, the following expansion holds for any x sufficiently far from ω:

$$v(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!} \nabla^k \Gamma(x) \cdot \int_{\partial\omega} \left[\!\left[\frac{\partial v}{\partial n} \right]\!\right] t^k \, \mathrm{d}\sigma(t). \tag{3.7}$$

Proof. (see also [33]) (i) The solution to (3.4) is given by $v := \widetilde{v} + \widetilde{f}$ where $\widetilde{f}$ is a lifting of the boundary condition (i.e. $\widetilde{f} \in \mathcal{D}^{1,2}(\mathbb{R}^d)$, vanishing outside a ball, and $\widetilde{f} = g$ on $\partial\omega$) and $\widetilde{v}$ is the unique Lax-Milgram solution to the variational problem

$$\text{find } \widetilde{v} \in \mathcal{D}_0^{1,2}(\mathbb{R}^d \backslash \partial\omega), \text{ such that } \forall v \in \mathcal{D}_0^{1,2}(\mathbb{R}^d \backslash \overline{\omega}), \int_{\mathbb{R}^d} \nabla v \cdot \nabla v \, \mathrm{d}x = \int_{\mathbb{R}^d} \Delta \widetilde{f} v \, \mathrm{d}x.$$

(ii) Using an integration by parts on (3.4), we can show that for any compactly supported function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} v\Delta\phi \,\mathrm{d}x = \int_{\partial\omega} \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] \phi \,\mathrm{d}\sigma, \tag{3.8}$$

i.e. $\Delta v = [\![\frac{\partial v}{\partial n}]\!]\,\mathrm{d}\sigma$ in the sense of distributions, where $\mathrm{d}\sigma$ is the surface measure of $\partial\omega$. Consider then the function $\hat{v}$ defined by

$$\hat{v}(x) := \left(\Gamma * \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right)(x) = \int_{\partial\omega} \Gamma(x-y) \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right](y)\,\mathrm{d}\sigma(y) = \mathcal{S}_\omega\left[\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right](x).$$

Due to (3.8) and the property of the fundamental solution, it holds $\Delta(v-\hat{v}) = 0$ in the sense of distributions, which implies that $v-\hat{v}$ is a harmonic function in $\mathbb{R}^d$ (see e.g. [22]). Furthermore, $v - \hat{v} \in \mathcal{D}^{1,2}(\mathbb{R}^d)$. Using some integration by part in $\mathbb{R}^d$, we find that $\nabla(v-\hat{v}) = 0$ in $\mathbb{R}^d$, which implies $v = \hat{v}$ and the representation formula (3.5) is proved. The asymptotic expansion and (3.7) follow from (2.5). □

Proposition 3.2 implies the existence of a function Φ solving the exterior problem

$$\begin{cases} -\Delta\Phi = 0 \text{ in } \mathbb{R}^d\backslash\overline{\omega}, \\ \quad \Phi = 1 \text{ on } \partial\omega, \\ \Phi(x) \to 0 \text{ as } |x| \to +\infty. \end{cases} \tag{3.9}$$

Such a function Φ vanishing at infinity does not exist in the two-dimensional setting (there may be either a constant or a logarithmic growth at infinity, see Section 3), which is one of the main differences with the case $d \geq 3$. We recall that the (harmonic) capacity of the set $\partial\omega$ is the positive number defined by

$$\mathrm{cap}(\omega) := -\int_{\partial\omega} \frac{\partial\Phi}{\partial n}\,\mathrm{d}\sigma = \int_{\mathbb{R}^d\backslash\overline{\omega}} |\nabla\Phi|^2\,\mathrm{d}x.$$

For the identification of the two-scale ansatz, we needs the following classical result [6], which can also be viewed as a consequence of the previous proposition.

Proposition 3.3. *The single layer potential $\mathcal{S}_\omega : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ is an invertible Fredholm operator for $d \geq 3$. Moreover, for any $\phi \in H^{-\frac{1}{2}}(\partial\omega)$, the function $\mathcal{S}_\omega[\phi]$ belongs to $\mathcal{D}^{1,2}(\mathbb{R}^d)$ and is the solution to an exterior Dirichlet problem of the form (3.4).*

3.2. Derivation of the two-scale ansatz based on the single layer potential representation

We now derive a two-scale ansatz for u_ϵ based on the representation (2.17) and the analytic expression (2.16). In this context where the dimension is greater than three, it is straightforward to obtain that the inverse of $\mathcal{S}_\omega(\epsilon)$ is analytic in ϵ.

Proposition 3.4. *Assume $d \geq 3$. Then $\mathcal{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ is an analytic operator in ϵ and we have further*

$$\mathcal{S}_\omega(\epsilon)^{-1} = \mathcal{S}_\omega^{-1} + O(\epsilon^{d-2}). \tag{3.10}$$

where $O(\epsilon^{d-2})$ is an analytic operators in ϵ estimated in operator norm.

Inserting the asymptotic formula (3.10) into (2.17), we read a two-scale ansatz for u_ϵ.

Corollary 3.1. *Assume $d \geq 3$. There exist functions $(v_p)_{p\geq 0}$ and $(w_p)_{p\geq d-2}$ such that the following ansatz holds for the solution u_ϵ to the perforated Laplace problem (1.1):*

$$u_\epsilon(x) = u(x) + \sum_{p=0}^{+\infty} \epsilon^p v_p\left(\frac{x-x_0}{\epsilon}\right) + \sum_{p=d-2}^{+\infty} \epsilon^p w_p(x), \qquad x \in \Omega\backslash\omega_\epsilon, \tag{3.11}$$

where:

(i) the series (3.11) converges for any fixed $x \in \Omega\backslash\{x_0\}$;
(ii) $v_p \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ is the solution of an exterior Dirichlet problem of the form (3.4) for any $p \geq 0$;
(iii) $w_p \in H^1(\Omega)$ for any $p \geq d-2$;
(iv) the series $\sum_{p=0}^{+\infty} \epsilon^p v_p$ is convergent in $\mathcal{D}^{1,2}(\mathbb{R}^d\backslash\overline{\omega})$;
(v) the series $\sum_{p=d-2}^{+\infty} \epsilon^p w_p$ is convergent in $H^1(\Omega)$;
(vi) the first term of the series is given by $v_0 = -u(x_0)\mathcal{S}_\omega[\mathcal{S}_\omega^{-1}[1_{\partial\omega}]] = -u(x_0)\Phi$ with Φ being the solution to the exterior problem (3.9).

Proof. We use the representation (2.17). Since $u_{|\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon} = u(x_0)1_{\partial\omega} + O(\epsilon)$, we find by using (3.10):

$$\mathcal{S}_\omega(\epsilon)^{-1}[u|_{\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon}] = u(x_0)\mathcal{S}_\omega^{-1}[1_{\partial\omega}] + O(\epsilon),$$

where $O(\epsilon)$ is analytic in ϵ. Noticing that (2.16) can be rewritten as

$$\mathcal{S}_\omega(\epsilon)[\phi]\left(\frac{x-x_0}{\epsilon}\right)=\mathcal{S}_\omega[\phi]\left(\frac{x-x_0}{\epsilon}\right)+\epsilon^{d-2}\int_{\partial\omega}R_\Omega(x,x_0+\epsilon t')\phi(t')\,\mathrm{d}\sigma(t'),$$

we obtain the ansatz (3.11) by inserting $\phi=\mathcal{S}_\omega(\epsilon)^{-1}[u_{|\partial\omega_\epsilon}\circ\tau_{x_0,\epsilon}]$, using a Taylor series and by identifying powers of ϵ. The convergence of the series results from the convergence of the far field expansion (2.5) for the fundamental solution. The properties (ii)-(v) are then easily verified. The property (vi) is obtained by computing explicitly the leading order asymptotic:

$$u_\epsilon(x)=u(x)-u(x_0)\mathcal{S}_\omega[\mathcal{S}_\omega^{-1}[1_{\partial\omega}]]\left(\frac{x-x_0}{\epsilon}\right)+O(\epsilon)=u(x)-u(x_0)\Phi\left(\frac{x-x_0}{\epsilon}\right)+O(\epsilon),$$

from where the value of v_0 is inferred. □

It would be feasible, in principle, to explicit the functions v_p and w_p from asymptotic series based on (2.17) and (3.10). However, such computations would involve Cauchy products and Neumann series which would be difficult to interpret directly. In the next subsection, we provide a comprehensive characterization of the functions v_p and w_p by identifying them as the solutions to some exterior and interior Laplace problems in $\mathbb{R}^d\backslash\overline{\omega}$ and Ω respectively.

3.3. The recursive system for computing the corrector functions

Inserting the ansatz (3.11) into the original perforated Laplace problem (1.1) and identifying identical powers of ϵ, the following proposition shows that the functions $(v_p)_{p\geq 0}$ and $(w_p)_{p\geq d-2}$ are sequences of correctors, correcting successive errors on the boundary and in the vicinity of the hole committed by the previous correctors.

Proposition 3.5. *The functions $(v_p)_{p\geq 0}$ and $(w_p)_{p\geq d-2}$ of (3.11) are uniquely characterized by the following recursive system of exterior and interior problems:*

$$\begin{cases}-\Delta v_p=0 & \text{in } \mathbb{R}^d\backslash\overline{\omega},\ p\geq 0\\ v_p(t)=-\dfrac{1}{p!}\nabla^p u(x_0)\cdot t^p & \text{for } t\in\partial\omega,\ 0\leq p<d-2,\\ v_p(t)=-\dfrac{1}{p!}\nabla^p u(x_0)\cdot t^p-\displaystyle\sum_{k=0}^{p-d+2}\frac{1}{k!}\nabla^k w_{p-k}(x_0)\cdot t^k & \text{for } t\in\partial\omega,\ p\geq d-2,\\ v_p(x)=O(|x|^{2-d}) & \text{as } |x|\to+\infty,\ p\geq 0,\end{cases}\tag{3.12}$$

and

$$\begin{cases}-\Delta w_p=0 & \text{in } \Omega,\\ w_p(x)=-\displaystyle\sum_{k=d-2}^{p}v_{p-k}^{(k)}(x) & \text{for } x\in\partial\Omega,\end{cases}\qquad \text{for all } p\geq d-2,\tag{3.13}$$

where for any $p\geq 0$ and $k\geq d-2$, $v_p^{(k)}$ are the functions occuring in the far field expansion of v_p, i.e.:

$$v_p\left(\frac{x-x_0}{\epsilon}\right)=\sum_{k=d-2}^{+\infty}\epsilon^k v_p^{(k)}(x)\tag{3.14}$$

with

$$v_p^{(k)}(x):=\frac{(-1)^{k-d+2}}{(k-d+2)!}\nabla^{k-d+2}\Gamma(x-x_0)\cdot\int_{\partial\omega}\left[\!\!\left[\frac{\partial v_p}{\partial n}\right]\!\!\right]t^{k-d+2}\,\mathrm{d}\sigma(t),\qquad x\in\partial\Omega,k\geq d-2.\tag{3.15}$$

Proof. Inserting $x=x_0+\epsilon t$ with $t\in\partial\omega$ in the ansatz (3.11), using the boundary condition satisfied by $u_\epsilon-u$ and some Taylor expansions in the vicinity of x_0, we obtain:

$$\begin{aligned}-\sum_{p=0}^{+\infty}\epsilon^p\frac{1}{p!}\nabla^p u(x_0)\cdot t^p=u_\epsilon(x_0+\epsilon t)-u(x_0+\epsilon t)&=\sum_{p=0}^{+\infty}\epsilon^p v_p(t)+\sum_{p=d-2}^{+\infty}\sum_{k=0}^{+\infty}\epsilon^{p+k}\frac{1}{k!}\nabla^k w_p(x_0)\cdot t^k\\&=\sum_{p=0}^{+\infty}\epsilon^p v_p(t)+\sum_{p=d-2}^{+\infty}\sum_{k=0}^{p-d+2}\epsilon^p\frac{1}{k!}\nabla^k w_{p-k}(x_0)\cdot t^k.\end{aligned}$$

Identifying identical powers of ϵ yields the system (3.12). Then, considering $x\in\partial\Omega$, we find by using the far field expansion (3.15):

$$0=u_\epsilon(x)-u(x)=\sum_{p=0}^{+\infty}\sum_{k=d-2}^{+\infty}\epsilon^{p+k}v_p^{(k)}(x)+\sum_{p=d-2}^{+\infty}\epsilon^p w_p(x)=\sum_{p=d-2}\sum_{k=d-2}^{p}\epsilon^p v_{p-k}^{(k)}(x)+\sum_{p=d-2}^{+\infty}\epsilon^p w_p(x).$$

Hence, (3.13) follows by identifying identical powers of ϵ. □

Remark 3.2. The system (3.12) and (3.13) determines completely the functions $(v_p)_{p\geq 0}$ and $(w_p)_{p\geq d-2}$. Indeed, the functions $(v_{p'})_{0\leq p'<d-2}$ are determined from (3.12). Then if $(v_{p'})_{0\leq p'\leq p-d+2}$ is determined for $p\geq d-2$, then w_p is determined from the boundary value problem (3.13), which determines in turn v_p through the exterior problem (3.12).

3.4. Quantitative error estimates for the truncated ansatz

The last step of our analysis is to provide error bounds in the space $H^1(\Omega\backslash\omega_\epsilon)$ for the truncation of the ansatz (3.11) determined from the cascade of equations (3.12) and (3.13). The main tool is the following norm estimate for the solution to a Poisson problem in the perforated cell.

Proposition 3.6. *Let $g\in H^{\frac{1}{2}}(\partial\omega_\epsilon)$ and $h\in H^{\frac{1}{2}}(\partial\Omega)$. Let $v_\epsilon\in H^1(\Omega\backslash\omega_\epsilon)$ be the solution to the boundary value problem*

$$\begin{cases}-\Delta v_\epsilon = 0 \text{ in } \Omega\backslash\overline{\omega}_\epsilon,\\ \quad v_\epsilon = g \text{ on } \partial\omega_\epsilon,\\ \quad v_\epsilon = h \text{ on } \partial\Omega.\end{cases}\tag{3.16}$$

There exists a constant $C>0$ independent of g, h and ϵ such that

$$||v_\epsilon||_{L^2(\Omega\backslash\overline{\omega}_\epsilon)}+||\nabla v_\epsilon||_{L^2(\Omega\backslash\overline{\omega}_\epsilon)}\leq C(||g||_{H^{\frac{1}{2}}(\partial\omega_\epsilon)}+||h||_{H^{\frac{1}{2}}(\partial\omega_\epsilon)}).$$

Proof. This is a classical result which can be proved by following the lines of Proposition 5.6 of the appendix. The key point is the Poincaré inequality $||v||_{L^2(\partial\Omega)}\leq C||\nabla v||_{L^2(\partial\Omega)}$ with a constant C independent of ϵ, which is valid for any $v\in H^1(\Omega\backslash\overline{\omega}_\epsilon)$ satisfying $v=0$ on $\partial\Omega$. □

Proposition 3.7. *For any $N\in\mathbb{N}$, let u_ϵ^N be the truncated ansatz*

$$u_\epsilon^N(x):=u(x)+\sum_{p=0}^N\epsilon^p v_p\left(\frac{x-x_0}{\epsilon}\right)+\sum_{p=d-2}^N\epsilon^p w_p(x),\quad x\in\Omega\backslash\overline{\omega}_\epsilon,$$

where the functions $(v_p)_{p\geq 0}$ and $(w_p)_{p\geq d-2}$ are defined from (3.12) and (3.13), and where second sum is null by convention if $N<d-2$. Then u_ϵ^N is an approximation of u_ϵ of order $O(\epsilon^{N+1})$ in the $H^1(\Omega\backslash\omega_\epsilon)$ norm in view of the following error bound:

$$||u_\epsilon-u_\epsilon^N||_{L^2(\Omega\backslash\omega_\epsilon)}+||\nabla u_\epsilon-\nabla u_\epsilon^N||_{L^2(\Omega\backslash\overline{\omega}_\epsilon)}\leq C_N\epsilon^{N+1},$$

for a constant C_N independent of ϵ (but which may depend on N).

