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Abstract

Phase retrieval refers to the problem of recovering some signal (which is often modelled as an element of a Hilbert space) from phaseless measurements. It has been shown that, in the deterministic setting, phase retrieval from frame coefficients is always unstable in infinite dimensional Hilbert spaces [7] and possibly severely ill-conditioned in finite dimensional Hilbert spaces [7].

Recently, it was also shown that phase retrieval from measurements induced by the Gabor transform with Gaussian window function is stable when one is willing to accept a more relaxed *semi-global* stability regime [1].

We present first evidence that this semi-global stability regime allows one to do phase retrieval from measurements induced by the discrete Gabor transform in such a way that the corresponding stability constant only scales linearly in the space dimension. To this end, we utilise well-known reconstruction formulae which have been used repeatedly in recent years [6, 12, 18, 20].

1 Introduction

Phase retrieval generally alludes to the non-linear inverse problem of recovering some signal (which in this paper will be modelled by $x \in \mathbb{C}^L$) from phaseless measurements. Some of its more well-known applications include ptychography for coherent diffraction imaging [15, 19, 23, 26] and audio processing [11, 13, 17]. It has been shown that the phase retrieval problem for frames in finite-dimensional Hilbert spaces [7] and a forteriori in finite-dimensional reflexive Banach spaces [2] is always stable, which elicits the question: Why are we still concerned with stability estimates for phase retrieval from discrete Gabor measurements? The reason is that phase retrieval for frames in infinite-dimensional spaces is always unstable [2, 7] and in addition one can construct sequences of finite-dimensional subspaces of infinite-dimensional Hilbert spaces along with frames for which the stability constant of phase retrieval increases exponentially in the dimension of the constructed subspaces [7]. Recent research [1] into the infinite dimensional phase retrieval problem has, however, led us to believe that the instability of phase retrieval is not an insurmountable obstacle to reconstruction. It was shown that stability can be restored for examples that exhibit a disconnectedness in the measurements by only reconstructing the phase semi-globally or in an *atoll sense*. Furthermore, it was shown in [14] that such disconnectedness in the measurements is the only source of instabilities for phase retrieval. The most prominent of these examples certainly consists of the Gaussian window $g(t) := \mathrm{e}^{-\pi t^2}$ in conjunction with the signals

$$f_\lambda^+(t) := g(t-\lambda) + g(t+\lambda) \qquad \text{and} \qquad f_\lambda^-(t) := g(t-\lambda) - g(t+\lambda)$$

depicted in figure 1. When λ increases, the Gaussian bumps in the signals $f_\lambda^\pm$ start to move further apart effectively generating what we call a time gap whose length depends linearly on λ. It can be shown, see [3], that the measurements generated by the continuous Gabor transform with Gaussian window of the signals $f_\lambda^\pm$ have distance on the order of

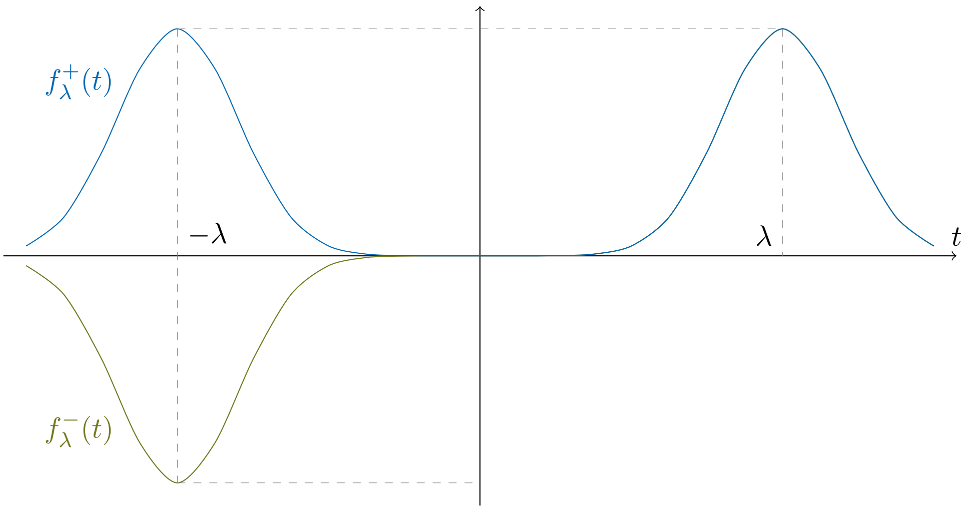


Figure 1: The most prominent example for instability of phase retrieval with continuous Gabor measurements.

$\mathrm{e}^{-\lambda^2}$ in the standard Sobolev space $W^{1,2}(\mathbb{R}^2)$ and that one can therefore not stably retrieve $f_\lambda^\pm$ from continuous Gabor transform measurements. Similar phenomena can be observed for the discrete setting considered in this paper and we do therefore propose the same paradigm as in [1] and try to recover signals in a more semi-global fashion than is usual in the phase retrieval literature up to this point.

One should note that in recent years a variety of stability result for phase retrieval have been proven. Some highlights of this research include:

i. The PhaseLift method [8, 10] which guarantees stable recovery from $\mathcal{O}(L \log L)$ randomly chosen Gaussian measurements with high probability.

ii. The research on polarisation for phase retrieval [4, 5, 21] in which the authors supplement an existing measurement ensemble in order to obtain a phase retrieval problem that is efficiently and stably solvable.

iii. Wirtinger flow and related methods [9, 24, 25] which offer stability guarantees for sufficiently many randomly chosen Gaussian measurements.

iv. The eigenvector-based angular synchronisation approach [16] which relies on a certain weak form of invertibility of the phase retrieval problem to prove a stability result for deterministic measurement systems.

In some way or another, all of these results are based on different setups than ours: As opposed to the papers referenced in item i. and iii., we will not work with a probabilistic measurement system but with a deterministic one. We will also not supplement our measurement ensemble as is done in the results referenced in item ii. and we will not work with the weak form of invertibility that is present in the paper referenced in item iv. In fact, we will consider the two well-known formulae (1) and (2) presented in section 2 which are heavily used to develop methods for exact phase retrieval from Gabor measurements in the literature [12, 18, 20]. We show that through a further analysis of the formulae (1) and (2), one can derive stability results for some of those methods and therefore also for phase retrieval in general. Our stability results come with constants that scale only linearly in the space dimension at the cost of relaxing the notion of stability to resemble the one proposed in [1].

Outline In section 2, we present the reader with the uniqueness results and the formulae on which our stability results hinge. In section 3, we utilise the ambiguity function relation

(2) in order to show that phase retrieval can be stably done based on the considerations in [6, 22]. In section 4, we use the autocorrelation relation (1) in order to show that phase retrieval can be done stably utilising results from [12,18]. As the proofs of our main results are a bit technical, they appear separately in section 5.

2 Prerequisites

Let $x, \varphi \in \mathbb{C}^L$. We define the *discrete Gabor transform (DGT)* of x with window function φ to be

$$\mathcal{V}_\varphi[x](m,n) := \frac{1}{\sqrt{L}} \cdot \sum_{\ell=0}^{L-1} x(\ell)\overline{\varphi(\ell-m)}\mathrm{e}^{-2\pi\mathrm{i}\frac{\ell n}{L}}, \qquad m,n = 0,\dots,L-1.$$

Here, and throughout this paper, the indexing is understood to be periodic. In particular, we use the convention $\varphi(\ell) = \varphi(\ell \mod L)$, for $\ell \in \mathbb{Z}$. Introducing the linear operator $\Pi_{(m,n)} : \mathbb{C}^L \to \mathbb{C}^L$, for $m, n \in \{0,\dots,L-1\}$, as

$$\Pi_{(m,n)}[\varphi](\ell) := \frac{1}{\sqrt{L}} \cdot \varphi(\ell-m)\mathrm{e}^{2\pi\mathrm{i}\frac{\ell n}{L}}, \qquad \ell = 0,\dots,L-1,$$

allows us to write

$$\mathcal{V}_\varphi[x](m,n) = \big(x, \Pi_{(m,n)}[\varphi]\big)$$

where $(\cdot,\cdot)$ denotes the standard inner product on $\mathbb{C}^L$. Another helpful way of looking at the DGT is to view it as a collection of windowed Fourier transforms. For this purpose, we denote $x_m(\ell) := x(\ell)\overline{\varphi(\ell-m)}$ and obtain

$$\mathcal{V}_\varphi[x](m,n) = \mathcal{F}[x_m](n),$$

where $\mathcal{F} : \mathbb{C}^L \to \mathbb{C}^L$ denotes the *discrete Fourier transform (DFT)*

$$\mathcal{F}[x](k) := \frac{1}{\sqrt{L}} \cdot \sum_{\ell=0}^{L-1} x(\ell)\mathrm{e}^{-2\pi\mathrm{i}\frac{\ell k}{L}}, \qquad k = 0,\dots,L-1,$$

with inverse

$$\mathcal{F}^{-1}[x](\ell) = \frac{1}{\sqrt{L}} \cdot \sum_{k=0}^{L-1} x(k)\mathrm{e}^{2\pi\mathrm{i}\frac{k\ell}{L}}, \qquad \ell = 0,\dots,L-1.$$

We are interested in the recovery of signals $x \in \mathbb{C}^L$ from the measurements

$$M_\varphi[x](m,n) := |\mathcal{V}_\varphi[x](m,n)|^2, \qquad m,n = 0,\dots,L-1.$$

It is immediately obvious that $x \in \mathbb{C}^L$ and any signal $x\mathrm{e}^{\mathrm{i}\alpha}$, with $\alpha \in \mathbb{R}$, will yield the same measurements $M_\varphi[x\mathrm{e}^{\mathrm{i}\alpha}] = M_\varphi[x]$. Therefore, to have any chance of recovery, we will actually view M_φ as an operator defined on the quotient space $\mathbb{C}^L/\mathcal{S}^1$, where $\mathcal{S}^1$ denotes the unit circle. Under various assumptions, which we will lay out in section 2, one can show that $M_\varphi : \mathbb{C}^L/\mathcal{S}^1 \to \mathbb{R}_+^{L\times L}$ is an injective operator and that phase retrieval is therefore possible *up to a global phase factor*. In addition, it was shown in [7] that M_φ has a uniformly continuous inverse whenever it is injective. In particular,

$$\inf_{\alpha\in\mathbb{R}} \|x - \mathrm{e}^{\mathrm{i}\alpha} y\|_2 \lesssim \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}},$$

for all $x, y \in \mathbb{C}^L$, where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm and the estimate depends on a constant which might increase exponentially in the space dimension L. Our phase retrieval problem is therefore possibly ill-conditioned.

