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Unitarization of the Horocyclic Radon Transform on Homogeneous Trees

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Abstract

Following previous work in the continuous setup, we construct the unitarization of the horocyclic Radon transform on a homogeneous tree X and we show that it intertwines the quasi regular representations of the group of isometries of X on the tree itself and on the space of horocycles.

Key words. Homogeneous trees, horocyclic Radon transform, dual pairs, quasi regular representations.

AMS subject classification. 44A12, 20E08, 22D10.

Introduction

The horocyclic Radon transform on homogeneous trees was introducted by P. Cartier [5] and studied by A. Figà-Talamanca and M.A. Picardello [9], W. Betori, J. Faraut and M. Pagliacci [3], M. Cowling, S. Meda and A.G. Setti [7], J. Cohen, F. Colonna and E. Tarabusi [6], and A. Veca [14], to name a few. Some of the typical issues considered are inversion formulæ and range problems. In this paper, we treat the unitarization problem, that is the determination of some kind of pseudo-differential operator such that the precomposition with the Radon transform yields a unitary operator. This is a classical problem in Radon theory, addressed first by Helgason in the case of the polar Radon transform in [12]. In [1], the authors consider a general setup that may be recast as a variation of the setup of dual pairs (X, Ξ) *à la* Helgason. They prove a general result concerning the unitarization of the Radon transform $\mathcal{R}$ from $L^2(X, \mathrm{d}x)$ to $L^2(\Xi, \mathrm{d}\xi)$ and then show that the resulting unitary operator intertwines the quasi regular representations of G on $L^2(X, \mathrm{d}x)$ and $L^2(\Xi, \mathrm{d}\xi)$. As already mentioned, this unitarization really means first composing (the closure of) $\mathcal{R}$ with a suitable pseudo-differential operator and then extending this composition to a unitary map, as it is done in the existing and well known precedecessors of this result [12], [13]. The techniques used in [1] cannot be transferred directly to the case of homogeneous trees primarily because the quasi regular representation is not irreducible, much less square integrable. Hence, we adopt here a combination of the classical approach followed by Helgason in the symmetric space case [11] and the techniques that have been exploited in [2]. The paper is organized in three sections. In Section 1, we present the main notions and the relevant results in the theory of homogeneous trees. Then, we give a brief overview of the Helgason-Fourier transform. In Section 2, we recall the horocyclic Radon transform on homogeneous trees, we present its link with the Helgason-Fourier transform and we show its intertwining properties with quasi regular representations. Finally, in Section 3, we prove the unitarization theorem for the horocyclic Radon transform.

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1 Preliminaries

In subsection 1.1, we recall the basic notions and results on homogeneous trees that will be used throughout and we focus on the space of horocycles. Subsection 1.2 is devoted to a brief overview of the Helgason-Fourier transform. We refer to [3], [7], [8] as standard references.

1.1 Homogeneous trees and horocycles

A *graph* is a pair $(X, \mathfrak{E})$, where X is the set of *vertices* and $\mathfrak{E}$ is the family of *edges*, where an edge is a two-element subset of X. We often think of an edge as a segment joining two vertices. If two vertices are joint by a segment, they are called *adjacent*. A *tree* is an undirected, connected, loop-free graph. In this paper we are interested in homogeneous trees. A *q-homogeneous* tree is a tree in which each vertex has exactly $q+1$ adjacent vertices. If $q \geqslant 1$, a q-homogeneous tree is infinite. From now on, we suppose $q \geqslant 2$ in order to exclude trivial cases, that is, segments and lines.

Given $u, v \in X$ with $u \neq v$, we denote by $[u, v]$ the unique ordered t-uple $(x_0 = u, x_1, \ldots, x_{t-1} = v) \in X^t$, where $\{x_i, x_{i+1}\} \in \mathfrak{E}$ and all the x_i are distinct. We call $[u, v]$ a (finite) *t-chain* and we think of it as a path starting at u and ending at v or, equivalently, as the finite sequence of consecutive 2-chains $[u, x_1], [x_1, x_2], \ldots, [x_{t-2}, v]$. With slight abuse of notation, if $[u, v] = (x_0, \ldots, x_{t-1})$ we write $u, v, x_i \in [u, v]$ and $[u, x_i] \cup [x_i, v]$, $i \in \{1, \ldots, t-2\}$. In particular, if u and v are adjacent, both $[u, v], [v, u] \in X^2$ are oriented, unlike the edge $\{u, v\} \in \mathfrak{E}$ which is not. A homogeneous tree X carries a natural distance $d\colon X \times X \to \mathbb{N}$, where for every $u, v \in X$ the distance $d(u, v)$ is the number of 2-chains in the path $[u, v]$.

Let G be the group of isometries on X. The group G is unimodular and locally compact, and acts transitively on X by the action

$$(g, x) \longmapsto g[x] := g(x), \quad g \in G.$$

We fix an arbitrary reference point $o \in X$ and we denote by K_o the corresponding stability group. Then, K_o is a maximal compact subgroup of G and $X \simeq G/K_o$ under the canonical isomorphism $gK_o \mapsto g[o]$. We endow X with the counting measure $\mathrm{d}x$ which is trivially G-invariant, and we denote by $L^2(X)$ the Hilbert space of square-integrable functions with respect to $\mathrm{d}x$. The group G acts on $L^2(X)$ by the quasi regular representation $\pi\colon G \longrightarrow \mathcal{U}(L^2(X))$ defined by

$$\pi(g)f(x) := f(g^{-1}[x]), \qquad f \in L^2(X),\, g \in G,$$

where $\mathcal{U}(L^2(X))$ denotes the group of unitary operators of $L^2(X)$. It is a well-known fact that π is not irreducible [8].

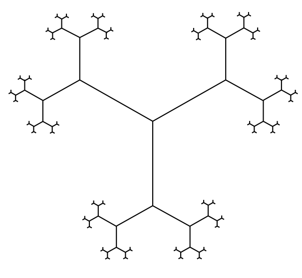

Figure 1: A portion of a 2-homogeneous tree

An *infinite chain* is an infinite sequence $(x_i)_{i\in\mathbb{N}}$ of vertices of X such that, for every $i \in \mathbb{N}$, $d(x_i, x_{i+1}) = 1$ and $x_i \neq x_{i+2}$. We denote by $c(X)$ the set of infinite chains on X. We say that two chains $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are equivalent if there exist $m \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $x_i = y_{i+m}$ for every $i \geqslant N$ and, in such case, we write $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$. The *boundary* of X is the space Ω of equivalence classes $c(X)/\sim$. We denote by p the canonical projection of $c(X)$ onto Ω and we write $[v,\omega) = (x_i)_{i\in\mathbb{N}}$ if $x_0 = v$ and $\omega = p((x_i)_{i\in\mathbb{N}})$. Furthermore, given $\omega_1, \omega_2 \in \Omega$ with $\omega_1 \neq \omega_2$, we denote by (ω_1, ω_2) the unique infinite sequence of vertices $(x_i)_{i\in\mathbb{Z}}$ such that $(x_{-i})_{i\in\mathbb{N}} \in \omega_1$ and $(x_i)_{i\in\mathbb{N}} \in \omega_2$, and we call it a doubly infinite chain.

A 2-chain $[v,u]$ is said to be positively oriented with respect to ω if $u \in [v,\omega)$, otherwise we say that $[v,u]$ is negatively oriented. For $\omega \in \Omega$ and $v,u \in X$, we denote by $\kappa_\omega(v,u) \in \mathbb{Z}$ the so-called *horocyclic index* of v and u w.r.t. ω, namely the number of positively oriented 2-chains (w.r.t. ω) in $[v,u]$ minus the number of negatively oriented 2-chains (w.r.t. ω) in $[v,u]$. Clearly, $|\kappa_\omega(v,u)| \leqslant d(v,u)$. It is easy to verify that, for every $v,u,x \in X$ and for every $\omega \in \Omega$,

$$\kappa_\omega(v,x) = \kappa_\omega(v,u) + \kappa_\omega(u,x). \tag{1}$$

Furthermore, we have the following result.

