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DIV-CURL PROBLEMS AND STREAM FUNCTIONS IN 3D LIPSCHITZ DOMAINS

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Abstract. We consider the problem of recovering the divergence-free velocity field $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of a given vorticity $\mathbf{F} = \operatorname{curl} \mathbf{U}$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. To that end, we solve the 'div-curl problem' for a given $\mathbf{F} \in \left[\mathbf{H}_0(\operatorname{curl}; \Omega)\right]'$. The solution is given in terms of a vector potential (or stream function) $\mathbf{A} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{U} = \operatorname{curl} \mathbf{A}$. After discussing existence and uniqueness of solutions and associated vector potentials, we propose a well-posed construction for the stream function. A numeral example of the construction is presented at the end.

Key words. div-curl system; stream function; vector potential; non-smooth domains

AMS subject classifications. 31B10; 35B65; 35C15; 35J56

1. Introduction.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Given a vorticity field $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^3$ defined over Ω, we are interested in solving the problem of *velocity recovery*:

$$\begin{cases} \operatorname{div} \mathbf{U} = 0 \\ \operatorname{curl} \mathbf{U} = \mathbf{F} \end{cases} \quad \text{in } \Omega. \tag{1.1}$$

This problem naturally arises in fluid mechanics when studying the vorticity formulation of the incompressible Navier–Stokes equations. Vortex methods, for example, are based on the vorticity formulation and require a solution of problem (1.1) in every time-step [8]. While our motivation lies in fluid dynamics, this 'div-curl problem' also is interesting in its own right.

On the whole space $\mathbb{R}^3$, this problem is a classical matter. Whenever $\mathbf{F}$ is smooth and compactly supported, the unique solution $\mathbf{U}$ of problem (1.1) that decays to zero at infinity is given by the Biot–Savart law [13, Proposition 2.16]. However, the case where Ω is a bounded domain is significantly more challenging.

In numerical simulations of the incompressible Navier–Stokes equations, it is common to fulfil the constraint $\operatorname{div} \mathbf{U} = 0$ only approximately, but it has recently been demonstrated that such a violation can cause significant instabilities. The importance for numerical methods to fulfil this constraint *exactly* was stressed by John et al. [12]. One way of achieving this requirement is the introduction of the *stream function*, or *vector potential*: instead of solving problem (1.1) directly, one seeks an approximation $\mathbf{A}_h$ of an auxiliary vector-field $\mathbf{A}$ such that $\mathbf{U} = \operatorname{curl} \mathbf{A}$. Because of the vector calculus identity $\operatorname{div} \circ \operatorname{curl} \equiv 0$, the velocity field $\mathbf{U}_h = \operatorname{curl} \mathbf{A}_h$ is always exactly divergence free.

1.1. Summary of Results.

In this work, we first collect results concerning the existence and uniqueness of the velocity fields $\mathbf{U}$ solving problem (1.1) and their associated stream functions $\mathbf{A}$. We will then derive a simple algorithm for solving problem (1.1) by means of a stream function $\mathbf{A}$. Stability will follow from the construction. Our results can be summarised as follows.

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1. **Existence of Velocity Fields.** (Theorem 3.1) Problem (1.1) has a solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ if and only if $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$ and $\langle \mathbf{F}, \mathbf{V}\rangle = 0$ for all $\mathbf{V} \in \mathbf{H}_0(\mathrm{curl};\Omega)$ with $\mathrm{curl}\,\mathbf{V} = \mathbf{0}$. In Lemma 3.4, we discuss equivalent alternative formulations of the latter condition.
2. **Existence of Stream Functions.** (Theorem 4.1) Let the velocity $\mathbf{U} \in \mathbf{L}^2(\Omega)$ solve problem (1.1). Then, $\mathbf{U}$ can be written in terms of a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ as $\mathbf{U} = \mathrm{curl}\,\mathbf{A}$ if and only if $\mathbf{U}$ fulfils $\int_{\Gamma_i} \mathbf{U}\cdot\mathbf{n}\,\mathrm{d}S = 0$ on each connected component Γ_i of the boundary $\Gamma := \partial\Omega$.
3. **Uniqueness.** (Theorems 3.7 and 4.2) If Ω is 'handle-free', the solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of problem (1.1) can be made unique by prescribing its normal trace $\mathbf{U}\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$. Moreover, if the prescribed boundary data fulfils $\int_{\Gamma_i} \mathbf{U}\cdot\mathbf{n}\,\mathrm{d}S = 0$ on each connected component $\Gamma_i \subset \Gamma$ of the boundary, there exist conditions that *uniquely* determine a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{U} = \mathrm{curl}\,\mathbf{A}$.
4. **Construction of Solutions.** (Section 5) The main novelty of this work lies in the explicit construction of solutions. Given a vorticity $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$ fulfilling the conditions of Item 1 and boundary data $\mathbf{U}\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$ fulfilling the conditions of Item 2, this construction will yield a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{U} = \mathrm{curl}\,\mathbf{A}$ solves problem (1.1). If the domain is handle-free, the obtained solution will be the uniquely defined stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$ from Item 3.
5. **Well-posedness.** (Theorem 5.3) From the structure of the construction one can directly infer its well-posedness. The functions $\mathbf{U}$ and $\mathbf{A}$ continuously depend on the given data $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$ and $\mathbf{U}\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$.
6. **Regularity.** (Theorem 6.2) If in addition to the above assumptions the given data fulfils $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\mathbf{U}\cdot\mathbf{n} \in L^2(\Gamma)$, then $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}(\Omega)$.

To the best of our knowledge, the existence result of Item 1 has not appeared in literature for a vorticity $\mathbf{F}$ of such low regularity. While the existence of stream functions $\mathbf{A} \in \mathbf{H}^1(\Omega)$ is already established, we are unaware of any previous results concerning their uniqueness or explicit construction. The $\mathbf{H}^{\frac{3}{2}}(\Omega)$ regularity of $\mathbf{A}$ will follow from classical results for the scalar Laplace equation.

1.2. Problematic Approaches. A naive approach for solving problem (1.1) relies on the observation that

$$-\boldsymbol{\Delta}\mathbf{U} = \mathrm{curl}(\underbrace{\mathrm{curl}\,\mathbf{U}}_{=\mathbf{F}}) - \nabla(\underbrace{\mathrm{div}\,\mathbf{U}}_{=0}) = \mathrm{curl}\,\mathbf{F}. \tag{1.2}$$

Based on this vector-calculus identity, it is tempting to solve three *decoupled* scalar Poisson problems $-\Delta U_i = (\mathrm{curl}\,\mathbf{F})_i$, $i = 1, 2, 3$, for the components of $\mathbf{U}$, say by prescribing the value of each one on the boundary. However, this approach is problematic. It is our aim to *integrate* $\mathbf{F}$, but instead this strategy asks that we *differentiate* first. Therefore, it needlessly requires to impose more regularity on the right-hand side. Moreover, there is no guarantee that its solution is divergence-free. Finally, since the tangential components of $\mathbf{U}$ allow us to compute $(\mathrm{curl}\,\mathbf{U})\cdot\mathbf{n}$ on the boundary, the boundary data must fulfil the compatibility condition $(\mathrm{curl}\,\mathbf{U})\cdot\mathbf{n} = \mathbf{F}\cdot\mathbf{n}$. We will later see that the solutions of problem (1.1) are *usually not* $\mathbf{H}^1(\Omega)$-regular, and thus the classic existence and uniqueness results in $H^1(\Omega)$ for the scalar Poisson problems $-\Delta U_i = (\mathrm{curl}\,\mathbf{F})_i$ are not applicable either.

Another straightforward approach assumes that $\mathbf{F} \in \mathbf{L}^2(\Omega)$. One may then extend $\mathbf{F}$ by zero to the whole space, yielding $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\mathbb{R}^3)$, and apply the Biot–Savart law to this extension. The normal trace $\mathbf{U} \cdot \mathbf{n}$ on Γ can then be prescribed by adding a suitable 'potential flow'. The problem here is that unless $\mathbf{F} \cdot \mathbf{n} = 0$ on the boundary, the zero extension $\widetilde{\mathbf{F}}$ will not be divergence-free. In this case the Biot–Savart law fails to yield the correct result. We will later see that this approach can in fact be fixed by introducing a suitable correction on the boundary.

1.3. Our Results in Context. A famous paper on vector potentials or stream functions in non-smooth domains is due to Amrouche, Bernardi, Dauge, and Girault [1]. They consider the closely related problem of finding a stream function $\mathbf{A}$ such that $\mathbf{U} = \operatorname{curl} \mathbf{A}$ for a *given velocity field* $\mathbf{U} \in \mathbf{L}^2(\Omega)$.

Clearly, the condition $\mathbf{U} = \operatorname{curl} \mathbf{A}$ alone does not uniquely determine $\mathbf{A}$: because of the vector calculus identity $\operatorname{curl} \circ (-\nabla) \equiv \mathbf{0}$, any gradient may be added to $\mathbf{A}$ without changing its curl. It is thus natural to enforce the gauge condition $\operatorname{div} \mathbf{A} = 0$. It is shown in the reference that under these hypotheses, a stream function $\mathbf{A}_1 \in \mathbf{H}^1(\Omega)$ satisfying $\operatorname{div} \mathbf{A}_1 = 0$ exists, but its boundary data is unknown. Amrouche et al. also prove the existence and uniqueness of a tangential stream function $\mathbf{A}_\mathrm{T} \in \mathbf{H}(\operatorname{curl}; \Omega) \cap \mathbf{H}_0(\operatorname{div}; \Omega)$ such that $\operatorname{div} \mathbf{A}_\mathrm{T} = 0$, and additionally describe a feasible finite element method for approximating $\mathbf{A}_\mathrm{T}$.

This is an interesting approach. Nevertheless, it has several drawbacks. In non-smooth domains, the tangential stream function $\mathbf{A}_\mathrm{T}$ may be significantly less regular than $\mathbf{A}_1$. In particular, functions from $\mathbf{H}(\operatorname{curl}; \Omega) \cap \mathbf{H}_0(\operatorname{div}; \Omega)$ may develop strong singularities near corners of the domain, which makes them difficult to approximate efficiently [2, Figure 1.3].

Our work builds on the above results and proposes natural conditions that uniquely determine a vector potential $\mathbf{A}_1 \in \mathbf{H}^1(\Omega)$ *without* explicitly involving boundary values of $\mathbf{A}_1$. We believe it is because previous approaches do prescribe boundary conditions like $\mathbf{A}_\mathrm{T} \cdot \mathbf{n} = 0$ that they yield less regular stream functions.

The stream function $\mathbf{A}_1$ can be explicitly constructed. The algorithm utilises the Newton operator: the bulk of the work lies in the explicit computation of a volume integral. Two companion *scalar* elliptic equations must also be solved on the boundary of the domain, but these are easily tackled using standard methods.

2. Basic Definitions and Notions.

2.1. Spaces Defined on Volumes. We denote by $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ the space of smooth compactly supported functions in Ω, and write $\mathcal{D}'(\Omega)$ for the space of distributions. Their vector-valued analogues $\boldsymbol{\mathcal{D}}(\Omega) := \left(C_0^\infty(\Omega)\right)^3$ and $\boldsymbol{\mathcal{D}}'(\Omega)$ are distinguished by a **bold** font. In the whole space $\mathbb{R}^3$, we will make use of the space of smooth functions $\mathcal{E}(\mathbb{R}^3) := C^\infty(\mathbb{R}^3)$ and its dual $\mathcal{E}'(\mathbb{R}^3)$—the space of compactly supported distributions. Their vector-valued analogues will be denoted by $\boldsymbol{\mathcal{E}}(\mathbb{R}^3)$ and $\boldsymbol{\mathcal{E}}'(\mathbb{R}^3)$, respectively.

We write $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ for the Hilbert spaces of square integrable scalar and vector-valued functions defined over Ω. $H^s(\Omega)$ and $\mathbf{H}^s(\Omega)$, $s > 0$, refer to the corresponding Sobolev spaces. The spaces $H_0^s(\Omega)$ and $\mathbf{H}_0^s(\Omega)$ are defined as the closures of $\mathcal{D}(\Omega)$ and $\boldsymbol{\mathcal{D}}(\Omega)$ in $H^s(\Omega)$ and $\mathbf{H}^s(\Omega)$, respectively. For $s < 0$ we set $H^s(\Omega) := \left[H_0^{-s}(\Omega)\right]'$ and $\mathbf{H}^s(\Omega) := \left[\mathbf{H}_0^{-s}(\Omega)\right]'$. We will always identify $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ with their duals, i. e., $\left[L^2(\Omega)\right]' = L^2(\Omega)$ and $\left[\mathbf{L}^2(\Omega)\right]' = \mathbf{L}^2(\Omega)$.

All differential operators are to be understood in the distributional sense. For example, for $\mathbf{U} \in \boldsymbol{\mathcal{D}}'(\Omega)$ the distribution $\operatorname{div} \mathbf{U} \in \mathcal{D}'(\Omega)$ is defined by:

$$\forall V \in \mathcal{D}(\Omega) : \langle \operatorname{div} \mathbf{U}, V \rangle := \langle \mathbf{U}, -\nabla V \rangle. \tag{2.1}$$

It is well-known that when one restricts the domain of a differential operator, the space of permitted test-functions for its range can be uniquely enlarged by continuity. Continuing with the example, for $\mathbf{U} \in \mathbf{L}^2(\Omega)$ one obtains:

$$\forall V \in \mathcal{D}(\Omega): \ \langle \operatorname{div} \mathbf{U}, V \rangle = \int_\Omega \mathbf{U} \cdot (-\nabla V) \, \mathrm{d}\mathbf{x} \leq \|\mathbf{U}\|_{\mathbf{L}^2(\Omega)} \|V\|_{H^1(\Omega)}. \tag{2.2}$$

In this case, the distribution $\operatorname{div} \mathbf{U} \in \mathcal{D}'(\Omega)$ can be uniquely extended by continuity to $\overline{\mathcal{D}(\Omega)}^{H^1(\Omega)} = H_0^1(\Omega)$. It is in this sense that we view $\operatorname{div} \mathbf{U} \in H^{-1}(\Omega)$ and write

$$\operatorname{div}\big|_{\mathbf{L}^2(\Omega)} : \mathbf{L}^2(\Omega) \to H^{-1}(\Omega). \tag{2.3}$$

We will often allow the different restrictions of the differential operators under consideration to carry the same name when their domains and ranges are clear.

