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Phase retrieval from sampled Gabor transform magnitudes: Counterexamples

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Abstract

We consider the recovery of square-integrable signals from discrete, equidistant samples of their Gabor transform magnitude and show that, in general, signals can not be recovered from such samples. In particular, we show that for any lattice, one can construct functions in $L^2(\mathbb{R})$ which do not agree up to global phase but whose Gabor transform magnitudes sampled on the lattice agree. These functions can be constructed to be either real-valued or complex-valued and have good concentration in both time and frequency.

1 Introduction

Let us consider the Gaussian $\phi(t) = \mathrm{e}^{-\pi t^2}$, for $t \in \mathbb{R}$. We may define the Gabor transform of a signal $f \in L^2(\mathbb{R})$ via

$$\mathcal{V}_\phi f(x,\omega) = \int_{\mathbb{R}} f(t)\phi(t-x)\mathrm{e}^{-2\pi \mathrm{i} t\omega}\,\mathrm{d}t, \qquad (x,\omega) \in \mathbb{R}^2.$$

In this note, we are interested in the uniqueness question of the *Gabor phase retrieval* problem which consists of recovering a function $f \in L^2(\mathbb{R})$ from the magnitude measurements

$$\left|\mathcal{V}_\phi f(x,\omega)\right|, \qquad (x,\omega) \in S, \tag{1}$$

where S is a subset of $\mathbb{R}^2$. We say that f is uniquely determined (up to a global phase factor) by the measurements (1) if for any $g \in L^2(\mathbb{R})$,

$$\left|\mathcal{V}_\phi f(x,\omega)\right| = \left|\mathcal{V}_\phi g(x,\omega)\right| \qquad \text{for all } (x,\omega) \in S,$$

implies that

$$f = \mathrm{e}^{\mathrm{i}\mu} g,$$

for some $\mu \in \mathbb{R}$. When $S = \mathbb{R}^2$, it is well-known that Gabor phase retrieval is uniquely solvable. In fact, this result holds true for any window function ψ for which $\mathcal{V}_\psi \psi$ is non-zero almost everywhere on $\mathbb{R}^2$. However, when S is a true subset of $\mathbb{R}^2$, the answer is less clear. In particular, measurements can only be collected on discrete sets S in applications. Thus, the question of uniqueness for Gabor phase retrieval is specifically interesting when S is discrete.

In recent work [1], we were able to show that *real-valued, bandlimited* signals in $L^2(\mathbb{R})$ are uniquely determined up to global phase from Gabor magnitude measurements (1) sampled on the discrete set $S = (4B)^{-1}\mathbb{Z} \times \{0\}$, where $B > 0$ is such that the bandwidth of the signal is contained in $[-B, B]$.

It is worth noting that while working on this paper, work by Grohs and Liehr [3] appeared showing that it is possible to recover *compactly supported* signals in $L^4([-C/2, C/2])$ up to global phase from Gabor magnitude measurements (1) sampled on the discrete set $S = \mathbb{Z} \times (2C)^{-1}\mathbb{Z}$.

In this paper, we focus on the general uniqueness question for Gabor phase retrieval:

Question 1.1. *Is there any lattice $S = a\mathbb{Z} \times b\mathbb{Z}$, $a, b > 0$, such that all functions in $L^2(\mathbb{R})$ are uniquely determined up to global phase from Gabor magnitude measurements* (1) *sampled on S?*

The main contribution of this paper is that we answer this question negatively: In particular, no matter how fine-grained the sampling set S, one will not be able to recover all functions in $L^2(\mathbb{R})$ from Gabor magnitude measurements (1).

In addition, our answer to Question 1.1 is constructive in the sense that we are able to explicitly give functions $f, g \in L^2(\mathbb{R})$ which do not agree up to global phase but which satisfy

$$|\mathcal{V}_\phi f(x,\omega)| = |\mathcal{V}_\phi g(x,\omega)|, \qquad \text{for all } (x,\omega) \in a\mathbb{Z} \times b\mathbb{Z}.$$

In particular, we show that there are such functions $f, g \in L^2(\mathbb{R})$ that are real-valued and well concentrated in both time and frequency. Moreover, reducing the signal class from $L^2(\mathbb{R})$ to the modulation space $M_m^{p,q}(\mathbb{R})$ or even to the class of real-valued signals in the modulation space $M_m^{p,q}(\mathbb{R})$ will not allow one to answer Question 1.1 positively.

To construct the functions $f, g \in L^2(\mathbb{R})$, we exploit the well-known relation of the Gabor and the Bargmann transform [2] and Hadamard's factorisation theorem.

Outline Section 1.1 introduces some basic notation. In Section 2.1, we show that for any set S of infinitely many equidistant parallel lines in the time-frequency plane, there exist functions $f, g \in L^2(\mathbb{R})$ which do not agree up to global phase but whose Gabor transform magnitudes agree on S. In Section 2.2, we apply the results from the previous section to show that functions in L^2 cannot be recovered from Gabor transform magnitudes sampled on the set $a\mathbb{Z} \times b\mathbb{Z}$, where $a, b > 0$, and thereby answer Question 1.1. In addition, we show that the same holds when we replace L^2 by any modulation space with exponentially growing weight function or even when only considering the real-valued functions in said modulation space. The proofs for our results can be found in Appendix A.