Proposition 1 ([7]). *Let $v \in X$ and $\omega \in \Omega$. If $[v,\omega) = (x_i)_{i\in\mathbb{N}}$, then for every $x \in X$*

$$\kappa_\omega(v,x) = \lim_{i\to\infty} (i - d(x,x_i)).$$

Proof. We fix $x \in X$ and we observe that

$$\lim_{i\to\infty} (i - d(x,x_i)) = \lim_{i\to\infty} (d(v,x_i) - d(x,x_i)).$$

Since $[v,\omega) \sim [x,\omega)$, then there exists $N \in \mathbb{N}$ such that $x_N \in [x,\omega)$ and $x_{N-1} \notin [x,\omega)$, with the understanding that if $v \in [x,\omega)$ then $N = 0$. Thus, for all $i \geqslant N$

$$d(v,x_i) - d(x,x_i) = d(v,x_N) - d(x,x_N)$$

and then

$$\lim_{i\to\infty} (d(v,x_i) - d(x,x_i)) = d(v,x_N) - d(x,x_N).$$

Furthermore, $[v,x] = [v,x_N] \cup [x_N,x]$, where $[v,x_N]$ is the union of positively oriented 2-chains and $[x_N,x]$ is the union of negatively oriented 2-chains. Hence,

$$\kappa_\omega(v,x) = d(v,x_N) - d(x,x_N) = \lim_{i\to\infty} (d(v,x_i) - d(x,x_i)) = \lim_{i\to\infty} (i - d(x,x_i))$$

and this concludes the proof. □

The boundary Ω is endowed with the compact topology (independent of the reference point o) generated by the open sets

$$\Omega(u) = \{\omega \in \Omega : u \in [o,\omega)\}, \qquad u \in X.$$

In every class $\omega \in \Omega$, there is a unique infinite chain $[o,\omega)$ starting at o and we denote by Γ_o the set of all infinite chains starting at o, i.e.

$$\Gamma_o = \{[o,\omega) : \omega \in \Omega\}.$$

It is easy to see that if $g \in G$ and $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$, then $(g[x_i])_{i\in\mathbb{N}} \sim (g[y_i])_{i\in\mathbb{N}}$ and this defines a transitive action of G on Ω. Indeed, if $(x_i)_{i\in\mathbb{N}} \in c(X)$, then $(g[x_i])_{i\in\mathbb{N}} \in c(X)$ as well, because

$$d(g[x_i], g[x_{i+1}]) = d(x_i, x_{i+1}) = 1$$

and $g[x_i] \neq g[x_{i+2}]$ since $g \in G$. Furthermore, $(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}}$ implies that there exist $m \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $d(g[x_i], g[y_{i+m}]) = d(x_i, y_{i+m}) = 0$ for every $i \geqslant N$ and then $(g[x_i])_{i\in\mathbb{N}} \sim (g[y_i])_{i\in\mathbb{N}}$. Precisely, the group G acts on Ω by the action

$$(g,\omega) \longmapsto g\cdot\omega := p((g[x_i])_{i\in\mathbb{N}}), \qquad \omega = p((x_i)_{i\in\mathbb{N}}).$$

This defines a transitive action of K_o on Γ_o given by

$$(k,[o,\omega)) \longmapsto [o, k\cdot\omega), \quad \omega \in \Omega.$$

We fix $\omega_0 \in \Omega$ and we denote by K_{o,ω_0} the stabilizer of $[o, \omega_0)$ in K_o, so that $\Gamma_o \simeq K_o/K_{o,\omega_0}$. Therefore, Γ_o admits a unique K_o-invariant probability measure μ^o, which is also G-quasi-invariant. We denote by ν^o the measure on Ω obtained as the push-forward of μ^o by means of the canonical projection $p_{|\Gamma_o} : \Gamma_o \to \Omega$. It has been shown in [8] that

$$\nu^o(\Omega(u)) = \frac{q}{(q+1)q^{d(o,u)}}, \qquad u \neq o.$$

By definition, the Poisson kernel $p_o(g, \omega)$ is the Radon-Nikodym derivative $d\nu^o(g^{-1} \cdot \omega)/d\nu^o(\omega)$, i.e.

$$\int_\Omega F(g^{-1} \cdot \omega) \mathrm{d}\nu^o(\omega) = \int_\Omega F(\omega) p_o(g^{-1}, \omega) \mathrm{d}\nu^o(\omega), \qquad F \in L^1(\Omega, \nu^o),\ g \in G, \tag{2}$$

and it is possible to prove [8] that

$$p_o(g, \omega) = q^{\kappa_\omega(o, g[o])}.$$

Since ν^o is K_o-invariant, we may write $p_o(gK_o, \omega)$ instead of $p_o(g, \omega)$. For every other choice of the reference vertex $v \in X$ the analogous objects $K_v, \Gamma_v, \mu_v, \nu_v, p_v$ can be introduced. It turns out that the measure ν^o is absolutely continuous with respect to ν^v. Precisely

$$\int_\Omega F(\omega) \mathrm{d}\nu^o(\omega) = \int_\Omega F(\omega) q^{\kappa_\omega(v,o)} \mathrm{d}\nu^v(\omega), \tag{3}$$

for every $F \in L^1(\Omega, \nu^o)$. Therefore, we can endow the boundary Ω with infinitely many measures which are absolutely continuous with respect to each other.

We are now in a position to introduce the space of horocycles.

Definition 2. For $\omega \in \Omega$, $v \in X$ and $n \in \mathbb{Z}$, the *horocycle* tangent to ω of index n with respect to the vertex v is the subset of X defined as

$$h^v_{\omega,n} = \{x \in X : \kappa_\omega(v, x) = n\}.$$

It follows immediately from (1) that for every $v, u \in X$, $n \in \mathbb{Z}$ and $\omega \in \Omega$

$$h^v_{\omega,n} = h^u_{\omega,n+\kappa_\omega(u,v)}. \tag{4}$$

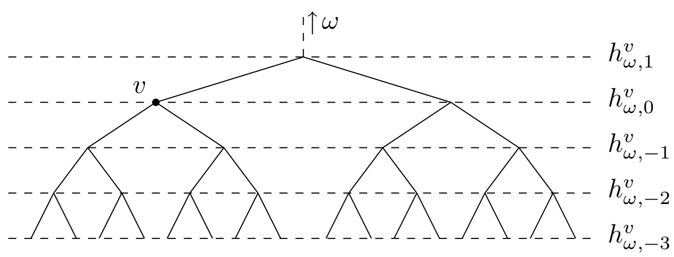

Figure 2: A part of a 2-homogeneous tree containing portions of horocycles (unions of vertices lying on dashed lines) which are tangent to ω.

We denote by Ξ the set of horocycles. The group G of isometries of X acts transitively on the space of horocycles Ξ through the action on vertices because the G-action maps horocycles in themselves. Indeed, if $\xi \in \Xi$, $\xi = h^v_{\omega,n}$, with $v \in X$, $\omega \in \Omega$, $n \in \mathbb{N}$ and

$[v, \omega) = (x_i)_{i\in\mathbb{N}}$, then for every $g \in G$

$$\begin{aligned} g[\xi] = \{g[x] : x \in X,\, \kappa_\omega(v,x) = n\} &= \{g[x] : x \in X,\, \lim_{i\to\infty}(i - d(x, x_i)) = n\} \\ &= \{x \in X : \lim_{i\to\infty}(i - d(g^{-1}[x], x_i)) = n\} \\ &= \{x \in X : \lim_{i\to\infty}(i - d(x, g[x_i])) = n\} \\ &= \{x \in X : \kappa_{g\cdot\omega}(g[v], x) = n\} \\ &= h^{g[v]}_{g\cdot\omega,n}, \end{aligned}$$

by Proposition 1. Therefore G acts transitively on Ξ by

$$(g, h^v_{\omega,n}) \longmapsto g.h^v_{\omega,n} := h^{g[v]}_{g\cdot\omega,n}.$$

Consider the horocycle

$$\xi_0 = h^o_{\omega_0,0} = \{x \in X : \kappa_{\omega_0}(o, x) = 0\}.$$

If $[o, \omega_0) = (x_i)_{i\in\mathbb{N}}$, then

$$g.\xi_0 = \{x \in X : \lim_{i\to\infty}(i - d(x, g[x_i])) = 0\}.$$

Hence, the isotropy at ξ_0 is $H = \bigcup_{j=0}^{\infty} H_j$, where H_j is the subgroup of isometries fixing the sub-path $[x_j, \omega_0) \in c(X)$. Therefore, $\Xi \simeq G/H$. Observe that H is the isotropy of G at $h^o_{\omega_0,n}$ for every $n \in \mathbb{Z}$. Thus, by (4) H is the isotropy of G at $h^v_{\omega_0,n}$ for every $n \in \mathbb{Z}$ and $v \in X$.

