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Analytic regularity for the Navier-Stokes equations in polygons with mixed boundary conditions

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Abstract

We prove weighted analytic regularity of Leray-Hopf variational solutions for the stationary, incompressible Navier-Stokes Equations (NSE) in plane polygonal domains, subject to analytic body forces. We admit mixed boundary conditions which may change type at each vertex. The weighted analytic regularity results are established in Hilbertian Kondrat'ev spaces. The proofs rely on a priori estimates for the corresponding linearized boundary value problem in sectors in corner-weighted Sobolev spaces and on an induction argument for the weighted norm estimates on the quadratic nonlinear term in the NSE, in a polar frame.

1 Introduction

The regularity properties of viscous, incompressible flow governed by the incompressible Navier-Stokes Equations (NSE) have attracted considerable attention since their introduction. Regularity results of weak, Leray-Hopf solutions in Sobolev and Besov scales in domains are at the core of the numerical analysis of the NSE. The *stationary NSE*, being for large values of the viscosity parameter, a perturbation of its linearization, the Stokes Equation, is an elliptic system in the sense of Agmon-Douglis-Nirenberg and affords analytic regularity at interior points of domains, for analytic forcing [1], see also [2]. This local analyticity of the velocity and the pressure extends to analytic parts of the boundary.

However, it is also classical that in the vicinity of corner points (in space dimension $d = 2$) and near edges and vertices (for polyhedra in space dimension $d = 3$), analyticity is lost, even if all other data of the stationary NSE is analytic. See, e.g., [3–7] and the references there. The reason is the appearance of *corner singularities* (in space dimension $d = 2$) and of *corner- and edge-singularities* (in polyhedra in space dimension $d = 3$). While singular solutions of the Stokes equation are well known to encode physically relevant effects (see, e.g., [6, 8]), they do obstruct large elliptic regularity shifts in standard (Besov or Triebel-Lizorkin) scales of function spaces and, consequently, high convergence rates of numerical discretizations. This has initiated the investigation of regularity of solutions in the presence of non smooth boundaries. One, in a sense, minimally regular situation is the assumption of mere Lipschitz regularity of the boundary. For the mixed boundary conditions of interest here, some regularity of velocity and pressure of Leray solutions in non-weighted Sobolev spaces have been obtained in [9]. In the mentioned polygonal and polyhedral domains, it has been known for some time that the velocity fields of stationary solutions for the incompressible NSE in plane, polygonal domains allow higher regularity in so-called *corner-weighted Sobolev spaces*. Here, weight functions which vanish in the corners of the polygon to a suitable power compensate for the loss of regularity in the vicinity of the corner. The corresponding Mellin calculus originated in [10]. See, e.g., [4, 5] and the references there. In [11], an authoritative account of these results, also for NSE in polyhedra, has been given. The results in [11, Chapter 11] provide regularity shifts in weighted spaces of *finite order*. To prove *weighted, analytic regularity* for velocity field $\boldsymbol{u}$ and the pressure field p in P of the stationary, incompressible NSE in polygons is the purpose of the present paper. Specifically, in a bounded polygon $P \subset \mathbb{R}^2$ whose boundary ∂P consists of a finite number n of straight sides, we consider the analytic regularity of solutions of the viscous, incompressible Navier-Stokes equations. Extending and revisiting our work [12] which addressed homogeneous Dirichlet ("no-slip") boundary conditions, we consider here the stationary and incompressible NSE in plane polygonal domains P with *mixed boundary conditions*, where now also slip and so-called "open" boundary parts are admitted. These conditions arise in numerous configurations in engineering and the sciences. Furthermore, our present proof of the weighted analytic regularity requires a proof technique which differs from the approach used in [12]. As the corresponding analysis for plane, linearized elasticity in

[13], it is based on regularity results for the linearization (the Stokes problem) in a sector built on the Agranovich-Vishik theory of complex-parametric operator pencils which was already used in [14] and [13] to obtain a priori estimates and shift theorems in corner-weighted spaces. See also [15] for a general exposition of the role of operator pencils for elliptic systems in conical domains.

The present paper provides a proof of weighted analytic regularity for the velocity $\boldsymbol{u}$ and the pressure field p of the stationary, incompressible Navier-Stokes equations in a polygon P, subject to possibly mixed boundary conditions on the sides of P. The details of the proof are distinct from the argument in our previous work [12] even for pure Dirichlet boundary conditions. In [12], a bootstrapping argument based on local, Caccioppoli estimates on balls contained in P and scaling was proposed. Furthermore, the proof proposed in [12] was incomplete; the gap is closed by the argument in the present paper, which provides in particular in the case of homogeneous Dirichlet (so-called "no-slip") boundary conditions, the weighted analytic regularity result in [12]. This was used in [16] to prove exponential rates of convergence of a certain *hp*-DGFEM discretization of the stationary NSE in polygons.

Analytic regularity results for solutions in corner-weighted Kondrat'ev-Sobolev spaces imply, as is well-known, *exponential convergence rate bounds* for numerical approximations by so-called *hp*-Finite Element Methods and also by model order reduction methods. We refer to [16] and to the references there for recent results on exponential convergence for the Navier-Stokes equations, for discontinuous Galerkin discretizations, and also to the discussion in [12, Section 2.2] for exponential rates for certain model order reduction approaches to the NSE in P.

1.1 Contributions

We establish weighted, analytic regularity results for Leray-Hopf solutions of the NSE in bounded, connected polygonal domains $P \subset \mathbb{R}^2$ with finitely many, straight sides. We generalize the analytic regularity results stated in [12] from the pure Dirichlet (also referred to as "no-slip") boundary conditions as studied in [12] to the case of mixed boundary conditions at any two sides of P which meet at one common vertex of ∂P. As in [12] we work under a small data hypothesis, ensuring in particular the uniqueness of weak solutions. We also develop the regularity theory based on a priori estimates of solutions for a linearization, the Stokes problem, in weighted, Hilbertian Sobolev spaces in a sector. The result contains the analytic regularity result in [12] as a special case, and its proof proceeds in a way that is fundamentally different from [12]. As mentioned, it is based on a regularity analysis in corner-weighted spaces and a novel bootstrapping argument in the quadratic nonlinearity in weighted Kondrat'ev spaces. As in [13, 14], the weighted a priori estimates for the velocity field and the bounds on the quadratic nonlinearity near corners $\mathfrak{c}$ are obtained for the projection of the velocity components in a polar frame centered at $\mathfrak{c}$, rather than for their Cartesian components.

The main result of the present paper is stated in Theorem 8. Specifically, under the small data hypothesis and the stated assumptions on the boundary conditions (see Assumption 1 for details), we show that there exist $A > 0$ and $\gamma \in (0, 1)$ such that the Leray-Hopf solutions $(\boldsymbol{u}, p)$ to the NSE satisfy, for all $j, k \in \{0, 1, \dots\}$, and for any corner $\mathfrak{c}$ of P

$$\left\| \left(\prod_{\mathfrak{c} \in \mathfrak{C}} |\cdot - \mathfrak{c}|^{j+k-\gamma} \right) \partial_{x_1}^j \partial_{x_2}^k \boldsymbol{u} \right\|_{L^2(P)} \le A^{j+k+1}(j+k)!,$$

and

$$\left\| \left(\prod_{\mathfrak{c} \in \mathfrak{C}} |\cdot - \mathfrak{c}|^{j+k-\gamma-1} \right) \partial_{x_1}^j \partial_{x_2}^k p \right\|_{L^2(P)} \le A^{j+k+1}(j+k)!.$$

1.2 Layout

As is well-known (e.g. [15] and the references there) the analysis of point singularities near corners of solutions of elliptic PDEs is based on polar coordinates centered at the corner. For elliptic systems of PDEs such as those of interest here, as in [13, 14] in addition we require projections of Cartesian components of the vector-valued solutions to a polar frame. In Section 1.3, we collect the corresponding notation for partial derivatives and solution fields. Section 2.4 presents the variational formulation, and a (classical) existence and uniqueness result. Section 2 presents strong formulations of the boundary value problems under consideration, detailing in particular also the boundary operators. Also, weak formulations are recapitulated, with statements on existence and, under small data hypothesis, uniqueness of solutions.

The corner-weighted, Kondrat'ev spaces that appear in the statement of the analytic regularity shifts are also introduced. Section 3 then presents a key technical step for the subsequent analytic regularity proof: a priori estimates in corner-weighted Sobolev norms in a sector for the linearized Stokes boundary value problem are recapitulated, from [14]. Importantly, they hold for several combinations of boundary conditions on the sides of the sector, and for the velocity field in a polar coordinate frame. With this in hand, Section 4 addresses the proof of the principal analytic regularity result for the NSE, Theorem 8, which is also the main result of the present paper. The key novel step in its proof is an inductive bootstrap argument for the quadratic nonlinear term in the NSE, in corner-weighted spaces and for the velocity field in a polar frame at each corner of P. This is developed in Section 4.1. Conclusions and a short discussion of the results, with some consequences and possible generalizations, are presented in Section 5. An appendix contains several lengthy calculations with appear in several of the proofs.

1.3 Notation

We define $\mathbb{N} = \{1, 2, \dots\}$ as the set of positive natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We refer to tuples $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ as multi-indices and we write $|\alpha| = \alpha_1 + \alpha_2$. For $k \in \mathbb{N}_0$, we write

$$\sum_{|\alpha|\leq k} = \sum_{\alpha\in\mathbb{N}_0^2:|\alpha|\leq k} .$$

Given Cartesian coordinates (x_1, x_2) and polar coordinates (r, ϑ), whose origin will be clear from the context, we denote Cartesian derivatives as $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}$ and polar derivatives as $\mathcal{D}^\alpha = \partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2}$. In the following text, we shall always use roman letters to denote function spaces describing Cartesian derivatives and calligraphy letters to denote function spaces treating polar derivatives, see Section 2.5.

For any vector field $\boldsymbol{u}$ with components in Cartesian coordinates

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

we denote its polar coordinate frame projection as

$$\overline{\boldsymbol{u}} := \begin{pmatrix} u_r \\ u_\vartheta \end{pmatrix} = A\boldsymbol{u}\ , \quad A := \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \tag{1}$$

where A shall be referred to as "transformation matrix". Here and throughout, vector-valued quantities such as $\boldsymbol{u}$ shall be understood as column vectors, with $\boldsymbol{u}^\top$ denoting the transpose vector, which accordingly denotes a row vector. The symbol L_{St} shall denote the Stokes operator, with various super- and subscripts indicating Cartesian or polar coordinates and frame, i.e. we write $\overline{L_{\mathrm{St}}}$ for its projection onto polar coordinates acting on the corresponding velocity components.

We observe that the projection (1) of the velocity field into a polar frame renders certain boundary conditions particularly simple: for example, the homogeneous slip boundary condition in a sector Q will amount to requiring the angular component u_ϑ to vanish on sides of Q.

All quantities which occur in this paper are real-valued. The overline symbol which will indicate polar-coordinate representation of vectors is therefore non-ambiguous.

We denote with an underline n-dimensional tuples $\underline{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and suppose arithmetic operations and inequalities such as $\underline{\gamma} < \underline{\beta}$ are understood component-wise: e.g., $\underline{\beta} + k = (\beta_1 + k, \dots, \beta_n + k)$ for all $k \in \mathbb{N}$; furthermore, we indicate, e.g., $\underline{\beta} > 0$ if $\beta_i > 0$ for all $i \in \{1, \dots, n\}$.

Finally, for $a \in \mathbb{R}$, we denote its nonnegative real part as $[a]_+ = \max(0, a)$.

For summability index $1 \leq r \leq \infty$, the usual Lebesgue spaces in P shall be denoted by $L^r(P)$, with norm defined also for vector fields $\boldsymbol{v} : P \to \mathbb{R}^2$ as

$\|\boldsymbol{v}\|^r_{L^r(P)} = \int_P \|\boldsymbol{v}\|^r_{\ell^r}$. We denote the usual Sobolev spaces of smoothness order $s > 0$ and summability index r by $W^{s,r}(P)$; we write $H^s(P)$ in the Hilbertian case $r = 2$.

2 The Navier-Stokes equations, functional setting, and main result

Following the introduction of the polygonal domain in Section 2.1, in Section 2.2 we state the strong form of the boundary value problems, and of the boundary operators, in Cartesian coordinates. Section 2.3 is devoted to the saddle point variational form of the boundary value problems of interest. Section 2.4 reviews statements on existence and uniqueness of weak solutions, under the small data hypothesis. In Section 2.5 we introduce the corner-weighted spaces on which the weighted analytic regularity results will be based. Finally, we state in Section 2.6 our main result.

2.1 Geometry of the domain

Let P be a polygon with $n \geq 3$ straight sides and n corners $\mathfrak{C} = \{\mathfrak{c}_1, \ldots, \mathfrak{c}_n\}$. Let Γ_D, Γ_N, and Γ_G be a disjoint partition of the boundary $\Gamma = \partial P$ of P comprising each of $n_D \geq 1$, $n_N \geq 0$ and $n_G \geq 0$ many sides of P, respectively, with $n = n_D + n_N + n_G$. We denote by $\boldsymbol{n} : \Gamma \to \mathbb{R}^2$ the exterior unit normal vector to P, defined almost everywhere on Γ, which belongs to $L^\infty(\Gamma;\ \mathbb{R}^2)$, and by $\boldsymbol{t} \in L^\infty(\Gamma;\ \mathbb{R}^2)$ correspondingly the unit tangent vector to Γ, pointing in counterclockwise tangential direction.

2.2 The Navier-Stokes boundary value problems

We assume that a kinematic viscosity $\nu > 0$ is given, which is constant throughout P. For a velocity field $\boldsymbol{u} : P \to \mathbb{R}^2$ and a scalar $p : P \to \mathbb{R}$, define

$$\varepsilon(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top\right), \qquad \sigma(\boldsymbol{u}, p) := 2\nu\varepsilon(\boldsymbol{u}) - p\,\mathrm{Id}_2,$$

where Id_2 is the 2×2 identity matrix.

With this notation, we consider the stationary, incompressible Navier-Stokes equations in P

$$\begin{aligned} -\nabla \cdot \sigma(\boldsymbol{u}, p) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= \boldsymbol{f} && \text{in } P \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{in } P \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D \\ \sigma(\boldsymbol{u}, p)\boldsymbol{n} &= \boldsymbol{0} && \text{on } \Gamma_N \\ (\sigma(\boldsymbol{u}, p)\boldsymbol{n}) \cdot \boldsymbol{t} = 0 \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \text{on } \Gamma_G. \end{aligned} \tag{2}$$

Here, Γ_D, Γ_N, and Γ_G correspond to so-called no-slip, open, and slip boundary conditions, respectively.

Remark 1 From the identity

$$2\nabla \cdot \varepsilon(\boldsymbol{u}) = \Delta \boldsymbol{u} + \nabla(\nabla \cdot \boldsymbol{u}), \tag{3}$$

the boundary value problem (2) is equivalent to

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } P\,, \\ \nabla\cdot\boldsymbol{u} = 0 \quad & \text{in } P\,, \\ \boldsymbol{u} = \boldsymbol{0} \quad & \text{on } \Gamma_D\,, \\ \sigma(\boldsymbol{u},p)\boldsymbol{n} = \boldsymbol{0} \quad & \text{on } \Gamma_N\,, \\ (\sigma(\boldsymbol{u},p)\boldsymbol{n})\cdot\boldsymbol{t} = 0 \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad & \text{on } \Gamma_G. \end{aligned} \tag{4}$$

2.3 Variational Formulation

Weak solutions of the NSE (2) in the sense of Leray-Hopf satisfy the NSE (2) in variational form. To state it, we introduce standard Sobolev spaces in P. *Throughout the remainder of this article, we shall work under*

Assumption 1 The boundary value problems (2), (4) satisfy the following conditions.

1. P is a bounded, connected polygon with a finite number of straight sides, and Lipschitz boundary $\Gamma = \partial P$.
2. $n_D \geq 1$.
3. All interior opening angles at corners of P are in $(0, 2\pi)$. In particular, slit domains which correspond to opening angle 2π are excluded.

Assumption 1 implies that the Dirichlet case considered in [12] is a special case of the present setting. Furthermore, Item 2 ensures that the linearization of the Navier-Stokes equations, i.e., the Stokes problem, admits unique variational velocity field solutions $\boldsymbol{u}$, possibly with pressure p unique up to constants if $\Gamma = \Gamma_D$.