Proof. The function $r_\epsilon^N:=u_\epsilon-u_\epsilon^N$ satisfies $-\Delta r_\epsilon^N=0$. Furthermore, in view of the definitions of v_p and w_p: it holds with $t\in\partial\omega$:

$$\begin{aligned}r_\epsilon(x_0+\epsilon t)&=u_\epsilon(x_0+\epsilon t)-u(x_0+\epsilon t)-\sum_{p=0}^N\epsilon^p v_p(t)-\sum_{p=d-2}^N\epsilon^p w_p(x_0+\epsilon t)\\&=-\sum_{p=0}^N\epsilon^p\frac{1}{p!}\nabla^p u(x_0)\cdot t^p-\sum_{p=0}^N\epsilon^p v_p(t)-\sum_{p=d-2}^N\sum_{k=0}^{N-p}\epsilon^{p+k}\frac{1}{k!}\nabla^k w_p(x_0)\cdot t^k+O(\epsilon^{N+1})\\&=\sum_{p=d-2}^N\sum_{k=0}^{p-d+2}\epsilon^p\frac{1}{k!}\nabla^k w_{p-k}(x_0)\cdot t^k-\sum_{k=0}^{N-d+2}\sum_{p=d-2+k}^N\epsilon^p\frac{1}{k!}\nabla^k w_{p-k}(x_0)\cdot t^k+O(\epsilon^{N+1})=O(\epsilon^{N+1}).\end{aligned}$$

Similarly, we find for $x\in\partial\Omega$:

$$\begin{aligned}r_\epsilon(x)&=u_\epsilon(x)-u(x)-\sum_{p=0}^N\epsilon^p v_p\left(\frac{x-x_0}{\epsilon}\right)-\sum_{p=d-2}^N w_p(x)\\&=-\sum_{p=0}^N\sum_{k=d-2}^{N-p}\epsilon^{p+k}v_p^{(k)}(x)+\sum_{p=d-2}^N\sum_{k=d-2}^{p}\epsilon^p v_{p-k}^{(k)}(x)+O(\epsilon^{N+1})=O(\epsilon^{N+1}).\end{aligned}$$

The result follows from Proposition 3.6. □

4. Asymptotic analysis of the perforated problem in dimension $d=2$

This section is devoted to the full asymptotic analysis of (1.1) in the more delicate case $d=2$. The main result is the two-scale expansion (4.22) (or equivalently, (4.25)), which is original and which is completely characterized in Proposition 4.6. This section follows the same structure as the previous one: the definition and the main properties of exterior Laplace solutions in dimension 2 are recalled in Section 4.1. We introduce an important (and classical) auxiliary function Φ vanishing on the hole ω and growing logarithmically at infinity. This enables, in Section 4.2, to compute full asymptotic expansions of the inverse of the operator $\mathcal{S}_\omega(\epsilon)$ arising in the representation formula (2.17), and then to obtain a complete power series expansion of $u_\epsilon(x)$. Finally, we characterize the terms of the series as the solutions to a cascade of explicit recursive interior and exterior problems in Section 4.3, before stating a quantitative error estimate in the $H^1(\Omega\backslash\omega_\epsilon)$–norm for the truncation of the ansatz in Section 4.4.

4.1. **Preliminaries: Deny-Lions space and the exterior Dirichlet problem in dimension $d = 2$**

In dimension 2, the Deny-Lions or Beppo-Levi space $\mathcal{D}^{1,2}(\mathbb{R}^2)$ defined by (3.1) has a significantly different characterization than in dimensions $d \geq 3$.

Proposition 4.1. *Assume $d = 2$. Then the following Poincaré inequality holds in $\mathcal{D}^{1,2}(\mathbb{R}^2)$:*

$$\forall v \in \mathcal{D}^{1,2}(\mathbb{R}^2), \qquad \left|\left|\frac{v}{(|x|+1)\log(|x|+2)}\right|\right|_{L^2(\mathbb{R}^2)} \leq C(||\nabla v||_{L^2(\mathbb{R}^2)} + ||v||_{L^2(B(0,1))}), \tag{4.1}$$

where $B(0,1)$ is the unit ball of $\mathbb{R}^2$ and for some constant $C > 0$. Reciprocally, $\mathcal{D}^{1,2}(\mathbb{R}^2)$ has the following characterization:

$$\mathcal{D}^{1,2}(\mathbb{R}^2) = \left\{ v \,\middle|\, \frac{v}{(|x|+1)\log(|x|+2)} \in L^2(\mathbb{R}^2) \text{ and } \nabla v \in L^2(\mathbb{R}^2) \right\}, \tag{4.2}$$

and

$$||v||_{\mathcal{D}^{1,2}(\mathbb{R}^2)} := \left(\left|\left|\frac{v}{(|x|+1)\log(|x|+2)}\right|\right|^2_{L^2(\mathbb{R}^2)} + ||\nabla v||^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}}, \tag{4.3}$$

or

$$||v||_{\mathcal{D}^{1,2}(\mathbb{R}^2)} := \left(||v||^2_{L^2(B(0,1))} + ||\nabla v||^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}}, \tag{4.4}$$

define two equivalent norms on $\mathcal{D}^{1,2}(\mathbb{R}^2)$.

Remark 4.1. One of the main differences between the cases $d = 2$ and $d \geq 3$ lies in the fact that $\mathcal{D}^{1,2}(\mathbb{R}^2)$ contains constant functions, but not $\mathcal{D}^{1,2}(\mathbb{R}^d)$ with $d \geq 3$. The condition $v/((|x|+1)\log(|x|+2))$ can be interpreted as $v(x) = o(\log(|x|))$. In particular and in contrast with $d \geq 3$, the fundamental solution does not belong to $\mathcal{D}^{1,2}(\mathbb{R}^2)$: $\Gamma \notin \mathcal{D}^{1,2}(\mathbb{R}^2)$.

The next proposition states the main properties of the solutions to the exterior Laplace problem with a Dirichlet boundary condition on the boundary of the set ω.

Proposition 4.2. *Let $g \in H^{\frac{1}{2}}(\partial\omega)$.*

(i) There exists a unique solution $v \in \mathcal{D}^{1,2}(\mathbb{R}^2)$ to the problem

$$\begin{cases} -\Delta v = 0 \text{ in } \mathbb{R}^2\backslash\partial\omega, \\ \quad v = g \text{ on } \partial\omega, \\ \quad v \in \mathcal{D}^{1,2}(\mathbb{R}^2). \end{cases} \tag{4.5}$$

(ii) The outer normal flux (as well as the inner normal flux) of v vanishes:

$$\int_{\partial\omega} \left.\frac{\partial v}{\partial n}\right|_+ \mathrm{d}\sigma = 0. \tag{4.6}$$

(iii) There exists a constant v^∞ such that the solution v admits the following single layer potential representation:

$$v = v^\infty + \mathcal{S}_\omega\left[\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right] \qquad \text{in } \mathbb{R}^2. \tag{4.7}$$

Consequently, the asymptotic behavior of v at infinity reads

$$v(x) = v^\infty + O(|x|^{-1}) \text{ as } |x| \to +\infty.$$

More precisely, the following expansion holds for any $x \in \mathbb{R}^2$ sufficiently far away from ω:

$$v(x) = v^\infty + \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} \nabla^k\Gamma(x) \cdot \int_{\partial\omega} \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] t^k \,\mathrm{d}\sigma(t).$$

Proof. (i) Existence and uniqueness of a solution to (4.5) is obtained as in the case $d \geq 3$: we use the Poincaré inequality which enables to write $||v||_{B(0,R)} \leq C||\nabla v||_{B(0,R)}$ for v satisfying $v = 0$ on $\partial\omega$ with R large enough to contain ω, and then using the definition (4.4) of the norm with $B(0,1)$ replaced with $B(0,R)$.
(ii) Integrating $-\Delta v = 0$ against the constant test function 1 in the whole set $\mathbb{R}^2\backslash\overline{\omega}$ implies (4.6) (this is possible because $1 \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{\omega})$).
(iii) Consider the function $\widetilde{v}$ defined by the single layer potential

$$\widetilde{v}(x) := \mathcal{S}_\omega\left[\left[\!\left[\frac{\partial v}{\partial n}\right]\!\right]\right](x) = \int_{\partial D} \Gamma(x-y) \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right](y) \,\mathrm{d}\sigma(y).$$

The function $\widetilde{v}$ satisfies $\Delta \widetilde{v} = \Delta v = \left[\!\left[\frac{\partial v}{\partial n}\right]\!\right] \mathrm{d}\sigma$ in the distributional sense. This implies that $v - \widetilde{v}$ is a harmonic function in $\mathbb{R}^2$. Since $v - \widetilde{v} \in \mathcal{D}^{1,2}(\mathbb{R}^2)$, an integration by parts in $\mathbb{R}^2$ yields $\nabla(v - \widetilde{v}) = 0$, which means that $v - \widetilde{v}$ is a constant, which we denote by v^∞. The asymptotic behavior follows from (4.6). □

We now introduce an important auxiliary function Φ, which solves an exterior boundary value problem in $\mathbb{R}^2\backslash\overline{\omega}$, but which does not belong to $\mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{\omega})$ due to a logarithmic growth.

Proposition 4.3. *(i) There exists a unique solution Φ to the problem*

$$\begin{cases} -\Delta \Phi = 0 \text{ in } \mathbb{R}^2\backslash\overline{\omega}, \\ \qquad \Phi = 0 \text{ on } \partial\omega, \\ \Phi(x) \sim \dfrac{1}{2\pi} \log|x| \text{ as } |x| \to +\infty, \end{cases} \tag{4.8}$$

satisfying $\Phi - \Gamma \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{\omega})$.

(ii) There exists a constant Φ^∞ such that Φ admits the following single layer potential representation:

$$\Phi(x) = \mathcal{S}_\omega\left[\frac{\partial \Phi}{\partial n}\bigg|_+\right](x) + \Phi^\infty, \qquad x \in \mathbb{R}^2\backslash\overline{\omega}. \tag{4.9}$$

Consequently, we have the asymptotic expansion

$$\Phi(x) = \frac{1}{2\pi}\log|x| + \Phi^\infty + O(|x|^{-1}). \tag{4.10}$$

(iii) Independently of the shape of the obstacle ω, the normal flux of Φ is equal to one:

$$\int_{\partial\omega} \frac{\partial \Phi}{\partial n}\bigg|_+ \mathrm{d}\sigma = 1. \tag{4.11}$$

Proof. (i) Using the fact that $\Delta \log|x| = 0$ in $\mathbb{R}^2\backslash\{0\}$, the solution Φ is given by $\Phi(x) = \frac{1}{2\pi}\log|x| + \Psi(x)$ where Ψ is the unique solution in $\mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{\omega})$ to the difference problem

$$\begin{cases} -\Delta \Psi = 0 \text{ in } \mathbb{R}^2\backslash\overline{\omega}, \\ \qquad \Psi = -\dfrac{1}{2\pi}\log|x| \text{ on } \partial\omega, \\ \qquad \Psi = \Psi^\infty + O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases}$$

(ii) The reasoning is the same as in point (iii) of Proposition 4.2, noticing that $\frac{\partial \Phi}{\partial n}\big|_- = 0$.
(iii) This result follows from (4.6) and Lemma 2.1 with $k = 0$. □

Remark 4.2. If ω is the unit disk, then the constant v^∞ arising in the far field asymptotic (4.7) is given by $v^\infty = \frac{1}{2\pi}\int_{\partial\omega} g \,\mathrm{d}\sigma$. Indeed, this is can be found by writing the solution to (4.5) explicitly in polar coordinates as

$$v(r,\theta) = \sum_{k\in\mathbb{Z}} r^{-|k|} g_k e^{\mathrm{i}k\theta} \text{ where } g_k := \frac{1}{2\pi}\int_0^{2\pi} g(t) e^{-\mathrm{i}kt}\,\mathrm{d}t.$$

This property is used crucially by Kozlov et. al. [28, section 1.4] for the asymptotic analysis of (1.1) with circular hole ω_ϵ. However, it does not hold for arbitrarily shaped obstacles. In fact, in the general case, integrating the boundary condition of (4.5) against $\frac{\partial \Phi}{\partial n}\big|_+$ yields

$$v^\infty = \int_{\partial\omega} g \frac{\partial \Phi}{\partial n}\bigg|_+ \mathrm{d}\sigma + \Phi^\infty \int_{\partial\omega} g \,\mathrm{d}\sigma. \tag{4.12}$$

4.2. Identification of the two-scale ansatz based on a single layer potential representation

We have now all the material to derive the two-scale ansatz based on the integral representation (2.17). As in the previous section, the main task is to determine the asymptotic expansion of the inverse $\mathcal{S}_\omega(\epsilon)^{-1}$. The identity (2.16) can be rewritten as

$$\mathcal{S}_\omega(\epsilon)[\phi](t) = \left(\frac{1}{2\pi}\log\epsilon + R_\Omega(x_0,x_0)\right)\int_{\partial\omega} \phi \,\mathrm{d}\sigma + \mathcal{S}_\omega[\phi] + O(\epsilon), \tag{4.13}$$

where $O(\epsilon)$ is analytic in ϵ. In order to compute the inverse of (4.13), we first need to invert the zero-th order part $\widetilde{\mathcal{S}}_\omega(\epsilon)$ defined by

$$\widetilde{\mathcal{S}}_\omega(\epsilon)[\phi] := \left(\frac{1}{2\pi}\log\epsilon + R_\Omega(x_0,x_0)\right)\int_{\partial\omega} \phi \,\mathrm{d}\sigma + \mathcal{S}_\omega[\phi]. \tag{4.14}$$

This operator is invertible despite the fact that $\mathcal{S}_\omega$ might not be invertible in two dimensions [8]. This point is established in the following proposition; it relies on Proposition 4.2 and the definition (4.8) of the auxiliary function Φ.

Proposition 4.4. *For $\epsilon > 0$ sufficiently small, the operator $\widetilde{S}_\omega(\epsilon) : H^{-\frac{1}{2}}(\partial\omega) \to H^{\frac{1}{2}}(\partial\omega)$ defined by (4.14) is invertible and its inverse reads explicitly*

$$(\widetilde{\mathcal{S}}_\omega(\epsilon))^{-1}[f] = \frac{v_f^\infty}{\frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0)} \left.\frac{\partial\Phi}{\partial n}\right|_+ + \left[\!\left[\frac{\partial v_f}{\partial n}\right]\!\right], \qquad \forall f \in H^{\frac{1}{2}}(\partial\omega), \tag{4.15}$$

where v_f is the unique solution to the problem

$$\begin{cases} \Delta v_f = 0 \text{ in } \mathbb{R}^2\backslash\overline{\omega}, \\ v_f = f \text{ on } \partial\omega, \\ v_f \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{\omega}), \end{cases} \tag{4.16}$$

and where v_f^∞ is the constant such that $v_f = v_f^\infty + O(|x|^{-1})$ as $|x| \to +\infty$. In particular, there exists a constant $C > 0$ independent of ϵ such that the operator norm of $\widetilde{\mathcal{S}}_\omega(\epsilon)^{-1}$ satisfies

$$|||(\widetilde{\mathcal{S}}_\omega(\epsilon))^{-1}|||_{H^{-\frac{1}{2}}(\partial\omega)\to H^{\frac{1}{2}}(\partial\omega)} \leq C.$$

Proof. Let us consider a right hand-side $f \in H^{\frac{1}{2}}(\partial\omega)$ and let us solve the equation

$$\widetilde{\mathcal{S}}_\omega(\epsilon)[\phi] = \left(\frac{1}{2\pi}\log\epsilon + R_\Omega(x_0,x_0)\right)\int_{\partial\omega}\phi\,\mathrm{d}\sigma + \mathcal{S}_\omega[\phi] = f. \tag{4.17}$$

The function $v := \mathcal{S}_\omega[\phi]$ has the following asymptotic behavior at infinity:

$$v(x) = \frac{1}{2\pi}\log|x|\int_{\partial\omega}\phi\,\mathrm{d}\sigma + O\left(\frac{1}{|x|}\right).$$

By the definition (4.17) of Φ, it follows that the function $w := v - \left(\int_{\partial\omega}\phi\,\mathrm{d}\sigma\right)\Phi$ solves the following exterior problem

$$\begin{cases} \Delta w = 0 \text{ in } \mathbb{R}^3\backslash\partial\omega, \\ w = f - \left(\frac{1}{2\pi}\log\epsilon + R_\Omega(x_0,x_0)\right)\int_{\partial\omega}\phi\,\mathrm{d}\sigma \text{ on } \partial\omega, \\ w = -\left(\int_{\partial\omega}\phi\,\mathrm{d}\sigma\right)\Phi^\infty + O\left(\frac{1}{|x|}\right) \text{ as } |x| \to +\infty. \end{cases}$$

Clearly, $w \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{\omega})$ and by uniqueness of the solution to a problem of the form (4.5), it must hold

$$w = v_f - \left(\frac{1}{2\pi}\log\epsilon + R_\Omega(x_0,x_0)\right)\int_{\partial\omega}\phi\,\mathrm{d}\sigma. \tag{4.18}$$

Then, from the fact that $w \to -\int_{\partial\omega}\phi\,\mathrm{d}\sigma\Phi^\infty$ as $|x| \to +\infty$, we deduce that

$$v_f^\infty - \left(\frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0)\right)\int_{\partial\omega}\phi\,\mathrm{d}\sigma = 0,$$

from where we obtain

$$\int_{\partial\omega}\phi\,\mathrm{d}\sigma = \frac{v_f^\infty}{\frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0)}.$$

Coming back to (4.18), we deduce

$$\mathcal{S}_\omega[\phi] = \frac{v_f^\infty}{\frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0)}\Phi + v_f - \frac{\frac{1}{2\pi}\log\epsilon + R_\Omega(x_0,x_0)}{\frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0)}v_f^\infty$$

The final expression (4.15) comes from the jump relation $\phi = [\![\frac{\partial\mathcal{S}_\omega[\phi]}{\partial n}]\!]$. ☐

In all what follows, we denote by a_ϵ the quantity

$$a_\epsilon := \frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0). \tag{4.19}$$

We infer the full asymptotic expansion of the inverse of $\mathcal{S}_\omega(\epsilon)$.