Let $\tau \in G$ be the one-step translation along (ω_1, ω_0), with $\omega_1 \in \Omega\backslash\{\omega_0\}$, where ω_0 is as in the definition of H, see [8] for further details on the one-step translations in G. Let $v \in (\omega_0, \omega_1)$ and assume $\tau(v) \in [v, \omega_0)$. Furthermore, we denote by A the subgroup of G generated by the powers of τ. It is easy to see that the group A acts on H by conjugation. Indeed, for every $m \in \mathbb{Z}$ and $g \in H$, g stabilizes every $h^v_{\omega_0,n}$ and hence

$$\tau^m g \tau^{-m}.\,\xi_0 = \tau^m g.\, h^{\tau^{-m}[o]}_{\omega_0,0} = \tau^m.\, h^{\tau^{-m}[o]}_{\omega_0,0} = h^{\tau^m\tau^{-m}[o]}_{\omega_0,0} = \xi_0,$$

where we use that $\tau^m \cdot \omega_0 = \tau^{-m} \cdot \omega_0 = \omega_0$. It has been proved in [14] that the resulting semidirect product $H \rtimes A$ has modular function

$$\Delta(h, \tau^m) = q^m.$$

With slight abuse of notation, we write $\Delta^{\frac{1}{2}}$ for the function $\Delta^{\frac{1}{2}} : \mathbb{Z} \to \mathbb{Z}$ defined by

$$\Delta^{\frac{1}{2}}(n) := q^{\frac{n}{2}},$$

and in what follows, the same notation is used for its trivial extension to $\Omega \times \mathbb{Z}$. The function $\Delta^{\frac{1}{2}}$ is the analogue of the function e^ρ in the theory of symmetric spaces (see [11]).

Following Helgason's approach, one could also realize the family of horocycles Ξ as follows. Once we have fixed the origin $o \in X$ and the closed subgroup H of G, we define the root horocycle ξ_0 as

$$\xi_0 = H[o] = h^o_{\omega_0,0}.$$

Then, for every $gH \in G/H$ we define

$$\xi = g[\xi_0] = gH[o],$$

which is independent of the choice of the representative g of $gH \in G/H$. By the definition of H, we have that

$$H = \{g \in G : g[\xi_0] = \xi_0\}. \tag{5}$$

Interchanging the roles of X and Ξ, we can define

$$\check{o} = K_o.\,\xi_0,$$

which is a closed K_o-invariant subset of Ξ by Lemma 1.1 in [12, Chapter II] and for every $x = gK_o \in G/K_o$ we set

$$\check{x} = g.\check{o} = gK_o.\xi_0,$$

which is independent of the choice of the representative g of $gK_o \in G/K_o$. By direct computation

$$\check{o} = \{h^o_{k\cdot\omega_0,0} : k \in K_o\},$$

which is the sheaf of horocycles passing through the origin $o \in X$. Similarly, $\check{x}$ is the sheaf of horocycles passing through $x \in X$. Furthermore, we can easily verify that

$$K_o = \{g \in G : g.\check{o} = \check{o}\}. \tag{6}$$

Conditions (5) and (6) are known as transversality conditions and (X, Ξ) is called a dual pair by Helgason [12, Chapter II]. The horocyclic Radon transform on homogeneous trees recalled in Section 2 is precisely the Radon transform *à la* Helgason between the dual pair (X, Ξ).

Clearly, the mapping $(v, \omega, n) \mapsto h^v_{\omega,n}$ is not injective and so Ξ is not well parametrized by $X \times \Omega \times \mathbb{Z}$. However, for fixed $v \in X$, the map $(\omega, n) \mapsto h^v_{\omega,n}$ is actually bijective and so Ξ may be identified with $\Omega \times \mathbb{Z}$. Formally, for every $v \in X$, there is a bijection

$$\Psi_v : \Omega \times \mathbb{Z} \to \Xi, \qquad \Psi_v(\omega, n) = h^v_{\omega,n}$$

and, for every fixed $\omega \in \Omega$, X can be covered disjointly as

$$X = \bigcup_{n\in\mathbb{Z}} h^v_{\omega,n}. \tag{7}$$

By equality (4), for each pair of vertices u, $v \in X$

$$\Psi_u^{-1} \circ \Psi_v(\omega, n) = (\omega, n + \kappa_\omega(u, v)).$$

So that, every function F on Ξ satisfies the relation

$$F \circ \Psi_v(\omega, n) = F \circ \Psi_u(\omega, n + \kappa_\omega(u, v)).$$

The topology that Ξ inherits as a product of Ω and $\mathbb{Z}$ is proved to be independent of the choice of $v \in X$. We denote by σ the measure on $\mathbb{Z}$ with density q^n with respect to the counting measure $\mathrm{d}n$. For every $v \in X$, we can endow Ξ with the measure λ obtained as the push-forward of the measure $\nu^v \otimes \sigma$ on $\Omega \times \mathbb{Z}$ by means of the map Ψ_v, i.e.

$$\lambda = \Psi_{v*}(\nu^v \otimes \sigma).$$

It turns out that λ is independent of the choice of the vertex v (see [3]). We denote by $L^1(\Xi)$ and $L^2(\Xi)$ the space of absolutely integrable functions and square-integrable functions with respect to λ, respectively. By definition of λ, for every $F \in L^1(\Xi)$

$$\int_\Xi F(\xi)\mathrm{d}\lambda(\xi) = \int_{\Omega\times\mathbb{Z}} (F \circ \Psi_v)(\omega, n)\mathrm{d}(\nu^v \otimes \sigma)(\omega, n) = \int_{\Omega\times\mathbb{Z}} (F \circ \Psi_v)(\omega, n)q^n\mathrm{d}\nu^v(\omega)\mathrm{d}n.$$

It is easy to verify that λ is G-invariant. So that, the group G acts on $L^2(\Xi)$ by the quasi regular representation $\hat{\pi} : G \longrightarrow \mathcal{U}(L^2(\Xi))$ defined by

$$\hat{\pi}(g)F(\xi) := F(g^{-1}.\xi), \qquad F \in L^2(\Xi),\ g \in G.$$

For every $v \in X$, let $L^2_v(\Omega \times \mathbb{Z})$ be the space of square-integrable function w.r.t. the measure $\nu^v \otimes \mathrm{d}n$. For every $F \in L^2(\Xi)$, we denote by $\Psi_v^* F$ the $(L^2(\Xi), L^2_v(\Omega \times \mathbb{Z}))$-pull-back of F by Ψ_v, i.e.

$$\Psi_v^* F(\omega, n) = (\Delta^{\frac{1}{2}} \cdot (F \circ \Psi_v))(\omega, n),$$

for almost every $(\omega, n) \in \Omega \times \mathbb{Z}$. Clearly, Ψ_v^* is a unitary operator from $L^2(\Xi)$ into $L_v^2(\Omega \times \mathbb{Z})$. Indeed, for every $F \in L^2(\Xi)$ we have that

$$\begin{aligned}\int_{\Omega\times\mathbb{Z}} |\Psi_v^* F(\omega, n)|^2 \, \mathrm{d}\nu^v(\omega)\mathrm{d}n &= \int_{\Omega\times\mathbb{Z}} \left|(\Delta^{\frac{1}{2}} \cdot (F \circ \Psi_v))(\omega, n)\right|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}n \\ &= \int_{\Omega\times\mathbb{Z}} |(F \circ \Psi_v)(\omega, n)|^2 \, q^n \mathrm{d}\nu^v(\omega)\mathrm{d}n \\ &= \int_{\Xi} |F(\xi)|^2 \, \mathrm{d}\lambda(\xi) = \|F\|^2_{L^2(\Xi)}\end{aligned}$$

and then Ψ_v^* is an isometry from $L^2(\Xi)$ into $L_v^2(\Omega \times \mathbb{Z})$. Surjectivity is also clear.

