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Counterexamples to local Lipschitz and local Hölder continuity with respect to the initial values for additive noise driven stochastic differential equations with smooth drift coefficient functions with at most polynomially growing derivatives

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Abstract

In the recent article [A. Jentzen, B. Kuckuck, T. Müller-Gronbach, and L. Yaroslavtseva, arXiv:1904.05963 (2019)] it has been proved that the solutions to every additive noise driven stochastic differential equation (SDE) which has a drift coefficient function with at most polynomially growing first order partial derivatives and which admits a

Lyapunov-type condition (ensuring the the existence of a unique solution to the SDE) depend in a logarithmically Hölder continuous way on their initial values. One might then wonder whether this result can be sharpened and whether in fact, SDEs from this class necessarily have solutions which depend locally Lipschitz continuously on their initial value. The key contribution of this article is to establish that this is not the case. More precisely, we supply a family of examples of additive noise driven SDEs which have smooth drift coefficient functions with at most polynomially growing derivatives whose solutions do not depend on their initial value in a locally Lipschitz continuous, nor even in a locally Hölder continuous way.

Contents

1 Introduction

The regularity of stochastic differential equations (SDEs) with respect to their initial values naturally arises as an important problem in stochastic analysis (cf., e.g., Chen & Li [1], Cox et al. [2], Fang et al. [3], Hairer et al. [4], Hairer & Mattingly [5], Hudde et al. [7], Krylov [12], Li [13], Li & Scheutzow [14], Liu & Röckner [15], and Scheutzow & Schulze [16]). At the same time this problem has strong links to the analysis of numerical approximations for SDEs (cf., e.g., Hudde et al. [6], Hutzenthaler & Jentzen [8], Hutzenthaler et al. [9], and Zhang [17]). There are several results in the scientific literature that provide sufficient or necessary conditions which ensure that SDEs have suitable regularity properties in the initial value (cf., e.g., [1, 2, 3, 4, 5, 7, 12, 13, 14, 15, 16]). In particular, in the recent article [10] it has been proved that every additive noise driven SDE which admits a Lyapunov-type condition (that ensures the existence of a unique solution of the SDE) and which has a drift coefficient function whose first order partial derivatives grow at most polynomially is at least logarithmically Hölder continuous in the initial value. This result shows that the solutions of additive noise driven SDEs which have a smooth drift coefficient function with at most polynomially growing derivative cannot have arbitrarily bad regularity properties with respect to the initial value (cf., e.g., the negative results in Hairer et al. [4] and [11] for SDEs without the restriction on the drift coefficient function). However, this result does not imply local Lipschitz continuity with respect to the initial value. Having this in mind, one may wonder whether the main result in [10] is actually sharp or whether, in fact, SDEs from this class necessarily have solutions which depend locally Lipschitz continuously on their initial value. The key contribution of this article is to establish that this is not the case. More precisely, the main result of this article, Theorem 4.4 in Subsection 4.3 below, shows that there are additive noise driven SDEs which have smooth drift coefficient functions with at most polynomially growing derivatives whose solutions do not depend on their initial value in a locally Lipschitz continuous, nor even in a locally Hölder continuous way. In order to illustrate the findings of this article in more detail, we now present in the following theorem a simplified version of Theorem 4.4 below.

Theorem 1.1. *Let $m \in \mathbb{N}$, $d \in \{5, 6, \dots\}$, $T \in (0,\infty)$, $\tau \in (0,T)$, $v \in \mathbb{R}^d$, $\delta \in \mathbb{R}^d \setminus \{0\}$ let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $|||\cdot||| \colon \mathbb{R}^m \to [0,\infty)$ be a norm, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then there exist $\mu \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $V \in C^\infty(\mathbb{R}^d, [0,\infty))$, $\kappa \in (0,\infty)$ such that*

(i) it holds for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |||z|||)V(x)$, and $\|x\| \leq V(x)$,

(ii) there exist unique stochastic processes $X^x \colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{1}$$

(iii) it holds for all $R, r \in (0,\infty)$ that

$$\sup_{x\in[-R,R]^d} \sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t)\|^r\big] < \infty, \tag{2}$$

(iv) it holds for all $R, q \in (0,\infty)$ that there exists $c \in (0,\infty)$ such that for all $x, y \in [-R,R]^d$ with $0 < \|x - y\| \neq 1$ it holds that

$$\sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] \leq c\,|\ln(\|x - y\|)|^{-q}, \tag{3}$$

and

(v) it holds for all $t \in (\tau, T)$, $\alpha \in (0, \infty)$ that there exists $c \in (0, \infty)$ such that for all $w \in \{v + r\delta : r \in [0, 1]\}$ it holds that

$$c \, \|v - w\|^{\alpha} \leq \mathbb{E}\big[\|X^v(t) - X^w(t)\|\big]. \tag{4}$$

Theorem 1.1 above is an immediate consequence of Theorem 4.4 below, the main result of this article. Theorem 4.4 in turn is proved by explicitly constructing a specific example of a family of SDEs with the desired properties (cf., e.g., (25), (44), (51), (63), and (120)) Observe that Theorem 1.1 establishes the existence of an additive noise driven SDE with a smooth drift coefficient function $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ and a diffusion coefficient $\sigma \in \mathbb{R}^{d \times m}$ such that the drift coefficient function has at most polynomially growing derivatives and admits a suitable Lyapunov-type condition (see item (i) above), such that there exist unique solution processes $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, to this SDE (see item (ii) above), such that for every $x \in \mathbb{R}^d$, $t \in [0, T]$ the absolute moments $\mathbb{E}\big[\|X^x(t)\|^r\big]$, $r \in (0, \infty)$, of the solution processes are finite (see item (iii) above), and such that the solution is regular with respect to the initial value in the sense of item (iv) above (cf. [10, Theorem 8.4]), yet fails to be locally Lipschitz or locally Hölder continuous in the initial values (see item (v) above).

The remainder of this article is organized as follows: In Section 2 we establish, roughly speaking, the existence of additive noise driven SDEs whose solutions depend non-locally Hölder continuously on their initial values. In Section 3 we establish, roughly speaking, the existence of solutions to certain additive noise driven SDEs whose solutions depend non-locally Hölder continuously on their initial values and whose drift coefficient functions are smooth with at most polynomially growing derivatives. In Section 4 we combine the results of Section 3 with an essentially well-known fact on affine linear transformations of solutions to SDEs in order to prove Theorem 4.4, the main result of this article.

2 On the existence of axis-aligned stochastic differential equations (SDEs) with non-locally Hölder continuous dependence on the initial values

In this section we establish in Proposition 2.3 below, roughly speaking, the existence of additive noise driven SDEs whose solutions depend non-locally Hölder continuously on their initial values. In our proof of Proposition 2.3 we employ the elementary lower bound for certain functionals of standard normal random variables in Lemma 2.1 below as well as the essentially well-known result on certain Lebesgue integrals involving standard Brownian motions in Lemma 2.2 below. For completeness we also provide in this section a detailed proof for Lemma 2.2. Our proof is elementary and avoids stochastic integration theory and the use of Itô's formula. Alternatively, Lemma 2.2 can be proved through a straightforward application of Itô's lemma.

2.1 Lower bounds for certain functionals of standard normal random variables

Lemma 2.1. *Let $p \in [1, \infty)$, $\kappa, c \in (0, \infty)$, $\varepsilon \in (0, 1/e]$ satisfy $c = p\kappa^{-2/p} + ([2\pi]^{1/2}p + 1)\kappa + 1$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $Z \colon \Omega \to \mathbb{R}$ be a standard normal random variable. Then*

$$\mathbb{E}\big[\varepsilon \exp\big(\kappa\,|Z|^p - \varepsilon^2\kappa \exp(2\kappa\,|Z|^p)\big)\big] \geq \exp\big(-c\,|\ln(\varepsilon)|^{2/p}\big). \tag{5}$$

Proof of Lemma 2.1. Throughout this proof let $\psi \colon \mathbb{R} \to (0, \infty)$ satisfy for all $z \in \mathbb{R}$ that

$$\psi(z) = \exp\big(z - \kappa \exp(2z)\big) \tag{6}$$

and let $a, b \in [0, \infty)$ satisfy

$$a = \big[\kappa^{-1}(\ln(1/\varepsilon) - 1)\big]^{1/p} \qquad \text{and} \qquad b = \big[\kappa^{-1}\ln(1/\varepsilon)\big]^{1/p}. \tag{7}$$

Note that (6) ensures that for all $z \in \mathbb{R}$ it holds that

$$\begin{aligned}\psi\big(\kappa\,|z|^p + \ln(\varepsilon)\big) &= \exp\big(\kappa\,|z|^p + \ln(\varepsilon) - \kappa \exp\big(2\kappa\,|z|^p + 2\ln(\varepsilon)\big)\big)\\ &= \varepsilon \exp\big(\kappa\,|z|^p - \varepsilon^2\kappa \exp\big(2\kappa\,|z|^p\big)\big).\end{aligned} \tag{8}$$

Combining this with the hypothesis that Z is a standard normal random variable, the fact that $0 \leq a < b$, and the fact that $[0, \infty) \ni z \mapsto \exp(-z^2/2) \in (0, \infty)$ is a decreasing function shows that

$$\begin{aligned}\mathbb{E}\big[\varepsilon \exp\big(\kappa\,|Z|^p - \varepsilon^2\kappa \exp(2\kappa\,|Z|^p)\big)\big] &= \mathbb{E}\big[\psi(\kappa\,|Z|^p + \ln(\varepsilon))\big]\\ &\geq \frac{1}{\sqrt{2\pi}} \int_a^b \psi(\kappa\,|z|^p + \ln(\varepsilon)) \exp\!\left(\frac{-z^2}{2}\right) \mathrm{d}z\\ &\geq \frac{\exp(\frac{-b^2}{2})}{\sqrt{2\pi}} \int_a^b \psi(\kappa\,|z|^p + \ln(\varepsilon))\, \mathrm{d}z\\ &\geq \frac{\exp(\frac{-b^2}{2})(b-a)}{\sqrt{2\pi}} \left[\inf_{z \in [a,b]} \psi(\kappa\,|z|^p + \ln(\varepsilon))\right].\end{aligned} \tag{9}$$

Next observe that the fact that $\{\kappa\,|z|^p + \ln(\varepsilon) \colon z \in [a, b]\} = [-1, 0]$ implies that

$$\inf_{z \in [a,b]} \psi(\kappa\,|z|^p + \ln(\varepsilon)) = \inf_{y \in [-1,0]} \psi(y) = \exp\!\left(\inf_{y \in [-1,0]} (y - \kappa \exp(2y))\right) \geq \exp(-1 - \kappa). \tag{10}$$

In the next step we note that the fact that for all $z \in (0, \infty)$ it holds that $\ln(z) \leq z$ proves that for all $z \in (0, \infty)$ it holds that

$$\exp\!\left(-\frac{(p-1)z^2}{2}\right) \leq \exp\!\left(-\frac{(p-1)\ln(z^2)}{2}\right) = (z^2)^{-\frac{p-1}{2}} = z^{-(p-1)}. \tag{11}$$