Corollary 4.1. *Assume $d = 2$. For any $f \in H^{\frac{1}{2}}(\partial\omega)$, there exists a family of functions $(v_{p,q})_{p\geq 0, 0\leq q\leq p}$ solutions to exterior Dirichlet problems of the form (4.16) such that the inverse $\mathcal{S}_\omega(\epsilon)^{-1}[f]$ of (4.13) admits the following series expansion as $\epsilon \to 0$:*

$$\mathcal{S}_\omega(\epsilon)^{-1}[f] = \sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}\left(\frac{v_{p,q}^\infty}{a_\epsilon}\left.\frac{\partial\Phi}{\partial n}\right|_+ + \left[\!\left[\frac{\partial v_{p,q}}{\partial n}\right]\!\right]\right). \tag{4.20}$$

Proof. Equation (4.13) can be rewritten as

$$\mathcal{S}_\omega(\epsilon) = \widetilde{\mathcal{S}}_\omega(\epsilon) + O(\epsilon) = \widetilde{\mathcal{S}}_\omega(\epsilon)(I + \widetilde{\mathcal{S}}_\omega(\epsilon)^{-1}O(\epsilon)),$$

Using a Neumann series, we obtain

$$\mathcal{S}_\omega(\epsilon)^{-1}[f] = \sum_{p=0}^{+\infty} \widetilde{\mathcal{S}}_\omega(\epsilon)^{-1}(O(\epsilon)^p \widetilde{\mathcal{S}}_\omega(\epsilon)^{-1})[f], \tag{4.21}$$

where $O(\epsilon)$ is an analytic operator valued function. Then, in view of the formula (4.15), there exists some functions $f_{p,q} \in H^{\frac{1}{2}}(\partial\omega)$ such that

$$\sum_{p=0}^{+\infty}(O(\epsilon)^p \widetilde{\mathcal{S}}_\omega(\epsilon)^{-1})[f] = \sum_{p=0}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} f_{p,q}.$$

Indeed, the factor a_ϵ appears with an exponent q in the term $(\widetilde{\mathcal{S}}_\omega(\epsilon)^{-1}O(\epsilon))^p[f]$ only if $p \geq q$. The result follows with $v_{p,q}$ being the solution to (4.16) with boundary datum $f_{p,q}$. □

Inserting (4.20) with $f := u|_{\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon}$ into (2.17), we obtain the following proposition.

Proposition 4.5. *Assume $d = 2$. There exist functions $(v_{p,q})_{p\geq 0, 0\leq q\leq p}$ and $(w_{p,q})_{p\geq 1, 0\leq q \leq p}$ such that the following ansatz holds for the solution u_ϵ to the perforated Laplace problem (1.1):*

$$\begin{aligned} u_\epsilon(x) = u(x) &+ \left(\sum_{p=0}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q}\frac{v_{p,q}^\infty}{a_\epsilon}\right)\left(\frac{1}{2\pi}\log\epsilon + R_\Omega(x,x_0) + \Phi\left(\frac{x-x_0}{\epsilon}\right) - \Phi^\infty\right) \\ &+ \sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}\left(v_{p,q}\left(\frac{x-x_0}{\epsilon}\right) - v_{p,q}^\infty\right) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q} w_{p,q}(x), \end{aligned} \tag{4.22}$$

where:

- *(i) the series (4.22) converges for any fixed $x \in \Omega\backslash\{x_0\}$;*
- *(ii) R_Ω is the solution of the difference problem (2.9);*
- *(iii) Φ is the auxiliary function with logarithmic growth at infinity defined by (4.8);*
- *(iv) $a_\epsilon := \frac{1}{2\pi}\log\epsilon + R_\Omega(x_0,x_0) - \Phi^\infty$;*
- *(v) for any $p \geq 0$ and $0 \leq q \leq p$, $v_{p,q} \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{\omega})$ is a function solving a boundary value problem of the form (4.16) which satisfies $v_{p,q}(x) = v_{p,q}^\infty + O\left(\frac{1}{|x|}\right)$ as $|x| \to +\infty$;*
- *(vi) $w_{p,q} \in H^1(\Omega)$ for any $p \geq 1$ and $0 \leq q \leq p$;*
- *(vii) the function $v_{0,0}$ is constant and equal to $v_{0,0}(x) = -u(x_0)$.*

Proof. Coming back to the representation (2.13), equation (2.16) can be rewritten

$$u_\epsilon(x) = u(x) - \frac{1}{2\pi}\log\epsilon\int_{\partial\omega}\phi_\epsilon\,\mathrm{d}\sigma - \mathcal{S}_\omega[\phi_\epsilon]\left(\frac{x-x_0}{\epsilon}\right) - \int_{\partial\omega} R_\Omega(x, x_0+\epsilon t')\phi_\epsilon(t')\,\mathrm{d}\sigma(t'), \tag{4.23}$$

where $\phi_\epsilon := \mathcal{S}_\omega(\epsilon)^{-1}[u_{|\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon}]$. By using a Taylor expansion, it is clear that the function $u_{|\partial\omega_\epsilon} \circ \tau_{x_0,\epsilon}$ is analytic in ϵ. Then, the representation (4.20) yields the existence of functions $(v_{p,q})_{p\geq 0, 0\leq q\leq p}$ satisfying (v) such that

$$\phi_\epsilon = -\sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}\left(\frac{v_{p,q}^\infty}{a_\epsilon}\left.\frac{\partial\Phi}{\partial n}\right|_+ + \left[\!\left[\frac{\partial v_{p,q}}{\partial n}\right]\!\right]\right). \tag{4.24}$$

The ansatz (4.22) follows by substituting (4.24) into (4.23), remarking that

$$\mathcal{S}_\omega\left[\left.\frac{\partial\Phi}{\partial n}\right|_+\right](x) = \Phi(x) - \Phi^\infty, \quad \mathcal{S}_\omega\left[\left[\!\left[\frac{\partial v_{p,q}}{\partial n}\right]\!\right]\right](x) = v_{p,q}(x) - v_{p,q^\infty},$$

and by using a Taylor expansion for $R_\Omega(x, x_0+\epsilon t')$ so as to identify the functions $w_{p,q}$.
Finally, we identify the leading order asymptotic. From (4.21), we find

$$\phi_\epsilon = u(x_0)\widetilde{\mathcal{S}}_\omega(\epsilon)^{-1}[1_{\partial\omega}] + O(\epsilon) = u(x_0)\frac{1}{\frac{1}{2\pi}\log\epsilon - \Phi^\infty + R_\Omega(x_0,x_0)}\left.\frac{\partial\Phi}{\partial n}\right|_+ + O(\epsilon),$$

which yields

$$u_\epsilon(x) = u(x) - \frac{u(x_0)}{a_\epsilon}\left[\frac{1}{2\pi}\log\epsilon + R_\Omega(x,x_0) + \Phi\left(\frac{x-x_0}{\epsilon}\right) - \Phi^\infty\right] + O(\epsilon),$$

from where we obtain $v_{0,0}(x) = -u(x_0)$. □

Remark 4.3. The first sum of the second line of (4.22) featuring $v_{p,q}((x-x_0)/\epsilon) - v_{p,q}^\infty$ starts at the rank $p = 1$ because $v_{0,0} = -u(x_0) = v_{p,q}^\infty$ is a constant.

Remark 4.4. Grouping the constants $v_{p,q}^{\infty}$, the ansatz (4.22) can also be written

$$\begin{aligned}u_\epsilon(x) = u(x) + &\left(\frac{1}{a_\epsilon}\sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}v_{p,q}^{\infty}\right)\left(\Phi\left(\frac{x-x_0}{\epsilon}\right) + R_\Omega(x,x_0) - R_\Omega(x_0,x_0)\right)\\ &+\sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}v_{p,q}\left(\frac{x-x_0}{\epsilon}\right) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}w_{p,q}(x).\end{aligned}\tag{4.25}$$

The form (4.22) makes visible the matching of the Dirichlet boundary condition on $\partial\Omega$ at order $O(\epsilon)$, while (4.25) makes more prominent the one on $\partial\omega_\epsilon$. Interestingly, (4.25) features the weighting function

$$\epsilon \mapsto \frac{1}{a_\epsilon}\sum_{p=0}^{+\infty}\sum_{q=0}^{p}\frac{\epsilon^p}{a_\epsilon^q}v_{p,q}^{\infty}$$

which depends only on ϵ (and not of x), and which is determined by the far field limits $(v_{p,q}^{\infty})_{0\le p,0\le q\le p}$ of the functions $(v_{p,q})_{p\ge 0,0\le q\le p}$.

4.3. **The recursive system for computing corrector functions**

Similarly to the derivation of Section 3.3, it is possible to characterize the functions $(v_{p,q})_{p\ge 0,0\le q\le p}$ and $(w_{p,q})_{p\ge 1,0\le q\le p}$ arising in the ansatz (4.22) as the solutions to a recursive system of exterior and interior Dirichlet problems in $\mathbb{R}^2\backslash\overline{\omega}$ and in Ω. This result is to our knowledge original, and illustrates well the relevance of the single layer potential representation (2.13), since the ansatz (4.22) could be otherwise difficult to guess.

Proposition 4.6. *The functions $(v_{p,q})_{p\ge 0,0\le q\le p}$ and $(w_{p,q})_{p\ge 1,0\le q\le p}$ are uniquely characterized as the solutions to the following recursive system of partial differential equations posed in the exterior domain $\mathbb{R}^2\backslash\overline{\omega}$ and in the interior set Ω:*

$$\left\{\begin{aligned}
-\Delta v_{p,q} &= 0, && \text{in } \mathbb{R}^2\backslash\overline{\omega}, && (4.26)\\
v_{0,0}(t) &= -u(x_0), && \text{for } t\in\partial\omega, && (4.27)\\
-\frac{1}{p!}\nabla^p u(x_0)\cdot t^p &= v_{p,0}(t) + \sum_{k=0}^{p-1}\frac{1}{k!}\nabla^k w_{p-k,0}(x_0)\cdot t^k, && \text{for } t\in\partial\omega,\ p\ge 1 && (4.28)\\
v_{p,q}(t) &= -\sum_{k=1}^{p-q+1}\frac{v_{p-k,q-1}^{\infty}}{k!}\nabla^k R(x_0)\cdot t^k, && &&\\
&\quad -\sum_{k=0}^{p-q}\frac{1}{k!}\nabla^k w_{p-k,q}(x_0)\cdot t^k, && \text{for } t\in\partial\omega,\ p\ge 1 \text{ and } 1\le q\le p, && (4.29)\\
v_{p,q}(x) &= v_{p,q}^{\infty} + O(|x|^{-1}), && && (4.30)
\end{aligned}\right.$$

where we denote by R the function defined by $R(x) := R_\Omega(x,x_0)$, and

$$\left\{\begin{aligned}
-\Delta w_{p,q} &= 0, && \text{in } \Omega, && (4.31)\\
w_{p,0}(x) &= -\sum_{k=1}^{p} v_{p-k,0}^{(k)}, && \text{on } \partial\Omega,\ p\ge 1, && (4.32)\\
w_{p,q} &= -\sum_{k=1}^{p-q+1}\Phi^{(k)} - \sum_{k=1}^{p-q} v_{p-k,q}^{(k)}, && \text{on } \partial\Omega \text{ for } p\ge 1 \text{ and } 1\le q\le p, && (4.33)
\end{aligned}\right.$$

where for any $k\in\mathbb{N}$, $\Phi_{k\ge 1}^{(k)}$ and $(v_{p,q}^{(k)})_{k\ge 1}$ are the functions occurring in the far field expansions of Φ and $v_{p,q}$:

$$v_{p,q}^{(k)}(x) = \frac{(-1)^k}{k!}\nabla^k\Gamma(x-x_0)\cdot\int_{\partial\omega}\left[\!\!\left[\frac{\partial v_{p,q}}{\partial n}\right]\!\!\right] t^k\,\mathrm{d}\sigma(t) \text{ and } \Phi^{(k)}(x) := \frac{(-1)^k}{k!}\nabla^k\Gamma(x-x_0)\cdot\int_{\partial\omega}\left.\frac{\partial\Phi}{\partial n}\right|_+ t^k\,\mathrm{d}\sigma(t),$$

for any $p\ge 0$, $0\le q\le p$ and $k\ge 1$ as $\epsilon\to 0$.

Proof. Inserting $x = x_0 + \epsilon t$ with $t \in \partial\omega$ in the boundary condition satisfied by $u_\epsilon - u$, and using a Taylor expansion, we read

$$
\begin{aligned}
&-\sum_{p=0}^{+\infty} \frac{\epsilon^p}{p!} \nabla^p u(x_0) \cdot t^p = u_\epsilon(x_0+\epsilon t) - u(x_0+\epsilon t) \\
&= \sum_{p=0}^{+\infty}\sum_{q=0}^{p} \left(\frac{\epsilon^p}{a_\epsilon^q} v_{p,q}^\infty + \frac{\epsilon^p}{a_\epsilon^{q+1}} v_{p,q}^\infty \sum_{k=1}^{+\infty} \frac{\epsilon^k}{k!} \nabla^k R(x_0)\cdot t^k \right) \\
&\quad + \sum_{p=0}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} \left(v_{p,q}(t) - v_{p,q}^\infty \right) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} \sum_{k=0}^{+\infty} \frac{\epsilon^k}{k!} \nabla^k w_{p,q}(x_0)\cdot t^k \\
&= \sum_{p=0}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} v_{p,q}(t) + \sum_{p=1}^{+\infty}\sum_{k=1}^{p}\sum_{q=1}^{p-k+1} \frac{\epsilon^p}{a_\epsilon^q} v_{p-k,q-1}^\infty \frac{1}{k!}\nabla^k R(x_0)\cdot t^k + \sum_{p=1}^{+\infty}\sum_{k=0}^{p-1}\sum_{q=0}^{p-k} \frac{\epsilon^p}{a_\epsilon^q}\frac{1}{k!}\nabla^k w_{p-k,q}(x_0)\cdot t^k \\
&= \sum_{p=0}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} v_{p,q}(t) + \sum_{p=1}^{+\infty}\sum_{q=1}^{p}\sum_{k=1}^{p-q+1} \frac{\epsilon^p}{a_\epsilon^q} v_{p-k,q-1}^\infty \frac{1}{k!}\nabla^k R(x_0)\cdot t^k + \sum_{p=1}^{+\infty}\sum_{q=0}^{p-1}\sum_{k=0}^{p-q} \frac{\epsilon^p}{a_\epsilon^q}\frac{1}{k!}\nabla^k w_{p-k,q}(x_0)\cdot t^k.
\end{aligned}
$$