The Hilbert spaces

$$\begin{aligned} \mathbf{H}(\operatorname{div}; \Omega) &:= \{\mathbf{U} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{U} \in L^2(\Omega)\}, \\ \mathbf{H}(\operatorname{curl}; \Omega) &:= \{\mathbf{U} \in \mathbf{L}^2(\Omega) \mid \operatorname{curl} \mathbf{U} \in \mathbf{L}^2(\Omega)\}, \end{aligned} \tag{2.4}$$

are equipped with the norms

$$\begin{aligned} \|\mathbf{U}\|^2_{\mathbf{H}(\operatorname{div};\Omega)} &:= \|\mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{div} \mathbf{U}\|^2_{L^2(\Omega)}, \\ \|\mathbf{U}\|^2_{\mathbf{H}(\operatorname{curl};\Omega)} &:= \|\mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \|\operatorname{curl} \mathbf{U}\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned} \tag{2.5}$$

respectively. Related 'homogeneous spaces' are defined as

$$\begin{aligned} \mathbf{H}_0(\operatorname{curl}; \Omega) &:= \overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\mathbf{H}(\operatorname{curl};\Omega)}, \\ \mathbf{H}_0(\operatorname{div}; \Omega) &:= \overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\mathbf{H}(\operatorname{div};\Omega)}. \end{aligned} \tag{2.6}$$

We refer to Amrouche et al. for a detailed exposition of the regularity and compactness properties of these spaces [1, Sections 2.2 and 2.3]. The definitions of this subsection will also be used with Ω replaced by $\mathbb{R}^3$.

2.2. Geometry. Throughout this article, we suppose that the Lipschitz domain $\Omega \subset \mathbb{R}^3$ of interest is bounded and connected. The so-called Betti numbers are defined by the dimensions of the cohomology groups associated to the de Rham complex

$$0 \to H^1(\Omega) \xrightarrow{-\nabla} \mathbf{H}(\operatorname{curl}; \Omega) \xrightarrow{\operatorname{curl}} \mathbf{H}(\operatorname{div}; \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega) \to 0 \tag{2.7}$$

as

$$\begin{aligned} &\beta_0 := \dim\left(\ker(-\nabla)\right), \\ &\beta_1 := \dim\left(\ker(\operatorname{curl}) / {-\nabla}\left(H^1(\Omega)\right)\right), \\ &\beta_2 := \dim\left(\ker(\operatorname{div}) / \operatorname{curl}\left(\mathbf{H}(\operatorname{curl}; \Omega)\right)\right). \end{aligned} \tag{2.8}$$

As a consequence of de Rham's theorem, these numbers are related to the topological properties of the domain [2, Theorem 2.2]. Since the kernel of the gradient is the space of constant functions, the zeroth Betti number indicates the number of connected components of Ω, which is therefore always assumed to be 1 in this work. β_1

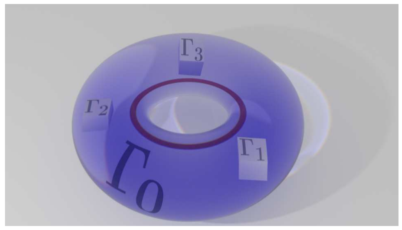

FIGURE 1. *A ring-shaped domain with three cubical holes is an example of a domain having non-trivial topology [1, Section 3]. There is one 'handle' through it: $\beta_1 = 1$. The red line is a representative of the equivalence class of non-bounding cycles. The three cubical inclusions ('holes') are not part of the domain: $\beta_2 = 3$. The boundary Γ has four connected components: $\Gamma_1, \Gamma_2, \Gamma_3$, and the domain's exterior boundary Γ_0.*

is the genus of the domain, in other words the number of 'handles'. β_2 is the number of 'holes' Θ_i, $i = 1, \dots, \beta_2$ in the domain. We define the *exterior domain* Θ_0 via:

$$\Theta_0 := \mathbb{R}^3 \setminus \overline{\Omega \cup \left(\bigcup_{i=1}^{\beta_2} \Theta_i \right)}. \tag{2.9}$$

The domain's boundary thus always has $\beta_2 + 1$ connected components $\Gamma_i := \partial\Theta_i$, $i = 0, \dots, \beta_2$. A domain with $\beta_1 = 0$ is called *handle-free*, *hole-free* if $\beta_2 = 0$, and we say that the topology of Ω is trivial or simple if both $\beta_1 = \beta_2 = 0$. We refer to Arnold and al. for more details [3, Section 2].

Their geometric interpretation is best illustrated through an example. In the domain depicted in Figure 1, $\beta_2 = 3$: three cube-like holes Θ_1, Θ_2, and Θ_3 were cut out of the toroidal volume. Their boundaries Γ_1, Γ_2, and Γ_3 are labelled in the figure. Together with the exterior boundary Γ_0, the boundary $\Gamma := \partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ thus has *four* $= \beta_2 + 1$ connected components.

On the one hand, the value of the second Betti number $\beta_2 \in \mathbb{N}$ is relevant to questions regarding the *existence* results stated in Item 1 and Item 2 of Section 1. These existence theorems will make use of arbitrary but fixed functions $T_i \in C_0^\infty(\mathbb{R}^3)$, $i = 0, \dots, \beta_2$, that act as indicators for the the connected components of the boundary:

$$T_i = \begin{cases} 1 & \text{in a neighbourhood of } \Gamma_i, \\ 0 & \text{in a neighbourhood of } \Gamma_j,\ j \in \{0, \dots, \beta_2\} \setminus \{i\}. \end{cases} \tag{2.10}$$

On the other hand, the value of β_1 is crucial to the *uniqueness* results of Item 3. For simplicity, we will restrict our attention to handle-free domains ($\beta_1 = 0$), that is domains for which every loop inside Ω is the boundary of a surface within Ω. The domain in Figure 1 is *not* handle-free, as the red loop is a representative of the equivalence class of non-bounding cycles. In that example, $\beta_1 = 1$. Nevertheless, we will make some remarks on what changes in the following results need to be anticipated in order to recover uniqueness of solutions when $\beta_1 > 0$.

2.3. Laplace and Newton Operator. The importance of the scalar Laplace operator $-\Delta := \operatorname{div} \circ (-\nabla)$ for potential theory is obvious, and we assume that the reader is well aware of the classical existence and uniqueness results for boundary value problems of the scalar Laplace equation. Its vector-valued analogue is defined component-wise: $-\boldsymbol{\Delta}\mathbf{V} := (-\Delta V_1, -\Delta V_2, -\Delta V_3)^\top$ for all $\mathbf{V} \in \boldsymbol{\mathcal{D}}'(\Omega)$. This operator is also known as Hodge–Laplacian and can equivalently be written as:

$$-\boldsymbol{\Delta}\mathbf{V} = \operatorname{curl}\operatorname{curl}\mathbf{V} - \nabla\operatorname{div}\mathbf{V}, \qquad \forall\mathbf{V} \in \boldsymbol{\mathcal{D}}'(\Omega). \tag{2.11}$$

The Newton potential is an inverse to the Laplacian on the whole space $\mathbb{R}^3$, and it will play a key role throughout this work. Let us denote by $G(\mathbf{x}) := (4\pi|\mathbf{x}|)^{-1}$ the fundamental solution of the Laplacian $-\Delta$ on the whole space $\mathbb{R}^3$, that is

$$-\Delta G(\mathbf{x}) = -\Delta\left(\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\right) = \delta, \tag{2.12}$$

where δ is the standard Dirac distribution centered at the origin. The Newton operator then is defined on the space of compactly supported distributions $\mathcal{E}'(\mathbb{R}^3)$ by convolution with G:

$$\mathcal{N} : \mathcal{E}'(\mathbb{R}^3) \to \mathcal{D}'(\mathbb{R}^3), \qquad U \mapsto G \star U. \tag{2.13}$$

In other words, for $U \in \mathcal{D}(\mathbb{R}^3)$ we have:

$$\big(\mathcal{N}U\big)(\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{U(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}\mathbf{y} \qquad \in \mathcal{E}(\mathbb{R}^3), \tag{2.14}$$

and for $U \in \mathcal{E}'(\mathbb{R}^3)$:

$$\langle\mathcal{N}U, V\rangle = \langle U, \mathcal{N}V\rangle, \qquad \forall V \in \mathcal{D}(\mathbb{R}^3). \tag{2.15}$$

Its vector-valued analogue $\boldsymbol{\mathcal{N}}$ is defined component-wise. For a given $U \in \mathcal{E}'(\mathbb{R}^3)$, the distribution $\mathcal{N}U$ is called the Newton potential of U.

This operator is an inverse for the Laplacian [11, Equations (4.4.2) and (4.4.3)]:

$$\forall U \in \mathcal{E}'(\mathbb{R}^3): \qquad -\Delta\mathcal{N}U = \mathcal{N}(-\Delta U) = U. \tag{2.16}$$

Moreover, because it is an operator of convolutional type, it commutes with differentiation [11, Equation (4.2.5)]. With these properties, it is an easy task to derive the Helmholtz decomposition.

Lemma 2.1 (Helmholtz decomposition). *Every compactly supported distribution* $\mathbf{U} \in \boldsymbol{\mathcal{E}}'(\mathbb{R}^3)$ *can be decomposed into a divergence-free and a curl-free part:*

$$\mathbf{U} = \operatorname{curl}\mathbf{A} - \nabla P, \tag{2.17}$$

where $\mathbf{A} := \boldsymbol{\mathcal{N}}\operatorname{curl}\mathbf{U} \in \boldsymbol{\mathcal{D}}'(\mathbb{R}^3)$ *and* $P := \mathcal{N}\operatorname{div}\mathbf{U} \in \mathcal{D}'(\mathbb{R}^3)$.

Proof. Using the above properties one readily obtains:

$$\begin{aligned}\mathbf{U} &= -\boldsymbol{\Delta}\boldsymbol{\mathcal{N}}\mathbf{U}\\ &= \operatorname{curl}(\operatorname{curl}\boldsymbol{\mathcal{N}}\mathbf{U}) - \nabla(\operatorname{div}\boldsymbol{\mathcal{N}}\mathbf{U})\\ &= \operatorname{curl}(\boldsymbol{\mathcal{N}}\operatorname{curl}\mathbf{U}) - \nabla(\mathcal{N}\operatorname{div}\mathbf{U}).\end{aligned} \tag{2.18}$$

∎

Application of this operator always increases the Sobolev regularity of a distribution $U \in H^s(\mathbb{R}^3) \cap \mathcal{E}'(\mathbb{R}^3)$ by two, that is $\mathcal{N}$ has the following mapping property and is continuous [16, Theorem 3.1.2]:

$$\mathcal{N} : H^s(\mathbb{R}^3) \cap \mathcal{E}'(\mathbb{R}^3) \to H^{s+2}_{\mathrm{loc}}(\mathbb{R}^3), \qquad s \in \mathbb{R}, \tag{2.19}$$

where

$$H^s_{\mathrm{loc}}(\mathbb{R}^3) := \left\{ U \in \mathcal{D}'(\mathbb{R}^3) \,\middle|\, U|_K \in H^s(K),\ \forall K \subset \mathbb{R}^3 \text{ compact} \right\}. \tag{2.20}$$

An analogous result for $\boldsymbol{\mathcal{N}}$ follows by component-wise application of the scalar mapping property.

The Newton potential $\mathcal{N}U$ of a compactly supported distribution $U \in \mathcal{E}'(\mathbb{R}^3)$ is regular outside $\operatorname{supp} U$ and decays to zero at infinity [9, Chapter II, §3.1, Proposition 2]:

$$\forall U \in \mathcal{E}'(\mathbb{R}^3): \ \big(\mathcal{N}U\big)(\mathbf{x}) = \mathcal{O}\left(|\mathbf{x}|^{-1}\right), \qquad |\mathbf{x}| \to \infty. \tag{2.21}$$

Even more importantly, the following result characterises the Newton potential [9, Chapter II, §3.1, Proposition 3].

Lemma 2.2. *Let $U \in \mathcal{D}'(\mathbb{R}^3)$ and $F \in \mathcal{E}'(\mathbb{R}^3)$. Then U is the Newton potential of F, that is $U = \mathcal{N}F$, if and only if:*

$$\begin{cases} -\Delta U = F & \text{on } \mathbb{R}^3, \\ U(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{2.22}$$

This characterisation allows for the derivation of representation formulæ for solutions of the Laplace equation on bounded domains, leading to boundary integral equations. This will appear at the end of this section. Again, analogous results hold for the vector-valued Newton operator $\boldsymbol{\mathcal{N}}$.

2.4. Conventional Trace Spaces. We collect some classic results to fix the notation. For a more complete discussion of the trace spaces, the reader is referred to the books of Sauter and Schwab [16, Chapter 2] and McLean [15, Chapter 3] and the references therein.

The boundary of a Lipschitz domain admits a surface measure S, allowing the definition of trace spaces $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma)$ that consist of square-integrable scalar and vector-valued fields. We again identify these spaces with their respective duals $\left[L^2(\Gamma)\right]' = L^2(\Gamma)$, $\left[\mathbf{L}^2(\Gamma)\right]' = \mathbf{L}^2(\Gamma)$, and use the notation $\langle \cdot, \cdot \rangle_\Gamma$ for their duality pairing on Γ.