The fundamental theorem of calculus, the fact that $0 \leq a^p < b^p$, and the fact that $(0,\infty) \ni z \mapsto z^{-(p-1)/p} \in \mathbb{R}$ is a decreasing function therefore ensure that

$$\begin{aligned} b - a &= \big[z^{\frac{1}{p}}\big]_{z=a^p}^{z=b^p} = p^{-1}\left[\int_{a^p}^{b^p} z^{-\frac{p-1}{p}}\,\mathrm{d}z\right] \geq p^{-1}(b^p - a^p)\left[\inf_{z\in[a^p,b^p]} z^{-\frac{p-1}{p}}\right] \\ &= p^{-1}\kappa^{-1}b^{-(p-1)} \geq (p\kappa)^{-1}\exp\!\left(-\frac{(p-1)b^2}{2}\right). \end{aligned} \tag{12}$$

This and the fact that for all $x \in (0,\infty)$ it holds that $\exp(-x) \leq \frac{1}{x}$ show that

$$\begin{aligned} \frac{\exp\!\big(\frac{-b^2}{2}\big)(b-a)}{\sqrt{2\pi}} &\geq \big[(2\pi)^{1/2}p\kappa\big]^{-1}\exp\!\left(-\frac{pb^2}{2}\right) \\ &= \big[(2\pi)^{1/2}p\kappa\big]^{-1}\exp\!\left(-\frac{p\big[\kappa^{-1}\ln(1/\varepsilon)\big]^{2/p}}{2}\right) \\ &\geq \big[(2\pi)^{1/2}p\kappa\big]^{-1}\exp\!\big(-p\kappa^{-2/p}[\ln(1/\varepsilon)]^{2/p}\big) \\ &\geq \exp\!\big(-(2\pi)^{1/2}p\kappa\big)\exp\!\big(-p\kappa^{-2/p}[\ln(1/\varepsilon)]^{2/p}\big) \\ &= \exp\!\big(-(2\pi)^{1/2}p\kappa - p\kappa^{-2/p}[\ln(1/\varepsilon)]^{2/p}\big). \end{aligned} \tag{13}$$

Combining this with (9) and (10) demonstrates that

$$\begin{aligned} &\mathbb{E}\big[\varepsilon\exp\!\big(\kappa\,|Z|^p - \varepsilon^2\kappa\exp(2\kappa\,|Z|^p)\big)\big] \\ &\geq \exp\!\big(-[2\pi]^{1/2}p\kappa - p\kappa^{-2/p}[\ln(1/\varepsilon)]^{2/p}\big)\exp(-1-\kappa) \\ &= \exp\!\big(-\big([2\pi]^{1/2}p\kappa + 1 + \kappa + p\kappa^{-2/p}[\ln(1/\varepsilon)]^{2/p}\big)\big). \end{aligned} \tag{14}$$

The fact that $\ln(1/\varepsilon)^{2/p} > 1$ hence implies that

$$\mathbb{E}\big[\varepsilon\exp\!\big(\kappa\,|Z|^p - \varepsilon^2\kappa\exp(2\kappa\,|Z|^p)\big)\big] \geq \exp\!\big(-\big([2\pi]^{1/2}p\kappa + 1 + \kappa + p\kappa^{-2/p}\big)[\ln(1/\varepsilon)]^{2/p}\big). \tag{15}$$

The proof of Lemma 2.1 is thus completed. □

2.2 On the distribution of certain integrals involving Brownian motions

Lemma 2.2. *Let $\tau \in (0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $g \in C^1(\mathbb{R},[0,\infty))$ satisfy $\{t \in \mathbb{R} \colon g(t) > 0\} \subseteq [0,\tau]$ and $\int_0^\tau |g(t)|^2\,\mathrm{d}t = 1$, let $W \colon [0,\tau]\times\Omega \to \mathbb{R}$ be a standard Brownian motion with continuous sample paths, and let $X \colon \Omega \to \mathbb{R}$ be a random variable which satisfies for all $\omega \in \Omega$ that*

$$X(\omega) = \int_0^\tau g'(s)W(s,\omega)\,\mathrm{d}s. \tag{16}$$

Then X is a standard normal random variable.

Proof of Lemma 2.2. Throughout this proof let $A \subseteq \mathbb{R}^2$ satisfy

$$A = \{(x,y) \in \mathbb{R}^2 \colon x < y\}. \tag{17}$$

Observe that (16) implies that for all $\omega \in \Omega$ it holds that

$$\begin{aligned} |X(\omega)|^2 &= \left(\int_0^\tau g'(s) W(s,\omega) \, \mathrm{d}s \right) \left(\int_0^\tau g'(u) W(u,\omega) \, \mathrm{d}u \right) \\ &= \int_0^\tau \int_0^\tau g'(s) g'(u) W(s,\omega) W(u,\omega) \, \mathrm{d}u \, \mathrm{d}s. \end{aligned} \tag{18}$$

Fubini's theorem hence shows that

$$\begin{aligned} \mathbb{E}\big[|X|^2\big] &= \int_0^\tau \int_0^\tau g'(s) g'(u) \mathbb{E}[W(s)W(u)] \, \mathrm{d}u \, \mathrm{d}s \\ &= \int_0^\tau \int_0^\tau g'(s) g'(u) \min(s,u) \, \mathrm{d}u \, \mathrm{d}s \\ &= \int_0^\tau \int_0^\tau g'(s) g'(u) \min(s,u) \big(\mathbb{1}_A(s,u) + \mathbb{1}_{\mathbb{R}^2 \setminus A}(s,u) \big) \, \mathrm{d}u \, \mathrm{d}s \\ &= \int_0^\tau \int_0^\tau g'(s) g'(u) \min(s,u) \mathbb{1}_A(s,u) \, \mathrm{d}u \, \mathrm{d}s \\ &\quad + \int_0^\tau \int_0^\tau g'(s) g'(u) \min(s,u) \mathbb{1}_{\mathbb{R}^2 \setminus A}(s,u) \, \mathrm{d}u \, \mathrm{d}s. \end{aligned} \tag{19}$$

This and Fubini's theorem demonstrate that

$$\begin{aligned} \mathbb{E}\big[|X|^2\big] &= \int_0^\tau \int_0^\tau g'(u) g'(s) \min(s,u) \mathbb{1}_A(s,u) \, \mathrm{d}s \, \mathrm{d}u \\ &\quad + \int_0^\tau \int_0^\tau g'(s) g'(u) \min(s,u) \mathbb{1}_{\mathbb{R}^2 \setminus A}(s,u) \, \mathrm{d}u \, \mathrm{d}s \\ &= \int_0^\tau \int_0^u g'(u) g'(s) s \, \mathrm{d}s \, \mathrm{d}u + \int_0^\tau \int_0^s g'(s) g'(u) u \, \mathrm{d}u \, \mathrm{d}s \\ &= 2 \left[\int_0^\tau \int_0^s g'(s) g'(u) u \, \mathrm{d}u \, \mathrm{d}s \right] = 2 \left(\int_0^\tau g'(s) \left[\int_0^s g'(u) u \, \mathrm{d}u \right] \mathrm{d}s \right). \end{aligned} \tag{20}$$

Furthermore, note that the hypothesis that $\{t \in \mathbb{R} \colon g(t) > 0\} \subseteq [0,\tau]$ and the hypothesis that g is a continuous function ensure that

$$g(0) = 0 = g(\tau). \tag{21}$$

Combining this, integration by parts, and the hypothesis that $\int_0^\tau |g(s)|^2 \, \mathrm{d}s = 1$ with (20) proves that

$$\begin{aligned}\mathbb{E}\big[|X|^2\big] &= 2\bigg(\bigg[g(s)\bigg(\int_0^s g'(u)u\,\mathrm{d}u\bigg)\bigg]_{s=0}^{s=\tau} - \int_0^\tau g(s)g'(s)s\,\mathrm{d}s\bigg)\\ &= -2\bigg[\int_0^\tau g(s)g'(s)s\,\mathrm{d}s\bigg] = -\int_0^\tau \big(\tfrac{\partial}{\partial s}\big[(g(s))^2\big]\big)s\,\mathrm{d}s\\ &= -\bigg(\big[(g(s))^2 s\big]_{s=0}^{s=\tau} - \int_0^\tau (g(s))^2\,\mathrm{d}s\bigg) = \int_0^\tau (g(s))^2\,\mathrm{d}s = 1.\end{aligned} \tag{22}$$

Combining this with (16) establishes that X is a standard normal random variable. The proof of Lemma 2.2 is thus completed. □

2.3 On SDEs with irregularities in the initial value

Proposition 2.3. *Let $T \in (0,\infty)$, $\tau \in (0,T)$, $n \in \mathbb{N}$, $f \in C^1(\mathbb{R},[0,\infty))$, $g \in C^2(\mathbb{R},[0,\infty))$ satisfy $\{t \in \mathbb{R}: g(t) > 0\} \subseteq [0,\tau]$, $\{t \in \mathbb{R}: f(t) > 0\} = (\tau,T)$, and $\int_0^\tau |g(s)|^2\,\mathrm{d}s = 1$, let $\sigma = (0,1,0,0,0) \in \mathbb{R}^5$, let $\|\cdot\|\colon \mathbb{R}^5 \to [0,\infty)$ be the standard norm on $\mathbb{R}^5$, let $\mu\colon \mathbb{R}^5 \to \mathbb{R}^5$ satisfy for all $x = (x_1,x_2,x_3,x_4,x_5) \in \mathbb{R}^5$ that*

$$\mu(x) = \big(1, 0, g'(x_1)x_2, f(x_1)x_4x_5, f(x_1)((x_3)^n - (x_4)^2)\big), \tag{23}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a standard Brownian motion with continuous sample paths, and let $X^x = (X_1^x, X_2^x, X_3^x, X_4^x, X_5^x)\colon [0,T]\times\Omega \to \mathbb{R}^5$, $x \in \mathbb{R}^5$, be stochastic processes with continuous sample paths which satisfy for all $x \in \mathbb{R}^5$, $t \in [0,T]$, $\omega \in \Omega$ that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega). \tag{24}$$

Then it holds for all $t \in (\tau,T)$ that there exists $c \in (0,\infty)$ such that for all $\varepsilon \in (0,1/e]$, $h = (0,0,0,\varepsilon,0) \in \mathbb{R}^5$ it holds that

$$\mathbb{E}\big[|X_4^h(t) - X_4^0(t)|\big] \geq \big|\mathbb{E}[X_4^h(t)] - \mathbb{E}[X_4^0(t)]\big| \geq \exp\!\big(-c\,|\ln(\varepsilon)|^{2/n}\big). \tag{25}$$

