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SHAPE HOLOMORPHY OF THE CALDERÓN PROJECTOR FOR THE LAPLACEAN IN $\mathbb{R}^2$

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Abstract. We establish the holomorphic dependence of the boundary integral operators (BIOs) comprising the Calderón projector for Laplacean in two dimensions on the shape of the boundary. More precisely, we show that the Calderón projector, as an element of the Banach space of bounded linear operators satisfying suitable mapping properties, depends holomorphically on a set of boundaries given by a collection of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$. In turn, this result implies that the solution of a well-posed first or second kind boundary integral equation (BIE) arising from the boundary reduction of the Laplace problem set on a domain of class $\mathscr{C}^2$ in two spatial dimensions depends holomorphically on the shape of the boundary, provided that the corresponding right-hand side does so as well. This property of *shape holomorphy* is of crucial significance to mathematically justify the construction of sparse surrogates of polynomial chaos type, and for dimension-independent convergence rates for the approximation of parametric solution families of BIEs in forward and inverse computational shape uncertainty quantification.

1. Introduction

Partial differential equations (PDEs) are ubiquitous as models of complex processes and phenomena in science and engineering, for instance: optimal shape design, inverse problems, biomedical imaging and non-destructive testing. These models are subject to the presence of sources of uncertainty, whose effects we would like to characterize. Computational uncertainty quantification (UQ) addresses mathematical models and numerical methods to assess in a quantitative manner how data and model uncertainty impact predictions furnished by scientific computing. Two types of uncertainty can be distinguished in a mathematical model: (i) *epistemic uncertainty*, which corresponds to uncertainty of the model itself and (ii) *aleatoric uncertainty*, which deals with propagation of uncertain parameters (e.g. domains of definition, material properties, external sources) into the so-called *Quantities of Interest* (QoI). In the ensuing discussion, we focus on the latter and assume that the former is negligible.

Following a parametric approach to represent uncertainty (*cp.* [14, 12]), one may write a parametric PDE together with its dependence on the uncertain parameters as follows: $\mathcal{A}(u, \boldsymbol{y}) = 0$, where $\boldsymbol{y} = (y_1, \dots, y_s) \in [-1,1]^s$, $s \in \mathbb{N}$, denotes the input parameter vector, $u \in X$ is the unknown solution of the problem and $\mathcal{A} : X \times [-1,1]^s \to W$ is a linear or nonlinear partial differential or integral operator, where X and W are Banach spaces. Assuming that for each parameter $\boldsymbol{y} \in [-1,1]^s$ there exists a unique solution $u = u_{\boldsymbol{y}}$, one may define the *uncertainty-to-solution map* $\boldsymbol{y} \in [-1,1]^s \mapsto u_{\boldsymbol{y}} \in X$.

Even if the operator $\mathcal{A}$ describes a well-understood problem (e.g. elliptic PDEs with diffusion coefficients depending on the parameters in an affine manner), the numerical approximation of the uncertainty-to-solution map becomes a challenge whenever the number of parameters $s \in \mathbb{N}$ is large or even infinity. This phenomenon corresponds to the so-called *curse of dimensionality*: the computational effort required for the numerical approximation of the uncertainty-to-solution map grows exponentially with the number of parameters. This issue also manifests itself as a deterioration of the convergence rates for the numerical approximation of the uncertainty-to-solution map as the parametric dimension increases.

Recently in [12], a strategy to obtain algebraic convergence rates for the approximation of $\boldsymbol{y} \mapsto u_{\boldsymbol{y}}$ for $s = \infty$ has been proposed. This approach relies on the construction of an *holomorphic extension* $\boldsymbol{z} \mapsto u_{\boldsymbol{z}}$ of the uncertainty-to-solution map on a certain tensor product of ellipses in the complex domain. The varying size of the so-called *Bernstein ellipses* quantifies the anisotropic dependence of the uncertainty-to-solution map on the corresponding parametric variables. This observation is crucial to achieve dimension-independent algebraic convergence rates for the approximation of the domain-to-solution map.

However, the holomorphic extension of the uncertainty-to-solution map has to be constructed and studied separately for each particular instance of the operator $\mathcal{A}$. So far, this analysis has been performed

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for elliptic diffusion equations with coefficients depending on the parameter vector in an affine and non-affine manner, parabolic diffusion equations and nonlinear, monotone elliptic PDEs [12, 14]. In particular, if the parameter vector $\boldsymbol{y}$ corresponds to a parametric representation for the *physical domain* of definition of $u_{\boldsymbol{y}}$, we refer to the uncertainty-to-solution map as the *domain-to-solution map* and to the property of holomorphic dependence of $u_{\boldsymbol{y}}$ on the parameter vector $\boldsymbol{y}$ as *shape holomorphy.*

