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Abstract

We consider the problem of phase retrieval from magnitudes of short-time Fourier transform (STFT) measurements. It is well-known that signals are uniquely determined (up to global phase) by their STFT magnitude when the underlying window has an ambiguity function that is nowhere vanishing. It is less clear, however, what can be said in terms of unique phase-retrievability when the ambiguity function of the underlying window vanishes on some of the time-frequency plane. In this short note, we demonstrate that by considering signals in Paley–Wiener spaces, it is possible to prove new uniqueness results for STFT phase retrieval. Among those, we establish a first uniqueness theorem for STFT phase retrieval *from magnitude-only samples* in a real-valued setting.

Keywords Phase retrieval, Short-time Fourier transform, Paley–Wiener space, Sampling theorem, Entire functions.

1 Introduction

The problem of phase retrieval has been around since the very early days of X-ray crystallography [3, 17]. To date, its applications include coherent diffraction imaging, astronomy and audio processing. The measurements in phase retrieval problems typically consist of phaseless Fourier-type data of the object of interest. Acquisition of magnitude-only measurements means loss of information that needs to be accounted for. One possible approach for phase retrieval is to collect redundant measurements as is done in ptychography for coherent diffraction imaging [5, 12, 16]. There, the idea is that instead of creating one set of measurements through diffraction, a sliding pinhole is added and many masked diffraction patterns are collected. Hence, the measurements can be thought of as magnitudes of windowed Fourier transforms. The same is true for measurements collected in audio processing such as the phase vocoder [8, 15]. Indeed, suppose one wants to alter an audio signal (for instance pitch-shift it). To do so, one can take its short-time Fourier transform (STFT), redistribute the magnitudes thereof in the time-frequency plane and look for a matching audio signal by performing phase retrieval. Motivated by these applications, we consider phaseless measurements of short-time Fourier transforms in this note. More precisely, we analyse the question of unique phase retrievability from STFT magnitudes when the underlying signal is bandlimited.

In general, rather little is known about the uniqueness of phase retrieval from STFT magnitude measurements. A known result is that one may recover signals up to global phase from phaseless STFT measurements when the ambiguity function of the underlying window function is nowhere vanishing [9, 11] (see Lemma 1.2). Additionally, the *complement property* [1, 2, 4] is a necessary condition for uniqueness. Finally, in the finite-dimensional setting, there is a plethora of results for phase retrieval from discrete short-time Fourier magnitudes [6, 14]. The aim of this note is to provide milder assumptions on the ambiguity function of the window that guarantee uniqueness of STFT phase retrieval when the considered signal class is that of bandlimited functions.

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Let us start by summarising relevant prerequisites and fixing notations: For a function $f \in L^2(\mathbb{R})$, we define the *short-time Fourier transform (STFT)* (with window function $\phi \in L^2(\mathbb{R})$) by

$$\mathcal{V}_\phi f(x, \omega) := \int_{\mathbb{R}} f(t) \overline{\phi(t-x)} e^{-2\pi i t \omega} dt, \quad x, \omega \in \mathbb{R}.$$

It can be shown that $\mathcal{V}_\phi f$ is uniformly continuous [9]. We consider the phase retrieval problem of recovering f from STFT magnitude measurements $|\mathcal{V}_\phi f|$. Note that it is impossible to distinguish between f and $e^{i\alpha} f$, for $\alpha \in \mathbb{R}$, from the STFT magnitude measurements alone. For this reason, we aim to reconstruct f *up to global phase*. That is, we attempt to recover the equivalence class

$$[f] := \{f e^{i\alpha} \mid \alpha \in \mathbb{R}\}.$$

One of the most important properties of the phase retrieval problem with STFT measurements is the ambiguity function relation. We use the convention

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}} f(t) e^{-2\pi i t \xi} dt, \quad \xi \in \mathbb{R},$$

for the Fourier transform on $L^1(\mathbb{R})$ and extend it to $L^2(\mathbb{R})$ by a density argument. In addition, we can define the *ambiguity function* of a signal $f \in L^2(\mathbb{R})$ via

$$\mathcal{A}f(x, \omega) := e^{\pi i x \omega} \mathcal{V}_f f(x, \omega).$$

The ambiguity function relation can now be stated as follows:

Lemma 1.1 (Ambiguity function relation). *Let $f, \phi \in L^2(\mathbb{R})$. Then,*

$$\mathcal{F}\left(|\mathcal{V}_\phi f|^2\right)(\omega, -x) = \mathcal{A}f(x, \omega) \cdot \overline{\mathcal{A}\phi(x, \omega)}, \quad x, \omega \in \mathbb{R}.$$

We included a proof of this well-known relation in appendix A, for the convenience of the reader. One direct corollary of the ambiguity function relation is that if the ambiguity function of $\phi \in L^2(\mathbb{R})$ is nowhere vanishing, then one may recover the ambiguity function of $f \in L^2(\mathbb{R})$ everywhere from the STFT magnitude measurements $|\mathcal{V}_\phi f|$. Furthermore, f is uniquely determined up to global phase by its ambiguity function. Combining these observations, one has (see e.g. [9, 11]):

Lemma 1.2. *Let $\phi \in L^2(\mathbb{R})$ be such that*

$$\mathcal{A}\phi(x, \omega) \neq 0, \quad \text{for a.e. } (x, \omega) \in \mathbb{R}^2.$$

Then, the following are equivalent for $f, g \in L^2(\mathbb{R})$:

1. $f = e^{i\alpha} g$, for some $\alpha \in \mathbb{R}$.
2. $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$.

Clearly, if $\phi \in L^2(\mathbb{R})$ is such that $\mathcal{A}\phi$ is zero in some region of \mathbb{C} , then, using the ambiguity function relation, one cannot recover $\mathcal{A}f$ everywhere. In this note, we ask whether under some additional assumptions, this scenario still enjoys unique phase recovery. In particular, we consider bandlimited functions f . It turns out that in this setting it suffices to assume that $\mathcal{A}\phi$ does not vanish on certain line segments in the time-frequency plane. More precisely, for $B > 0$ and $p \in \{1, 2\}$, we consider the *Paley–Wiener space* of bandlimited functions defined as

$$\text{PW}_B^p := \left\{ f : \mathbb{C} \rightarrow \mathbb{C} \mid \exists F \in L^p([-B, B]) \forall z \in \mathbb{C} : f(z) = \int_{-B}^B F(\xi) e^{2\pi i \xi z} d\xi \right\}.$$

We record the following classical results on functions in the Paley–Wiener space that we will make use of:

Theorem 1.3 (Paley–Wiener theorem). *Let $B > 0$. Then, the following are equivalent:*

1. $f \in \text{PW}_B^2$.