As mentioned before, the number of known uniqueness results has seen a stark rise in the past few years. In the following, we want to mention those that inspired our stability estimates. Let us start by remarking that almost all uniqueness results can be traced back to two consequential formulae which are well-known in the literature. The first of these relates the Gabor measurements to the autocorrelation of x_m.

Lemma 2.1 *Let $x, \varphi \in \mathbb{C}^L$. Then,*

$$\mathcal{F}^{-1}\left[M_\varphi[x](m,\cdot)\right](n) = \frac{1}{\sqrt{L}} \cdot \left(x_m * x_m^{\#}\right)(n), \qquad m,n = 0,\dots,L-1, \tag{1}$$

holds with $x_m^{\#}(\ell) = \overline{x_m(-\ell)}$ and $x_m(\ell) = x(\ell)\overline{\varphi(\ell-m)}$.

Proof See appendix B. □

The right-hand side in the above result is the aforementioned autocorrelation of x_m

$$\left(x_m * x_m^{\#}\right)(n) = \sum_{\ell=0}^{L-1} x(\ell)\overline{x(\ell-n)}\varphi(\ell-n-m)\overline{\varphi(\ell-m)}.$$

The second of these formulae relates the Gabor measurements to the ambiguity function of x and the ambiguity function of φ.

Lemma 2.2 *Let $x, \varphi \in \mathbb{C}^L$. Then,*

$$\mathcal{F}\left[M_\varphi[x]\right](m,n) = \left(x, \Pi_{(-n,m)}[x]\right) \cdot \left(\Pi_{(-n,m)}[\varphi], \varphi\right), \qquad m,n = 0,\dots,L-1. \tag{2}$$

holds, where $\mathcal{F}[M_\varphi[x]]$ denotes the two-dimensional Fourier transform which is the composition of two standard DFTs.

Proof See appendix B. □

The right-hand side in the above result is the multiplication of the ambiguity function of x with the ambiguity function of φ.

First, we want to consider the uniqueness results from [6, 22] which are based solely on equation (2).

Corollary 2.3 (Theorem 2.2 in [6], p. 547) *Let $x, \varphi \in \mathbb{C}^L$ with*

$$\left(\varphi, \Pi_{(m,n)}[\varphi]\right) \neq 0, \qquad m,n = 0,\dots,L-1.$$

Then, x is uniquely determined by the measurements $M_\varphi[x]$ up to global phase.

While this result is exceptionally nice in the sense that it does not impose any requirements on the signal, it is quite restrictive in its requirements on the window function φ. For instance, windows φ with support length $|\operatorname{supp}\varphi|$ smaller than $L/2$ will always have zero entries in their ambiguity function.

Corollary 2.4 (Theorem 2.4 in [6], p. 549) *Let $x, \varphi \in \mathbb{C}^L$ with x nowhere-vanishing, i.e.* $\operatorname{supp} x = \{0,\dots,L-1\}$, *and*

$$\left(\varphi, \Pi_{(m,n)}[\varphi]\right) \neq 0, \qquad m = 0,1,\ n = 0,\dots,L-1.$$

Then, x is uniquely determined by the measurements $M_\varphi[x]$ up to global phase.

This result is in some sense orthogonal to corollary 2.3: Its requirements on the window function are moderate while its requirements on the signal are rather restrictive. Of course, we might also infer a variety of results that are based on different trade-offs between restrictions on the window and restrictions on the signal.

Corollary 2.5 *Let $x, \varphi \in \mathbb{C}^L$ and let $\Delta_{\text{time}} \in \{0, \dots, \lfloor L/2 \rfloor - 1\}$ be a maximum time separation parameter. Suppose that x has at most Δ_{time} consecutive zeroes in between two non-zero entries and assume that*

$$\left(\varphi, \Pi_{(m,n)}[\varphi]\right) \neq 0, \qquad m = 0, \dots, \Delta_{\text{time}} + 1, \; n = 0, \dots, L-1.$$

Then, x is uniquely determined by the measurements $M_\varphi[x]$ up to global phase.

Proof According to equation (2), we can reconstruct $\left(x, \Pi_{(m,n)}[x]\right)$, for $m = 0, \dots, \Delta_{\text{time}} + 1$, $n = 0, \dots, L-1$, from the measurements $M_\varphi[x]$. Applying the inverse Fourier transform in n yields

$$\mathcal{F}^{-1}\left[\left(x, \Pi_{(m,\cdot)}[x]\right)\right](n) = x(n)\overline{x(n-m)}, \qquad m = 0, \dots, \Delta_{\text{time}} + 1, \; n = 0, \dots, L-1.$$

As x has at most Δ_{time} consecutive zeros between two non-zero entries, we can recover it up to global phase by phase propagation. We start with any non-zero entry $x(\ell_0)$ of x which we reconstruct up to global phase by computing its absolute value

$$|x(\ell_0)|^2 = \mathcal{F}^{-1}\left[\left(x, \Pi_{(0,\cdot)}[x]\right)\right](\ell_0)$$

and setting its phase value to be zero. Then, we propagate phase information to the next non-zero entry $x(\ell_1)$ (which by assumption satisfies $\ell_1 - \ell_0 \leq \Delta_{\text{time}} + 1$) by the formula

$$\mathcal{F}^{-1}\left[\left(x, \Pi_{(m,\cdot)}[x]\right)\right](n) = x(n)\overline{x(n-m)}$$

with $m \leq \Delta_{\text{time}} + 1$. $\square$

Secondly, we will work with a uniqueness result first proven in [12] and later generalised in [18] based mostly on equation (1). Consider the following statement.

Corollary 2.6 (Theorem 1 in [12], p. 639) *Let $x, \varphi \in \mathbb{C}^L$. Suppose that*

$$\operatorname{supp} \varphi = [n_0, n_0 + \ell_\varphi] \mod L,$$

with $n_0, \ell_\varphi \in \{0, \dots, L-1\}$ and $\ell_\varphi < L/2$. Suppose additionally that $\mathcal{F}[|\varphi|^2]$ and x are nowhere-vanishing as well as that $\ell_\varphi - 1$ and L are coprime. Then, x is uniquely determined by the measurements $M_\varphi[x]$ up to global phase.

The work in [18] shows that one can also derive this result as part of a graph-theoretical formulation for phase retrieval. Consider the graph $G = (V, E)$ with vertex set $V := \operatorname{supp} x$ and edge between the vertices $\ell \in V$ and $\ell' \in V$ if $|\ell - \ell'| = \ell_\varphi \mod L$. The connectedness of this graph will then allow us to do phase retrieval under some mild additional assumptions.

Corollary 2.7 (Theorem 3.1 in [18], p. 373) *Let $x, \varphi \in \mathbb{C}^L$. Suppose that*

$$\operatorname{supp} \varphi = [n_0, n_0 + \ell_\varphi] \mod L,$$

with $n_0, \ell_\varphi \in \{0, \dots, L-1\}$ and $\ell_\varphi < L/2$ as well as that $\mathcal{F}[|\varphi|^2]$ is nowhere-vanishing. If $G = (V, E)$ is connected, then x is uniquely determined by the measurements $M_\varphi[x]$ up to global phase.

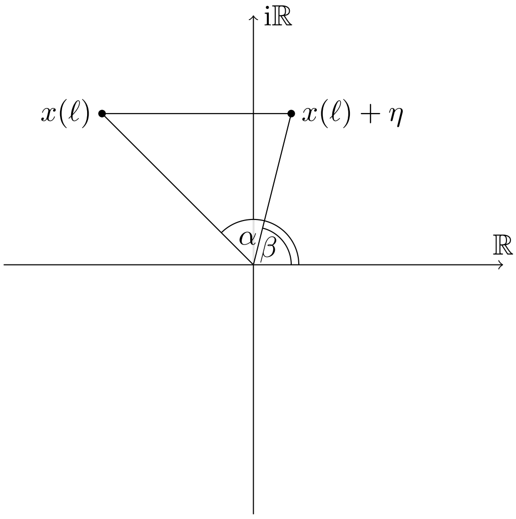


Figure 2: For $x(\ell), \eta \in \mathbb{C}$, the difference in absolute values satisfies $||x(\ell)| - |x(\ell)+\eta|| \leq |\eta|$ such that the map $|\cdot| : \mathbb{C} \to \mathbb{R}_+$ can be seen to be stable. At the same time, we could choose $x(\ell) = -\varepsilon/2$, $\eta = \varepsilon$ such that $|\alpha - \beta| = \pi \geq \pi/\varepsilon \cdot |\eta|$, where $\alpha, \beta \in (-\pi, \pi]$ denote the principal values of the arguments of $x(\ell), x(\ell + \eta) \in \mathbb{C}$, respectively. Evidently, the function which maps complex numbers to their phase is unstable at the origin.

3 Stability estimates based on the ambiguity function relation

3.1 The simplest case: Mild restrictions on the window

First, we will derive stability estimates by employing equation (2) and corollaries 2.3–2.5. In doing this, we want to start with the very simple setup of corollary 2.4.