1.2 The Helgason-Fourier transform on homogeneous trees

The Helgason-Fourier transform can be defined on homogeneous trees (see [7], [8], [9]) in analogy with the setup of symmetric spaces [11]. We briefly recall its definition and its main features. We put $T = 2\pi/\log(q)$, $\mathbb{T} = \mathbb{R}/T\mathbb{Z} \simeq [0, T)$ and we denote by $\mathrm{d}t$ the normalized Lebesgue measure on $\mathbb{T}$. Let $C_c(X)$ be the space of compactly supported functions on X.

Definition 3. The *Helgason-Fourier transform* of $f \in C_c(X)$ with respect to the vertex $v \in X$ is the function $\mathcal{H}_v f : \Omega \times \mathbb{T} \longrightarrow \mathbb{C}$ defined by

$$\mathcal{H}_v f(\omega, t) = \sum_{x\in X} f(x) q^{(\frac{1}{2}+it)\kappa_\omega(v,x)}, \qquad (\omega, t) \in \Omega \times \mathbb{T}.$$

As the Euclidean Fourier transform, the Helgason-Fourier transform extends to a unitary operator on $L^2(X)$ (see [8], [9]). The Plancherel measure involves a version of the Harish-Chandra **c**-function inspired by the symmetric space construction [10], namely the meromorphic function

$$\mathbf{c}(z) = \frac{1}{q+1} \frac{q^{1-z} - q^{z-1}}{q^{-z} - q^{z-1}}, \qquad z \in \mathbb{C} \text{ with } q^{2z-1} \neq 1.$$

We put

$$c_q = \frac{q}{2(q+1)} \tag{8}$$

and we denote by $L^2_{v,\mathbf{c}}(\Omega \times \mathbb{T})^\sharp$ the space of functions F in $L^2_{v,\mathbf{c}}(\Omega \times \mathbb{T})$, the space of square-integrable functions on $\Omega \times \mathbb{T}$ w.r.t. the measure $c_q \, |\mathbf{c}(1/2 + it)|^{-2} \, \mathrm{d}\nu^v \mathrm{d}t$, satisfying the symmetry

$$\int_\Omega p_v(x, \omega)^{\frac{1}{2}-it} F(\omega, t)\mathrm{d}\nu^v(\omega) = \int_\Omega p_v(x, \omega)^{\frac{1}{2}+it} F(\omega, -t)\mathrm{d}\nu^v(\omega), \tag{9}$$

for every $x \in X$ and for almost every $t \in \mathbb{T}$.

Theorem 4 ([8])**.** *The Helgason-Fourier transform $\mathcal{H}_v$ extends to a unitary operator $\mathscr{H}_v$ from $L^2(X)$ onto $L^2_{v,\mathbf{c}}(\Omega \times \mathbb{T})^\sharp$.*

2 The horocyclic Radon transform

In this section we recall the definition of the horocyclic Radon transform on homogeneous trees and its fundamental properties. As already mentioned, the horocyclic Radon transform is precisely the Radon transform *à la* Helgason between the dual pair (X, Ξ). The case of homogeneous trees is not covered by the general setup considered in [1] by the authors since the quasi regular representation π of G on $L^2(X)$ is not irreducible. For this reason, we can not apply the results presented in [1] in order to obtain a unitarization theorem and we therefore adopt an approach which mimics the one used in [12] and [2] in the case of the polar and the affine Radon transforms, respectively.

Definition 5. The horocyclic Radon transform $\mathcal{R}f$ of a function $f \in C_c(X)$ is the map $\mathcal{R}f : \Xi \to \mathbb{C}$ defined by

$$\mathcal{R}f(\xi) = \sum_{x \in \xi} f(x).$$

We recall that for every $v \in X$ there exists a bijection $\Psi_v \colon \Omega \times \mathbb{Z} \to \Xi$ given by $(\omega, n) \mapsto h^v_{\omega,n}$ and we shall write $\mathcal{R}_v f = \mathcal{R}f \circ \Psi_v$.

Definition 6. Let $v \in X$. The Abel transform $\mathcal{A}_v f$ of a function $f \in C_c(X)$ is the map $\mathcal{A}_v f : \Omega \times \mathbb{Z} \to \mathbb{C}$ defined by

$$\mathcal{A}_v f(\omega, n) = \Psi_v^*(\mathcal{R}f)(\omega, n) = (\Delta^{\frac{1}{2}} \cdot \mathcal{R}_v f)(\omega, n).$$

We need to introduce the Fourier transform on $L^2(\mathbb{Z})$. We denote by L^2_T the space of T-periodic functions f on $\mathbb{R}$ such that

$$\|f\|^2_{L^2_T} = \int_0^T |f(t)|^2 \mathrm{d}t < +\infty.$$

Let $s \in L^2(\mathbb{Z})$, the Fourier transform $\mathcal{F}s$ of s is defined as the Fourier series of the T-periodic function with Fourier coefficients $(s(n))_{n \in \mathbb{Z}}$. Precisely,

$$\mathcal{F}s = \sum_{n \in \mathbb{Z}} s(n) q^{in\cdot},$$

where the series converges in L^2_T. The Parseval identity reads

$$\|\mathcal{F}s\|^2_{L^2_T} = \sum_{n \in \mathbb{Z}} |s(n)|^2.$$

Furthermore, if $s \in L^1(\mathbb{Z})$, for almost every $t \in \mathbb{T}$

$$\mathcal{F}s(t) = \sum_{n \in \mathbb{Z}} s(n) q^{int}.$$

We are now ready to state the result which relates the Helgason-Fourier transform with the horocyclic Radon transform. For the reader's convenience, we include the proof.

Proposition 7 ([4, 7])**.** *Let $v \in X$. For every $f \in C_c(X)$ and $\omega \in \Omega$, $\mathcal{A}_v f(\omega, \cdot) \in L^1(\mathbb{Z})$ and*

$$(I \otimes \mathcal{F})\mathcal{A}_v f(\omega, t) = \mathcal{H}_v f(\omega, t), \tag{10}$$

for almost every $t \in \mathbb{T}$.

Proof. Let $f \in C_c(X)$ and $\omega \in \Omega$. By formula (7)

$$\sum_{n \in \mathbb{Z}} |\mathcal{A}_v f(\omega, n)| = \sum_{n \in \mathbb{Z}} q^{\frac{n}{2}} |\mathcal{R}_v f(\omega, n)| \leqslant \sum_{n \in \mathbb{Z}} q^{\frac{n}{2}} \sum_{x \in h^v_{\omega,n}} |f(x)| = \sum_{x \in \mathrm{supp} f} |f(x)| q^{\frac{\kappa_\omega(v,x)}{2}} < +\infty.$$

Then, $\mathcal{A}_v f(\omega, \cdot)$ is in $L^1(\mathbb{Z})$ and applying again (7) we have that for almost every $t \in \mathbb{T}$

$$\begin{aligned}(I \otimes \mathcal{F})\mathcal{A}_v f(\omega, t) &= \sum_{n \in \mathbb{Z}} \mathcal{A}_v f(\omega, n) q^{itn} = \sum_{n \in \mathbb{Z}} q^{\frac{n}{2}} \mathcal{R}_v f(\omega, n) q^{itn} \\ &= \sum_{n \in \mathbb{Z}} q^{\frac{n}{2}} \sum_{x \in h^v_{\omega,n}} f(x) q^{itn} = \sum_{x \in X} f(x) q^{(\frac{1}{2}+it)\kappa_\omega(v,x)} = \mathcal{H}_v f(\omega, t),\end{aligned}$$

and this concludes the proof. □

We refer to Theorem 7 as the Fourier Slice Theorem for the horocyclic Radon transform in analogy with the polar Radon transform, see [12] as a classical reference.