Shape holomorphy has been already established for different classes of *differential* operators $\mathcal{A}$: time-harmonic electromagnetic wave scattering by perfectly conducting and dielectric obstacles [35, 2], stationary Stokes and Navier-Stokes equation [15] and volume formulations for acoustic wave scattering by a single penetrable obstacle in a low frequency regime [31]. To our knowledge, there are to date no results on holomorphic dependence of the BIOs on the boundary shape. We recall that the integral equation method allows the boundary reduction of certain classes of PDEs into BIEs by means of BIOs. This approach offers advantages over domain methods for PDEs, such as the finite element and finite difference methods. Hence the increasing interest during the last decades in this technique. Among these advantages we highlight:

(i) The capability to numerically treat more complex geometries than domain methods. Only boundary meshes are required for the numerical resolution of BIEs, as opposed to volume ones in domain methods, thus making the process of mesh generation and refinement easier.
(ii) The use of the integral equation method and BIEs is particularly well-suited to deal with problems in unbounded domains, such as acoustic and electromagnetic wave scattering.

The significance of holomorphic dependence of differential and integral operators, and of their inverses, on the shape of the domain lies in the classical result on exponential convergence of polynomial approximation for holomorphic functions. Upon a suitable domain or boundary parametric representation, shape holomorphy enables the construction of polynomial surrogates of operators and of solution manifolds, which can be used to accelerate computational engineering design. The fact that many parameters might be required for realistic modelling implies *high dimensionality* of the parametric surrogates. Recently developed interpolation and quadrature processes will overcome the curse of dimensionality inherent in classical numerical approaches (see, e.g., [10] for sensitivity-based surrogates, [12] for sparse-grid interpolation construction of surrogates, [22] for Quasi-Monte Carlo quadratures, [21, 29] for implications in Bayesian shape identification and [2] for computational electromagnetics). The mathematical and algorithmic development of these applications is beyond the scope of the present article, and will be reported elsewhere.

We establish the holomorphic dependence of the Calderón projector for the Laplace equation on a collection of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$. Specifically, we establish *holomorphy of the domain-to-operator map* associated to the Calderón projector. Again, this entails the holomorphic dependence of solutions of first and second kind boundary integral formulation on the shape of the boundary, *within* $\mathscr{C}^2$ *boundaries.* We emphasize that this holomorphic dependence does not require smoothness or even analytic regularity of the boundary.

In the line of previous works in this subject (e.g. [15, 35]), we establish shape holomorphy by proving *complex* Fréchet differentiability of the Calderón projector (viewed as an element of the complex Banach space of bounded linear operators) with respect to a collection of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$. The roadmap of our argument reads as follows. Firstly, let Γ_r be a Jordan curve in $\mathbb{R}^2$ and let $r : [0,1] \to \mathbb{R}^2$ be a $\mathscr{C}^2$–smooth boundary representation of Γ_r. Using a suitable pullback operator defined by means of the boundary representation $r \in [0,1] \to \mathbb{R}^2$, we transform the BIOs contained in the Calderón projector into 1-periodic integral operators, with mapping properties between appropriate periodic Sobolev spaces in the interval $[0,1]$, usually referred to as the *reference domain.* The presence of the boundary representation $r : [0,1] \to \mathbb{R}^2$ is completely isolated in the integrand of the arising 1-periodic integral operator. Then, we proceed to calculate and analyze the *complex* Fréchet derivative of the domain-to-operator map associated to the Calderón projector with respect to the boundary representation $r : \mathrm{I} \to \mathbb{R}^2$. However, the exact meaning of complex differentiability with respect to the boundary representation of a Jordan curve is not properly defined, as the BIOs are only defined for boundary representations with values in $\mathbb{R}^2$. Consequently, the main difficulty in the shape holomorphy analysis and the central achievement of the present work is to provide a meaningful description of the complex Fréchet derivative of the BIOs contained in the Calderón projector with respect to a suitable collection of *complex-valued boundary representations.*

Establishing holomorphy by verification of Fréchet differentiability of (suitably "complexified") domain-to-operator maps is closely related to the so-called *material derivatives* of the BIOs. This concept naturally appears in shape optimization [49]. In fact, the closest results to shape holomorphy of the BIOs that one may find in the literature are related to *shape differentiability* of the BIOs. Shape differentiability of the BIOs has been studied in shape inverse problems in acoustic, elastic and electromagnetic wave scattering [38, 39, 5, 51], shape optimization [49, 20, 26, 25, 28], electrical impedance tomography [27, 24], shape sensitivity analysis [34, 13] and crack detection [33, 36], among others.