One can immediately see that there are some intricacies to the phase retrieval problem for signals $x \in \mathbb{C}^L$. One of those is dealing with entries $x(\ell)$ of x which have small (or even vanishing) magnitude. For these entries, extracting the phase of $x(\ell)$ is unstable (or even impossible). See figure 2 for a depiction of this situation. Because of this, we will mostly work with a graph capturing only the larger entries of our signals. For $x, y \in \mathbb{C}^L$ and a tolerance parameter $\delta_0 > 0$, we define the vertex set

$$V := V(\delta_0, x, y) := \{\ell \in \{0, \ldots, L-1\} \mid |x(\ell)|, |y(\ell)| > \delta_0\}. \tag{3}$$

In the following, we will work with the ℓ^2-norm on subsets of $\{0, \ldots, L-1\}$. For the vertex set V, we define

$$\|x\|_{\ell^2(V)} := \left(\sum_{\ell \in V} |x(\ell)|^2 \right)^{\frac{1}{2}}.$$

We may now prove the following result on the stability of magnitude retrieval.

Lemma 3.1 (Stability of magnitude retrieval) *Let $x, y \in \mathbb{C}^L$ be two signals. Let $\delta_0, \delta_1 > 0$ be tolerance parameters, and let $V = V(\delta_0, x, y)$ be as in equation (3). Then,*

$$\||x| - |y|\|_{\ell^2(V)} \leq \frac{1}{2\delta_0\delta_1} \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} + \frac{1}{2\delta_0} \cdot \varepsilon$$

holds, with

$$\varepsilon := \varepsilon(\delta_1, x, y, \varphi) := \left(\sum_{\substack{\ell=0 \\ |\mathcal{F}[|\varphi|^2](\ell)| \leq \delta_1}}^{L-1} \left| \mathcal{F}\left[|x|^2 - |y|^2 \right](\ell) \right|^2 \right)^{\frac{1}{2}}.$$

Proof Let us start with the simple fact

$$||x(\ell)| - |y(\ell)||^2 = \frac{\left||x(\ell)|^2 - |y(\ell)|^2\right|^2}{||x(\ell)| + |y(\ell)||^2}.$$

Summing over $\ell \in V$ and taking the square root gives the ℓ^2-norm which we might estimate by

$$\||x| - |y|\|_{\ell^2(V)} = \left(\sum_{\ell \in V} \frac{\left||x(\ell)|^2 - |y(\ell)|^2\right|^2}{(|x(\ell)| + |y(\ell)|)^2}\right)^{\frac{1}{2}} \leq \frac{1}{2\delta_0} \cdot \left\||x|^2 - |y|^2\right\|_{\ell^2(V)}.$$

According to Plancherel's theorem, we have

$$\left\||x|^2 - |y|^2\right\|_{\ell^2(V)} \leq \left\||x|^2 - |y|^2\right\|_2 = \left\|\mathcal{F}\left[|x|^2 - |y|^2\right]\right\|_2.$$

We are now ready to bring in equation (2) in the form of

$$\mathcal{F}\left[M_\varphi[x]\right](m, 0) = \mathcal{F}\left[|x|^2\right](m) \cdot \overline{\mathcal{F}\left[|\varphi|^2\right](m)}.$$

To do this stably, the norm $\|\mathcal{F}[|x|^2 - |y|^2]\|_2$ may be split into a region where $|\mathcal{F}[|\varphi|^2]|$ is small and one where it is not. We get

$$\left\|\mathcal{F}\left[|x|^2 - |y|^2\right]\right\|_2 \leq \left(\sum_{\substack{\ell=0 \\ |\mathcal{F}[|\varphi|^2](\ell)| > \delta_1}}^{L-1} \frac{\left|\mathcal{F}\left[M_\varphi[x] - M_\varphi[y]\right](\ell, 0)\right|^2}{\left|\mathcal{F}\left[|\varphi|^2\right](\ell)\right|^2}\right)^{\frac{1}{2}} + \left(\sum_{\substack{\ell=0 \\ |\mathcal{F}[|\varphi|^2](\ell)| \leq \delta_1}}^{L-1} \left|\mathcal{F}\left[|x|^2 - |y|^2\right](\ell)\right|^2\right)^{\frac{1}{2}},$$

which immediately gives rise to the estimate

$$\left\|\mathcal{F}\left[|x|^2 - |y|^2\right]\right\|_2 \leq \frac{1}{\delta_1} \cdot \left\|\mathcal{F}\left[M_\varphi[x] - M_\varphi[y]\right](\cdot, 0)\right\|_2 + \varepsilon.$$

Finally, we can use a quite crude estimate and Plancherel's theorem in two dimensions to see that

$$\left\|\mathcal{F}\left[M_\varphi[x] - M_\varphi[y]\right](\cdot, 0)\right\|_2 \leq \left\|\mathcal{F}\left[M_\varphi[x] - M_\varphi[y]\right]\right\|_2 = \left\|M_\varphi[x] - M_\varphi[y]\right\|_{\mathrm{F}}$$

holds. Putting all our calculations together yields

$$\||x| - |y|\|_{\ell^2(V)} \leq \frac{1}{2\delta_0\delta_1} \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} + \frac{1}{2\delta_0} \cdot \varepsilon$$

as desired. □

Remark 3.2 *We arrive at the stability constant*

$$\frac{1}{2\delta_0\delta_1}$$

for magnitude retrieval. Note that δ_0 *enters the constant as the map* $x \mapsto x^2$ *is not Lipschitz continuous close to zero and we do therefore need a lower bound on the absolute value of our signal entries. Note also that* δ_1 *and the error term* ε *encapsulate the instability of recovering* $\mathcal{F}[|x|^2]$ *through equation* (2)*. In particular, the division by* $\mathcal{F}[|\varphi|^2]$ *can only be stable if* $\mathcal{F}[|x|^2]$ *is small whenever the divisor is small. In the setting of the above theorem this can (somewhat heuristically) be rephrased to: 'Whenever* $\mathcal{F}[|\varphi|^2]$ *falls below the threshold* δ_1*, the signals must have well-aligned frequency supports in the sense that* $\mathcal{F}[|x|^2 - |y|^2]$ *is small'.*

Next, we can deal with the retrieval of the phases. First, in accordance with corollary 2.4, we will only use the entries $(x, \Pi_{(1,m)}x)$ and $(\varphi, \Pi_{(1,m)}\varphi)$, where $m \in \{0, \dots, L-1\}$, for our recovery which allows us to do phase propagation on adjacent entries. To be precise, we can propagate the phase from $x(\ell)$ to $x(\ell+1)$ (or back), for any $\ell \in \{0, \dots, L-1\}$. Mathematically this fact can probably best be described by supplying an edge set E in addition to the vertex set V from (3). We will put an edge between $\ell \in V$ and $\ell' \in V$ if

$$|\ell - \ell'| = 1 \mod L.$$

The resulting graph will subsequently be denoted by $G = (V, E)$ and the connected components of G will be called *temporal islands.*

Theorem 3.3 (Stability of phase retrieval on a single temporal island) *Consider two signals $x, y \in \mathbb{C}^L$, a window $\varphi \in \mathbb{C}^L$ and two tolerance parameters $\delta_0, \delta_1 > 0$. Let $G = (V, E)$ be defined as above (with $V = V(\delta_0, x, y)$ as in equation (3)) and assume that* G is connected. *Then,*

$$\inf_{\alpha\in\mathbb{R}} \left\|x - \mathrm{e}^{\mathrm{i}\alpha} y\right\|_{\ell^2(V)} \le \frac{1}{\delta_0\delta_1} \cdot \left(\frac{1}{2} + \frac{\min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0} \cdot |V|\right) \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}}$$
$$+ \frac{1}{\delta_0} \cdot \left(\frac{1}{2} + \frac{\min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0} \cdot |V|\right) \cdot \varepsilon.$$

holds, with

$$\varepsilon := \varepsilon(\delta_1, x, y, \varphi) := \left(\sum_{\substack{\ell=0 \\ |(\varphi, \Pi_{(0,\ell)}\varphi)| \le \delta_1}}^{L-1} \left|(x, \Pi_{(0,\ell)}x) - (y, \Pi_{(0,\ell)}y)\right|^2 + 2 \cdot \sum_{\substack{\ell=0 \\ |(\varphi, \Pi_{(1,\ell)}\varphi)| \le \delta_1}}^{L-1} \left|(x, \Pi_{(1,\ell)}x) - (y, \Pi_{(1,\ell)}y)\right|^2 \right)^{\frac{1}{2}}.$$

Proof See section 5. □

Remark 3.4 *The stability constant derived in the above result is*

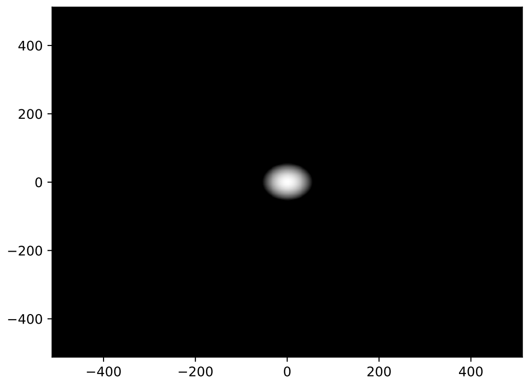
$$\frac{1}{\delta_0\delta_1} \cdot \left(\frac{1}{2} + \frac{\min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0} \cdot |V|\right)$$

and consists of a contribution from the magnitude retrieval estimate in lemma 3.1 and the phase retrieval estimate presented in section 5. In addition to the dependencies on δ_0 and δ_1, the new constant depends on the supremum norm of our signals and the size of our graph G. The latter should be seen as a mild ill-conditioning as $|V|$ is potentially on the order of L such that the stability constant might increase linearly in the space dimension. As mentioned before, the constant $\delta_1 > 0$ and the error ε reflect a trade-off: Do we use the ambiguity function of the window $(\Pi_{(\sigma,\ell)}[\varphi], \varphi)$ with possibly small values at the expense of a larger stability constant? As before our stability result is only meaningful if the difference of the ambiguity functions of the signals is small on the region where the ambiguity function of the window is small.