We denote henceforth the space of velocity fields of variational solutions to the Navier-Stokes equations (2) as

$$\boldsymbol{W} = \left\{ \boldsymbol{v} \in [H^1(P)]^2 : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_G \right\}. \tag{5}$$

We denote by $\boldsymbol{W}^* \subset [H^{-1}(P)]^2$ its dual, with identification of $L^2(P)^2 \simeq [L^2(P)^2]^*$. We also define $Q = L^2(P)$ if $|\Gamma_D| < |\Gamma|$ (i.e., if not the entire boundary is a Dirichlet boundary) and set $Q = L^2_0(P) := L^2(P)/\mathbb{R}$ in the case that $\Gamma = \Gamma_D$.

We are interested in variational solutions $(\boldsymbol{u}, p)$ of (2). To state the corresponding variational formulation, we introduce the usual bi- and trilinear forms:

$$a(\boldsymbol{u},\boldsymbol{v}) := 2\nu \int_P \sum_{i,j=1}^{2} [\varepsilon(\boldsymbol{u})]_{ij}[\varepsilon(\boldsymbol{v})]_{ij}\, d\boldsymbol{x} \,,$$
$$b(\boldsymbol{u},p) := -\int_P p\nabla\cdot \boldsymbol{u}\, d\boldsymbol{x} \,, \tag{6}$$
$$t(\boldsymbol{w};\ \boldsymbol{u},\boldsymbol{v}) := \int_P ((\boldsymbol{w}\cdot\nabla)\boldsymbol{u})\cdot \boldsymbol{v}\, d\boldsymbol{x} \,.$$

With these forms, we state the variational formulation of (2): find $(\boldsymbol{u},p) \in \boldsymbol{W}\times Q$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) + t(\boldsymbol{u};\ \boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = \int_P \boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x} \,,$$
$$b(\boldsymbol{u},q) = 0 \,, \tag{7}$$

for all $\boldsymbol{v}\in\boldsymbol{W}$ and all $q\in Q$.

2.4 Existence and uniqueness of solutions

We recapitulate results on existence and uniqueness of variational solutions of the NSE (7). As is well-known, uniqueness of such solutions in the stationary case requires a small data hypothesis. To state it,

we introduce the coercivity constant of the viscous (diffusion) term

$$C_{\mathrm{coer}} := \inf_{\substack{\boldsymbol{v}\in\boldsymbol{W}\\ \|\boldsymbol{v}\|_{H^1(P)}=1}} 2\int_P \sum_{i,j=1}^{2} [\varepsilon(\boldsymbol{v})]_{ij}[\varepsilon(\boldsymbol{v})]_{ij}\, d\boldsymbol{x}$$

and the continuity constant for the trilinear transport term

$$C_{\mathrm{cont}} := \sup_{\substack{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{W}\\ \|\boldsymbol{u}\|_{H^1(P)}=\|\boldsymbol{v}\|_{H^1(P)}=\|\boldsymbol{w}\|_{H^1(P)}=1}} \int_P ((\boldsymbol{u}\cdot\nabla)\boldsymbol{v})\cdot\boldsymbol{w}\, d\boldsymbol{x} \,.$$

The following existence and uniqueness result is then classical, see e.g. [5, Theorem 3.2]. It is valid under a small data hypothesis. To state it, we introduce

$$\mathbf{M} := \left\{ \boldsymbol{v}\in\boldsymbol{W} : \|\boldsymbol{v}\|_{H^1(P)} \le \frac{C_{\mathrm{coer}}\nu}{2C_{\mathrm{cont}}} \right\} \,.$$

Theorem 2 *Suppose that Assumption 1 holds and assume that* $\|\boldsymbol{f}\|_{\boldsymbol{W}^*} \le \frac{C_{\mathrm{coer}}^2\nu^2}{4C_{\mathrm{cont}}}$*. There exists a solution* $(\boldsymbol{u},p)\in\boldsymbol{W}\times L^2(P)$ *to* (2) *with right hand side* $\boldsymbol{f}$*. The velocity field* $\boldsymbol{u}$ *is unique in* $\mathbf{M}$*.*

As we assumed above $n_D \ge 1$, there is always at least one side of P where homogeneous Dirichlet ("no-slip") BCs are imposed.

2.5 Functional setting

For $x \in P$ and for $i \in \{1, \dots, n\}$, let $r_i(x) := \mathrm{dist}(x, \mathfrak{c}_i)$. We define the corner weight function

$$\Phi_{\underline{\beta}}(x) := \prod_{i=1}^{n} r_i^{\beta_i}(x).$$

We next introduce the corner-weighted function spaces to be used for the regularity analysis. As the notation used in the literature dealing with weighted Sobolev spaces is not always uniform, we present here several definitions of corner-weighted spaces and discuss how they relate for the range of weight exponents that is relevant to the present work.

2.5.1 Corner-weighted function spaces of finite order in P

In the polygon P, for $j, k \in \mathbb{N}_0$ and $\underline{\gamma} \in \mathbb{R}^n$, we introduce homogeneous corner-weighted seminorms and associated norms given by

$$|v|^2_{K^j_{\underline{\gamma}}(P)} := \sum_{|\alpha|=j} \|\Phi_{|\alpha|-\underline{\gamma}} \partial^\alpha v\|^2_{L^2(P)}, \qquad \|v\|^2_{K^k_{\underline{\gamma}}(P)} := \sum_{j=0}^{k} |v|^2_{K^j_{\underline{\gamma}}(P)}. \tag{8}$$

Furthermore, we also require non-homogeneous, corner-weighted Sobolev norms. They are, for $\ell \in \mathbb{N}_0$, $k \in \mathbb{N}$ with $k > \ell$, and $\underline{\beta} \in \mathbb{R}^n$ given by

$$\|v\|^2_{H^{k,\ell}_{\underline{\beta}}(P)} := \|v\|^2_{H^{\ell-1}(P)} + \sum_{\ell \le |\alpha| \le k} \|\Phi_{\underline{\beta}+|\alpha|-\ell} \partial^\alpha v\|^2_{L^2(P)}, \tag{9}$$

with the convention that the first term is omitted when $\ell = 0$. We therefore define the homogeneous, corner-weighted Sobolev spaces $K^k_{\underline{\gamma}}(P)$ and the non-homogeneous, corner-weighted Sobolev spaces $H^{k,\ell}_{\underline{\beta}}(P)$ as the spaces of, respectively, weakly differentiable functions with bounded $K^k_{\underline{\gamma}}(P)$ and $H^{k,\ell}_{\underline{\beta}}(P)$ norms.

2.5.2 Corner-weighted analytic classes $B^\ell_{\underline{\beta}}(P)$ and $K^\varpi_{\underline{\gamma}}(P)$

With the weighted, Kondrat'ev-type spaces at hand, we now introduce *weighted analytic classes* which will quantify the loss of analyticity of velocity and pressure in a vicinity of the corner points.

$$B^\ell_{\underline{\beta}}(P) := \Big\{ v \in \bigcap_{k \ge \ell} H^{k,\ell}_{\underline{\beta}}(P) : \exists C, A > 0 \text{ s. t.}$$
$$\|\Phi_{\underline{\beta}+|\alpha|-\ell} \partial^\alpha v\|_{L^2(P)} \le C A^{|\alpha|-\ell} (|\alpha|-\ell)!, \ \forall |\alpha| \ge \ell \Big\}, \tag{10}$$

and

$$K^{\varpi}_{\underline{\gamma}}(P) := \Bigg\{ v \in \bigcap_{k\in\mathbb{N}_0} K^k_{\underline{\gamma}}(P) : \exists C, A > 0 \text{ s. t.} \\ \forall \alpha \in \mathbb{N}_0^2 : \ \ \|\Phi_{|\alpha|-\underline{\gamma}} \partial^\alpha v\|_{L^2(P)} \le C A^{|\alpha|} |\alpha|! \Bigg\}. \tag{11}$$

The aforementioned weighted analytic classes are defined in terms of two constants $C > 0$ and $A > 0$. The constant $C > 0$ quantifies the size of a function in terms of linear scaling of norms, whereas the constant $A > 0$ relates to the size of the domain of analyticity.

2.5.3 Corner-weighted spaces in sectors

We shall require function spaces in plane sectors $Q_{\delta,\omega}(\mathfrak{c})$ of opening $\omega \in (0, 2\pi)$, radius $\delta \in (0, \infty]$ and with vertex $\mathfrak{c} \in \mathbb{R}^2$, defined as

$$Q_{\delta,\omega}(\mathfrak{c}) = \left\{ x \in \mathbb{R}^2 : r(x, \mathfrak{c}) \in (0, \delta),\ \vartheta(x) \in (0, \omega) \right\}.$$

We do not indicate the dependence on the vertex $\mathfrak{c}$ when this is clear from the context.

For all $k \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, we introduce the (homogeneous) corner-weighted, Hilbertian Kondrat'ev space $\mathcal{W}^k_\beta(Q_{\delta,\omega})$ of functions v in $Q_{\delta,\omega}(\mathfrak{c})$ with bounded norm given by

$$\|v\|^2_{\mathcal{W}^k_\beta(Q_{\delta,\omega})} = \sum_{|\alpha|\le k} \|r^{\beta-k+\alpha_1} \mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega})}. \tag{12}$$

Here, $\mathcal{D}^\alpha = \partial_r^{\alpha_1} \partial_\vartheta^{\alpha_2}$ denotes the partial derivative of order $\alpha \in \mathbb{N}_0^2$ in polar coordinates. We write $\mathcal{L}_\beta = \mathcal{W}^0_\beta$. For $k, \ell \in \mathbb{N}_0$ with $k \ge \ell$ and for $\beta \in \mathbb{R}$, $\mathcal{H}^{k,l}_\beta(Q_{\delta,\omega})$ denotes the space of functions with finite norm

$$\|v\|^2_{\mathcal{H}^{k,\ell}_\beta(Q_{\delta,\omega})} := \|v\|^2_{H^{\ell-1}(Q_{\delta,\omega})} + \sum_{\ell\le|\alpha|\le k} \|r^{\alpha_1+\beta-\ell} \mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega})},$$

where the first term is dropped if $\ell = 0$. For $\ell \in \mathbb{N}_0$ and $\beta \in \mathbb{R}$, the corner-weighted analytic class with weak derivatives in polar coordinates is given by

$$\mathcal{B}^\ell_\beta(Q_{\delta,\omega}) = \Bigg\{ v \in \bigcap_{k=\ell}^{\infty} \mathcal{H}^{k,\ell}_\beta(Q_{\delta,\omega}) : \exists C, A > 0 \\ \text{s. t. } \|r^{\alpha_1+\beta-\ell} \mathcal{D}^\alpha v\|_{L^2(Q_{\delta,\omega})} \le C A^{|\alpha|-\ell} (|\alpha| - \ell)!,\ \forall |\alpha| \ge \ell \Bigg\}. \tag{13}$$

In the sector $Q_{\delta,\omega}(\mathfrak{c})$, the definition of the spaces $H^{k,\ell}_\beta(Q_{\delta,\omega}(\mathfrak{c}))$ and $B^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))$ follows from (10) by replacing $\Phi_{\underline{\beta}+|\alpha|-\ell}$ in (9) and (10) with $r(\cdot,\mathfrak{c})^{\beta+|\alpha|-\ell}$. Similarly, the corner-weighted spaces $K^k_\gamma(Q_{\delta,\omega}(\mathfrak{c}))$ and $K^\varpi_\gamma(Q_{\delta,\omega}(\mathfrak{c}))$ can be defined by replacing $\Phi_{|\alpha|-\underline{\gamma}}$ in (8) and (11) with $r(\cdot,\mathfrak{c})^{|\alpha|-\gamma}$.

2.5.4 Relation between corner-weighted spaces

In this section we collect results on embeddings between some of the corner-weighted spaces we introduced. They are of independent interest, and will be required at various stages in the ensuing proof of the analytic regularity shifts.

For ease of reading, we either cite or postpone all proofs to Appendix A. The following implication between polar frame velocity $\overline{\boldsymbol{u}}$ in (1) and Cartesian frame velocity components $\boldsymbol{u}$ holds.

Lemma 3 *For all $0<\delta\leq 1$, $\omega\in(0,2\pi)$, $\mathfrak{c}\in\mathbb{R}^2$, $\ell\in\{0,1,2\}$, and $\beta\in(0,1)$, if $\overline{\boldsymbol{u}}\in\mathcal{B}^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))^2$ and $\overline{\boldsymbol{u}}(\mathfrak{c})=\boldsymbol{0}$ when $\ell=2$, then $\boldsymbol{u}\in B^\ell_\beta(Q_{\delta,\omega})^2$.*

The reverse implication, in the case $\ell=0$, is treated in the following statement.

Lemma 4 *For all $0<\delta\leq 1$, $\omega\in(0,2\pi)$, $\mathfrak{c}\in\mathbb{R}^2$, and $\beta\in(0,1)$, if $\boldsymbol{v}\in B^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))^2$ then $\overline{\boldsymbol{v}}\in\mathcal{B}^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))^2$.*

The corner-weighted spaces in Cartesian and polar frames are equivalent: the following lemmas on equivalence and embedding between weighted spaces state this formally.

Lemma 5 *Let $0<\delta\leq 1$, $\omega\in(0,2\pi)$, $\beta\in(0,1)$, $\mathfrak{c}\in\mathbb{R}^2$. Then the following equivalence relations hold for any $\ell\in\{0,1,2\}$ and $\mathbb{N}_0\ni k\geq\ell$:*

1. $v\in H^{k,\ell}_\beta(Q_{\delta,\omega}(\mathfrak{c}))\iff v\in\mathcal{H}^{k,\ell}_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.
2. $v\in B^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))\iff v\in\mathcal{B}^\ell_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.
3. $v\in H^{1,1}_\beta(Q_{\delta,\omega}(\mathfrak{c}))\iff v\in\mathcal{W}^1_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.

Lemma 6 *Let $0<\delta\leq 1$, $\omega\in(0,2\pi)$, $\beta\in(0,1)$, $\mathfrak{c}\in\mathbb{R}^2$. Then the following embeddings are continuous:*

1. $\mathcal{W}^2_\beta(Q_{\delta,\omega}(\mathfrak{c}))\hookrightarrow H^{2,2}_\beta(Q_{\delta,\omega}(\mathfrak{c}))\hookrightarrow C^0(\overline{Q_{\delta,\omega}(\mathfrak{c})})$.
2. *If $v\in H^{2,2}_\beta(Q_{\delta,\omega}(\mathfrak{c}))$ and $v(\mathfrak{c})=0$, then $v\in\mathcal{W}^2_\beta(Q_{\delta,\omega}(\mathfrak{c}))$.*

For the proof of Lemma 5, see [17, Theorem 1.1, Theorem 2.1, Lemma A.2]. For the proof of Lemma 6, see [17, Lemma 1.1, Lemma A.1, Lemma A.2] and [18, Section 2]. The following lemma asserts that functions that belong to corner-weighted Kondrat'ev spaces with non-homogeneous weights for a certain range of indices, with the additional requirement of the function vanishing at the vertex for second order spaces, also belong to the corresponding spaces with homogeneous weights. We refer to [19, Chapter 7] for an in-depth presentation.

Lemma 7 *Let* $0 < \delta \leq 1$, $\omega \in (0, 2\pi)$, $\beta \in (0,1)$, $\mathfrak{c} \in \mathbb{R}^2$, $k \in \{1, 2\}$, *and* $v \in H^{k,k}_{\beta}(Q_{\delta,\omega}(\mathfrak{c}))$. *Let furthermore* $v(\mathfrak{c}) = 0$ *when* $k = 2$. *Then,* $v \in K^k_{k-\beta}(Q_{\delta,\omega}(\mathfrak{c}))$.

2.6 Statement of the main result

We are ready to present our main result on the weighted analytic regularity of the solution to (2) here using the corner-weighted spaces introduced in Section 2.5. The explicit form of the operator pencil $\mathcal{A}(\lambda)$ which arises for the presently considered Stokes problem and its boundary conditions is given in Appendix B.

Theorem 8 *Let* $\underline{\beta}_f = (\beta_1, \ldots, \beta_n) \in (0,1)^n$ *be such that at each corner* $\mathfrak{c}_i$, *the operator pencil* $\mathcal{A}(\lambda)$ *for the linearized (Stokes) boundary value problem defined in* (16) *has no eigenvalue on* $\mathrm{Im}(\lambda) = 1 - \beta_i$. *Let further* $\boldsymbol{f} \in [B^0_{\underline{\beta}_f}(P)]^2 \cap \boldsymbol{W}^*$ *be such that* $\|\boldsymbol{f}\|_{\boldsymbol{W}^*} \leq \frac{C^2_{\mathrm{coer}}\nu^2}{4C_{\mathrm{cont}}}$. *Suppose in addition that Assumption 1 holds and let* $(\boldsymbol{u}, p)$ *be the weak solution to* (7) *with right hand side* $\boldsymbol{f}$.