Although present in many applications, the literature on shape differentiability of the BIOs is limited. In [41, 43] and [6] shape differentiability of the BIOs in the Fréchet sense for acoustic and elastic problems is analyzed, respectively, by using a setting based on Hölder continuous and differentiable function spaces and assuming boundaries of class $\mathscr{C}^2$. Subsequently, this technique is extended to the BIOs arising in electromagnetic wave scattering [42]. The approach used to obtain these results has been used as well in [18] to prove that a collection of BIOs characterized by a class of pseudo-homogeneous kernels (for a precise definition of this concept we refer to [18, Section 2] and references therein) are infinitely differentiable with respect to the boundary. This analysis is later used to compute shape derivatives of the BIOs in electromagnetic wave scattering [19, 51]. Nevertheless, these results are obtained under the assumption that the boundaries taken into consideration in the analysis are smooth. Hence, only derivatives in the Gâteaux sense were computed by means of this approach. Fréchet differentiability can not be obtained directly, as one would need normed spaces as opposed to Fréchet ones. Regardless of these issues, the above results do not imply an holomorphic dependence of the BIOs on the shape of the boundary. To the best of our knowledge, shape holomophy of the BIOs has not been addressed so far.

This manuscript is structured as follows. In Section 2 we establish the notation to be used throughout this work and also introduce the BIOs for the Laplace equation. In Section 3 we describe the class of boundary representations to be considered in our analysis. In Section 4, we analyze the *complex* Fréchet differentiability of a collection of 1-periodic integral operators (as elements of the Banach space of bounded linear operators) with respect to a set of complex-valued boundary representations. This result is built upon mathematical tools concerning holomorphic maps between Banach spaces, which are properly introduced. The abstract framework presented therein will allow us not only to analyze the BIOs for the Laplacean in $\mathbb{R}^2$ but is also a stepping-stone to obtain shape holomorphy of the BIOs associated, for example, to the Helmholtz and Stokes- as well as to the BIOs arising in the Lamé-system in linear elasticity. Section 5 is devoted to our main result: shape holomorphy of the Calderón projector. We prove the holomorphic dependence of the Calderón projector for the Laplace operator on a collection of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$. Using the pullback operator, introduced in Section 3.3, we transport the BIOs to the reference domain $[0, 1]$. As the result of this operation, we obtain a collection of 1-periodic integral operators fitting the framework of Section 4. In Section 6 we introduce a key concept for the approximation of a maps depending on a countable number of parameters, the so-called $(\boldsymbol{b}, \varepsilon)$-holomorphy. We prove $(\boldsymbol{b}, \varepsilon)$-holomorphy of the BIOs, after the boundary representations are parametrized in an affine manner. Moreover, we discuss implications of this notion in the construction of surrogates of the polynomial type for the approximation of parametric maps, with *convergence rates that are independent of the number of parameters* (i.e., of the dimension of the parameter space). We further elaborate on the importance of this result as a foundational step in the analysis of state-of-the-art techniques usually employed in forward and inverse computational UQ, which are capable of achieving dimension-independent convergence rates in the approximation of parametric maps with high-dimensional shape parametrizations. Finally, in Section 7 we provide concluding remarks and sketch directions of future research.

2. Preliminaries

2.1. **Notation.** Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2$ be a domain, let $k \in \mathbb{N}_0$ and $\lambda \in (0, 1]$. We denote by $\mathscr{C}^k(\Omega)$ the space of k times continuously differentiable functions in Ω. Furthermore, we denote by $\mathscr{C}^{k,\lambda}(\Omega)$ the subspace of k times continuously differentiable functions in Ω with Hölder continuous partial derivatives of order λ. Throughout, we adopt the reference domain $\mathrm{I} := [0, 1]$ for the closed curve $\Gamma = \partial\Omega$. On I, we consider the closed subspace of $\mathscr{C}^0(\mathbb{R})$ of 1-periodic, complex-valued continuous functions, defined as

$$\mathscr{C}^0_{\text{per}}(\mathrm{I}) := \left\{ u \in \mathscr{C}^0(\mathbb{R}) : \; u(x) = u(x+1), \quad x \in \mathbb{R} \right\}.$$

Recursively, we define

$$\mathscr{C}^k_{\text{per}}(\mathrm{I}) := \left\{ u \in \mathscr{C}^{k-1}_{\text{per}}(\mathrm{I}) : \text{ such that } u^{(k)} \in \mathscr{C}^0_{\text{per}}(\mathrm{I}) \right\}, \quad k \in \mathbb{N},$$

where $u^{(k)}$ denotes the k-th derivative of the function $u : \mathrm{I} \to \mathbb{C}$. We adopt the notation u' to denote the first derivative. The space $\mathscr{C}^k_{\text{per}}(\mathrm{I})$ endowed with the norm

$$\|u\|_{\mathscr{C}^k_{\text{per}}(\mathrm{I})} \coloneqq \sum_{\ell=0}^{k} \left\| u^{(\ell)} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I})} \quad \text{for} \quad u \in \mathscr{C}^k_{\text{per}}(\mathrm{I}, \mathbb{K}),$$

is a (complex) Banach space, where $\|u\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I})} \coloneqq \max\limits_{t \in \mathrm{I}} |u(t)|$. We also introduce the semi-norm

$$|u|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \coloneqq \max_{t \in \mathrm{I}} |u'(t)|$$

Equivalently, for a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ one may define $\mathscr{C}^k_{\text{per}}(\mathrm{I}, \mathbb{K}^2)$, but replacing the absolute value $|\cdot|$ by the Euclidean norm $\|\cdot\|$.