1.1 Basic notions

We want to emphasise that the notation ϕ is reserved for the Gaussian $\phi(t) = \mathrm{e}^{-\pi t^2}$, for $t \in \mathbb{R}$, throughout this paper.

In addition, as is standard in the literature, we will denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of smooth functions with rapid decrease and by $\mathcal{S}'(\mathbb{R})$ its dual, the space of tempered distributions. It is natural to extend the definition of the Gabor transform on $L^2(\mathbb{R})$ to the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ by making use of the dual pairing $\langle \cdot, \cdot \rangle$ on $\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$:

$$\mathcal{V}_\phi T(x,\omega) = \left\langle T, \phi(\cdot - x)\mathrm{e}^{-2\pi \mathrm{i} \cdot \omega} \right\rangle, \qquad (x,\omega) \in \mathbb{R}^2.$$

We will encounter the modulation spaces which can be used to quantify the decay of the Gabor transform of signals [2]:

Definition 1.2. *Let $1 \leq p, q \leq \infty$ and let $m : \mathbb{R}^2 \to [0,\infty)$. The modulation space $M_m^{p,q}(\mathbb{R})$ is given by*

$$\left\{ T \in \mathcal{S}'(\mathbb{R}) \, ; \, \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathcal{V}_\phi T(x,\omega)|^p \, m(x,\omega)^p \, \mathrm{d}x \right)^{q/p} \mathrm{d}\omega \right)^{1/q} < \infty \right\}.$$

Finally, we will say that a weight function $m : \mathbb{R}^2 \to [0,\infty)$ grows at most exponentially if there exist constants $\sigma_0, \sigma_1 > 0$ such that for all $(x,\omega) \in \mathbb{R}^2$,

$$m(x,\omega) \leq \sigma_0 \mathrm{e}^{\sigma_1 \sqrt{x^2+\omega^2}}.$$

2 Counterexamples and what to learn from them

2.1 Examples in semidiscrete settings

Using the relation of the Gabor transform and the Bargmann transform [2] as well as Hadamard's factorisation theorem, allows us to come up with functions $f, g \in L^2(\mathbb{R})$ which do not agree up to global phase but which do generate measurements (1) that agree when $S \subset \mathbb{R}^2$ is chosen to be any set of infinitely many equidistant parallel lines in the time-frequency plane. In the following, we will

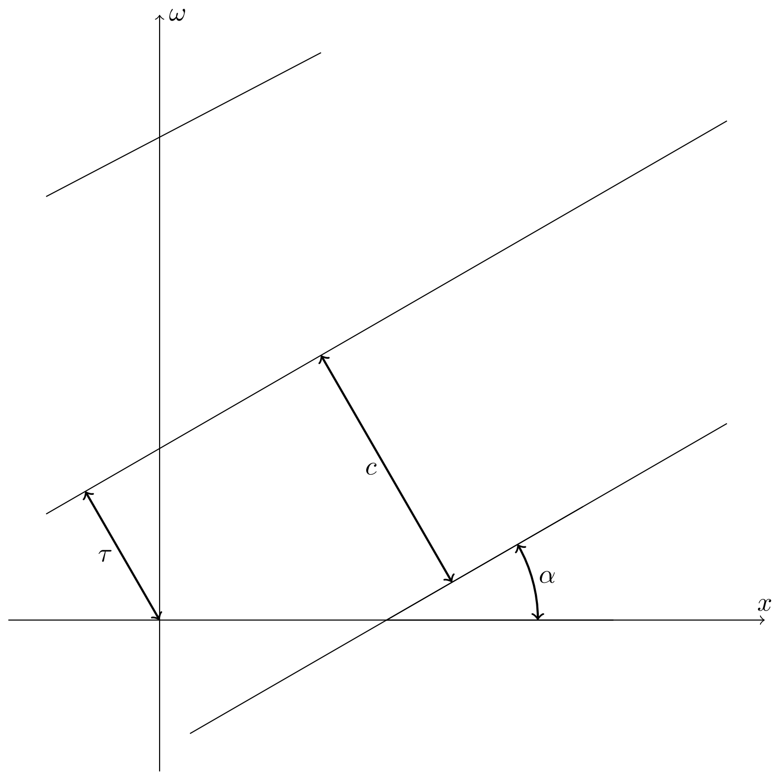


Figure 1: The set S is defined by the angle α, and the distances τ as well as c.

work with $\alpha \in \mathbb{R}$, $c \in (0, \infty)$ as well as $\tau \in [0, c)$ and model the infinitude of equidistant parallel lines in the time-frequency plane by

$$S = \left\{ \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} t + \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} (\tau + kc) \,;\, t \in \mathbb{R},\ k \in \mathbb{Z} \right\} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \mathbb{R} + \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} (\tau + c\mathbb{Z}). \tag{2}$$