Then

$$(\boldsymbol{u}, p) \in [B^2_{\underline{\beta}_f}(P)]^2 \times B^1_{\underline{\beta}_f}(P).$$

The remainder of the paper is devoted to the proof of Theorem 8. It is based on inductive bootstrapping elliptic regularity for the linearized boundary value problem in corner-weighted Sobolev spaces of finite order, of Kondrat'ev type. Such estimates are in principle known (e.g. [5, 11, 14, 20]). They are recapitulated for the readers' convenience in the form required subsequently in the next Section 3. The weighted a priori estimates are then combined with novel analytic estimates of the quadratic nonlinearity in polar frame in corner-weighted spaces that will be developed in Section 4.

3 The Stokes equation in a sector

Consider, for $\mathfrak{c} \in \mathbb{R}^2$, $\delta \in (0,1)$ and $\omega \in (0, 2\pi)$, the sector $Q_{\delta,\omega}(\mathfrak{c})$. Denote by $\Gamma_1 := \{x \in \mathbb{R}^2 : r(x, \mathfrak{c}) \in (0, \delta), \vartheta(x) = 0\}$ and $\Gamma_2 := \{x \in \mathbb{R}^2 : r(x, \mathfrak{c}) \in (0, \delta), \vartheta(x) = \omega\}$ the two edges meeting at $\mathfrak{c}$. Let also $\Gamma_\delta = \Gamma_1 \cup \Gamma_2$. As all the results in this section are independent of $\mathfrak{c}$, we omit the dependence of the sector in the notation and write $Q_{\delta,\omega} = Q_{\delta,\omega}(\mathfrak{c})$.

We consider variational solutions to the Stokes problem in $Q_{\delta,\omega}$

$$L^{\sigma}_{\mathrm{St}}(\boldsymbol{u},p) = \begin{pmatrix} \boldsymbol{f} \\ h \end{pmatrix} \quad \text{in } Q_{\delta,\omega}\,, \qquad B(\boldsymbol{u},p) = \boldsymbol{0} \quad \text{on } \Gamma_\delta\,. \tag{14}$$

The system (14) reads in components

$$\begin{aligned} -\nabla\cdot\sigma(\boldsymbol{u},p) = \boldsymbol{f} &\quad \text{in } Q_{\delta,\omega} \\ \nabla\cdot\boldsymbol{u} = h &\quad \text{in } Q_{\delta,\omega} \\ \boldsymbol{u} = \boldsymbol{0} &\quad \text{on } \Gamma^S_D \\ \sigma(\boldsymbol{u},p)\boldsymbol{n} = \boldsymbol{g} &\quad \text{on } \Gamma^S_N \\ (\sigma(\boldsymbol{u},p)\boldsymbol{n})\cdot\boldsymbol{t} = 0 \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 &\quad \text{on } \Gamma^S_G, \end{aligned} \tag{15}$$

where $\Gamma^S_D, \Gamma^S_N, \Gamma^S_G \in \{\varnothing, \Gamma_1, \Gamma_2\}$ are pairwise disjoint and such that $\Gamma^S_D \cup \Gamma^S_N \cup \Gamma^S_G = \Gamma_\delta$, and where $B(\boldsymbol{u},p)$ in (14) denotes any of the three boundary operators in (15). We observe that in (15) we did not include inhomogeneous boundary data on Γ^S_G, as this is the physical case of the "no-slip" BCs. We also observe that the nonzero boundary data $\boldsymbol{g}$ on Γ^S_N will appear in the analytic regularity shift argument in the proof of Lemma 17.

For the Stokes problem in $Q_{\delta,\omega}$, the following regularity result in corner-weighted spaces is a slight extension of [14, Theorem 5.2]. The proof proceeds along the lines of that of the cited theorem, by localizing $\boldsymbol{u}$ and p near each corner $\mathfrak{c}$ and by rewriting (15) by using polar frame $\overline{\boldsymbol{u}}$, followed by an application of the Fourier transform in radial direction to obtain the following parameterized system of ordinary differential equations.

$$\mathcal{A}(\lambda)(\overline{\boldsymbol{u}},p) = [\widehat{L}(\lambda)(\overline{\boldsymbol{u}},p), \widehat{B}(\lambda)(\overline{\boldsymbol{u}},p)] = [(\boldsymbol{0},0),(\boldsymbol{g}\cdot i_1, \boldsymbol{g}\cdot i_2)] \tag{16}$$

in the interval $(0,\omega)$, where ω denotes opening angle of the sector at $\mathfrak{c}$ (see Appendix B for details and [15] for general theory of such pencils in connection with elliptic boundary value problems in conical domains). Here, for $j=1,2$, $i_j = 1$ if Neumann boundary condition is prescribed on Γ_j and $i_j = 0$ otherwise.

Theorem 9 *Let $\omega \in (0,2\pi)$ and $\beta \in (0,1)$ be such that the pencil $\mathcal{A}(\lambda)$ in (16) does not have eigenvalues on the line $\{\mathrm{Im}(\lambda) = 1-\beta\}$.*

Then for any $\delta > 0$, there exists a constant $C_{\mathrm{sec}} = C_{\mathrm{sec}}(\beta,\delta) > 0$ such that for all $\boldsymbol{f} \in L_\beta(Q_{\delta,\omega})$ and $(\boldsymbol{u},p)$ satisfying (15) in $Q_{\delta,\omega}$,

$$\begin{aligned} &\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}(\mathfrak{c})}\|_{\mathcal{W}^2_\beta(Q_{\delta/2,\omega})} + \|p\|_{\mathcal{W}^1_\beta(Q_{\delta/2,\omega})} \\ &\le C_{\mathrm{sec}}\left(\|\overline{\boldsymbol{f}}\|_{\mathcal{L}_\beta(Q_{\delta,\omega})} + \|\boldsymbol{u}\|_{H^1(Q_{\delta,\omega}\setminus Q_{\delta/2,\omega})} + \|p\|_{L^2(Q_{\delta,\omega}\setminus Q_{\delta/2,\omega})} + \|\overline{\boldsymbol{g}}\|_{\mathcal{W}^{1/2}_\beta(\Gamma^S_N)}\right) \end{aligned} \tag{17}$$

For a detailed development, see [21, Lemma 5.1.1]. We remark here that this result could also be derived using [20, Theorem 5.1] or [15, Chapter 5.1] if only homogeneous Dirichlet (so-called "no-slip") boundary conditions are considered.

Remark 10 By relation (3), if $(\boldsymbol{u}, p) \in [H^{2,2}_{\beta}(Q_{\delta,\omega})]^2 \times \mathcal{W}^1_{\beta}(Q_{\delta,\omega})$ is a solution of

$$L^{\Delta}_{\mathrm{St}}(\boldsymbol{u}, p) = \begin{pmatrix} \boldsymbol{f} + \nu\nabla h \\ h \end{pmatrix} \quad \text{in } Q_{\delta,\omega}\,, \qquad B(\boldsymbol{u}, p) = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{g} \\ \boldsymbol{0} \end{pmatrix} \quad \text{on } \Gamma^S_D \times \Gamma^S_N \times \Gamma^S_G,$$

or, in components,

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} + \nu\nabla h \quad & \text{in } Q_{\delta,\omega} \\ \nabla\cdot\boldsymbol{u} = h \quad & \text{in } Q_{\delta,\omega} \\ \boldsymbol{u} = \boldsymbol{0} \quad & \text{on } \Gamma^S_D \\ \sigma(\boldsymbol{u}, p)\boldsymbol{n} = \boldsymbol{g} \quad & \text{on } \Gamma^S_N \\ (\sigma(\boldsymbol{u}, p)\boldsymbol{n})\cdot\boldsymbol{t} = 0 \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad & \text{on } \Gamma^S_G, \end{aligned} \tag{18}$$

then it is also a solution of (15). Estimate (17) therefore also holds for solutions of (18) when $h = 0$.

4 Proof of the main result

We now prove Theorem 8, which, as our main result, ensures analytic regularity in scales of weighted spaces of solutions to the Navier-Stokes equations (2). First, we will devote our attention to the nonlinear transport term, as treating this term is the main difference in comparison to the weighted analytic regularity proof for the Stokes problem in P in [14].

4.1 Estimate of the nonlinear term

We start by rewriting the quadratic nonlinearity $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ in polar coordinates and projecting its Cartesian components into the polar frame as in (1). We note here that the gradient operator in Cartesian coordinates is projected to a polar frame by (cf. the definition of A in (1))

$$\nabla = A^{-1}\begin{pmatrix} \partial_r \\ r^{-1}\partial_\vartheta \end{pmatrix}\,. \tag{19}$$

Lemma 11 *For any constant vector field* $\mathbf{c}$ *taking value* $(c_1, c_2)^\top \in \mathbb{R}^2$*, the following equality holds:*

$$\overline{((\boldsymbol{u}+\mathbf{c})\cdot\nabla)(\boldsymbol{u}+\mathbf{c})} = \begin{pmatrix} (u_r + c_r)\partial_r u_r + \frac{1}{r}((u_\vartheta + c_\vartheta)\partial_\vartheta u_r - (u_\vartheta + c_\vartheta)u_\vartheta) \\ (u_r + c_r)\partial_r u_\vartheta + \frac{1}{r}((u_\vartheta + c_\vartheta)\partial_\vartheta u_\vartheta + (u_\vartheta + c_\vartheta)u_r) \end{pmatrix}. \tag{20}$$

Proof We have

$$\overline{((\boldsymbol{u}+\mathbf{c})\cdot\nabla)(\boldsymbol{u}+\mathbf{c})} = \overline{((\boldsymbol{u}+\mathbf{c})\cdot\nabla)\boldsymbol{u}} = A\left(\left((\overline{\boldsymbol{u}}+\overline{\mathbf{c}})\cdot(A^{-\top}A^{-1}\begin{pmatrix}\partial_r\\ r^{-1}\partial_\vartheta\end{pmatrix})\right)A^{-1}\overline{\boldsymbol{u}}\right)$$
$$= A\left(\left((\overline{\boldsymbol{u}}+\overline{\mathbf{c}})\cdot\begin{pmatrix}\partial_r\\ r^{-1}\partial_\vartheta\end{pmatrix}\right)A^{-1}\overline{\boldsymbol{u}}\right)$$
$$= A\Bigg[\begin{pmatrix}\cos\vartheta(u_r+c_r)\partial_r u_r - \sin\vartheta(u_r+c_r)\partial_r u_\vartheta\\ \sin\vartheta(u_r+c_r)\partial_r u_r + \cos\vartheta(u_r+c_r)\partial_r u_\vartheta\end{pmatrix}$$
$$+\frac{1}{r}\begin{pmatrix}\cos\vartheta(u_\vartheta+c_\vartheta)\partial_\vartheta u_r - \sin\vartheta(u_\vartheta+c_\vartheta)u_r - \sin\vartheta(u_\vartheta+c_\vartheta)\partial_\vartheta u_\vartheta - \cos\vartheta(u_\vartheta+c_\vartheta)u_\vartheta\\ \sin\vartheta(u_\vartheta+c_\vartheta)\partial_\vartheta u_r + \cos\vartheta(u_\vartheta+c_\vartheta)u_r + \cos\vartheta(u_\vartheta+c_\vartheta)\partial_\vartheta u_\vartheta - \sin\vartheta(u_\vartheta+c_\vartheta)u_\vartheta\end{pmatrix}\Bigg]$$
$$= \begin{pmatrix}(u_r+c_r)\partial_r u_r + \frac{1}{r}((u_\vartheta+c_\vartheta)\partial_\vartheta u_r - (u_\vartheta+c_\vartheta)u_\vartheta)\\ (u_r+c_r)\partial_r u_\vartheta + \frac{1}{r}((u_\vartheta+c_\vartheta)\partial_\vartheta u_\vartheta + (u_\vartheta+c_\vartheta)u_r)\end{pmatrix}.$$

□

In order to treat the individual nonlinear terms arising from the polar representation of the transport term of the Navier-Stokes equation obtained above, we need a technical result on weighted interpolation estimates in plane sectors. The following statement is the polar version of [12, Lemma 1.10].

Lemma 12 *Let $\delta,\omega\in\mathbb{R}$ such that $0<\delta\le 1$ and $\omega\in(0,2\pi)$. For all $\tilde\beta_1,\tilde\beta_2\in\mathbb{R}$ such that $\tilde\beta_2>\tilde\beta_1+1/2$, there exists a constant $C_{\mathrm{int}}=C_{\mathrm{int}}(\delta,\omega,\tilde\beta_1,\tilde\beta_2)>0$ such that, for all $\alpha\in\mathbb{N}_0^2$ and all functions φ such that*

$$\max_{|\eta|\le 1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|_{L^2(Q_{\delta,\omega})}<\infty,$$

the following bound holds:

$$\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^4(Q_{\delta,\omega})}\le C_{\mathrm{int}}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(Q_{\delta,\omega})}$$
$$\times\left(\sum_{|\eta|\le 1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|^{1/2}_{L^2(Q_{\delta,\omega})}+\alpha_1^{1/2}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|^{1/2}_{L^2(Q_{\delta,\omega})}\right).$$

Proof We set $\delta=1$. Consider the dyadic partition of $Q_{1,\omega}$ given by the sets

$$S^j:=\left\{x\in Q_{1,\omega}:2^{-j-1}<r(x)<2^{-j}\right\},\qquad j\in\mathbb{N}_0,$$

and denote the linear maps $\Psi_j:S^j\to S^0$. Denote $\widehat\varphi_j:=\varphi\circ\Psi_j^{-1}:S^0\to\mathbb{R}$ and write $\widehat{\mathcal{D}}^\alpha$ for derivation with respect to polar coordinates (r,ϑ) in S^0. Then, by scaling, for any $q\in[1,\infty)$,

$$\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^q(S^j)}=2^{-j(\tilde\beta_2+2/q)}\|r^{\tilde\beta_2+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat\varphi_j\|_{L^q(S^0)}.\tag{21}$$

Furthermore, the following interpolation inequality holds in S^0: there exists $C_0>0$ such that

$$\|v\|_{L^4(S^0)}\le C_0\|v\|^{1/2}_{H^1(S^0)}\|v\|^{1/2}_{L^2(S^0)}\tag{22}$$

holds for all $v\in H^1(S^0)$. In addition, by (19), for all $v\in H^1(S^0)$,

$$\|v\|^2_{H^1(S^0)}\le 16\left(\|v\|^2_{L^2(S^0)}+\|\partial_r v\|^2_{L^2(S^0)}+\|\partial_\vartheta v\|^2_{L^2(S^0)}\right).\tag{23}$$

Combining (22) and (23) and choosing $v = r^{\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j$ gives

$$
\begin{aligned}
&\|r^{\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^4(S^0)}\\
&\quad\leq 2C_0\|r^{\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^2(S^0)}^{1/2}\left(\sum_{|\eta|\leq 1}\|\mathcal{D}^\eta(r^{\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j)\|_{L^2(S^0)}^2\right)^{1/4}\\
&\quad\leq 2C_0\|r^{\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^2(S^0)}^{1/2}\left(\sum_{|\eta|\leq 1}\|r^{\alpha_1}\widehat{\mathcal{D}}^{\alpha+\eta}\widehat{\varphi}_j\|_{L^2(S^0)}^2+\alpha_1^2\|r^{\alpha_1-1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^2(S^0)}^2\right)^{1/4}.
\end{aligned}
$$

Therefore, using the bound $2^{-|a|}\leq r(x)^a\leq 2^{|a|}$ valid for all $x\in S^0$ and all $a\in\mathbb{R}$,

$$
\begin{aligned}
\|r^{\tilde\beta_2+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^4(S^0)} &\leq 2^{|\tilde\beta_2|+|\tilde\beta_1|+1/2}2C_0\|r^{\tilde\beta_1+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^2(S^0)}^{1/2}\\
&\quad\times\left(\sum_{|\eta|\leq 1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\widehat{\mathcal{D}}^{\alpha+\eta}\widehat{\varphi}_j\|_{L^2(S^0)}^2+\alpha_1^2\|r^{\tilde\beta_1+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^2(S^0)}^2\right)^{1/4}.
\end{aligned}
$$

We denote $C_1 := 2^{|\tilde\beta_2|+|\tilde\beta_1|+1/2}2C_0$. Using this last inequality and (21) twice,

$$
\begin{aligned}
&\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^4(S^j)}\\
&\quad\leq 2^{-j(\tilde\beta_2+1/2)}\|r^{\tilde\beta_2+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^4(S^0)}\\
&\quad\leq 2^{-j(\tilde\beta_2+1/2)}C_1\|r^{\tilde\beta_1+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^2(S^0)}^{1/2}\\
&\qquad\times\left(\sum_{|\eta|\leq 1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\widehat{\mathcal{D}}^{\alpha+\eta}\widehat{\varphi}_j\|_{L^2(S^0)}^2+\alpha_1^2\|r^{\tilde\beta_1+\alpha_1}\widehat{\mathcal{D}}^\alpha\widehat{\varphi}_j\|_{L^2(S^0)}^2\right)^{1/4}\\
&\quad\leq C_1 2^{-j(\tilde\beta_2-\tilde\beta_1-1/2)}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^2(S^j)}^{1/2}\\
&\qquad\times\left(\sum_{|\eta|\leq 1}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|_{L^2(S^j)}^2+\alpha_1^2\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^2(S^j)}^2\right)^{1/4}.
\end{aligned}
$$

Since $\tilde\beta_2-\tilde\beta_1-1/2>0$, we can conclude that

$$
\begin{aligned}
\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_2+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^4(S^j)}^4 &\leq C_1^4\left(\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^2(S^j)}^2\right)\\
&\quad\times\left(\sum_{|\eta|\leq 1}\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}\varphi\|_{L^2(S^j)}^2+\alpha_1^2\sum_{j\in\mathbb{N}_0}\|r^{\tilde\beta_1+\alpha_1}\mathcal{D}^\alpha\varphi\|_{L^2(S^j)}^2\right).
\end{aligned}
$$

Taking the fourth root of both sides of the inequality above concludes the proof for the case $\delta = 1$. The general case $\delta\in(0,1]$ follows by scaling (with constant C_{int} depending on δ). □

Using the interpolation result obtained above, we can estimate, under a regularity assumption on $\boldsymbol{u}$, the individual terms appearing in (20). This is done in the following Lemma 13 and Corollary 14.