Let $\mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$ be the closed subspace of 1-biperiodic (i.e., 1-periodic in each variable), complex-valued functions which are continuous in $\mathrm{I} \times \mathrm{I}$

$$\mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I}) \coloneqq \left\{ u \in \mathscr{C}^0(\mathbb{R} \times \mathbb{R}) : \begin{array}{ll} u(t, x) = u(t, x+1), & \forall t \in \mathrm{I} \text{ and} \\ u(x, s) = u(x+1, s), & \forall s \in \mathrm{I} \end{array} , \quad \forall x \in \mathbb{R} \right\}.$$

Again, we define recursively

$$\mathscr{C}^k_{\text{per}}(\mathrm{I} \times \mathrm{I}) \coloneqq \left\{ u \in \mathscr{C}^{k-1}_{\text{per}}(\mathrm{I} \times \mathrm{I}) : \frac{\partial^{|\alpha|} u}{\partial t^{\alpha_2} \partial s^{\alpha_1}}(t, s) \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I}), \quad |\alpha| = k \right\}, \quad k \in \mathbb{N},$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ and $|\alpha| = \alpha_1 + \alpha_2$. We adopt the notation $\frac{\partial u}{\partial t}$ and $\frac{\partial u}{\partial s}$ to denote the first derivative with respect to the first and second variable, respectively, of the function $u : \mathrm{I} \times \mathrm{I} \to \mathbb{C}$. Equipped with the norm

$$\|u\|_{\mathscr{C}^k_{\text{per}}(\mathrm{I} \times \mathrm{I})} \coloneqq \sum_{\ell=0}^{k} \sum_{\substack{\alpha \in \mathbb{N}_0^2 \\ |\alpha| \leq \ell}} \left\| \frac{\partial^{|\alpha|} u}{\partial t^{\alpha_2} \partial s^{\alpha_1}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})}, \quad u \in \mathscr{C}^k_{\text{per}}(\mathrm{I} \times \mathrm{I}),$$

where $\|u\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})} \coloneqq \max\limits_{(t,s) \in \mathrm{I} \times \mathrm{I}} |u(t, s)|$, $\mathscr{C}^k_{\text{per}}(\mathrm{I} \times \mathrm{I})$ is a (complex) Banach space.

Let $\mathrm{D} \subset \mathbb{R}^2$ be a bounded domain of class $\mathscr{C}^k$, for $k \in \mathbb{N}$ [32, Definition 3.3.1]. We denote by $L^2(\partial\mathrm{D})$ the set of scalar-valued, square integrable functions over $\partial\mathrm{D}$ and by $\mathscr{C}^{k,\lambda}(\partial\mathrm{D})$ the set of Hölder continuous functions in $\partial\mathrm{D}$ [32, Section 1.2]. Let $H^s(\partial\mathrm{D})$, for $s \in [0, k]$, be the the Sobolev space of traces on $\partial\mathrm{D}$ ([46, Section 2.4], [32, Section 4.2]). As is customary, we identify $H^0(\partial\mathrm{D})$ with $L^2(\partial D)$ and, for $s \in [0, k]$, $H^{-s}(\partial\mathrm{D})$ with the dual space of $H^s(\partial\mathrm{D})$. The duality pairing between $H^{-s}(\partial\mathrm{D})$ and $H^s(\partial\mathrm{D})$ is denote by $\langle \cdot, \cdot \rangle_{\partial\mathrm{D}}$, with the subscript accounting for the domain of definition. Finally, for complex Banach spaces X and Y, we denote by $\mathscr{L}(X, Y)$ the space of bounded linear operators from X into Y and by $\mathscr{L}_{\text{iso}}(X, Y)$ the (open) subset of isomorphisms, i.e. bounded linear operators with a bounded inverse. Recall that $\mathscr{L}(X, Y)$ is a (complex) Banach space equipped with the standard operator norm [44, Theorem III.2].

2.2. **1-Periodic Sobolev spaces.** We recall results concerning 1-periodic Sobolev spaces that are required in the ensuing analysis. More details may found in [37, Chapter 8], [45, Setion 5.3] and [1, Section 6.5]. Define

$$L^2(\mathrm{I}) \coloneqq \left\{ u : \mathrm{I} \to \mathbb{C} \text{ measurable such that} : \int_0^1 |u(s)|^2 \, ds < \infty \right\}.$$

The Fourier expansion of $u \in L^2(\mathrm{I})$ is given by

$$u(t) = \sum_{\ell \in \mathbb{Z}} a_\ell(u) \exp(\imath 2\pi \ell t), \quad t \in \mathrm{I}, \tag{2.1}$$

where the Fourier coefficients $a_\ell(u)$ in (2.1) are given by

$$a_\ell(u) = \int_0^1 u(s) \exp(-\imath 2\pi \ell s) ds.$$

We denote by $(\cdot,\cdot)_{L^2(\mathrm{I})}$ the $L^2(\mathrm{I})$-inner product understood in the bilinear sense, i.e.

$$(u,v)_{L^2(\mathrm{I})} = \int_0^1 u(s)v(s)ds, \quad \text{for all} \quad u,v \in L^2(\mathrm{I}).$$

Therefore, the $L^2(\mathrm{I})$-norm is given by $\|u\|_{L^2(\mathrm{I})} \coloneqq (u,\overline{u})^{\frac{1}{2}}_{L^2(\mathrm{I})}$.