2. f is an entire function such that there exists a constant $c > 0$ for which

$$|f(z)| \leq c e^{2\pi B|z|}, \quad z \in \mathbb{C},$$

and

$$\int_{\mathbb{R}} |f(t)|^2 dt < \infty.$$

Theorem 1.4 (WSK sampling theorem). *Let $B > 0$ and $f \in \text{PW}_B^2$. Then, we have*

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2B}\right) \text{sinc}(2Bt - n), \quad t \in \mathbb{R}.$$

Theorem 1.5 (see Theorem 1 in [18], p. 723). *Let $p \in \{1, 2\}$, let $B > 0$ and let $f \in \text{PW}_B^p$ be real-valued on the real line. Then, f can be uniquely determined up to global sign from $\left\{ \left| f\left(\frac{n}{4B}\right) \right| \mid n \in \mathbb{Z} \right\}$.*

Another property of bandlimited functions that we will employ is that their ambiguity function is compactly supported in frequency domain.

Lemma 1.6. *Let $B > 0$ and $f \in \text{PW}_B^2$. Then, $\mathcal{A}f$ is uniformly continuous and $\text{supp } \mathcal{A}f \subset \mathbb{R} \times (-2B, 2B)$.*

Proof. See appendix B. □

Therefore, we can consider $\phi \in L^2(\mathbb{R})$ such that $\mathcal{A}\phi$ does not vanish on $\mathbb{R} \times (-2B, 2B)$ and reconstruct all $f \in \text{PW}_B^2$ up to global phase from the STFT magnitude measurements $|\mathcal{V}_\phi f|$. In what follows, we will show that, in fact, the STFT phase retrieval problem is uniquely solvable for signals in PW_B^2 under weaker assumptions on $\mathcal{A}\phi$. We remark that our uniqueness results are mainly of theoretical interest and do not suggest a method for stable phase recovery.

Outline This paper is divided into two main parts. First, in Section 2, we consider signals in the Paley–Wiener space which are real-valued on the real line and develop two uniqueness results for this case: In particular, we show that if the ambiguity function of the window is non-zero almost everywhere on a certain line segment in the time-frequency plane, then all bandlimited signals are uniquely determined by their STFT magnitudes (Subsection 2.1). In addition, we show that if the Fourier transform of the window is non-zero almost everywhere on an open interval around the origin and if the window is real-valued itself, then all bandlimited signals are uniquely determined by *samples* of their STFT magnitudes (Subsection 2.2). Secondly, in Section 3, we consider general signals in the Paley–Wiener space and develop two uniqueness results in this setting. More precisely, we show that if the ambiguity function of the window is non-zero almost everywhere on two parallel line segments in the time-frequency plane that are sufficiently close together, then uniqueness of phase retrieval from STFT magnitudes holds for all signals in PW_B^2 (Subsection 3.1). In addition, we show that if the ambiguity function of the window does not vanish on a single line segment in the time-frequency plane, then all bandlimited signals are uniquely determined by their STFT magnitudes (Subsection 3.2). In Section 4, we discuss our results for some examples of window classes.

2 Real-valued signals

2.1 Reconstruction from full measurements

If $f \in \text{PW}_B^2$, for some $B > 0$, and the window $\phi \in L^2(\mathbb{R})$ is such that $\mathcal{A}\phi(0, \omega) \neq 0$, for $\omega \in (-2B, 2B)$, then one can use the Ambiguity Function Relation to obtain $\mathcal{A}f(0, \cdot)$ everywhere. Therefore, one can recover $|f|$ on the real line via Fourier inversion. If f is real-valued on the real line, this is enough to recover f everywhere up to global phase [18] (see Theorem 1.5). The last insight is particularly important such that we state it as a lemma.

Lemma 2.1. *Let $B > 0$ and $f \in \text{PW}_B^2$ be real-valued on the real line. Then, f is uniquely determined by $\{|f(t)| \mid t \in \mathbb{R}\}$ up to global sign.*

Proof. This follows immediately from Theorem 1.5. Alternatively, one might consider the following argument: Let us assume without loss of generality that f is non-trivial. By the Paley–Wiener theorem (see Theorem 1.3), f is an entire function. The roots of a non-zero entire function are isolated and therefore there must exist an interval $I \subset \mathbb{R} \subset \mathbb{C}$ such that for all $t \in I$, $f(t) \neq 0$. Therefore, $|f|$ agrees with f up to global sign on I . In other words, $|f|$ is the restriction of f or $-f$ to the interval I and thus analytically extending $|f|$ from I to \mathbb{C} yields f or $-f$. \square

Note that the same does not hold for general signals. Indeed, consider the following counterexample.

Example 2.2. *Let $B > 0$, $f(z) = \text{sinc}(Bz)$ and $g(z) = \text{sinc}(Bz)e^{\pi i B z}$, for $z \in \mathbb{C}$. One can readily show that*

$$\begin{aligned} f(z) &= \frac{1}{B} \int_{-B}^B \chi_{[-B/2, B/2]}(\xi) e^{2\pi i \xi z} d\xi, \quad z \in \mathbb{C}, \\ g(z) &= \frac{1}{B} \int_{-B}^B \chi_{[0, B]}(\xi) e^{2\pi i \xi z} d\xi, \quad z \in \mathbb{C}. \end{aligned}$$

Therefore, we have $f, g \in \text{PW}_B^2$. In addition,

$$|f(t)| = |\text{sinc}(Bt)| = |g(t)|, \quad t \in \mathbb{R},$$

but f and g do not agree up to global phase.

Many more counterexamples may be constructed using Hadamard’s factorisation theorem and ideas similar to the ones in [13]. We may now combine the lemma above with the ambiguity function relation to derive the following theorem.

Theorem 2.3. *Let $B > 0$ and $\phi \in L^2(\mathbb{R})$ such that*

$$\mathcal{A}\phi(0, \omega) \neq 0, \quad \text{for a.e. } \omega \in (-2B, 2B).$$

Then, the following are equivalent for $f, g \in \text{PW}_B^2$ real-valued on the real line:

1. $f = \pm g$.
2. $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$.