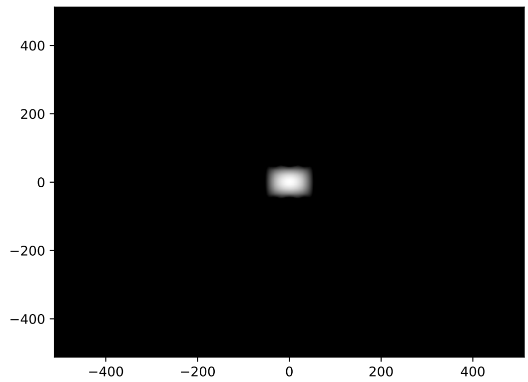
To make this theorem and, in particular, the definition of the error ε more palpable, we plot the ambiguity functions $(\varphi, \Pi_{(m,n)}[\varphi])$ of four commonly used window functions $\varphi \in \mathbb{C}^L$ in figure 3. For reference, we use $L = 1024$ and the windows

$$\varphi_{\text{gauss}}(\ell) = \mathrm{e}^{-\pi\frac{(\ell-512)^2}{32^2}}, \qquad \varphi_{\text{hamming}}(\ell) := \begin{cases} \frac{25}{46} - \frac{21}{46}\cos\left(\frac{2\pi\ell}{63}\right) & \text{if } \ell = 0, \dots, 63, \\ 0 & \text{else,} \end{cases}$$

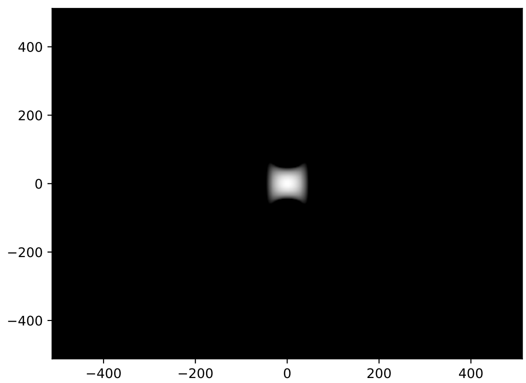
$$\varphi_{\text{hann}}(\ell) := \begin{cases} \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi\ell}{63}\right) & \text{if } \ell = 0, \dots, 63, \\ 0 & \text{else,} \end{cases} \qquad \varphi_{\text{rectangular}}(\ell) := \begin{cases} 1 & \text{if } \ell = 0, \dots, 63, \\ 0 & \text{else.} \end{cases}$$



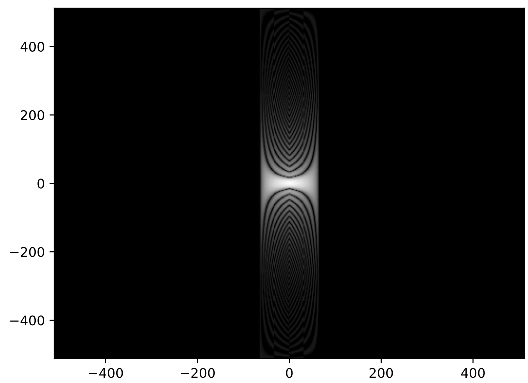
(a) The ambiguity function of the Gaussian window.



(b) The ambiguity function of the Hamming window.



(c) The ambiguity function of the Hann window.



(d) The ambiguity function of the rectangular window.

Figure 3: Visualisation of the ambiguity functions $(\varphi, \Pi_{(m,n)}[\varphi])$ of some commonly used window functions.

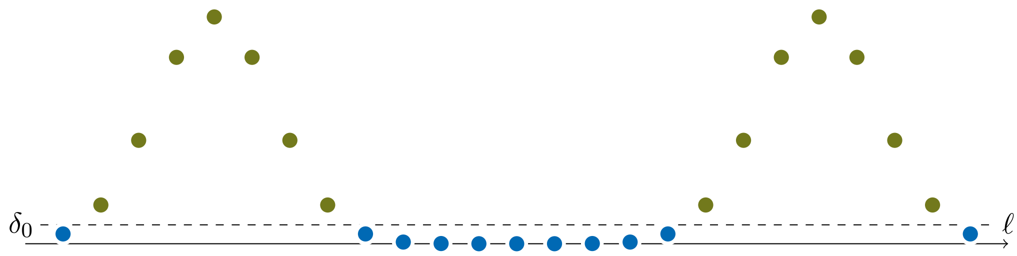


Figure 4: The function f_λ^+ from the introduction after discretisation. Entries of the resulting signal that fall below a certain threshold $\delta_0 > 0$ are coloured in blue. The remaining entries are coloured in green and make up the vertex set V. In this picture, we can clearly see the two temporal islands.

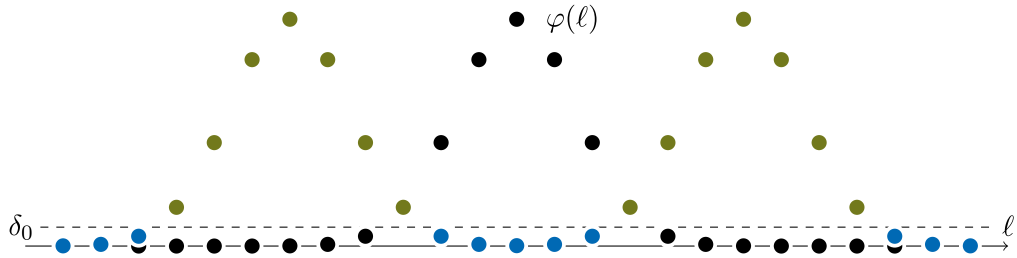


Figure 5: A similar picture to figure 4. Here, we see how the Gaussian window φ can be used to 'build a bridge' from one island to the other.

One can immediately see that the ambiguity functions $(\varphi, \Pi_{(m,n)}[\varphi])$ are always concentrated around the origin $(0,0)$ of the time-frequency plane. In particular, it is necessary for stable recovery of signals $x, y \in \mathbb{C}^L$ through equation (2) that the quantity $|(x, \Pi_{(m,n)}[x]) - (y, \Pi_{(m,n)}[y])|$ is not too big away from the origin.

3.2 Multiple islands and different restrictions on the window

The phase propagation procedure presented as part of the proof of theorem 3.3 carries over quite naturally to the case where the graph $G = (V, E)$ is disconnected rather than connected. We say the graph G has multiple *temporal islands*. It is of course interesting to consider this case, as there is a wide range of signals for which G will be disconnected. For instance, recordings of human speech will typically consist of multiple temporal islands as speakers tend to leave short gaps (i.e. modes of silence) in between words. In addition, a discretisation of the signal f_λ^+ from the introduction as depicted in figure 4 will yield two temporal islands.

In addition, one should note that until now we have only worked with minimal restrictions on the ambiguity function $(\varphi, \Pi_{(m,n)}[\varphi])$ of the window φ, i.e. we have only utilised the ambiguity function for $m = 0, 1$. In the following, we want to generalise our result to be able to use $(\varphi, \Pi_{(m,n)}[\varphi])$ for $m = 0, \ldots, \Delta_{\text{time}}+1$, where $\Delta_{\text{time}} < L/2$. The application for this more general theorem is signals which have multiple temporal islands (i.e. connected components of the graph G from subsection 3.1) together with a window φ whose ambiguity function has a rather large support. In this case, we may be able to harness corollary 2.5 in order to propagate phase stably across a temporal gap. See figure 5 for a visualisation.

To precisely formulate this bridging procedure, we will redefine our graph G. Note that the graph from the prior two theorems may be recovered from our new construction by using the right parameters and, in particular, the following construction is more general than the one in subsection 3.1. We let the vertex set V of our new graph be identical to the one of our old graph and thus be defined by equation (3). Furthermore, we draw an edge between $\ell \in V$ and $\ell' \in V$ if

$$1 \leq |\ell - \ell'| \leq \Delta_{\text{time}} + 1 \mod L.$$

With this graph at our hands, we can formulate our stability result in full generality.

Theorem 3.5 (Main theorem) *Consider two signals $x, y \in \mathbb{C}^L$, a window $\varphi \in \mathbb{C}^L$, two tolerance parameters $\delta_0, \delta_1 > 0$ and a maximum time separation parameter $\Delta_{\text{time}} \in \{0, \ldots, \lfloor L/2 \rfloor - 1\}$ as in corollary 2.5. Let $G = (V, E)$ be defined as above (with $V = V(\delta_0, x, y)$ as in equation (3)) and assume that G has K connected components with vertex sets $\{V_k\}_{k=1}^K$. Then,*

$$\begin{aligned}\inf_{\alpha_1,\ldots,\alpha_K \in \mathbb{R}} \sum_{k=1}^{K} \left\| x - \mathrm{e}^{\mathrm{i}\alpha_k} y \right\|_{\ell^2(V_k)} \leq{}& \frac{1}{\delta_0 \delta_1} \\ &\cdot \left(\frac{1}{2} + \frac{\min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0} \cdot \sum_{k=1}^{K} |V_k| \right) \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} \\ &+ \frac{1}{\delta_0} \cdot \left(\frac{1}{2} + \frac{\min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0} \cdot \sum_{k=1}^{K} |V_k| \right) \cdot \varepsilon\end{aligned}$$

holds, with

$$\varepsilon := \varepsilon(\delta_1, x, y, \varphi) := \left(2 \cdot \sum_{k=0}^{\Delta_{\text{time}}+1} \sum_{\substack{\ell=0 \\ |(\varphi, \Pi_{(k,\ell)}\varphi)| \leq \delta_1}}^{L-1} \left| (x, \Pi_{(k,\ell)} x) - (y, \Pi_{(k,\ell)} y) \right|^2 \right)^{\frac{1}{2}}.$$

Proof See section 5. □

3.3 Frequency islands

It is well known that the time (or space) domain and the frequency domain are intimately related through the Fourier transform. A textbook example of this relation is Parseval's formula

$$(x, y) = (\mathcal{F}[x], \mathcal{F}[y])$$

which effectively states that the DFT $\mathcal{F} : \mathbb{C}^L \to \mathbb{C}^L$ is a unitary operator. In this light, it is not surprising that we can derive stability results for recovering $\mathcal{F}[x]$ from the measurements $M_\varphi[x]$ resembling the theorems derived above by instrumentalising Parseval's identity. We will refer to these results as *dual results* to the theorems proven before.