Remark 1. The operation of complex conjugation is not holomorphic. To show holomorphy of certain maps, we have to avoid it in analytic continuation of real-valued functions. For that reason, we define inner products and duality pairings in the bilinear sense, rather than in the sesquilinear one.

Definition 2.1 ([37, Definition 8.1])**.** *Let $0 \le s < \infty$. By $H^s_{\mathrm{per}}(\mathrm{I})$ we denote the space of all $u \in L^2(\mathrm{I})$ with the property*

$$\|u\|_{H^s_{\mathrm{per}}(\mathrm{I})} \coloneqq \left(\sum_{\ell\in\mathbb{Z}} (1+\ell^2)^s \left|a_\ell(u)\right|^2\right)^{\frac{1}{2}} < \infty,$$

for the Fourier coefficients $a_\ell(u)$ of $u \in L^2(\mathrm{I})$.

Theorem 2.2 ([37, Theorem 8.2])**.** *Let $0 \le s < \infty$. The Sobolev space $H^s_{\mathrm{per}}(\mathrm{I})$ is a (complex) Hilbert space with the following scalar product (to be understood in the bilinear sense)*

$$(u,v)_{H^s_{\mathrm{per}}(\mathrm{I})} = \sum_{\ell\in\mathbb{Z}} (1+\ell^2)^s a_\ell(u) a_\ell(v), \quad u,v \in H^s_{\mathrm{per}}(\mathrm{I}),$$

The inner product $(\cdot,\cdot)_{H^s_{\mathrm{per}}(\mathrm{I})}$ induces the norm $\|u\|_{H^s_{\mathrm{per}}(\mathrm{I})}$, i.e. $\|u\|_{H^s_{\mathrm{per}}(\mathrm{I})} = (u,\overline{u})^{\frac{1}{2}}_{H^s_{\mathrm{per}}(\mathrm{I})}$, for $u \in H^s_{\mathrm{per}}(\mathrm{I})$.

By the Riesz representation theorem, we identify $H^{-s}_{\mathrm{per}}(\mathrm{I})$, for $0 \le s < \infty$, with the dual space of $H^s_{\mathrm{per}}(\mathrm{I})$ (i.e. the set all bounded linear functionals acting on $H^s_{\mathrm{per}}(\mathrm{I})$). We denote by $\langle\cdot,\cdot\rangle_{\mathrm{per}}$ the $H^{-s}_{\mathrm{per}}(\mathrm{I})$-$H^s_{\mathrm{per}}(\mathrm{I})$ duality pairing, again in the bilinear sense (without conjugation in the second argument). Adjoint operators in the $H^{-s}_{\mathrm{per}}(\mathrm{I})$-$H^s_{\mathrm{per}}(\mathrm{I})$ duality pairing are labeled with a † superscript. For $u,v \in L^2(\mathrm{I})$, we have

$$\langle u,v\rangle_{\mathrm{per}} = \int_0^1 u(s)v(s)ds.$$

We endow $H^{-s}_{\mathrm{per}}(\mathrm{I})$, for $0 \le s < \infty$, with the norm

$$\|u\|_{H^{-s}_{\mathrm{per}}(\mathrm{I})} \coloneqq \sup_{0\neq v\in H^s_{\mathrm{per}}(\mathrm{I})} \frac{\left|\langle u,v\rangle_{\mathrm{per}}\right|}{\|v\|_{H^s_{\mathrm{per}}(\mathrm{I})}}, \quad u \in H^{-s}_{\mathrm{per}}(\mathrm{I}).$$

Finally, we remark that $H^0_{\mathrm{per}}(\mathrm{I})$ can be identified with $L^2(\mathrm{I})$.