Proof. First, note that if $f = \pm g$, then it follows immediately that $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$. Secondly, suppose that $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$. It follows from the ambiguity function relation that

$$\mathcal{A}f(x, \omega) \cdot \overline{\mathcal{A}\phi(x, \omega)} = \mathcal{A}g(x, \omega) \cdot \overline{\mathcal{A}\phi(x, \omega)}, \quad x, \omega \in \mathbb{R}.$$

Hence, by the assumption on the ambiguity function of the window, $\mathcal{A}f(0, \omega) = \mathcal{A}g(0, \omega)$, for a.e. $\omega \in (-2B, 2B)$. By the Paley–Wiener theorem, f and g are square integrable and thus $\mathcal{A}f$ and $\mathcal{A}g$ are (uniformly) continuous. Therefore, $\mathcal{A}f(0, \omega) = \mathcal{A}g(0, \omega)$, for all $\omega \in (-2B, 2B)$. We know from Lemma 1.6 that $\text{supp } \mathcal{A}f, \text{supp } \mathcal{A}g \subset \mathbb{R} \times (-2B, 2B)$ and consequently, that $\mathcal{A}f(0, \cdot) = \mathcal{A}g(0, \cdot)$. Since

$$\mathcal{A}f(0, \cdot) = \mathcal{F}\left(|f|^2\right), \quad \mathcal{A}g(0, \cdot) = \mathcal{F}\left(|g|^2\right),$$

we have $|f| = |g|$. Applying Lemma 2.1 yields the assertion. \square

Remark 2.4. *Using basic Fourier-analytic results, one can show a statement which is similar to Theorem 2.3 for compactly supported, even, real-valued functions:*

Let $B > 0$ and $\phi \in L^2(\mathbb{R})$ such that

$$\mathcal{A}\phi(x, 0) \neq 0, \quad \text{for a.e. } x \in (-2B, 2B).$$

Then, the following are equivalent for $f, g \in L^2([-B, B])$ even and real-valued:

1. $f = \pm g$.
2. $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$.

2.2 Reconstruction from sampled STFT measurements

So far, we have not used the sampling theorem in [18] for the reconstruction from *samples* of the STFT magnitudes. Let us consider the same setup as before and ask whether the phase is uniquely determined *by sampled data*. To the best of our knowledge, the following is the first uniqueness result for phase retrieval from sampled STFT magnitude measurements.

Theorem 2.5. *Let $B > 0$ and let $\phi \in L^2(\mathbb{R})$ be a real-valued window such that*

$$(\mathcal{F}\phi)(\xi) \neq 0, \quad \text{for a.e. } \xi \in (-B, B).$$

Then, the following are equivalent for $f, g \in \text{PW}_B^2$ real-valued on the real line:

1. $f = \pm g$.
2. $|\mathcal{V}_\phi f(\frac{n}{4B}, 0)| = |\mathcal{V}_\phi g(\frac{n}{4B}, 0)|$, for all $n \in \mathbb{Z}$.

Proof. First, note that if $f = \pm g$, then it follows immediately that

$$\left| \mathcal{V}_\phi f\left(\frac{n}{4B}, 0\right) \right| = \left| \mathcal{V}_\phi g\left(\frac{n}{4B}, 0\right) \right|, \quad n \in \mathbb{Z}.$$

Secondly, assume that the above equation holds. Let us define $\phi^\#(t) := \overline{\phi(-t)}$, for $t \in \mathbb{R}$ (this simplifies to $\phi^\#(t) := \phi(-t)$ because ϕ is real-valued). Note that

$$\mathcal{V}_\phi f(x, 0) = \int_{\mathbb{R}} f(t) \overline{\phi(t-x)} dt = (f * \phi^\#)(x), \quad x \in \mathbb{R},$$

and hence

$$\left| (f * \phi^\#)\left(\frac{n}{4B}\right) \right| = \left| (g * \phi^\#)\left(\frac{n}{4B}\right) \right|, \quad n \in \mathbb{Z}.$$

As $f, g \in \text{PW}_B^2$, it follows from the convolution theorem that $f * \phi^\#$ and $g * \phi^\#$ extend to functions in PW_B^1 . Indeed, as $f \in \text{PW}_B^2$, there exists $F' \in L^2([-B, B])$ such that

$$f(z) = \int_{-B}^B F'(\xi) e^{2\pi i \xi z} d\xi, \quad z \in \mathbb{C}.$$

Now, consider $F = F' \overline{\mathcal{F}\phi} \in L^1([-B, B])$. Then, by the convolution theorem,

$$(f * \phi^\#)(t) = \int_{-B}^B F(\xi) e^{2\pi i \xi t} d\xi, \quad t \in \mathbb{R}.$$

Therefore, the analytic extensions of $f * \phi^\#$ and $g * \phi^\#$ belong to PW_B^1 . In addition, $f * \phi^\#$ and $g * \phi^\#$ are real-valued such that it follows from Theorem 1.5 that $f * \phi^\# = \pm(g * \phi^\#)$. Consequently, we have

$$\mathcal{F}(f * \phi^\#) = \pm \mathcal{F}(g * \phi^\#).$$

By the assumption on the Fourier transform of the window, it follows that $\mathcal{F}f = \pm \mathcal{F}g$ and hence, $f = \pm g$. \square

Remark 2.6. *We observe the following:*

1. *The sampling rate only depends on the bandwidth of the signals and is exactly twice the Nyquist rate.*
2. *For the Gaussian $\phi(t) := e^{-\pi t^2}$, $t \in \mathbb{R}$, we can readily see that $\mathcal{F}\phi$ is non-zero everywhere. In addition, the Gaussian is real-valued such that the theorem above implies that all bandlimited signals are uniquely determined by samples of their STFT magnitudes with Gaussian window (also called Gabor transform magnitudes).*
3. *We use the uniform sampling sequence $X = \{\frac{n}{4B}\}_{n \in \mathbb{Z}} \subset \mathbb{R}$, for convenience of notation. In fact, our result still holds if one replaces $X = \{\frac{n}{4B}\}_{n \in \mathbb{Z}}$ by any separated, uniformly dense sampling sequence with density lower bounded by $4B$ [18].*
4. *While the STFT is complex-valued, we employ only real-valued information on one line of the time-frequency plane. On this line, sign retrieval suffices for the uniqueness result to hold.*

3 Complex-valued signals

3.1 Using the ambiguity function on two line segments

We have seen that for complex-valued $f \in \text{PW}_B^2$, it is not true that f is uniquely determined up to global phase by $\{|f(t)| \mid t \in \mathbb{R}\}$. Therefore, we need to change our strategy to deduce a general uniqueness result. One can, for instance, impose slightly stronger assumptions on the ambiguity function of the window to show that in this case one may recover f uniquely up to global phase from $|\mathcal{V}_\phi f|$.