Let us start by applying Parseval's equality to equation (2) which yields

$$\mathcal{F}\left[M_\varphi[x]\right](m, n) = \left(\mathcal{F}[x], \mathcal{F}[\Pi_{(-n,m)}[x]]\right) \cdot \left(\mathcal{F}[\Pi_{(-n,m)}[\varphi]], \mathcal{F}[\varphi]\right).$$

In addition,

$$\mathcal{F}[\Pi_{(-n,m)}[x]] = \Pi_{(m,n)}[\mathcal{F}[x]] \cdot \mathrm{e}^{-2\pi\mathrm{i}\frac{mn}{L}}$$

holds, such that

$$\mathcal{F}\left[M_\varphi[x]\right](m, n) = \left(\mathcal{F}[x], \Pi_{(m,n)}[\mathcal{F}[x]]\right) \cdot \left(\Pi_{(m,n)}[\mathcal{F}[\varphi]], \mathcal{F}[\varphi]\right). \tag{4}$$

From this equation, which can be viewed as dual to equation (2), we can deduce the stability of phase retrieval on frequency islands. Let us introduce the graph $\widehat{G} = (\widehat{V}, \widehat{E})$ with vertex set

$$\widehat{V} := \widehat{V}(\delta_0, x, y) := \{\ell \in \{0, \ldots, L-1\} \mid |\mathcal{F}[x](\ell)|, |\mathcal{F}[y](\ell)| > \delta_0\}$$

and edge between $\ell \in \widehat{V}$ and $\ell' \in \widehat{V}$ if

$$1 \leq |\ell - \ell'| \leq \Delta_{\text{freq}} + 1 \mod L,$$

for a fixed (but arbitrary) $\Delta_{\text{freq}} \in \{0, \dots, \lfloor L/2 \rfloor - 1\}$. Now, consider the following result.

Theorem 3.6 (Stability for frequency gaps) *Consider two signals $x, y \in \mathbb{C}^L$, a window $\varphi \in \mathbb{C}^L$, two tolerance parameters $\delta_0, \delta_1 > 0$ and a maximum frequency separation parameter $\Delta_{\text{freq}} \in \{0, \dots, \lfloor L/2 \rfloor - 1\}$. Let $\widehat{G} = (\widehat{V}, \widehat{E})$ be defined as above and assume that $\widehat{G}$ has K connected components with vertex sets $\{\widehat{V}_k\}_{k=1}^K$. Then,*

$$\begin{aligned}
\inf_{\alpha_1, \dots, \alpha_K \in \mathbb{R}} \sum_{k=1}^{K} \left\| \mathcal{F}[x] - \mathrm{e}^{\mathrm{i}\alpha_k} \mathcal{F}[y] \right\|_{\ell^2(\widehat{V}_k)} \leq \frac{1}{\delta_0 \delta_1} & \\
\cdot \left(\frac{1}{2} + \frac{\min\{\|\mathcal{F}[x]\|_\infty, \|\mathcal{F}[y]\|_\infty\}}{\delta_0} \cdot \sum_{k=1}^{K} |\widehat{V}_k| \right) \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} & \\
+ \frac{1}{\delta_0} \cdot \left(\frac{1}{2} + \frac{\min\{\|\mathcal{F}[x]\|_\infty, \|\mathcal{F}[y]\|_\infty\}}{\delta_0} \cdot \sum_{k=1}^{K} |\widehat{V}_k| \right) \cdot \varepsilon &
\end{aligned}$$

holds, with

$$\varepsilon := \varepsilon(\delta_1, x, y, \varphi) := \left(2 \cdot \sum_{k=0}^{\Delta_{\text{freq}}+1} \sum_{\substack{\ell=0 \\ |(\varphi, \Pi_{(\ell,k)}\varphi)| \leq \delta_1}}^{L-1} \left| (x, \Pi_{(\ell,k)} x) - (y, \Pi_{(\ell,k)} y) \right|^2 \right)^{\frac{1}{2}}.$$

Proof See section 5. $\square$

Remark 3.7 *In the preceding pages, we have presented approaches to dealing with time and frequency gaps in signals when doing phase retrieval. Unfortunately, it is not so clear how to extend this work to the more general case of time-frequency gaps considered in [1]. It is likely that one has to come up with a different approach that allows one to do phase propagation in frequency and time direction simultaneously to actually handle time-frequency gaps.*

We want to end this section by remarking that from our proof strategy for the frequency result a straight-forward dual version of corollary 2.5 follows.

Corollary 3.8 *Let $x, \varphi \in \mathbb{C}^L$ and let $\Delta_{\text{freq}} \in \{0, \dots, \lfloor L/2 \rfloor - 1\}$ be a maximum frequency separation parameter. Suppose that $\mathcal{F}[x]$ has at most Δ_{freq} consecutive zeroes in between two non-zero entries and assume that*

$$\big(\varphi, \Pi_{(m,n)}[\varphi]\big) \neq 0, \qquad m = 0, \dots, L-1, \ n = 0, \dots, \Delta_{\text{freq}} - 1.$$

Then, x is uniquely determined by the measurements $M_\varphi[x]$ up to global phase.

4 Stability estimates based on the autocorrelation relation

The goal of this section is to apply the techniques we have developed thus far to the setup proposed in [18]. Our approach will be designed to work with signals $x \in \mathbb{C}^L$ which potentially have very small entries and with window functions $\varphi \in \mathbb{C}^L$ for which the Fourier transform of the magnitude squared $\mathcal{F}[|\varphi|^2]$ potentially has zero entries. We emphasise these two particularities as the stability result developed in [18] (theorem 4.2 on p. 375) relies on window functions for which $\mathcal{F}[|\varphi|^2]$ is nowhere-vanishing, and while one may apply it to signals $x \in \mathbb{C}^L$ with very small entries, the resulting stability constant will be ill-behaved as it depends inversely on $\min_{\ell \in \operatorname{supp} x} |x(\ell)|^2$.

Remember from section 2 that the authors of [18] consider the graph $G = (V, E)$ with vertex set $V := \operatorname{supp}(x)$ and an edge between $\ell \in V$ and $\ell' \in V$ if

$$|\ell - \ell'| = \ell_\varphi \mod L,$$

where $\ell_\varphi + 1$ denotes the length of the window function. They do then propose to reconstruct the magnitude of a signal $x \in \mathbb{C}^L$ using the ambiguity function relation (2) (as in section 3) and propagate phase from one entry of x to another if their indices have distance ℓ_φ using equation (1). As before, we will work with local lower bounds on the ambiguity function of the window and the signal in order to ascertain that all the aforedescribed steps can be carried out stably.

In order to introduce the local lower bounds on the signal, we will have to modify the graphs presented in [18] slightly: Let us consider two signals $x, y \in \mathbb{C}^L$, a tolerance parameter $\delta_0 > 0$ and a window function $\varphi \in \mathbb{C}^L$ such that

$$\operatorname{supp}(\varphi) = [n_0, n_0 + \ell_\varphi] \mod L,$$

for $n_0 \in \{0, \ldots, L-1\}$ and $\ell_\varphi \in \{0, \ldots, \lceil L/2 \rceil - 1\}$. Introduce the graph $G = (V, E)$ with vertex set

$$V := V(\delta_0, x, y) := \{\ell \in \{0, \ldots, L-1\} \mid |x(\ell)|, |y(\ell)| > \delta_0\} \tag{5}$$

and edge between $\ell \in V$ and $\ell' \in V$ if

$$|\ell - \ell'| = \ell_\varphi \mod L. \tag{6}$$

We can then state the following stability estimate whose proof is inspired by the proof of corollary 2.7.

Theorem 4.1 *Consider two signals $x, y \in \mathbb{C}^L$ and a window $\varphi \in \mathbb{C}^L$ such that*

$$\operatorname{supp}(\varphi) = [n_0, n_0 + \ell_\varphi] \mod L,$$

for $n_0 \in \{0, \ldots, L-1\}$ and $\ell_\varphi \in \{0, \ldots, \lceil L/2 \rceil - 1\}$. Let $\delta_0, \delta_1 > 0$ be two tolerance parameters and assume that the graph $G = (V, E)$ constructed as in (5), (6) is connected. Then,

$$\inf_{\alpha \in \mathbb{R}} \left\| x - \mathrm{e}^{\mathrm{i}\alpha} y \right\|_{\ell^2(V)} \leq \frac{1}{\delta_0} \cdot \left(\frac{1}{2\delta_1} + \frac{(2L)^{\frac{1}{2}} \min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0 |\varphi(n_0)\varphi(n_0 + \ell_\varphi)|} \cdot |V| \right) \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} + \frac{1}{2\delta_0} \cdot \varepsilon$$

holds, with

$$\varepsilon := \varepsilon(\delta_1, x, y, \varphi) := \left(\sum_{\substack{\ell=0 \\ |\mathcal{F}[|\varphi|^2](\ell)| \leq \delta_1}}^{L-1} \left| \mathcal{F}\left[|x|^2 - |y|^2 \right](\ell) \right|^2 \right)^{\frac{1}{2}}.$$

Proof See section 5. □

Remark 4.2 *The stability constant in this result is*

$$\frac{1}{\delta_0} \cdot \left(\frac{1}{2\delta_1} + \frac{(2L)^{\frac{1}{2}} \min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0 |\varphi(n_0)\varphi(n_0 + \ell_\varphi)|} \cdot |V| \right).$$

The part of the constant coming from the magnitude retrieval is exactly the same as in the results in section 3 and was explained in section 3. The part of the constant coming from phase retrieval is a slight modification from the constants in section 3. It is mostly the term $|\varphi(n_0)\varphi(n_0+\ell_\varphi)|$ in the denominator that deserves some attention. It is clear that phase propagation based on the relation

$$\mathcal{F}^{-1}\left[M_\varphi[x](n,\cdot)\right](\ell_\varphi) = x(n_0+\ell_\varphi+n)\overline{x(n_0+n)}\varphi(n_0)\overline{\varphi(n_0+\ell_\varphi)}$$

will be unstable whenever the ends of the window $\varphi(n_0)$ and $\varphi(n_0+\ell_\varphi)$ are close to zero. In particular, the reconstruction method proposed by the authors of [12, 18] benefits from windows for which $|\varphi(n_0)\varphi(n_0+\ell_\varphi)|$ is large such as the Hamming or rectangular windows.