Lemma 2.3 ([37, Theorem 8.5 & Theorem 8.6])**.** *We have the following norm equivalences.*

(i) *Let $k \in \mathbb{N}_0$. For $u \in \mathscr{C}^k_{\mathrm{per}}(\mathrm{I})$, the norm $\|u\|_{H^k_{\mathrm{per}}(\mathrm{I})}$ is equivalent to*

$$\|u\|_{k,\mathrm{per}} \coloneqq \left(\sum_{\ell=0}^{k} \left\|u^{(\ell)}\right\|^2_{L^2(\mathrm{I})}\right)^{\frac{1}{2}}.$$

(ii) *Let $\lambda \in (0,1)$. For $u \in \mathscr{C}^1_{\mathrm{per}}(\mathrm{I})$, the norm $\|u\|_{H^\lambda_{\mathrm{per}}(\mathrm{I})}$ is equivalent to*

$$\|u\|_{\lambda,\mathrm{per}} \coloneqq \left(\|u\|^2_{L^2(\mathrm{I})} + \int_0^1\int_0^1 \frac{|u(t)-u(s)|^2}{|\sin(\pi(t-s))|^{2\lambda+1}} dsdt\right)^{\frac{1}{2}}.$$

Lemma 2.4 ([45, Lemma 5.12.2])**.** *Assume that for an operator A there holds $\mathsf{A} \in \mathscr{L}\left(H^{\lambda_1}_{\mathrm{per}}(\mathrm{I}), H^{\mu_1}_{\mathrm{per}}(\mathrm{I})\right)$ and $\mathsf{A} \in \mathscr{L}\left(H^{\lambda_2}_{\mathrm{per}}(\mathrm{I}), H^{\mu_2}_{\mathrm{per}}(\mathrm{I})\right)$ with some $\lambda_1 \le \lambda_2$ and $\mu_1 \le \mu_2$. Then for $\varrho \in [0,1]$,*

$$\|\mathsf{A}\|_{\mathscr{L}\left(H^{\varrho\lambda_1+(1-\varrho)\lambda_2}_{\mathrm{per}}(\mathrm{I}), H^{\varrho\mu_1+(1-\varrho)\mu_2}_{\mathrm{per}}(\mathrm{I})\right)} \le \|\mathsf{A}\|^{\varrho}_{\mathscr{L}\left(H^{\lambda_1}_{\mathrm{per}}(\mathrm{I}), H^{\mu_1}_{\mathrm{per}}(\mathrm{I})\right)} \|\mathsf{A}\|^{1-\varrho}_{\mathscr{L}\left(H^{\lambda_2}_{\mathrm{per}}(\mathrm{I}), H^{\mu_2}_{\mathrm{per}}(\mathrm{I})\right)}.$$

Lemma 2.5 ([45, Lemma 5.13.1] & [37, Corollary 8.8])**.** *Let* $\lambda \in [-1,1]$, $u \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ *and* $v \in H^\lambda_{\text{per}}(\mathrm{I})$. *Then, we have* $u\,v \in H^\lambda_{\text{per}}(\mathrm{I})$ *and*

$$\|u\,v\|_{H^\lambda_{\text{per}}(\mathrm{I})} \leq C_\lambda \|u\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \|v\|_{H^\lambda_{\text{per}}(\mathrm{I})},$$

for a constant $C_\lambda > 0$ *depending only on* λ.

2.3. **Boundary Integral Operators.** Let $\mathrm{G}(\mathbf{x},\mathbf{y})$ be the fundamental solution to the Laplace equation in $\mathbb{R}^2$ (*cf.* [50, Chapter 5] or [46, Section 3.1]), given by

$$\mathrm{G}(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\log\|\mathbf{x}-\mathbf{y}\|, \quad \mathbf{x},\mathbf{y} \in \mathbb{R}^2 \quad \text{and} \quad \mathbf{x} \neq \mathbf{y}.$$

Let D be an open bounded domain of class $\mathscr{C}^2$ with boundary $\Gamma = \partial\mathrm{D}$. Denote by $\mathrm{D^c} := \mathbb{R}^2\backslash\overline{\mathrm{D}}$ its complement. We define the *single layer potential*

$$(\mathcal{S}_\Gamma\psi)(\mathbf{x}) := \int_\Gamma \mathrm{G}(\mathbf{x},\mathbf{y})\psi(\mathbf{y})ds_{\mathbf{y}}, \quad \mathbf{x} \in \mathrm{D}\cup\mathrm{D^c}, \tag{2.2}$$

and the *double layer potential*

$$(\mathcal{D}_\Gamma\phi)(\mathbf{x}) := \int_\Gamma \boldsymbol{\nu}_\Gamma(\mathbf{y})\cdot\mathbf{grad_y}\mathrm{G}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})ds_{\mathbf{y}}, \quad \mathbf{x} \in \mathrm{D}\cup\mathrm{D^c}, \tag{2.3}$$

where the densities ψ and ϕ are defined on the boundary Γ and $\boldsymbol{\nu}_\Gamma$ denotes the outer normal vector to Γ. Furthermore, we define the *boundary integral operators* (BIOs) on Γ and for $\mathbf{x}\in\Gamma$ as follows

$$(\mathsf{V}_\Gamma\psi)(\mathbf{x}) := \lim_{\mathbf{z}\in\mathrm{D}\to\mathbf{x}\in\Gamma}(\mathcal{S}_\Gamma\psi)(\mathbf{z}), \tag{2.4a}$$