Theorem 3.1. *Let $B > 0$, $c \in (0, \frac{1}{2B}]$ and $\phi \in L^2(\mathbb{R})$ such that*

$$\mathcal{A}\phi(0, \omega) \neq 0 \quad \text{and} \quad \mathcal{A}\phi(c, \omega) \neq 0,$$

for a.e. $\omega \in (-2B, 2B)$. Then, the following are equivalent for $f, g \in \text{PW}_B^2$:

1. $f = e^{i\alpha}g$, for some $\alpha \in \mathbb{R}$.
2. $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$.

Proof. First, note that if $f = e^{i\alpha}g$, for some $\alpha \in \mathbb{R}$, then it follows immediately that $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$. Secondly, suppose that $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$. Let us also assume without loss of generality that f and g are non-zero. As in the proof of Theorem 2.3, we find that $\mathcal{A}f(0, \cdot) = \mathcal{A}g(0, \cdot)$ and $\mathcal{A}f(c, \cdot) = \mathcal{A}g(c, \cdot)$. Therefore, $|f| = |g|$ on \mathbb{R} and

$$f(t)\overline{f(t-c)} = g(t)\overline{g(t-c)}, \quad t \in \mathbb{R}.$$

By the Paley–Wiener theorem, f and g are entire functions. Therefore, we know that f and g have a countable number of roots (as their roots are isolated). In particular, there exists some $t_0 \in \mathbb{R}$ such that for all $n \in \mathbb{Z}$, we have

$$f(t_0 + nc) \neq 0 \quad \text{and} \quad g(t_0 + nc) \neq 0.$$

Now, let us set $\alpha \in (-\pi, \pi]$ to be such that

$$f(t_0) = e^{i\alpha}g(t_0).$$

Then, we can use the relation

$$f(t)\overline{f(t-c)} = g(t)\overline{g(t-c)}, \quad t \in \mathbb{R},$$

to recursively find that

$$f(t_0 + nc) = e^{i\alpha}g(t_0 + nc), \quad n \in \mathbb{Z}.$$

Finally, since $f, g \in \text{PW}_B^2$ and $c \leq \frac{1}{2B}$, it follows that $f(t_0 + \cdot), g(t_0 + \cdot) \in \text{PW}_{\frac{1}{2c}}^2$. Therefore, we deduce from the WSK sampling theorem (see Theorem 1.4) that

$$f(t_0 + t) = \sum_{n \in \mathbb{Z}} f(t_0 + nc) \text{sinc}\left(\frac{t}{c} - n\right) = \sum_{n \in \mathbb{Z}} e^{i\alpha}g(t_0 + nc) \text{sinc}\left(\frac{t}{c} - n\right) = e^{i\alpha}g(t_0 + t),$$

for all $t \in \mathbb{R}$. Hence, we conclude that $f = e^{i\alpha}g$. □

Remark 3.2. *We can apply basic Fourier analysis to develop a result similar to Theorem 3.1 for compactly supported functions:*

Let $B > 0$, $c \in (0, \frac{1}{2B}]$ and $\phi \in L^2(\mathbb{R})$ such that

$$\mathcal{A}\phi(x, 0) \neq 0 \quad \text{and} \quad \mathcal{A}\phi(x, c) \neq 0,$$

for a.e. $x \in (-2B, 2B)$. Then, the following are equivalent for $f, g \in L^2([-B, B])$:

1. $f = e^{i\alpha}g$, for some $\alpha \in \mathbb{R}$.

$$2. |\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|.$$

We note that the requirement $c \leq \frac{1}{2B}$ is linked to the use of the WSK sampling theorem in the proof of the above result. The following example shows that it is necessary in order for $f \in \text{PW}_B^2$ to be uniquely determined up to global phase by $|f(t)|$ and $f(t)\overline{f(t-c)}$, for $t \in \mathbb{R}$.

Example 3.3. Let $\epsilon, B > 0$ and $c = \frac{1}{2B-\epsilon} > \frac{1}{2B}$. Consider $f(z) = \text{sinc}(\epsilon z)e^{\pi i(2B-\epsilon)z}$ as well as $g(z) = \text{sinc}(\epsilon z)e^{-\pi i(2B-\epsilon)z}$, for $z \in \mathbb{C}$. Then, it is readily seen that

$$f(z) = \frac{1}{\epsilon} \int_{-B}^B \chi_{[B-\epsilon, B]}(\xi) e^{2\pi i \xi z} d\xi, \quad z \in \mathbb{C},$$

as well as

$$g(z) = \frac{1}{\epsilon} \int_{-B}^B \chi_{[-B, -B+\epsilon]}(\xi) e^{2\pi i \xi z} d\xi, \quad z \in \mathbb{C}.$$

Therefore, we have $f, g \in \text{PW}_B^2$. In addition,

$$|f(t)| = |\text{sinc}(\epsilon t)| = |g(t)|,$$

as well as

$$\begin{aligned} f(t)\overline{f(t-c)} &= \text{sinc}(\epsilon t) \text{sinc}(\epsilon(t-c)) e^{\pi i(2B-\epsilon)c} = -\text{sinc}(\epsilon t) \text{sinc}(\epsilon(t-c)), \\ &= \text{sinc}(\epsilon t) \text{sinc}(\epsilon(t-c)) e^{-\pi i(2B-\epsilon)c} = g(t)\overline{g(t-c)}, \end{aligned}$$

hold, for $t \in \mathbb{R}$. However, f and g do not agree up to global phase.

Many more examples may be constructed using Hadamard's factorisation theorem and ideas similar to the ones in [13].

3.2 Using the ambiguity function on a single line segment

We can approach the reconstruction of general bandlimited functions from their STFT magnitude measurements from a slightly different angle and obtain another uniqueness result. The following statement is neither stronger nor weaker than Theorem 3.1: Indeed, it has the advantage that one only needs to assume that the ambiguity function of the window does not vanish on a single line segment while having the disadvantage that one has to make this assumption pointwise and not in an L^2 -sense.

Theorem 3.4. Let $B > 0$ and $\phi \in L^2(\mathbb{R})$ be such that

$$\mathcal{A}\phi(0, \omega) \neq 0, \quad \omega \in [-2B, 2B].$$

Then, the following are equivalent for $f, g \in \text{PW}_B^2$:

1. $f = e^{i\alpha}g$, for some $\alpha \in \mathbb{R}$.
2. $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$.