As a consequence of Rademacher's theorem, the boundary of a Lipschitz domain admits an atlas of regularity $W^{1,\infty}$. For $s \in [0, 1]$ this allows for intrinsic definitions of Sobolev trace spaces $H^s(\Gamma)$ and $\mathbf{H}^s(\Gamma)$. Their duals are again denoted by $H^{-s}(\Gamma)$ and $\mathbf{H}^{-s}(\Gamma)$.

For functions $V \in C(\overline{\Omega})$ the boundary trace γV is defined as the restriction of V to Γ:

$$\gamma V := V|_\Gamma. \tag{2.23}$$

This operator admits a unique continuous extension to Sobolev spaces $H^s(\Omega)$ and is bounded [16, Theorem 2.6.8]:

$$\gamma : H^{s+\frac{1}{2}}(\Omega) \to H^s(\Gamma), \quad \|\gamma V\|_{H^s(\Gamma)} \le C \|V\|_{H^{s+\frac{1}{2}}(\Omega)}, \qquad s \in (0, 1), \tag{2.24}$$

where $C > 0$ is a constant that only depends on s and Ω. Analogous results hold for the vector-valued trace operator $\boldsymbol{\gamma}$.

The trace operator is surjective, and it is possible to construct bounded and continuous right-inverses γ^{-1} [16, Theorem 2.6.11]:

$$\gamma^{-1} : H^s(\Gamma) \to H^{s+\frac{1}{2}}(\mathbb{R}^3), \quad \|\gamma^{-1} v\|_{H^{s+\frac{1}{2}}(\mathbb{R}^3)} \le C \|v\|_{H^s(\Gamma)}, \qquad s \in (0, 1), \tag{2.25}$$

where $C > 0$ again is a constant that only depends on s and Ω. Such right-inverses moreover fulfil $\|\gamma^{-1}v|_\Omega\|_{H^{s+\frac{1}{2}}(\Omega)} \le C\|v\|_{H^s(\Gamma)}$ and are sometimes called *lifting maps*. Again, analogous results hold for the vector-valued liftings $\boldsymbol{\gamma}^{-1}$.

2.5. Normal Traces. It is guaranteed from Rademacher's theorem that the boundary of a Lipschitz domain has an essentially bounded unit normal vector field $\mathbf{n} \in \mathbf{L}^\infty(\Gamma)$, directed towards the exterior of Ω [16, Theorem 2.7.1]. This allows us to define the normal component an $\nu\mathbf{u}$ of arbitrary boundary vector-field $\mathbf{u} \in \mathbf{L}^2(\Gamma)$:

$$\nu : \mathbf{L}^2(\Gamma) \to L^2(\Gamma),\ \mathbf{u} \mapsto \mathbf{u} \cdot \mathbf{n}. \tag{2.26}$$

By abuse of notation, we will also write ν for the composition $\nu \circ \boldsymbol{\gamma}$. For sufficiently smooth functions $\mathbf{U}$ and P on Ω a divergence theorem holds [16, Theorem 2.7.3]:

$$\langle \nu\mathbf{U}, \gamma P\rangle_\Gamma = \int_\Omega \mathbf{U} \cdot \nabla P\, + P \operatorname{div} \mathbf{U} \,\mathrm{d}\mathbf{x}. \tag{2.27}$$

For $p \in H^{\frac{1}{2}}(\Gamma)$ arbitrary, we can thus write:

$$\langle \nu\mathbf{U}, p\rangle_\Gamma = \int_\Omega \mathbf{U} \cdot \nabla(\gamma^{-1}p)\, + (\gamma^{-1}p) \operatorname{div} \mathbf{U} \,\mathrm{d}\mathbf{x}. \tag{2.28}$$

By means of this formula we see that the operator ν admits a unique and continuous extension to $\mathbf{H}(\operatorname{div};\Omega)$, that is:

$$\nu : \mathbf{H}(\operatorname{div};\Omega) \to H^{-\frac{1}{2}}(\Gamma). \tag{2.29}$$

This operator is surjective, and its kernel is exactly the space $\mathbf{H}_0(\operatorname{div};\Omega)$. It is common to write $\mathbf{U} \cdot \mathbf{n}$ instead of $\nu\mathbf{U}$ where applicable.

2.6. Tangential Traces. The theory of tangential trace spaces for Lipschitz domains has been developed by Buffa et al. [5] Here we will recall the ingredients of their theory that we will need later on. For $\mathbf{u} \in \mathbf{L}^2(\Gamma)$, one defines its tangential trace $\boldsymbol{\tau}\mathbf{u}$ and rotated tangential trace $\boldsymbol{\rho}\mathbf{u}$ via:

$$\begin{aligned} \boldsymbol{\tau} &: \mathbf{L}^2(\Gamma) \to \mathbf{L}^2(\Gamma), \quad \mathbf{u} \mapsto \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n} = \mathbf{n} \times (\mathbf{u} \times \mathbf{n}), \\ \boldsymbol{\rho} &: \mathbf{L}^2(\Gamma) \to \mathbf{L}^2(\Gamma), \quad \mathbf{u} \mapsto \mathbf{u} \times \mathbf{n} = -\mathbf{n} \times \boldsymbol{\tau}\mathbf{u}. \end{aligned} \tag{2.30}$$

Because of the normal vector's lack of regularity, one will usually only have $\boldsymbol{\tau}\mathbf{u} \in \mathbf{L}^2(\Gamma)$, $\boldsymbol{\rho}\mathbf{u} \in \mathbf{L}^2(\Gamma)$ and $\nu\mathbf{u} \in L^2(\Gamma)$, even if $\mathbf{u} \in \mathbf{H}^1(\Gamma)$. This makes it difficult to assess the regularity of tangential traces using the intrinsic definitions of trace Sobolev spaces. Of particular importance is the case of regularity $\frac{1}{2}$, and so one instead defines:

$$\begin{aligned} \mathbf{H}_\mathrm{T}^{\frac{1}{2}}(\Gamma) &:= \{\boldsymbol{\tau}\mathbf{v} \mid \mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)\}, \\ \mathbf{H}_\mathrm{R}^{\frac{1}{2}}(\Gamma) &:= \{\boldsymbol{\rho}\mathbf{v} \mid \mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)\}, \end{aligned} \tag{2.31}$$

with associated quotient norms:

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{H}_\mathrm{T}^{\frac{1}{2}}(\Gamma)} &:= \inf_{\boldsymbol{\tau}\mathbf{v}=\mathbf{u}} \|\mathbf{v}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}, \\ \|\mathbf{u}\|_{\mathbf{H}_\mathrm{R}^{\frac{1}{2}}(\Gamma)} &:= \inf_{\boldsymbol{\rho}\mathbf{v}=\mathbf{u}} \|\mathbf{v}\|_{\mathbf{H}^{\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{2.32}$$

These are Banach spaces, and it is important to note that these spaces will usually differ, unless the domain Ω is smooth. They are, however, isomorphic to one another, through the relation $\boldsymbol{\rho}\mathbf{u} = -\mathbf{n} \times \boldsymbol{\tau}\mathbf{u}$.

For reasons that become more clear below, it is a useful convention to define their normed duals as:

$$\begin{aligned} \mathbf{H}_{\mathrm{T}}^{-\frac{1}{2}}(\Gamma) &:= \big[\mathbf{H}_{\mathrm{R}}^{\frac{1}{2}}(\Gamma)\big]', \\ \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\Gamma) &:= \big[\mathbf{H}_{\mathrm{T}}^{\frac{1}{2}}(\Gamma)\big]'. \end{aligned} \tag{2.33}$$

We allow ourselves to reuse the symbols $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ for the composite maps $\boldsymbol{\tau} \circ \boldsymbol{\gamma}$ and $\boldsymbol{\rho} \circ \boldsymbol{\gamma}$. For $\mathbf{U} \in \mathbf{H}^1(\Omega)$, $\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$, one can show that the following integration by parts formulæ hold [5, Equation (27)]:

$$\begin{aligned} \langle \boldsymbol{\tau}\mathbf{U}, \boldsymbol{\rho}\mathbf{v} \rangle_\Gamma &= \int_\Omega \big(\boldsymbol{\gamma}^{-1}\mathbf{v}\big) \cdot \operatorname{curl} \mathbf{U} - \mathbf{U} \cdot \operatorname{curl}\big(\boldsymbol{\gamma}^{-1}\mathbf{v}\big) \,\mathrm{d}\mathbf{x}, \\ \langle \boldsymbol{\rho}\mathbf{U}, \boldsymbol{\tau}\mathbf{v} \rangle_\Gamma &= \int_\Omega \mathbf{U} \cdot \operatorname{curl}\big(\boldsymbol{\gamma}^{-1}\mathbf{v}\big) - \big(\boldsymbol{\gamma}^{-1}\mathbf{v}\big) \cdot \operatorname{curl} \mathbf{U} \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{2.34}$$

Via these formulæ one can see that the composite operators $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ admit unique continuous extensions to the space $\mathbf{H}(\mathrm{curl};\Omega)$:

$$\begin{aligned} \boldsymbol{\tau} &: \mathbf{H}(\mathrm{curl};\Omega) \to \mathbf{H}_{\mathrm{T}}^{-\frac{1}{2}}(\Gamma), \\ \boldsymbol{\rho} &: \mathbf{H}(\mathrm{curl};\Omega) \to \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\Gamma), \end{aligned} \tag{2.35}$$

making clear the reasoning behind convention (2.33). The kernel of these operators is exactly the space $\mathbf{H}_0(\mathrm{curl};\Omega)$. It is common to write $\mathbf{U} \times \mathbf{n}$ for $\boldsymbol{\rho}\mathbf{U}$ and $\mathbf{n} \times (\mathbf{U} \times \mathbf{n})$ for $\boldsymbol{\tau}\mathbf{U}$ where applicable.

These operators, however, are *not surjective*. One of the main results of the work of Buffa et al. is the precise characterisation of their ranges [5]. This will require some more tools, that will be described in the following subsection.

2.7. Trace Calculus. As mentioned before, the boundary of a Lipschitz domain admits an atlas of regularity $W^{1,\infty}$. This allows for the direct definition of the *trace gradient* $-\nabla_\Gamma : H^1(\Gamma) \to \mathbf{L}^2(\Gamma)$ through differentiation of the charts of the atlas [5, Equation (21)]. The *trace divergence* div_Γ then is defined as the operator dual to the trace gradient: $\mathrm{div}_\Gamma := (-\nabla_\Gamma)'$. Their composition yields the Laplace–Beltrami operator $-\Delta_\Gamma := \mathrm{div}_\Gamma \circ (-\nabla_\Gamma)$, which continuously maps $H^1(\Gamma) \to H^{-1}(\Gamma)$.

The *trace curl* $\mathbf{curl}_\Gamma : H^1(\Gamma) \to \mathbf{L}^2(\Gamma)$ is defined as the rotated gradient: $\mathbf{curl}_\Gamma := \mathbf{n} \times (-\nabla_\Gamma)$. Denoting its dual operator by curl_Γ, one can verify that the Laplace–Beltrami operator can also be written as $-\Delta_\Gamma = \mathrm{curl}_\Gamma \circ \mathbf{curl}_\Gamma$.

We will later make use of the fact that this operator is coercive on $H^1(\Gamma)/\mathbb{R}$. In other words, for any $f \in H^{-1}(\Gamma)$ that satisfies $\langle f, 1 \rangle_{\Gamma_i} = 0$, $i = 0, \ldots, \beta_2$, the Laplace–Beltrami equation $-\Delta_\Gamma q = f$ has a unique solution $q \in H^1(\Gamma)/\mathbb{R}$. This solution continuously depends on f: $\|q\|_{H^1(\Gamma)} \leq C_\Gamma \|f\|_{H^{-1}(\Gamma)}$.

These operators can be extended in several ways. It can be shown that for $U \in H^2(\Omega)$ one has $\gamma U \in H^1(\Gamma)$, and furthermore that the surface gradient satisfies: [5, Proposition 3.4]

$$-\nabla_\Gamma \gamma U = \boldsymbol{\tau}(-\nabla U). \tag{2.36}$$

On the one hand, the right side of this equation is also well-defined for $U \in H^1(\Omega)$, because one then has $-\nabla U \in \mathbf{H}(\mathrm{curl};\Omega)$. Through this relation $-\nabla_\Gamma$ and $\mathbf{curl}_\Gamma$ can be extended to operators acting on $H^{\frac{1}{2}}(\Gamma)$:

$$\begin{aligned} -\nabla_\Gamma &: H^{\frac{1}{2}}(\Gamma) \to \mathbf{H}_{\mathrm{T}}^{-\frac{1}{2}}(\Gamma), \quad u \mapsto \boldsymbol{\tau}\big(-\nabla(\gamma^{-1}u)\big), \\ \mathbf{curl}_\Gamma &: H^{\frac{1}{2}}(\Gamma) \to \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\Gamma), \quad u \mapsto \mathbf{n} \times (-\nabla_\Gamma u), \end{aligned} \tag{2.37}$$

giving rise to restricted dual operators div_Γ and curl_Γ:

$$\begin{aligned} \mathrm{div}_\Gamma &: \mathbf{H}_{\mathrm{R}}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \quad \langle \mathrm{div}_\Gamma \mathbf{u}, v\rangle_\Gamma := \langle \mathbf{u}, -\nabla_\Gamma v\rangle_\Gamma, \\ \mathrm{curl}_\Gamma &: \mathbf{H}_{\mathrm{T}}^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \quad \langle \mathrm{curl}_\Gamma \mathbf{u}, v\rangle_\Gamma := \langle \mathbf{u}, \mathbf{curl}_\Gamma v\rangle_\Gamma. \end{aligned} \tag{2.38}$$

All these operators are bounded [5, Proposition 3.6].