$$(\mathsf{K}_\Gamma\phi)(\mathbf{x}) := \lim_{\mathbf{z}\in\mathrm{D}\to\mathbf{x}\in\Gamma}(\mathcal{D}_\Gamma\phi)(\mathbf{z}) + \frac{1}{2}\phi(\mathbf{x}), \tag{2.4b}$$

$$(\mathsf{K}'_\Gamma\psi)(\mathbf{x}) := \lim_{\mathbf{z}\in\mathrm{D}\to\mathbf{x}\in\Gamma}\boldsymbol{\nu}_\Gamma(\mathbf{z})\cdot\mathbf{grad_z}(\mathcal{S}_\Gamma\psi)(\mathbf{z}) - \frac{1}{2}\psi(\mathbf{x}), \tag{2.4c}$$

$$(\mathsf{W}_\Gamma\phi)(\mathbf{x}) := -\lim_{\mathbf{z}\in\mathrm{D}\to\mathbf{x}\in\Gamma}\boldsymbol{\nu}_\Gamma(\mathbf{z})\cdot\mathbf{grad_z}(\mathcal{D}_\Gamma\phi)(\mathbf{z}), \tag{2.4d}$$

In the following we refer to V_Γ, K_Γ, K'_Γ and W_Γ as the *single layer*, *double layer*, *adjoint double layer* and *hypersingular* BIOs, respectively. The following result provides the explicit representation of the BIOs on domain of class $\mathscr{C}^2$.

Lemma 2.6 ([32, Lemma 1.2.1])**.** *Let* Γ *be a boundary of class* $\mathscr{C}^2$ *and let* ϕ *and* ψ *be continuous. Then the limits* (2.4a), (2.4b), (2.4c) *and* (2.4d) *exist uniformly with respect to all* $\mathbf{x}\in\Gamma$ *and all* ϕ *and* ψ *with* $\sup_{\mathbf{x}\in\Gamma}|\phi(\mathbf{x})| \leq 1$, $\sup_{\mathbf{x}\in\Gamma}|\psi(\mathbf{x})| \leq 1$. *Furthermore, these limits can be expressed by*

$$(\mathsf{V}_\Gamma\phi)(\mathbf{x}) = \int_\Gamma \mathrm{G}(\mathbf{x},\mathbf{y})\phi(y)ds_{\mathbf{y}},$$

$$(\mathsf{K}_\Gamma\psi)(\mathbf{x}) = \int_\Gamma \boldsymbol{\nu}_\Gamma(\mathbf{y})\cdot\mathbf{grad_y}\mathrm{G}(\mathbf{x},\mathbf{y})\psi(\mathbf{y})ds_{\mathbf{y}},$$

$$(\mathsf{K}'_\Gamma\phi)(\mathbf{x}) = \int_\Gamma \boldsymbol{\nu}_\Gamma(\mathbf{x})\cdot\mathbf{grad_x}\mathrm{G}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})ds_{\mathbf{y}},$$

for $\mathbf{x}\in\Gamma$, *where these integrals are understood in the improper sense with weakly singular kernels.*

For a vector $\boldsymbol{v} = (v_1,v_2)\in\mathbb{R}^2$ we introduce the notation $[\boldsymbol{v}]^\perp := (v_2,-v_1)$. For a smooth function φ defined on Γ, we define

$$(\mathrm{curl}_\Gamma\varphi)(\mathbf{x}) := \boldsymbol{\nu}(\mathbf{x})\cdot[\nabla\widetilde{\varphi}(\mathbf{x})]^\perp, \quad \mathbf{x}\in\mathbb{R}^2,$$

where $\widetilde{\varphi}$ is a smooth extension of φ to a neighborhood of Γ. The following lemma addresses the explicit representation of the hypersingular operator on boundaries of class $\mathscr{C}^2$.

Lemma 2.7 (Maue's formula, [32, Lemma 1.2.2] & [50, Theorem 6.15])**.** *Let* Γ *be a boundary of class* $\mathscr{C}^2$ *and let* φ *be a Hölder continuous differentiable function. Then the limit in* (2.4d) *exists uniformly with respect to all* $\mathbf{x}\in\Gamma$ *and all* φ *with* $\|\varphi\|_{\mathscr{C}^{1,\alpha}(\Gamma)} \leq 1$. *Moreover, the operator* W_Γ *satisfies*

$$\mathsf{W}_\Gamma\,\varphi = -\mathrm{curl}_\Gamma\circ\mathsf{V}_\Gamma\circ\mathrm{curl}_\Gamma\,\varphi$$

for $\varphi\in\mathscr{C}^1(\Gamma)$.

The BIOs defined in (2.4) possess the following mapping properties between Sobolev spaces.