Identifying identical powers of ϵ and a_ϵ, we obtain the boundary conditions (4.27) to (4.29). We then consider $x \in \partial\Omega$. Using the far field expansions

$$
\Phi\left(\frac{x-x_0}{\epsilon}\right) = \frac{1}{2\pi}\log|x-x_0| - \frac{1}{2\pi}\log(\epsilon) + \Phi^\infty + \sum_{k=1}^{+\infty} \epsilon^k \Phi^{(k)}(x), \tag{4.34}
$$

$$
v_{p,q}\left(\frac{x-x_0}{\epsilon}\right) = v_{p,q}^\infty + \sum_{k=1}^{+\infty} \epsilon^k v_{p,q}^{(k)}(x), \tag{4.35}
$$

and recalling $R_\Omega(x,x_0) = -\frac{1}{2\pi}\log|x-x_0|$ for $x \in \partial\Omega$, we obtain

$$
\begin{aligned}
0 = u_\epsilon(x) - u(x) &= \sum_{p=0}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^{q+1}} \sum_{k=1}^{+\infty} \epsilon^k \Phi^{(k)}(x) + \sum_{p=0}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} \sum_{k=1}^{+\infty} \epsilon^k v_{p,q}^{(k)}(x) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} w_{p,q}(x) \\
&= \sum_{p=1}^{+\infty}\sum_{k=1}^{p}\sum_{q=1}^{p-k+1} \frac{\epsilon^p}{a_\epsilon^q} \Phi_k(x) + \sum_{p=1}^{+\infty}\sum_{k=1}^{p}\sum_{q=0}^{p-k} \frac{\epsilon^p}{a_\epsilon^q} v_{p-k,q}^{(k)}(x) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} w_{p,q}(x) \\
&= \sum_{p=1}^{+\infty}\sum_{q=1}^{p}\sum_{k=1}^{p-q+1} \frac{\epsilon^p}{a_\epsilon^q} \Phi^{(k)}(x) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p-1}\sum_{k=1}^{p-q} \frac{\epsilon^p}{a_\epsilon^q} v_{p-k,q}^{(k)}(x) + \sum_{p=1}^{+\infty}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} w_{p,q}(x).
\end{aligned}
$$

Identifying identical powers of ϵ and a_ϵ yield finally the boundary conditions (4.32) and (4.33). $\square$

Remark 4.5. The recursive system (4.26) to (4.33) characterizes completely the functions $(v_{p,q})_{p\geq 0, 0\leq q\leq p}$ and $(w_{p,q})_{p\geq 1, 0\leq q\leq p}$. Indeed:

- the function $v_{0,0}$ is determined by (4.27);
- once the functions $v_{p',q'}$ are known for all $0 \leq p' \leq p-1$ and $0 \leq q' \leq p'$, one can compute the functions $w_{p,q}$ for $0 \leq q \leq p$ from the boundary conditions (4.32) and (4.33);
- once the functions $v_{p',q'}$ are known for all $0 \leq p' \leq p-1$ and $0 \leq q' \leq p'$, and once the functions $w_{p',q'}$ are known for all $0 \leq p' \leq p$ and $0 \leq q' \leq p'$, it is possible to compute the functions $v_{p,q}$ for all $0 \leq q \leq p$ from the boundary conditions (4.28) and (4.29).

4.4. Quantitative error estimates for the truncated ansatz

We finally provide an error bound for the truncation of the ansatz (4.22) determined from the cascade of equations of Proposition 4.6.

Proposition 4.7. *For any $N \in \mathbb{N}$, let u_ϵ^N be the truncated ansatz at rank N:*

$$
\begin{aligned}
u_\epsilon^N(x) := u(x) &+ \sum_{p=0}^{N}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q}\frac{v_{p,q}^\infty}{a_\epsilon} \left(\frac{1}{2\pi}\log\epsilon + R_\Omega(x,x_0) + \Phi\left(\frac{x-x_0}{\epsilon}\right) - \Phi^\infty \right) \\
&+ \sum_{p=0}^{N}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} \left(v_{p,q}\left(\frac{x-x_0}{\epsilon}\right) - v_{p,q}^\infty \right) + \sum_{p=1}^{N}\sum_{q=0}^{p} \frac{\epsilon^p}{a_\epsilon^q} w_{p,q}(x).
\end{aligned}
$$

The function u_ϵ^N is an approximation of u_ϵ at order $O(\epsilon^{N+1})$ in the $H^1(\Omega\backslash\omega_\epsilon)$–norm in view of the following error bound:

$$
||u_\epsilon - u_\epsilon^N||_{L^2(\Omega\backslash\omega_\epsilon)} + ||\nabla u_\epsilon - \nabla u_\epsilon^N||_{L^2(\Omega\backslash\overline{\omega}_\epsilon)} \leq C_N \epsilon^{N+1},
$$

for a constant C_N independent of ϵ (but which may depend on N).

Proof. The proof follows from the definition of the functions $(v_{p,q})_{p\geq 0, 0\leq q\leq p}$ and $(w_{p,q})_{p\geq 1, 0\leq q\leq p}$ and is similar to that of Proposition 3.7. We note that the key estimate of Proposition 3.6 remains true in this two-dimensional setting $d=2$. □

5. High-order asymptotics for a perforated periodic cell problem

In this section, we show how the ideas of the previous sections can be adapted to derive in a similar manner arbitrary order asymptotics for the solution $\mathcal{X}_\eta$ to the periodic cell problem (1.2). The section outlines as follows. Section 5.1 introduces an appropriate variational framework for precising the meaning of "periodic" boundary conditions for the solution to a Laplace problem of the form (1.2). The periodic Green function is introduced and key estimates are established. The next Section 5.2 establishes a characterization of the normal derivative $\frac{\partial \mathcal{X}_\eta}{\partial n}$ as the generator of the kernel of a suitable periodic single layer potential. This property is exploited in Section 5.3 and Section 5.4 to read full asymptotic expansions for $\mathcal{X}_\eta$ in the cases $d\geq 3$ and $d=2$ respectively. As previously, we derive cascade of well posed systems of exterior and periodic problems for the terms of the obtained two-scale expansion, and we prove some error estimates in energy norm.

5.1. **Preliminaries: the Poisson equation with periodic boundary conditions**

This part introduces the mathematical background for manipulating solutions to Poisson problems with periodic boundary conditions with or without a hole. Section 5.1.1 recalls some facts about the Sobolev space of periodic functions. A variational framework for the existence and uniqueness of solutions to the Laplace equation with periodic boundary conditions (without a hole) is then provided in Section 5.1.2. Section 5.1.3 introduces the periodic Green function and states its main properties. Finally, Section 5.1.4 states an existence and uniqueness result for the solutions to the Laplace equation with periodic boundary conditions perforated with a small hole, and states uniform norm estimates for the solution.

5.1.1. *Sobolev space of periodic functions*

Let $\mathcal{C}^\infty_{\mathrm{per}}(P)$ be the space of smooth P–periodic functions:

$$\mathcal{C}^\infty_{\mathrm{per}}(P) = \{f\in\mathcal{C}^\infty(\mathbb{R}^d)\,|\,\forall 1\leq i\leq d,\, f(x+\boldsymbol{e}_i)=f(x)\}, \tag{5.1}$$

where $(\boldsymbol{e}_i)_{1\leq i\leq d}$ is the canonical basis of $\mathbb{R}^d$.

Definition 5.1. We define the space $H^1_{\mathrm{per}}(P)$ as the completion of the space of smooth periodic functions for the H^1 norm:

$$H^1_{\mathrm{per}}(P) := \overline{\{v\in\mathcal{C}^\infty(P)|\ v \text{ is } P\text{–periodic}\}}^{||\cdot||+||\nabla\cdot||}.$$

In what follows, we denote by $\hat{v}(\xi)$ the trigonometric coefficients of any function $v\in L^2(P)$ for any $\xi\in\mathbb{Z}^d$:

$$\forall\xi\in\mathbb{Z}^d,\ \hat{v}(\xi) := \int_P e^{-2i\pi\xi\cdot x}v(x)\,\mathrm{d}x.$$

The following result is well-known, see e.g. Proposition 5.38 in [35].

Proposition 5.1. *The space $H^1_{\mathrm{per}}(P)$ admits the following equivalent characterization:*

$$H^1_{\mathrm{per}}(P) = \{v\in L^2(P)\,|\,\sum_{\xi\in\mathbb{Z}^d}(1+4\pi^2|\xi|^2)|\hat{v}(\xi)|^2<+\infty\}. \tag{5.2}$$

Four our applications whereby we need to estimates the solution of a problem of the form (1.2) from the boundary data, we introduce a different characterization of $H^1_{\mathrm{per}}(P)$ suitable for the formulation of elliptic problems with periodicity conditions. Functions of $H^1_{\mathrm{per}}(P)$ are $H^1(P)$ functions whose trace on ∂P is periodic in the following weak sense.

Definition 5.2. A function $v\in L^2(\partial P)$ is said to be P–periodic if for any smooth periodic function $\phi\in\mathcal{C}^\infty_{\mathrm{per}}(P)$ it satisfies

$$\int_{\partial P} v\phi\boldsymbol{n}\,\mathrm{d}\sigma = 0, \tag{5.3}$$

or equivalently, v coincide on opposite matching faces: $v|_{y_i=-1/2} = v|_{y_i=1/2}$ for any $1\leq i\leq d$.

Proposition 5.2. *The space $H^1_{\mathrm{per}}(P)$ can alternatively be characterized as the subspace of functions of $H^1(P)$ whose traces are P–periodic in the sense of (5.3):*

$$H^1_{\mathrm{per}}(P) = \{v\in H^1(P)\,|\,v_{|\partial P} \textit{ is } P\textit{–periodic}\}. \tag{5.4}$$

Proof. Obviously any function $v \in \mathcal{C}^\infty_{\text{per}}(P)$ satisfies (5.3). By density and the $L^2(\partial P)$ continuity of the trace operator of $H^1_{\text{per}}(P)$, (5.3) is also true for any function $v \in H^1_{\text{per}}(P)$. This implies the direct inclusion $\subset$.

Let us prove the reverse inclusion $\supset$. We use the characterization (5.2). Let $v \in H^1(P)$ satisfy (5.3). By using Green's identity, the Fourier coefficients of v satisfy for any $1 \le k \le d$:

$$\hat{v}(\xi) = \int_P e^{-2\mathrm{i}\pi\xi\cdot x} v(x)\,\mathrm{d}x = \int_P \operatorname{div}\left(-\frac{\boldsymbol{e}_k}{2\mathrm{i}\pi\xi_k} e^{-2\mathrm{i}\pi\xi\cdot x}\right) v(x)\,\mathrm{d}x$$

$$= \int_P \frac{\boldsymbol{e}_k}{2\mathrm{i}\pi\xi_k}\cdot\nabla v(x) e^{-2\mathrm{i}\pi\xi\cdot x}\,\mathrm{d}x - \int_{\partial P} \frac{\boldsymbol{e}_k\cdot\boldsymbol{n}}{2\mathrm{i}\pi\xi_k} e^{-2\mathrm{i}\pi\xi\cdot y} v(y)\,\mathrm{d}\sigma(y). \quad (5.5)$$

By using (5.3) with $\phi(x) = e^{-2i\pi\xi\cdot x}$, the last integral of (5.5) vanishes and we obtain

$$\forall 1 \le k \le d,\ \hat{v}(\xi) = \frac{1}{2i\pi\xi_k}\widehat{\partial_k v}(\xi).$$

Since $\nabla v \in L^2(P)$, the series $\sum_{\xi\in\mathbb{Z}^d}\sum_{k=1}^d |\widehat{\partial_k v}(\xi)|^2$ converges and therefore we obtain

$$\sum_{\xi\in\mathbb{Z}^d}(1+4\pi^2|\xi|^2)|\hat{v}(\xi)|^{2\mathbb{Z}^d}|\hat{v}(\xi)|^2 + \sum_{\xi\in\mathbb{Z}^d}\sum_{k=1}^d |\widehat{\partial_k v}(\xi)|^2 < +\infty.$$

This implies that v belongs to $H^1_{\text{per}}(P)$ by (5.2). $\square$

5.1.2. *The Poisson equation with periodic boundary conditions*

In what follows, we give a variational meaning to a Laplace problem in $H^1_{\text{per}}(P)$ equipped with suitable periodic boundary conditions. We denote by $H^{-1}_{\text{per}}(P)$ the dual space of $H^1_{\text{per}}(P)$.

Proposition 5.3. *Let $f \in H^{-1}_{\text{per}}(P)$, $h_0 \in H^{\frac{1}{2}}(\partial P)$ and $h_1 \in L^2(\partial P)$. The problem*

$$\begin{cases} -\Delta u = f \text{ in } P, \\ u - h_0 \text{ is } P\text{–periodic}, \\ \dfrac{\partial u}{\partial \boldsymbol{n}}\boldsymbol{n} - h_1\boldsymbol{n} \text{ is } P\text{–periodic}, \end{cases} \quad (5.6)$$

admits a unique solution $u \in H^1(P)$, defined up to a constant, if and only if the compatibility condition

$$\int_P f\,\mathrm{d}x + \int_{\partial P} h_1\,\mathrm{d}\sigma = 0 \quad (5.7)$$

is satisfied. Furthermore, the solution u satisfies the following properties:

(i) $u - \widetilde{h}_0 \in H^1_{\text{per}}(P)$ for any extension $\widetilde{h}_0 \in H^1(P)$ of h_0 satisfying $\widetilde{h}_0 = h_0$ on ∂P;

(ii) there exists a constant C independent of f, h_0 and h_1 such that

$$||\nabla u||_{L^2(P)} \le C(||f||_{H^{-1}(P)} + ||h_0||_{H^{\frac{1}{2}}(\partial P)} + ||h_1||_{L^2(\partial P)}). \quad (5.8)$$

Proof. Let us consider an extension $\widetilde{h}_0 \in H^1(P)$ of h_0 satisfying $\widetilde{h}_0 = h_0$ on ∂P and $||\widetilde{h}_0||_{H^1(P)} \le C||h_0||_{H^{\frac{1}{2}}(\partial P)}$. We consider the following variational problem: find $v \in H^1_{\text{per}}(P)$ such that for any $v' \in H^1_{\text{per}}(P)$, it holds:

$$\int_P \nabla v\cdot\nabla v'\,\mathrm{d}x = \int_P fv'\,\mathrm{d}x + \int_{\partial P} h_1 v'\,\mathrm{d}\sigma - \int_P \nabla\widetilde{h}_0\cdot\nabla v'\,\mathrm{d}x. \quad (5.9)$$

By standard arguments involving the Poincaré-Wirtinger inequality ($||v - \int_P v\,\mathrm{d}x||_{L^2(P)} \le C||\nabla v||_{L^2(P)}$) and the Lax-Milgram theorem, this problem admits a unique solution defined up to a constant if and only if (5.7) is satisfied. We define then $u := v + \widetilde{h}_0$. By integration by parts, we obtain that (5.9) is a weak formulation of (5.6). The estimate (5.8) is obtained by setting $v' = v$ in (5.9). Uniqueness comes from the fact that if v is another solution to (5.6), then $u - v$ is harmonic and periodic, hence it is constant. $\square$

Remark 5.1. The last boundary condition of (5.6) can be interpreted more conveniently as

$$\forall v' \in H^1_{\text{per}}(P),\ \int_{\partial P}\frac{\partial u}{\partial \boldsymbol{n}}v'\,\mathrm{d}\sigma = \int_{\partial P} h_1 v'\,\mathrm{d}\sigma.$$

If u is smooth, this means that the values of $\partial u/\partial n$ on matching faces of ∂P satisfy, for any $1 \le i \le d$,

$$\left.\left(\frac{\partial u}{\partial n} - h_1\right)\right|_{y_i=-\frac{1}{2}} = -\left.\left(\frac{\partial u}{\partial n} - h_1\right)\right|_{y_i=\frac{1}{2}}.$$

We also need regularity estimates. We assume now that h_0 and h_1 are respectively the boundary trace and the normal derivative of a smooth (possibly non-periodic) function h living in the vicinity of ∂P. Without loss of generality, we assume that h is an element of $\mathcal{C}^\infty(P\backslash T)$ where we recall that $T \Subset P$.