Let us consider the signals

$$f(t) = \mathrm{e}^{-\pi t^2} \left(\cosh\left(\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c} + \frac{\pi \mathrm{i}\tau}{2c} \right) + \mathrm{i} \sinh\left(\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c} + \frac{\pi \mathrm{i}\tau}{2c} \right) \right), \tag{3}$$

$$g(t) = \mathrm{e}^{-\pi t^2} \left(\cosh\left(\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c} + \frac{\pi \mathrm{i}\tau}{2c} \right) - \mathrm{i} \sinh\left(\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c} + \frac{\pi \mathrm{i}\tau}{2c} \right) \right), \tag{4}$$

for $t \in \mathbb{R}$, which are both in $L^2(\mathbb{R})$. In the following, we will show that while f and g do not agree up to global phase, it holds that

$$\left| \mathcal{V}_\phi f(x, \omega) \right| = \left| \mathcal{V}_\phi g(x, \omega) \right|, \qquad \text{for all } (x, \omega) \in S. \tag{5}$$

Remark 2.1. *The functions f and g were constructed by considering what equation* (5) *implies for the Bargmann transforms of f and g in the complex plane. In particular, it is possible to apply Hadamard's factorisation theorem to the Bargmann transform of f and g and then follow ideas similar to the ones presented in [4, 5].*

Let us start by computing the Gabor transforms of f and g:

Lemma 2.2. *Let $\alpha \in \mathbb{R}$, $c \in (0, \infty)$, $\tau \in [0, c)$, and let $f, g \in L^2(\mathbb{R})$ be defined as in equations* (3) *and* (4)*. Then, we have*

$$\mathcal{V}_\phi f(x, -\omega) = \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}|z|^2 + \pi \mathrm{i} x\omega + \frac{\pi \mathrm{e}^{2\mathrm{i}\alpha}}{8c^2}} \left(\cosh\left(\frac{\pi}{2c} \left(\mathrm{e}^{\mathrm{i}\alpha} z + \mathrm{i}\tau \right) \right) + \mathrm{i} \sinh\left(\frac{\pi}{2c} \left(\mathrm{e}^{\mathrm{i}\alpha} z + \mathrm{i}\tau \right) \right) \right),$$

$$\mathcal{V}_\phi g(x, -\omega) = \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}|z|^2 + \pi \mathrm{i} x\omega + \frac{\pi \mathrm{e}^{2\mathrm{i}\alpha}}{8c^2}} \left(\cosh\left(\frac{\pi}{2c} \left(\mathrm{e}^{\mathrm{i}\alpha} z + \mathrm{i}\tau \right) \right) - \mathrm{i} \sinh\left(\frac{\pi}{2c} \left(\mathrm{e}^{\mathrm{i}\alpha} z + \mathrm{i}\tau \right) \right) \right),$$

for $(x,\omega)\in\mathbb{R}^2$ *and* $z = x + \mathrm{i}\omega$.

Proof. See Appendix A.

With this, we can show that $\mathcal{V}_\phi f$ and $\mathcal{V}_\phi g$ do not agree up to global phase:

Proposition 2.3. *Let* $\alpha\in\mathbb{R}$*,* $c\in(0,\infty)$*,* $\tau\in[0,c)$*, and let* $f,g\in L^2(\mathbb{R})$ *be defined as in equations* (3) *and* (4)*. Then, the roots of* $\mathcal{V}_\phi f$ *are given by* $\{(2ck-\frac{c}{2}+\tau)\cdot(-\sin\alpha,\cos\alpha)\,;\,k\in\mathbb{Z}\}$ *and the roots of* $\mathcal{V}_\phi g$ *are given by* $\{(2ck+\frac{c}{2}+\tau)\cdot(-\sin\alpha,\cos\alpha)\,;\,k\in\mathbb{Z}\}$.

Proof. For $(x,\omega)\in\mathbb{R}^2$, let us consider $\mathcal{V}_\phi f$ which consists of two factors: The first factor

$$\frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)+\pi\mathrm{i}x\omega+\frac{\pi\mathrm{e}^{2\mathrm{i}\alpha}}{8c^2}}$$

is non-zero. The second factor

$$\cosh\left(\frac{\pi}{2c}\left(\mathrm{e}^{\mathrm{i}\alpha}(x-\mathrm{i}\omega)+\mathrm{i}\tau\right)\right)+\mathrm{i}\sinh\left(\frac{\pi}{2c}\left(\mathrm{e}^{\mathrm{i}\alpha}(x-\mathrm{i}\omega)+\mathrm{i}\tau\right)\right)$$

might alternatively be written as

$$\cos\left(\frac{\pi\mathrm{i}}{2c}\left(\mathrm{e}^{\mathrm{i}\alpha}(x-\mathrm{i}\omega)+\mathrm{i}\tau\right)\right)+\sin\left(\frac{\pi\mathrm{i}}{2c}\left(\mathrm{e}^{\mathrm{i}\alpha}(x-\mathrm{i}\omega)+\mathrm{i}\tau\right)\right).$$

The latter is zero if and only if

$$\frac{\pi\mathrm{i}}{2c}\left(\mathrm{e}^{\mathrm{i}\alpha}(x-\mathrm{i}\omega)+\mathrm{i}\tau\right)=\pi k-\frac{\pi}{4},$$

for some $k\in\mathbb{Z}$. The equation above is equivalent to

$$\omega+\mathrm{i}x=\left(2ck-\frac{c}{2}+\tau\right)\mathrm{e}^{-\mathrm{i}\alpha}$$

and thus the root set of $\mathcal{V}_\phi f$ must have the form $\{(2ck-\frac{c}{2}+\tau)\cdot(-\sin\alpha,\cos\alpha)\,;\,k\in\mathbb{Z}\}$. The argument for $\mathcal{V}_\phi g$ is similar. □

Having that $\mathcal{V}_\phi f$ and $\mathcal{V}_\phi g$ do not agree up to global phase and the Gabor transform is a linear operator, it follows that also f and g do not agree up to global phase.