Proof. First, note that if $f = e^{i\alpha}g$, for some $\alpha \in \mathbb{R}$, then it follows immediately that $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$. Secondly, suppose that $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$ and assume without loss of generality that f and g are non-zero. Note that $\mathcal{A}\phi$ is continuous such that $|\mathcal{A}\phi|$ is continuous and by assumption $|\mathcal{A}\phi(0, \omega)| > 0$, for $\omega \in [-2B, 2B]$. By the extreme value theorem, there exists a positive constant $\Delta > 0$ such that $|\mathcal{A}\phi(0, \omega)| \geq \Delta$, for $\omega \in [-2B, 2B]$. As $|\mathcal{A}\phi|$ is uniformly continuous, there exists a $\delta > 0$ such that

$$|\mathcal{A}\phi(x, \omega)| > \frac{\Delta}{2}, \quad (x, \omega) \in (-\delta, \delta) \times [-2B, 2B].$$

In particular, it follows that

$$\mathcal{A}\phi(x, \omega) \neq 0, \quad (x, \omega) \in (-\delta, \delta) \times [-2B, 2B].$$

Lemma 1.1 further implies

$$\mathcal{A}f(x, \omega) = \mathcal{A}g(x, \omega), \quad (x, \omega) \in (-\delta, \delta) \times [-2B, 2B].$$

Hence, by employing Lemma 1.6, we deduce that $\mathcal{A}f(x, \omega) = \mathcal{A}g(x, \omega)$, for $(x, \omega) \in (-\delta, \delta) \times \mathbb{R}$. By Fourier inversion, we have

$$f(t)\overline{f(t-c)} = g(t)\overline{g(t-c)}, \quad t \in \mathbb{R},$$

for $c \in (-\delta, \delta)$. As f and g are entire functions and assumed to be non-zero, there exists a $t_0 \in \mathbb{R}$ such that $f(t_0), g(t_0) \neq 0$. As $|f(t_0)| = |g(t_0)|$, it follows that there exists an $\alpha \in \mathbb{R}$ such that $f(t_0) = e^{i\alpha}g(t_0)$. This implies that $f(t) = e^{i\alpha}g(t)$ for all $t \in (t_0 - \delta, t_0 + \delta)$. As f and g are entire, we conclude that $f = e^{i\alpha}g$. \square

Remark 3.5. *As before, we can make a similar statement as the above for compactly supported functions:*

Let $B > 0$ and $\phi \in L^2(\mathbb{R})$ be such that

$$\mathcal{A}\phi(x, 0) \neq 0, \quad x \in [-2B, 2B].$$

Then, the following are equivalent for $f, g \in L^2([-B, B])$:

1. $f = e^{i\alpha}g$, for some $\alpha \in \mathbb{R}$.
2. $|\mathcal{V}_\phi f| = |\mathcal{V}_\phi g|$.

4 Examples

In the following, we want to consider different windows and their ambiguity functions in order to put the results which we have developed in context. We start by considering the most well-known window in time-frequency analysis: The *Gaussian window* $\phi(t) := e^{-\pi t^2}$, for $t \in \mathbb{R}$ (see Figure 1). For the Gaussian, one can show that

$$\mathcal{A}\phi(x, \omega) = \frac{1}{\sqrt{2}}e^{-\frac{\pi}{2}(x^2 + \omega^2)}, \quad x, \omega \in \mathbb{R}.$$

Therefore, $\mathcal{A}\phi$ is nowhere vanishing and all $f \in L^2(\mathbb{R})$ may be uniquely recovered up to global phase from their STFT magnitude measurements $|\mathcal{V}_\phi f|$. Note that this already follows from the classical theory about uniqueness of STFT phase retrieval (Lemma 1.2). One could be tempted to believe that the only windows for which the ambiguity functions are non-zero everywhere are the generalised Gaussians e^q , where q is a polynomial of degree two. This belief is wrong, however, as was recently shown in [10], and one can in fact construct more functions $\phi \in L^2(\mathbb{R})$ such that

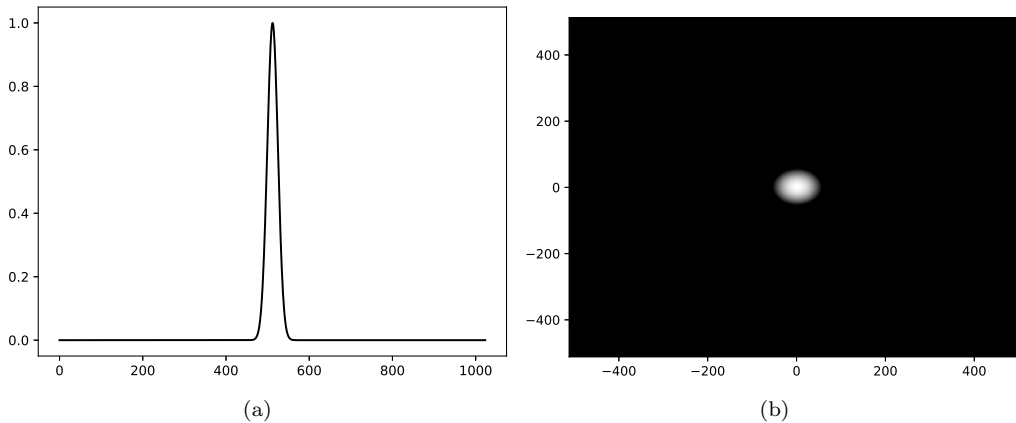


Figure 1: Picture of a discretisation of the Gaussian window and its ambiguity function.

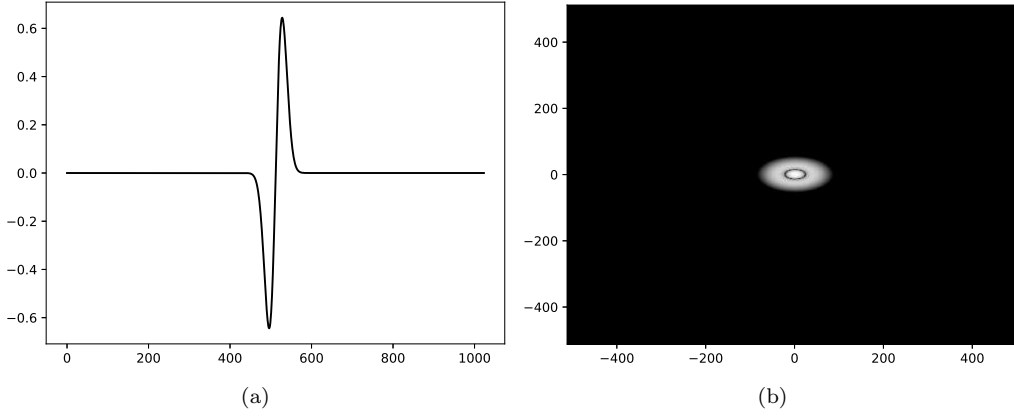


Figure 2: Picture of a discretisation of the Hermite function H_1 and its ambiguity function.