Proposition 5.4. *Let $f \in \mathcal{C}^\infty_{\text{per}}(P)$ be a smooth periodic function. Assume that there exists an open set T such that $T \Subset P$ and let $h \in \mathcal{C}^\infty(P\backslash T)$ be such that:*

(H1) the Laplacian of h is a smooth P–periodic function:

$$\Delta h \in \mathcal{C}^\infty_{\text{per}}(P\backslash T),$$

(H2) h satisfies the compatibility condition

$$\int_P f \,\mathrm{d}x + \int_{\partial P} \frac{\partial h}{\partial n} \,\mathrm{d}y = 0. \tag{5.10}$$

Then the unique solution u, defined up to a constant, to

$$\begin{cases} -\Delta u = f \text{ in } P, \\ u - h \text{ is } P\text{–periodic}, \\ \dfrac{\partial u}{\partial \boldsymbol{n}}\boldsymbol{n} - \dfrac{\partial h}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic}, \end{cases} \tag{5.11}$$

satisfies the following properties:

(i) $u - h \in \mathcal{C}^\infty_{per}(P\backslash T)$ is a smooth P–periodic function on $P\backslash T$,

(ii) for any $k > d/2 - 1$, there exists a constant $C_k > 0$ which depends only on k such that

$$||u - \langle u\rangle||_{L^\infty(P)} + ||\nabla u||_{L^\infty(P)} \leq C_k(||f||_{H^k(P)} + ||h||_{H^{k+2}(P\backslash T)}). \tag{5.12}$$

Proof. Let $\theta \in \mathcal{C}^\infty(P)$ be a cut-off function satisfying $\theta = 1$ in the vicinity of ∂P and compactly supported in $P\backslash T$. Then $\widetilde{h} = \theta h \in \mathcal{C}^\infty(P)$ is a smooth function (extended by 0 in T) satisfying $\widetilde{h} = h$ in the vicinity of ∂P and $\widetilde{h} = 0$ on T. The function $v := u - \widetilde{h}$ is the solution to

$$\begin{cases} -\Delta v = f + \Delta\widetilde{h}, \\ v \text{ is } P\text{–periodic}, \\ \dfrac{\partial v}{\partial \boldsymbol{n}}\boldsymbol{n} \text{ is } P\text{–periodic}. \end{cases} \tag{5.13}$$

Since $\Delta h \in \mathcal{C}^\infty_{per}(P\backslash T)$, it holds that $\Delta\widetilde{h} \in \mathcal{C}^\infty_{per}(P)$ is a smooth periodic function. Therefore, by solving (5.13) explicitly with trigonometric expansions, we obtain the standard regularity estimate

$$||u - \langle u\rangle||_{H^{k+2}(P)} \leq C_k(||f||_{H^k(P)} + ||h||_{H^{k+2}(P)}),$$

for any k and a constant $C_k > 0$ depending only on k. Then (5.12) follows by the Sobolev embedding theorem (see e.g. Theorem 5.29 in [35]). □

5.1.3. *The periodic Green function*

We have now all the material to introduce the periodic Green function which is defined, up to the choice of an additive constant, as the unique function $G_\#(x, \cdot)$ satisfying

$$\begin{cases} \Delta G_\# = \delta_x - 1 \text{ in } P, \\ G_\# \text{ is } P\text{–periodic}. \end{cases} \tag{5.14}$$

The periodic Green function G is a classical object of solid-state physics, see [8, 6, 37]. See also [24] in the context of the Stokes system, and [27, Appendix A.2] in homogenization. A common and rather straightforward definition of G is possible in terms of a singular Fourier series expansion, e.g.

$$G(x) = -\sum_{\xi \in \mathbb{Z}^d\backslash\{0\}} \frac{e^{2\mathrm{i}\pi\xi\cdot x}}{4\pi^2|\xi|^2}, \qquad x \in P\backslash\{0\}. \tag{5.15}$$

For our purpose, we prefer to rely on a definition of $G_\#$ making use of a suitable difference problem.

Proposition 5.5. *The periodic Green kernel $G_{\#,\eta}$ of (5.14) is given by*

$$G_\#(x, y) = \Gamma(x - y) + R_\#(x - y),$$

where $R_\#$ is the unique solution, defined up to a constant, to the difference problem

$$\begin{cases} -\Delta R_\# = 1 \text{ in } P, \\ R_\# + \Gamma \text{ is } P\text{–periodic }, \\ \dfrac{\partial R_\#}{\partial n}\boldsymbol{n} + \dfrac{\partial \Gamma}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic }. \end{cases} \tag{5.16}$$

The function $R_\# \in \mathcal{C}^\infty(P)$ is smooth in P and $R_\# + \Gamma \in \mathcal{C}^\infty(\mathbb{R}^d\backslash\mathbb{Z}^d)$ is a smooth P–periodic function.

Proof. Existence and uniqueness of the function $R_\#$ is obtained by applying Proposition 5.4. Assumptions (H1) and (H2) are satisfied with $f = 1$ and $h = -\Gamma$, since with e.g. $T = B(0, 1/4)$:

(H1) $\Delta\Gamma = 0$ in $\mathcal{C}^\infty_{per}(P\backslash T)$;

(H2) the compatibility condition is satisfied due to the identity $\int_{\partial P} \frac{\partial \Gamma}{\partial n}\,\mathrm{d}\sigma = 1$ resulting from Lemma 2.1.

□

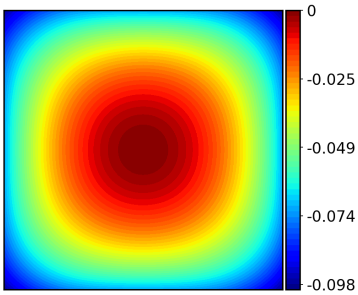
In view of the symmetry of P, it is easy to verify that

$$\nabla R_\#(0) = 0 \text{ and } \nabla^2 R_\#(0) = -\frac{1}{d}I. \tag{5.17}$$

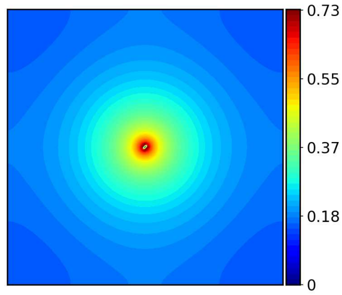
Remark 5.2. Observing that Γ is already a P–periodic function in the sense of Definition 5.1, we deduce that $R_\#$ belongs to $H^1_{\text{per}}(P)$ and that it can be conveniently computed with the finite-element method by solving the variational problem

$$\text{find } R_\# \in H^1_{\text{per}}(P) \text{ such that } \forall v \in H^1_{\text{per}}(P),\ \int_P \nabla R_\# \cdot \nabla v\,\mathrm{d}x = \int_P v\,\mathrm{d}x - \int_{\partial P} \frac{\partial \Gamma}{\partial n} v\,\mathrm{d}\sigma. \tag{5.18}$$

We plot on Figure 2 below the function $R_\#$ computed with this method in dimension $d = 2$.



(A) R



(B) G

FIGURE 2. Periodic Green function $G_\# = \Gamma + R_\#$ in the two-dimensional domain $P = (-\frac{1}{2}, \frac{1}{2})^2$ computed by solving (5.18) with the finite-element method.

5.1.4. *The Poisson problem in the perforated periodic cell*

We now state results regarding existence and uniqueness of solutions to the periodic Laplace equation perforated with a small hole ηT. Throughout this part, the average of a function $v \in L^1(P)$ is denoted by $\langle v \rangle$:

$$\langle v \rangle := \int_P v(x)\,\mathrm{d}x.$$

If v vanishes on $\partial(\eta T)$, we still denote by $\langle v \rangle$ the same quantity where we assume that v is extended by 0 in ηT.

Lemma 5.1. *There exists a constant $C > 0$ independent of η such that the following Poincaré inequalities hold for any $v \in H^1(P\backslash(\eta T))$ vanishing on $\partial(\eta T)$:*

$$||v||_{L^2(P\backslash(\eta T))} \leq \begin{cases} C|\log\eta|^{\frac{1}{2}}\,||\nabla v||_{L^2(P\backslash(\eta T))} & \text{if } d = 2, \\ C\eta^{1-d/2}||\nabla v||_{L^2(P\backslash(\eta T))} & \text{if } d \geq 3, \end{cases} \tag{5.19}$$

and

$$|\langle v \rangle| \leq \begin{cases} C|\log\eta|^{\frac{1}{2}}\,||\nabla v||_{L^2(P\backslash(\eta T))} & \text{if } d = 2, \\ C\eta^{1-d/2}||\nabla v||_{L^2(P\backslash(\eta T))} & \text{if } d \geq 3, \end{cases} \tag{5.20}$$

and

$$||v - \langle v \rangle||_{L^2(P\backslash(\eta T))} \leq C||\nabla v||_{L^2(P\backslash(\eta T))}. \tag{5.21}$$

Proof. The first inequality is classical [29, 2]. A detailed proof is found in Theorem A.1 of [27]. The second inequality is obtained from the first by the Cauchy-Schwarz inequality. The third one is just the Poincaré-Wirtinger inequality in the hypercube $P = (-\frac{1}{2}, \frac{1}{2})^d$. □

Proposition 5.6. *Let $h \in \mathcal{C}^\infty(P\backslash T)$ be a function satisfying the assumption (H1) of Proposition 5.4, namely*

$$\Delta h \in \mathcal{C}^\infty_{\text{per}}(P\backslash T).$$

For $f \in \mathcal{C}^\infty_{\mathrm{per}}(P)$, $g \in H^{\frac{1}{2}}(\partial(\eta T))$, there exists a unique solution $u \in H^1(P\backslash(\eta T))$ to

$$\begin{cases} -\Delta u = f \text{ in } P\backslash(\eta T), \\ u = g \text{ on } \partial(\eta T), \\ u - h \text{ is } P\text{–periodic}, \\ \dfrac{\partial u}{\partial n}\boldsymbol{n} - \dfrac{\partial h}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic.} \end{cases} \tag{5.22}$$

Furthermore, for any $k > \frac{d}{2} - 1$, there exists a constant C_k independent of η, f and h such that if $d \geq 3$,

$$||\nabla u||_{L^2(P\backslash(\eta T))} \leq C_k\left[|\alpha(f,h)|\eta^{1-d/2} + ||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)}\eta^{d/2-1} + ||f - \alpha(f,h)||_{H^k(P)} + ||h||_{H^{k+2}(P\backslash T)}\right], \tag{5.23}$$

and

$$||u||_{L^2(P\backslash(\eta T))} \leq C_k\left[|\alpha(f,h)|\eta^{2-d} + ||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)} + ||f - \alpha(f,h)||_{H^k(P)} + ||h||_{H^{k+2}(P\backslash T)}\right], \tag{5.24}$$

where $\alpha(f,h)$ is the "default of compatibility" defined by

$$\alpha(f,h) := \int_P f \, \mathrm{d}x + \int_{\partial P} \frac{\partial h}{\partial n} \, \mathrm{d}\sigma. \tag{5.25}$$

If $d = 2$ the same bounds hold by replacing $\eta^{1-d/2}$ with $|\log\eta|^{\frac{1}{2}}$:

$$||\nabla u||_{L^2(P\backslash(\eta T))} \leq C_k\left[|\alpha(f,h)|\,|\log\eta|^{\frac{1}{2}} + ||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)} + ||f - \alpha(f,h)||_{H^k(P)} + ||h||_{H^{k+2}(P\backslash T)}\right], \tag{5.26}$$

and

$$||u||_{L^2(P\backslash(\eta T))} \leq C_k\left[|\alpha(f,h)|\,|\log\eta| + ||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)}|\log\eta|^{\frac{1}{2}} + ||f - \alpha(f,h)||_{H^k(P)} + ||h||_{H^{k+2}(P\backslash T)}\right]. \tag{5.27}$$

Proof. Existence and uniqueness of a solution $u \in H^1(P\backslash(\eta T))$ can be obtained by adapting the proof of Proposition 5.3, where no compatibility condition is required because of the Dirichlet boundary condition on $\partial(\eta T)$. Let us prove the estimates (5.23) and (5.24). We decompose $u = u_1 + u_2$ in terms of the solutions to two distinct problems: $u_1 \in H^1(P)$ is defined as the unique solution to

$$\begin{cases} -\Delta u_1 = f - \alpha(f,h) \text{ in } P, \\ u_1 - h \text{ is } P\text{–periodic}, \\ \dfrac{\partial u_1}{\partial n}\boldsymbol{n} - \dfrac{\partial h}{\partial \boldsymbol{n}}\boldsymbol{n} \text{ is } P\text{–periodic}, \\ u_1(0) = 0. \end{cases} \tag{5.28}$$

The problem (5.28) is of the type of (5.11). Existence is ensured by the compatibility condition

$$\int_P f \, \mathrm{d}x - \alpha(f,h) + \int_{\partial P} \frac{\partial h}{\partial n} \, \mathrm{d}y = 0,$$

which is satisfied by the definition (5.25) of $\alpha(f,h)$. Uniqueness is obtained by the condition $u_1(0) = 0$, which can be enforced by considering the solution $\widetilde{u}_1$ of the same problem satisfying $\langle \widetilde{u}_1 \rangle = 0$ before setting $u_1 := \widetilde{u}_1 - \widetilde{u}_1(0)$, which makes sense due to the smoothness of u_1. By Propositions 5.3 and 5.4, u_1 satisfies the following bounds:

$$\begin{aligned} ||u_1||_{L^2(P)} + ||\nabla u_1||_{L^2(P)} &\leq ||\widetilde{u}_1||_{L^2(P)} + ||\nabla \widetilde{u}_1||_{L^2(P)} + |\widetilde{u}_1(0)| \\ &\leq C_k(||f - \alpha(f,h)||_{H^k(P)} + ||h||_{H^{k+2}(P\backslash T)}), \end{aligned} \tag{5.29}$$

and

$$||\nabla u_1||_{L^\infty(P)} \leq C_k(||f - \alpha(f,h)||_{H^k(P)} + ||h||_{H^{k+2}(P\backslash T)}). \tag{5.30}$$

The function u_2 is defined as the solution to the difference problem

$$\begin{cases} -\Delta u_2 = \alpha(f,h) \text{ in } P, \\ u_2 = g - u_1 \text{ on } \partial(\eta T), \\ u_{2|\partial P} \text{ is } P\text{–periodic}, \\ \dfrac{\partial u_2}{\partial n}\boldsymbol{n}\Big|_{\partial P} \text{ is } P\text{–periodic.} \end{cases}$$

Let K be an open set containing the obstacle T: $T \Subset K$. Consider an extension $\widetilde{g}_\eta$ in K of the function $g(\eta\cdot)$ compactly supported in K, satisfying $\widetilde{g}_\eta = g(\eta\cdot)$ on ∂T and $||\widetilde{g}_\eta||_{H^1(K)} \leq C||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)}$ for a constant C independent of g and η. We define $\hat{g} := \widetilde{g}_\eta(\cdot/\eta)$, which yields an extension of g compactly supported in P (extending by 0 in $P\backslash(\eta K)$) satisfying $\hat{g} = g$ on $\partial(\eta T)$ and

$$\begin{aligned} ||\hat{g}||_{H^1(P)} = ||\widetilde{g}_\eta(\cdot/\eta)||_{H^1(\eta K)} &= \eta^{d/2-1}||\nabla\widetilde{g}_\eta||_{L^2(K)} + \eta^{d/2}||\widetilde{g}_\eta||_{L^2(K)} \\ &\leq C\eta^{d/2-1}||\widetilde{g}_\eta||_{H^1(K)} \leq C\eta^{d/2-1}||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)}. \end{aligned}$$

Similarly, by setting $\hat{u}_1(x) = u_1(x)\theta(x/\eta)$ for a cutoff function $\theta \in \mathcal{C}_c^\infty(K)$ satisfying $\theta = 1$ on ∂T, we construct an extension $\hat{u}_1 \in H^1(P)$ compactly supported in P such that $\hat{u}_1 = u_1$ on $\partial(\eta T)$ and

$$||\hat{u}_1||_{H^1(P)} = \eta^{d/2}||\theta u_1(\eta\cdot)||_{L^2(K)} + \eta^{d/2-1}||\nabla(\theta u_1(\eta\cdot))||_{L^2(K)} \le C\eta\eta^{d/2-1}||\nabla u_1||_{L^\infty(P)},$$

where we have used $u_1(0) = 0$ in the last estimate (which implies $||u_1(\eta\cdot)||_{L^\infty(K)} \le C\eta||\nabla u_1||_{L^\infty(P)}$). Since the function $v_2 := u_2 - \hat{g} - \hat{u}_1$ satisfies $v_2 = 0$ on $\partial(\eta T)$, an integration by parts yields, for $d \ge 3$,

$$\begin{aligned}
\int_{P\setminus(\eta T)} |\nabla v_2|^2 \,\mathrm{d}x &= \int_{P\setminus(\eta T)} \nabla u_2 \cdot \nabla v_2 \,\mathrm{d}x - \int_{P\setminus(\eta T)} \nabla v_2 \cdot (\nabla\hat{g} + \nabla\hat{u}_1) \\
&= \int_{P\setminus(\eta T)} \alpha(f,h) v_2 \,\mathrm{d}x - \int_{P\setminus(\eta T)} \nabla v_2 \cdot (\nabla\hat{g} + \nabla\hat{u}_1) \\
&\le C(|\alpha(f,h)|\eta^{1-d/2} + ||\nabla\hat{g}||_{L^2(P\setminus(\eta T))} + ||\nabla\hat{u}_1||_{L^2(P\setminus(\eta T))})||\nabla v_2||_{L^2(P\setminus(\eta T))} \\
&\le C(|\alpha(f,h)|\eta^{1-d/2} + \eta^{d/2-1}||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)} + \eta^{d/2}||\nabla u_1||_{L^\infty(P)})||\nabla v_2||_{L^2(P\setminus(\eta T))}.
\end{aligned} \tag{5.31}$$