Finally, we can show that the Gabor transforms of f and g agree on the parallel lines S as parametrised in equation (2):

Theorem 2.4 (Main result)**.** *Let* $\alpha\in\mathbb{R}$*,* $c\in(0,\infty)$*,* $\tau\in[0,c)$ *and let* $S\subset\mathbb{R}^2$ *be defined as in equation* (2)*. Furthermore we define* $f,g\in L^2(\mathbb{R})$ *as in equations* (3) *and* (4)*. Then, for all* $\mu\in\mathbb{R}$

$$f\neq e^{i\mu}g$$

while

$$|\mathcal{V}_\phi f(x,\omega)|=|\mathcal{V}_\phi g(x,\omega)|,\qquad \textit{for all } (x,\omega)\in S.$$

Proof. See Appendix A.

2.2 The fully discrete setting

Next, we consider the fully discrete setting, i.e. measurements on the sampling set $a\mathbb{Z}\times b\mathbb{Z}$, where $a,b>0$. Based on the examples constructed in Subsection 2.1, there are two ways of finding $f,g\in L^2(\mathbb{R})$ that do not agree up to global phase but such that

$$|\mathcal{V}_\phi f(x,\omega)|=|\mathcal{V}_\phi g(x,\omega)|,\qquad \text{for all } (x,\omega)\in a\mathbb{Z}\times b\mathbb{Z}.$$

First, one could consider $\alpha=0$, $c=b$ and $\tau=0$. In this case, we have

$$f(t)=\mathrm{e}^{-\pi t^2}\left(\cosh\left(\frac{\pi t}{b}\right)+\mathrm{i}\sinh\left(\frac{\pi t}{b}\right)\right),$$

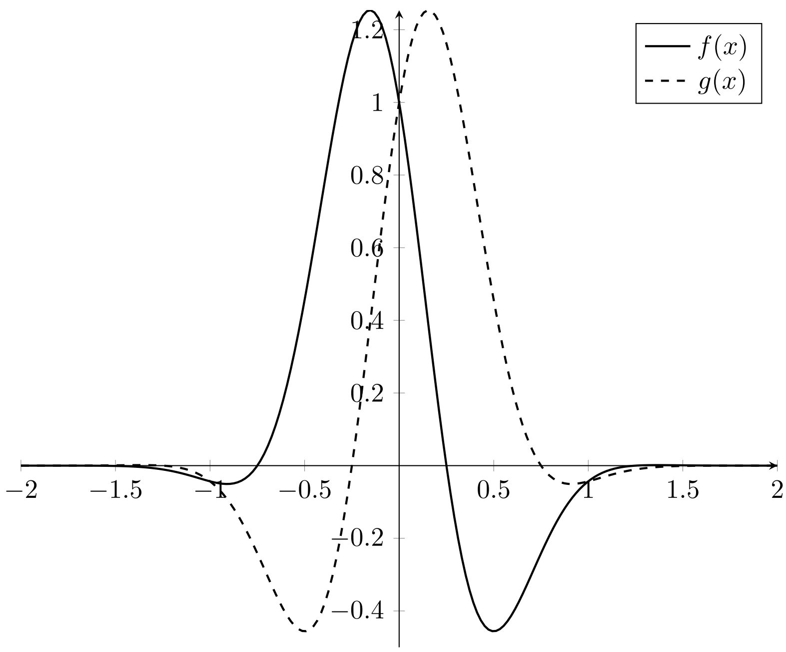


Figure 2: The functions defined in equations (6) and (7) with $a = 1$.

$$g(t) = \mathrm{e}^{-\pi t^2}\left(\cosh\left(\frac{\pi t}{b}\right) - \mathrm{i}\sinh\left(\frac{\pi t}{b}\right)\right),$$

for $t \in \mathbb{R}$, and by Lemma 2.2, we find that the Gabor transform magnitudes of f and g agree on $S = \mathbb{R} \times b\mathbb{Z}$ and thus, in particular, on $a\mathbb{Z} \times b\mathbb{Z}$.