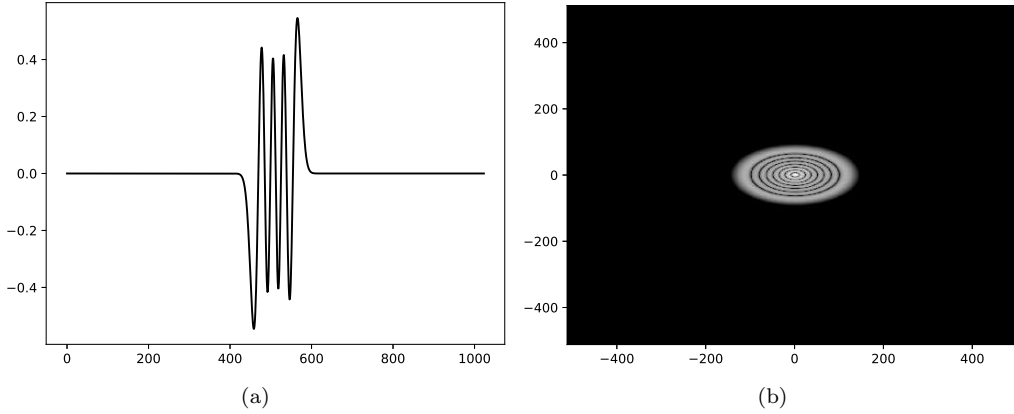


Figure 3: Picture of a discretisation of the Hermite function H_7 and its ambiguity function.

$\mathcal{A}\phi \neq 0$ everywhere. Finally, note that ϕ is real-valued and that the Fourier transform of ϕ is also a Gaussian. In particular, $\mathcal{F}\phi$ vanishes nowhere and it follows from Theorem 2.5 that for all $B > 0$, one can recover all $f \in \text{PW}_B^2$ that are real-valued on the real line up to global sign from the sampled measurements $|\mathcal{V}_\phi f(\frac{n}{4B}, 0)|$, for $n \in \mathbb{Z}$.

The next class of window functions we want to study is that of the Hermite functions. We define the monomials

$$e_n(z) := \sqrt{\frac{\pi^n}{n!}} z^n, \quad z \in \mathbb{C},$$

for $n \in \mathbb{Z}_{\geq 0}$. One can show that these monomials form an orthonormal basis of the Fock space $\mathcal{F}^2(\mathbb{C})$ [9]. The pre-images of these monomials under the Bargmann transform $\mathcal{B} : L^2(\mathbb{R}) \rightarrow \mathcal{F}^2(\mathbb{C})$ are called *Hermite functions* and we write $H_n := \mathcal{B}^{-1}e_n$, for $n \in \mathbb{Z}_{\geq 0}$. The ambiguity function of the Hermite functions can be expressed in terms of the *Laguerre polynomials*

$$L_k^{(j)}(t) := \sum_{m=0}^k \frac{(k+j)!}{(k-m)!(j+m)!} \frac{(-t)^m}{m!}, \quad t \in \mathbb{R},$$

where $k, j \in \mathbb{Z}_{\geq 0}$ [7]. In particular, we have for all $x, \omega \in \mathbb{R}$ and $z = x + i\omega$ that (see Figure 2 and Figure 3 for an illustration):

$$\mathcal{A}H_n(x, \omega) = e^{-\frac{\pi}{2}|z|^2} L_n^{(0)}(\pi|z|^2) = e^{-\frac{\pi}{2}|z|^2} \sum_{m=0}^n \binom{n}{m} \frac{(-\pi)^m |z|^{2m}}{m!}.$$

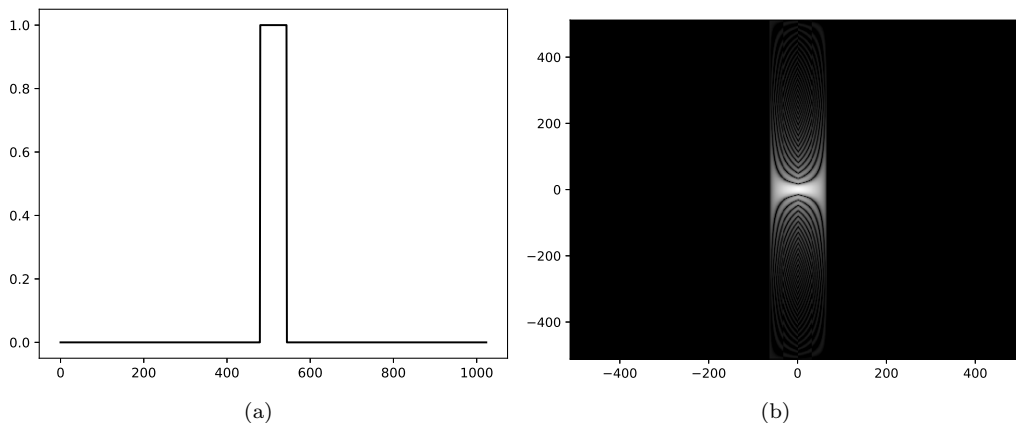


Figure 4: Picture of a discretisation of the rectangular window and its ambiguity function.

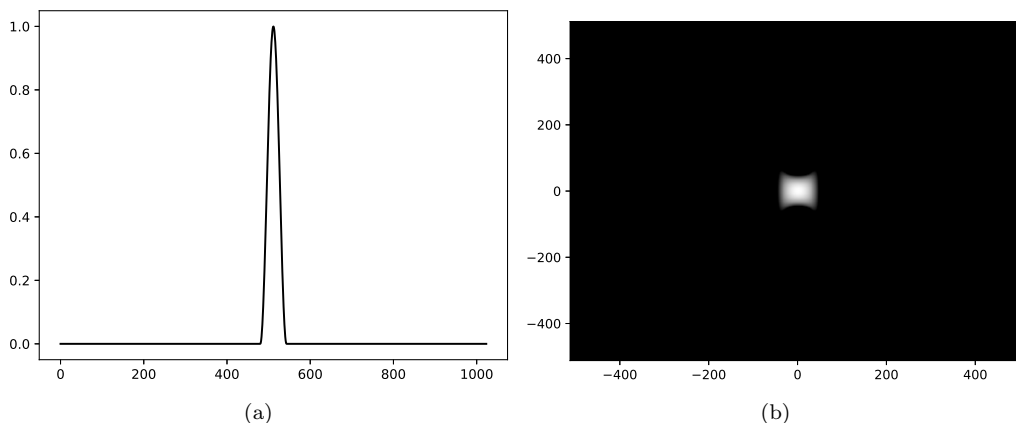


Figure 5: Picture of a discretisation of the Hanning window and its ambiguity function.