We obtain thus if $d \ge 3$,

$$\begin{aligned}
||\nabla u_2||_{L^2(P\setminus(\eta T))} &\le ||\nabla v_2||_{L^2(P\setminus(\eta T))} + C\eta^{d/2-1}||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)} + \eta^{d/2}||\nabla u_1||_{L^\infty(P)} \\
&\le C_k(|\alpha(f,h)|\eta^{1-d/2} + \eta^{d/2-1}||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)} + \eta^{d/2}||f - \alpha(f,h)||_{H^k(P)} + \eta^{d/2}||h||_{H^{k+2}(P\setminus T)}).
\end{aligned} \tag{5.32}$$

The Poincaré inequality (5.19) yields then

$$||u_2||_{L^2(P\setminus(\eta T))} \le C_k(|\alpha(f,h)\eta^{2-d} + ||g(\eta\cdot)||_{H^{\frac{1}{2}}(\partial T)} + \eta||f - \alpha(f,h)||_{H^k(P)} + \eta||h||_{H^{k+2}(P\setminus T)}). \tag{5.33}$$

The final result for $d \ge 3$ follows by summing the two estimates (5.29) and (5.33). In the case $d = 2$, the estimates (5.26) and (5.27) are similarly obtained by replacing the quantities $\eta^{1-d/2}$ with $|\log\eta|^{\frac{1}{2}}$ coming from the Poincaré inequality (5.19) in (5.31) to (5.33). □

Remark 5.3. The estimates (5.23) and (5.24) highlight that in the limit $\eta \to 0$, the blowing up of a solution u to (5.22) may come from either the compatibility condition $\alpha(f,h) = 0$ not being satisfied (as in the problem (1.2)), or from a boundary datum g of order greater than $\eta^{1-d/2}$.

5.2. A single layer potential characterization of the periodic perforated solution

We now focus on the asymptotic analysis of the solution $\mathcal{X}_\eta$ to the periodic cell problem (1.2), which is well-posed in the sense provided by Proposition 5.6. We start by deriving a characterization of $\mathcal{X}_\eta$ in terms of an appropriate single layer potential operator. Since the choice of constant for the periodic Green function $G_\#$ matters in the definition of this single layer potential, we choose this constant to be set such that

$$R_\#(0) = 0, \tag{5.34}$$

and we keep this choice in the whole remainder of our analysis. The mapping $\eta \mapsto R_\#(\eta\cdot)$ is analytic in η and since $\nabla R_\#(0) = 0$ (equation (5.17)), it holds $R_\#(\eta t) = O(\eta^2)$ for $t \in \partial T$. We now are now able to introduce a specific single layer potential parameterized by a constant $\kappa \in \mathbb{R}$.

Definition 5.3. For a given real $\kappa \in \mathbb{R}$, we define $\mathcal{S}^\kappa_{\#,\eta T}$ to be the single layer potential defined by

$$\mathcal{S}^\kappa_{\#,\eta T}[\phi](x) := \int_{\partial(\eta T)} G_\#(x,y)\phi(y)\,\mathrm{d}\sigma(y) + \eta^{2-d}\left(-\frac{\log\eta}{2\pi}\delta_{d=2} + \kappa\right)\int_{\partial(\eta T)} \phi\,\mathrm{d}\sigma, \tag{5.35}$$

for any $\phi \in H^{-\frac{1}{2}}(\partial(\eta T))$ and $x \in P$.

The single layer potential $\mathcal{S}^\kappa_{\#,\eta T}$ is obtained by adding to the periodic Green kernel a particular constant depending on η and κ. It is a compact perturbation of the standard single layer potential $\mathcal{S}_{\eta T}$, and hence satisfies jump relations analogous to (2.12). The constant κ is to be set below to a special value κ_η (depending on η), making $\mathcal{S}^\kappa_{\#,\eta T}$ not invertible. The scaling η^{2-d} and the additive term $-\log\eta/(2\pi)$ when $d = 2$ are motivated by simplifications occurring in Proposition 5.7 below.

We have the following property showing how the cell solution $\mathcal{X}_\eta$ to (1.2) is related to $\mathcal{S}^\kappa_{\#,\eta T}$.

Lemma 5.2. *The kernel of $\mathcal{S}^\kappa_{\#,\eta}$ is either trivial or is the space of functions proportional to $\frac{\partial\mathcal{X}_\eta}{\partial n}$:*

$$\mathrm{Ker}(\mathcal{S}^\kappa_{\#,\eta T}) \subset \mathrm{span}\left(\frac{\partial\mathcal{X}_\eta}{\partial n}\right).$$

Moreover, $\mathcal{X}_\eta$ has the following single layer potential representation when this kernel is not trivial:

$$\mathcal{X}_\eta(x) = \frac{1}{-1+\eta^d|T|}\mathcal{S}^\kappa_{\#,\eta T}\left[\frac{\partial\mathcal{X}_\eta}{\partial n}\right]. \tag{5.36}$$

Proof. If $\mathcal{S}^{\kappa}_{\#,\eta T}[\phi]=0$ on $\partial(\eta T)$, then the function $u:=\mathcal{S}^{\kappa}_{\#,\eta T}[\phi]$ satisfies

$$\begin{cases} -\Delta u = \displaystyle\int_{\partial(\eta T)}\phi\,\mathrm{d}\sigma(y) \text{ in } P\backslash\eta T,\\ \quad u=0 \text{ on } \partial T,\\ \quad u \text{ is } P\text{–periodic},\\ \dfrac{\partial u}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic}.\end{cases}$$

Therefore $u=\left(\int_{\partial(\eta T)}\phi\,\mathrm{d}\sigma\right)\mathcal{X}_\eta$ and $\phi=\left(\int_{\partial(\eta T)}\phi\,\mathrm{d}\sigma\right)\frac{\partial\mathcal{X}_\eta}{\partial n}\Big|_+$ by using the jump relation of $\mathcal{S}^{\kappa}_{\#,\eta T}$. Reciprocally, if $\phi=\frac{\partial\mathcal{X}_\eta}{\partial n}$ is an element of the kernel, then (5.36) holds since

$$\int_{\partial(\eta T)}\frac{\partial\mathcal{X}_\eta}{\partial n}\,\mathrm{d}\sigma=-|P\backslash(\eta T)|=-1+\eta^d|T|.$$

□

Motivated by the characterization (5.36), we show in the next Sections 5.3 and 5.4 how to choose the constant κ in (5.35) in such a way $\mathcal{S}_{\#,\eta T}$ has a non trivial kernel.

Before proceeding, we introduce the rescaling function τ_η defined by

$$\tau_\eta(t):=\eta t \text{ for any } t\in\ \partial T,$$

and the rescaling operator $\mathcal{P}_\eta : H^s(\partial(\eta T))\to H^s(\partial T)$ defined by

$$\mathcal{P}_\eta[\phi]:=\phi\circ\tau_\eta \text{ for any } \phi\in H^s(\partial(\eta T)), \text{ for any } s\in\mathbb{R}.$$

Using the same argument as in Proposition 2.5, we can prove that $\mathcal{S}^{\kappa}_{\#,\eta T}$ can be factorized in terms of some analytic operator $\mathcal{S}^{\kappa}_T(\eta)$.

Proposition 5.7. *The following factorization holds:*

$$\mathcal{S}^{\kappa}_{\#,\eta T}=\eta\mathcal{P}_\eta^{-1}\mathcal{S}^{\kappa}_T(\eta)\mathcal{P}_\eta, \tag{5.37}$$

where $\mathcal{S}^{\kappa}_T(\eta) : H^{-\frac12}(\partial\omega)\to H^{\frac12}(\partial\omega)$ is given by

$$\mathcal{S}^{\kappa}_T(\eta)[t]=\mathcal{S}_T[\phi](t)+\kappa\int_{\partial T}\phi\,\mathrm{d}\sigma+\eta^{d-2}\int_{\partial T}R_\#(\eta(t-t'))\phi(t')\,\mathrm{d}\sigma(t'),\qquad t\in\partial T. \tag{5.38}$$

Proof. The proof is identical to that of Proposition 2.5, where simplifications occur in both cases $d\geq 3$ and $d=2$ owing to the choice of the additive constants in the definition (5.35). □

5.3. Full asymptotic expansions in dimension $d\geq 3$

Throughout this part, we denote by $\phi^*\in L^2(\partial T)$ the function

$$\phi^*:=\mathcal{S}_T^{-1}[1_{\partial T}]=\frac{\partial\Phi}{\partial n}\Big|_+, \tag{5.39}$$

where $\Phi\in\mathcal{D}^{1,2}(\mathbb{R}^d\backslash T)$ is the solution to the exterior problem (3.9). We denote by $\mathrm{cap}(T)$ the capacity of the obstacle T which is the positive number defined by [8]:

$$\mathrm{cap}(T):=-\int_{\partial T}\phi^*\,\mathrm{d}\sigma=-\int_{\partial T}\frac{\partial\Phi}{\partial n}\,\mathrm{d}\sigma=\int_{\mathbb{R}^d\backslash T}|\nabla\Phi|^2\,\mathrm{d}x. \tag{5.40}$$

Proposition 5.8. *There exists a real analytic function $\eta\mapsto\kappa_\eta$ such that the operator $\mathcal{S}^{\kappa_\eta}_{\#,\eta T}$ defined according to (5.35) has a non-trivial kernel, given by*

$$\mathrm{Ker}(\mathcal{S}^{\kappa_\eta}_{\#,\eta T})=\mathrm{span}\left(\frac{\partial\mathcal{X}_\eta}{\partial n}\right).$$

Moreover, κ_η and $\frac{\partial\mathcal{X}_\eta}{\partial n}$ admit the following convergent series representations:

$$\kappa_\eta:=\frac{1}{\mathrm{cap}(T)}+\sum_{p\geq d}\eta^p c_p,\ \text{ and }\ \frac{\partial\mathcal{X}_\eta}{\partial n}=(1-\eta^d|T|)\left(\frac{\eta^{1-d}}{\mathrm{cap}(T)}\phi^*\circ\tau_\eta^{-1}+\sum_{p\geq1}\eta^p\phi_p\circ\tau_\eta^{-1}\right), \tag{5.41}$$

for some constants $(c_p)_{p\geq d}$ and functions $(\phi_p)_{p\geq1}$ of $L^2(\partial T)$ satisfying

$$\int_{\partial T}\phi_p\,\mathrm{d}\sigma=0 \text{ for all } p\geq1.$$

Proof. Let us consider the functional

$$F((\kappa,\phi),\eta) := \left(\mathcal{S}_T[\phi] + \kappa\int_{\partial T}\phi\,\mathrm{d}\sigma + \eta^{d-2}\int_{\partial T} R_\#(\eta(\cdot - t'))\phi(t')\,\mathrm{d}\sigma(t'), \int_{\partial T}\phi\,\mathrm{d}\sigma + \operatorname{cap}(T)\right). \tag{5.42}$$

Clearly, $F((1/\operatorname{cap}(T),\phi^*),0) = 0$. Moreover, let us solve the following linear system with right-hand side $(\psi, b) \in L^2(\partial T)\times\mathbb{R}$ and unknown $(\delta\phi,\delta\kappa)\in L^2(\partial T)\times\mathbb{R}$:

$$\mathrm{D}_{(\kappa,\phi)}F((1/\operatorname{cap}(T),\phi^*),0)\cdot(\delta\phi,\delta\kappa) = \left(\mathcal{S}_T[\delta\phi] + \delta\kappa\int_{\partial T}\phi^*\,\mathrm{d}\sigma + \frac{1}{\operatorname{cap}(T)}\int_{\partial T}\delta\phi\,\mathrm{d}\sigma, \int_{\partial T}\delta\phi\,\mathrm{d}\sigma\right) = (\psi, b).$$

The above equation is equivalent to

$$\delta\phi - \operatorname{cap}(T)\delta\kappa\phi^* + \frac{1}{\operatorname{cap}(T)}b\phi^* = \mathcal{S}_T^{-1}[\psi] \text{ and } \int_{\partial T}\delta\phi\,\mathrm{d}\sigma = b.$$

Integrating over ∂T, we obtain

$$\delta\kappa = \frac{1}{\operatorname{cap}(T)^2}\int_{\partial T}\mathcal{S}_T^{-1}[\psi]\,\mathrm{d}\sigma \text{ and } \delta\phi = \mathcal{S}_T^{-1}[\psi] + \frac{1}{\operatorname{cap}(T)}\left(\int_{\partial T}\mathcal{S}_T^{-1}[\psi]\,\mathrm{d}\sigma - b\right)\phi^*.$$

Consequently, $\mathrm{D}_{(\kappa,\phi)}F((1/\operatorname{cap}(T),\phi^*),0)$ is invertible, and the inverse reads

$$\begin{aligned}\mathrm{D}_{(\kappa,\phi)}&F((1/\operatorname{cap}(T),\phi^*),0)^{-1}\cdot(\psi,b)\\ &= \left(\mathcal{S}_D^{-1}[\psi] + \frac{1}{\operatorname{cap}(T)}\int_{\partial T}\psi\,\mathrm{d}\sigma\phi^* - \frac{1}{\operatorname{cap}(T)}b\phi^*, -\frac{1}{\operatorname{cap}(T)}\int_{\partial T}\psi\,\mathrm{d}\sigma\right).\end{aligned} \tag{5.43}$$

Therefore, the (analytic) implicit function theorem yields analytic κ_η and ϕ_η such that $F((\kappa_\eta,\phi_\eta),\eta)=0$ for η belonging to a neighborhood of zero with $c_0 = 1/\operatorname{cap}(T)$ and $\phi_0=\phi^*$. Obviously, with such choice of k_η and ϕ_η, it holds $\mathcal{S}_T^{\kappa_\eta}(\eta)[\phi_\eta]=0$ and hence $\mathcal{S}^{\kappa_\eta}_{\#,\eta T}[\phi_\eta\circ\tau_\eta^{-1}]=0$ (due to the factorization (5.37)), showing that $\operatorname{Ker}(\mathcal{S}_{\#,\eta T})$ is not trivial. Let us estimate the magnitude of the first order variation $(\delta\phi,\delta\kappa) := (\phi_\eta - \phi^*, \kappa_\eta - 1/\operatorname{cap}(T))$. We have by using (5.17):

$$\begin{aligned}0 &= F((\kappa_\eta,\phi_\eta),\eta)\\ &= \mathrm{D}_{(\kappa,\phi)}F((c_0,\phi^*),0)\cdot(\delta\phi,\delta\kappa) + \left(\eta^{d-2}\int_{\partial T}R_\#(\eta(\cdot - t'))\phi_\eta(t')\,\mathrm{d}\sigma(t'), 0\right)\\ &= \mathrm{D}_{(\kappa,\phi)}F((c_0,\phi^*),0)\cdot(\delta\phi,\delta\kappa) + \left(O(\eta^d),0\right).\end{aligned}$$

Solving this equation with (5.43), we obtain that $(\delta\phi,\delta\kappa)$ is of order $O(\eta^d)$. We obtain therefore the existence of $(c_p)_{p\geq d}$ and $(\phi_p)_{p\geq d}$ such that

$$\kappa_\eta = \frac{1}{\operatorname{cap}(T)} + \sum_{p\geq d}\eta^p c_p \text{ and } \phi_\eta = \phi^* + \sum_{p\geq d}\eta^p\phi_p.$$

Finally, Lemma 5.2 implies that there exists a constant α_η such that $\frac{\partial\mathcal{X}_\eta}{\partial n} = \alpha_\eta\phi_\eta\circ\tau_\eta^{-1}$. The constant can be identified by computing

$$\int_{\partial(\eta T)}\frac{\partial\mathcal{X}_\eta}{\partial n}\,\mathrm{d}\sigma = -\int_{P\backslash(\eta T)}\Delta\mathcal{X}_\eta\,\mathrm{d}x = -\int_{P\backslash(\eta T)}1\,\mathrm{d}x = -1 + \eta^d|T|;$$

while

$$\alpha_\eta\int_{\partial(\eta T)}\phi_\eta\circ\tau_\eta^{-1}\,\mathrm{d}\sigma = \alpha_\eta\eta^{d-1}\int_{\partial T}\phi_\eta\,\mathrm{d}\sigma = -a_\eta\eta^{d-1}\operatorname{cap}(T).$$

The result follows with $\alpha_\eta = (1-\eta^d|T|)\eta^{1-d}/\operatorname{cap}(T)$. ☐

We are now ready to read an ansatz for $\mathcal{X}_\eta$ from the series expansion (5.41).