On the other hand, one may define $H^{\frac{3}{2}}(\Gamma) := \gamma(H^2(\Omega))$, equipped with the quotient norm and normed dual $H^{-\frac{3}{2}}(\Gamma)$. The restrictions of $-\nabla_\Gamma$ and $\mathbf{curl}_\Gamma$ to $H^{\frac{3}{2}}(\Gamma)$ then are continuous as operators [5, Proposition 3.4]:

$$\begin{aligned} -\nabla_\Gamma &: H^{\frac{3}{2}}(\Gamma) \to \mathbf{H}_{\mathrm{T}}^{\frac{1}{2}}(\Gamma), \\ \mathbf{curl}_\Gamma &: H^{\frac{3}{2}}(\Gamma) \to \mathbf{H}_{\mathrm{R}}^{\frac{1}{2}}(\Gamma), \end{aligned} \tag{2.39}$$

giving rise to extended dual operators:

$$\begin{aligned} \mathrm{div}_\Gamma &: \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma), \quad \langle \mathrm{div}_\Gamma \mathbf{u}, v\rangle_\Gamma := \langle \mathbf{u}, -\nabla_\Gamma v\rangle_\Gamma, \\ \mathrm{curl}_\Gamma &: \mathbf{H}_{\mathrm{T}}^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{3}{2}}(\Gamma), \quad \langle \mathrm{curl}_\Gamma \mathbf{u}, v\rangle_\Gamma := \langle \mathbf{u}, \mathbf{curl}_\Gamma v\rangle_\Gamma. \end{aligned} \tag{2.40}$$

One can show that for all $\mathbf{U} \in \mathbf{H}(\mathrm{curl};\Omega)$ it holds that [5, Equation (30)]:

$$\mathrm{curl}_\Gamma \boldsymbol{\tau}\mathbf{U} = \mathbf{n} \cdot \mathrm{curl}\,\mathbf{U}, \tag{2.41}$$

justifying the notation. But because $\mathrm{curl}\,\mathbf{U} \in \mathbf{H}(\mathrm{div};\Omega)$, we also have $\mathbf{n} \cdot \mathrm{curl}\,\mathbf{U} \in H^{-\frac{1}{2}}(\Gamma)$. This gives rise to the following definitions:

$$\begin{aligned} \mathbf{H}_{\mathrm{T}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma) &:= \{\mathbf{s} \in \mathbf{H}_{\mathrm{T}}^{-\frac{1}{2}}(\Gamma) \mid \mathrm{curl}_\Gamma \mathbf{s} \in H^{-\frac{1}{2}}(\Gamma)\}, \\ \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma) &:= \{\mathbf{s} \in \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\Gamma) \mid \mathrm{div}_\Gamma \mathbf{s} \in H^{-\frac{1}{2}}(\Gamma)\}. \end{aligned} \tag{2.42}$$

From (2.41) it then follows that the ranges of the trace maps $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ from (2.35) can be narrowed down to:

$$\begin{aligned} \boldsymbol{\tau} &: \mathbf{H}(\mathrm{curl};\Omega) \to \mathbf{H}_{\mathrm{T}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma;\Gamma), \\ \boldsymbol{\rho} &: \mathbf{H}(\mathrm{curl};\Omega) \to \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma). \end{aligned} \tag{2.43}$$

It is one of the main results of Buffa et al. that these operators are indeed surjective [5, Theorem 4.1]. This allowed them to derive Hodge decompositions for these spaces, which can be seen as trace analogues of the Helmholtz decomposition in Lemma 2.1. We will only make use of the decomposition for $\mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$.

LEMMA 2.3 (Hodge Decomposition [5, Theorem 5.5]). *Let $\Gamma = \partial\Omega$ be the boundary of a handle-free, bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. Then any $\mathbf{s} \in \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma)$ can be uniquely decomposed as:*

$$\mathbf{s} = \mathbf{curl}_\Gamma\, p - \nabla_\Gamma\, q, \tag{2.44}$$

where $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ and $q \in H^1(\Gamma)/\mathbb{R}$ with $-\Delta_\Gamma\, q \in H^{-\frac{1}{2}}(\Gamma)$ are uniquely determined up to a constant on each connected component Γ_i, $i = 0, \ldots, \beta_2$ of the boundary.

2.8. Trace Jumps and a Representation Formula. The trace operators introduced in the previous sections were all defined with respect to the domain Ω. One can instead also consider the corresponding traces with respect to the complementary domain $\Omega^C := \mathbb{R}^3 \setminus \overline{\Omega}$. These *one-sided* traces exist whenever the restriction of a vector field $\mathbf{U} \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ to the domains Ω and Ω^C is sufficiently smooth. If $\mathbf{U}$ is smooth *across* Γ, then the one-sided traces coincide. Otherwise the difference of these traces is denoted by the jump operator $[\![\cdot]\!]$. For example, for $\boldsymbol{\rho}$ one writes: $[\![\boldsymbol{\rho}\mathbf{U}]\!] := \boldsymbol{\rho}\mathbf{U}|_\Omega - \boldsymbol{\rho}^C\mathbf{U}|_{\Omega^C}$. The importance of the jump operator lies in the following representation formula.

LEMMA 2.4. *Let* $\mathbf{U} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ *fulfil:*

$$\begin{cases} -\boldsymbol{\Delta}\mathbf{U} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \Gamma, \\ \operatorname{div} \mathbf{U} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{U}(\mathbf{x}) \to \mathbf{0} & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{2.45}$$

Then $\mathbf{U} = \boldsymbol{\mathcal{N}}(-\boldsymbol{\Delta}\mathbf{U})$ *and* $-\boldsymbol{\Delta}\mathbf{U} = \boldsymbol{\tau}'[\![\boldsymbol{\rho}\operatorname{curl}\mathbf{U}]\!]$.

Proof. The fact that $\mathbf{U} = \boldsymbol{\mathcal{N}}(-\boldsymbol{\Delta}\mathbf{U})$ directly follows from Lemma 2.2. Because $\operatorname{div}\mathbf{U} = 0$ globally, we have $-\boldsymbol{\Delta}\mathbf{U} = \operatorname{curl}(\operatorname{curl}\mathbf{U})$ and $\operatorname{curl}(\operatorname{curl}\mathbf{U})|_{\mathbb{R}^3\setminus\Gamma} = \mathbf{0}$. We thus obtain for all $\mathbf{V} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$:

$$\langle -\boldsymbol{\Delta}\mathbf{U}, \mathbf{V}\rangle = \langle \operatorname{curl}\mathbf{U}, \operatorname{curl}\mathbf{V}\rangle = \int_\Omega \operatorname{curl}\mathbf{U}\cdot\operatorname{curl}\mathbf{V}\,\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3\setminus\overline{\Omega}} \operatorname{curl}\mathbf{U}\cdot\operatorname{curl}\mathbf{V}\,\mathrm{d}\mathbf{x}, \tag{2.46}$$

where we used that $\operatorname{curl}\mathbf{U} \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ since $\mathbf{U} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$. Now, by definition of the rotated tangential trace:

$$\langle \boldsymbol{\rho}\operatorname{curl}\mathbf{U}, \boldsymbol{\tau}\mathbf{V}\rangle_\Gamma = \int_\Omega \operatorname{curl}\mathbf{U}\cdot\operatorname{curl}\mathbf{V} - \underbrace{\operatorname{curl}(\operatorname{curl}\mathbf{U})}_{=\mathbf{0}}\cdot\mathbf{V}\,\mathrm{d}\mathbf{x} = \int_\Omega \operatorname{curl}\mathbf{U}\cdot\operatorname{curl}\mathbf{V}\,\mathrm{d}\mathbf{x}. \tag{2.47}$$

Applying the same methodology to the integral over $\mathbb{R}^3 \setminus \overline{\Omega}$ and using the definition of $\boldsymbol{\rho}^C$ yields the desired result, because the fact that $\mathbf{V}$ is smooth across Γ guarantees that $[\![\boldsymbol{\tau}\mathbf{V}]\!] = \mathbf{0}$. ∎

3. Velocity Fields. In this section we prove the existence of velocity fields solving (1.1) as claimed in Item 1. The abstract integrability condition is reformulated in Lemma 3.4. The uniqueness result for velocity fields presented in Item 3 is also covered.

3.1. Existence of Velocity Fields.

THEOREM 3.1. *Suppose that* $\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz domain and let* $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$. *The div-curl system*

$$\begin{cases} \operatorname{curl}\mathbf{U} = \mathbf{F} \\ \operatorname{div}\mathbf{U} = 0 \end{cases} \quad \text{in } \Omega \tag{3.1}$$

has a solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ *if and only if* $\mathbf{F}$ *fulfils the following integrability condition:*

$$\langle \mathbf{F}, \mathbf{V}\rangle = 0 \qquad \forall \mathbf{V} \in \mathbf{K}(\Omega), \tag{3.2}$$

where

$$\mathbf{K}(\Omega) := \{\mathbf{V} \in \mathbf{H}_0(\mathrm{curl};\Omega) \mid \operatorname{curl}\mathbf{V} = \mathbf{0}\}. \tag{3.3}$$

Remark 3.2. The condition $\mathbf{F} \in \big[\mathbf{H}_0(\mathrm{curl};\Omega)\big]'$ is natural. To see this, note that for an arbitrary vector-field $\mathbf{U} \in \mathbf{L}^2(\Omega)$ it holds that

$$\forall \mathbf{V} \in \boldsymbol{\mathcal{D}}(\Omega): \quad \langle \mathrm{curl}\,\mathbf{U}, \mathbf{V}\rangle = \int_\Omega \mathbf{U}\cdot \mathrm{curl}\,\mathbf{V}\,\mathrm{d}\mathbf{x} \leq \|\mathbf{U}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{V}\|_{\mathbf{H}(\mathrm{curl};\Omega)}. \tag{3.4}$$

The distribution $\mathrm{curl}\,\mathbf{U} \in \boldsymbol{\mathcal{D}}'(\Omega)$ thus admits a unique continuous extension to $\overline{\boldsymbol{\mathcal{D}}(\Omega)}^{\mathbf{H}(\mathrm{curl};\Omega)} = \mathbf{H}_0(\mathrm{curl};\Omega)$ and the associated operator

$$\mathrm{curl}\,\big|_{\mathbf{L}^2(\Omega)} : \mathbf{L}^2(\Omega) \to \big[\mathbf{H}_0(\mathrm{curl};\Omega)\big]' \tag{3.5}$$

is continuous. Therefore, any solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of (3.1) must necessarily fulfil $\mathbf{F} = \mathrm{curl}\,\mathbf{U} \in \big[\mathbf{H}_0(\mathrm{curl};\Omega)\big]'$ and this is the reason why we *cannot* simply demand $\mathbf{F} \in \mathbf{H}^{-1}(\Omega)$.

The integrability condition (3.2) will be enlightened by the alternative formulations discussed at the end of this section.

LEMMA 3.3. *Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain and let $\mathbf{F} \in [\mathbf{H}_0(\mathrm{curl};\Omega)]'$. The equation*

$$\mathrm{curl}\,\mathbf{W} = \mathbf{F} \tag{3.6}$$

has a solution $\mathbf{W} \in \mathbf{L}^2(\Omega)$ if and only if $\mathbf{F}$ fulfils the integrability condition (3.2).

Proof. The continuous operator

$$\mathrm{curl}\,\big|_{\mathbf{H}_0(\mathrm{curl};\Omega)} : \mathbf{H}_0(\mathrm{curl};\Omega) \to \mathbf{L}^2(\Omega) \tag{3.7}$$

has closed range [2, Box 3.1]. The curl operator is symmetric and the dual of the mapping (3.7) is the operator $\mathrm{curl}\,\big|_{\mathbf{L}^2(\Omega)}$ given in (3.5). Hence, Banach's closed range theorem yields

$$\mathrm{Range}\left(\mathrm{curl}\,\big|_{\mathbf{L}^2(\Omega)}\right) = \left(\ker \mathrm{curl}\,\big|_{\mathbf{H}_0(\mathrm{curl};\Omega)}\right)^0 = \big(\mathbf{K}(\Omega)\big)^0. \tag{3.8}$$

That is,

$$\mathrm{Range}\left(\mathrm{curl}\,\big|_{\mathbf{L}^2(\Omega)}\right) = \left\{\mathbf{F} \in \big[\mathbf{H}_0(\mathrm{curl};\Omega)\big]' \,\middle|\, \langle \mathbf{F}, \mathbf{V}\rangle = 0 \;\forall\, \mathbf{V} \in \mathbf{K}(\Omega)\right\}. \tag{3.9}$$

Evidently, problem (3.6) has a solution if and only if $\mathbf{F} \in \mathrm{Range}\left(\mathrm{curl}\,\big|_{\mathbf{L}^2(\Omega)}\right)$. This is precisely the statement of the Lemma. □

Proof of Theorem 3.1. Lemma 3.3 guarantees the existence of a $\mathbf{W} \in \mathbf{L}^2(\Omega)$ such that $\mathrm{curl}\,\mathbf{W} = \mathbf{F}$. This function does not necessarily fulfil $\mathrm{div}\,\mathbf{W} = 0$. But in this case we let $P \in H_0^1(\Omega)$ denote the unique solution to the Poisson problem

$$-\Delta P = \mathrm{div}\,\mathbf{W} \qquad \text{in } \Omega, \tag{3.10}$$

and note that $\mathbf{U} := \mathbf{W} - \nabla P$ solves the div-curl system (3.1). □

The integrability condition (3.2) is most natural for the chosen method of proof. However, it is hard to verify in practice. For this reason, it is worthwhile considering equivalent alternative conditions.