Therefore, the set of roots of $\mathcal{A}H_n$ consists of concentric rings around the origin of the time-frequency plane. The radius of these rings is determined by the positive roots of the n -th Laguerre polynomial $L_n^{(0)}$. In particular, $\mathcal{A}H_n$ is non-zero almost everywhere and it follows from Lemma 1.2 that all $f \in L^2(\mathbb{R})$ are uniquely determined up to global phase by their STFT measurements. As for the Gaussian case, uniqueness of STFT phase retrieval already follows from the ambiguity function relation. Note that the Fourier transform of the Hermite function H_n is $(-i)^n H_n$ [9]. In addition, one can show that $h_n(t) := e^{\pi t^2} H_n(t)$, $t \in \mathbb{R}$, is a polynomial of degree n [7]. It follows that $\mathcal{F}H_n$ has only finitely many roots and thus $\mathcal{F}H_n$ is non-zero almost everywhere. In addition, as H_n is real-valued, it follows from Theorem 2.5 that for all $B > 0$, it holds that all $f \in \text{PW}_B^2$ that are real-valued on the real line can be recovered up to global sign from the sampled measurements $|\mathcal{V}_{H_n} f(\frac{n}{4B}, 0)|$, for $n \in \mathbb{Z}$.

The third class of window functions, we consider is that of compactly supported window functions. This class includes all windows commonly used in practice. Consider for instance the *rectangular window*

$$\phi(t) = \begin{cases} 1 & \text{if } t \in [-1, 1], \\ 0 & \text{else,} \end{cases}$$

or the *Hanning window* $\phi := \cos^2 \chi_{[-\pi/2, \pi/2]}$. If $\phi \in L^2(\mathbb{R})$ is compactly supported, then for any fixed $x \in \mathbb{R}$, the function $\omega \mapsto \mathcal{A}\phi(x, \omega)$ is bandlimited. In particular, $\omega \mapsto \mathcal{A}\phi(x, \omega)$ extends to an analytic function on the complex plane. Therefore, $z \mapsto \mathcal{A}\phi(x, z)$ is either zero or has merely isolated zeroes on the real line. Hence, we can conclude that for all $x \in \mathbb{R}$ such that $\omega \mapsto \mathcal{A}\phi(x, \omega)$ is not the trivial map, it holds that

$$\mathcal{A}\phi(x, \omega) \neq 0, \quad \text{for a.e. } \omega \in \mathbb{R}.$$

For a depiction of the ambiguity functions of the rectangular and Hanning windows see Figure 4 and Figure 5, respectively. As $\mathcal{A}\phi(0, \omega) = \mathcal{F}(|\phi|^2)(\omega)$, it follows that $\mathcal{A}\phi(0, \omega) \neq 0$ for a.e. $\omega \in \mathbb{R}$, as long as ϕ is not the trivial window. Therefore, Theorem 2.3 implies that for all $B > 0$, it holds that all $f \in \text{PW}_B^2$ which are real-valued on the real line are uniquely determined up to global sign by their STFT measurements. We can say more, however. If ϕ is not the trivial window, then $\mathcal{A}\phi(0, 0) = \|\phi\|_2 > 0$. As $\mathcal{A}\phi$ is continuous, it follows that for all $x > 0$ which are small enough, $\mathcal{A}\phi(x, 0) > 0$. Therefore, $\mathcal{A}\phi(x, \omega) \neq 0$ for almost every $\omega \in \mathbb{R}$. By Theorem 3.1, we find that for all $B > 0$, it holds that all $f \in \text{PW}_B^2$ are uniquely determined up to global phase by their STFT magnitudes.

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A The ambiguity function relation

Proof of the ambiguity function relation. Using the family $\{f_x \in L^1(\mathbb{R}) \mid x \in \mathbb{R}\}$ given by

$$f_x(t) = f(t)\overline{\phi(t-x)}, \quad t \in \mathbb{R},$$

for $x \in \mathbb{R}$, allows us to write

$$\mathcal{V}_\phi f(x, \omega) = \mathcal{F}f_x(\omega), \quad x, \omega \in \mathbb{R}.$$

In addition, we can readily see that

$$\overline{\mathcal{F}f_x(\omega)} = \mathcal{F}f_x^\#(\omega), \quad x, \omega \in \mathbb{R},$$

where $f_x^\#(t) = \overline{f_x(-t)}$, for $t, x \in \mathbb{R}$. Let $x \in \mathbb{R}$ be fixed but arbitrary. As $f_x \in L^1(\mathbb{R})$, it follows from the convolution theorem that

$$|\mathcal{V}_\phi f(x, \omega)|^2 = \mathcal{V}_\phi f(x, \omega)\overline{\mathcal{V}_\phi f(x, \omega)} = \mathcal{F}(f_x * f_x^\#)(\omega), \quad \omega \in \mathbb{R}.$$

As $\mathcal{V}_\phi f \in L^2(\mathbb{R}^2)$ (by the orthogonality relations of the STFT [9]), it follows that the STFT magnitude measurements squared are in L^1 and thus the Fourier inversion theorem implies that

$$\mathcal{F}\left(|\mathcal{V}_\phi f(x, \cdot)|^2\right)(-x') = (f_x * f_x^\#)(x') = \int_{\mathbb{R}} f(t)\overline{f(t-x')\phi(t-x)}\phi(t-x-x') dt,$$

for $-x' \in \mathbb{R}$. Finally, we can see the above as a function in $x \in \mathbb{R}$ and note that

$$\begin{aligned} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(t)\overline{f(t-x')\phi(t-x)}\phi(t-x-x') dt \right| dx &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t)f(t-x')\phi(t-x)\phi(t-x-x')| dt dx \\ &= \int_{\mathbb{R}} |f(t)f(t-x')| \int_{\mathbb{R}} |\phi(t-x)\phi(t-x-x')| dx dt \\ &= \int_{\mathbb{R}} |f(t)f(t-x')| \int_{\mathbb{R}} |\phi(x)\phi(x-x')| dx dt \\ &\leq \|f\|_2^2 \|\phi\|_2^2 < \infty, \end{aligned}$$

for $x' \in \mathbb{R}$, by the triangle inequality, Tonelli's theorem, a change of variables and Cauchy–Schwarz. Therefore, we may take the Fourier transform in x and obtain