Proof of Proposition 2.3. Throughout this proof let $v = (0,0,0,1,0) \in \mathbb{R}^5$ and let $\kappa_t \in [0,\infty)$, $t \in [\tau,T]$, satisfy for all $t \in [\tau,T]$ that

$$\kappa_t = \int_\tau^t \int_\tau^s f(u)f(s)\,\mathrm{d}u\,\mathrm{d}s. \tag{26}$$

Observe that (23) and (24) imply that for all $\varepsilon \in [0,\infty)$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} &X_1^{\varepsilon v}(t,\omega) = \int_0^t 1\,\mathrm{d}s = t\\ \text{and}\qquad &X_2^{\varepsilon v}(t,\omega) = \int_0^t 0\,\mathrm{d}s + W(t,\omega) = W(t,\omega).\end{aligned} \tag{27}$$

This, (23), and (24) show that for all $\varepsilon \in [0,\infty)$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$X_3^{\varepsilon v}(t,\omega) = \int_0^t g'(X_1^{\varepsilon v}(s,\omega)) X_2^{\varepsilon v}(s,\omega)\,\mathrm{d}s = \int_0^t g'(s) W(s,\omega)\,\mathrm{d}s. \tag{28}$$

The hypothesis that $\{t \in \mathbb{R} \colon g(t) > 0\} \subseteq [0,\tau]$ hence ensures that for all $\varepsilon \in [0,\infty)$, $t \in [\tau,T]$, $\omega \in \Omega$ it holds that

$$X_3^{\varepsilon v}(t,\omega) = \int_0^\tau g'(s) W(s,\omega)\,\mathrm{d}s = X_3^{\varepsilon v}(\tau,\omega). \tag{29}$$

Next note that (23), (24), and (27) prove that for all $\varepsilon \in [0,\infty)$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} X_4^{\varepsilon v}(t,\omega) &= \varepsilon + \int_0^t f(X_1^{\varepsilon v}(s,\omega)) X_4^{\varepsilon v}(s,\omega) X_5^{\varepsilon v}(s,\omega)\,\mathrm{d}s \\ &= \varepsilon + \int_0^t f(s) X_4^{\varepsilon v}(s,\omega) X_5^{\varepsilon v}(s,\omega)\,\mathrm{d}s. \end{aligned} \tag{30}$$

Hence, we obtain that for all $\varepsilon \in [0,\infty)$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$X_4^{\varepsilon v}(t,\omega) = \varepsilon \exp\!\left(\int_0^t f(s) X_5^{\varepsilon v}(s,\omega)\,\mathrm{d}s \right). \tag{31}$$

The hypothesis that $\{t \in \mathbb{R} \colon f(t) > 0\} = (\tau,T)$ hence implies that for all $\varepsilon \in [0,\infty)$, $t \in [\tau,T]$, $\omega \in \Omega$ it holds that

$$X_4^{\varepsilon v}(t,\omega) = \varepsilon \exp\!\left(\int_\tau^t f(s) X_5^{\varepsilon v}(s,\omega)\,\mathrm{d}s \right). \tag{32}$$

Moreover, observe that (23), (24), and (27) show that for all $\varepsilon \in [0,\infty)$, $s \in [0,T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} X_5^{\varepsilon v}(s,\omega) &= \int_0^s f(X_1^{\varepsilon v}(u,\omega)) \big([X_3^{\varepsilon v}(u,\omega)]^n - [X_4^{\varepsilon v}(u,\omega)]^2 \big)\,\mathrm{d}u \\ &= \int_0^s f(u) \big([X_3^{\varepsilon v}(u,\omega)]^n - [X_4^{\varepsilon v}(u,\omega)]^2 \big)\,\mathrm{d}u. \end{aligned} \tag{33}$$

Combining this with the hypothesis that $\{t \in \mathbb{R} \colon f(t) > 0\} = (\tau,T)$ and (29) demonstrates that for all $\varepsilon \in [0,\infty)$, $s \in [\tau,T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} X_5^{\varepsilon v}(s,\omega) &= \int_\tau^s f(u) \big([X_3^{\varepsilon v}(u,\omega)]^n - [X_4^{\varepsilon v}(u,\omega)]^2 \big)\,\mathrm{d}u \\ &= \int_\tau^s f(u) \big([X_3^{\varepsilon v}(\tau,\omega)]^n - [X_4^{\varepsilon v}(u,\omega)]^2 \big)\,\mathrm{d}u \\ &= [X_3^{\varepsilon v}(\tau,\omega)]^n \left[\int_\tau^s f(u)\,\mathrm{d}u \right] - \left[\int_\tau^s f(u) [X_4^{\varepsilon v}(u,\omega)]^2\,\mathrm{d}u \right]. \end{aligned} \tag{34}$$

This and (32) prove that for all $\varepsilon \in [0, \infty)$, $t \in [\tau, T]$, $\omega \in \Omega$ it holds that

$$
\begin{aligned}
&X_4^{\varepsilon v}(t, \omega) \\
&= \varepsilon \exp\biggl(\int_\tau^t f(s) \biggl[[X_3^{\varepsilon v}(\tau, \omega)]^n \biggl(\int_\tau^s f(u)\, \mathrm{d}u\biggr) - \biggl(\int_\tau^s f(u)[X_4^{\varepsilon v}(u, \omega)]^2\, \mathrm{d}u\biggr)\biggr]\, \mathrm{d}s\biggr) \\
&= \varepsilon \exp\biggl(\int_\tau^t f(s)[X_3^{\varepsilon v}(\tau, \omega)]^n \biggl(\int_\tau^s f(u)\, \mathrm{d}u\biggr) \mathrm{d}s - \int_\tau^t f(s) \biggl(\int_\tau^s f(u)[X_4^{\varepsilon v}(u, \omega)]^2\, \mathrm{d}u\biggr) \mathrm{d}s\biggr) \\
&= \varepsilon \exp\biggl([X_3^{\varepsilon v}(\tau, \omega)]^n \int_\tau^t \int_\tau^s f(u) f(s)\, \mathrm{d}u\, \mathrm{d}s - \int_\tau^t \int_\tau^s f(s) f(u) [X_4^{\varepsilon v}(u, \omega)]^2\, \mathrm{d}u\, \mathrm{d}s\biggr) \\
&= \varepsilon \exp\biggl(\kappa_t [X_3^{\varepsilon v}(\tau, \omega)]^n - \int_\tau^t \int_\tau^s f(s) f(u) [X_4^{\varepsilon v}(u, \omega)]^2\, \mathrm{d}u\, \mathrm{d}s\biggr).
\end{aligned} \tag{35}
$$

Therefore, we obtain that for all $\varepsilon \in [0, \infty)$, $u \in [\tau, T]$, $\omega \in \Omega$ it holds that

$$
X_4^{\varepsilon v}(u, \omega) \le \varepsilon \exp\bigl(\kappa_u [X_3^{\varepsilon v}(\tau, \omega)]^n\bigr). \tag{36}
$$

Combining this with the fact that for all $s \in [\tau, T]$, $t \in [s, T]$ it holds that $\kappa_t \ge \kappa_s$ ensures that for all $\varepsilon \in [0, \infty)$, $t \in [\tau, T]$, $\omega \in \Omega$ it holds that

$$
\begin{aligned}
&\int_\tau^t \int_\tau^s f(s) f(u) [X_4^{\varepsilon v}(u, \omega)]^2\, \mathrm{d}u\, \mathrm{d}s \le \int_\tau^t \int_\tau^s f(s) f(u) \varepsilon^2 \exp\bigl(2\kappa_u [X_3^{\varepsilon v}(\tau, \omega)]^n\bigr)\, \mathrm{d}u\, \mathrm{d}s \\
&\le \varepsilon^2 \exp\bigl(2\kappa_t [X_3^{\varepsilon v}(\tau, \omega)]^n\bigr) \int_\tau^t \int_\tau^s f(s) f(u)\, \mathrm{d}u\, \mathrm{d}s = \varepsilon^2 \kappa_t \exp\bigl(2\kappa_t [X_3^{\varepsilon v}(\tau, \omega)]^n\bigr).
\end{aligned} \tag{37}
$$

This and (35) establish that for all $\varepsilon \in [0, \infty)$, $t \in [\tau, T]$, $\omega \in \Omega$ it holds that

$$
X_4^{\varepsilon v}(t, \omega) \ge \varepsilon \exp\Bigl(\kappa_t [X_3^{\varepsilon v}(\tau, \omega)]^n - \varepsilon^2 \kappa_t \exp\bigl(2\kappa_t [X_3^{\varepsilon v}(\tau, \omega)]^n\bigr)\Bigr). \tag{38}
$$

Furthermore, note that (32) implies that for all $t \in [\tau, T]$, $\omega \in \Omega$ it holds that

$$
X_4^0(t, \omega) = 0. \tag{39}
$$

This, (29), and (38) show that for all $\varepsilon \in [0, \infty)$, $t \in [\tau, T]$ it holds that

$$
\begin{aligned}
\bigl|\mathbb{E}[X_4^{\varepsilon v}(t)] - \mathbb{E}[X_4^0(t)]\bigr| &= \bigl|\mathbb{E}[X_4^{\varepsilon v}(t)]\bigr| \\
&\ge \mathbb{E}\bigl[\varepsilon \exp\bigl(\kappa_t [X_3^{\varepsilon v}(\tau)]^n - \varepsilon^2 \kappa_t \exp\bigl(2\kappa_t [X_3^{\varepsilon v}(\tau)]^n\bigr)\bigr)\bigr] \\
&= \mathbb{E}\bigl[\varepsilon \exp\bigl(\kappa_t [X_3^{\varepsilon v}(t)]^n - \varepsilon^2 \kappa_t \exp\bigl(2\kappa_t [X_3^{\varepsilon v}(t)]^n\bigr)\bigr)\bigr].
\end{aligned} \tag{40}
$$

In the next step we observe that the fact that $\{t \in \mathbb{R} \colon f(t) > 0\} = (\tau, T)$ ensures that for all $t \in (\tau, T]$ it holds that

$$
\kappa_t > 0. \tag{41}
$$

In addition, note that (29) and Lemma 2.2 (with $\tau \leftarrow \tau$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $g \leftarrow g$, $W \leftarrow W|_{[0,\tau]\times\Omega}$, $X \leftarrow (\Omega \ni \omega \mapsto X_3^{\varepsilon v}(t,\omega) \in \mathbb{R})$ for $\varepsilon \in [0,\infty)$, $t \in [\tau, T]$ in the notation of Lemma 2.2) demonstrate that for all $\varepsilon \in [0,\infty)$, $t \in [\tau, T]$ it holds that $X_3^{\varepsilon v}(t)$ is a standard normal random variable. Combining this and (41) with Lemma 2.1 (with $p \leftarrow n$, $\kappa \leftarrow \kappa_t$, $c \leftarrow n(\kappa_t)^{-2/n} + (\sqrt{2\pi}n + 1)\kappa_t + 1$, $\varepsilon \leftarrow \varepsilon$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $Z \leftarrow (\Omega \ni \omega \mapsto X_3^{\varepsilon v}(t,\omega) \in \mathbb{R})$ for $\varepsilon \in (0, 1/e]$, $t \in (\tau, T)$ in the notation of Lemma 2.1) proves that for all $t \in (\tau, T)$ there exist $c \in (0,\infty)$ such that for all $\varepsilon \in (0, 1/e]$ it holds that

$$\mathbb{E}\big[\varepsilon \exp\big(\kappa_t [X_3^{\varepsilon v}(t)]^n - \varepsilon^2 \kappa_t \exp\big(2\kappa_t [X_3^{\varepsilon v}(t)]^n\big)\big)\big] \geq \exp\big(-c\,|\ln(\varepsilon)|^{2/n}\big). \tag{42}$$

This and (40) establish (25). The proof of Proposition 2.3 is thus completed. □

3 On the existence of solutions to axis-aligned SDEs with non-locally Hölder continuous dependence on the initial values

In this section we establish in Proposition 3.5 below, roughly speaking, the existence of solutions to certain additive noise driven SDEs whose solutions depend non-locally Hölder continuously on their initial values and whose drift coefficient functions are smooth with at most polynomially growing derivatives. In our proof of Proposition 3.5 we employ Proposition 2.3 above, the elementary fact in Lemma 3.2 below that certain drift coefficient functions have at most polynomially growing derivatives, the elementary fact in Lemma 3.3 below that appropriate drift coefficient functions satisfy a suitable Lyapunov-type condition, as well as the well-known result on the existence of certain smooth bump functions in Lemma 3.4 below. In our proof of Lemma 3.2 below we employ the well-known fact in Lemma 3.1 below that the Frobenius norm is an upper bound for the operator norm induced by the standard norm.