As remarked upon before, in contrast to the stability result in [18], our result is applicable even when the Fourier transform of the magnitude squared of the window function $\mathcal{F}[|\varphi|^2]$ has vanishing entries. We pay for this indulgence by having $\frac{1}{2\delta_0\delta_1}$ as the stability constant for magnitude retrieval and incurring the extra error term $\frac{\varepsilon}{2\delta_0}$ when compared to the result from [18]. We should also note that in [18] the stability constant for the phase retrieval estimate scales like $\sqrt{|\operatorname{supp} x|}\cdot L^3$ (which becomes $L^{7/2}$ for nowhere-vanishing signals) whereas our stability constant merely scales like $|V|\cdot L^{1/2}$ (which becomes $L^{3/2}$ for signals whose entries have absolute values in excess of the threshold δ_0).

Remark 4.3 (On disconnected graphs and duality results) *Finally, we would like to remark that one can prove a result resembling theorem 4.1 in the case where G has $K\in\mathbb{N}$ connected components whose vertex sets are denoted by $V_1,\dots,V_K\subset V$. In fact, the proof works by repeating the proof of theorem 4.1 with every mention of α and V replaced by α_k and V_k, respectively, for a fixed $k\in\{1,\dots,K\}$, and then summing the resulting estimates over k.*

Similarly, we may utilise lemma 2.1 in order to deduce that

$$\mathcal{F}\left[M_\varphi[x](\cdot,n)\right](m) = \frac{1}{\sqrt{L}}\cdot\sum_{k=0}^{L-1}\mathcal{F}[x](k)\overline{\mathcal{F}[x](k-m)}\mathcal{F}[\varphi](k-m-n)\overline{\mathcal{F}[\varphi](k-n)}$$

holds, for $x,\varphi\in\mathbb{C}^L$ and $m,n=0,\dots,L-1$. This in turn can be used to deduce a stability result which is essentially the Fourier-dual of theorem 4.1.

5 Proofs of the main results

Proof of theorem 3.3 Let $\alpha\in\mathbb{R}$ be arbitrary. According to proposition A.1, we have

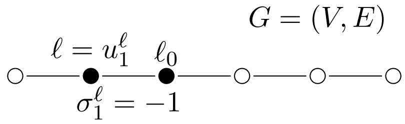
$$|x(\ell)-\mathrm{e}^{\mathrm{i}\alpha}y(\ell)| \le ||x(\ell)|-|y(\ell)|| + \min\{|y(\ell)|,|x(\ell)|\}\cdot\left|\frac{x(\ell)}{|x(\ell)|}-\mathrm{e}^{\mathrm{i}\alpha}\frac{y(\ell)}{|y(\ell)|}\right|$$

whenever $x(\ell)$ and $y(\ell)$ do not vanish. Squaring and summing over $\ell\in V$ yields the estimate

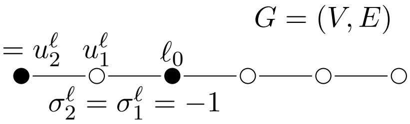
$$\left\|x-\mathrm{e}^{\mathrm{i}\alpha}y\right\|_{\ell^2(V)} \le \||x|-|y|\|_{\ell^2(V)} + \left(\sum_{\ell\in V}\min\{|y(\ell)|,|x(\ell)|\}^2\cdot\left|\frac{x(\ell)}{|x(\ell)|}-\mathrm{e}^{\mathrm{i}\alpha}\frac{y(\ell)}{|y(\ell)|}\right|^2\right)^{\frac{1}{2}}.$$

The magnitude difference $\||x|-|y|\|_{\ell^2(V)}$ was already estimated in lemma 3.1. The phase difference can be estimated by

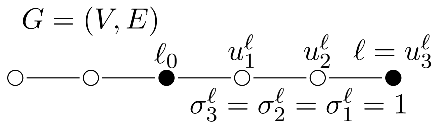
$$\min\{\|x\|_\infty,\|y\|_\infty\}\cdot\left(\sum_{\ell\in V}\left|\frac{x(\ell)}{|x(\ell)|}-\mathrm{e}^{\mathrm{i}\alpha}\frac{y(\ell)}{|y(\ell)|}\right|^2\right)^{\frac{1}{2}}.$$



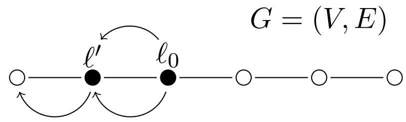
(a) For the left neighbour of ℓ_0, we obtain the one element sequences $(u_1^\ell = \ell)$ and $(\sigma_1^\ell = -1)$.



(b) An example with two element sequences in a graph with $|V| = 6$.



(c) An example with three element sequences in a graph with $|V| = 6$.



(d) The arrows indicate the realisations of ℓ' as part of a sequence $(u_j^\ell)_{j=1}^{M(\ell)}$. The upper arrow indicates that ℓ' comes up as $u_1^{\ell'}$. The lower arrow indicates that ℓ' comes up as $u_1^{\ell'-1}$.

Figure 6: Visualisation of our graph construction.

As in the proof of corollary 2.5, we will make use of a phase propagation procedure. We start with a vertex $\ell_0 \in V$ that has graph distance at most $|V|/2$ to all other vertices in V. Then, we pick $\alpha \in \mathbb{R}$ such that

$$\left| \frac{x(\ell_0)}{|x(\ell_0)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell_0)}{|y(\ell_0)|} \right| = 0.$$

According to proposition A.2, we have

$$\begin{aligned} \left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right| &\leq \left| \frac{x(\ell_0)}{|x(\ell_0)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell_0)}{|y(\ell_0)|} \right| + \frac{2\left| x(\ell)\overline{x(\ell_0)} - y(\ell)\overline{y(\ell_0)} \right|}{\max\{|x(\ell)x(\ell_0)|, |y(\ell)y(\ell_0)|\}} \\ &= \frac{2\left| x(\ell)\overline{x(\ell_0)} - y(\ell)\overline{y(\ell_0)} \right|}{\max\{|x(\ell)x(\ell_0)|, |y(\ell)y(\ell_0)|\}} \leq \frac{2}{\delta_0^2} \cdot \left| x(\ell)\overline{x(\ell_0)} - y(\ell)\overline{y(\ell_0)} \right|, \end{aligned}$$

for all $\ell \in V$. Additionally, we have

$$\mathcal{F}^{-1}\left[\left(x, \Pi_{(\sigma,\cdot)}[x]\right)\right](\ell) = x(\ell)\overline{x(\ell - \sigma)},$$

for $x \in \mathbb{C}^L$ and $\sigma \in \mathbb{Z}$, which will finally allow us to bring equation (2) into the proof. If $\ell \in V$ is a neighbour of ℓ_0, then $\ell - \sigma = \ell_0$ for some $\sigma \in \{\pm 1\}$. Thus,

$$\left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right| \leq \frac{2}{\delta_0^2} \cdot \left| \mathcal{F}^{-1}\left[\left(x, \Pi_{(\sigma,\cdot)}[x]\right) - \left(y, \Pi_{(\sigma,\cdot)}[y]\right)\right](\ell) \right|.$$

In the same way, one may prove that for any $\ell \in V \setminus \{\ell_0\}$, there exist sequences $(u_j^\ell)_{j=1}^{M(\ell)} \subset V$, $(\sigma_j^\ell)_{j=1}^{M(\ell)} \subset \{\pm 1\}$, with $M(\ell) \leq |V|/2$, such that

$$\left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right| \leq \frac{2}{\delta_0^2} \cdot \sum_{j=1}^{M(\ell)} \left| \mathcal{F}^{-1}\left[\left(x, \Pi_{(\sigma_j^\ell,\cdot)}[x]\right) - \left(y, \Pi_{(\sigma_j^\ell,\cdot)}[y]\right)\right](u_j^\ell) \right|.$$

For a visualisation of the sequences $(u_j^\ell)_{j=1}^{M(\ell)} \subset V$ and $(\sigma_j^\ell)_{j=1}^{M(\ell)} \subset \{\pm 1\}$ see figure 6. Squaring and summing over $\ell \in V$ yields the estimate

$$\left\| \frac{x}{|x|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y}{|y|} \right\|_{\ell^2(V)} \leq \frac{2}{\delta_0^2} \cdot \left(\sum_{\ell \in V} \left(\sum_{j=1}^{M(\ell)} \left| \mathcal{F}^{-1}\left[\left(x, \Pi_{(\sigma_j^\ell,\cdot)}[x]\right) - \left(y, \Pi_{(\sigma_j^\ell,\cdot)}[y]\right)\right](u_j^\ell) \right| \right)^2 \right)^{\frac{1}{2}}.$$