$$\begin{aligned} \mathcal{F}\left(|\mathcal{V}_\phi f|^2\right)(\omega', -x') &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t)\overline{f(t-x')\phi(t-x)}\phi(t-x-x')e^{-2\pi i x \omega'} dt dx \\ &= \int_{\mathbb{R}} f(t)\overline{f(t-x')}e^{-2\pi i t \omega'} \int_{\mathbb{R}} \overline{\phi(t-x)}\phi(t-x-x')e^{2\pi i(t-x)\omega'} dx dt \\ &= \mathcal{V}_f f(x', \omega')\overline{\mathcal{V}_\phi \phi(x', \omega')}, \end{aligned}$$

for $x', \omega' \in \mathbb{R}$, by Fubini's theorem and a change of variables. Finally, note that all the equalities in this proof are actual equalities as all functions that we compare are continuous functions by virtue of them being Fourier transforms of L^1 functions. \square

B The ambiguity function of a Paley–Wiener function

Proof of Lemma 1.6. It follows from Plancherel’s theorem that the restriction of f to the real line is in $L^2(\mathbb{R})$. Therefore, $\mathcal{A}f(x, \omega) = e^{\pi i x \omega} \mathcal{V}_f f(x, \omega)$ is uniformly continuous [9]. Now, define $f_x(t) := f(t - x)$, for $t, x \in \mathbb{R}$, and let $x \in \mathbb{R}$ be arbitrary but fixed. We compute

$$e^{-\pi i x \omega} \mathcal{A}f(x, \omega) = \int_{\mathbb{R}} f(t) \overline{f(t - x)} e^{-2\pi i t \omega} dt = \mathcal{F}(f \cdot f_x)(\omega) = (\mathcal{F}f * \mathcal{F}f_x)(\omega),$$

using the convolution theorem. Furthermore, we find that

$$\begin{aligned} \mathcal{F}f_x(\omega) &= \int_{\mathbb{R}} \overline{f(t - x)} e^{-2\pi i t \omega} dt = e^{-2\pi i x \omega} \int_{\mathbb{R}} \overline{f(t)} e^{2\pi i t(-\omega)} dt = e^{-2\pi i x \omega} \overline{\mathcal{F}f(-\omega)} \\ &= e^{-2\pi i x \omega} \overline{\mathcal{F}f(-\omega)}. \end{aligned}$$

Therefore, we have

$$e^{-\pi i x \omega} \mathcal{A}f(x, \omega) = \int_{\mathbb{R}} \mathcal{F}f(\xi) \mathcal{F}f_x(\omega - \xi) d\xi = \int_{-B}^B \mathcal{F}f(\xi) \overline{\mathcal{F}f(\xi - \omega)} e^{-2\pi i x(\omega - \xi)} d\xi.$$

If $\omega \in \mathbb{R}$ is such that $|\omega| \geq 2B$, then we can readily see that $\xi - \omega \notin [-B, B]$, for $\xi \in (-B, B)$. It follows that $\mathcal{A}f(x, \omega) = 0$. \square

References

- [1] Radu Balan, Pete Casazza, and Dan Edidin. On signal reconstruction without phase. *Applied and Computational Harmonic Analysis*, 20(3):345–356, 2006.
- [2] Afonso S Bandeira, Jameson Cahill, Dustin G Mixon, and Aaron A Nelson. Saving phase: Injectivity and stability for phase retrieval. *Applied and Computational Harmonic Analysis*, 37(1):106–125, 2014.
- [3] William Lawrence Bragg. The Rutherford memorial lecture 1960: The development of x-ray analysis. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 262(1309):145–158, 1961.
- [4] Jameson Cahill, Peter Casazza, and Ingrid Daubechies. Phase retrieval in infinite-dimensional Hilbert spaces. *Transactions of the American Mathematical Society, Series B*, 3(3):63–76, 2016.
- [5] Henry N Chapman. Phase-retrieval x-ray microscopy by Wigner-distribution deconvolution. *Ultramicroscopy*, 66(3-4):153–172, 1996.
- [6] Yonina C Eldar, Pavel Sidorenko, Dustin G Mixon, Shaby Barel, and Oren Cohen. Sparse phase retrieval from short-time Fourier measurements. *IEEE Signal Processing Letters*, 22(5):638–642, 2014.
- [7] Gerald B Folland. *Harmonic analysis in phase space*. Number 122. Princeton University Press, 1989.
- [8] Daniel Griffin and Jae Lim. Signal estimation from modified short-time Fourier transform. *IEEE Transactions on Acoustics, Speech, and Signal Processing*, 32(2):236–243, 1984.
- [9] Karlheinz Gröchenig. *Foundations of time-frequency analysis*. Birkhäuser, 2001.
- [10] Karlheinz Gröchenig, Philippe Jaming, and Eugenia Malinnikova. Zeros of the Wigner distribution and the short-time Fourier transform. *Revista Matemática Complutense*, pages 1–22, 2019.
- [11] Philipp Grohs and Martin Rathmair. Stable Gabor phase retrieval and spectral clustering. *Communications on Pure and Applied Mathematics*, 72(5):981–1043, 2019.

- [12] Reiner Hegerl and W Hoppe. Dynamische Theorie der Kristallstrukturanalyse durch Elektronenbeugung im inhomogenen Primärstrahlwellenfeld. *Berichte der Bunsengesellschaft für physikalische Chemie*, 74(11):1148–1154, 1970.
- [13] John N Mc Donald. Phase retrieval and magnitude retrieval of entire functions. *Journal of Fourier Analysis and Applications*, 10(3):259–267, 2004.
- [14] Götz E Pfander and Pavel Zheltov. Estimation of overspread scattering functions. *IEEE Transactions on Signal Processing*, 63(10):2451–2463, 2015.
- [15] Zdeněk Průša, Peter Balazs, and Peter Lempel Søndergaard. A noniterative method for reconstruction of phase from STFT magnitude. *IEEE/ACM Transactions on Audio, Speech, and Language Processing*, 25(5):1154–1164, 2017.
- [16] John M Rodenburg. Ptychography and related diffractive imaging methods. *Advances in imaging and electron physics*, 150:87–184, 2008.
- [17] David Sayre. X-ray crystallography: The past and present of the phase problem. *Structural Chemistry*, 13(1):81–96, 2002.
- [18] Gaurav Thakur. Reconstruction of bandlimited functions from unsigned samples. *Journal of Fourier Analysis and Applications*, 17(4):720–732, 2011.