3.1 On drift coefficient functions with at most polynomially growing derivatives

Lemma 3.1. *Let $d \in \mathbb{N}$, $A = (a_{i,j})_{i,j\in\{1,2,\ldots,d\}} \in \mathbb{R}^{d\times d}$ and let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$. Then it holds for all $x \in \mathbb{R}^d$ that*

$$\|Ax\| \leq \left[\sum_{i,j=1}^{d} |a_{i,j}|^2\right]^{\frac{1}{2}} \|x\|. \tag{43}$$

Lemma 3.2. *Let $n \in \mathbb{N} \cap [2,\infty)$, $c \in [0,\infty)$, let $\|\cdot\| \colon \mathbb{R}^5 \to [0,\infty)$ be the standard norm on $\mathbb{R}^5$, let $\mu \colon \mathbb{R}^5 \to \mathbb{R}^5$, $f \in C^1(\mathbb{R},[0,\infty))$, $g \in C^2(\mathbb{R},[0,\infty))$ satisfy for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ that*

$$\mu(x) = \big(1, 0, g'(x_1)x_2, f(x_1)x_4x_5, f(x_1)\big[(x_3)^n - (x_4)^2\big]\big), \tag{44}$$

and assume $c \geq \sup_{t\in\mathbb{R}}\big[\max\{|f(t)|, |f'(t)|, |g'(t)|, |g''(t)|\}\big]$. *Then it holds for all* $x, h \in \mathbb{R}^5$ *that* $\mu \in C^1(\mathbb{R}^5, \mathbb{R}^5)$ *and*

$$\|\mu'(x)h\| \leq 4nc(1 + \|x\|^n)\|h\|. \tag{45}$$

Proof of Lemma 3.2. Observe that (44), the hypothesis that $f \in C^1(\mathbb{R}, [0, \infty))$, and the hypothesis that $g \in C^2(\mathbb{R}, [0, \infty))$ imply that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ it holds that $\mu \in C^1(\mathbb{R}^5, \mathbb{R}^5)$ and

$$\mu'(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ g''(x_1)x_2 & g'(x_1) & 0 & 0 & 0 \\ f'(x_1)x_4x_5 & 0 & 0 & f(x_1)x_5 & f(x_1)x_4 \\ f'(x_1)\big[(x_3)^n - (x_4)^2\big] & 0 & nf(x_1)(x_3)^{n-1} & -2f(x_1)x_4 & 0 \end{pmatrix}. \tag{46}$$

Moreover, note that the hypothesis that $n \geq 2$ and the triangle inequality show that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ it holds that

$$\begin{aligned} \big|(x_3)^n - (x_4)^2\big|^2 &= (x_3)^{2n} - 2(x_3)^n(x_4)^2 + (x_4)^4 \leq |x_3|^{2n} + 2|x_3|^n|x_4|^2 + |x_4|^4 \\ &\leq \|x\|^{2n} + 2\|x\|^{n+2} + \|x\|^4 \leq 4(1 + \|x\|^{2n}). \end{aligned} \tag{47}$$

Combining this with (46) and Lemma 3.1 shows that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, $h \in \mathbb{R}^5$ it holds that

$$\begin{aligned} \|\mu'(x)h\| &\leq \big[|g''(x_1)|^2|x_2|^2 + |g'(x_1)|^2 + |f'(x_1)|^2|x_4|^2|x_5|^2 + |f(x_1)|^2|x_5|^2 + |f(x_1)|^2|x_4|^2 \\ &\qquad + |f'(x_1)|^2\big|(x_3)^n - (x_4)^2\big|^2 + n^2|f(x_1)|^2|x_3|^{2n-2} + 4|f(x_1)|^2|x_4|^2\big]^{1/2}\|h\| \\ &\leq \big[c^2\|x\|^2 + c^2 + c^2\|x\|^2\|x\|^2 + c^2\|x\|^2 + c^2\|x\|^2 + 4c^2(1 + \|x\|^{2n}) \\ &\qquad + n^2c^2\|x\|^{2n-2} + 4c^2\|x\|^2\big]^{1/2}\|h\| \\ &= c\big[7\|x\|^2 + 1 + \|x\|^4 + 4(1 + \|x\|^{2n}) + n^2\|x\|^{2n-2}\big]^{1/2}\|h\| \\ &\leq c\big[(13 + n^2)(1 + \|x\|^{2n})\big]^{1/2}\|h\| \\ &\leq c\big[14n^2(1 + \|x\|^{2n})\big]^{1/2}\|h\| \leq 4cn(1 + \|x\|^n)\|h\|. \end{aligned} \tag{48}$$

This completes the proof of Lemma 3.2. □

3.2 On suitable Lyapunov-type functions for additive noise driven SDEs

Lemma 3.3. *Let* $n \in \mathbb{N}$, $p \in [1, \infty)$, $q \in [2pn, \infty)$, $f \in C^1(\mathbb{R}, [0, \infty))$, $g \in C^2(\mathbb{R}, [0, \infty))$, $\sigma = (0, 1, 0, 0, 0) \in \mathbb{R}^5$, *let* $\|\cdot\|\colon \mathbb{R}^5 \to [0, \infty)$ *be the standard norm on* $\mathbb{R}^5$, *let* $\mu\colon \mathbb{R}^5 \to \mathbb{R}^5$ *satisfy for all* $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ *that*

$$\mu(x) = \big(1, 0, g'(x_1)x_2, f(x_1)x_4x_5, f(x_1)\big[(x_3)^n - (x_4)^2\big]\big), \tag{49}$$

and let $V\colon \mathbb{R}^5 \to [0,\infty)$ *satisfy for all* $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ *that*

$$V(x) = \big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^p + |x_2|^q + |x_3|^q + 1. \tag{50}$$

Then it holds for all $x, h \in \mathbb{R}^5$*,* $z \in \mathbb{R}$ *that* $V \in C^1(\mathbb{R}^5, [0,\infty))$*,* $\|x\| \leq V(x)$*, and*

$$V'(x)\mu(x + \sigma z) \leq \big(2p + (2p+q)\big(\sup\nolimits_{t\in\mathbb{R}}\big[\max\{|f(t)|, |g'(t)|\}\big]\big)\big)(1 + |z|)V(x). \tag{51}$$

Proof of Lemma 3.3. Throughout this proof let $c \in [0,\infty]$ satisfy

$$c = \sup\nolimits_{t\in\mathbb{R}}\big[\max\{|f(t)|, |g'(t)|\}\big] \tag{52}$$

and assume w.l.o.g. that $c < \infty$. Observe that the hypothesis that $p \geq 1$ and the hypothesis that $q \geq 2pn \geq 2$ show that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ it holds that

$$\begin{aligned} \|x\|^2 &= |x_1|^2 + |x_2|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 \\ &\leq |x_1|^2 + \big(1 + |x_2|^q\big) + \big(1 + |x_3|^q\big) + |x_4|^2 + |x_5|^2 \\ &\leq \big(1 + |x_1|^2 + |x_4|^2 + |x_5|^2\big)^p + |x_2|^q + |x_3|^q + 1 = V(x). \end{aligned} \tag{53}$$

The fact that for all $x \in \mathbb{R}^5$ it holds that $V(x) \geq 1$ hence ensures that

$$\|x\| \leq [V(x)]^{1/2} \leq V(x). \tag{54}$$

Furthermore, note that (50), the triangle inequality, the hypothesis that $q \geq 2pn \geq 2$, and the fact that for all $r \in (1,\infty)$, $f \in C(\mathbb{R},\mathbb{R})$, $x \in \mathbb{R}$ with $f = (\mathbb{R} \ni y \mapsto |y|^r \in \mathbb{R})$ it holds that $|f'(x)| = r|x|^{r-1}$ imply that for all $x = (x_1, x_2, x_3, x_4, x_5)$, $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{R}^5$ it holds that $V \in C^1(\mathbb{R}^5, [0,\infty))$ and

$$\begin{aligned} |V'(x)v| \leq \big|2p\big(1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big)^{p-1}(x_1 v_1 + x_4 v_4 + x_5 v_5)\big| \\ + q|x_2|^{q-1}|v_2| + q|x_3|^{q-1}|v_3|. \end{aligned} \tag{55}$$

This proves that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, $z \in \mathbb{R}$ it holds that

$$\begin{aligned} &|V'(x)\mu(x + \sigma z)| \\ &= |V'(x)\mu(x_1, x_2 + z, x_3, x_4, x_5)| \\ &= \big|V'(x)\big(1, 0, g'(x_1)(x_2 + z), f(x_1)x_4x_5, f(x_1)((x_3)^n - (x_4)^2)\big)\big| \\ &\leq \big|2p\big(1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big)^{p-1}\big(x_1 + f(x_1)(x_4)^2 x_5 + f(x_1)((x_3)^n - (x_4)^2)x_5\big)\big| \\ &\qquad + q|x_3|^{q-1}|g'(x_1)(x_2 + z)| \\ &\leq 2p\big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^{p-1}\big(|x_1| + |f(x_1)||x_3|^n|x_5|\big) \\ &\qquad + q|x_3|^{q-1}|g'(x_1)|(|x_2| + |z|). \end{aligned} \tag{56}$$

Next observe that (50) ensures that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ it holds that

$$|x_1| \big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^{p-1} \leq \big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^p \leq V(x). \tag{57}$$