Secondly, one could also take $\alpha = \frac{\pi}{2}$, $c = a$ and $\tau = 0$ resulting in

$$f(t) = \mathrm{e}^{-\pi t^2}\left(\cos\left(\frac{\pi t}{a}\right) - \sin\left(\frac{\pi t}{a}\right)\right), \tag{6}$$

$$g(t) = \mathrm{e}^{-\pi t^2}\left(\cos\left(\frac{\pi t}{a}\right) + \sin\left(\frac{\pi t}{a}\right)\right), \tag{7}$$

for $t \in \mathbb{R}$ (see Figures 2 and 3 for visualisations of f, g and the magnitudes of their Gabor transforms, respectively). As in the previous case, Lemma 2.2 implies that the Gabor transform magnitudes of f and g are equal on $S = a\mathbb{Z} \times \mathbb{R}$ and hence also on $a\mathbb{Z} \times b\mathbb{Z}$.

This second example is particularly nice, since both f and g are real-valued. In addition, both f and g are well concentrated in time and frequency:

Proposition 2.5. *Let $a > 0$ and let $f, g \in L^2(\mathbb{R})$ be defined by equations* (6) *and* (7)*. Then, for all $1 \leq p, q \leq \infty$ and all $m : \mathbb{R}^2 \to [0, \infty)$ that grow at most exponentially, one has that $f, g \in M^{p,q}_m(\mathbb{R})$.*

Proof. See Appendix A.

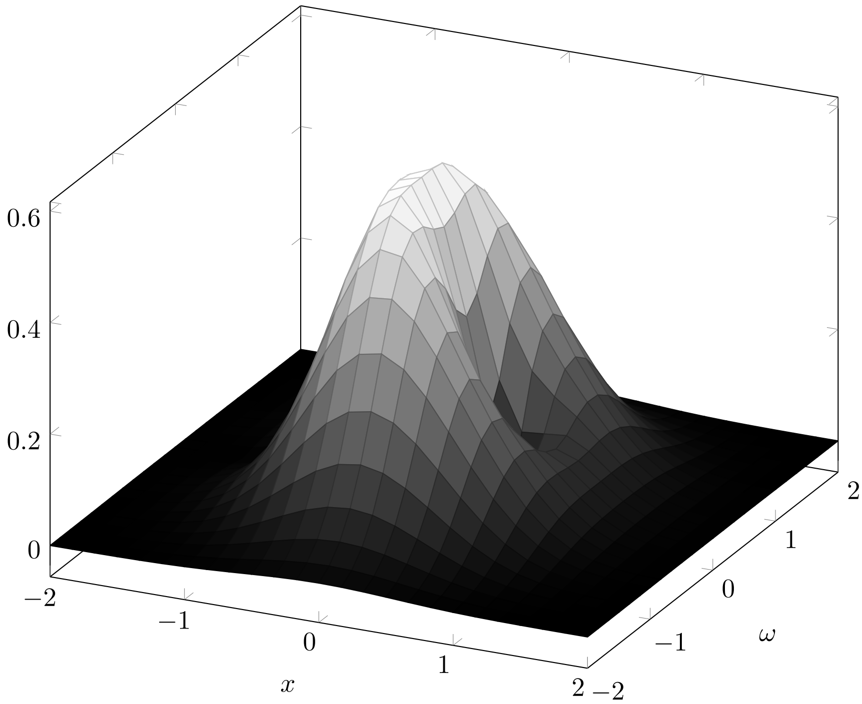
Remark 2.6. *The attentive reader will note that one can also prove the above proposition under even weaker assumptions on the weight function m. To be precise, the growth condition*

$$m(x, \omega) \leq C\mathrm{e}^{\left(\frac{\pi}{2} - \epsilon\right)\left(x^2 + \omega^2\right)}, \qquad \text{for all } (x, \omega) \in \mathbb{R}^2,$$

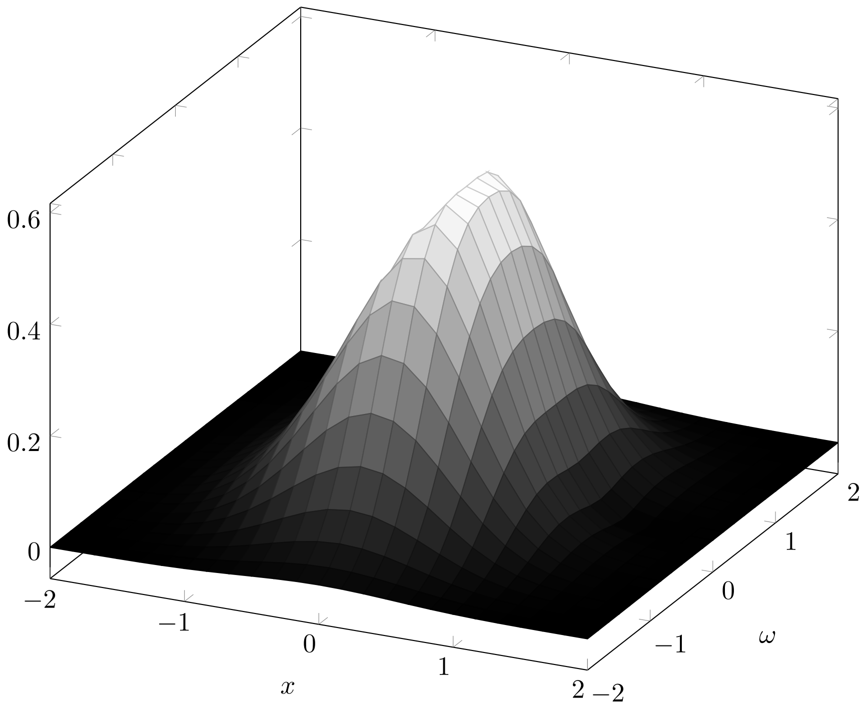
where $C, \epsilon > 0$ are constants, is sufficient for the above proposition to hold.