LEMMA 3.4. *Suppose that* $\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz domain and let* $\mathbf{F} \in [\mathbf{H}_0(\mathrm{curl};\Omega)]'$. *Together, the following conditions are equivalent to the integrability condition* (3.2)*:*

$$\operatorname{div}\mathbf{F} = 0 \qquad in\ \Omega, \tag{3.11a}$$

$$\langle \mathbf{F}, -\nabla T_i\big|_\Omega \rangle = 0 \qquad i = 1,\dots,\beta_2. \tag{3.11b}$$

If in particular $\mathbf{F} \in \mathbf{L}^2(\Omega)$ *and* $\operatorname{div}\mathbf{F} = 0$, *condition* (3.11b) *is equivalent to:*

$$\int_{\Gamma_i} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = 0 \qquad i = 1,\dots,\beta_2. \tag{3.12}$$

Remark 3.5. Notice that definition (2.10) guarantees that $-\nabla T_i\big|_\Omega \in \boldsymbol{\mathcal{D}}(\Omega)$.

Remark 3.6. Together (3.11a) and (3.11b) also imply that $\langle \mathbf{F}, -\nabla T_0|_\Omega\rangle = 0$ holds. If in particular $\beta_2 = 0$, it suffices to demand $\operatorname{div}\mathbf{F} = 0$.

Proof. ($\Rightarrow$) Since $\mathrm{curl}\circ(-\nabla) \equiv \mathbf{0}$, conditions (3.11a) and (3.11b) are immediately seen to be necessary from the definitions.

($\Leftarrow$) To see that they also are sufficient, let $\mathbf{V} \in \mathbf{K}(\Omega)$ be arbitrary. We may extend this function by zero outside Ω:

$$\widetilde{\mathbf{V}} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{x} \mapsto \begin{cases} \mathbf{V}(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \mathbf{0} & \text{else.} \end{cases} \tag{3.13}$$

Since $\boldsymbol{\rho}(\mathbf{V}) = \mathbf{0}$, we have $\mathrm{curl}\,\widetilde{\mathbf{V}} = \mathbf{0}$ on all of $\mathbb{R}^3$. Since its support is compact, we may use the Helmholtz decomposition (2.17) to rewrite this extension in terms of

$$\widetilde{\mathbf{V}} = \mathrm{curl}\,\boldsymbol{\mathcal{N}}\underbrace{\mathrm{curl}\,\widetilde{\mathbf{V}}}_{=\mathbf{0}} - \nabla\underbrace{\mathcal{N}\operatorname{div}\widetilde{\mathbf{V}}}_{=:\widetilde{P}} = -\nabla\widetilde{P}. \tag{3.14}$$

The restriction $P := \widetilde{P}\big|_\Omega$ belongs to $H^1(\Omega)$, because $-\nabla P = \mathbf{V} \in \mathbf{L}^2(\Omega)$. Moreover, we see from $\boldsymbol{\tau}(\mathbf{V}) = \mathbf{0}$ that $P = C_i$ for some constant $C_i \in \mathbb{R}$ on each connected component Γ_i of the boundary, $i = 0,1,\dots,\beta_2$. Because $\widetilde{P} \to 0$ at infinity, $-\nabla\widetilde{P} = \mathbf{0}$ outside Ω, and $\widetilde{P} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ we have $C_0 = 0$. From the decomposition

$$P = \underbrace{\left(P - \sum_{i=1}^{\beta_2} C_i T_i\big|_\Omega\right)}_{=:P_0 \in H^1_0(\Omega)} + \sum_{i=1}^{\beta_2} C_i T_i\big|_\Omega, \tag{3.15}$$

we obtain

$$\langle \mathbf{F}, \mathbf{V}\rangle = \langle \mathbf{F}, -\nabla P\rangle = \underbrace{\langle \mathbf{F}, -\nabla P_0\rangle}_{=0,\ (3.11a)} + \sum_{i=1}^{\beta_2} C_i \underbrace{\langle \mathbf{F}, -\nabla T_i\big|_\Omega\rangle}_{=0,\ (3.11b)} = 0. \tag{3.16}$$

Thus (3.2) is equivalent to the combination of (3.11a) and (3.11b).

Finally, the equivalence of (3.11b) and (3.12) directly follows from the definition of the normal trace: if $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div}\mathbf{F} = 0$, we also have $\mathbf{F} \in \mathbf{H}(\mathrm{div};\Omega)$. Thus $\mathbf{F}$

has a well-defined normal trace and

$$\begin{aligned}\int_{\Gamma_i} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S &= \int_{\Gamma} (\mathbf{F}\cdot\mathbf{n})\gamma T_i\,\mathrm{d}S = \langle \nu\mathbf{F}, \gamma T_i\rangle_{\Gamma} \\ &= \int_{\Omega} \mathbf{F}\cdot\nabla T_i\big|_{\Omega}\,\mathrm{d}\mathbf{x} + \int_{\Omega} \underbrace{\operatorname{div}\mathbf{F}}_{=0}\, T_i\big|_{\Omega}\,\mathrm{d}\mathbf{x} \\ &= \langle \mathbf{F}, \nabla T_i\big|_{\Omega}\rangle.\end{aligned} \tag{3.17}$$

∎

3.2. Uniqueness of Velocity Fields.

THEOREM 3.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, handle-free Lipschitz domain and let $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$ fulfil the integrability condition (3.2). Additionally, let $g \in H^{-\frac{1}{2}}(\Gamma)$ be given such that $\langle g, 1\rangle_\Gamma = 0$. Then the div-curl system (3.1) has exactly one solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ with $\mathbf{U}\cdot\mathbf{n} = g$ on Γ.*

Remark 3.8. In Theorem 3.7, the normal trace $\mathbf{U}\cdot\mathbf{n}$ is well-defined because a square-integrable solution $\mathbf{U}$ of the div-curl system satisfies $\operatorname{div}\mathbf{U} = 0$.

Proof. Let us first remark that the condition $\langle g, 1\rangle_\Gamma = 0$ is necessary. To see this, note that because $\operatorname{div}\mathbf{U} = 0$, any solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of the div-curl system (3.1) must fulfil:

$$\int_{\Gamma} \mathbf{U}\cdot\mathbf{n}\,\mathrm{d}S = \int_{\Omega} \operatorname{div}\mathbf{U}\,\mathrm{d}\mathbf{x} = 0. \tag{3.18}$$

Now let $\mathbf{W} \in \mathbf{L}^2(\Omega)$ denote any solution of the div-curl system, whose existence is guaranteed by Theorem 3.1. Let $P \in H^1(\Omega)/\mathbb{R}$ be the unique solution of the Neumann problem:

$$\begin{cases} -\Delta P = 0 & \text{in } \Omega, \\ -\nabla P\cdot\mathbf{n} = g - \mathbf{W}\cdot\mathbf{n} & \text{on } \Gamma. \end{cases} \tag{3.19}$$

Then the function $\mathbf{U} := \mathbf{W} - \nabla P$ fulfils the conditions of the theorem.

To see that it is unique, let $\mathbf{U}_1, \mathbf{U}_2 \in \mathbf{L}^2(\Omega)$ denote two solutions of the div-curl system (3.1) that fulfil $\mathbf{U}_1\cdot\mathbf{n} = \mathbf{U}_2\cdot\mathbf{n} = g$ on the boundary Γ. Then their difference $\mathbf{D} := \mathbf{U}_1 - \mathbf{U}_2$ solves

$$\begin{cases} \operatorname{div}\mathbf{D} = 0 & \text{in } \Omega, \\ \operatorname{curl}\mathbf{D} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{D}\cdot\mathbf{n} = 0 & \text{on } \Gamma. \end{cases} \tag{3.20}$$

In other words, $\mathbf{D}$ is a so-called Neumann harmonic field. These functions form a space of dimension β_1 [1, Proposition 3.14][2, Section 4.3]. By hypothesis $\beta_1 = 0$, and thus $\mathbf{D} = \mathbf{0}$. ∎

The above proof hints at what needs to be done in order to recover uniqueness in the case $\beta_1 \neq 0$. One needs to prescribe β_1 functionals that determine the Neumann harmonic components of $\mathbf{U}$. A construction of these fields and corresponding functionals can be found in the work of Amrouche et al. [1]

4. Stream Functions. We prove the existence result for stream functions of Item 2 in this section. The related uniqueness statement of Item 3 is proven in Theorem 4.2.

4.1. Existence of Stream Functions. The following theorem is a variant of a result by Girault and Raviart [10, Theorem 3.4]. We give a different proof, which uses the Newton operator instead of Fourier transforms.

THEOREM 4.1. *Let $\Omega \subset \mathbb{R}^3$ denote a bounded Lipschitz domain. Then $\mathbf{U} \in \mathbf{L}^2(\Omega)$ satisfies*

$$\operatorname{div} \mathbf{U} = 0, \qquad \text{in } \Omega, \tag{4.1a}$$

$$\int_{\Gamma_i} \mathbf{U} \cdot \mathbf{n} \, \mathrm{d}S = 0, \qquad i = 1, \dots, \beta_2, \tag{4.1b}$$

if and only if there exists a vector-field $\mathbf{A} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{curl} \mathbf{A} = \mathbf{U} & \text{in } \Omega, \\ -\boldsymbol{\Delta} \mathbf{A} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ \operatorname{div} \mathbf{A} = 0 & \text{in } \mathbb{R}^3, \\ \mathbf{A}(\mathbf{x}) \to \mathbf{0} & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{4.2}$$

Proof. ($\Leftarrow$) Note that the conditions (4.1a) and (4.1b) are exactly the integrability conditions (3.11a) and (3.12). Because of Lemma 3.3 and Lemma 3.4, these conditions are necessary to ensure the existence of a vector-field $\mathbf{A} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{U} = \operatorname{curl} \mathbf{A}$ in Ω.

($\Rightarrow$) In order to show sufficiency, the idea is to extend $\mathbf{U}$ to $\mathbb{R}^3$ by 'potential flows' matching $\mathbf{U} \cdot \mathbf{n}$ on Γ, then use the Newton operator.

We want to exploit the following scalar functions. For $i = 0$, we let $P_0 \in H^1_{\mathrm{loc}}(\Theta_0)$ denote the solution of the problem:

$$\begin{cases} -\Delta P_0 = 0 & \text{in } \Theta_0, \\ -\nabla P_0 \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n} & \text{on } \Gamma_0, \\ P_0(\mathbf{x}) \to 0 & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{4.3}$$

whereas for $i = 1, \dots, \beta_2$ we define $P_i \in H^1(\Omega_i)/\mathbb{R}$ as the solution of:

$$\begin{cases} -\Delta P_i = 0 & \text{in } \Theta_i, \\ -\nabla P_i \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n} & \text{on } \Gamma_i. \end{cases} \tag{4.4}$$

Because of condition (4.1b), it is well-known that these problems are well-posed.

We are now ready to extend $\mathbf{U}$ to the whole space as

$$\widetilde{\mathbf{U}} : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \mathbf{x} \mapsto \begin{cases} \mathbf{U}(\mathbf{x}) & \mathbf{x} \in \Omega, \\ -\nabla P_i(\mathbf{x}) & \mathbf{x} \in \Theta_i,\ i = 0, ..., \beta_2. \end{cases} \tag{4.5}$$

Because $[\![\widetilde{\mathbf{U}} \cdot \mathbf{n}]\!] = 0$, we have $\operatorname{div} \widetilde{\mathbf{U}} = 0$ on $\mathbb{R}^3$. Since $\operatorname{supp}\left(\operatorname{curl} \widetilde{\mathbf{U}}\right) \subset \overline{\Omega}$, $\operatorname{curl} \widetilde{\mathbf{U}} \in \mathbf{H}^{-1}(\mathbb{R}^3)$ is compactly supported, and we may define $\mathbf{A} := \boldsymbol{\mathcal{N}} \operatorname{curl} \widetilde{\mathbf{U}}$.