Lemma 13 *Let $\beta \in (0,1)$, $0 < \delta \le 1$, $\omega \in (0, 2\pi)$. Then, there exists a constant $C_{\mathrm{d}} = C_{\mathrm{d}}(\beta, \delta, \omega) > 0$ such that for all $u \in \mathcal{W}^2_\beta(Q_{\delta,\omega})$ with $\|u\|_{\mathcal{W}^2_\beta(Q_{\delta,\omega})} \le 1$ such that there exist constants $A_u, E_u > 1$, and $k \in \mathbb{N}$ satisfying*

$$\|r^{\beta+\alpha_1-2}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta,\omega})} \le A_u^{|\alpha|-2}E_u^{\alpha_2}(|\alpha|-2)!, \quad \forall \alpha \in \mathbb{N}_0^2 : 2 \le |\alpha| \le k+1, \tag{24}$$

it holds for all $\alpha, \eta \in \mathbb{N}_0^2$ such that $|\eta| \le 1$ and $|\alpha| \le k - |\eta|$ that

$$\begin{aligned}&\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(r^{\eta_1}\mathcal{D}^\eta u)\|_{L^4(Q_{\delta,\omega})}\\ &\qquad \le C_{\mathrm{d}}(|\alpha|+1)^{1/2}A_u^{[|\alpha|+|\eta|-3/2]_+}E_u^{\alpha_2+\eta_2+1/2}[|\alpha|+|\eta|-2]_+!.\end{aligned} \tag{25}$$

Proof We start by proving the theorem in the case $|\eta| = 0$. Applying Lemma 12 with $\tilde\beta_2 = \beta/2 - 1$ and $\tilde\beta_1 = \beta - 2$ (note that $\beta \in (0,1)$ implies $\tilde\beta_2 > \tilde\beta_1 + 1/2$), for all $|\alpha| \le k$,

$$\begin{aligned}\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} &\le C_{\mathrm{int}}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha u\|^{1/2}_{L^2(Q_{\delta,\omega})}\\ &\times\left(\sum_{|\eta|\le 1}\|r^{\beta-2+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}u\|^{1/2}_{L^2(Q_{\delta,\omega})} + \alpha_1^{1/2}\|r^{\beta-2+\alpha_1}\mathcal{D}^\alpha u\|^{1/2}_{L^2(Q_{\delta,\omega})}\right).\end{aligned} \tag{26}$$

When $|\alpha| \ge 2$, using (24), we have

$$\begin{aligned}&\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})}\\ &\quad \le C_{\mathrm{int}}A_u^{|\alpha|-3/2}E_u^{\alpha_2+1/2}(2(|\alpha|-1)!^{1/2} + (1+\alpha_1^{1/2})(|\alpha|-2)!^{1/2})(|\alpha|-2)!^{1/2}\\ &\quad \le C_{\mathrm{int}}A_u^{|\alpha|-3/2}E_u^{\alpha_2+1/2}(2(|\alpha|-1)^{1/2} + 1 + \alpha_1^{1/2})(|\alpha|-2)!\\ &\quad \le C_{\mathrm{int}}A_u^{|\alpha|-3/2}E_u^{\alpha_2+1/2}4|\alpha|^{1/2}(|\alpha|-2)!.\end{aligned}$$

If $|\alpha| \le 1$, instead, it follows from $\|u\|_{\mathcal{W}^2_\beta(Q_{\delta,\omega})} \le 1$ and (26) that

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} \le C_{\mathrm{int}}(2+\alpha_1^{1/2}) \le 4C_{\mathrm{int}}.$$

This proves (25) for $|\eta| = 0$, i.e., that for all $|\alpha| \le k$,

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} \le 4C_{\mathrm{int}}A_u^{[|\alpha|-3/2]_+}E_u^{\alpha_2+1/2}(|\alpha|+1)^{1/2}[|\alpha|-2]_+!. \tag{27}$$

Consider now the case $|\eta| = 1$. We have

$$\begin{aligned}&\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(r^{\eta_1}\mathcal{D}^\eta u)\|_{L^4(Q_{\delta,\omega})}\\ &\qquad \le \|r^{\beta/2-1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}u\|_{L^4(Q_{\delta,\omega})} + \alpha_1\eta_1\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})}.\end{aligned}$$

For all $|\alpha| \le k-1$, we can apply (27) to the two terms in the right hand side above:

$$\begin{aligned}\alpha_1\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} &\le 4C_{\mathrm{int}}A_u^{[|\alpha|-3/2]_+}E_u^{\alpha_2+1/2}(|\alpha|+1)^{1/2}\alpha_1[|\alpha|-2]_+!\\ &\le 4C_{\mathrm{int}}A_u^{[|\alpha|-1/2]_+}E_u^{\alpha_2+\eta_2+1/2}(|\alpha|+1)^{1/2}2[|\alpha|-1]_+!,\end{aligned}$$

and

$$\|r^{\beta/2-1+\alpha_1+\eta_1}\mathcal{D}^{\alpha+\eta}u\|_{L^4(Q_{\delta,\omega})} \le 4C_{\mathrm{int}}A_u^{[|\alpha|-1/2]_+}E_u^{\alpha_2+\eta_2+1/2}(|\alpha|+1)^{1/2}[|\alpha|-1]_+!.$$

Hence, for all $|\alpha| \le k-1$ and all $|\eta| = 1$,

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(r^{\eta_1}\mathcal{D}^\eta u)\|_{L^4(Q_{\delta,\omega})} \le 12C_{\mathrm{int}}A_u^{[|\alpha|-1/2]_+}E_u^{\alpha_2+\eta_2+1/2}(|\alpha|+1)^{1/2}[|\alpha|-1]_+!,$$

which concludes the proof, with $C_{\mathrm{d}} = 12C_{\mathrm{int}}$. □

Corollary 14 *Let $\beta \in (0,1)$, $0 < \delta \leq 1$, $\omega \in (0, 2\pi)$, and let $u \in \mathcal{W}^2_\beta(Q_{\delta,\omega})$ satisfies $\|u\|_{\mathcal{W}^2_\beta(Q_{\delta,\omega})} \leq 1$. Suppose that there exist $A_u, E_u > 1$ and $k \in \mathbb{N}$ such that*

$$\|r^{\beta+\alpha_1-2}\mathcal{D}^\alpha u\|_{L^2(Q_{\delta,\omega})} \leq A_u^{|\alpha|-2} E_u^{\alpha_2}(|\alpha|-2)!, \quad \forall \alpha \in \mathbb{N}^2_0 : 2 \leq |\alpha| \leq k+1.$$

Then, with the constant C_{d} from Lemma 13, for all $\alpha \in \mathbb{N}^2_0$ such that $|\alpha| \leq k$,

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(ru)\|_{L^4(Q_{\delta,\omega})} \leq 4C_{\mathrm{d}}(|\alpha|+1)^{1/2} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}[|\alpha|-2]_+!. \quad (28)$$

Proof We start from the bound

$$\|r^{\beta/2-1+\alpha_1}\mathcal{D}^\alpha(ru)\|_{L^4(Q_{\delta,\omega})} \leq \|r^{\beta/2+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} + \alpha_1\|r^{\beta/2-1+\alpha_1}\mathcal{D}^{(\alpha_1-1,\alpha_2)}u\|_{L^4(Q_{\delta,\omega})},$$

where the second term is absent if $\alpha_1 = 0$. From Lemma 13, it follows that

$$\|r^{\beta/2+\alpha_1}\mathcal{D}^\alpha u\|_{L^4(Q_{\delta,\omega})} \leq \delta C_{\mathrm{d}}(|\alpha|+1)^{1/2} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}[|\alpha|-2]_+!$$

and that (when $\alpha_1 \geq 1$)

$$\begin{aligned}
&\alpha_1\|r^{\beta/2-1+\alpha_1}\mathcal{D}^{(\alpha_1-1,\alpha_2)}u\|_{L^4(Q_{\delta,\omega})} \\
&\quad \leq \delta\alpha_1|\alpha|^{1/2} A_u^{[|\alpha|-5/2]_+} E_u^{\alpha_2+1/2}[|\alpha|-3]_+! \\
&\quad \leq \max_{j\in\mathbb{N}}\left(\frac{j^{3/2}}{(j+1)^{1/2}\max(j-2,1)}\right)(|\alpha|+1)^{1/2} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}[|\alpha|-2]_+! \\
&\quad \leq \frac{3}{2}\sqrt{3}(|\alpha|+1)^{1/2} A_u^{[|\alpha|-3/2]_+} E_u^{\alpha_2+1/2}[|\alpha|-2]_+!
\end{aligned}$$

Equation (28) follows from the above, bounding $1 + \frac{3}{2}\sqrt{3} \leq 4$ for ease of notation. □

We are now in position to estimate the weighted norms of the nonlinear term in the sector $Q_{\delta,\omega}(\mathfrak{c})$, under the assumptions of analytic bounds on the weighted norms of $\boldsymbol{u}$. Initially, we do this under the assumption that $\overline{\boldsymbol{u}} \in \mathcal{W}^2_\beta(Q_{\delta,\omega}(\mathfrak{c}))^2$ (which implies that $\boldsymbol{u}$ vanishes at the vertex of the sector) in Lemma 15.

Lemma 15 (Weighted analytic estimates for the quadratic nonlinearity in polar frame)
Assume that $\beta \in (0,1)$, $0 < \delta \leq 1$, $\omega \in (0, 2\pi)$ and $c > 0$ are given fixed.

Then, there exists a constant $C_{\mathrm{t}} = C_{\mathrm{t}}(\beta, \delta, \omega, c) > 0$ such that for all constant vector fields $\mathbf{c} = (c_1, c_2)^\top \in \mathbb{R}^2$ such that $|c_1| + |c_2| < c$ and all $\boldsymbol{w} : Q_{\delta,\omega} \to \mathbb{R}^2$ with $\|\overline{\boldsymbol{w}}\|_{\mathcal{W}^2_\beta(Q_{\delta,\omega})} \leq 1$ such that there exist $k \in \mathbb{N}$ and constants $A_w, E_w \geq 1$ satisfying

$$\begin{cases} \|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha w_r\|_{L^2(Q_{\delta,\omega})} \leq A_w^{|\alpha|-2} E_w^{\alpha_2}(|\alpha|-2)! \\ \|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha w_\vartheta\|_{L^2(Q_{\delta,\omega})} \leq A_w^{|\alpha|-2} E_w^{\alpha_2}(|\alpha|-2)!, \end{cases} \quad \text{for all } 2 \leq |\alpha| \leq k+1,$$

the following inequality holds:

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^2\overline{((\boldsymbol{w}+\mathbf{c})\cdot\nabla)(\boldsymbol{w}+\mathbf{c})})\|_{L^2(Q_{\delta,\omega})} \\
&\qquad \leq C_{\mathrm{t}} A_w^{|\alpha|-1} E_w^{\alpha_2+2}|\alpha|!, \qquad \forall \alpha \in \mathbb{N}^2_0 : 1 \leq |\alpha| \leq k.
\end{aligned} \quad (29)$$

Proof By Lemma 6, there exists a constant $C_{\mathrm{emb}} = C_{\mathrm{emb}}(\beta,\delta,\omega) > 0$ such that $\|\overline{\boldsymbol{w}}\|_{\mathcal{W}^2_\beta(Q_{\delta,\omega})} \leq 1$ implies $\overline{\boldsymbol{w}} \in [C^0(\overline{Q_{\delta,\omega}})]^2$ and

$$\|\overline{\boldsymbol{w}}\|_{L^\infty(Q_{\delta,\omega})} \leq C_{\mathrm{emb}}. \tag{30}$$

Next, we recall from Lemma 11 that

$$r^2\overline{((\boldsymbol{w}+\mathbf{c})\cdot\nabla)(\boldsymbol{w}+\mathbf{c})} = \begin{pmatrix} r^2(w_r+c_r)\partial_r w_r + r((w_\vartheta+c_\vartheta)\partial_\vartheta w_r - (w_\vartheta+c_\vartheta)w_\vartheta) \\ r^2(w_r+c_r)\partial_r w_\vartheta + r((w_\vartheta+c_\vartheta)\partial_\vartheta w_\vartheta + (w_\vartheta+c_\vartheta)w_r) \end{pmatrix}. \tag{31}$$

We will estimate the individual terms.

Estimate of rw_ϑ^2 *and* rw_rw_ϑ

Let $v \in \{w_r, w_\vartheta\}$. From (28), Lemma 13 and Corollary 14 it follows that for any α as in (29)

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta v))\|_{L^2(Q_{\delta,\omega})} \\
&\quad\leq \sum_{j=0}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}\|r^{\eta_1+\beta/2-1}\mathcal{D}^\eta(rv)\|_{L^4(Q_{\delta,\omega})}\|r^{\alpha_1-\eta_1+\beta/2-1}\mathcal{D}^{\alpha-\eta}w_\vartheta\|_{L^4(Q_{\delta,\omega})} \\
&\quad\leq \sum_{j=0}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}4C_{\mathrm{d}}^2(|\eta|+1)^{1/2}A_w^{[|\eta|-3/2]_+}E_w^{\eta_2+1/2}[|\eta|-2]_+! \\
&\qquad\qquad \times (|\alpha|-|\eta|+1)^{1/2}A_w^{[|\alpha|-|\eta|-3/2]_+}E_w^{\alpha_2-\eta_2+1/2}[|\alpha|-|\eta|-2]_+! \\
&\quad\leq 4C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1} \\
&\qquad\times\sum_{j=0}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}j!(|\alpha|-j)!\frac{(j+1)^{1/2}(|\alpha|-j+1)^{1/2}}{\max(j(j-1),1)\max((|\alpha|-j)(|\alpha|-j-1),1)}.
\end{aligned}$$

Here we have used $[|\eta|-3/2]_+ + [|\alpha|-|\eta|-3/2]_+ \leq [|\alpha|-3/2]_+$ for all $\eta\leq\alpha$.