Proposition 5.9. *There exist functions $(v_p)_{p\geq 2}$ and $(w_p)_{p\geq 0}$ such that the following ansatz holds:*

$$\mathcal{X}_\eta(x) = \frac{\eta^{2-d}}{\operatorname{cap}(T)}(1-\Phi(x/\eta)) + \sum_{p\geq 2}\eta^p v_p(x/\eta) + \sum_{p\geq 0}\eta^p w_p(x), \tag{5.44}$$

where:

(i) the series are convergent for any fixed $x\in P\backslash\{0\}$;

(ii) Φ is the solution to the exterior problem (3.9);

(iii) v_p is the solution to an exterior Dirichlet problem in $\mathbb{R}^d\backslash T$ for any $p\geq 2$. Moreover, $\int_{\partial T}\left[\!\left[\frac{\partial v_p}{\partial n}\right]\!\right]\mathrm{d}\sigma = 0$ or equivalently, $v_p(x) = O(|x|^{1-d})$ as $|x|\to+\infty$ for $p\geq 2$;

(iv) $w_p\in H^1(P)$ is a function of the interor domain, and $w_0(0)=0$ and $w_1(0)=0$.

Proof. From the result of Lemma 5.2 and the factorization (5.37), we can infer that

$$\mathcal{X}_\eta(x) = \left(\int_{\partial(\eta T)} \frac{\partial \mathcal{X}_\eta}{\partial n} \,\mathrm{d}\sigma \right)^{-1} \mathcal{S}^{\kappa_\eta}_{\#,\eta T}\left[\frac{\partial \mathcal{X}_\eta}{\partial n}\right](x) = \frac{1}{-1+\eta^d|T|}\mathcal{S}^{\kappa_\eta}_{T}(\eta)\left[\frac{\partial \mathcal{X}_\eta}{\partial n}\circ\tau_\eta\right](x/\eta). \tag{5.45}$$

The ansatz follows by inserting (5.41) in this expression and by using the formula (5.38). □

Similarly as in Sections 3.3 and 4.3, we characterize the corrector functions $(v_p)_{p\geq 2}$ and $(w_p)_{p\geq 0}$ as the solutions to a recursive system of exterior Dirichlet problems in $\mathbb{R}^d\backslash T$ and periodic Laplace problems in P in the form of (5.11).

Proposition 5.10. *The functions $(v_p)_{p\geq 2}$ and $(w_p)_{p\geq 0}$ of (5.44) are uniquely characterized as the solutions to the following recursive systems of exterior and interior problems:*

$$\begin{cases} -\Delta w_p = \begin{cases} 1 \text{ if } p=0, \\ 0 \text{ if } p\geq 1, \end{cases} & \text{in } P, \\ \begin{cases} w_p - \dfrac{1}{\mathrm{cap}(T)}\Phi^{(d-2+p)} \text{ is } P\text{–periodic}, \\ \dfrac{\partial w_p}{\partial n}\boldsymbol{n} - \dfrac{1}{\mathrm{cap}(T)}\dfrac{\partial \Phi^{(d-2+p)}}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic}, \end{cases} & \text{for } 0\leq p\leq d, \\ \begin{cases} w_p - \dfrac{1}{\mathrm{cap}(T)}\Phi^{(d-2+p)} + \displaystyle\sum_{k=d-1}^{p-2} v^{(k)}_{p-k} \quad \text{is } P\text{–periodic}, \\ \dfrac{\partial w_p}{\partial n}\boldsymbol{n} - \dfrac{1}{\mathrm{cap}(T)}\dfrac{\partial \Phi^{(d-2+p)}}{\partial n}\boldsymbol{n} + \displaystyle\sum_{k=d-1}^{p-2}\dfrac{\partial v^{(k)}_{p-k}}{\partial n}\boldsymbol{n} \quad \text{is } P\text{–periodic}, \end{cases} & \text{for } p>d, \end{cases} \tag{5.46}$$

and

$$\begin{cases} -\Delta v_p = 0 \text{ in } \mathbb{R}^d\backslash T, \\ v_p(t) = -w_p(0) - \displaystyle\sum_{k=1}^{p}\frac{1}{k!}\nabla^k w_{p-k}(0)\cdot t^k \text{ for } t\in\partial T, \quad p\geq 2, \\ v_p(x) = O(|x|^{1-d}) \text{ as } |x|\to+\infty, \end{cases} \tag{5.47}$$

where $w_0(0)=w_1(0)=0$ and $w_p(0)$ is determined for $p\geq 2$ by the condition

$$\int_{\partial T}\left.\frac{\partial v_p}{\partial n}\right|_+ \mathrm{d}\sigma = 0 \text{ for } p\geq 2. \tag{5.48}$$

and $(\Phi^{(k)})_{k\in\mathbb{N}}$ and $(v^{(k)}_p)_{k\in\mathbb{N}}$ denote the functions arising in the far field expansion of $(v_p)_{p\geq 2}$ and Φ:

$$v^{(k)}_p(x) := \frac{(-1)^{k-d+2}}{(k-d+2)!}\nabla^{k-d+2}\Gamma(x)\cdot\int_{\partial T}\left[\!\left[\frac{\partial v_p}{\partial n}\right]\!\right] t^{k-d+2}\,\mathrm{d}\sigma(t), \quad k\geq d-1, \tag{5.49}$$

and

$$\Phi^{(k)}(x) := \frac{(-1)^{k-d+2}}{(k-d+2)!}\nabla^{k-d+2}\Gamma(x)\cdot\int_{\partial T}\left.\frac{\partial \Phi}{\partial n}\right|_+ t^{k-d+2}\,\mathrm{d}\sigma(t), \quad k\geq d-2. \tag{5.50}$$

Proof. The identity $-\Delta\mathcal{X}_\eta = 1 = -\sum_{p\geq 0}\eta^p\Delta w_p$, yields by identification $-\Delta w_0=1$, and $-\Delta w_p=0$ for $p\geq 1$ in P. Then, setting $x=\eta t$ with $t\in\partial T$, we obtain

$$\mathcal{X}_\eta(t) = 0 = \sum_{p\geq 2}\eta^p v_p(t) + \sum_{p=0}^{+\infty}\eta^p\sum_{k=0}^{+\infty}\eta^k\frac{1}{k!}\nabla^k w_p(0)\cdot t^k = \sum_{p\geq 2}\eta^p v_p(t) + \sum_{p=0}^{+\infty}\sum_{k=0}^{p}\eta^p\frac{1}{k!}\nabla^k w_{p-k}(0)\cdot t^k,$$

from where we infer (5.47). For $x\in\partial P$, we find that

$$\begin{aligned}\mathcal{X}_\eta(x) &= \frac{\eta^{2-d}}{\mathrm{cap}(T)}\left(1-\sum_{k=d-2}^{+\infty}\eta^k\Phi^{(k)}(x)\right) + \sum_{p\geq 2}\eta^p\sum_{k=d-1}^{+\infty}\eta^k v^{(k)}_p(x) + \sum_{p\geq 0}\eta^p w_p(x) \\ &= \frac{\eta^{2-d}}{\mathrm{cap}(T)} - \sum_{p=0}^{+\infty}\frac{\eta^p}{\mathrm{cap}(T)}\Phi^{(d-2+p)}(x) + \sum_{p=d+1}^{+\infty}\sum_{k=d-1}^{p-2}\eta^p v^{(k)}_{p-k}(x) + \sum_{p\geq 0}\eta^p w_p(x).\end{aligned}$$

The boundary conditions of (5.46) follow from the fact that $\mathcal{X}_\eta$ must be P–periodic.
Finally, let us verify the well-posedness of (5.46) and (5.47), namely the compatibility condition (5.10) which ensures the well-posedness of (5.46). Since the solution Φ of (3.9) satisfies $\Phi^{(d-2)}(x) = -\mathrm{cap}(T)\Gamma(x)$, it follows that the compatibility condition of (5.46) is satisfied for $p=0$, and we have even $w_0=R_\#$. Due to the identities (2.6) of Lemma 2.1, it is also satisfied for $p\geq 1$. Therefore, (5.46) admits a unique solution w_p defined up to

the choice of the value of $w_p(0)$ as soon as the functions $(v'_p)_{0\le p'\le p-(d-1)}$ are known. Finally, let us clarify how the constant $w_p(0)$ in (5.47) is determined by (5.48). Clearly, $v_p = -w_p(0)\Phi + \widetilde{v}_p$ where $\widetilde{v}_p$ is the solution to

$$\begin{cases} -\Delta \widetilde{v}_p = 0 \text{ in } \mathbb{R}^d \setminus \overline{T}, \\ \widetilde{v}_p(t) = -\displaystyle\sum_{k=1}^{p} \frac{1}{k!} \nabla^k w_{p-k}(0) \cdot t^k \text{ for } t \in \partial T, \\ \widetilde{v}_p(x) = O(|x|^{2-d}) \text{ as } |x| \to +\infty. \end{cases}$$

From (5.48), we infer that $w_p(0)$ is given by the formula $w_p(0) = -\frac{1}{\mathrm{cap}(T)} \int_{\partial T} \frac{\partial \widetilde{v}_p}{\partial n}\Big|_+ \mathrm{d}\sigma$. □

We conclude our analysis by providing the following error bounds for the truncation of the ansatz (5.44). The proof requires significant adaptations of the one of Proposition 3.7 due to the peculiarity of the variational framework accounting for the P–periodicity.

Proposition 5.11. *For any $N \in \mathbb{N}$, let $\mathcal{X}_\eta^N$ be the truncated ansatz at rank N:*

$$\mathcal{X}_\eta^N(x) := \frac{\eta^{2-d}}{\mathrm{cap}(T)}(1 - \Phi(x/\eta)) + \sum_{p=2}^{N} \eta^p v_p(x/\eta) + \sum_{p=0}^{N} \eta^p w_p(x),$$

where the functions $(v_p)_{p\ge 2}$ and $(w_p)_{p\ge 0}$ are defined according to Proposition 5.10. Then $\mathcal{X}_\eta^N$ is an approximation of the solution $\mathcal{X}_\eta$ to (1.2) at the order $O(\eta^{N+1})$ in the $H^1(P\setminus(\eta T))$–norm according to the following error bound:

$$||\mathcal{X}_\eta - \mathcal{X}_\eta^N||_{L^2(P\setminus(\eta T))} + ||\nabla \mathcal{X}_\eta - \nabla \mathcal{X}_\eta^N||_{L^2(P\setminus(\eta T))} \le C_N \eta^{N+1}.$$

Proof. The function $r_\eta := \mathcal{X}_\eta - \mathcal{X}_\eta^N$ satisfies $-\Delta r_\eta = 0$ (because the term $w_0 = R_\#$ is included in the truncated ansatz), as well as the following boundary conditions: for any $t \in \partial T$,

$$\begin{aligned} r_\eta(\eta t) &= -\mathcal{X}_\eta^N(\eta t) = -\sum_{p=2}^{N} \eta^p v_p(t) - \sum_{p=0}^{N} \eta^p w_p(\eta t) = -\sum_{p=2}^{N} \eta^p v_p(t) - \sum_{p=0}^{N}\sum_{k=0}^{N-p} \eta^{p+k} \frac{1}{k!} \nabla^k w_p(0)\cdot t^k + O(\eta^{N+1}) \\ &= -\sum_{p=2}^{N} \eta^p v_p(t) - \sum_{p=0}^{N}\sum_{k=0}^{p} \eta^p \frac{1}{k!} \nabla^k w_{p-k}(0)\cdot t^k + O(\eta^{N+1}) = -w_0(0) - \eta(w_1(0) + \nabla w_0(0)\cdot t) + O(\eta^{N+1}) \\ &= O(\eta^{\min(2,N+1)}), \end{aligned}$$

and for any $x \in \partial P$:

$$\begin{aligned} r_\eta(x) &= \mathcal{X}_\eta(x) - \frac{\eta^{2-d}}{\mathrm{cap}(T)} + \frac{\eta^{2-d}}{\mathrm{cap}(T)}\Phi(x/\eta) - \sum_{p=2}^{N} \eta^p v_p(x/\eta) - \sum_{p=0}^{N} \eta^p w_p(x) \\ &= \frac{\eta^{2-d}}{\mathrm{cap}(T)}\Phi(x/\eta) - \sum_{p=2}^{N} \eta^p v_p(x/\eta) - \frac{1}{\mathrm{cap}(T)} \sum_{p=0}^{N} \eta^p \Phi^{(d-2+p)}(x) + \sum_{p=0}^{N}\sum_{k=d+1}^{p-2} \eta^p v_{p-k}^{(k)} + H, \end{aligned}$$

where H is a P–periodic function due to the boundary conditions of (5.46). Let then h_η to be the function defined on $P\setminus\overline{T}$ (a neighborhood of ∂T) by:

$$h_\eta(x) := \frac{\eta^{2-d}}{\mathrm{cap}(T)}\Phi(x/\eta) - \sum_{p=2}^{N} \eta^p v_p(x/\eta) - \frac{1}{\mathrm{cap}(T)} \sum_{p=0}^{N} \eta^p \Phi^{(d-2+p)}(x) + \sum_{p=0}^{N}\sum_{k=d+1}^{p-2} \eta^p v_{p-k}^{(k)}(x).$$

Using the asymptotic expansion in the far field of Φ and v_p, we find that $||h_\eta||_{H^k(P\setminus\overline{T})} \le C_k \eta^{N+1}$ for any $k > d/2 - 1$. Furthermore, observing that $\Delta h_\eta = 0$, we can use the result of Proposition 5.6 to estimate the function r_η, which satisfies

$$\begin{cases} -\Delta r_\eta = 0 \text{ in } P\setminus(\eta T), \\ r_\eta(\eta t) = O(\eta^{\min(2,N+1)}) \text{ for } t \in \partial T, \\ r_\eta - h_\eta \text{ is } P\text{–periodic}, \\ \dfrac{\partial r_\eta}{\partial n}\boldsymbol{n} - \dfrac{\partial h_\eta}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic}. \end{cases}$$

By using Lemma 2.1 and (5.50), we evaluate the default of compatibility which is given in this context by

$$\begin{aligned}
\alpha(0,h_\eta) = \int_{\partial P} \frac{\partial h_\eta}{\partial n}\,\mathrm{d}\sigma &= \frac{\eta^{2-d}}{\mathrm{cap}(T)}\int_{\partial P}\frac{\partial \Phi(x/\eta)}{\partial n}\,\mathrm{d}\sigma - \sum_{p=2}^{N}\eta^p\int_{\partial P}\frac{\partial v_p(x/\eta)}{\partial n}\,\mathrm{d}\sigma + 1\\
&= \frac{\eta^{2-d}}{\mathrm{cap}(T)}\int_{\partial(\eta T)}\frac{\partial \Phi(x/\eta)}{\partial n}\,\mathrm{d}\sigma - \sum_{p=2}^{N}\eta^p\int_{\partial(\eta T)}\frac{\partial v_p(x/\eta)}{\partial n}\,\mathrm{d}\sigma + 1\\
&= \frac{1}{\mathrm{cap}(T)}\int_{\partial T}\frac{\partial \Phi}{\partial n}\,\mathrm{d}\sigma - \sum_{p=2}^{N}\eta^{p+d-2}\int_{\partial T}\frac{\partial v_p}{\partial n}\,\mathrm{d}\sigma + 1 = 0,
\end{aligned}$$

where we used the harmonicity of Φ and v_p at the second line, and the definition (5.40) of the capacity at the third line. The final estimate follows by using the result of estimates of Proposition 5.6. □

5.4. Full asymptotic expansions in dimension $d = 2$

In the two-dimensional case, we consider the function

$$\phi^* := \left.\frac{\partial \Phi}{\partial n}\right|_+,$$

where Φ is the solution to the exterior boundary value problem (4.8) with logarithmic growth at infinity. In this two-dimensional context, we can adapt the proof of Proposition 5.8 to obtain the following result.