In addition, note that (50), (52), the fact that $\frac{1}{2p} + \frac{p-(1/2)}{p} = 1$, the Young inequality, and the hypothesis that $q \geq 2pn$ demonstrate that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ it holds that

$$\begin{aligned}
&|f(x_1)||x_3|^n|x_5|\big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^{p-1} \\
&\leq c\,|x_3|^n \big|(x_5)^2\big|^{1/2}\big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^{p-1} \\
&\leq c\,|x_3|^n \big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^{p-1/2} \\
&\leq c\left[\frac{|x_3|^{2pn}}{2p} + \frac{\big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^p}{\frac{p}{p-\frac{1}{2}}}\right] \\
&\leq c\big[1 + |x_3|^q + \big|1 + (x_1)^2 + (x_4)^2 + (x_5)^2\big|^p\big] \leq cV(x).
\end{aligned} \tag{58}$$

Next observe that (50), the fact that $\frac{1}{q} + \frac{q-1}{q} = 1$, and the Young inequality show that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ it holds that

$$\begin{aligned}
|x_3|^{q-1}|g'(x_1)||x_2| \leq c\,|x_2||x_3|^{q-1} &\leq c\left[\frac{|x_2|^q}{q} + \frac{(|x_3|^{q-1})^{\frac{q}{q-1}}}{\frac{q}{q-1}}\right] \\
&\leq c\big[|x_2|^q + |x_3|^q\big] \leq cV(x).
\end{aligned} \tag{59}$$

Moreover, note that the fact that $\frac{1}{q} + \frac{q-1}{q} = 1$ implies that for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$, $z \in \mathbb{R}$ it holds that

$$|x_3|^{q-1}|g'(x_1)||z| \leq c\,|z||x_3|^{q-1} \leq c\,|z|\big(1 + |x_3|^q\big) \leq c\,|z|V(x). \tag{60}$$

Combining this, (57), (58), and (59) with (56) proves that

$$\begin{aligned}
V'(x)\mu(x + \sigma z) \leq |V'(x)\mu(x + \sigma z)| &\leq 2pV(x) + 2pcV(x) + qcV(x) + qc\,|z|V(x) \\
&= (2p + 2pc)V(x) + qc(1 + |z|)V(x) \leq (2p + 2pc + qc)(1 + |z|)V(x).
\end{aligned} \tag{61}$$

This and (54) establish (51). The proof of Lemma 3.3 is thus completed. □

3.3 On solutions to SDEs with irregularities in the initial value

Lemma 3.4. *Let $a \in \mathbb{R}$, $b \in (a, \infty)$. Then there exists a function $f \in C^\infty(\mathbb{R}, [0, \infty))$ which satisfies that $\{t \in \mathbb{R} \colon f(t) > 0\} = (a, b)$ and $\int_a^b |f(t)|^2 \, dt = 1$.*

Proposition 3.5. *Let $d \in \{5, 6, \dots\}$, $n \in \{2, 3, \dots\}$, $T \in (0, \infty)$, $\tau \in (0, T)$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0, T] \times \Omega \to \mathbb{R}$ be a standard Brownian motion with continuous sample paths. Then there exist $\mu \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^d$, $V \in C^\infty(\mathbb{R}^d, [0, \infty))$, $\kappa \in [1, \infty)$ such that*

(i) it holds for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}$ that $\|\mu'(x)h\| \le \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \le \kappa(1 + |z|)V(x)$, and $\|x\| \le V(x)$,

(ii) there exist unique stochastic processes $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$X^x(t, \omega) = x + \int_0^t \mu(X^x(s, \omega))\, \mathrm{d}s + \sigma W(t, \omega), \tag{62}$$

and

(iii) it holds for all $t \in (\tau, T)$ that there exists $c \in (0, \infty)$ such that for all $\varepsilon \in (0, 1/e]$, $h = (0, 0, 0, \varepsilon, 0, 0, \dots, 0) \in \mathbb{R}^d$, it holds that

$$\exp\big(-c\,|\ln(\|h\|)|^{2/n}\big) = \exp\big(-c\,|\ln(\varepsilon)|^{2/n}\big) \le \mathbb{E}\big[\|X^h(t) - X^0(t)\|\big]. \tag{63}$$

Proof of Proposition 3.5. Throughout this proof let $|||\cdot||| \colon \mathbb{R}^5 \to [0, \infty)$ be the standard norm on $\mathbb{R}^5$, let $f, g \in C^\infty(\mathbb{R}, [0, \infty))$ satisfy $\{t \in \mathbb{R} \colon f(t) > 0\} = (\tau, T)$, $\{t \in \mathbb{R} \colon g(t) > 0\} = (0, \tau)$, and $\int_0^\tau |g(t)|^2 = 1$, let $\rho = (0, 1, 0, 0, 0) \in \mathbb{R}^5$, let $\sigma = (0, 1, 0, 0, \dots, 0) \in \mathbb{R}^d$, let $C, \kappa \in [0, \infty)$ satisfy

$$C = \sup\nolimits_{t \in \mathbb{R}} \big[\max\{1, |f(t)|, |f'(t)|, |g'(t)|, |g''(t)|\}\big] \qquad \text{and} \qquad \kappa = 2 + 8(n + 1)C, \tag{64}$$

let $\varpi \colon \mathbb{R}^d \to \mathbb{R}^5$ satisfy for all $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ that $\varpi(x) = (x_1, x_2, x_3, x_4, x_5)$, let $U \colon \mathbb{R}^5 \to [0, \infty)$ satisfy for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ that

$$U(x) = 1 + (x_1)^2 + (x_4)^2 + (x_5)^2 + (x_2)^{2n} + (x_3)^{2n} + 1, \tag{65}$$

let $V \colon \mathbb{R}^d \to [0, \infty)$ satisfy for all $x \in \mathbb{R}^d$ that

$$V(x) = U(\varpi(x)) + \left[\sum_{i \in \mathbb{N} \cap (5, d+1)} (x_i)^2\right] + 1, \tag{66}$$

let $\nu \colon \mathbb{R}^5 \to \mathbb{R}^5$ satisfy for all $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ that

$$\nu(x) = \big(1, 0, g'(x_1)x_2, f(x_1)x_4x_5, f(x_1)\big[(x_3)^n - (x_4)^2\big]\big), \tag{67}$$

and let $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ satisfy for all x, $y = (y_1, y_2, \dots, y_d) \in \mathbb{R}^d$ with $\mu(x) = y$ that

$$\varpi(y) = \nu(\varpi(x)) \qquad \text{and} \qquad \forall\, i \in \mathbb{N} \cap (5, d+1) \colon y_i = 0 \tag{68}$$

(cf. Lemma 3.4). Observe that (67), the fact that $f, g \in C^\infty(\mathbb{R}, [0, \infty))$, and Lemma 3.2 (with $n \leftarrow n$, $\|\cdot\| \leftarrow |||\cdot|||$, $\mu \leftarrow \nu$, $f \leftarrow f$, $g \leftarrow g$, $c \leftarrow C$ in the notation of Lemma 3.2) establish that for all $x, h \in \mathbb{R}^5$ it holds that $\nu \in C^\infty(\mathbb{R}^5, \mathbb{R}^5)$ and

$$\begin{aligned} |||\nu'(x)h||| &\leq 4nC(1 + |||x|||^n)\, |||h||| \leq 4nC(2 + |||x|||^\kappa)\, |||h||| \\ &\leq 8nC(1 + |||x|||^\kappa)\, |||h||| \leq \kappa(1 + |||x|||^\kappa)\, |||h|||. \end{aligned} \tag{69}$$

This and (68) show that for all $x, h \in \mathbb{R}^d$ it holds that $\mu \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ and

$$\|\mu'(x)h\| = \big|\big|\big| \big[\nu'(\varpi(x))\big]\varpi(h) \big|\big|\big| \leq \kappa(1 + |||\varpi(x)|||^\kappa)\, |||\varpi(h)||| \leq \kappa(1 + \|x\|^\kappa)\, \|h\|. \tag{70}$$

In the next step we note that (65), the fact that $f, g \in C^\infty(\mathbb{R}, [0, \infty))$, and Lemma 3.3 (with $n \leftarrow n$, $p \leftarrow 1$, $q \leftarrow 2n$, $f \leftarrow f$, $g \leftarrow g$, $\sigma \leftarrow \rho$, $\|\cdot\| \leftarrow |||\cdot|||$, $\mu \leftarrow \nu$, $V \leftarrow U$ in the notation of Lemma 3.3) prove that for all $x, h \in \mathbb{R}^5$, $z \in \mathbb{R}$ it holds that

$$U \in C^\infty(\mathbb{R}^5, [0, \infty)), \;\; |||x||| \leq U(x), \;\text{ and }\; U'(x)\nu(x+\rho z) \leq 2(1+C+nC)(1+|z|)U(x). \tag{71}$$

Combining this with (66) demonstrates for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that

$$\begin{aligned} \|x\| &= \left[|||\varpi(x)|||^2 + \left(\sum_{i \in \mathbb{N} \cap (5, d+1)} (x_i)^2 \right) \right]^{\frac{1}{2}} \leq |||\varpi(x)||| + \left[\sum_{i \in \mathbb{N} \cap (5, d+1)} (x_i)^2 \right]^{\frac{1}{2}} \\ &\leq U(\varpi(x)) + \max\left\{ 1, \left[\sum_{i \in \mathbb{N} \cap (5, d+1)} (x_i)^2 \right] \right\} \leq V(x). \end{aligned} \tag{72}$$

Moreover, observe that (66) and (71) imply that for all $x = (x_1, x_2, \ldots, x_d)$, $h = (h_1, h_2, \ldots, h_d) \in \mathbb{R}^d$ it holds that $V \in C^\infty(\mathbb{R}^d, [0, \infty))$ and

$$V'(x)h = \big[U'(\varpi(x))\big]\varpi(h) + \left[\sum_{i \in \mathbb{N} \cap (5, d+1)} (2x_i h_i) \right]. \tag{73}$$

This, (68), and (71) ensure that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}$ it holds that

$$\begin{aligned} &V'(x)\mu(x + \sigma z) = U'(\varpi(x))\varpi(\mu(x + \sigma z)) = U'(\varpi(x))\nu\big(\varpi(x) + \rho z\big) \\ &\leq 2(1 + C + nC)(1 + |z|)U(\varpi(x)) \leq 2(1 + C + nC)(1 + |z|)V(x) \\ &\leq \kappa(1 + |z|)V(x). \end{aligned} \tag{74}$$