We summarise our results in the following theorem:

Theorem 2.7. *Let $a, b > 0$, let $1 \leq p, q \leq \infty$ and let $m : \mathbb{R}^2 \to [0, \infty)$ grow at most exponentially. The following three statements are true:*



(a) $|\mathcal{V}_\phi f|$



(b) $|\mathcal{V}_\phi g|$

Figure 3: Spectrograms of the functions defined in equations (6) and (7) with $a = 1$.

1. *There exist* complex-valued $f, g \in L^2(\mathbb{R})$ *which do not agree up to global phase but for which*

$$|\mathcal{V}_\phi f(x,\omega)| = |\mathcal{V}_\phi g(x,\omega)|, \qquad \text{for all } (x,\omega) \in a\mathbb{Z} \times b\mathbb{Z}, \tag{8}$$

holds.

2. *There exist* real-valued $f, g \in L^2(\mathbb{R})$ *which do not agree up to global sign but for which equation* (8) *holds.*

3. *The above statements remain true if* $L^2(\mathbb{R})$ *is replaced by* $M^{p,q}_m(\mathbb{R})$.

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A Proofs

To prove Lemma 2.2, it is useful to show the following proposition first.

Proposition A.1. *Let* $a, b \in \mathbb{R}$ *and*

$$h(t) = \mathrm{e}^{-\pi t^2} \mathrm{e}^{2\pi(a+\mathrm{i}b)t}, \qquad t \in \mathbb{R}.$$

Then, it holds that

$$\mathcal{V}_\phi h(x,\omega) = \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right) - \pi \mathrm{i} x\omega + \frac{\pi}{2}(a+\mathrm{i}b)^2 + \pi(a+\mathrm{i}b)(x-\mathrm{i}\omega)}, \qquad (x,\omega) \in \mathbb{R}^2.$$

Proof. Let $(x,\omega) \in \mathbb{R}^2$. We compute

$$\begin{aligned}
\mathcal{V}_\phi h(x,\omega) &= \int_{\mathbb{R}} \mathrm{e}^{-\pi t^2} \mathrm{e}^{2\pi(a+\mathrm{i}b)t} \mathrm{e}^{-\pi(t-x)^2} \mathrm{e}^{-2\pi \mathrm{i} t\omega} \,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-\pi\left(t^2+(t-x)^2-2at\right)} \mathrm{e}^{-2\pi \mathrm{i} t(\omega-b)} \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \mathrm{e}^{-2\pi\left(t^2-(x+a)t+\frac{x^2}{2}\right)} \mathrm{e}^{-2\pi \mathrm{i} t(\omega-b)} \,\mathrm{d}t = \int_{\mathbb{R}} \mathrm{e}^{-2\pi\left(\left(t-\frac{1}{2}(x+a)\right)^2+\frac{x^2}{4}-\frac{ax}{2}-\frac{a^2}{4}\right)} \mathrm{e}^{-2\pi \mathrm{i} t(\omega-b)} \,\mathrm{d}t \\
&= \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}x^2+\pi a x+\frac{\pi}{2}a^2} \int_{\mathbb{R}} \mathrm{e}^{-\pi s^2} \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{s}{\sqrt{2}}+\frac{1}{2}(x+a)\right)(\omega-b)} \,\mathrm{d}s \\
&= \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}x^2+\pi a x+\frac{\pi}{2}a^2-\pi \mathrm{i}(x+a)(\omega-b)} \int_{\mathbb{R}} \mathrm{e}^{-\pi s^2} \mathrm{e}^{-2\pi \mathrm{i} s\frac{\omega-b}{\sqrt{2}}} \,\mathrm{d}s \\
&= \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}x^2-\pi \mathrm{i} x\omega+\pi(a+\mathrm{i}b)x+\pi \mathrm{i} ab-\pi \mathrm{i} a\omega+\frac{\pi}{2}a^2-\frac{\pi}{2}(\omega-b)^2} \\
&= \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)-\pi \mathrm{i} x\omega+\frac{\pi}{2}\left(a^2-b^2+2\mathrm{i}ab\right)+\pi(a+\mathrm{i}b)x-\pi(\mathrm{i}a-b)\omega} \\
&= \frac{1}{\sqrt{2}} \mathrm{e}^{-\frac{\pi}{2}\left(x^2+\omega^2\right)-\pi \mathrm{i} x\omega+\frac{\pi}{2}(a+\mathrm{i}b)^2+\pi(a+\mathrm{i}b)(x-\mathrm{i}\omega)}.
\end{aligned}$$