Now, for all $j\in\mathbb{N}_0$,

$$\frac{(j+1)^{1/2}}{\max(j(j-1),1)} = \frac{(j+1)^{1/2}\max(j,1)^{1/2}}{\max(j-1,1)}\frac{1}{\max(j,1)^{3/2}} \leq \sqrt{6}\frac{1}{\max(j,1)^{3/2}}.$$

In addition,

$$\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta} = \binom{|\alpha|}{j}.$$

Therefore,

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta v))\|_{L^2(Q_{\delta,\omega})} \\
&\quad\leq 24C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1}\sum_{j=0}^{|\alpha|}j!(|\alpha|-j)!\frac{1}{\max(j,1)^{3/2}\max(|\alpha|-j,1)^{3/2}}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}. \\
&\quad\leq 24C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1}|\alpha|!\sum_{j=0}^{|\alpha|}\frac{1}{\max(j,1)^{3/2}\max(|\alpha|-j,1)^{3/2}}.
\end{aligned}$$

We have, by the Cauchy-Schwarz inequality,

$$\sum_{j=0}^{|\alpha|}\frac{1}{\max(j,1)^{3/2}\max(|\alpha|-j,1)^{3/2}} \leq \sum_{j=0}^{|\alpha|}\frac{1}{\max(j,1)^3} \leq 1+\zeta(3) \leq \frac{5}{2}.$$

We conclude that for any α as in (29),

$$\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta^2))\|_{L^2(Q_{\delta,\omega})} \leq 60C_{\mathrm{d}}^2 A_w^{[|\alpha|-3/2]_+} E_w^{\alpha_2+1}|\alpha|! \tag{32}$$

and

$$\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(rw_\vartheta w_r))\|_{L^2(Q_{\delta,\omega})} \leq 60C_{\mathrm{d}}^2 A_w^{[|\alpha|-3/2]_+} E_w^{\alpha_2+1}|\alpha|!. \tag{33}$$

Estimate of $r^2 c_r\partial_r v$, $rc_\vartheta\partial_\vartheta v$ and $rc_\vartheta v$ for $v\in\{w_r, w_\vartheta\}$

Let $\xi\in\mathbb{N}_0^2$ such that $|\xi|\leq 1$ and let $\varphi\in\{c_r, c_\vartheta\}$. We have

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{1+\xi_1}\varphi\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}\\
&\quad\leq \sum_{\eta=(0,j),j\in\{0,\dots,\alpha_2\}}\binom{\alpha_2}{j}\|\partial_\vartheta^j\varphi\|_{L^\infty(Q_{\delta,\omega})}\|r^{\alpha_1+\beta-2}\mathcal{D}^{\alpha-\eta}(r^{1+\xi_1}\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}\\
&\quad\leq c\sum_{\eta=(0,j),j\in\{0,\dots,\alpha_2\}}\binom{\alpha_2}{j}\|r^{\alpha_1+\beta-2}\mathcal{D}^{\alpha-\eta}(r^{1+\xi_1}\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}.
\end{aligned}$$

If $\alpha_1=0$, then

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{1+\xi_1}\varphi\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}\\
&\leq c\sum_{\eta=(0,j),j\in\{0,\dots,\alpha_2\}}\binom{|\alpha|}{j}\|r^{\xi_1+1+\beta-2}\mathcal{D}^{\alpha-\eta}\mathcal{D}^\xi v\|_{L^2(Q_{\delta,\omega})}\\
&\leq c\sum_{j=0}^{|\alpha|}\binom{|\alpha|}{j}A_w^{[|\alpha|-j-1]_+}E_w^{\alpha_2-j+\xi_2}[|\alpha|-j-1]_+!\\
&\leq c\sum_{j=0}^{|\alpha|}\frac{|\alpha|!}{j!}A_w^{[|\alpha|-j-1]_+}E_w^{\alpha_2-j+\xi_2}\\
&\leq ecA_w^{|\alpha|-1}E_w^{\alpha_2+1}|\alpha|!
\end{aligned}$$

since $\sum_{j=0}^{|\alpha|}\frac{1}{j!}\leq\sum_{j=0}^{+\infty}\frac{1}{j!}=e$. If $\alpha_1>0$,

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{1+\xi_1}\varphi\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}\\
&\leq c\sum_{j\in\{0,\dots,\alpha_2\},\eta=(0,j)}\binom{\alpha_2}{j}\Big(\|r^{\alpha_1+\xi_1+1+\beta-2}\mathcal{D}^{\alpha-\eta}\mathcal{D}^\xi v\|_{L^2(Q_{\delta,\omega})}\\
&\qquad+(1+\xi_1)\alpha_1\|r^{\alpha_1+\xi_1+\beta-2}\mathcal{D}^{\alpha-\eta-(1,0)}\mathcal{D}^\xi v\|_{L^2(Q_{\delta,\omega})}\\
&\qquad+(1+\xi_1)\xi_1\frac{\alpha_1(\alpha_1-1)}{2}\|r^{\alpha_1+\beta-2}\mathcal{D}^{\alpha-\eta-(2,0)}\mathcal{D}^\xi v\|_{L^2(Q_{\delta,\omega})}\Big)\\
&\leq c\sum_{j\in\{0,\dots,\alpha_2\}}\binom{\alpha_2}{j}4A_w^{[|\alpha|-j-1]_+}E_w^{\alpha_2-j+1}[|\alpha|-j-1]_+!\\
&\leq 4c\sum_{j=0}^{|\alpha|}\frac{|\alpha|!}{j!}A_w^{|\alpha|-1}E_w^{\alpha_2+\xi_2}\\
&\leq 4ecA_w^{|\alpha|-1}E_w^{\alpha_2+\xi_2}|\alpha|!.
\end{aligned}$$

In conclusion, we have that for any $\varphi \in \{c_r, c_\vartheta\}$, any $v \in \{w_r, w_\vartheta\}$ and any $\xi \in \mathbb{N}_0^2$ with $|\xi| \leq 1$,

$$\begin{aligned} &\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{1+\xi_1}\varphi\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})} \\ &\qquad \leq 4ecA_w^{|\alpha|-1}E_w^{\alpha_2+1}|\alpha|!, \qquad \forall \alpha \in \mathbb{N}_0^2 : 1 \leq |\alpha| \leq k. \end{aligned} \tag{34}$$

Estimate of the remaining terms

Let $v, w \in \{w_r, w_\vartheta\}$ and let $\xi \in \mathbb{N}_0^2$ such that $|\xi| = 1$. We have, for any $|\alpha| > 0$,

$$\begin{aligned} &\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{1+\xi_1}w\mathcal{D}^\xi v))\|_{L^2(Q_{\delta,\omega})} \\ &\leq \sum_{j=1}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}\|r^{\eta_1+\beta/2-1}\mathcal{D}^\eta(rw)\|_{L^4(Q_{\delta,\omega})}\|r^{\alpha_1-\eta_1+\beta/2-1}\mathcal{D}^{\alpha-\eta}(r^{\xi_1}\mathcal{D}^\xi v)\|_{L^4(Q_{\delta,\omega})} \\ &\quad + \|r^{\alpha_1+\beta-1}w\mathcal{D}^\alpha(r^{\xi_1}\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})} \\ &= (I) + (II). \end{aligned} \tag{35}$$

We bound the sum in term (I) by similar techniques as above, using Lemma 13 and Corollary 14:

$$\begin{aligned} (I) &\leq \sum_{j=1}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}4C_{\mathrm{d}}^2(|\eta|+1)^{1/2}A_w^{[|\eta|-3/2]_+}E_w^{\eta_2+1/2}[|\eta|-2]_+! \\ &\qquad\qquad \times(|\alpha|-|\eta|+1)^{1/2}A_w^{[|\alpha|-|\eta|-1/2]_+}E_w^{\alpha_2-\eta_2+\xi_2+1/2}[|\alpha|-|\eta|-1]_+! \\ &\leq 4C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+1+\xi_2} \\ &\qquad\qquad \times\sum_{j=1}^{|\alpha|}\sum_{|\eta|=j,\eta\leq\alpha}\binom{\alpha}{\eta}j!(|\alpha|-j)!\frac{(j+1)^{1/2}(|\alpha|-j+1)^{1/2}}{\max(j(j-1),1)\max(|\alpha|-j,1)}, \end{aligned}$$

where we have used that

$$[|\eta|-3/2]_+ + [|\alpha|-|\eta|-1/2]_+ \leq [|\alpha|-3/2]_+, \qquad \forall \eta\leq\alpha : |\eta|\geq 1.$$

Since

$$\frac{(j+1)^{1/2}}{\max(j,1)} = \frac{(j+1)^{1/2}}{\max(j,1)^{1/2}}\frac{1}{\max(j,1)^{1/2}} \leq \sqrt{2}\frac{1}{\max(j,1)^{1/2}},$$

and using Hölder's inequality, we obtain

$$\begin{aligned} (I) &\leq 8C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+\xi_2+1}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{\max(j-1,1)\max(j,1)^{1/2}\max(|\alpha|-j,1)^{1/2}} \\ &\leq 8C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+\xi_2+1}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{\max(j-1,1)^{3/2}\max(|\alpha|-j,1)^{1/2}} \\ &\leq 8C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+\xi_2+1}|\alpha|!\left(1+\sum_{j=1}^{|\alpha|-1}j^{-2}\right)^{3/4}\left(1+\sum_{j=1}^{|\alpha|-1}j^{-2}\right)^{1/4} \\ &\leq 24C_{\mathrm{d}}^2A_w^{[|\alpha|-3/2]_+}E_w^{\alpha_2+\xi_2+1}|\alpha|!, \end{aligned} \tag{36}$$

where we have used $1+\zeta(2)\leq 3$. We now estimate term (II) in (35). Remark that

$$(II) \leq \|rw\|_{L^\infty(Q_{\delta,\omega})}\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{\xi_1}\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}. \tag{37}$$

In addition,

$$\begin{aligned}
&\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{\xi_1}\mathcal{D}^\xi v)\|_{L^2(Q_{\delta,\omega})}\\
&\quad\leq \|r^{\alpha_1+\xi_1+\beta-2}\mathcal{D}^{\alpha+\xi}v\|_{L^2(Q_{\delta,\omega})} + \alpha_1\xi_1\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha v\|_{L^2(Q_{\delta,\omega})}\\
&\quad\leq A_w^{|\alpha|-1}E_w^{\alpha_2+\xi_2}(|\alpha|-1)! + \xi_1|\alpha|A_w^{|\alpha|-1}E_w^{\alpha_2}[|\alpha|-2]_+!\\
&\quad\leq 3A_w^{|\alpha|-1}E_w^{\alpha_2+\xi_2}(|\alpha|-1)!.
\end{aligned}$$

Hence, from (30) and (37), for any α as in (29),

$$(II) \leq 3\delta C_{\mathrm{emb}}A_w^{|\alpha|-1}E_w^{\alpha_2+\xi_2}(|\alpha|-1)!. \tag{38}$$

It follows from (35), (36), and (38) that, for any $v, w \in \{w_r, w_\vartheta\}$ and any multi-index ξ such that $|\xi|=1$,

$$\|r^{\alpha_1+\beta-2}\mathcal{D}^\alpha(r^{1+\xi_1}w\mathcal{D}^\xi v))\|_{L^2(Q_{\delta,\omega})} \leq (24C_{\mathrm{d}}^2+3C_{\mathrm{emb}})A_w^{|\alpha|-1}E_w^{\alpha_2+1+\xi_2}|\alpha|!. \tag{39}$$

The combination of the formula (31) and of the bounds (32), (33), (34), and (39) concludes the proof, with

$$C_{\mathrm{t}} = 6\max\left(60C_{\mathrm{d}}^2+4ec, 24C_{\mathrm{d}}^2+3C_{\mathrm{emb}}+4ec\right).$$

□

4.2 Analytic regularity in the polygon *P*

We can now prove the main result of this paper. With analyticity in the interior and up to edges of P being classical, we concentrate on the sectors near the corners $\mathfrak{c}_i$ of the domain P. We define for $\delta\in(0,1)$,

$$S_\delta^i := Q_{\delta,\omega_i}(\mathfrak{c}_i), \qquad i=1,\dots,n. \tag{40}$$

We prepare the bootstrapping argument required for establishing analytic regularity by proving that the solution $(\boldsymbol{u}, p)$ as is given in Theorem 2 satisfies that $(\boldsymbol{u}-\boldsymbol{u}(\mathfrak{c}_i), p) \in [\mathcal{W}^2_{\beta_i}(S^i_\delta)]^2\times\mathcal{W}^1_{\beta_i}(S^i_\delta)$.

Lemma 16 *Let* $\underline{\beta}_f=(\beta_1,\dots,\beta_n)\in(0,1)^n$ *be such that around each corner* $\mathfrak{c}_i$*, the operator pencil* $\mathcal{A}(\lambda)$ *defined as in* (16) *has no eigenvalue on* $\mathrm{Im}(\lambda)=1-\beta_i$ *for* $i=1,...,n$ *and let* $\boldsymbol{f}\in[L_{\underline{\beta}_f}(P)]^2\cap\boldsymbol{W}^*$ *such that* $\|\boldsymbol{f}\|_{\boldsymbol{W}^*}\leq\frac{C_{\mathrm{coer}}^2\nu^2}{4C_{\mathrm{cont}}}$*. Suppose that Assumption 1 holds. Let* $(\boldsymbol{u}, p)$ *be the solution to* (2) *with right hand side* $\boldsymbol{f}$*.*

Then, the following results hold:

1. *For all* $0<\delta\leq 1$ *with* $\delta<\frac{1}{4}\min_{i,j}|\mathfrak{c}_j-\mathfrak{c}_i|$*,*

$$(\boldsymbol{u}-\boldsymbol{u}(\mathfrak{c}), p)\in[\mathcal{W}^2_{\beta_i}(S^i_{\delta/2})]^2\times\mathcal{W}^1_{\beta_i}(S^i_{\delta/2}), \qquad \forall i\in\{1,\dots,n\}.$$

2. *For any corner* $\mathfrak{c}_i$ *which touches a complete side* $\Gamma_i\subset\Gamma_G$*,* $\boldsymbol{u}(\mathfrak{c}_i)\cdot\boldsymbol{n}=0$ *on* Γ_i *where* $\boldsymbol{n}$ *is the outer normal vector of* Γ_i*.*

Proof We start by showing the first assertion. For all $s \in (1,2)$ and for $t = (1/s - 1/2)^{-1}$,

$$\|\boldsymbol{f}\|_{L^s(P)} \leq \|\Phi_{-\underline{\beta}_f}\|_{L^t(P)} \|\Phi_{\underline{\beta}_f} \boldsymbol{f}\|_{L^2(P)}.$$

Therefore $\boldsymbol{f} \in [L_{\underline{\beta}_f}(P)]^2$ implies

$$\boldsymbol{f} \in [L^s(P)]^2, \qquad \forall s \in \left(1, \frac{2}{1+\max \underline{\beta}_f}\right).$$

In addition, $\boldsymbol{u} \in [H^1(P)]^2$ implies by Sobolev embedding $\boldsymbol{u} \in [L^t(P)]^2$ for all $t \in [1,\infty)$. By Hölder's inequality, choosing $t \in [1,\infty)$ and $s = (1/2 + 1/t)^{-1}$,

$$\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{L^s(P)} \leq \|\boldsymbol{u}\|_{L^t(P)} \|\nabla \boldsymbol{u}\|_{L^2(P)} < \infty$$

which implies $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in [L^s(P)]^2$, for all $s \in [1,2)$. It follows from [5, Corollary 4.2] that there exists $q > 1$ such that $(\boldsymbol{u}, p) \in [W^{2,q}(P)]^2 \times W^{1,q}(P)$. This in turn implies, by Sobolev embedding, $\boldsymbol{u} \in [L^\infty(P)]^2$. Hence $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in [L^2(P)]^2$. We conclude by applying Theorem 9 to each corner sector to obtain that there exists a constant C_{sec} such that for each $i \in \{1,\dots,n\}$,

$$\begin{aligned}\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}(\mathfrak{c})}\|_{\mathcal{W}^2_{\beta_i}(S^i_{\delta/2})} + \|p\|_{\mathcal{W}^1_{\beta_i}(S^i_{\delta/2})} \leq C_{\mathrm{sec}}\Big(&\|\overline{\boldsymbol{f}}\|_{\mathcal{L}_{(\underline{\beta}_f)_i}(S^i_\delta)} \\ &+ \|\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}}\|_{\mathcal{L}_{(\underline{\beta}_f)_i}(S^i_\delta)} + \|\boldsymbol{u}\|_{H^1(P)} + \|p\|_{L^2(P)}\Big).\end{aligned}$$

Now, since $\boldsymbol{f} \in [\mathcal{L}_{\underline{\beta}_f}(P)]^2$ and $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in [L^2(P)]^2$, it holds that $\overline{\boldsymbol{f}} \in [\mathcal{L}_{\underline{\beta}_f}(S^i_\delta)]^2$ and $\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}} \in [\mathcal{L}_{\underline{\beta}_f}(S^i_\delta)]^2$; hence, the right hand side of the inequality above is bounded. Using [13, Corollary 4.2] to bound the norm of the Cartesian version of the flux concludes the proof of the regularity result.