This, (70), and (72) establish item (i). Next we combine (71) and [10, Lemma 5.4] (with $d \leftarrow 5$, $m \leftarrow 1$, $T \leftarrow T$, $\mu \leftarrow \nu$, $\sigma \leftarrow \rho$, $\varphi \leftarrow (\mathbb{R} \ni z \mapsto 2(1 + C + nC)(1 + |z|) \in [0, \infty))$, $V \leftarrow U$, $\|\cdot\| \leftarrow |||\cdot|||$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$ in the notation of [10, Lemma 5.4]) to obtain that there exist unique stochastic processes $Y^x\colon [0,T]\times\Omega\to\mathbb{R}^5$, $x\in\mathbb{R}^5$, with continuous sample paths which satisfy for all $x\in\mathbb{R}^5$, $t\in[0,T]$, $\omega\in\Omega$ that

$$Y^x(t,\omega) = x + \int_0^t \nu(Y^x(s,\omega))\,\mathrm{d}s + \rho W(t,\omega). \tag{75}$$

In addition, note that (72), (74), and [10, Lemma 5.4] (with $d\leftarrow d$, $m\leftarrow 1$, $T\leftarrow T$, $\mu\leftarrow\mu$, $\sigma\leftarrow\sigma$, $\varphi\leftarrow(\mathbb{R}\ni z\mapsto\kappa(1+|z|)\in[0,\infty))$, $V\leftarrow V$, $\|\cdot\|\leftarrow\|\cdot\|$, $(\Omega,\mathcal{F},\mathbb{P})\leftarrow(\Omega,\mathcal{F},\mathbb{P})$, $W\leftarrow W$ in the notation of [10, Lemma 5.4]) ensure that there exist unique stochastic processes $X^x\colon[0,T]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, with continuous sample paths which satisfy for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega). \tag{76}$$

This proves item (ii). In the next step let $Z^x\colon[0,T]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, satisfy for all $x=(x_1,x_2,\dots,x_d)$, $y=(y_1,y_2,\dots,y_d)\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ with $Z^x(t,\omega)=y$ that

$$\varpi(y) = Y^{\varpi(x)}(t,\omega) \qquad \text{and} \qquad \forall\, i\in\mathbb{N}\cap(5,d+1)\colon y_i = x_i. \tag{77}$$

Observe that (68) and (75) demonstrate that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ it holds that

$$\begin{aligned}
\varpi(Z^x(t,\omega)) = Y^{\varpi(x)}(t,\omega) &= \varpi(x) + \int_0^t \nu(Y^{\varpi(x)}(s,\omega))\,\mathrm{d}s + \rho W(t,\omega)\\
&= \varpi(x) + \int_0^t \nu\big(\varpi(Z^x(s,\omega))\big)\,\mathrm{d}s + \rho W(t,\omega)\\
&= \varpi(x) + \int_0^t \varpi\big(\mu(Z^x(s,\omega))\big)\,\mathrm{d}s + \rho W(t,\omega)\\
&= \varpi(x) + \varpi\bigg(\int_0^t \mu(Z^x(s,\omega))\,\mathrm{d}s\bigg) + \varpi(\sigma)W(t,\omega)\\
&= \varpi\bigg(x + \int_0^t \mu(Z^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega)\bigg).
\end{aligned} \tag{78}$$

This and the fact that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$, $y,z\in\mathbb{R}^d$, $i\in\mathbb{N}\cap(5,d+1)$ with $y=\int_0^t\mu(Z^x(s,\omega))\,\mathrm{d}s$ and $z=\rho W(t,\omega)$ it holds that $y_i=0=z_i$ establishes that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ it holds that

$$Z^x(t,\omega) = x + \int_0^t \mu(Z^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega). \tag{79}$$

Combining this with (76) shows that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ it holds that

$$X^x(t,\omega) = Z^x(t,\omega). \tag{80}$$

Next note that (75) and Proposition 2.3 (with $T \leftarrow T$, $\tau \leftarrow \tau$, $n \leftarrow n$, $f \leftarrow f$, $g \leftarrow g$, $\sigma \leftarrow \rho$, $\|\cdot\| \leftarrow |||\cdot|||$, $\mu \leftarrow \nu$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$, $(X^x)_{x\in\mathbb{R}^5} \leftarrow (Y^x)_{x\in\mathbb{R}^5}$ in the notation of Proposition 2.3) prove that for all $t \in (\tau, T)$ there exists $c \in (0, \infty)$ such that for all $\varepsilon \in (0, 1/e]$, $h = (0, 0, 0, \varepsilon, 0) \in \mathbb{R}^5$ it holds that

$$\exp(-c\,|\ln(\varepsilon)|^{2/n}) \leq \mathbb{E}\big[|||Y^h(t) - Y^0(t)|||\big]. \tag{81}$$

Note that (77) implies that for all $x = (x_1, x_2, \ldots, x_d)$, $y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} &\|Z^x(t,\omega) - Z^y(t,\omega)\| \\ &= \left[|||Y^{\varpi(x)}(t,\omega) - Y^{\varpi(y)}(t,\omega)|||^2 + \left(\sum_{i\in\mathbb{N}\cap(5,d+1)} (x_i - y_i)^2\right)\right]^{\frac{1}{2}} \\ &\geq |||Y^{\varpi(x)}(t,\omega) - Y^{\varpi(y)}(t,\omega)|||. \end{aligned} \tag{82}$$

This, (80) and (81) demonstrate that for all $t \in (\tau, T)$ there exists $c \in (0, \infty)$ such that for all $\varepsilon \in (0, 1/e]$, $h = (0, 0, 0, \varepsilon, 0, 0, \ldots, 0) \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &\exp\bigl(-c\,|\ln(\varepsilon)|^{2/n}\bigr) \leq \mathbb{E}\big[\|Y^{\varpi(h)}(t) - X^0(t)\|\big] \\ &\leq \mathbb{E}\big[\|Z^h(t) - Z^0(t)\|\big] = \mathbb{E}\big[\|X^h(t) - X^0(t)\|\big]. \end{aligned} \tag{83}$$

This establishes item (iii). The proof of Proposition 3.5 is thus completed. □

4 On the existence of solutions to SDEs with non-locally Hölder continuous dependence on the initial values

In this section we establish in Theorem 4.4 below the existence of solutions to certain additive noise driven SDEs whose solutions depend non-locally Hölder continuously on their initial values and whose drift coefficient functions are smooth with at most polynomially growing derivatives. In our proof of Theorem 4.4 we employ Lemma 4.2 below as well as the elementary estimate for certain real-valued functions in Lemma 4.3 below. Lemma 4.2 is a strengthened version of Proposition 3.5 above. Our proof of Lemma 4.2 employs the essentially well-known fact on affine linear transformations of solutions to SDEs in Lemma 4.1 below. For completeness we also provide in this section a detailed proof of Lemma 4.1.

4.1 On affine linear transformations of SDEs

Lemma 4.1. *Let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, $\nu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\rho \in \mathbb{R}^{d\times m}$, $v \in \mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0, T] \times \Omega \to \mathbb{R}^m$ and $Y^x\colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic* *processes with continuous sample paths, assume for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that*

$$Y^x(t,\omega) = x + \int_0^t \nu(Y^x(s,\omega))\,\mathrm{d}s + \rho W(t,\omega), \tag{84}$$

let $L \in \mathbb{R}^{d\times d}$ be an invertible matrix, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfy for all $x \in \mathbb{R}^d$ that $\mu(x) = L\nu(L^{-1}(x-v))$, and let $X^x\colon [0,T]\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$X^x(t,\omega) = LY^{L^{-1}(x-v)}(t,\omega) + v. \tag{85}$$

Then it holds for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + L\rho W(t,\omega). \tag{86}$$

Proof of Lemma 4.1. Note that (85) ensures that for all $x \in \mathbb{R}^d$, $s \in [0,T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} \mu(X^x(s,\omega)) &= L\nu(L^{-1}(X^x(s,\omega)-v)) \\ &= L\nu\big(L^{-1}([LY^{L^{-1}(x-v)}(s,\omega)+v]-v)\big) \\ &= L\nu\big(L^{-1}LY^{L^{-1}(x-v)}(s,\omega)\big) \\ &= L\nu\big(Y^{L^{-1}(x-v)}(s,\omega)\big). \end{aligned} \tag{87}$$

Therefore, it holds for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$\begin{aligned} X^x(t,\omega) &= LY^{L^{-1}(x-v)}(t,\omega) + v \\ &= L\left[L^{-1}(x-v) + \int_0^t \nu(Y^{L^{-1}(x-v)}(s,\omega))\,\mathrm{d}s + \rho W(t,\omega)\right] + v \\ &= [x-v] + L\left[\int_0^t \nu(Y^{L^{-1}(x-v)}(s,\omega))\,\mathrm{d}s\right] + L\rho W(t,\omega) + v \\ &= x + \int_0^t L\nu(Y^{L^{-1}(x-v)}(s,\omega))\,\mathrm{d}s + L\rho W(t,\omega) \\ &= x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + L\rho W(t,\omega). \end{aligned} \tag{88}$$

This completes the proof of Lemma 4.1. □

4.2 On solutions to SDEs with irregularities in the initial value

Lemma 4.2. *Let $d \in \{5,6,\dots\}$, $n \in \{2,3,\dots\}$, $T \in (0,\infty)$, $\tau \in (0,T)$, $v \in \mathbb{R}^d$, $\delta \in \mathbb{R}^d \setminus \{0\}$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a standard Brownian motion with continuous sample paths. Then there exist $\mu \in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$, $\sigma \in \mathbb{R}^d$, $V \in C^\infty(\mathbb{R}^d,[0,\infty))$, $\kappa \in (0,\infty)$ such that* *(i)* *it holds for all* $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}$ *that* $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^{\kappa}\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |z|)V(x)$, *and* $\|x\| \leq V(x)$,

(ii) *there exist unique stochastic processes* $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, *with continuous sample paths such that for all* $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ *it holds that*

$$X^x(t, \omega) = x + \int_0^t \mu(X^x(s, \omega))\, \mathrm{d}s + \sigma W(t, \omega), \tag{89}$$

and

(iii) *it holds for all* $t \in (\tau, T)$ *that there exists* $c \in (0, \infty)$ *such that for all* $w \in \{v + r\delta \colon r \in (0, 1/e]\}$ *it holds that*

$$\|\delta\| \exp\big(-c\, |\ln(\|v - w\|)|^{2/n}\big) \leq \mathbb{E}\big[\|X^v(t) - X^w(t)\|\big]. \tag{90}$$

Proof of Lemma 4.2. Throughout this proof, let $u = (0, 0, 0, 1, 0, \ldots, 0) \in \mathbb{R}^d$, let $A \in \mathbb{R}^{d \times d}$ be an orthogonal matrix which satisfies

$$Au = \frac{\delta}{\|\delta\|}, \tag{91}$$

and let $B \in \mathbb{R}^{d \times d}$ satisfy

$$B = \|\delta\| A. \tag{92}$$

Note that Proposition 3.5 (with $d \leftarrow d$, $n \leftarrow n$, $T \leftarrow T$, $\tau \leftarrow \tau$, $\|\cdot\| \leftarrow \|\cdot\|$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$ in the notation of Proposition 3.5) shows that there exist $\nu \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $\rho \in \mathbb{R}^d$, $U \in C^\infty(\mathbb{R}^d, [0, \infty))$, $\varkappa \in [1, \infty)$ which satisfy that