We now claim that $\operatorname{curl} \mathbf{A} = \widetilde{\mathbf{U}}$ on $\mathbb{R}^3$. From the properties of $\boldsymbol{\mathcal{N}}$ it follows that $\mathbf{A} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ and $\mathbf{A}(\mathbf{x}) \to \mathbf{0}$ at infinity. Because $\boldsymbol{\mathcal{N}}$ commutes with differentiation, we have $\operatorname{div} \mathbf{A} = \mathcal{N} \operatorname{div} \operatorname{curl} \widetilde{\mathbf{U}} = 0$ on $\mathbb{R}^3$, and thus:

$$\begin{cases} \operatorname{curl}(\operatorname{curl} \mathbf{A}) = & -\boldsymbol{\Delta} \mathbf{A} & = \operatorname{curl} \widetilde{\mathbf{U}} \\ \operatorname{div}(\operatorname{curl} \mathbf{A}) = & 0 & = \operatorname{div} \widetilde{\mathbf{U}} \end{cases} \qquad \text{on } \mathbb{R}^3. \tag{4.6}$$

The difference $\mathbf{D} := \operatorname{curl}\mathbf{A} - \widetilde{\mathbf{U}}$ therefore fulfils:

$$\begin{cases} -\boldsymbol{\Delta}\mathbf{D} = \mathbf{0} & \text{on } \mathbb{R}^3, \\ \mathbf{D}(\mathbf{x}) \to \mathbf{0} & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{4.7}$$

so that from Lemma 2.2 we conclude $\mathbf{D} = \boldsymbol{\mathcal{N}}\mathbf{0} = \mathbf{0}$, that is $\operatorname{curl}\mathbf{A} = \widetilde{\mathbf{U}}$. ∎

4.2. Uniqueness of Stream Functions. When the domain under consideration is handle-free, then the following uniqueness result holds.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a handle-free, bounded Lipschitz domain ($\beta_1 = 0$) and let $\mathbf{U} \in \mathbf{L}^2(\Omega)$ fulfil the conditions of Theorem 4.1. Then there exists exactly one vector-field $\mathbf{A} \in \mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$ satisfying (4.2).*

Proof. Suppose that $\mathbf{A}_1$ and $\mathbf{A}_2$ are two vector-fields in $\mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$ satisfying (4.2). Then their difference $\mathbf{D} := \mathbf{A}_1 - \mathbf{A}_2 \in \mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$ fulfils:

$$\begin{cases} -\boldsymbol{\Delta}\mathbf{D} = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \Gamma, \\ \operatorname{div}\mathbf{D} = 0 & \text{in } \mathbb{R}^3, \\ \operatorname{curl}\mathbf{D} = \mathbf{0} & \text{in } \Omega, \\ \mathbf{D}(\mathbf{x}) \to \mathbf{0} & \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{4.8}$$

Thus, Lemma 2.4 is applicable, yielding:

$$\begin{cases} \mathbf{D} = \boldsymbol{\mathcal{N}}(-\boldsymbol{\Delta}\mathbf{D}), \\ -\boldsymbol{\Delta}\mathbf{D} = \boldsymbol{\tau}'[\![\boldsymbol{\rho}\operatorname{curl}\mathbf{D}]\!]. \end{cases} \tag{4.9}$$

From the mapping properties of $\boldsymbol{\rho}$ it follows that $\mathbf{s} := [\![\boldsymbol{\rho}\operatorname{curl}\mathbf{D}]\!] \in \mathbf{H}_{\text{R}}^{-\frac{1}{2}}(\operatorname{div}_\Gamma;\Gamma)$. The Hodge Decomposition from Lemma 2.3 furthermore yields

$$\mathbf{s} := [\![\boldsymbol{\rho}\operatorname{curl}\mathbf{D}]\!] = \mathbf{curl}_\Gamma\, p - \nabla_\Gamma\, q \tag{4.10}$$

for some functions $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$, $q \in H^1(\Gamma)/\mathbb{R}$ that are uniquely determined up to a constant on each connected part Γ_i, $i = 0, \dots, \beta_2$ of the boundary. It thus suffices to establish that $p = q = 0$.

We first consider q. The fact that $\operatorname{div}\mathbf{D} = 0$ on $\mathbb{R}^3$ implies that for all $V \in \mathcal{D}(\mathbb{R}^3)$:

$$\begin{aligned} \langle \operatorname{div}_\Gamma \mathbf{s}, \gamma V\rangle_\Gamma = \langle \mathbf{s}, -\nabla_\Gamma\, \gamma V\rangle_\Gamma = \langle \mathbf{s}, \boldsymbol{\tau}(-\nabla V)\rangle_\Gamma = \\ \langle -\boldsymbol{\Delta}\mathbf{D}, -\nabla V\rangle = \langle \operatorname{div}(-\boldsymbol{\Delta}\mathbf{D}), V\rangle = \langle -\Delta \operatorname{div}\mathbf{D}, V\rangle = 0, \end{aligned} \tag{4.11}$$

that is $\operatorname{div}_\Gamma \mathbf{s} = 0$. This in turn implies for q:

$$-\Delta_\Gamma\, q = \underbrace{\operatorname{div}_\Gamma(\mathbf{curl}_\Gamma\, p)}_{=0} - \operatorname{div}_\Gamma \nabla_\Gamma\, q = \operatorname{div}_\Gamma \mathbf{s} = 0. \tag{4.12}$$

But because the Laplace–Beltrami operator $-\Delta_\Gamma$ is coercive on $H^1(\Gamma)/\mathbb{R}$, we have have the implication $-\Delta_\Gamma\, q = 0 \Longrightarrow q = 0$, and thus also $\mathbf{s} = \mathbf{curl}_\Gamma\, p$.

Considering p, we note that because $\operatorname{curl}\mathbf{D} = \mathbf{0}$ in Ω, we have $0 = (\operatorname{curl}\mathbf{D}) \cdot \mathbf{n} = \operatorname{curl}_\Gamma \boldsymbol{\tau}\mathbf{D}$ on Γ from (2.41). Thus for all $v \in H^{\frac{1}{2}}(\Gamma)$:

$$\begin{aligned} 0 = \langle \operatorname{curl}_\Gamma \boldsymbol{\tau}\mathbf{D}, v\rangle_\Gamma &= \langle \boldsymbol{\tau}\mathbf{D}, \mathbf{curl}_\Gamma\, v\rangle_\Gamma \\ &= \langle \boldsymbol{\tau}\boldsymbol{\mathcal{N}}\boldsymbol{\tau}'\mathbf{s}, \mathbf{curl}_\Gamma\, v\rangle_\Gamma = \langle \boldsymbol{\tau}\boldsymbol{\mathcal{N}}\boldsymbol{\tau}'\, \mathbf{curl}_\Gamma\, p, \mathbf{curl}_\Gamma\, v\rangle_\Gamma. \end{aligned} \tag{4.13}$$

The last expression can be enlightened using a more explicit representation. Following Claeys and Hiptmair [6, Equations (41) and (42)], under the additional assumption that $\mathbf{curl}_\Gamma\, p$, $\mathbf{curl}_\Gamma\, v \in \mathbf{L}^\infty(\Gamma)$, we have:

$$\langle \boldsymbol{\tau}\boldsymbol{\mathcal{N}}\boldsymbol{\tau}'\, \mathbf{curl}_\Gamma\, p, \mathbf{curl}_\Gamma\, v\rangle_\Gamma = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{\mathbf{curl}_\Gamma\, p(\mathbf{y})\cdot \mathbf{curl}_\Gamma\, v(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}\,\mathrm{d}S(\mathbf{y})\,\mathrm{d}S(\mathbf{x}). \tag{4.14}$$

Here one clearly recognises the hypersingular boundary integral operator for the scalar Laplace equation [16, Section 3.3.4]. This operator is known to be coercive on $H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ [16, Theorem 3.5.3], and we conclude that $p = 0$. ☐

Let us now make some remarks on the case $\beta_1 \neq 0$. We define $\Omega^C := \mathbb{R}^3 \setminus \overline{\Omega}$ as the complementary domain of Ω, and $\mathbf{B} := \operatorname{curl}\mathbf{D}|_\Omega$ and $\mathbf{B}^C := \operatorname{curl}\mathbf{D}|_{\Omega^C}$. These functions are Neumann harmonic fields:

$$\operatorname{div}\mathbf{B} = 0, \qquad \operatorname{curl}\mathbf{B} = \mathbf{0} \qquad \text{in } \Omega, \qquad \mathbf{B}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma, \tag{4.15}$$

$$\operatorname{div}\mathbf{B}^C = 0, \qquad \operatorname{curl}\mathbf{B}^C = \mathbf{0} \qquad \text{in } \Omega^C, \qquad \mathbf{B}^C\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma. \tag{4.16}$$

Ultimately, the idea is to rely on the fact that in handle-free domains the space of Neumann harmonic fields only contains the zero element, and thus $\mathbf{B} = \mathbf{0}$ and $\mathbf{B}^C = \mathbf{0}$. In the case $\beta_1 \neq 0$, however, neither Ω nor Ω^C are handle-free, and in fact we have $\beta_1^C = \beta_1$. The spaces of Neumann harmonic fields on Ω and Ω^C then each have dimension β_1.

Buffa has derived the analogue of Lemma 2.3 for the case of Lipschitz polyhedra with $\beta_1 \neq 0$ [4]. Because $\beta_1 = \beta_1^C$, it contains an additional term from the $2\beta_1$-dimensional space of harmonic tangential fields. Half of these components are fixed because of the condition $\operatorname{curl}\mathbf{A}|_\Omega = \mathbf{U}$, the other half concerns the external harmonic fields. To ensure uniqueness of $\mathbf{A}$, one additionally needs to prescribe the Neumann harmonic components of $\mathbf{U}^C := \operatorname{curl}\mathbf{A}|_{\Omega^C}$.

5. Construction of Solutions.

In this section, we first provide a construction for a stream function $\mathbf{A} \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ for the general case of a given vorticity field $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$. This construction may also be considered an alternative proof of the existence results of Theorems 3.1 and 4.1.

In computational practice, one will usually have $\mathbf{F} \in \mathbf{L}^2(\Omega)$. Under this assumption we can simplify the construction, yielding an algorithm that is more easily implementable.

5.1. The General Case.

Let $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$ be given and suppose that it fulfils the integrability condition (3.2), or the equivalent conditions (3.11a) and (3.11b). Furthermore, let $g \in H^{-\frac{1}{2}}(\Gamma)$ be given boundary data such that $\langle g, 1\rangle_{\Gamma_i} = 0$, $i = 0, \ldots, \beta_2$.

Our approach is to first find a suitable extension $\widetilde{\mathbf{F}} \in \left[\mathbf{H}(\mathrm{curl};\mathbb{R}^3)\right]'$ of $\mathbf{F} \in \left[\mathbf{H}_0(\mathrm{curl};\Omega)\right]'$ and then apply the Newton operator to it. Let $\mathbf{R} \in \mathbf{H}_0(\mathrm{curl};\Omega)$ denote the Riesz representative of $\mathbf{F}$, i. e., the uniquely determined function $\mathbf{R}$ such that for all $\mathbf{V} \in \mathbf{H}_0(\mathrm{curl};\Omega)$:

$$\underbrace{\int_\Omega \mathbf{R}\cdot\mathbf{V} \,+\, \operatorname{curl}\mathbf{R}\cdot\operatorname{curl}\mathbf{V}\,\mathrm{d}\mathbf{x}}_{=:\mathfrak{B}(\mathbf{R},\mathbf{V})} = \langle \mathbf{F}, \mathbf{V}\rangle. \tag{5.1}$$

The expression $\mathfrak{B}(\mathbf{R},\mathbf{V})$ is not only well-defined for $\mathbf{V} \in \mathbf{H}_0(\mathrm{curl};\Omega)$, but also for any

$\mathbf{V} \in \mathbf{H}(\mathrm{curl};\mathbb{R}^3)$. We thus define $\widetilde{\mathbf{F}} \in \left[\mathbf{H}(\mathrm{curl};\mathbb{R}^3)\right]'$ as follows:

$$\forall \mathbf{V} \in \mathbf{H}(\mathrm{curl};\mathbb{R}^3): \; \langle \widetilde{\mathbf{F}}, \mathbf{V}\rangle := \mathfrak{B}(\mathbf{R},\mathbf{V}). \tag{5.2}$$

Obviously $\widetilde{\mathbf{F}} \in \left[\mathbf{H}(\mathrm{curl};\mathbb{R}^3)\right]' \subset \mathbf{H}^{-1}(\mathbb{R}^3)$ extends $\mathbf{F}$ and is compactly supported with $\operatorname{supp} \widetilde{\mathbf{F}} \subset \overline{\Omega}$.

This extension does not necessarily fulfil $\operatorname{div} \widetilde{\mathbf{F}} = 0$ on all of $\mathbb{R}^3$. However, the following result is useful.

LEMMA 5.1. *One has* $\operatorname{div} \widetilde{\mathbf{F}} \in H^{-1}(\mathbb{R}^3)$. *Moreover, there exists a uniquely determined surface functional* $f \in H^{-\frac{1}{2}}(\Gamma)$ *such that:*

$$\langle \operatorname{div} \widetilde{\mathbf{F}}, V\rangle = -\langle f, \gamma V\rangle_\Gamma \qquad \forall V \in H^1(\mathbb{R}^3), \tag{5.3}$$

and

$$\begin{cases} \langle f, 1\rangle_{\Gamma_i} = 0, & i = 0,\dots,\beta_2, \\ \|f\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'}. \end{cases} \tag{5.4}$$

Proof. First note that $\forall V \in \mathcal{D}(\mathbb{R}^3)$:

$$\begin{aligned} \langle \operatorname{div} \widetilde{\mathbf{F}}, V\rangle = \langle \widetilde{\mathbf{F}}, -\nabla V\rangle = \mathfrak{B}(\mathbf{R}, -\nabla V) &= \int_\Omega \mathbf{R}\cdot(-\nabla V)\,\mathrm{d}\mathbf{x} \\ &\le \|\mathbf{R}\|_{\mathbf{L}^2(\Omega)}\|-\nabla V\|_{\mathbf{L}^2(\Omega)} \\ &\le \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'}\|V\|_{H^1(\Omega)}. \end{aligned} \tag{5.5}$$

The distribution $\operatorname{div} \widetilde{\mathbf{F}} \in \mathcal{D}'(\mathbb{R}^3)$ thus admits a unique continuous extension to $\overline{\mathcal{D}(\mathbb{R}^3)}^{\|\cdot\|_{H^1(\mathbb{R}^3)}} = H^1(\mathbb{R}^3)$, and we may write $\operatorname{div} \widetilde{\mathbf{F}} \in H^{-1}(\mathbb{R}^3)$ with $\|\operatorname{div} \widetilde{\mathbf{F}}\|_{H^{-1}(\mathbb{R}^3)} \le \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'}$.