An application of Jensen's inequality implies

$$\left\| \frac{x}{|x|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y}{|y|} \right\|_{\ell^2(V)} \leq \frac{2^{\frac{1}{2}} |V|^{\frac{1}{2}}}{\delta_0^2} \cdot \left(\sum_{\ell \in V} \sum_{j=1}^{M(\ell)} \left| \mathcal{F}^{-1} \left[\left(x, \Pi_{(\sigma_j^\ell, \cdot)}[x] \right) - \left(y, \Pi_{(\sigma_j^\ell, \cdot)}[y] \right) \right] (u_j^\ell) \right|^2 \right)^{\frac{1}{2}},$$

as $M(\ell) \leq |V|/2$. Next, we will consider the sequences $(u_j^\ell)_{j=1}^{M(\ell)} \subset V$ and $(\sigma_j^\ell)_{j=1}^{M(\ell)} \subset \{\pm 1\}$ a little bit more precisely. One can see that any $\ell' \in V$ is realised at most $|V|/2$ times as an element in a sequence $(u_j^\ell)_{j=1}^{M(\ell)}$: Once when $\ell = \ell'$ in which case $u_{M(\ell')}^{\ell'} = \ell'$ by the construction of our sequence, and subsequently at most $|V|/2 - 1$ times when propagating phase to vertices $\ell \in V$ that are on the other side of ℓ' when viewed from ℓ_0. Consider subfigure 6d for a visualisation of this argument. Consequently, we might very crudely estimate

$$\left\| \frac{x}{|x|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y}{|y|} \right\|_{\ell^2(V)} \leq \frac{|V|}{\delta_0^2} \cdot \left(\sum_{\sigma \in \{\pm 1\}} \sum_{\ell \in V} \left| \mathcal{F}^{-1} \left[\left(x, \Pi_{(\sigma, \cdot)}[x] \right) - \left(y, \Pi_{(\sigma, \cdot)}[y] \right) \right] (\ell) \right|^2 \right)^{\frac{1}{2}}.$$

Using Plancherel's theorem yields

$$\left\| \frac{x}{|x|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y}{|y|} \right\|_{\ell^2(V)} \leq \frac{|V|}{\delta_0^2} \cdot \left(\sum_{\sigma \in \{\pm 1\}} \sum_{\ell=0}^{L-1} \left| \left(x, \Pi_{(\sigma, \ell)}[x] \right) - \left(y, \Pi_{(\sigma, \ell)}[y] \right) \right|^2 \right)^{\frac{1}{2}}. \tag{7}$$

According to equation (2), we have

$$\mathcal{F}\left[M_\varphi[x] \right] (\ell, -\sigma) = \left(x, \Pi_{(\sigma, \ell)}[x] \right) \cdot \left(\Pi_{(\sigma, \ell)}[\varphi], \varphi \right)$$

which suggests splitting the estimate (7) into a term where the ambiguity function of the window $(\Pi_{(\sigma,\ell)}[\varphi], \varphi)$ is lower bounded and one where it is upper bounded. The former may then be estimated by

$$\frac{|V|}{\delta_0^2 \delta_1} \cdot \left(\sum_{\sigma \in \{\pm 1\}} \sum_{\substack{\ell=0 \\ |(\Pi_{(\sigma,\ell)}[\varphi], \varphi)| > \delta_1}}^{L-1} \left| \mathcal{F}\left[M_\varphi[x] - M_\varphi[y] \right] (\ell, -\sigma) \right|^2 \right)^{\frac{1}{2}}$$

while the latter is upper bounded by $|V|/\delta_0^2 \cdot \varepsilon$. The proof is finished after another crude estimate and an application of Plancherel's theorem. Namely,

$$\begin{aligned} &\left(\sum_{\sigma \in \{\pm 1\}} \sum_{\substack{\ell=0 \\ |(\Pi_{(\sigma,\ell)}[\varphi], \varphi)| > \delta_1}}^{L-1} \left| \mathcal{F}\left[M_\varphi[x] - M_\varphi[y] \right] (\ell, -\sigma) \right|^2 \right)^{\frac{1}{2}} \\ &\qquad \leq \left(\sum_{\ell, k=0}^{L-1} \left| \mathcal{F}\left[M_\varphi[x] - M_\varphi[y] \right] (\ell, k) \right|^2 \right)^{\frac{1}{2}} = \left\| M_\varphi[x] - M_\varphi[y] \right\|_{\mathrm{F}} \end{aligned}$$

yields

$$\begin{aligned} \left\| x - \mathrm{e}^{\mathrm{i}\alpha} y \right\|_{\ell^2(V)} \leq{}& \frac{1}{\delta_0 \delta_1} \cdot \left(\frac{1}{2} + \frac{\min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0} \cdot |V| \right) \cdot \left\| M_\varphi[x] - M_\varphi[y] \right\|_{\mathrm{F}} \\ &+ \frac{1}{\delta_0} \cdot \left(\frac{1}{2} + \frac{\min\{\|x\|_\infty, \|y\|_\infty\}}{\delta_0} \cdot |V| \right) \cdot \varepsilon. \qquad \square \end{aligned}$$

Proof of theorem 3.5 The proof is highly similar to the one of theorem 3.3. In particular, we can follow the proof of the latter (while replacing every mention of α and V by α_k and V_k, respectively, for a fixed $k \in \{1, \ldots, K\}$) up to the point where we find the sequences $(u_j^\ell)_{j=1}^{M(\ell)} \subset V$ and $(\sigma_j^\ell)_{j=1}^{M(\ell)} \subset \{\pm 1\}$. These must be replaced by appropriately defined sequences $(u_j^\ell)_{j=1}^{M(\ell)} \subset V$ and $(m_j^\ell)_{j=1}^{M(\ell)} \subset \{-\Delta_{\text{time}} - 1, \ldots, \Delta_{\text{time}} + 1\} \setminus \{0\}$. The ensuing estimates work in the exact same way as the ones in the proof of theorem 3.3. $\square$

Proof of theorem 3.6 The proof of this theorem is overwhelmingly similar to the proof of theorem 3.5. Therefore, we will not present it in full but rather focus on the few estimates were the two proofs differ. Replacing x by $\mathcal{F}[x]$ and y by $\mathcal{F}[y]$ in the opening lines of the proof of lemma 3.1 allows us to deduce

$$\| |\mathcal{F}[x]| - |\mathcal{F}[y]| \|_{\ell^2(V)} \leq \frac{1}{2\delta_0} \cdot \left\| \mathcal{F}\left[|\mathcal{F}[x]|^2 \right] - \mathcal{F}\left[|\mathcal{F}[y]|^2 \right] \right\|_2. \tag{8}$$

Here, we bring in equation (4) in the form of

$$\mathcal{F}\left[M_\varphi[x]\right](0, \ell) = \mathcal{F}\left[|\mathcal{F}[x]|^2 \right](\ell) \cdot \mathcal{F}\left[|\mathcal{F}[\varphi]|^2 \right](\ell)$$

and split the right-hand side of inequality (8) into a region on which the autocorrelation (divided by $\sqrt{L}$) of the window $\mathcal{F}[|\mathcal{F}[\varphi]|^2] = (\varphi * \varphi^\#)/\sqrt{L}$ is smaller than the threshold δ_1 and one on which it is not. On the part, on which the autocorrelation of the window falls below δ_1, our right-hand side is certainly bounded by ε. On the other part, we may use equation (4) in the above form to find that the right-hand side is bounded by

$$\frac{1}{2\delta_0\delta_1} \cdot \left(\sum_{\substack{\ell=0 \\ |(\varphi*\varphi^\#)(\ell)|/\sqrt{L} \leq \delta_1}}^{L-1} |\mathcal{F}[M_\varphi[x] - M_\varphi[y]](0, \ell)|^2 \right)^{\frac{1}{2}}.$$

As in the proof of lemma 3.1, we deduce that

$$\| |\mathcal{F}[x]| - |\mathcal{F}[y]| \|_{\ell^2(\widehat{V})} \leq \frac{1}{2\delta_0\delta_1} \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} + \frac{1}{2\delta_0} \cdot \varepsilon.$$

Using proposition A.1, allows us to conclude that it remains to estimate

$$\left\| \frac{\mathcal{F}[x]}{|\mathcal{F}[x]|} - e^{i\alpha_k} \frac{\mathcal{F}[x]}{|\mathcal{F}[x]|} \right\|_{\ell^2(\widehat{V}_k)},$$

for some $\alpha_k \in (-\pi, \pi]$ and $k \in \{1, \ldots, K\}$. This estimate is obtained by employing equation (4) with $m \in \{-\Delta_{\text{freq}} - 1, \ldots, \Delta_{\text{freq}} + 1\} \setminus \{0\}$. Considerations resembling the one in the proof of theorem 3.5 lead us to the estimate

$$\left\| \frac{\mathcal{F}[x]}{|\mathcal{F}[x]|} - e^{i\alpha_k} \frac{\mathcal{F}[x]}{|\mathcal{F}[x]|} \right\|_{\ell^2(\widehat{V}_k)} \leq \frac{|\widehat{V}_k|}{\delta_0^2 \delta^1} \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} + \frac{|\widehat{V}_k|}{\delta_0^2} \cdot \varepsilon$$

and thus to the result in theorem 3.6. $\square$

Proof of theorem 4.1 We split the proof into a magnitude retrieval and a phase retrieval estimate. For the magnitude retrieval, we obtain

$$\| |x| - |y| \|_{\ell^2(V)} \leq \frac{1}{2\delta_0\delta_1} \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}} + \frac{1}{2\delta_0} \cdot \varepsilon$$

according to lemma 3.1. For the phase retrieval, we need to consider a new strategy based on equation (1). According to proposition A.1, we have to deal with

$$\left(\sum_{\ell \in V} \min\{|y(\ell)|, |x(\ell)|\}^2 \cdot \left| \frac{x(\ell)}{|x(\ell)|} - e^{i\alpha} \frac{y(\ell)}{|y(\ell)|} \right|^2 \right)^{\frac{1}{2}},$$

where $\alpha \in \mathbb{R}$. Using Hölder's inequality, this is easily estimated by

$$\min\{\|x\|_\infty, \|y\|_\infty\} \cdot \left\| \frac{x}{|x|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y}{|y|} \right\|_{\ell^2(V)}.$$

It remains to estimate the ℓ^2-norm of the phase difference on V. Let us start with a vertex $\ell_0 \in V$ that has graph distance at most $|V|/2$ from any other vertex in V. We pick $\alpha \in \mathbb{R}$ such that

$$\left| \frac{x(\ell_0)}{|x(\ell_0)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell_0)}{|y(\ell_0)|} \right| = 0.$$