To show the second point, we fix $i \in \{1,\dots,n\}$ and assume that $\Gamma \subset \Gamma_G$ abuts $\mathfrak{c}_i$. Then, for any point $\boldsymbol{x} \in \Gamma$ we have, due to the boundary condition, $\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{n} = 0$, where $\boldsymbol{n}$ is the outer normal vector of Γ. In addition, Lemma 6 implies that $\boldsymbol{u} \in C^0(\overline{S^i_\delta})^2$ since $\boldsymbol{u} - \boldsymbol{u}(\mathfrak{c}) \in \mathcal{W}^2_{\beta_i}(S^i_{\delta/2})^2 \subset C^0(\overline{S^i_{\delta/2}})^2$. Therefore, by letting $\boldsymbol{x} \to \mathfrak{c}_i$ along Γ, we have $\boldsymbol{u}(\mathfrak{c}_i)\cdot\boldsymbol{n} = \lim_{\boldsymbol{x}\to\mathfrak{c}_i} \boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{n} = 0$. □

We prove weighted analytic estimates for Leray-Hopf weak solutions in each corner sector.

Lemma 17 *Let $\underline{\beta}_f = (\beta_1,\dots,\beta_n) \in (0,1)^n$ be such that in a vicinity of each corner $\mathfrak{c}_i$, the operator pencil $\mathcal{A}(\lambda)$ defined as in (16) has no eigenvalue on $\mathrm{Im}(\lambda) = 1 - \beta_i$ and $\boldsymbol{f} \in [B^0_{\underline{\beta}_f}(P)]^2 \cap \boldsymbol{W}^*$ such that $\|\boldsymbol{f}\|_{\boldsymbol{W}^*} \leq \frac{C^2_{\mathrm{coer}}\nu^2}{4C_{\mathrm{cont}}}$. Suppose that Assumption 1 holds and let $(\boldsymbol{u}, p)$ be the solution to (2) with right hand side $\boldsymbol{f}$.*

Then there exists $\delta_P \in (0,1]$ such that for all $i \in \{1,2,\dots,n\}$, $(\boldsymbol{u}, p) \in [B^2_{\beta_i}(S^i_{\delta_P/2})]^2 \times B^1_{\beta_i}(S^i_{\delta_P/2})$.

Remark 18 Lemma 17 implies in particular that if $\boldsymbol{u}(\mathfrak{c}_i) = \boldsymbol{0}$ (this happens when at least one straight edge of $S^i_{\delta_P}$ is a zero Dirichlet edge or both edges are equipped with homogeneous Slip boundary condition), then $\boldsymbol{u} \in [B^2_{\beta_i}(S^i_{\delta_P/2})]^2 \subset$

$[H^{2,2}_{\underline{\beta}_f}(S^i_{\delta_P/2})]^2$ and $p \in B^1_{\beta_i}(S^i_{\delta_P/2}) \subset H^{1,1}_{\underline{\beta}_f}(S^i_{\delta_P/2})$ implies by Lemma 7 that $\boldsymbol{u} \in [K^2_{2-\beta_i}(S^i_{\delta_P/2})]^2$ and that $p \in K^1_{1-\beta_i}(S^i_{\delta_P/2})$. Furthermore, by definition $B^\ell_{\beta_i}(S^i_{\delta_P/2}) \cap K^\ell_{\ell-\beta_i}(S^i_{\delta_P/2}) = K^\varpi_{\ell-\underline{\beta}_i}(S^i_{\delta_P/2})$. Therefore, $\boldsymbol{u} \in [K^\varpi_{2-\beta_i}(S^i_{\delta_P/2})]^2$ and $p \in K^\varpi_{1-\beta_i}(S^i_{\delta_P/2})$ in this case.

Proof Write $\underline{\beta}_f = (\beta_1, \dots, \beta_n) \in (0,1)^n$. Fix $0 < \delta_P \leq 1$ such that $\delta_P < \frac{1}{4}\min_{i,j} |\mathfrak{c}_j - \mathfrak{c}_i|$ and such that

$$\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{u}(\mathfrak{c}_i)}\|_{\mathcal{W}^2_{\beta_i}(S^i_{\delta_P})} \leq 1, \qquad \|p\|_{\mathcal{W}^1_{\beta_i}(S^i_{\delta_P})} \leq 1, \qquad \forall i \in \{1, \dots, n\}. \tag{41}$$

Note that this condition is meaningful thanks to Lemma 16. The proof proceeds by induction, in each of the corner sectors. Fix $i \in \{1, \dots, n\}$. We write $r(x) \coloneqq r_i(x) = |x - \mathfrak{c}_i|$ for compactness.

Let $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}(\mathfrak{c}_i)$. Before setting up the inductive bootstrap argument, we rewrite the NSE with $\widetilde{\boldsymbol{u}}$ in polar coordinates and rearrange the equations in the sector $S^i_{\delta_P}$ as

$$\overline{L^\Delta_{\mathrm{St}}}(\overline{\widetilde{\boldsymbol{u}}}, p) = \begin{pmatrix} A(\boldsymbol{f} - ((\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)) \cdot \nabla)(\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i))) \\ 0 \end{pmatrix} \quad \text{in } S^i_{\delta_P}, \tag{42a}$$

$$\overline{B}(\overline{\widetilde{\boldsymbol{u}}}, p) = \boldsymbol{0} \quad \text{on } \Gamma_\delta. \tag{42b}$$

This set of equations has the following component-wise form:

$$-\frac{1}{r^2}\begin{pmatrix} \nu((r\partial_r)^2 + \partial_\vartheta^2 - 1) & -2\nu\partial_\vartheta \\ 2\nu\partial_\vartheta & \nu((r\partial_r)^2 + \partial_\vartheta^2 - 1) \end{pmatrix}\begin{pmatrix} \widetilde{u}_r \\ \widetilde{u}_\vartheta \end{pmatrix} + \frac{1}{r}\begin{pmatrix} r\partial_r \\ \partial_\vartheta \end{pmatrix} p = \widehat{\boldsymbol{f}} \text{ in } S^i_{\delta_P}, \tag{43}$$

$$\frac{1}{r}\left((r\partial_r + 1)\,\widetilde{u}_r + \partial_\vartheta \widetilde{u}_\vartheta\right) = 0 \text{ in } S^i_{\delta_P}, \tag{44}$$

$$\overline{\widetilde{\boldsymbol{u}}} = \boldsymbol{0} \text{ on } \partial S^i_{\delta_P} \cap \Gamma_D. \tag{45}$$

Here $\widehat{\boldsymbol{f}} = \overline{\boldsymbol{f}} - \overline{(\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)) \cdot \nabla)(\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i))}$. On $\partial S^i_{\delta_P} \cap \Gamma_N$ and $\partial S^i_{\delta_P} \cap \Gamma_G$, respectively,

$$\begin{pmatrix} \nu(r^{-1}\partial_\vartheta \widetilde{u}_r + \partial_r \widetilde{u}_\vartheta - r^{-1}\widetilde{u}_\vartheta \\ -p + 2\nu r^{-1}(\partial_\vartheta \widetilde{u}_\vartheta + \widetilde{u}_r)) \end{pmatrix} = \boldsymbol{0} \tag{46}$$

and

$$\begin{pmatrix} \widetilde{u}_\vartheta \\ \nu(\partial_r \widetilde{u}_\vartheta + \frac{1}{r}\partial_\vartheta \widetilde{u}_r - \frac{1}{r}\widetilde{u}_\vartheta) \end{pmatrix} = \boldsymbol{0}\,. \tag{47}$$

See Appendix C for details of the derivation.

The analyticity of $\boldsymbol{u}$ and p in $\overline{P \setminus \left(\bigcup_{i=1}^n S^i_{\delta_P/2}\right)}$ and the analyticity assumption on $\boldsymbol{f}$, i.e., $\boldsymbol{f} \in [B^0_{\underline{\beta}_f}(P)]^2$ (whence $\overline{\boldsymbol{f}} \in [\mathcal{B}^0_{\beta_i}(S^i_{\delta_P})]^2$ by Lemma 4), imply that there exists $A_1 > 0$ such that, for all $|\alpha| \geq 1$,

$$\|r^{\beta_i + \alpha_1 - 2}\mathcal{D}^\alpha(r^2\overline{\boldsymbol{f}})\|_{L^2(S^i_{\delta_P})} \leq A_1^{|\alpha|}|\alpha|!, \tag{48a}$$

$$\|r^{\beta_i + \alpha_1 - 2}\mathcal{D}^\alpha(r^2\overline{((\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)) \cdot \nabla)(\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)))}\|_{L^2(S^i_{\delta_P} \setminus S^i_{\delta_P/2})} \leq A_1^{|\alpha|}|\alpha|!, \tag{48b}$$

$$\|r^{\beta_i+\alpha_1-1}\mathcal{D}^\alpha p\|_{L^2(S^i_{\delta_P}\setminus S^i_{\delta_P/2})} \leq A_1^{|\alpha|-1}(|\alpha|-1)!, \tag{48c}$$

and, for all $k \in \mathbb{N}$,

$$\|r^k\partial_r^k\widetilde{\overline{\boldsymbol{u}}}\|_{H^1(S^i_{\delta_P}\setminus S^i_{\delta_P/2})} \leq A_1^k k!. \tag{48d}$$

For the ensuing induction argument, we define the constants

$$E_u = \max\left(2, 8\left(1+\frac{1}{\nu}\right)^{3/2}, (8\nu)^{3/2}\right), \tag{49a}$$

and

$$A_u = \max\left(22C_{\mathrm{sec}}A_1, 2C_{\mathrm{sec}}(C_{\mathrm{t}}+9)E_u^2, \frac{4}{\nu}A_1, 4\left(\frac{1}{\nu}(C_{\mathrm{t}}+2)+4\right)E_u^{4/3}, 4A_1, 4(C_{\mathrm{t}}+1+3\nu)E_u, 2\right). \tag{49b}$$

We now formulate our induction assumption.

Induction assumption

We say $H_{\hat{k},k_2}$ holds for $\hat{k} \in \mathbb{N}$ and $k_2 \in \mathbb{N}$ with $k_2 \leq \hat{k}$, if

$$\begin{aligned} \|r_i^{\beta_i+\alpha_1-2}\mathcal{D}^\alpha\widetilde{u}_r\|_{L^2(S^i_{\delta_P/2})} &\leq A_u^{|\alpha|-2}E_u^{[\alpha_2-4/3]_+}(|\alpha|-2)!, \\ \|r_i^{\beta_i+\alpha_1-2}\mathcal{D}^\alpha\widetilde{u}_\vartheta\|_{L^2(S^i_{\delta_P/2})} &\leq A_u^{|\alpha|-2}E_u^{[\alpha_2-4/3]_+}(|\alpha|-2)!, \end{aligned} \qquad \forall\alpha\in\mathbb{N}_0^2 : \begin{cases} 2 \leq |\alpha| \leq \hat{k}+1, \\ \alpha_2 \leq k_2+1, \end{cases} \tag{50a}$$

and

$$\|r_i^{\beta_i+\alpha_1-1}\mathcal{D}^\alpha p\|_{L^2(S^i_{\delta_P/2})} \leq A_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!, \qquad \forall\alpha\in\mathbb{N}_0^2 : \begin{cases} 1 \leq |\alpha| \leq \hat{k}, \\ \alpha_2 \leq k_2, \end{cases} \tag{50b}$$

where A_u and E_u are the constants in (49b) and (49a).

Strategy of the proof

We start the induction by noting that $H_{1,1}$ holds due to Lemma 16 and to (41).

The induction proof of the statement will be composed of two main steps. In the first step, we show

$$\forall k \in \mathbb{N}, \qquad H_{k,k} \implies H_{k+1,1}. \tag{51}$$

Then, in the following step, we will show that, for all $k \in \mathbb{N}$ and all $j \in \mathbb{N}$ such that $j \leq k$,

$$H_{k,k} \text{ and } H_{k+1,j} \implies H_{k+1,j+1}. \tag{52}$$

Combining (51) and (52), we obtain that

$$H_{k,k} \implies H_{k+1,k+1}, \tag{53}$$

We infer from (53) that $H_{k,k}$ is verified for all $k \in \mathbb{N}$. This will conclude the proof.

***Step 1: proof of* (51)**

We fix $k \in \mathbb{N}$ and suppose that $H_{k,k}$ holds. Define

$$\overline{\boldsymbol{v}} := r^k \partial_r^k \overline{\widetilde{\boldsymbol{u}}}, \qquad q := r^k \partial_r^k p. \tag{54}$$

Then, for all $|\eta| \leq 2$,

$$r^{\eta_1} \mathcal{D}^\eta \overline{\boldsymbol{v}} = r^k \partial_r^k (r^{\eta_1} \mathcal{D}^\eta \overline{\widetilde{\boldsymbol{u}}}) \tag{55}$$

and

$$\mathcal{D}^\eta q = r^{k-2} \partial_r^k (r^{\eta_1+1} \mathcal{D}^\eta p) - k r^{k-1} \partial_r^{k-1} \mathcal{D}^\eta p - \eta_1 k(k-1) r^{k-2} \partial_r^{k-1} p. \tag{56}$$

Furthermore, multiplying (44) by r and differentiating by ∂_r^k we obtain

$$(r\partial_r + (k+1))\partial_r^k \widetilde{u}_r + \partial_r^k \partial_\vartheta \widetilde{u}_\vartheta = 0,$$

hence

$$0 = r^{k-1}(r\partial_r + (k+1))\partial_r^k \widetilde{u}_r + r^{k-1}\partial_\vartheta \partial_r^k \widetilde{u}_\vartheta = \frac{1}{r}\left((r\partial_r + 1)v_r + \partial_\vartheta v_\vartheta\right). \tag{57}$$

From (55), (56), and (57), it follows that the pair $(\overline{\boldsymbol{v}}, q)$ as defined in (54) satisfies, with $\overline{L^{\Delta}_{\mathrm{St}}}$ and $\overline{B}$ in polar frame and acting on the velocity field $\overline{\widetilde{\boldsymbol{u}}}$ in polar frame as defined in (42a) and (42b) formally the Stokes boundary value problem

$$\begin{aligned} &\overline{L^{\Delta}_{\mathrm{St}}}(\overline{\boldsymbol{v}}, q) = \begin{pmatrix} \widetilde{\boldsymbol{f}} \\ 0 \end{pmatrix}, \quad \text{in } S^i_{\delta_P}, \\ &\overline{B}(\overline{\boldsymbol{v}}, q) = \begin{pmatrix} \boldsymbol{0} \\ \widetilde{\boldsymbol{g}} \\ \boldsymbol{0} \end{pmatrix}, \quad \text{on } (\partial S^i_{\delta_P} \cap \Gamma_D) \times (\partial S^i_{\delta_P} \cap \Gamma_N) \times (\partial S^i_{\delta_P} \cap \Gamma_G),. \end{aligned} \tag{58}$$

Here, $\widetilde{\boldsymbol{f}}$ and (assuming that $\partial S^i_{\delta_P} \cap \Gamma_N \neq \varnothing$) $\widetilde{\boldsymbol{g}}$ are defined by

$$\begin{aligned} &\widetilde{\boldsymbol{f}} = r^{k-2}\partial_r^k (r^2 (\overline{\boldsymbol{f}} - \overline{((\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)) \cdot \nabla)(\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i))})) - k r^{k-2} \begin{pmatrix} r\partial_r^k p + (k-1)\partial_r^{k-1} p \\ \partial_r^{k-1}\partial_\vartheta p \end{pmatrix}, \\ &\widetilde{\boldsymbol{g}} = \begin{pmatrix} 0 \\ k r^{k-1} \partial_r^{k-1} p \end{pmatrix}. \end{aligned} \tag{59}$$