(A) it holds for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}$ that $\|\nu'(x)h\| \leq \varkappa\big(1 + \|x\|^{\varkappa}\big)\|h\|$, $U'(x)\nu(x + \rho z) \leq \varkappa(1 + |z|)U(x)$, and $\|x\| \leq U(x)$,

(B) there exist unique stochastic processes $Y^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$Y^x(t, \omega) = x + \int_0^t \nu(Y^x(s, \omega))\, \mathrm{d}s + \rho W(t, \omega), \tag{93}$$

and

(C) it holds for all $t \in (\tau, T)$ that there exists $c \in (0, \infty)$ such that for all $w \in \{ru \colon r \in (0, 1/e]\}$ it holds that

$$\exp\big(-c\, |\ln(\|w\|)|^{2/n}\big) \leq \mathbb{E}\big[\|Y^0(t) - Y^w(t)\|\big]. \tag{94}$$

Next let $\mu \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^d$, and $Z^x \colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$\mu(x) = B\nu(B^{-1}(x-v)), \qquad \sigma = B\rho, \qquad \text{and} \qquad Z^x(t,\omega) = BY^{B^{-1}(x-v)}(t,\omega) + v. \tag{95}$$

Observe that Lemma 4.1 (with $d \leftarrow d$, $m \leftarrow 1$, $T \leftarrow T$, $\nu \leftarrow \nu$, $\rho \leftarrow \rho$, $v \leftarrow v$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$, $(Y^x)_{x\in\mathbb{R}^d} \leftarrow (Y^x)_{x\in\mathbb{R}^d}$, $L \leftarrow B$, $\mu \leftarrow \mu$, $(X^x)_{x\in\mathbb{R}^d} \leftarrow (Z^x)_{x\in\mathbb{R}^d}$ in the notation of Lemma 4.1) proves that for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$Z^x(t,\omega) = x + \int_0^t \mu(Z^x(s,\omega))\,ds + \sigma W(t,\omega). \tag{96}$$

Furthermore, note that the chain rule implies that for all $x \in \mathbb{R}^d$ it holds that

$$\mu'(x) = B\nu'(B^{-1}(x-v))B^{-1}. \tag{97}$$

In addition, observe that the assumption that A is an orthogonal matrix ensures that for all $x \in \mathbb{R}^d$ it holds that

$$\|Bx\| = \|\|\delta\|Ax\| = \|\delta\|\|Ax\| = \|\delta\|\|x\|. \tag{98}$$

Combining this with (97) and item (A) proves that for all $x, h \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
\|\mu'(x)h\| &= \|B\nu'(B^{-1}(x-v))B^{-1}h\| = \|\delta\|\|\nu'(B^{-1}(x-v))B^{-1}h\| \\
&\le \|\delta\|\varkappa\big(1 + \|B^{-1}(x-v)\|^\varkappa\big)\|B^{-1}h\| \\
&= \|\delta\|\varkappa\big(1 + \|\delta\|^{-1}\|x-v\|^\varkappa\big)\|\delta\|^{-1}\|h\| \\
&= \varkappa\big(1 + \|\delta\|^{-1}\|x-v\|^\varkappa\big)\|h\| \\
&\le \varkappa\big(1 + \|\delta\|^{-1}2^\varkappa(\|x\|^\varkappa + \|v\|^\varkappa)\big)\|h\| \\
&= \varkappa\big(1 + \|\delta\|^{-1}2^\varkappa\|v\|^\varkappa + \|\delta\|^{-1}2^\varkappa\|x\|^\varkappa\big)\|h\| \\
&\le \varkappa\big(1 + \|\delta\|^{-1}2^\varkappa \max\{1, \|v\|^\varkappa\}\big)\big(1 + \|x\|^\varkappa\big)\|h\|.
\end{aligned} \tag{99}$$

In the next step let $\kappa \in [1,\infty)$ satisfy

$$\kappa = 2\varkappa\big(1 + \|\delta\|^{-1}2^\varkappa \max\{1, \|v\|^\varkappa\}\big). \tag{100}$$

Note that (99) demonstrates that for all $x, h \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
\|\mu'(x)h\| &\le \varkappa\big(1 + \|\delta\|^{-1}2^\varkappa \max\{1, \|v\|^\varkappa\}\big)\big(2 + \|x\|^\kappa\big)\|h\| \\
&\le 2\varkappa\big(1 + \|\delta\|^{-1}2^\varkappa \max\{1, \|v\|^\varkappa\}\big)\big(1 + \|x\|^\kappa\big)\|h\| \\
&= \kappa\big(1 + \|x\|^\kappa\big)\|h\|.
\end{aligned} \tag{101}$$

Next let $V \colon \mathbb{R}^d \to [0,\infty)$ satisfy for all $x \in \mathbb{R}^d$ that

$$V(x) = \|\delta\|U(B^{-1}(x-v)) + \|v\|. \tag{102}$$

Observe that the chain rule implies that for all $x \in \mathbb{R}^d$ it holds that

$$V'(x) = \|\delta\| U'(B^{-1}(x-v))B^{-1}. \tag{103}$$

Hence, we obtain that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}$ it holds that

$$\begin{aligned} V'(x)\mu(x+\sigma z) &= \|\delta\| U'(B^{-1}(x-v))B^{-1}\mu(x+\sigma z) \\ &= \|\delta\| U'(B^{-1}(x-v))B^{-1}\big[B\nu(B^{-1}(x+\sigma z - v))\big] \\ &= \|\delta\| U'(B^{-1}(x-v))\nu(B^{-1}(x-v)+\rho z)) \\ &\leq \|\delta\| \varkappa(1+|z|)U(B^{-1}(x-v)) \\ &= \varkappa(1+|z|)(V(x)-\|v\|) \\ &\leq \varkappa(1+|z|)V(x) \leq \kappa(1+|z|)V(x). \end{aligned} \tag{104}$$

Furthermore, note that (102) and item (A) ensure that for all $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} V(x) &\geq \|\delta\|\|B^{-1}(x-v)\| + \|v\| = \|\delta\|\|\delta\|^{-1}\|x-v\| + \|v\| \\ &= \|x-v\| + \|v\| \geq \|x\|. \end{aligned} \tag{105}$$

Combining this, (101), and (104) establishes item (i). This and [10, Lemma 5.4] shows that there exist unique stochastic processes $X^x \colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega). \tag{106}$$

Therefore, we obtain item (ii). In addition, observe that (106) and (96) imply that for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$X^x(t,\omega) = Z^x(t,\omega). \tag{107}$$

This, (91), (92), and (95) prove that for all $r \in \mathbb{R}$, $t \in (\tau, T)$, $\omega \in \Omega$ it holds that

$$\begin{aligned} X^v(t,\omega) - X^{v+r\delta}(t,\omega) &= \big[BY^{B^{-1}(v-v)}(t,\omega) + v\big] - \big[BY^{B^{-1}(v+r\delta-v)}(t,\omega)+v\big] \\ &= B\big(Y^0(t,\omega) - Y^{r\|\delta\|^{-1}A^{-1}\delta}(t,\omega)\big) \\ &= B\big(Y^0(t,\omega) - Y^{ru}(t,\omega)\big). \end{aligned} \tag{108}$$

Combining this with item (C) shows that for all $t \in (\tau, T)$ there exists $c \in (0,\infty)$ such that for all $r \in (0, 1/e]$ it holds that

$$\begin{aligned} \mathbb{E}\big[\|X^v(t) - X^{v+r\delta}(t)\|\big] &= \mathbb{E}\big[\|B\big(Y^0(t) - Y^{ru}(t)\big)\|\big] \\ &= \|\delta\|\mathbb{E}\big[\|Y^0(t) - Y^{ru}(t)\|\big] \\ &\geq \|\delta\| \exp\big(-c\,|\ln(r)|^{2/n}\big). \end{aligned} \tag{109}$$

This establishes item (iii). The proof of Lemma 4.2 is thus completed. □

4.3 On solutions to SDEs with non-locally Hölder continuous dependence on the initial values

Lemma 4.3. *Let* $c, R, \alpha \in (0, \infty)$, $K \in [0, \infty)$, $\beta \in (0, 1)$ *satisfy*

$$K = \inf\left[\{1\} \cup \left\{ \frac{\exp(-c\,|\ln(r)|^{\beta})}{r^{\alpha}} \colon r \in \left[\exp\!\left(-[\alpha^{-1}c]^{\frac{1}{1-\beta}}\right), \infty\right) \cap (0, R] \right\}\right]. \tag{110}$$

Then

(i) it holds for all $r \in \left(0, \exp\!\left(-[\alpha^{-1}c]^{\frac{1}{1-\beta}}\right)\right]$ *that* $\exp(-c\,|\ln(r)|^{\beta}) \geq r^{\alpha}$ *and*

(ii) it holds for all $r \in (0, R]$ *that* $K > 0$ *and* $\exp(-c\,|\ln(r)|^{\beta}) \geq K r^{\alpha}$.

Proof of Lemma 4.3. Throughout this proof let $C \in (0, \infty)$ satisfy

$$C = \exp\!\left(-[\alpha^{-1}c]^{\frac{1}{1-\beta}}\right). \tag{111}$$

Note that the fact that $C < 1$ shows that for all $r \in (0, C]$ it holds that

$$\begin{aligned} \ln(r) &= -|\ln(r)| = -|\ln(r)|^{1-\beta}|\ln(r)|^{\beta} \leq -|\ln(C)|^{1-\beta}|\ln(r)|^{\beta} \\ &= -\left|-[\alpha^{-1}c]^{\frac{1}{1-\beta}}\right|^{1-\beta}|\ln(r)|^{\beta} = -\alpha^{-1}c\,|\ln(r)|^{\beta}. \end{aligned} \tag{112}$$

Hence, we obtain that for all $r \in (0, C]$ it holds that

$$r^{\alpha} = \exp(\alpha \ln(r)) \leq \exp(-c\,|\ln(r)|^{\beta}). \tag{113}$$

Next observe that (110) implies that for all $r \in [C, \infty) \cap (0, R]$ it holds that

$$K r^{\alpha} \leq \exp(-c\,|\ln(r)|^{\beta}). \tag{114}$$

Furthermore, note that the fact that $(0, \infty) \ni r \mapsto \exp(-c\,|\ln(r)|^{\beta}) r^{-\alpha} \in (0, \infty)$ is a continuous function and the fact that $[C, \infty) \cap (0, R]$ is a compact set ensure that $K > 0$. Combining this with (113), (114), and the fact that $K \leq 1$ establishes items (i) and (ii). The proof of Lemma 4.3 is thus completed. ☐