Let $f \in C_c(X)$ and $v \in X$. By Parseval identity and Proposition 7 we have that

$$\begin{aligned}\int_{\Xi} |\mathcal{R}f(\xi)|^2 \mathrm{d}\lambda(\xi) &= \int_{\Omega\times\mathbb{Z}} |\Psi_v^*(\mathcal{R}f)(\omega,n)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}n \\ &= \int_{\Omega\times\mathbb{T}} |(I\otimes\mathcal{F})(\Psi_v^*(\mathcal{R}f))(\omega,t)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t \\ &= \int_{\Omega\times\mathbb{T}} |\mathcal{H}_v f(\omega,t)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t.\end{aligned}$$

Since f has finite support, then by the definition of the Helgason-Fourier transform, the inequality $|\kappa_\omega(v,x)| \leqslant d(v,x)$ and $\nu^v(\Omega) = 1$ the above leads to

$$\begin{aligned}\int_{\Xi} |\mathcal{R}f(\xi)|^2 \mathrm{d}\lambda(\xi) &= \int_{\Omega\times\mathbb{T}} |\sum_{x\in\mathrm{supp}f} f(x) q^{(\frac{1}{2}+it)\kappa_\omega(v,x)}|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t \\ &\leqslant \int_{\Omega} (\sum_{x\in\mathrm{supp}f} |f(x)| q^{\frac{\kappa_\omega(v,x)}{2}})^2 \mathrm{d}\nu^v(\omega) \\ &\leqslant \int_{\Omega} \sum_{x\in\mathrm{supp}f} |f(x)|^2 \sum_{x\in\mathrm{supp}f} q^{\kappa_\omega(v,x)} \mathrm{d}\nu^v(\omega) \\ &= \sum_{x\in\mathrm{supp}f} |f(x)|^2 \sum_{x\in\mathrm{supp}f} \int_{\Omega} q^{\kappa_\omega(v,x)} \mathrm{d}\nu^v(\omega) \\ &\leqslant \sum_{x\in\mathrm{supp}f} |f(x)|^2 \sum_{x\in\mathrm{supp}f} \int_{\Omega} q^{d(v,x)} \mathrm{d}\nu^v(\omega) \\ &= \sum_{x\in\mathrm{supp}f} |f(x)|^2 \sum_{x\in\mathrm{supp}f} q^{d(v,x)} < +\infty.\end{aligned}$$

Therefore, $\mathcal{R}f \in L^2(\Xi)$ for every $f \in C_c(X)$. The horocyclic Radon transform intertwines the regular representations of G on $L^2(X)$ and $L^2(\Xi)$. This result is a direct consequence of the fact that X and Ξ carry G-invariant measures $\mathrm{d}x$ and $\mathrm{d}\lambda$.

Proposition 8. *For every $g \in G$ and $f \in C_c(X)$*

$$\mathcal{R}(\pi(g)f) = \hat{\pi}(g)(\mathcal{R}f).$$

Proof. For all $g \in G$ and $f \in C_c(X)$

$$\mathcal{R}(\pi(g)f)(\xi) = \sum_{x\in\xi} f(g^{-1}[x]) = \sum_{y\in g^{-1}.\xi} f(y) = \hat{\pi}(g)(\mathcal{R}f)(\xi),$$

for every $\xi \in \Xi$. □

We now introduce a closed subspace of $L^2(\Xi)$ which will play a crucial role because it is the range of the unitarization of the horocyclic Radon transform.

Let $v \in X$. For every $F \in L^2(\Xi)$

$$\|F\|^2_{L^2(\Xi)} = \int_{\Omega} \sum_{n\in\mathbb{Z}} |\Psi_v^* F(\omega,n)|^2 \mathrm{d}\nu^v(\omega) < +\infty.$$

So that, the function $\Psi_v^* F(\omega,\cdot)$ is in $L^2(\mathbb{Z})$ for almost every $\omega \in \Omega$. Moreover, by Parseval identity and Fubini theorem

$$\begin{aligned}\|F\|^2_{L^2(\Xi)} &= \int_{\Omega\times\mathbb{Z}} |\Psi_v^* F(\omega,n)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}n \\ &= \int_{\Omega\times\mathbb{T}} |(I\otimes\mathcal{F})\Psi_v^* F(\omega,t)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t \\ &= \int_{\mathbb{T}}\int_{\Omega} |(I\otimes\mathcal{F})\Psi_v^* F(\omega,t)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t < +\infty.\end{aligned}$$

Then, for almost every $t \in \mathbb{T}$ the function $(I \otimes \mathcal{F})\Psi_v^* F(\cdot, t)$ is in $L^2(\Omega, \nu^v)$ and

$$|\int_\Omega (I \otimes \mathcal{F})\Psi_v^* F(\omega, t) \mathrm{d}\nu^v(\omega)| \leqslant \int_\Omega |(I \otimes \mathcal{F})\Psi_v^* F(\omega, t)| \mathrm{d}\nu^v(\omega) < +\infty.$$

We denote by $L^2_\flat(\Xi)$ the space of functions in $L^2(\Xi)$ satisfying the symmetry condition

$$\int_\Omega (I \otimes \mathcal{F})\Psi_v^* F(\omega, t) \mathrm{d}\nu^v(\omega) = \int_\Omega (I \otimes \mathcal{F})\Psi_v^* F(\omega, -t) \mathrm{d}\nu^v(\omega) \tag{11}$$

for every $v \in X$ and for almost every $t \in \mathbb{T}$.

Our main results in Section 3 are based on the following characterization of $L^2_\flat(\Xi)$. For every $v \in X$, we denote by $L^2_v(\Omega \times \mathbb{T})$ the space of square-integrable functions on $\Omega \times \mathbb{T}$ w.r.t. the measure $\nu^v \otimes \mathrm{d}t$.

Proposition 9. *Let $v \in X$. The operator Φ_v defined on $F \in L^2(\Xi)$ by*

$$\Phi_v F(\omega, t) = (I \otimes \mathcal{F})\Psi_v^* F(\omega, t) = (I \otimes \mathcal{F})(\Delta^{\frac{1}{2}} \cdot (F \circ \Psi_v))(\omega, t), \qquad a.e.\ (\omega, t) \in \Omega \times \mathbb{T},$$

is an isometry from $L^2(\Xi)$ into $L^2_v(\Omega \times \mathbb{T})$. Furthermore, for every other $u \in X$

$$\Phi_u F(\omega, t) = p_u(v, \omega)^{\frac{1}{2} + it} \Phi_v F(\omega, t), \tag{12}$$

for almost every $(\omega, t) \in \Omega \times \mathbb{T}$. Finally, a function F belongs to $L^2_\flat(\Xi)$ if and only if $\Phi_v F$ satisfies (9) for every $x \in X$ and almost every $t \in \mathbb{T}$.

By Proposition 9, $F \in L^2_\flat(\Xi)$ implies that $\Phi_v F$ satisfies (9) for every $v \in X$. Conversely, if we want to prove that a function $F \in L^2(\Xi)$ satisfies (11) it is enough to verify that (9) holds true for at least one, hence every, $v \in X$. This last remark will prove very useful in our proofs.