Next, we find that the value $\langle \operatorname{div} \widetilde{\mathbf{F}}, V\rangle$ only depends on the Dirichlet trace $\gamma V \in H^{\frac{1}{2}}(\Gamma)$ of the trial function $V \in H^1(\mathbb{R}^3)$. To see this, let $V_1, V_2 \in H^1(\mathbb{R}^3)$ have the same Dirichlet trace, $\gamma V_1 = \gamma V_2$. Because $\gamma(V_1 - V_2) = 0$, one finds that $-\nabla(V_1 - V_2)|_\Omega \in \mathbf{H}_0(\mathrm{curl};\Omega)$, and thus:

$$\langle \operatorname{div} \widetilde{\mathbf{F}}, V_1\rangle - \langle \operatorname{div} \widetilde{\mathbf{F}}, V_2\rangle = \mathfrak{B}\big(\mathbf{R}, -\nabla(V_1 - V_2)\big) = \langle \mathbf{F}, -\nabla(V_1 - V_2)|_\Omega\rangle \overset{(3.2)}{=} 0. \tag{5.6}$$

We may thus define $f \in H^{-\frac{1}{2}}(\Gamma)$ as follows:

$$\forall v \in H^{\frac{1}{2}}(\Gamma): \qquad \langle f, v\rangle_\Gamma := -\langle \operatorname{div} \widetilde{\mathbf{F}}, \gamma^{-1} v\rangle, \tag{5.7}$$

where $\gamma^{-1}: H^{\frac{1}{2}}(\Gamma) \to H^1(\mathbb{R}^3)$ is fixed, but may be any linear and bounded lifting operator. Clearly, we have $\|f\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'}$, because

$$\begin{aligned} \langle f, v\rangle_\Gamma &= -\langle \operatorname{div} \widetilde{\mathbf{F}}, \gamma^{-1} v\rangle \\ &\le \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'}\|\gamma^{-1} v\|_{H^1(\mathbb{R}^3)} \\ &\le \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'}\|\gamma^{-1}\|_{H^{\frac{1}{2}}(\Gamma)\to H^1(\mathbb{R}^3)}\|v\|_{H^{\frac{1}{2}}(\Gamma)} \end{aligned} \tag{5.8}$$

for all $v \in H^{\frac{1}{2}}(\Gamma)$.

Finally, for $i = 0,\dots,\beta_2$, we have

$$\langle f, 1\rangle_{\Gamma_i} = \langle f, \gamma T_i\rangle_\Gamma = -\langle \operatorname{div} \widetilde{\mathbf{F}}, \gamma^{-1} T_i\rangle = \langle \mathbf{F}, \nabla \gamma^{-1} T_i|_\Omega\rangle \overset{(3.2)}{=} 0. \tag{5.9}$$

∎

As a consequence of the preceding lemma we may define $q \in H^1(\Gamma)/\mathbb{R}$, uniquely up to a constant on each connected component of the boundary Γ_i, $i = 0, \ldots, \beta_2$, as the solution to the Laplace–Beltrami equation:

$$-\Delta_\Gamma q = f \qquad \text{on } \Gamma, \tag{5.10}$$

and furthermore define:

$$\begin{aligned} \widehat{\mathbf{F}} &:= \widetilde{\mathbf{F}} - \boldsymbol{\tau}' \nabla_\Gamma q, \\ \widehat{\mathbf{A}} &:= \boldsymbol{\mathcal{N}} \widehat{\mathbf{F}}. \end{aligned} \tag{5.11}$$

LEMMA 5.2. *One has* $\widehat{\mathbf{F}} \in \mathbf{H}^{-1}(\mathbb{R}^3) \cap \boldsymbol{\mathcal{E}}'(\mathbb{R}^3)$, $\widehat{\mathbf{A}} \in \mathbf{H}^1_{\text{loc}}(\mathbb{R}^3)$ *decaying to zero at infinity, and moreover:*

$$\begin{cases} \|\widehat{\mathbf{F}}\|_{\mathbf{H}^{-1}(\mathbb{R}^3)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\text{curl};\Omega)'}, \\ \|\widehat{\mathbf{A}}\|_{\mathbf{H}^1(\Omega)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\text{curl};\Omega)'}, \\ -\boldsymbol{\Delta}\widehat{\mathbf{A}} = \widehat{\mathbf{F}} = \mathbf{0} & \text{on } \mathbb{R}^3 \setminus \overline{\Omega}, \\ -\boldsymbol{\Delta}\widehat{\mathbf{A}} = \widehat{\mathbf{F}} = \mathbf{F} & \text{on } \Omega, \\ \operatorname{div} \widehat{\mathbf{A}} = \operatorname{div} \widehat{\mathbf{F}} = 0 & \text{on } \mathbb{R}^3. \end{cases} \tag{5.12}$$

Proof. We first consider the properties of $\widehat{\mathbf{F}}$; the corresponding properties of $\widehat{\mathbf{A}}$ then easily follow from the properties of $\boldsymbol{\mathcal{N}}$.

We already established that $\widetilde{\mathbf{F}} \in \mathbf{H}^{-1}(\mathbb{R}^3)$ and $\|\widetilde{\mathbf{F}}\|_{\mathbf{H}^{-1}(\mathbb{R}^3)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\text{curl};\Omega)'}$. For the surface functional,

$$\|\nabla_\Gamma q\|_{\mathbf{L}^2(\Gamma)} \le \|q\|_{H^1(\Gamma)} \lesssim \|f\|_{H^{-1}(\Gamma)} \lesssim \|f\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\text{curl};\Omega)'}, \tag{5.13}$$

so that $\forall \mathbf{V} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$:

$$\langle \nabla_\Gamma q, \boldsymbol{\tau}\mathbf{V}\rangle_\Gamma \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\text{curl};\Omega)'} \|\mathbf{V}\|_{\mathbf{H}^1(\mathbb{R}^3)}. \tag{5.14}$$

Thus $-\boldsymbol{\tau}'\nabla_\Gamma q \in \mathbf{H}^{-1}(\mathbb{R}^3)$ with

$$\| -\boldsymbol{\tau}'\nabla_\Gamma q\|_{\mathbf{H}^{-1}(\mathbb{R}^3)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\text{curl};\Omega)'}. \tag{5.15}$$

The fact that $\widehat{\mathbf{F}} = \mathbf{0}$ on $\mathbb{R}^3 \setminus \overline{\Omega}$ is obvious. We have $\widehat{\mathbf{F}} = \mathbf{F}$ on Ω, because the surface functional vanishes there.

For the divergence, we note that $\forall V \in \mathcal{D}(\mathbb{R}^3)$:

$$\langle \nabla_\Gamma q, \boldsymbol{\tau}\nabla V\rangle_\Gamma = \langle \nabla_\Gamma q, \nabla_\Gamma \gamma V\rangle_\Gamma = \langle -\Delta_\Gamma q, \gamma V\rangle_\Gamma = \langle f, \gamma V\rangle_\Gamma = -\langle \operatorname{div} \widetilde{\mathbf{F}}, V\rangle, \tag{5.16}$$

and therefore $\operatorname{div} \widehat{\mathbf{F}} = 0$.

The properties of $\widehat{\mathbf{A}}$ now directly follow from the mapping properties of the Newton operator, the fact that it is an inverse to the vector Laplacian, and that $\boldsymbol{\mathcal{N}}$ commutes with differentiation. ∎

With these properties in place, one immediately verifies that $\widehat{\mathbf{U}} := \operatorname{curl} \widehat{\mathbf{A}}$ solves the div-curl system (3.1), but does not necessarily fulfil $\widehat{\mathbf{U}} \cdot \mathbf{n} = g$ on Γ. To fix its normal component, it then suffices to solve the hypersingular boundary integral equation:

$$\forall v \in H^{\frac{1}{2}}(\Gamma): \quad \langle \boldsymbol{\tau}\boldsymbol{\mathcal{N}}\boldsymbol{\tau}' \, \mathbf{curl}_\Gamma \, p, \mathbf{curl}_\Gamma \, v\rangle_\Gamma = \langle g - \operatorname{curl} \widehat{\mathbf{A}} \cdot \mathbf{n}, v\rangle_\Gamma, \tag{5.17}$$

for the unknown $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$. This problem is known to be well-posed, and its solution continuously depends on $\widehat{\mathbf{U}} \cdot \mathbf{n}$ and g [16, Theorem 3.5.3]:

$$\begin{aligned} \|p\|_{H^{\frac{1}{2}}(\Gamma)} &\lesssim \|\widehat{\mathbf{U}} \cdot \mathbf{n} - g\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\widehat{\mathbf{U}}\|_{\mathbf{H}(\mathrm{div};\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\lesssim \|\widehat{\mathbf{A}}\|_{\mathbf{H}^1(\Omega)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{5.18}$$

We now finally define:

$$\begin{cases} \mathbf{s} := \mathbf{curl}_\Gamma\, p - \nabla_\Gamma q \in \mathbf{H}_{\mathrm{R}}^{-\frac{1}{2}}(\mathrm{div}_\Gamma;\Gamma), \\ \mathbf{A} := \boldsymbol{\mathcal{N}}\big(\widetilde{\mathbf{F}} + \boldsymbol{\tau}'\mathbf{s}\big) \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3), \\ \mathbf{U} := \mathrm{curl}\,\mathbf{A} \in \mathbf{L}^2(\mathbb{R}^3). \end{cases} \tag{5.19}$$

Then $\mathbf{U}$ solves the div-curl system and $\mathbf{U}\cdot\mathbf{n} = g$ on Γ, and $\mathbf{A}$ is a stream function for $\mathbf{U}$. In the case of a handle-free domain, Theorems 3.7 and 4.2 guarantee that these functions are unique. Moreover, the solution continuously depends on $\mathbf{F}$ and g. In total we have therefore proven the following theorem.

THEOREM 5.3. *Let* $\mathbf{F} \in \big[\mathbf{H}_0(\mathrm{curl};\Omega)\big]'$ *be given and fulfil the integrability condition* (3.2)*, or the equivalent conditions* (3.11a) *and* (3.11b)*. Furthermore, let* $g \in H^{-\frac{1}{2}}(\Gamma)$ *be given, such that* $\langle g, 1\rangle_{\Gamma_i} = 0$ *for all* $i = 0, \ldots, \beta_2$*.*

Then a solution to the div-curl system (3.1) *with* $\mathbf{U}\cdot\mathbf{n} = g$ *on* Γ*, and its associated stream function* $\mathbf{A}$ *are given by* (5.19)*. In case of a handle-free domain* $\mathbf{U}$ *and* $\mathbf{A}$ *are the uniquely determined functions from Theorems* 3.7 *and* 4.2*.*

These functions linearly and continuously depend on the data $\mathbf{F}$ *and* g*, and we have:*

$$\|\mathbf{U}\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{A}\|_{\mathbf{H}^1(\Omega)} \lesssim \|\mathbf{F}\|_{\mathbf{H}_0(\mathrm{curl};\Omega)'} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{5.20}$$

It is this well-posedness result which makes the construction accessible in practice. We conclude this subsection with a summary of the construction. In the subsequent section we give some remarks on its practical implementation.

1. Define $\widetilde{\mathbf{F}}$ via (5.2).
2. Solve the Laplace–Beltrami equation for $q \in H^1(\Gamma)/\mathbb{R}$:

$$\langle \nabla_\Gamma q, \nabla_\Gamma v\rangle_\Gamma = \langle \widetilde{\mathbf{F}}, \nabla\gamma^{-1} v\rangle \qquad \forall v \in H^1(\Gamma).$$

3. Solve the hypersingular boundary integral equation (5.17) for $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$.
4. Define $\mathbf{A}$ and $\mathbf{U}$ as in (5.19).

5.2. A remark on square-integrable vorticity fields. The vorticity is often known in practice to be square-integrable. When that is the case, the construction can be slightly simplified. Thus, let $\mathbf{F} \in \mathbf{L}^2(\Omega)$ fulfil the integrability conditions (3.11a) and (3.12).

We first note that $\mathbf{F}$ now possesses a natural extension by zero:

$$\widetilde{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{x} \mapsto \begin{cases} \mathbf{F}(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \mathbf{0} & \text{else.} \end{cases} \tag{5.21}$$

Next, we note that because $\mathbf{F} \in \mathbf{L}^2(\Omega)$ and $\operatorname{div} \mathbf{F} = 0$ in Ω, we also have $\mathbf{F} \in \mathbf{H}(\operatorname{div}; \Omega)$. Thus $\mathbf{F}$ has a normal trace $\mathbf{F} \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$, that by condition (3.12) satisfies $\langle \mathbf{F} \cdot \mathbf{n}, 1 \rangle_{\Gamma_i} = 0$, $i = 0, \dots, \beta_2$. We may thus instead define $q \in H^1(\Gamma)/\mathbb{R}^3$ as the solution to the Laplace–Beltrami equation

$$-\Delta_\Gamma\, q = \mathbf{F} \cdot \mathbf{n} \qquad \text{on } \Gamma. \tag{5.22}$$

Defining $\widehat{\mathbf{F}} := \widetilde{\mathbf{F}} - \boldsymbol{\tau}' \nabla_\Gamma q$, and $\widehat{\mathbf{A}} := \boldsymbol{\mathcal{N}} \widehat{\mathbf{F}}$ one verifies that Lemma 5.2 holds. The term $\boldsymbol{\tau}'(-\nabla_\Gamma\, q)$ is the correction to the Biot–Savart law mentioned subsection 1.2. From this point the construction then proceeds as before.

6. Regularity.

We begin this section by recalling a result of Costabel [7].

LEMMA 6.1. *Let $\Omega \subset \mathbb{R}^3$ denote a handle-free, bounded Lipschitz domain and let $\mathbf{U} \in \mathbf{L}^2(\Omega)$ fulfil:*

$$\operatorname{div} \mathbf{U} \in L^2(\Omega),\ \operatorname{curl} \mathbf{U} \in \mathbf{L}^2(\Omega). \tag{6.1}$$

Then $\mathbf{U}$ satisfies $\mathbf{U} \cdot \mathbf{n} \in L^2(\Gamma)$ on Γ if and only if $\mathbf{U} \times \mathbf{n} \in \mathbf{L}^2(\Gamma)$; and in this case $\mathbf{U}$ fulfils $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$.

This result can directly be applied to solutions of the div-curl system (3.1). In the following, we show that it also implies higher regularity of the associated stream functions.