Using (48a), Lemma 15 with $\boldsymbol{w} = \boldsymbol{u}$, the inductive hypothesis $H_{k,k}$, and the fact that for all $v \in L^2(S^i_{\delta_P})$

$$\|v\|_{L^2(S^i_{\delta_P})} \leq \|v\|_{L^2(S^i_{\delta_P/2})} + \|v\|_{L^2(S^i_{\delta_P} \setminus S^i_{\delta_P/2})},$$

we find from (59)

$$\begin{aligned} \|\widetilde{\boldsymbol{f}}\|_{\mathcal{L}_{\beta_i}(S^i_{\delta_P})} &\leq \|r^{\beta_i+k-2}\partial_r^k (r^2 \overline{\boldsymbol{f}})\|_{L^2(S^i_{\delta_P})} \\ &\quad + \|r^{\beta_i+k-2}\partial_r^k (r^2 \overline{((\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)) \cdot \nabla)(\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i))})\|_{L^2(S^i_{\delta_P})} \\ &\quad + k\|r^{\beta_i+k-1}\partial_r^k p\|_{L^2(S^i_{\delta_P})} + k(k-1)\|r^{\beta_i+k-2}\partial_r^{k-1} p\|_{L^2(S^i_{\delta_P})} \\ &\quad + k\|r^{\beta_i+k-2}\partial_r^{k-1}\partial_\vartheta p\|_{L^2(S^i_{\delta_P})} \\ &\leq A_1^k k! + \left(C_{\mathrm{t}} A_u^{k-1} E_u^2 + A_1^k\right) k! + k\left(A_u^{k-1} + A_1^{k-1}\right)(k-1)! \\ &\quad + k(k-1)\left(A_u^{k-2} + A_1^{k-2}\right)(k-2)! + k\left(A_u^{k-1} E_u + A_1^{k-1}\right) \\ &\leq \left(5A_1^k + (C_{\mathrm{t}} + 3)A_u^{k-1} E_u^2\right) k!. \end{aligned}$$

Furthermore,

$$
\begin{aligned}
\|\widetilde{\boldsymbol{g}}\|_{\mathcal{W}^{1/2}_{\beta}(\partial S^i_{\delta_P}\cap\Gamma_N)} &\le k\|r^{k-1}\partial_r^{k-1}p\|_{W^1_\beta(S^i_{\delta_P})} \\
&\le k\Big(\|r^{k-2+\beta}\partial_r^{k-1}p\|_{L^2(S^i_{\delta_P})} \\
&\qquad + \|r^{k-2+\beta}\partial_r^{k-1}\partial_\vartheta p\|_{L^2(S^i_{\delta_P})} + \|r^{k-1+\beta}\partial_r^{k}p\|_{L^2(S^i_{\delta_P})} \\
&\qquad + (k-1)\|r^{k-2+\beta}\partial_r^{k-1}p\|_{L^2(S^i_{\delta_P})}\Big) \\
&\le 4k\left(A_1^{k-1}+A_u^{k-1}E_u\right)(k-1)! \\
&= 4\left(A_1^{k-1}+A_u^{k-1}E_u\right)k!.
\end{aligned}
$$

It follows from (58), Theorem 9, (48d), (48c), and the two inequalities above that

$$
\begin{aligned}
&\|\overline{\boldsymbol{v}}-\overline{\boldsymbol{v}(\mathfrak{c}_i)}\|_{\mathcal{W}^2_{\beta_i}(S^i_{\delta_P/2})} + \|q\|_{\mathcal{W}^1_{\beta_i}(S^i_{\delta_P/2})} \\
&\quad\le C_{\mathrm{sec}}\left(\|\widetilde{\boldsymbol{f}}\|_{\mathcal{L}_{\beta_i}(S^i_{\delta_P})} + \|\overline{\boldsymbol{v}}\|_{H^1(S^i_{\delta_P}\setminus S^i_{\delta_P/2})} + \|q\|_{L^2(S^i_{\delta_P}\setminus S^i_{\delta_P/2})} + \|\widetilde{\boldsymbol{g}}\|_{\mathcal{W}^{1/2}_{\beta}(\partial S^i_{\delta_P}\cap\Gamma_N)}\right) \\
&\quad\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+7)A_u^{k-1}E_u^2\right)k!.
\end{aligned}
\tag{60}
$$

We claim that $\overline{\boldsymbol{v}(\mathfrak{c}_i)} = \mathbf{0}$. This means that this term in (60) could be omitted. To prove the claim, we observe that the validity of $H_{k,k}$ implies that $\|r^{k+\beta_i-2}\partial_r^k\overline{\widetilde{\boldsymbol{u}}}\|_{L^2(S^i_{\delta_P/2})} < +\infty$ and thus $\overline{\boldsymbol{v}} \in \mathcal{L}_{\beta_i-2}(S^i_{\delta_P/2})^2$. This is equivalent to $\boldsymbol{v} \in \mathcal{L}_{\beta_i-2}(S^i_{\delta_P/2})^2$. Using (60), [13, Corollary 4.2, Page 322] and Lemma 6 we have that $\boldsymbol{v} \in C^0(\overline{S^i_{\delta_P/2}})^2$. Then the condition $\boldsymbol{v} \in \mathcal{L}_{\beta_i-2}(S^i_{\delta_P/2})^2$ forces $\boldsymbol{v}$ (and $\overline{\boldsymbol{v}}$) to vanish at $\mathfrak{c}_i$ since otherwise $r^{2(\beta_i-2)}v_i^2$ would not be integrable on $S^i_{\delta_P/2}$.

Now, for all $|\eta| = 2$,

$$
\mathcal{D}^\eta\overline{\boldsymbol{v}} = r^k\partial_r^k\mathcal{D}^\eta\overline{\widetilde{\boldsymbol{u}}} + \eta_1 k r^{k-1}\partial_r^{k+\eta_1-1}\partial_\vartheta^{\eta_2}\overline{\widetilde{\boldsymbol{u}}} + [\eta_1-1]_+k(k-1)r^{k-2}\partial_r^k\overline{\widetilde{\boldsymbol{u}}}.
$$

Therefore, for all $|\eta| = 2$,

$$
\begin{aligned}
&\|r^{\beta_i+k+\eta_1-2}\partial_r^k\mathcal{D}^\eta\overline{\widetilde{\boldsymbol{u}}}\|_{L^2(S^i_{\delta_P/2})} \\
&\quad\le \|\overline{\boldsymbol{v}}\|_{\mathcal{W}^2_{\beta_i}(S^i_{\delta_P/2})} + \eta_1 k\|r^{\beta_i+k+\eta_1-3}\partial_r^{k+\eta_1-1}\partial_\vartheta^{\eta_2}\overline{\widetilde{\boldsymbol{u}}}\|_{L^2(S^i_{\delta_P/2})} \\
&\qquad\qquad + k(k-1)\|r^{\beta_i+k-2}\partial_r^k\overline{\widetilde{\boldsymbol{u}}}\|_{L^2(S^i_{\delta_P/2})} \\
&\quad\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+7)A_u^{k-1}E_u^2\right)k! + 2kA_u^{k-1}(k-1)! + k(k-1)A_u^{k-2}(k-2)! \\
&\quad\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+9)A_u^{k-1}E_u^2\right)k!.
\end{aligned}
$$

For all $|\eta| = 1$,

$$
\mathcal{D}^\eta q = r^k\partial_r^k\mathcal{D}^\eta q + \eta_1 k r^{k-1}\partial_r^k p,
$$

hence

$$
\begin{aligned}
\|r^{\beta_i+k+\eta_1-1}\partial_r^k\mathcal{D}^\eta p\|_{L^2(S^i_{\delta_P/2})} &\le \|q\|_{\mathcal{W}^1_{\beta_i}(S^i_{\delta_P/2})} + k\|r^{\beta_i+k-1}\partial_r^k p\|_{L^2(S^i_{\delta_P/2})} \\
&\le C_{\mathrm{sec}}\left(11A_1^k + (C_{\mathrm{t}}+7)A_u^{k-1}E_u^2\right)k! + kA_u^{k-1}(k-1)!
\end{aligned}
$$

$$\leq C_{\rm sec}\left(11A_1^k + (C_{\rm t}+8)A_u^{k-1}E_u^2\right)k!.$$

From (49b) it follows that for every $k\in\mathbb{N}$

$$\max_{|\eta|=2}\|r^{\beta_i+k+\eta_1-2}\partial_r^k\mathcal{D}^\eta\overline{\widetilde{\boldsymbol{u}}}\|_{L^2(S^i_{\delta_P/2})}\leq A_u^k k!,\quad \max_{|\eta|=1}\|r^{\beta_i+k+\eta_1-1}\partial_r^k\mathcal{D}^\eta p\|_{L^2(S^i_{\delta_P/2})}\leq A_u^k k!,$$

i.e., that $H_{k+1,1}$ holds. We have shown implication (51).

Step 2: proof of (52)

We now fix $j\in\{1,\dots,k\}$ and we assume that $H_{k,k}$ and $H_{k+1,j}$ hold true.

Multiply (44) by r and differentiate by $\partial_r^{k-j}\partial_\vartheta^{j+1}$ to obtain

$$r\partial_r^{k+1-j}\partial_\vartheta^{j+1}\widetilde{u}_r + (k+1-j)\partial_r^{k-j}\partial_\vartheta^{j+1}\widetilde{u}_r + \partial_r^{k-j}\partial_\vartheta^{j+2}\widetilde{u}_\vartheta = 0.$$

Therefore, using $H_{k+1,j}$,

$$\begin{aligned}
&\|r^{\beta_i+k-j-2}\partial_r^{k-j}\partial_\vartheta^{j+2}\widetilde{u}_\vartheta\|_{L^2(S^i_{\delta_P/2})}\\
&\quad\leq \|r^{\beta_i+k-j-1}\partial_r^{k+1-j}\partial_\vartheta^{j+1}\widetilde{u}_r\|_{L^2(S^i_{\delta_P/2})} + k\|r^{\beta_i+k-j-2}\partial_r^{k-j}\partial_\vartheta^{j+1}\widetilde{u}_r\|_{L^2(S^i_{\delta_P/2})}\\
&\quad\leq A_u^kE_u^{j-1/3}k! + kA_u^{k-1}E_u^{j-1/3}(k-1)!\\
&\quad\leq 2A_u^kE_u^{j-1/3}k!\\
&\quad\leq A_u^kE_u^{j+2/3}k!.
\end{aligned}\tag{61}$$

This proves the estimate for $\widetilde{u}_\vartheta$.

To prove the bound on $\widetilde{u}_r$, multiply the first equation in (43) by r^2 and differentiate by $\partial_r^{k-j}\partial_\vartheta^j$, to obtain

$$\begin{aligned}
\nu\partial_r^{k-j}\partial_\vartheta^{j+2}\widetilde{u}_r = &-\nu\left(r^2\partial_r^2 + (2(k-j)+1)r\partial_r + (k-j)^2-1\right)\partial_r^{k-j}\partial_\vartheta^j\widetilde{u}_r - 2\nu\partial_r^{k-j}\partial_\vartheta^{j+1}\widetilde{u}_\vartheta\\
&+ (r^2\partial_r^2 + 2(k-j)r\partial_r + (k-j)(k-j-1))\partial_r^{k-j-1}\partial_\vartheta^j p\\
&- \partial_r^{k-j}\partial_\vartheta^j\left(r^2(\overline{\boldsymbol{f}} - \overline{((\widetilde{\boldsymbol{u}}+\boldsymbol{u}(\mathfrak{c}_i))\cdot\nabla)(\widetilde{\boldsymbol{u}}+\boldsymbol{u}(\mathfrak{c}_i))})_r\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|r^{\beta_i+k-j-2}\partial_r^{k-j}\partial_\vartheta^{j+2}\widetilde{u}_r\|_{L^2(S^i_{\delta_P/2})}\\
&\quad\leq\left(A_u^2k! + 2kA_u(k-1)! + k(k-2)(k-2)!\right)A_u^{k-2}E_u^{[j-4/3]_+} + 2A_u^{k-1}E_u^{j-1/3}(k-1)!\\
&\qquad+\frac{1}{\nu}\left(A_u^kk! + 2(k-1)A_u^{k-1}(k-1)! + (k-1)(k-2)A_u^{k-2}(k-2)!\right)E_u^j\\
&\qquad+\frac{1}{\nu}A_1^kk! + \frac{1}{\nu}C_{\rm t}A_u^{k-1}E_u^{j+2}k!\\
&\quad\leq\left(\frac{1}{\nu}A_1^k + \left(1+\frac{1}{\nu}\right)A_u^kE_u^j + \left(\frac{1}{\nu}(C_{\rm t}+2)+4\right)A_u^{k-1}E_u^{j+2} + \left(1+\frac{1}{\nu}\right)A_u^{k-2}E_u^j\right)k!.\\
&\quad\leq A_u^kE_u^{j+2/3}k!
\end{aligned}\tag{62}$$

This provides the estimate for $\widetilde{u}_r$.

Last, consider the second equation of (43): multiplying by r^2 and differentiating by $\partial_r^{k-j}\partial_\vartheta^j$ we obtain

$$r\partial_r^{k-j}\partial_\vartheta^{j+1}p = \nu\left(r^2\partial_r^2 + (2(k-j)+1)r\partial_r + (k-j)^2 - 1 + \partial_\vartheta^2\right)\partial_r^{k-j}\partial_\vartheta^j\widetilde{u}_\vartheta$$

$$+ 2\nu\partial_r^{k-j}\partial_\vartheta^{j+1}\widetilde{u}_r - (k-j)\partial_r^{k-j-1}\partial_\vartheta^{j+1}p$$
$$+ \partial_r^{k-j}\partial_\vartheta^{j}\Big(r^2(\overline{\boldsymbol{f}} - \overline{((\widetilde{\boldsymbol{u}}+\boldsymbol{u}(\mathfrak{c}_i))\cdot\nabla)(\widetilde{\boldsymbol{u}}+\boldsymbol{u}(\mathfrak{c}_i)))_\vartheta}\Big).$$

Hence,

$$\begin{aligned}
&\|r^{\beta_i+k-j-1}\partial_r^{k-j}\partial_\vartheta^{j+1}p\|_{L^2(S^i_{\delta_P/2})}\\
&\leq \nu\Big(A_u^2k! + 2kA_u(k-1)! + k(k-2)(k-2)!\Big)A_u^{k-2}E_u^{[j-4/3]_+}\\
&\quad + \nu A_u^k E_u^{j+1/3}k! + 2\nu A_u^{k-1}E_u^{j-1/3}(k-1)! + (k-1)A_u^{k-2}E_u^{j+1}(k-2)!\\
&\quad + A_1^k k! + C_{\mathrm{t}}A_u^{k-1}E_u^{j+2}k!\\
&\leq \Big(A_1^k + 2\nu A_u^k E_u^{j+1/3} + (C_{\mathrm{t}}+1+3\nu)A_u^{k-1}E_u^{j+2} + A_u^{k-2}E_u^{j+1}\Big)k!\\
&\leq A_u^k E_u^{j+1}k!.
\end{aligned} \tag{63}$$

Then, the estimates in (61), (62), and (63) imply that $H_{k+1,j+1}$ holds true. By the strategy outlined above, this shows implication (53) and thus verifies $H_{k,k}$ for all $k \in \mathbb{N}$. Therefore $(\overline{\widetilde{\boldsymbol{u}}}, p) \in [\mathcal{B}^2_{\beta_i}(S^i_{\delta_P/2})]^2 \times \mathcal{B}^1_{\beta_i}(S^i_{\delta_P/2})$, which leads to $(\widetilde{\boldsymbol{u}}, p) \in [B^2_{\beta_i}(S^i_{\delta_P/2})]^2 \times B^1_{\beta_i}(S^i_{\delta_P/2})$ due to $\widetilde{\boldsymbol{u}}(\mathfrak{c}_i) = \mathbf{0}$ and Lemma 3. The proof is concluded by noting that $\boldsymbol{u} - \widetilde{\boldsymbol{u}}$ is a constant vector field. □

Combining the estimates in each sector with classical results on the analyticity of the solution in the interior of the domain and on regular parts of the boundary from [1], this implies the weighted analytic regularity in P of solutions to the stationary, incompressible Navier-Stokes equations, stated in Theorem 8.