Proof. By Parseval identity, for every $F \in L^2(\Xi)$ we have that

$$\begin{aligned}\int_{\Omega \times \mathbb{T}} |\Phi_v F(\omega, t)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t &= \int_\Omega \int_\mathbb{T} |(I \otimes \mathcal{F})\Psi_v^* F(\omega, t)|^2 \mathrm{d}t\mathrm{d}\nu^v(\omega) \\ &= \int_{\Omega \times \mathbb{Z}} |\Psi_v^* F(\omega, n)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}n = \|F\|^2_{L^2(\Xi)},\end{aligned}$$

so that Φ_v is an isometry from $L^2(\Xi)$ into $L^2_v(\Omega \times \mathbb{T})$. Now, let $u \in X$ and $F \in L^2(\Xi)$. For almost every $\omega \in \Omega$ we have that

$$\begin{aligned}0 &= \lim_{N \to +\infty} \int_0^T |\sum_{n=-N}^{N} F \circ \Psi_u(\omega, n) q^{\frac{n}{2}} q^{int} - \Phi_u F(\omega, t)|^2 \mathrm{d}t \\ &= \lim_{N \to +\infty} \int_0^T |\sum_{n=-N}^{N} F \circ \Psi_v(\omega, n + \kappa_\omega(v, u)) q^{\frac{n}{2}} q^{int} - \Phi_u F(\omega, t)|^2 \mathrm{d}t \\ &= \lim_{N \to +\infty} \int_0^T |\sum_{m=-N+\kappa_\omega(v,u)}^{N+\kappa_\omega(v,u)} F \circ \Psi_v(\omega, m) q^{\frac{1}{2}(m - \kappa_\omega(v,u))} q^{it(m - \kappa_\omega(v,u))} - \Phi_u F(\omega, t)|^2 \mathrm{d}t \\ &= \lim_{N \to +\infty} \int_0^T |q^{(\frac{1}{2} + it)\kappa_\omega(u,v)} \sum_{m=-N+\kappa_\omega(v,u)}^{N+\kappa_\omega(v,u)} F \circ \Psi_v(\omega, m) q^{\frac{m}{2}} q^{imt} - \Phi_u F(\omega, t)|^2 \mathrm{d}t\end{aligned}$$

and, since

$$\Phi_v F(\omega, t) = \lim_{N \to +\infty} \sum_{m=-N+\kappa_\omega(v,u)}^{N+\kappa_\omega(v,u)} F \circ \Psi_v(\omega, m) q^{\frac{m}{2}} q^{imt}$$

in L^2_T, we conclude that relation (12) holds true. Finally, let $F \in L^2(\Xi)$. For every $x \in X$ and for almost every $t \in \mathbb{T}$, (12) yields

$$\begin{aligned}\int_\Omega p_v(x,\omega)^{\frac{1}{2}-it}\Phi_v F(\omega,t)\mathrm{d}\nu^v(\omega) &= \int_\Omega p_v(x,\omega)^{\frac{1}{2}-it}p_v(x,\omega)^{\frac{1}{2}+it}\Phi_x F(\omega,t)\mathrm{d}\nu^v(\omega)\\ &= \int_\Omega \Phi_x F(\omega,t)p_v(x,\omega)\mathrm{d}\nu^v(\omega)\\ &= \int_\Omega \Phi_x F(\omega,t)\mathrm{d}\nu^x(\omega).\end{aligned}$$

Then, for every $x \in X$ and almost every $t \in \mathbb{T}$

$$\int_\Omega p_v(x,\omega)^{\frac{1}{2}-it}\Phi_v F(\omega,t)\mathrm{d}\nu^v(\omega) = \int_\Omega (I\otimes\mathcal{F})\Psi_x^* F(\omega,t)\mathrm{d}\nu^x(\omega).$$

This equality allows us to conclude that $F \in L^2_\flat(\Xi)$ if and only if $\Phi_v F$ satisfies (9) and this concludes our proof.

□

Corollary 10. *For every $f \in C_c(X)$,*

$$\Phi_v(\mathcal{R}f) = \mathcal{H}_v f \tag{13}$$

in $L^2_v(\Omega\times\mathbb{T})$ and $\mathcal{R}f \in L^2_\flat(\Xi)$.

Proof. The proof follows immediately by Proposition 7 and the fact that the Helgason-Fourier transform satisfies (9). □

Some comments are in order. Proposition 9 with Corollary 10 show that $\mathcal{R}(C_c(X)) \subseteq L^2_\flat(\Xi)$ and it highlights the link between the range of the Radon transform with the range of the Helgason-Fourier transform, which will play a crucial role in our main result. The range $\mathcal{R}(C_c(X))$ has already been completely characterized in [6]. We recall the result in [6] for completeness and in order to understand the relation with $L^2_\flat(\Xi)$.

Theorem 11 (Theorem 1, [6])**.** *The range of the horocyclic Radon transform on the space of functions with finite support on X is the space of continuous compactly supported functions on Ξ satisfying the following two conditions*

(i) for some $v \in X$, hence for every $v \in X$, $\sum_{n\in\mathbb{Z}} F\circ\Psi_v(\omega,n)$ is independent of $\omega \in \Omega$;

(ii) for every $v \in X$ and $n \in \mathbb{Z}$

$$\int_\Omega \Psi_v^* F(\omega,n)\mathrm{d}\nu^v(\omega) = \int_\Omega \Psi_v^* F(\omega,-n)\mathrm{d}\nu^v(\omega). \tag{14}$$

It is worth observing that condition (11) is the equivalent on the frequency side of equation (14) for continuous compactly supported functions on Ξ. As it will be made clear in the next section, condition (11) better suits our needs.

3 Unitarization and Intertwining

In order to obtain the unitarization for the horocyclic Radon transform that we are after, we need some technicalities. Figure 3 below might help the reader to take track of all the spaces and operators involved in our construction.

Let $v \in X$. We set

$$\mathcal{D}_v = \{\varphi \in L^2_v(\Omega\times\mathbb{Z}) : (I\otimes\mathcal{F})\varphi \in L^2_{v,\boldsymbol{c}}(\Omega\times\mathbb{T})\}$$

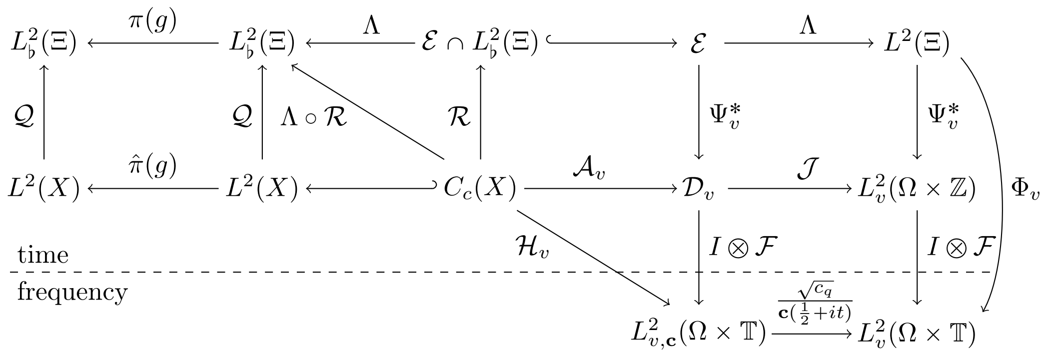

Figure 3: Spaces and operators that come into play in our construction.