Proposition 5.12. *There exists a real analytic function $\eta \mapsto \kappa_\eta$ such that the operator $\mathcal{S}^{\kappa_\eta}_{\#,\eta T}$ defined according to (5.35) has a non-trivial kernel, given by*

$$\mathrm{Ker}(\mathcal{S}^{\kappa_\eta}_{\#,\eta T}) = \mathrm{span}\left(\frac{\partial \mathcal{X}_\eta}{\partial n}\right).$$

Moreover, κ_η and $\frac{\partial \mathcal{X}_\eta}{\partial n}$ admit the following series representations:

$$\kappa_\eta = \Phi_\infty + \sum_{p\geq 2}\eta^p c_p \text{ and } \frac{\partial \mathcal{X}_\eta}{\partial n} = (1-\eta^2|T|)\left(-\eta^{-1}\phi^*\circ\tau_\eta^{-1} + \sum_{p\geq 1}\eta^p\phi_p\circ\tau_\eta^{-1}\right),$$

where Φ_∞ is the constant of (4.9), and for some constants $(c_p)_{p\geq 2}$ and functions $(\phi_p)_{p\geq 1}$ of $L^2(\partial T)$ satisfying

$$\int_{\partial T}\phi_p\,\mathrm{d}\sigma = 0 \text{ for all } p \geq 1.$$

Proof. We apply once again the implicit function theorem to the functional $F((\kappa,\phi),\eta)$ with $\mathrm{cap}(T)$ replaced by -1 in (5.42). Since $F((\Phi_\infty, \frac{\partial\Phi}{\partial n}|_+),0) = 0$, we obtain by proceeding similarly the existence of coefficients $(c_p)_{p\geq 2}$ and $(\phi_p)_{p\geq 2}$ such that $F((\kappa_\eta,\phi_\eta),\eta) = 0$ for small $\eta \geq 0$ with

$$\kappa_\eta = \Phi_\infty + \sum_{p\geq 2}\eta^p c_p \text{ and } \phi_\eta = \phi^* + \sum_{p\geq 2}\eta^p\phi_p.$$

Then, there exists a constant α_η such that $\frac{\partial \mathcal{X}}{\partial n} = \alpha_\eta\phi_\eta\circ\tau_\eta^{-1}$, and the same identification process yields $\alpha_\eta = -(1-\eta^2|T|)\eta^{-1}$. □

Repeating the proof of Proposition 5.9, we infer as such the following ansatz for $\mathcal{X}_\eta$ in dimension $d = 2$.

Proposition 5.13. *There exist functions $(v_p)_{p\geq 2}$ and $(w_p)_{p\geq 0}$ such that the following ansatz holds:*

$$\mathcal{X}_\eta(x) = \Phi(x/\eta) + \sum_{p=2}^{+\infty}\eta^p v_p(x/\eta) + \sum_{p=0}^{+\infty}\eta^p w_p(x), \tag{5.51}$$

where:

- *(i) the series of (5.51) converge for any fixed $x \in P\backslash\{0\}$;*
- *(ii) Φ is the solution to the exterior problem (4.8) with logarithmic growth;*
- *(iii) $v_p \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash T)$ is the solution to an exterior Dirichlet problem in $\mathbb{R}^d\backslash\overline{T}$ satisfying $v_p(x) = O(|x|^{-1})$ as $|x| \to +\infty$, for $p \geq 2$ (namely, satisfying additionally $v_p^\infty = 0$);*
- *(iv) $w_p \in H^1(P)$ is a function of the interior domain satisfying $w_0(0) = w_0(1) = 0$.*

Remark 5.4. It is remarkable that no logarithm of η occurs in the ansatz (5.51), in contrast to the one (4.22) obtained for the perforated problem (1.1). This further highlights the strength of the layer potential method to derive the correct form of the two scale asymptotic expansion.

Proposition 5.14. *The functions $(v_p)_{p\geq 2}$ and $(w_p)_{p\geq 0}$ of (5.51) are uniquely characterized as the solutions to the following recursive systems of exterior and interior problems:*

$$\begin{cases} -\Delta w_p = \begin{cases} 1 \text{ if } p=0, \\ 0 \text{ if } p\geq 1, \end{cases} & \text{in } P, \\ \begin{cases} w_p + \Phi^{(k)} \text{ is } P\text{–periodic}, \\ \dfrac{\partial w_p}{\partial n}\boldsymbol{n} + \dfrac{\partial \Phi^{(k)}}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic}, \end{cases} & \text{for } 0 \leq p \leq 2, \\ \begin{cases} w_p + \Phi^{(k)} + \displaystyle\sum_{k=1}^{p-2} v_{p-k}^{(k)} \text{ is } P\text{–periodic}, \\ \dfrac{\partial w_p}{\partial n}\boldsymbol{n} + \dfrac{\partial \Phi^{(k)}}{\partial n}\boldsymbol{n} + \displaystyle\sum_{k=1}^{p-2} \dfrac{\partial v_{p-k}^{(k)}}{\partial n}\boldsymbol{n} \text{ is } P\text{–periodic}, \end{cases} & \text{for } 0 \leq p \leq 2, \end{cases} \tag{5.52}$$

and

$$\begin{cases} -\Delta v_p = 0 \text{ in } \mathbb{R}^d \backslash T, \\ v_p(t) = -w_p(0) - \displaystyle\sum_{k=1}^{p} \frac{1}{k!}\nabla^k w_{p-k}(0)\cdot t^k \text{ for } t \in \partial T, \quad p \geq 2, \\ v_p(x) = O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases} \tag{5.53}$$

Here, $w_0(0) = 0$, $w_1(0) = 0$ and $w_p(0)$ is determined by the condition $v_p(x) = O(|x|^{-1})$ as $|x| \to +\infty$ for $p \geq 2$, and $v_p^{(k)}$ and $\Phi^{(k)}$ are defined as in (5.49) and (5.50) for $k \geq 1$.

Proof. First, $-\Delta \mathcal{X}_\eta = 1$ yields similarly as in Proposition 5.10 the identity $-\Delta w_0 = 1$ and $-\Delta w_p = 0$ for $p \geq 1$, as well as the boundary condition of (5.53) on ∂T. Then, for $x \in \partial P$, we have

$$\begin{aligned} \mathcal{X}_\eta(x) &= -\frac{1}{2\pi}\log\eta + \Phi^\infty + \sum_{k=0}^{+\infty} \eta^k \Phi^{(k)}(x) + \sum_{p=2}^{+\infty}\sum_{k=1}^{+\infty} \eta^{p+k} v_p^{(k)}(x) + \sum_{p=0}^{+\infty} \eta^p w_p(x) \\ &= -\frac{1}{2\pi}\log\eta + \Phi^\infty + \sum_{k=0}^{+\infty} \eta^k \Phi^{(k)}(x) + \sum_{p=3}^{+\infty}\sum_{k=1}^{p-2} \eta^{p} v_{p-k}^{(k)}(x) + \sum_{p=0}^{+\infty} \eta^p w_p(x), \end{aligned}$$

which yields the periodic boundary condition of (5.52). It is clear that the compatibility conditions of (5.52) are satisfied (due to Lemma 2.1). Finally, $w_p(0)$ is determined as $w_p(0) = \widetilde{v}_p^\infty$ where $\widetilde{v}_p \in \mathcal{D}^{1,2}(\mathbb{R}^2\backslash\overline{T})$ is the unique solution to the Dirichlet problem

$$\begin{cases} -\Delta \widetilde{v}_p = 0 \text{ in } \mathbb{R}^2\backslash\overline{T}, \\ \widetilde{v}_p = -\displaystyle\sum_{k=1}^{p} \frac{1}{k!}\nabla w_{p-k}(0)\cdot t^k \text{ for } t \in \partial T, \\ \widetilde{v}_p(x) = \widetilde{v}_p^\infty + O(|x|^{-1}) \text{ as } |x| \to +\infty. \end{cases}$$

□

Adapting the proof of Proposition 5.11 yields the following final result.

Proposition 5.15. *For any $N \in \mathbb{N}$, let $\mathcal{X}_\eta^N$ be the truncated ansatz at rank N:*

$$\mathcal{X}_\eta^N(x) := \Phi(x/\eta) + \sum_{p=2}^{N} \eta^p v_p(x/\eta) + \sum_{p=0}^{N} \eta^p w_p,$$

where $(v_p)_{p\geq 2}$ and $(w_p)_{p\geq 0}$ are the functions defined by the recursive system of Proposition 5.14. Then $\mathcal{X}_\eta^N$ is an approximation of the solution $\mathcal{X}_\eta$ to (1.2) at order $O(\eta^{N+1})$ in the $H^1(P\backslash(\eta T))$ norm according to the following error bounds:

$$||\mathcal{X}_\eta - \mathcal{X}_\eta^0||_{L^2(P\backslash(\eta T))} + ||\nabla\mathcal{X}_\eta - \nabla\mathcal{X}_\eta^0||_{L^2(P\backslash(\eta T))} \leq C_N \eta,$$

and

$$|\log\eta|^{-\frac{1}{2}}||\mathcal{X}_\eta - \mathcal{X}_\eta^N||_{L^2(P\backslash(\eta T))} + ||\nabla\mathcal{X}_\eta - \nabla\mathcal{X}_\eta^N||_{L^2(P\backslash(\eta T))} \leq C_N \eta^{N+1} \qquad \text{for any } N \geq 1.$$

Proof. The proof is identical to that of Proposition 5.11, the only difference coming from the factor $|\log\eta|^{\frac{1}{2}}$ coming from (5.27), which affects the $L^2(P\backslash(\eta T))$ bound on $\mathcal{X}_\eta - \mathcal{X}_\eta^N$. □

References

[1] Allaire, G. Homogenization of the Stokes flow in a connected porous medium. *Asymptotic Analysis 2*, 3 (1989), 203–222.

[2] Allaire, G. Continuity of the Darcy's law in the low-volume fraction limit. *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie IV 18*, 4 (1991), 475–499.

[3] Allaire, G. Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes I. Abstract framework, a volume distribution of holes. *Archive for Rational Mechanics and Analysis 113*, 3 (1991), 209–259.

[4] Allaire, G., Jouve, F., and Van Goethem, N. Damage and fracture evolution in brittle materials by shape optimization methods. *Journal of Computational Physics 230*, 12 (2011), 5010–5044.

[5] Ammari, H., Calmon, P., and Iakovleva, E. Direct elastic imaging of a small inclusion. *SIAM Journal on Imaging Sciences 1*, 2 (2008), 169–187.

[6] Ammari, H., Fitzpatrick, B., Kang, H., Ruiz, M., Yu, S., and Zhang, H. *Mathematical and Computational Methods in Photonics and Phononics*. American Mathematical Society, oct 2018.

[7] Ammari, H., Garnier, J., Jugnon, V., and Kang, H. Stability and resolution analysis for a topological derivative based imaging functional. *SIAM Journal on Control and Optimization 50*, 1 (2012), 48–76.

[8] Ammari, H., and Kang, H. *Polarization and moment tensors: with applications to inverse problems and effective medium theory*, vol. 162. Springer Science & Business Media, 2007.

[9] Ammari, H., Kang, H., and Lee, H. Asymptotic expansions for eigenvalues of the Lamé system in the presence of small inclusions. *Communications in Partial Differential Equations 32*, 11 (2007), 1715–1736.

[10] Ammari, H., Kang, H., Nakamura, G., and Tanuma, K. Complete asymptotic expansions of solutions of the system of elastostatics in the presence of an inclusion of small diameter and detection of an inclusion. *Journal of elasticity and the physical science of solids 67*, 2 (2002), 97–129.

[11] Auroux, D., Jaafar-Belaid, L., and Rjaibi, B. Application of the topological gradient method to tomography. *Revue Africaine de la Recherche en Informatique et Mathématiques Appliquées 13* (2010), 91–104.

[12] Auroux, D., and Masmoudi, M. Image processing by topological asymptotic analysis. In *ESAIM: Proceedings* (2009), vol. 26, EDP Sciences, pp. 24–44.

[13] Bensoussan, A., Lions, J.-L., and Papanicolaou, G. *Asymptotic analysis for periodic structures*, vol. 374. American Mathematical Soc., 2011.

[14] Capdeboscq, Y., and Vogelius, M. S. A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction. *ESAIM: Mathematical Modelling and Numerical Analysis 37*, 1 (2003), 159–173.

[15] Caubet, F., and Dambrine, M. Localization of small obstacles in Stokes flow. *Inverse Problems 28*, 10 (2012), 105007.

[16] Dalla Riva, M., Musolino, P., and Rogosin, S. Series expansions for the solution of the Dirichlet problem in a planar domain with a small hole. *Asymptotic Analysis 92*, 3-4 (2015), 339–361.

[17] de Cristoforis, M. L. Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach. *Analysis 28*, 1 (2008), 63–93.

[18] de Faria, J. R., and Novotny, A. A. On the second order topological asymptotic expansion. *Structural and Multidisciplinary Optimization 39*, 6 (2009), 547–555.

[19] Feppon, F. High order homogenization of the Poisson equation in a perforated periodic domain. *To appear in the Radon Series on Computational and Applied Mathematics* (mar 2020).

[20] Feppon, F., and Ammari, H. Homogenization of sound-absorbing and high-contrast acoustic metamaterials in subcritical regimes. *HAL preprint hal-03372593* (Oct. 2021). working paper or preprint.

[21] Folland, G. B. *Introduction to Partial Differential Equations*, second ed., vol. 102. Princeton University Press, Princeton, NJ, 1976.

[22] Golse, F. MAT431 Distributions, analyse de Fourier, équations aux dérivées partielles, 2012.

[23] Guillaume, P., and Idris, K. S. The topological asymptotic expansion for the Dirichlet problem. *SIAM journal on Control and Optimization 41*, 4 (2002), 1042–1072.

[24] Hasimoto, H. On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres. *Journal of Fluid Mechanics 5*, 2 (1959), 317–328.

[25] Hassine, M., and Khelifi, K. On the high-order topological asymptotic expansion for shape functions. *Electronic journal of differential equations 2016*, 110 (2016), 1–16.

[26] Hintermüller, M., Laurain, A., and Novotny, A. A. Second-order topological expansion for electrical impedance tomography. *Advances in Computational Mathematics 36*, 2 (2012), 235–265.

[27] Jing, W. A unified homogenization approach for the Dirichlet problem in perforated domains. *SIAM Journal on Mathematical Analysis 52*, 2 (2020), 1192–1220.

[28] Kozlov, V., Maz'ya, V., and Movchan, A. *Asymptotic analysis of fields in multi-structures*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1999. Oxford Science Publications.

[29] Ladyzhenskaya, O. A. *The mathematical theory of viscous incompressible flow*, vol. 2. Gordon and Breach New York, 1969.

[30] Lions, J.-L. Some methods in the mathematical analysis of systems and their control. *Beijing, Science Press* (1981).

[31] Maz'ya, V., Movchan, A., Nieves, M., et al. *Green's kernels and meso-scale approximations in perforated domains*, vol. 2077. Springer, 2013.

[32] McLean, W. C. H. *Strongly elliptic systems and boundary integral equations*, vol. 86. Cambridge university press, 2000.

[33] Nédélec, J.-C. *Acoustic and electromagnetic equations: integral representations for harmonic problems*, vol. 144. Springer Science & Business Media, 2013.

[34] Pommier, J., and Samet, B. The topological asymptotic for the Helmholtz equation with Dirichlet condition on the boundary of an arbitrarily shaped hole. *SIAM journal on control and optimization 43*, 3 (2004), 899–921.

[35] Robinson, J. C. *Infinite-dimensional dynamical systems: an introduction to dissipative parabolic PDEs and the theory of global attractors*, vol. 28. Cambridge University Press, 2001.

[36] Samet, B., Amstutz, S., and Masmoudi, M. The topological asymptotic for the Helmholtz equation. *SIAM Journal on Control and Optimization 42*, 5 (2003), 1523–1544.

[37] Zimmerman, R. W. Elastic moduli of a solid containing spherical inclusions. *Mechanics of Materials 12*, 1 (1991), 17–24.