Proof of Theorem 8 The analyticity of weak solutions $(\boldsymbol{u}, p)$ in the interior and up to analytic parts of the boundary is classical, see, e.g., [2, 22]. The weighted analytic regularity near corners of P is provided in Lemma 17. □

Remark 19 If for each corner $\mathfrak{c} \in \mathfrak{C}$, either at least one of the two sides of P meeting in $\mathfrak{c}$ is a Dirichlet side with no-slip BCs or both sides of P meeting in $\mathfrak{c}$ are equipped with homogeneous slip boundary condition, then by repeating the argument in Remark 18 near each corner and using again the analyticity of $(\boldsymbol{u}, p)$ in the interior and up to analytic parts of the boundary, the present analysis will imply

$$(\boldsymbol{u}, p) \in [K^{\varpi}_{2-\underline{\beta}_f}(P)]^2 \times K^{\varpi}_{1-\underline{\beta}_f}(P).$$

5 Conclusion and Discussion

We have shown analytic regularity of Leray-Hopf solutions of the stationary, viscous and incompressible Navier-Stokes equations in polygonal domains P, subject to sufficiently small and analytic in $\overline{P}$ forcing. We proved analytic regularity of the velocity and pressure in scales of corner-weighted, Kondrat'ev spaces. The present setting of mixed BCs covers most examples of interest in applications, such as, e.g., channel flow with homogeneous Neumann condition at the outflow boundary. With the argument in [12] containing a gap, in the particular case of homogeneous Dirichlet ("no-slip") boundary conditions on all of ∂P the present result implies that the result in [16] stands under the assumptions stated in [16]. The analytic regularity in homogeneous weighted spaces implies, as explained in the discussion in [16, Section 5], corresponding bounds on n-widths of solution sets which, in turn, imply exponential convergence of reduced basis and of Model Order Reduction methods. Corresponding remarks apply also in the present, more general situation, and we do not spell them out here.

We note that our weighted analytic regularity arguments were local, at each sector. Assumption 1, item 2 was only imposed to ease the presentation of (known) results on existence and uniqueness of variational ("Leray-Hopf") solutions.

The present results also imply, along the lines of [16] (where only the case of no-slip BCs on all of ∂P was considered), exponential rates of convergence of *hp*-approximations. Details on the exponential convergence rate bounds for further discretizations in the case of the presently considered mixed boundary conditions shall be elaborated elsewhere.

Declarations

The authors declare that data sharing in this article is not applicable as no datasets were generated or analysed during the current study.

A Proofs of Section 2.5.4

Proof of Lemma 3 The third item of Lemma 5 and the second item of Lemma 6 give that for any $\ell \in \{0, 1, 2\}$ there exists a constant $A_0 > 1$ such that for any $\alpha \in \mathbb{N}_0^2$,

$$\|r^{\beta+\alpha_1-\ell}\mathcal{D}^\alpha \overline{\boldsymbol{u}}\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \leq A_0^{|\alpha|+1}|\alpha|!.$$

Then we have

$$\|r^{\beta-\ell}\boldsymbol{u}\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \leq 4\|r^{\beta-\ell}\overline{\boldsymbol{u}}\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))},$$

and for all $|\alpha| \geq 1$,

$$\begin{aligned}
&\|r^{\beta+\alpha_1-\ell}\mathcal{D}^\alpha u_1\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \\
&\leq \sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial_\vartheta^j \cos\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\beta+\alpha_1-\ell}\partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2-j}u_r\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \\
&\quad + \sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial_\vartheta^j \sin\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\beta+\alpha_1-\ell}\partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2-j}u_\vartheta\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \\
&\leq 2A_0^{|\alpha|+1}|\alpha|!\sum_{j=0}^{\alpha_2}A_0^{-j}\binom{\alpha_2}{j} \leq 2(2A_0)^{|\alpha|+1}|\alpha|!.
\end{aligned}$$

A similar estimate holds for u_2. By the above results and using the third item of Lemma 5 and the first item of Lemma 6 we have $\boldsymbol{u} \in [\mathcal{B}_\beta^\ell(Q_{\delta,\omega}(\mathfrak{c}))]^2$, which, by the second item of Lemma 5, is equivalent to $\boldsymbol{u} \in [B_\beta^\ell(Q_{\delta,\omega}(\mathfrak{c}))]^2$. □

Proof of Lemma 4 From $\boldsymbol{v} \in [B^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))]^2$ it follows that $\boldsymbol{v} \in [\mathcal{B}^0_\beta(Q_{\delta,\omega}(\mathfrak{c}))]^2$ by [17, Theorem 1.1]. Then, there exists $A_0 > 1$ such that, for all $|\alpha| \geq 1$,

$$\begin{aligned}\|r^{\alpha_1+\beta}\mathcal{D}^\alpha v_r\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} &\leq \sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial^j_\vartheta \cos\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\alpha_1+\beta}\partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2-j}v_1\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))}\\ &+\sum_{j=0}^{\alpha_2}\binom{\alpha_2}{j}\|\partial^j_\vartheta \sin\vartheta\|_{L^\infty(Q_{\delta,\omega}(\mathfrak{c}))}\|r^{\alpha_1+\beta}\partial_r^{\alpha_1}\partial_\vartheta^{\alpha_2-j}v_2\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))}\\ &\leq 2A_0^{|\alpha|}|\alpha|!\sum_{j=0}^{\alpha_2}A_0^{-j}\binom{\alpha_2}{j} \leq 2(2A_0)^{|\alpha|}|\alpha|!.\end{aligned}$$

The estimate for v_ϑ follows by the same argument. □

Proof of Lemma 7 Lemma 6 implies that $v \in \mathcal{W}^k_\beta(Q_{\delta,\omega}(\mathfrak{c}))$. Elementary calculus yields

$$\begin{aligned}\partial_{x_1} &= \cos\vartheta\partial_r - \frac{\sin\vartheta}{r}\partial_\vartheta,\\ \partial_{x_2} &= \sin\vartheta\partial_r + \frac{\cos\vartheta}{r}\partial_\vartheta,\\ \partial^2_{x_1} &= \cos^2\vartheta\partial_r^2 + \frac{2\cos\vartheta\sin\vartheta}{r^2}\partial_\vartheta + \frac{\sin^2\vartheta}{r}\partial_r - \frac{2\cos\vartheta\sin\vartheta}{r}\partial_{r\vartheta} + \frac{\sin^2\vartheta}{r^2}\partial^2_\vartheta,\\ \partial^2_{x_2} &= \sin^2\vartheta\partial_r^2 - \frac{2\cos\vartheta\sin\vartheta}{r^2}\partial_\vartheta + \frac{\cos^2\vartheta}{r}\partial_r + \frac{2\cos\vartheta\sin\vartheta}{r}\partial_{r\vartheta} + \frac{\cos^2\vartheta}{r^2}\partial^2_\vartheta,\\ \partial_{x_1}\partial_{x_2} &= \cos\vartheta\sin\vartheta\partial_r^2 + \frac{\sin^2\vartheta-\cos^2\vartheta}{r^2}\partial_\vartheta + \frac{\cos^2\vartheta-\sin^2\vartheta}{r}\partial_{r\vartheta} - \frac{\sin\vartheta\cos\vartheta}{r}\partial_r - \frac{\sin\vartheta\cos\vartheta}{r^2}\partial^2_\vartheta.\end{aligned}$$

Therefore there exists $C > 0$ ($C = 7$ when $k = 2$ and $C = 2$ when $k = 1$ will suffice) such that for any $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \leq k$,

$$\|r^{\beta-k+|\alpha|}\partial^\alpha v\|_{L^2(Q_{\delta,\omega}(\mathfrak{c}))} \leq C\left(\sum_{|\alpha|\leq k}\|r^{\beta-k+\alpha_1}\mathcal{D}^\alpha v\|^2_{L^2(Q_{\delta,\omega}(\mathfrak{c}))}\right)^{1/2} = C\|v\|_{\mathcal{W}^k_\beta(Q_{\delta,\omega}(\mathfrak{c}))}.$$

By definition, it follows that $v \in K^k_{k-\beta}(Q_{\delta,\omega}(\mathfrak{c}))$. □

B Parametric Operator Pencil for Stokes-Problem

In this appendix, we give details about the parametrized system (16). Recall that r and ϑ are polar coordinates in the sector $Q_{\delta,\omega}$. Set $D = -i\partial_\vartheta$. The parametric differential operator $\widehat{L}(\lambda)$ in (16) is defined as

$$\widehat{L}(\lambda)(\overline{\boldsymbol{u}}, p)$$
$$=\left(\begin{pmatrix}\nu D^2 + 2\nu(1+\lambda^2) & \nu(3+i\lambda)iD & -(1+i\lambda)\\ -\nu(3-i\lambda)iD & \nu 2D^2+\nu(1+\lambda^2) & iD\end{pmatrix}\begin{pmatrix}u_r\\ u_\vartheta\\ p\end{pmatrix}, \begin{pmatrix}1-i\lambda & iD\end{pmatrix}\begin{pmatrix}u_r\\ u_\vartheta\end{pmatrix}\right). \tag{64}$$

We also define the boundary operator $\widehat{B}(\lambda)$ in (16) as

$$\widehat{B}(\lambda)(\overline{\boldsymbol{u}}, p) = \left(A_0(\lambda) \begin{pmatrix} u_r \\ u_\vartheta \\ p \end{pmatrix}, A_\omega(\lambda) \begin{pmatrix} u_r \\ u_\vartheta \\ p \end{pmatrix} \right), \tag{65}$$

where, for any $\mu \in \{0, \omega\}$,

$$A_\mu(\lambda)(\overline{\boldsymbol{u}}, p) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & \text{if } \{\vartheta = \mu\} \text{ corresponds to a Dirichlet edge,} \\ \begin{pmatrix} \nu i D & -\nu(1+i\lambda) & 0 \\ 2\nu & 2\nu i D & -1 \end{pmatrix}, & \text{if } \{\vartheta = \mu\} \text{ corresponds to a Neumann edge,} \\ \begin{pmatrix} 0 & 1 & 0 \\ iD & -(1+i\lambda) & 0 \end{pmatrix}, & \text{if } \{\vartheta = \mu\} \text{ corresponds to a Slip edge.} \end{cases} \tag{66}$$

For the derivation of this parametric system, see [21, Chapter 4.2]

C Stokes operator in polar coordinates

In this appendix we provide the elementary calculations to verify (43)-(47), which describe the NSE with boundary conditions in polar coordinates and polar components. We recall the representation of the NSE in the Cartesian reference frame

$$L^{\Delta}_{\mathrm{St}}(\boldsymbol{u}, p) = \begin{pmatrix} \boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \\ 0 \end{pmatrix} \quad \text{in } S^i_{\delta_P}, \tag{67}$$

$$B(\boldsymbol{u}, p) = \boldsymbol{0} \quad \text{on } \Gamma_\delta. \tag{68}$$

Using $\widetilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}(\mathfrak{c}_i)$ we rewrite this set of equations as

$$L^{\Delta}_{\mathrm{St}}(\widetilde{\boldsymbol{u}}, p) = \begin{pmatrix} \boldsymbol{f} - ((\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)) \cdot \nabla)(\widetilde{\boldsymbol{u}} + \boldsymbol{u}(\mathfrak{c}_i)) \\ 0 \end{pmatrix} \quad \text{in } S^i_{\delta_P}, \tag{69}$$

$$B(\widetilde{\boldsymbol{u}}, p) = -B(\boldsymbol{u}(\mathfrak{c}_i), 0) = \boldsymbol{0} \quad \text{on } \Gamma_\delta. \tag{70}$$

(69) follows directly from (67). We justify that the right-hand side of (70) is a zero vector. To this end, we note firstly that due to Lemma 16, $\boldsymbol{u} - \boldsymbol{u}(\mathfrak{c}_i) \in \mathcal{W}^2_{\beta_i}(S^i_\delta)^2 \subset C^0(\overline{S^i_\delta})^2$ and thus $\boldsymbol{u} \in C^0(\overline{S^i_\delta})^2$, which implies the continuity of $\boldsymbol{u}|_{\overline{\Gamma}_\delta}$ along $\overline{\Gamma}_\delta$. On a Dirichlet side, we use the homogeneous Dirichlet boundary condition and the continuity of $\boldsymbol{u}$ to derive $\boldsymbol{u}(\mathfrak{c}_i) = \boldsymbol{0}$, which implies $B(\boldsymbol{u}(\mathfrak{c}_i), 0) = \boldsymbol{0}$ on this side. On a Neumann side, $B(\boldsymbol{u}(\mathfrak{c}_i), 0) = \boldsymbol{0}$ as all entries of $\varepsilon(\boldsymbol{u}(\mathfrak{c}_i))$ equal zero. For a side equipped with Slip boundary condition, Lemma 16 shows that the first component of $B(\boldsymbol{u}(\mathfrak{c}_i), 0)$ equals 0 and the second component also vanishes with the same reasoning as in the case of a Neumann side. The right-hand sides of (45), (46) and (47) are thus verified.

The vector Laplacian in a polar reference frame reads [23, Equation (3.151)]

$$\overline{\Delta\widetilde{\boldsymbol{u}}} = \frac{1}{r^2}\begin{pmatrix}(r\partial_r)^2+\partial_\vartheta^2-1 & -2\partial_\vartheta \\ 2\partial_\vartheta & (r\partial_r)^2+\partial_\vartheta^2-1\end{pmatrix}\overline{\widetilde{\boldsymbol{u}}}$$

and [24, Equation (II.4.C3)]

$$\overline{\nabla p} = \begin{pmatrix}\partial_r p \\ r^{-1}\partial_\vartheta p\end{pmatrix}.$$

The divergence of $\widetilde{\boldsymbol{u}}$, which equals to $\nabla\cdot\boldsymbol{u}$, is [24, Equation (II.4.C5)] $\nabla\cdot\widetilde{\boldsymbol{u}} = \frac{1}{r}\left((r\partial_r+1)\,\widetilde{u}_r + \partial_\vartheta\widetilde{u}_\vartheta\right)$, whence (43) and (44).

Regarding the boundary conditions (70), we start from the expression of the stress tensor in polar coordinates and polar frame, see [24, Equation (II.4.C9)],

$$\overline{\varepsilon(\widetilde{\boldsymbol{u}})} = \begin{pmatrix}\partial_r u_r & \frac{1}{2}(\partial_r\widetilde{u}_\vartheta + r^{-1}(\partial_\vartheta\widetilde{u}_r - \widetilde{u}_\vartheta)) \\ \frac{1}{2}(\partial_r\widetilde{u}_\vartheta + r^{-1}(\partial_\vartheta\widetilde{u}_r - \widetilde{u}_\vartheta)) & r^{-1}(\partial_\vartheta\widetilde{u}_\vartheta + \widetilde{u}_r)\end{pmatrix} \tag{71}$$

hence the stress tensor in a polar reference frame reads

$$\overline{\sigma(\widetilde{\boldsymbol{u}},p)} = 2\nu\overline{\varepsilon(\widetilde{\boldsymbol{u}})} - p\,\mathrm{Id}_2 = \nu\begin{pmatrix}2\partial_r\widetilde{u}_r & \partial_r\widetilde{u}_\vartheta + r^{-1}(\partial_\vartheta\widetilde{u}_r - \widetilde{u}_\vartheta) \\ \partial_r\widetilde{u}_\vartheta + r^{-1}(\partial_\vartheta\widetilde{u}_r - \widetilde{u}_\vartheta) & 2r^{-1}(\partial_\vartheta\widetilde{u}_\vartheta + \widetilde{u}_r)\end{pmatrix} - p\,\mathrm{Id}_2\,. \tag{72}$$

We have furthermore

$$\overline{\boldsymbol{n}} = \pm\begin{pmatrix}0\\1\end{pmatrix}, \qquad \overline{\boldsymbol{t}} = \mp\begin{pmatrix}1\\0\end{pmatrix},$$

where the sign depends on the side of the sector being considered. Then, by matrix-vector multiplication,

$$\overline{\sigma(\widetilde{\boldsymbol{u}},p)\boldsymbol{n}} = \pm\nu\begin{pmatrix}\partial_r\widetilde{u}_\vartheta + r^{-1}(\partial_\vartheta\widetilde{u}_r - \widetilde{u}_\vartheta) \\ 2r^{-1}(\partial_\vartheta\widetilde{u}_\vartheta + \widetilde{u}_r) - p\end{pmatrix}$$

and consequently

$$(\sigma(\widetilde{\boldsymbol{u}},p)\boldsymbol{n})\cdot\boldsymbol{t} = \overline{\sigma(\widetilde{\boldsymbol{u}},p)\boldsymbol{n}}\cdot\overline{\boldsymbol{t}} = -\partial_r\widetilde{u}_\vartheta - \frac{1}{r}(\partial_\vartheta\widetilde{u}_r - \widetilde{u}_\vartheta).$$

Also, it follows from the definition that $\widetilde{\boldsymbol{u}}\cdot\boldsymbol{n} = \overline{\widetilde{\boldsymbol{u}}}\cdot\overline{\boldsymbol{n}} = \pm\widetilde{u}_\vartheta$, thus verifying (45), (46), and (47).

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