and we define the operator $\mathcal{J}_v \colon \mathcal{D}_v \subseteq L^2_v(\Omega \times \mathbb{Z}) \to L^2_v(\Omega \times \mathbb{Z})$ as the Fourier multiplier

$$(I \otimes \mathcal{F})(\mathcal{J}_v \varphi)(\omega, t) = \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2} + it)\right|}(I \otimes \mathcal{F})\varphi(\omega, t), \quad \text{a.e. } (\omega, t) \in \Omega \times \mathbb{T},$$

where c_q is given by (8). We define the set of functions

$$\mathcal{E} = \{F \in L^2(\Xi) : \Phi_v F \in L^2_{v,\mathbf{c}}(\Omega \times \mathbb{T})\}$$

and we consider the operator $\Lambda \colon \mathcal{E} \subseteq L^2(\Xi) \to L^2(\Xi)$ given by

$$\Lambda F = \Psi_v^{*\,-1} \mathcal{J}_v \Psi_v^* F.$$

It is worth observing that the operator Λ is independent of the choice of the vertex $v \in X$ in its definition. Indeed, take $u \in X$ and put

$$\tilde{\Lambda} F = \Psi_u^{*\,-1} \mathcal{J}_u \Psi_u^* F.$$

We verify next that $\Lambda = \tilde{\Lambda}$. By Proposition 9, it is sufficient to prove that

$$\Phi_v(\tilde{\Lambda} F) = \Phi_v(\Lambda F)$$

for every $F \in L^2(\Xi)$. For almost every $(\omega, t) \in \Omega \times \mathbb{T}$ (12) yields

$$\begin{aligned}
\Phi_v(\tilde{\Lambda} F)(\omega, t) &= p_v(u, \omega)^{\frac{1}{2}+it} \Phi_u(\tilde{\Lambda} F)(\omega, t) \\
&= p_v(u, \omega)^{\frac{1}{2}+it} (I \otimes \mathcal{F})(\mathcal{J}_u \Psi_u^* F)(\omega, t) \\
&= p_v(u, \omega)^{\frac{1}{2}+it} \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2} + it)\right|}(I \otimes \mathcal{F})\Psi_u^* F(\omega, t) \\
&= \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2} + it)\right|}(I \otimes \mathcal{F})\Psi_v^* F(\omega, t) \\
&= (I \otimes \mathcal{F})(\mathcal{J}_v \Psi_v^* F)(\omega, t) = \Phi_v(\Lambda F)(\omega, t)
\end{aligned}$$

and we can conclude that $\Lambda = \tilde{\Lambda}$.

Let $v \in X$. As a direct consequence of the definition of Λ and $\mathcal{J}_v$, for every $F \in \mathcal{E}$ and for almost every $(\omega, t) \in \Omega \times \mathbb{T}$ we have that

$$\begin{aligned}
\Phi_v(\Lambda F)(\omega, t) &= (I \otimes \mathcal{F})(\mathcal{J}_v \Psi_v^* F)(\omega, t) \\
&= \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2} + it)\right|}(I \otimes \mathcal{F})(\Psi_v^* F)(\omega, t) \\
&= \frac{\sqrt{c_q}}{\left|\mathbf{c}(\frac{1}{2} + it)\right|}\Phi_v F(\omega, t). \qquad (15)
\end{aligned}$$

The operator Λ intertwines the regular representation $\hat{\pi}$ as shown by the next proposition.

Proposition 12. *The subspace $\mathcal{E}$ is $\hat{\pi}$-invariant and for all $F \in \mathcal{E}$ and $g \in G$*

$$\hat{\pi}(g)\Lambda F = \Lambda\hat{\pi}(g)F. \tag{16}$$

Proof. We consider $F \in \mathcal{E}$, $g \in G$ and we prove that $\hat{\pi}(g)F \in \mathcal{E}$. We observe that

$$\hat{\pi}(g)F \circ \Psi_v(\omega, n) = F \circ \Psi_{g^{-1}[v]}(g^{-1} \cdot \omega, n)$$

for almost every $(\omega, n) \in \Omega \times \mathbb{Z}$. Therefore, we have

$$\Psi_v^*(\hat{\pi}(g)F)(\omega, n) = \Psi_{g^{-1}[v]}^* F(g^{-1} \cdot \omega, n)$$

and consequently

$$\Phi_v(\hat{\pi}(g)F)(\omega, t) = \Phi_{g^{-1}[v]}F(g^{-1} \cdot \omega, t) \tag{17}$$

for almost every $(\omega, t) \in \Omega \times \mathbb{T}$. By equations (2), (3) and (17)

$$\begin{aligned}
&\int_{\Omega\times\mathbb{T}} |\Phi_v(\hat{\pi}(g)F)(\omega,t)|^2 \frac{c_q \mathrm{d}\nu^v(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} \\
&= \int_{\mathbb{T}}\int_{\Omega} |\Phi_{g^{-1}[v]}F(g^{-1}\cdot\omega,t)|^2 \frac{c_q \mathrm{d}\nu^v(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} \\
&= \int_{\mathbb{T}}\int_{\Omega} |\Phi_{g^{-1}[v]}F(\omega,t)|^2 p_v(g^{-1}[v],\omega) \frac{c_q \mathrm{d}\nu^v(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} \\
&= \int_{\Omega\times\mathbb{T}} |\Phi_{g^{-1}[v]}F(\omega,t)|^2 \frac{c_q \mathrm{d}\nu^{g^{-1}[v]}(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2} < +\infty
\end{aligned}$$

and we conclude that $\hat{\pi}(g)F \in \mathcal{E}$. We next prove the intertwining property (16). We have already observed that, by Proposition 9, it is enough to prove that

$$\Phi_v(\hat{\pi}(g)\Lambda F) = \Phi_v(\Lambda\hat{\pi}(g)F)$$

for every $g \in G$ and $F \in \mathcal{E}$. By equations (12), (15) and (17), for almost every $(\omega, t) \in \Omega\times\mathbb{T}$, we have the chain of equalities

$$\begin{aligned}
\Phi_v(\hat{\pi}(g)\Lambda F)(\omega,t) &= \Phi_{g^{-1}[v]}(\Lambda F)(g^{-1}\cdot\omega,t) \\
&= \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\Phi_{g^{-1}[v]}F(g^{-1}\cdot\omega,t) \\
&= \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2}+it)|}\Phi_v(\hat{\pi}(g)F)(\omega,t) = \Phi_v(\Lambda\hat{\pi}(g)F)(\omega,t),
\end{aligned}$$

which proves the intertwining relation. ☐

The next result follows directly by Proposition 9 and equation (15).

Corollary 13. *For every $F \in \mathcal{E}$, $\Lambda F \in L^2_\flat(\Xi)$ if and only if $F \in L^2_\flat(\Xi)$.*

Proof. By Proposition 9, $\Lambda F \in L^2_\flat(\Xi)$ if and only if $\Phi_v(\Lambda F)$ satisfies (9). By (15) and since $t \mapsto |\mathbf{c}(1/2+it)|$ is even, $\Phi_v(\Lambda F)$ satisfies (9) if and only if $\Phi_v(F)$ satisfies (9), which is equivalent to $F \in L^2_\flat(\Xi)$. This concludes the proof. ☐

We are now in a position to prove our main result.

Theorem 14. *The composite operator $\Lambda\mathcal{R}$ extends to a unitary operator*

$$\mathcal{Q}\colon L^2(X) \longrightarrow L^2_\flat(\Xi)$$

which intertwines the representations π and $\hat{\pi}$, i.e.

$$\hat{\pi}(g)\mathcal{Q} = \mathcal{Q}\pi(g), \qquad g \in G. \tag{18}$$

Theorem 14 implies that $\hat{\pi}$ is not irreducible, too.