Theorem 4.4. *Let* $m \in \mathbb{N}$, $d \in \{5, 6, \dots\}$, $T \in (0, \infty)$, $\tau \in (0, T)$, $v \in \mathbb{R}^d$, $\delta \in \mathbb{R}^d \setminus \{0\}$ *let* $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ *be the standard norm on* $\mathbb{R}^d$, *let* $|||\cdot||| \colon \mathbb{R}^m \to [0, \infty)$ *be a norm, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, and let* $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ *be a standard Brownian motion with continuous sample paths. Then there exist* $\mu \in C^{\infty}(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d \times m}$, $V \in C^{\infty}(\mathbb{R}^d, [0, \infty))$, $\kappa \in (0, \infty)$ *such that*

(i) it holds for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ *that* $\|\mu'(x)h\| \leq \kappa\left(1 + \|x\|^{\kappa}\right)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |||z|||)V(x)$, *and* $\|x\| \leq V(x)$,

(ii) there exist unique stochastic processes $X^x\colon [0,T]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, with continuous sample paths such that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ it holds that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{115}$$

(iii) it holds for all $\omega\in\Omega$ that $\big([0,T]\times\mathbb{R}^d \ni (t,x)\mapsto X^x(t,\omega)\in\mathbb{R}^d\big)\in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$,

(iv) it holds for all $x,h\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ that

$$\big(\tfrac{\partial}{\partial x}X^x(t,\omega)\big)(h) = h + \int_0^t \mu'(X^x(s,\omega))\big(\big(\tfrac{\partial}{\partial x}X^x(s,\omega)\big)(h)\big)\,\mathrm{d}s, \tag{116}$$

(v) it holds for all $R,r\in(0,\infty)$ that $\Omega\ni\omega\mapsto \sup_{x\in[-R,R]^d}\sup_{t\in[0,T]}\big(\|X^x(t,\omega)\|^r\big)\in[0,\infty]$ is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function,

(vi) it holds for all $R,r\in(0,\infty)$ that

$$\mathbb{E}\Big[\sup_{x\in[-R,R]^d}\sup_{t\in[0,T]}\big(\|X^x(t)\|^r\big)\Big] < \infty, \tag{117}$$

(vii) it holds for all $R,q\in(0,\infty)$ that there exists $c\in(0,\infty)$ such that for all $x,y\in[-R,R]^d$ with $0<\|x-y\|\neq 1$ it holds that

$$\sup_{t\in[0,T]}\mathbb{E}\big[\|X^x(t)-X^y(t)\|\big] \le c\,|\ln(\|x-y\|)|^{-q}, \tag{118}$$

(viii) there exists $K\in(0,\infty)$ such that for all $t\in(\tau,T)$ there exists $c\in(0,\infty)$ such that for all $w\in\{v+r\delta\colon r\in(0,1]\}$ it holds that

$$K\exp\big(-c\,|\ln(\|v-w\|)|^{2/n}\big) \le \mathbb{E}\big[\|X^v(t)-X^w(t)\|\big], \tag{119}$$

and

(ix) it holds for all $t\in(\tau,T)$, $\alpha\in(0,\infty)$ that there exists $c\in(0,\infty)$ such that for all $w\in\{v+r\delta\colon r\in[0,1]\}$ it holds that

$$c\,\|v-w\|^\alpha \le \mathbb{E}\big[\|X^v(t)-X^w(t)\|\big]. \tag{120}$$

Proof of Theorem 4.4. Throughout this proof let $K\in(0,\infty)$ satisfy

$$K = \max\bigg\{1,\ \sup\nolimits_{z=(z_1,z_2,\dots,z_m)\in\mathbb{R}^m\setminus\{0\}}\frac{\sum_{i=1}^m|z_i|}{|\!|\!|z|\!|\!|}\bigg\}. \tag{121}$$

and let $W_i\colon [0,T]\times\Omega\to\mathbb{R}$, $i\in\{1,2,\dots,m\}$, satisfy for all $t\in[0,T]$, $\omega\in\Omega$ that

$$W(t,\omega)=\big(W_1(t,\omega),W_2(t,\omega),\dots,W_m(t,\omega)\big). \tag{122}$$

Note that Lemma 4.2 (with $d\leftarrow d$, $T\leftarrow T$, $\tau\leftarrow\tau$, $v\leftarrow v$, $\delta\leftarrow e\delta$, $\|\cdot\|\leftarrow\|\cdot\|$, $(\Omega,\mathcal{F},\mathbb{P})\leftarrow(\Omega,\mathcal{F},\mathbb{P})$, $W\leftarrow W_1$ in the notation of Lemma 4.2) establishes that there exist $\mu\in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$, $\rho=(\rho_1,\rho_2,\dots,\rho_d)\in\mathbb{R}^d$, $V\in C^\infty(\mathbb{R}^d,[0,\infty))$, $\varkappa\in(0,\infty)$ which satisfy that

(A) it holds for all $x,h\in\mathbb{R}^d$, $z\in\mathbb{R}$ that $\|\mu'(x)h\|\leq\varkappa\big(1+\|x\|^\varkappa\big)\|h\|$, $V'(x)\mu(x+\rho z)\leq\varkappa(1+|z|)V(x)$, and $\|x\|\leq V(x)$,

(B) there exist unique stochastic processes $X^x\colon[0,T]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, with continuous sample paths such that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ it holds that

$$X^x(t,\omega)=x+\int_0^t\mu(X^x(s,\omega))\,\mathrm{d}s+\rho W_1(t,\omega), \tag{123}$$

and

(C) it holds for all $t\in(\tau,T)$ that there exists $c\in(0,\infty)$ such that for all $w\in\{v+r\delta\colon r\in(0,1]\}$ it holds that

$$e\|\delta\|\exp\big(-c\,|\ln(\|v-w\|)|^{2/n}\big)\leq\mathbb{E}\big[\|X^v(t)-X^w(t)\|\big]. \tag{124}$$

In the next step let $\kappa\in(0,\infty)$ and $\sigma=(\sigma_{i,j})_{i\in\{1,2,\dots,d\},\,j\in\{1,2,\dots,m\}}\in\mathbb{R}^{d\times m}$ satisfy for all $i\in\{1,2,\dots,d\}$, $j\in\{1,2,\dots,m\}$ that

$$\kappa=2K\varkappa\qquad\text{and}\qquad\sigma_{i,j}=\begin{cases}\rho_i & : j=1\\ 0 & : j>1.\end{cases} \tag{125}$$

Observe that item (A) ensures that for all $x\in\mathbb{R}^d$, $z=(z_1,z_2,\dots,z_m)\in\mathbb{R}^m$ it holds that

$$\begin{aligned}V'(x)\mu(x+\sigma z)&=V'(x)\mu(x+\rho z_1)\leq\varkappa(1+|z_1|)V(x)\\&\leq\varkappa(1+K|\!|\!|z|\!|\!|)V(x)\leq\kappa(1+|\!|\!|z|\!|\!|)V(x).\end{aligned} \tag{126}$$

Furthermore, note that item (A) shows that for all $x,h\in\mathbb{R}^d$, $z\in\mathbb{R}$ it holds that

$$\|\mu'(x)h\|\leq\varkappa(2+\|x\|^\kappa)\,\|h\|\leq2\varkappa(1+\|x\|^\kappa)\,\|h\|\leq\kappa(1+\|x\|^\kappa)\,\|h\|. \tag{127}$$

This and item (A) prove item (i). In the next step we observe that (125) implies that for all $t\in[0,T]$, $\omega\in\Omega$ it holds that

$$\sigma W(t,\omega)=\rho W_1(t,\omega). \tag{128}$$

This and (123) establish item (ii). Next note that [10, Lemma 5.4] proves items (iii) and (iv). In addition, observe that [10, Lemma 6.6] demonstrates items (v) and (vi). Moreover, note that [10, Lemma 8.4] establishes item (vii). Next observe that item (C) implies item (viii). Combining item (viii) with Lemma 4.3 proves item (ix). The proof of Theorem 4.4 is thus completed. □

References

[1] Chen, X., and Li, X.-M. Strong completeness for a class of stochastic differential equations with irregular coefficients. *Electron. J. Probab. 19*, 91 (2014), 1–34.

[2] Cox, S., Hutzenthaler, M., and Jentzen, A. Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations. *arXiv:1309.5595* (2013), 84 pages. Revision requested from Mem. Amer. Math. Soc.

[3] Fang, S., Imkeller, P., and Zhang, T. Global flows for stochastic differential equations without global Lipschitz conditions. *Ann. Probab. 35*, 1 (2007), 180–205.

[4] Hairer, M., Hutzenthaler, M., and Jentzen, A. Loss of regularity for Kolmogorov equations. *Ann. Probab. 43*, 2 (2015), 468–527.

[5] Hairer, M., and Mattingly, J. C. Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Ann. Math. (2) 164*, 3 (2006), 993–1032.

[6] Hudde, A., Hutzenthaler, M., Jentzen, A., and Mazzonetto, S. On the Itô-Alekseev-Gröbner formula for stochastic differential equations. *arXiv:1812.09857* (2018), 29 pages.

[7] Hudde, A., Hutzenthaler, M., and Mazzonetto, S. A stochastic Gronwall inequality and applications to moments, strong completeness, strong local Lipschitz continuity, perturbations. *arXiv:1903.08727* (2019), 21 pages.

[8] Hutzenthaler, M., and Jentzen, A. On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients. *arXiv:1401.0295* (2014), 41 pages. To appear in Ann. Probab.

[9] Hutzenthaler, M., Jentzen, A., Lindner, F., and Pušnik, P. Strong convergence rates on the whole probability space for space-time discrete numerical approximation schemes for stochastic Burgers equations. *arXiv:1911.01870* (2019), 60 pages.

[10] Jentzen, A., Kuckuck, B., Müller-Gronbach, T., and Yaroslavtseva, L. On the strong regularity of degenerate additive noise driven stochastic differential equations with respect to their initial values. *arXiv:1904.05963* (2019), 59 pages.

[11] Jentzen, A., Müller-Gronbach, T., and Yaroslavtseva, L. On stochastic differential equations with arbitrary slow convergence rates for strong approximation. *Commun. Math. Sci. 14*, 6 (2016), 1477–1500.

[12] Krylov, N. V. On Kolmogorov's equations for finite-dimensional diffusions. In *Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998)*, vol. 1715 of *Lecture Notes in Math.* Springer, Berlin, 1999, pp. 1–63.

[13] Li, X.-M. Strong p-completeness of stochastic differential equations and the existence of smooth flows on noncompact manifolds. *Probab. Theory Related Fields 100*, 4 (1994), 485–511.

[14] Li, X.-M., and Scheutzow, M. Lack of strong completeness for stochastic flows. *Ann. Probab. 39*, 4 (2011), 1407–1421.

[15] Liu, W., and Röckner, M. *Stochastic partial differential equations: an introduction.* Universitext. Springer, Cham, 2015.

[16] Scheutzow, M., and Schulze, S. Strong completeness and semi-flows for stochastic differential equations with monotone drift. *J. Math. Anal. Appl. 446*, 2 (2017), 1555–1570.

[17] Zhang, X. Stochastic flows and Bismut formulas for stochastic Hamiltonian systems. *Stochastic Process. Appl. 120*, 10 (2010), 1929–1949.