□

Proof of Lemma 2.2. Let $(x,\omega) \in \mathbb{R}^2$. By definition of cosh and sinh and the linearity of the Gabor transform, it follows that we can assemble the Gabor transforms of f and g by summing Gabor transforms of functions of the form

$$h(t) = \mathrm{e}^{-\pi t^2} \mathrm{e}^{2\pi(a+\mathrm{i}b)t}, \qquad t \in \mathbb{R},$$

where $a, b \in \mathbb{R}$. Therefore, we can make use of proposition A.1 here. In particular, we find that

$$\begin{aligned}
\cosh\left(\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c} + \frac{\pi \mathrm{i}\tau}{2c}\right) &= \cosh\left(\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c}\right) \cosh\left(\frac{\pi \mathrm{i}\tau}{2c}\right) + \sinh\left(\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c}\right) \sinh\left(\frac{\pi \mathrm{i}\tau}{2c}\right) \\
&= \frac{\cosh\left(\frac{\pi \mathrm{i}\tau}{2c}\right)}{2}\left(\mathrm{e}^{\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c}} + \mathrm{e}^{-\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c}}\right) + \frac{\sinh\left(\frac{\pi \mathrm{i}\tau}{2c}\right)}{2}\left(\mathrm{e}^{\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c}} - \mathrm{e}^{-\frac{\pi \mathrm{e}^{\mathrm{i}\alpha} t}{c}}\right)
\end{aligned}$$

along with

$$\sinh\left(\frac{\pi e^{i\alpha}t}{c}+\frac{\pi i\tau}{2c}\right)=\frac{\cosh\left(\frac{\pi i\tau}{2c}\right)}{2}\left(e^{\frac{\pi e^{i\alpha}t}{c}}-e^{-\frac{\pi e^{i\alpha}t}{c}}\right)+\frac{\sinh\left(\frac{\pi i\tau}{2c}\right)}{2}\left(e^{\frac{\pi e^{i\alpha}t}{c}}+e^{-\frac{\pi e^{i\alpha}t}{c}}\right).$$

Therefore, we have

$$\begin{aligned}
\mathcal{V}_\phi f(x,\omega) &= \frac{1+i}{2}\left(\cosh\left(\frac{\pi i\tau}{2c}\right)+\sinh\left(\frac{\pi i\tau}{2c}\right)\right)\mathcal{V}_\phi e^{-\pi\cdot^2}e^{\frac{\pi e^{i\alpha}\cdot}{c}}(x,\omega)\\
&\quad+\frac{1-i}{2}\left(\cosh\left(\frac{\pi i\tau}{2c}\right)-\sinh\left(\frac{\pi i\tau}{2c}\right)\right)\mathcal{V}_\phi e^{-\pi\cdot^2}e^{-\frac{\pi e^{i\alpha}\cdot}{c}}(x,\omega)\\
&=\frac{1}{\sqrt{2}}e^{-\frac{\pi}{2}(x^2+\omega^2)-\pi i x\omega+\frac{\pi e^{2i\alpha}}{8c^2}}\left(\frac{1+i}{2}\left(\cosh\left(\frac{\pi i\tau}{2c}\right)+\sinh\left(\frac{\pi i\tau}{2c}\right)\right)e^{\frac{\pi e^{i\alpha}}{2c}(x-i\omega)}\right.\\
&\qquad\left.+\frac{1-i}{2}\left(\cosh\left(\frac{\pi i\tau}{2c}\right)-\sinh\left(\frac{\pi i\tau}{2c}\right)\right)e^{-\frac{\pi e^{i\alpha}}{2c}(x-i\omega)}\right)\\
&=\frac{1}{\sqrt{2}}e^{-\frac{\pi}{2}(x^2+\omega^2)-\pi i x\omega+\frac{\pi e^{2i\alpha}}{8c^2}}\left(\cosh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)\right.\\
&\qquad\left.+i\sinh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)\right).
\end{aligned}$$

Similarly, we can compute

$$\begin{aligned}
\mathcal{V}_\phi g(x,\omega)=\frac{1}{\sqrt{2}}e^{-\frac{\pi}{2}(x^2+\omega^2)-\pi i x\omega+\frac{\pi e^{2i\alpha}}{8c^2}}&\left(\cosh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)\right.\\
&\left.-i\sinh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)\right).
\end{aligned}$$

□

Proof of Lemma 2.4. From Lemma 2.2, it is clear that it suffices to show

$$\begin{aligned}
&\left|\cosh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)+i\sinh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)\right|\\
&\qquad=\left|\cosh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)-i\sinh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)\right|,
\end{aligned}$$

for $(x,\omega)\in S$. Hence, let $t\in\mathbb{R}$ and $k\in\mathbb{Z}$ be arbitrary but fixed and consider

$$\begin{pmatrix}x\\ \omega\end{pmatrix}=\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix}t+\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}(\tau+kc).$$