Proof. We first show that $\Lambda\mathcal{R}$ extends to a unitary operator $\mathcal{Q}$ from $L^2(X)$ onto $L^2(\Xi)$. Let $f \in C_c(X)$ and $v \in X$. By the Fourier Slice Theorem (10), the Parseval identity and the definition of Λ, we have that

$$\begin{aligned}
\|f\|^2_{L^2(X)} &= \|\mathcal{H}_v f\|^2_{L^2_{v,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp} \\
&= \|(I\otimes\mathcal{F})(\Psi_v^*(\mathcal{R}f))\|^2_{L^2_{v,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp} \\
&= \int_{\Omega\times\mathbb{T}} |(I\otimes\mathcal{F})(\mathcal{J}_v\Psi_v^*(\mathcal{R}f))(\omega,t)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t \\
&= \int_{\Omega\times\mathbb{T}} |(I\otimes\mathcal{F})(\Psi_v^*(\Lambda\mathcal{R}f))(\omega,t)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}t \\
&= \int_{\Omega\times\mathbb{Z}} |\Psi_v^*(\Lambda\mathcal{R}f)(\omega,n)|^2 \mathrm{d}\nu^v(\omega)\mathrm{d}n \\
&= \|\Lambda\mathcal{R}f\|^2_{L^2(\Xi)}.
\end{aligned}$$

Hence, $\Lambda\mathcal{R}$ is an isometric operator from $C_c(X)$ into $L^2(\Xi)$. Since $C_c(X)$ is dense in $L^2(X)$, $\Lambda\mathcal{R}$ extends to a unique isometry from $L^2(X)$ onto the closure of $\mathrm{Ran}(\Lambda\mathcal{R})$ in $L^2(\Xi)$. We must show that $\Lambda\mathcal{R}$ has dense image in $L^2_\flat(\Xi)$. The inclusion $\mathrm{Ran}(\Lambda\mathcal{R}) \subseteq L^2_\flat(\Xi)$ follows immediately from Corollary 10 and Corollary 13. Let $F \in L^2_\flat(\Xi)$ be such that $\langle F, \Lambda\mathcal{R}f\rangle_{L^2(\Xi)} = 0$ for every $f \in C_c(X)$. By the Parseval identity and the Fourier Slice Theorem (10) we have that

$$\begin{aligned}
0 &= \langle F, \Lambda\mathcal{R}f\rangle_{L^2(\Xi)} \\
&= \int_{\Omega\times\mathbb{Z}} (F\circ\Psi_v)(\omega,n)\overline{(\Lambda\mathcal{R}f\circ\Psi_v)(\omega,n)}q^n \mathrm{d}\nu^v(\omega)\mathrm{d}n \\
&= \int_{\Omega\times\mathbb{Z}} (\Psi_v^*F)(\omega,n)\overline{(\mathcal{J}_v\Psi_v^*(\mathcal{R}f))(\omega,n)} \mathrm{d}\nu^v(\omega)\mathrm{d}n \\
&= \int_{\Omega\times\mathbb{T}} \Phi_v(F)(\omega,t)\overline{(I\otimes\mathcal{F})(\mathcal{J}_v\Psi_v^*(\mathcal{R}f))(\omega,t)} \mathrm{d}\nu^v(\omega)\mathrm{d}t \\
&= \int_{\Omega\times\mathbb{T}} \Phi_v(F)(\omega,t)\overline{(I\otimes\mathcal{F})(\Psi_v^*(\mathcal{R}f))(\omega,t)} \frac{\sqrt{c_q}\mathrm{d}\nu^v(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|} \\
&= \int_{\Omega\times\mathbb{T}} \frac{|\mathbf{c}(\frac{1}{2}+it)|}{\sqrt{c_q}} \Phi_v(F)(\omega,t)\overline{\mathcal{H}_v f(\omega,t)} \frac{c_q\mathrm{d}\nu^v(\omega)\mathrm{d}t}{|\mathbf{c}(\frac{1}{2}+it)|^2}.
\end{aligned}$$

For simplicity in the notation, we denote by $\Theta_v F$ the function on $\Omega\times\mathbb{T}$ defined as

$$\Theta_v F(\omega,t) = \frac{|\mathbf{c}(\frac{1}{2}+it)|}{\sqrt{c_q}}\Phi_v(F)(\omega,t), \qquad \text{a.e. } (\omega,t)\in\Omega\times\mathbb{T}.$$

Hence we have proved that $\langle\Theta_v F, \mathcal{H}_v f\rangle = 0$ for every $f \in C_c(X)$. The following two facts follow immediately by Proposition 9. Since Φ_v is an isometry from $L^2(\Xi)$ into $L^2_v(\Omega\times\mathbb{T})$, then $\Theta_v F$ belongs to $L^2_{v,\mathbf{c}}(\Omega\times\mathbb{T})$. Furthermore, since $F \in L^2_\flat(\Xi)$ and since $t \mapsto |\mathbf{c}(1/2+it)|$ is even, then $\Theta_v F \in L^2_{v,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$. By Theorem 4, $\mathcal{H}_v(C_c(X))$ is dense in $L^2_{v,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$. So that, $\Theta_v F = 0$ in $L^2_{v,\mathbf{c}}(\Omega\times\mathbb{T})^\sharp$ and then $\Phi_v(F) = 0$ in $L^2_v(\Omega\times\mathbb{T})$. Since Φ_v is an isometry from $L^2(\Xi)$ into $L^2_v(\Omega\times\mathbb{T})$, then $F = 0$ in $L^2(\Xi)$. Therefore, $\overline{\mathrm{Ran}(\Lambda\mathcal{R})} = L^2_\flat(\Xi)$ and $\Lambda\mathcal{R}$ extends uniquely to a surjective isometry

$$\mathcal{Q}\colon L^2(X) \longrightarrow L^2_\flat(\Xi).$$

Observe that $\mathcal{Q}f = \Lambda\mathcal{R}f$ for every $f \in C_c(X)$. Then, the intertwining property (18) follows immediately from Proposition 8 and Proposition 12. ∎

As a byproduct, one obtains an extended Fourier Slice Theorem.

Proposition 15 (Fourier Slice Theorem, version II)**.** *Let $v \in X$. For every $f \in L^2(X)$*

$$(I \otimes \mathcal{F})(\Psi_v^*(\mathcal{Q}f))(\omega, t) = \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|} \mathscr{H}_v f(\omega, t)$$

for almost every $(\omega, t) \in \Omega \times \mathbb{T}$.

Proof. Let $v \in X$. For every $f \in C_c(X)$, by (13) and (15) we have that

$$\begin{aligned}(I \otimes \mathcal{F})(\Psi_v^*(\mathcal{Q}f))(\omega, t) &= \Phi_v(\mathcal{Q}f)(\omega, t) \\ &= \Phi_v(\Lambda \mathcal{R} f)(\omega, t) \\ &= \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|} \Phi_v(\mathcal{R} f)(\omega, t) \\ &= \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|} \mathcal{H}_v f(\omega, t),\end{aligned}$$

for almost every $(\omega, t) \in \Omega \times \mathbb{T}$. Let $f \in L^2(X)$, since $C_c(X)$ is dense in $L^2(X)$, then there exists a sequence $(f_m)_m \subseteq C_c(X)$ such that $f_m \to f$ in $L^2(X)$. Then, since $\mathcal{Q}$ is a unitary operator from $L^2(X)$ onto $L^2_\flat(\Xi)$ and Φ_v is an isometry from $L^2(\Xi)$ into $L^2_v(\Omega \times \mathbb{T})$, then $\Phi_v(\mathcal{Q}f_m) \to \Phi_v(\mathcal{Q}f)$ in $L^2_v(\Omega \times \mathbb{T})$. Since $f_m \in C_c(X)$ for every $m \in \mathbb{N}$,

$$(I \otimes \mathcal{F})(\Psi_v^*(\mathcal{Q}f_m))(\omega, t) = \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|} \mathcal{H}_v f_m(\omega, t),$$

for almost every $(\omega, t) \in \Omega \times \mathbb{T}$. Hence, passing to a subsequence if necessary, for almost every $(\omega, t) \in \Omega \times \mathbb{T}$

$$\lim_{m \to +\infty} \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|} \mathcal{H}_v f_m(\omega, t) = (I \otimes \mathcal{F})(\Psi_v^*(\mathcal{Q}f))(\omega, t).$$

Therefore, passing to a subsequence if necessary, for almost every $(\omega, t) \in \Omega \times \mathbb{T}$

$$(I \otimes \mathcal{F})(\Psi_v^*(\mathcal{Q}f))(\omega, t) = \lim_{m \to +\infty} \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|} \mathcal{H}_v f_m(\omega, t) = \frac{\sqrt{c_q}}{|\mathbf{c}(\frac{1}{2} + it)|} \mathscr{H}_v f(\omega, t)$$

and this concludes our proof. □

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