Then, for any neighbour $\ell \in V$ of ℓ_0, we have

$$\left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right| \leq \frac{2\left| x(\ell)\overline{x(\ell_0)} - y(\ell)\overline{y(\ell_0)} \right|}{\max\{|x(\ell)x(\ell_0)|, |y(\ell)y(\ell_0)|\}} \leq \frac{2}{\delta_0^2} \cdot \left| x(\ell)\overline{x(\ell_0)} - y(\ell)\overline{y(\ell_0)} \right|$$

according to proposition A.2 and the definition of the vertex set. By the definition of the edge set, we find that $\ell = \ell_0 + \ell_\varphi$ or $\ell = \ell_0 - \ell_\varphi$ holds modulo L. In the former case, it follows from equation (1) that

$$\mathcal{F}^{-1}\left[M_\varphi[x](\ell_0 - n_0, \cdot)\right](\ell_\varphi) = \frac{1}{\sqrt{L}} \cdot x(\ell)\overline{x(\ell_0)}\varphi(n_0)\overline{\varphi(n_0 + \ell_\varphi)},$$

due to the support properties of the window φ and $\ell_\varphi < L/2$. In the latter case, we need to switch the sign of ℓ_φ in the formula above to obtain

$$\mathcal{F}^{-1}\left[M_\varphi[x](\ell_0 - n_0, \cdot)\right](-\ell_\varphi) = \frac{1}{\sqrt{L}} \cdot x(\ell_0)\overline{x(\ell)}\varphi(n_0 + \ell_\varphi)\overline{\varphi(n_0)}.$$

Iteratively, one might show that for any $\ell \in V \setminus \{\ell_0\}$, there exist sequences $(u_j^\ell)_{j=1}^{M(\ell)} \subset V$ and $(\sigma_j^\ell)_{j=1}^{M(\ell)} \subset \{\pm 1\}$, $j = 1, \ldots, M(\ell)$, where $M(\ell) \leq |V|/2$, such that

$$\begin{aligned}\left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right| \leq{}& \frac{2 \cdot L^{\frac{1}{2}}}{\delta_0^2 |\varphi(n_0)\varphi(n_0 + \ell_\varphi)|} \\ & \cdot \sum_{j=1}^{M(\ell)} \left| \mathcal{F}^{-1}\left[M_\varphi[x](u_j^\ell - n_0, \cdot) - M_\varphi[y](u_j^\ell - n_0, \cdot)\right](\sigma_j^\ell \ell_\varphi) \right|.\end{aligned}$$

Much like in the proof of theorem 3.3, squaring, summing over ℓ, applying Jensen's inequality and analysing the structure of the graph such that we see that each vertex of G comes up at most $|V| - 1$ times as part of a sequence $(u_j^\ell)_{j=1}^{M(\ell)}$ yields an upper bound for the phase differences in terms of

$$\frac{|V|}{2^{\frac{1}{2}}} \cdot \left(\sum_{\ell,k=0}^{L-1} \left| \mathcal{F}^{-1}\left[M_\varphi[x](\ell, \cdot) - M_\varphi[y](\ell, \cdot)\right](k) \right|^2 \right)^{\frac{1}{2}} = \frac{|V|}{2^{\frac{1}{2}}} \cdot \|M_\varphi[x] - M_\varphi[y]\|_{\mathrm{F}},$$

where the equality comes straight from Plancherel's theorem. Putting the thus obtained phase estimate and the magnitude estimate together yields the statement. ☐

A Pointwise estimates for the stability results

Let us start by the typical splitting of signal differences into phase and magnitude part.

Proposition A.1 *Let $x, y \in \mathbb{C}^L$. Then, we have for all $\alpha \in \mathbb{R}$ that*

$$|x(\ell) - \mathrm{e}^{\mathrm{i}\alpha} y(\ell)| \leq ||x(\ell)| - |y(\ell)|| + \min\{|y(\ell)|, |x(\ell)|\} \cdot \left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right|$$

holds for all $\ell \in \{0, \dots, L-1\}$ such that $x(\ell) \neq 0$ and $y(\ell) \neq 0$, and that

$$|x(\ell) - \mathrm{e}^{\mathrm{i}\alpha} y(\ell)| = ||x(\ell)| - |y(\ell)||$$

holds for all $\ell \in \{0, \dots, L-1\}$ such that $x(\ell) = 0$ or $y(\ell) = 0$.

Proof Let $\alpha \in \mathbb{R}$ and $\ell \in \{0, \dots, L-1\}$ such that $x(\ell) \neq 0$ and $y(\ell) \neq 0$. Then,

$$\begin{aligned}
|x(\ell) - \mathrm{e}^{\mathrm{i}\alpha} y(\ell)| &= \left| |x(\ell)| \cdot \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} |y(\ell)| \cdot \frac{y(\ell)}{|y(\ell)|} \right| \\
&= \left| (|x(\ell)| - |y(\ell)|) \cdot \frac{x(\ell)}{|x(\ell)|} + \left(\frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right) \cdot |y(\ell)| \right| \\
&\leq ||x(\ell)| - |y(\ell)|| + |y(\ell)| \cdot \left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right|.
\end{aligned}$$

If $x(\ell) = 0$, then

$$|x(\ell) - \mathrm{e}^{\mathrm{i}\alpha} y(\ell)| = |y(\ell)| = ||x(\ell)| - |y(\ell)||.$$

□

In addition, a result about phase propagation will be handy.

Proposition A.2 *Let $x, y \in \mathbb{C}^L$ and $\alpha \in \mathbb{R}$. Then, we have that*

$$\left| \frac{x(k)}{|x(k)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(k)}{|y(k)|} \right| \leq \left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right| + \frac{2\left| x(k)\overline{x(\ell)} - y(k)\overline{y(\ell)} \right|}{\max\{|x(k)x(\ell)|, |y(k)y(\ell)|\}}$$

holds, for all $\ell, k \in \{0, \dots, L-1\}$ such that $x(k), x(\ell), y(k), y(\ell) \neq 0$.

Proof We compute

$$\begin{aligned}
\left| \frac{x(k)}{|x(k)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(k)}{|y(k)|} \right| &= \left| \frac{x(k)\overline{x(\ell)}x(\ell)}{|x(k)||x(\ell)|^2} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(k)\overline{y(\ell)}y(\ell)}{|y(k)||y(\ell)|^2} \right| \\
&= \left| \left(\frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right) \frac{x(k)\overline{x(\ell)}}{|x(k)x(\ell)|} + \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \left(\frac{x(k)\overline{x(\ell)}}{|x(k)x(\ell)|} - \frac{y(k)\overline{y(\ell)}}{|y(k)y(\ell)|} \right) \right| \\
&\leq \left| \frac{x(\ell)}{|x(\ell)|} - \mathrm{e}^{\mathrm{i}\alpha} \frac{y(\ell)}{|y(\ell)|} \right| + \left| \frac{x(k)\overline{x(\ell)}}{|x(k)x(\ell)|} - \frac{y(k)\overline{y(\ell)}}{|y(k)y(\ell)|} \right|.
\end{aligned}$$

Then, we make use of the well-known trick

$$\left| \frac{z_0}{|z_0|} - \frac{z_1}{|z_1|} \right| = \frac{|z_0|z_1| - |z_0|z_1|}{|z_0 z_1|} \leq \frac{|z_0 - z_1| + ||z_0| - |z_1||}{|z_0|} \leq \frac{2|z_0 - z_1|}{|z_0|}$$

and its symmetry in z_0 and z_1.

□

B Proofs of the most fundamental formulae

For convenience of the reader, we want to present the proofs of the two formulae presented in section 2. We want to stress that these formulae are well-known in the literature and have repeatedly been used to prove uniqueness results in recent years [6, 12, 18, 20]. We start with lemma 2.1.

Proof of lemma 2.1 We consider

$$\begin{aligned} M_\varphi[x](m,n) &= |\mathcal{F}[x_m](n)|^2 = \mathcal{F}[x_m](n) \cdot \overline{\mathcal{F}[x_m](n)} = \mathcal{F}[x_m](n) \cdot \mathcal{F}[x_m^\#](n) \\ &= \frac{1}{\sqrt{L}} \cdot \mathcal{F}[x_m * x_m^\#](n), \end{aligned}$$

where we used the definitions $x_m(\ell) = x(\ell)\overline{\varphi(\ell - m)}$ and $x_m^\#(\ell) = \overline{x_m(-\ell)}$ together with the well-known convolution theorem for the DFT. Applying the inverse Fourier transform concludes the proof. □

Next, we prove lemma 2.2.

Proof of lemma 2.2 According to lemma 2.1, we have

$$\mathcal{F}^{-1}\left[M_\varphi[x](m,\cdot)\right](n) = \frac{1}{\sqrt{L}} \cdot \sum_{\ell=0}^{L-1} x(\ell)\overline{x(\ell-n)}\varphi(\ell-n-m)\overline{\varphi(\ell-m)}$$

and therefore

$$\mathcal{F}\left[M_\varphi[x](m,\cdot)\right](n) = \frac{1}{\sqrt{L}} \cdot \sum_{\ell=0}^{L-1} x(\ell)\overline{x(\ell+n)}\varphi(\ell+n-m)\overline{\varphi(\ell-m)}$$

holds. Taking the DFT in m yields

$$\begin{aligned} \mathcal{F}\left[M_\varphi[x]\right](m,n) &= \frac{1}{L} \cdot \sum_{\ell,k=0}^{L-1} x(\ell)\overline{x(\ell+n)}\varphi(\ell+n-k)\overline{\varphi(\ell-k)}\mathrm{e}^{-2\pi\mathrm{i}\frac{km}{L}} \\ &= \frac{1}{L} \cdot \sum_{\ell=0}^{L-1} x(\ell)\overline{x(\ell+n)}\mathrm{e}^{-2\pi\mathrm{i}\frac{\ell m}{L}} \cdot \sum_{k=0}^{L-1} \varphi(\ell+n-k)\overline{\varphi(\ell-k)}\mathrm{e}^{2\pi\mathrm{i}\frac{(\ell-k)m}{L}} \\ &= \left(x, \Pi_{(-n,m)}[x]\right) \cdot \left(\Pi_{(-n,m)}[\varphi], \varphi\right). \end{aligned}$$

□

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