Then, we have

$$\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}=\frac{\pi t}{2c}-\frac{\pi i k}{2}$$

and therefore also

$$\begin{aligned}
&\left|\cosh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)+i\sinh\left(\frac{\pi e^{i\alpha}}{2c}(x-i\omega)+\frac{\pi i\tau}{2c}\right)\right|\\
&=\left|\cosh\left(\frac{\pi t}{2c}-\frac{\pi i k}{2}\right)+i\sinh\left(\frac{\pi t}{2c}-\frac{\pi i k}{2}\right)\right|\\
&=\left|\cosh\left(\frac{\pi i k}{2}\right)\left(\cosh\left(\frac{\pi t}{2c}\right)+i\sinh\left(\frac{\pi t}{2c}\right)\right)-\sinh\left(\frac{\pi i k}{2}\right)\left(\sinh\left(\frac{\pi t}{2c}\right)+i\cosh\left(\frac{\pi t}{2c}\right)\right)\right|\\
&=\left|\cos\left(\frac{\pi k}{2}\right)\left(\cosh\left(\frac{\pi t}{2c}\right)+i\sinh\left(\frac{\pi t}{2c}\right)\right)-\sin\left(\frac{\pi k}{2}\right)\left(i\sinh\left(\frac{\pi t}{2c}\right)-\cosh\left(\frac{\pi t}{2c}\right)\right)\right|.
\end{aligned}$$

Next, we note that either the summand involving $\cos(\frac{\pi k}{2})$ or the summand involving $\sin(\frac{\pi k}{2})$ is non-zero. Therefore, we can use that $|\overline{z}| = |z|$, for $z \in \mathbb{C}$, to see that

$$\begin{aligned}
&\left|\cosh\left(\frac{\pi e^{i\alpha}}{2c}(x - i\omega) - \frac{\pi i\tau}{2c}\right) + i\sinh\left(\frac{\pi e^{i\alpha}}{2c}(x - i\omega) - \frac{\pi i\tau}{2c}\right)\right| \\
&\quad = \left|\cos\left(\frac{\pi k}{2}\right)\left(\cosh\left(\frac{\pi t}{2c}\right) - i\sinh\left(\frac{\pi t}{2c}\right)\right) - \sin\left(\frac{\pi k}{2}\right)\left(i\sinh\left(\frac{\pi t}{2c}\right) + \cosh\left(\frac{\pi t}{2c}\right)\right)\right| \\
&\quad = \left|\cosh\left(\frac{\pi ik}{2}\right)\left(\cosh\left(\frac{\pi t}{2c}\right) - i\sinh\left(\frac{\pi t}{2c}\right)\right) - \sinh\left(\frac{\pi ik}{2}\right)\left(\sinh\left(\frac{\pi t}{2c}\right) - i\cosh\left(\frac{\pi t}{2c}\right)\right)\right| \\
&\quad = \left|\cosh\left(\frac{\pi t}{2c} - \frac{\pi ik}{2}\right) - i\sinh\left(\frac{\pi t}{2c} - \frac{\pi ik}{2}\right)\right| \\
&\quad = \left|\cosh\left(\frac{\pi e^{i\alpha}}{2c}(x - i\omega) + \frac{\pi i\tau}{2c}\right) - i\sinh\left(\frac{\pi e^{i\alpha}}{2c}(x - i\omega) + \frac{\pi i\tau}{2c}\right)\right|.
\end{aligned}$$

□

Proof of Proposition 2.5. Let $1 \leq p, q < \infty$ (the cases in which p or q are infinite are similar) and let $m : \mathbb{R}^2 \to [0, \infty)$ grow at most exponentially. Let us start by noting that according to Lemma 2.2, we have that

$$|\mathcal{V}_\phi f(x, \omega)| = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{8a^2}} \left|\cos\left(\frac{\pi}{2a}(x - i\omega)\right) - \sin\left(\frac{\pi}{2a}(x - i\omega)\right)\right| e^{-\frac{\pi}{2}(x^2+\omega^2)},$$

for $(x, \omega) \in \mathbb{R}^2$. We can now use $|\cos(z)| \leq e^{|\operatorname{Im} z|}$ and $|\sin(z)| \leq e^{|\operatorname{Im} z|}$, for $z \in \mathbb{C}$, to see that

$$|\mathcal{V}_\phi f(x, \omega)| \leq \sqrt{2} e^{-\frac{\pi}{8a^2}} e^{\frac{\pi}{2a}|\omega|} e^{-\frac{\pi}{2}(x^2+\omega^2)}.$$

According to the above inequality and because m grows at most exponentially which implies that there must exist $\sigma_0, \sigma_1 > 0$ such that

$$m(x, \omega) \leq \sigma_0 e^{\sigma_1(|x|+|\omega|)}, \qquad (x, \omega) \in \mathbb{R}^2,$$

we have that

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathcal{V}_\phi f(x, \omega)|^p \, m(x, \omega)^p \, dx\right)^{q/p} d\omega \\
&\qquad \leq \sqrt{2}^q \sigma_0^q e^{-\frac{\pi q}{8a^2}} \int_{\mathbb{R}} \left(e^{\left(\sigma_1 + \frac{\pi}{2a}\right)|\omega| - \frac{\pi}{2}\omega^2} \int_{\mathbb{R}} \left(e^{\sigma_1|x| - \frac{\pi}{2}x^2}\right)^p dx\right)^{q/p} d\omega < \infty.
\end{aligned}$$

Therefore, $f \in M_m^{p,q}(\mathbb{R})$. The same arguments can be employed to deduce that $g \in M_m^{p,q}(\mathbb{R})$. □

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