THEOREM 6.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded, handle-free Lipschitz domain. Let $\mathbf{F} \in \mathbf{L}^2(\Omega)$ be given and fulfil the integrability condition (3.2), or the equivalent conditions (3.11a) and (3.12). Furthermore, let $g \in L^2(\Gamma)$ be given, such that $\langle g, 1 \rangle_{\Gamma_i} = 0$ for all $i = 0, \dots, \beta_2$.*

Then, the unique solution $\mathbf{U} \in \mathbf{L}^2(\Omega)$ of the div-curl system (3.1) with $\mathbf{U} \cdot \mathbf{n} = g$ on Γ fulfils $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$, and its uniquely determined stream function $\mathbf{A}$ from Theorem 4.2 fulfils $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma)$.

Proof. The regularity of $\mathbf{U}$ is exactly Costabel's result Lemma 6.1.

For the regularity of $\mathbf{A}$, we first consider the case $g = 0$. Thus, let $\mathbf{U}_0$ denote the unique solution of the div-curl system (3.1) that satisfies $\mathbf{U}_0 \cdot \mathbf{n} = 0$ on Γ, and let $\mathbf{A}_0 \in \mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ denote its associated stream function. An application of the representation formula for the vector Laplacian then yields:

$$\begin{aligned} \mathbf{A}_0 &= \boldsymbol{\mathcal{N}}(\widetilde{\mathbf{F}} + \boldsymbol{\tau}' \mathbf{s}_0) \quad && \text{on } \mathbb{R}^3, \\ \mathbf{s}_0 &= [\![\operatorname{curl} \mathbf{A}_0 \times \mathbf{n}]\!] \quad && \text{on } \Gamma, \end{aligned} \tag{6.2}$$

where $\widetilde{\mathbf{F}} \in \mathbf{L}^2(\mathbb{R}^3)$ is $\mathbf{F}$'s zero extension as defined in (5.21). Clearly, from the mapping properties of the Newton operator, it follows that $\boldsymbol{\mathcal{N}} \widetilde{\mathbf{F}} \in \mathbf{H}^2_{\mathrm{loc}}(\mathbb{R}^3)$. For the boundary term we note that from the construction of $\mathbf{A}_0$ in the proof of Theorem 4.1 it is clear that $\operatorname{curl} \mathbf{A}_0 = \mathbf{0}$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. This implies that

$$\mathbf{s}_0 = \mathbf{U}_0 \times \mathbf{n} \qquad \text{on } \Gamma, \tag{6.3}$$

and because of Lemma 6.1 this yields $\mathbf{s}_0 \in \mathbf{L}^2(\Gamma)$. The boundary term $\boldsymbol{\mathcal{N}} \boldsymbol{\tau}' \mathbf{s}_0$ may thus alternatively be interpreted as a component-wise application of the scalar single layer potential operator $\mathcal{N} \gamma'$ to the components of $\mathbf{s}_0$. For this operator the following mapping property it is known [16, Remark 3.1.18b]:

$$\mathcal{N} \gamma' : L^2(\Gamma) \to H^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma), \tag{6.4}$$

and thus $\boldsymbol{\mathcal{N}}\boldsymbol{\tau}'\mathbf{s}_0 \in \mathbf{H}^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma)$.

For general boundary data $\mathbf{U}\cdot\mathbf{n} = g \in L^2(\Gamma)$, one then needs to solve the hypersingular boundary integral equation:

$$\forall v \in H^{\frac{1}{2}}(\Gamma): \qquad \langle \boldsymbol{\tau}\boldsymbol{\mathcal{N}}\boldsymbol{\tau}'\,\mathbf{curl}_\Gamma\, p, \mathbf{curl}_\Gamma\, v\rangle_\Gamma = \langle g, v\rangle_\Gamma, \tag{6.5}$$

for the unknown $p \in H^{\frac{1}{2}}(\Gamma)/\mathbb{R}$ and set $\mathbf{s} := \mathbf{s}_0 + \mathrm{curl}_\Gamma\, p$, $\mathbf{A} := \boldsymbol{\mathcal{N}}(\widetilde{\mathbf{F}} + \boldsymbol{\tau}'\mathbf{s})$. For the integral equation the following regularity result is known [16, Theorem 3.2.3b]:

$$g \in L^2(\Gamma) \Longrightarrow p \in H^1(\Gamma). \tag{6.6}$$

Thus $\mathrm{curl}_\Gamma\, p \in \mathbf{L}^2(\Gamma)$ and by the same arguments as above one obtains that $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Gamma)$. ∎

7. Numerical Illustration of Increased Regularity. We consider the domain $\Omega := (0,1)^3 \setminus [0.1, 0.8]^3$ and the smooth velocity field $\mathbf{U} \in \mathbf{C}^\infty(\Omega)$ associated to the vorticity $\mathbf{F} \in \mathbf{C}^\infty(\Omega)$ given by:

$$\mathbf{U}(\mathbf{x}) := \frac{1}{2}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \qquad \mathbf{F}(\mathbf{x}) := \mathrm{curl}\,\mathbf{U}(\mathbf{x}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{7.1}$$

The domain Ω was chosen to be asymmetric, non-smooth, non-convex, and topologically non-trivial ($\beta_2 = 1$), while at the same time being 'easy' from the viewpoint of meshing. An illustration is given in Figure 2.

Neither for the tangential potential $\mathbf{A}_\mathrm{T}$, nor for the potential introduced in this work explicit expressions are known. Thus, the finite element method by Amrouche et al. has been implemented to compute $\mathbf{A}_\mathrm{T}$. [1] For the other stream function, notice that in this case the Newton potential:

$$\boldsymbol{\mathcal{N}}\widetilde{\mathbf{F}}(\mathbf{x}) = \frac{1}{4\pi}\int_\Omega \frac{\mathrm{d}\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{7.2}$$

can be easily evaluated analytically. For the Laplace–Beltrami equation for q as well as the hypersingular boundary integral equation for p standard Galerkin methods were used. Note that the method by Amrouche et al. requires to use the velocity field $\mathbf{U}$ as input, while the approach discussed in this work only requires $\mathbf{F}$ and the boundary data $\mathbf{U}\cdot\mathbf{n}$.

The numerical results along the line $(x_1, 0.5, 0.8)^\top$ are shown in Figure 3. One clearly sees that close to the corners of the interior boundary Γ_1 neither solutions are smooth. But while the tangential potential $\mathbf{A}_\mathrm{T}$ first shows a very steep increase at $x_1 = 0.1$, followed by a jump-discontinuity, the new potential $\mathbf{A}$ only exhibits a small kink, which suggests more regularity.

8. Conclusions and Outlook. In this work, we have established precise conditions under which a divergence-free velocity field $\mathbf{U} \in \mathbf{L}^2(\Omega)$ can be recovered from its given curl and boundary data $\mathbf{U}\cdot\mathbf{n}$. Additionally, minor complementary assumptions on the boundary data guarantees that this velocity field can be represented in terms of a stream function $\mathbf{A} \in \mathbf{H}^1(\Omega)$, which can be explicitly constructed. This stream function is more regular than the tangential vector potential suggested by Amrouche et al. [1]

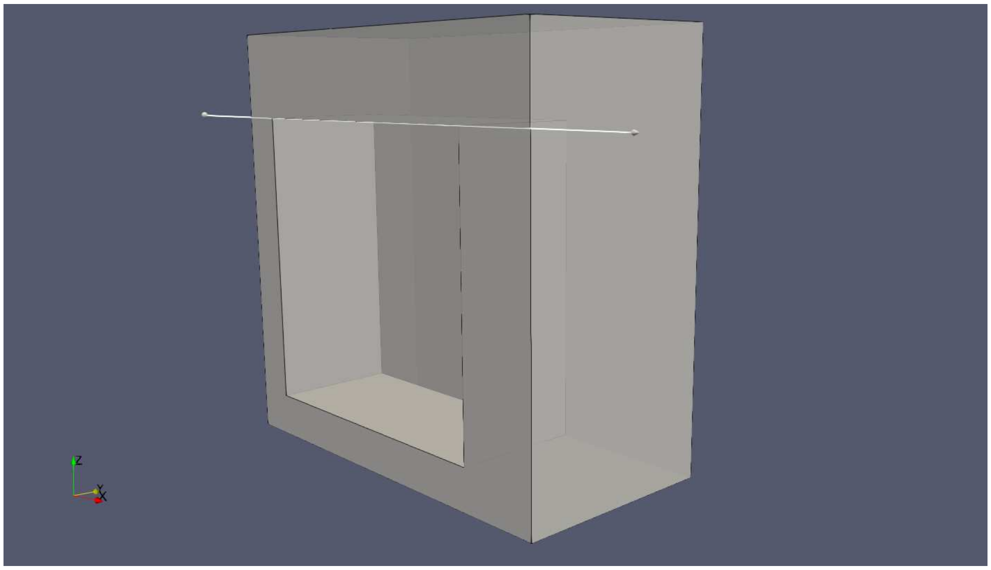

FIGURE 2. *A graphical illustration of the domain used in the numerical experiment, $\Omega = (0,1)^3 \setminus [0.1, 0.8]^3$. Here only the part where $x_2 \geq 0.5$ is shown. The domain is asymmetric, has non-trivial topology, and is not convex. In Figure 3 plots along the indicated line are given.*

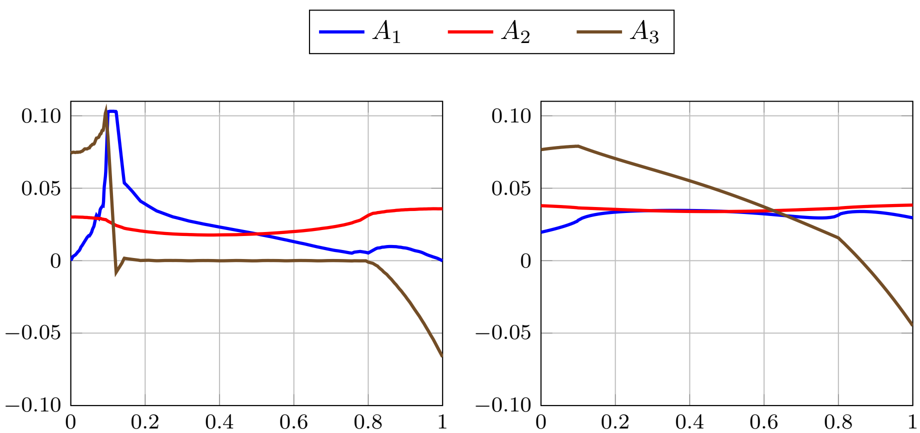

FIGURE 3. *Two different vector potentials for the same velocity field $\mathbf{U} = \frac{1}{2}(-x_2, x_1, 0)^\top$ on the domain $\Omega = (0,1)^3 \setminus [0.1, 0.8]^3$, plotted along the line $(x_1, 0.5, 0.8)^\top$. On the left: the tangential vector potential $\mathbf{A}_\mathrm{T} \in \mathbf{H}_0(\mathrm{div};\Omega) \cap \mathbf{H}(\mathrm{curl};\Omega)$ by Amrouche et al. [1] Especially the third component A_3 shows a very steep increase at $x_1 = 0.1$, followed by a jump-type discontinuity. On the right: the new vector potential $\mathbf{A}$ presented in this work. In this case we have $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}(\Omega)$; only a small kink in the components is visible at $x_1 = 0.1$ and $x_1 = 0.8$.*

The regularity result of Theorem 6.2 is sharp in several ways. Let us for example consider the case of a handle-free domain Ω with $\beta_1 = 0$ and suppose that $\mathbf{F} \equiv \mathbf{0}$. It is then a classical result that the velocity field $\mathbf{U}$ can be written in terms of the gradient of a scalar potential: $\mathbf{U} = -\nabla P$, where $-\Delta P = 0$ in Ω. Even if the given boundary data $\mathbf{U} \cdot \mathbf{n}$ is smooth, it is known from regularity theory for the scalar

Laplace equation that, on bounded Lipschitz domains, the highest regularity one can expect is $P \in H^{\frac{3}{2}}(\Omega)$, which therefore only leads to $\mathbf{U} \in \mathbf{H}^{\frac{1}{2}}(\Omega)$. There are indeed examples of domains Ω and boundary data $\mathbf{U} \cdot \mathbf{n}$ where $P \notin H^{\frac{3}{2}+\varepsilon}(\Omega)$ for any $\varepsilon > 0$. In this sense, the vector potential $\mathbf{A} \in \mathbf{H}^{\frac{3}{2}}(\Omega)$ introduced in this work has the highest possible regularity one can expect for arbitrary Lipschitz domains.

However, an interesting question remains. Suppose that the given data $\mathbf{F}$ and $\mathbf{U} \cdot \mathbf{n}$ are such that the velocity field $\mathbf{U}$ does happen to have higher regularity, say $\mathbf{U} \in \mathbf{H}^{s}(\Omega)$ for some $s > \frac{1}{2}$. McIntosh and Costabel have proven that in this case, another vector potential $\mathbf{A}_s \in \mathbf{H}^{1+s}(\Omega)$ exists. [14, Corollary 4.7] In other words, there always exists a stream function that is more regular than its velocity field by one order. From a numerical point of view, this would be desirable, because the price paid to approximate $\mathbf{A}_s$ instead of $\mathbf{U}$ could be compensated by higher order approximations. In the numerical example discussed in section 7, such a smooth vector potential is given by:

$$\mathbf{A}_\infty(\mathbf{x}) = -\frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ x_1^2 + x_2^2 \end{pmatrix}. \tag{8.1}$$

However, numerical experiments indicate that the vector potential proposed in this work is *not smooth*. Therefore, the problem of devising an algorithm to approximate reliably and efficiently the

smoothest possible stream function remains open.

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