Theorem 2.8 ([46, Theorem 3.1.16] & [17, Theorem 1]). *Let* D *be a bounded Lipschitz domain with boundary* $\Gamma \coloneqq \partial \mathrm{D}$*. The BIOs defined in* (2.4) *are linear and bounded according to*

$$\begin{aligned} \mathsf{V}_\Gamma &: H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \\ \mathsf{K}_\Gamma &: H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \\ \mathsf{K}'_\Gamma &: H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \\ \mathsf{W}_\Gamma &: H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma). \end{aligned}$$

Ahead, for $\mathbf{V}_\Gamma \coloneqq H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$, we shall also require the *Calderón projector*

$$\mathsf{C}_\Gamma \coloneqq \begin{pmatrix} \frac{1}{2}\mathsf{I} - \mathsf{K}_\Gamma & \mathsf{V}_\Gamma \\ \mathsf{W}_\Gamma & \frac{1}{2}\mathsf{I} + \mathsf{K}'_\Gamma \end{pmatrix} \in \mathscr{L}\left(\mathbf{V}_\Gamma, \mathbf{V}_\Gamma\right). \tag{2.6}$$

3. Boundary Representations

3.1. **Jordan Curves.** In the following, we recall some concepts concerning arcs and curves in two dimensions. These definitions will allow us to describe precisely the collection of boundaries of class $\mathscr{C}^2$ to be considered throughout this work. For further details we refer to [45, Chapter 2].

Definition 3.1. *($\mathscr{C}^k$–smooth, regular boundary representation)*

(i) *A point set* $\Gamma \subset \mathbb{R}^2$ *is an* arc or curve *if there exists a continuous vector-valued function* $r : \mathrm{I} \to \mathbb{R}^2$ *such that* Γ *is the image of the interval* I *through the function* $r : \mathrm{I} \to \mathbb{R}^2$*. We say that the map* $r : \mathrm{I} \to \mathbb{R}^2$ *is a* boundary representation *of* Γ*.*

(ii) *A point set* $\Gamma \subset \mathbb{R}^2$ *is a* Jordan arc *(or a simple arc) if there exists a one-to-one, continuous function* $r : \mathrm{I} \to \mathbb{R}^2$ *such that* Γ *is the image of the interval* I *through the function* $r : \mathrm{I} \to \mathbb{R}^2$*. If* $r : \mathrm{I} \to \mathbb{R}^2$ *is continuous, one-to-one in* $[0, 1)$ *and* $r(0) = r(1)$*, the point set* Γ *is referred to as a* Jordan curve *(or a simple closed curve).*

(iii) *For* $k \in \mathbb{N}$ *given, we say that a boundary representation* $r : \mathrm{I} \to \mathbb{R}^2$ *of a Jordan arc* Γ *is* $\mathscr{C}^k$*–smooth (we also say that* Γ *is a* $\mathscr{C}^k$*–smooth Jordan arc) if* $r \in \mathscr{C}^k(\mathrm{I}, \mathbb{R}^2)$*. A boundary representation* $r : \mathrm{I} \to \mathbb{R}^2$ *of a Jordan curve* Γ *is said to be* $\mathscr{C}^k$*–smooth (we also say that* Γ *is a* $\mathscr{C}^k$*–smooth Jordan curve) if* $r \in \mathscr{C}^k_{\mathrm{per}}(\mathrm{I}, \mathbb{R}^2)$*.*

(iv) *A boundary representation is said to be* regular *if it is* $\mathscr{C}^1$*-smooth and if* $r'(t) \neq \mathbf{0}$*, for* $t \in \mathrm{I}$*.*

Proposition 3.2 (Jordan's Theorem, [45, Theorem 2.4.2]). *Let* Γ *be a Jordan curve in* $\mathbb{R}^2$*. Then* $\mathbb{R}^2 \backslash \Gamma = \mathrm{D} \cup \mathrm{D}^{\mathrm{c}}$*, where* D *and* D^{c} *are two domains, exactly one of which is bounded. Furthermore, the curve* Γ *is the boundary of* D *and of* D^{c}*.*

We say that the bounded domain D defined by a Jordan curve Γ, according to Proposition 3.2, is called the *interior* of Γ and the unbounded domain $\mathrm{D}^{\mathrm{c}} = \mathbb{R}^2 \backslash \overline{\mathrm{D}}$ is the *exterior* of Γ. Let $r = (r_1, r_2)^\top : \mathrm{I} \to \mathbb{R}^2$ be a regular boundary representation of a Jordan curve Γ in the sense of Definition 3.1. Let us set $\check{r}(t) \coloneqq r_1(t) + \imath r_2(t)$, where $\imath \coloneqq \sqrt{-1}$ denotes the imaginary unit. Now, set $\check{r}(t) : \mathrm{I} \to \mathbb{C}$, i.e. $\check{r} \in \mathscr{C}^2_{\mathrm{per}}(\mathrm{I})$ defines a curve in the complex plane. We define the *winding number* of a Jordan curve Γ corresponding to a boundary representation $r : \mathrm{I} \to \mathbb{R}$ at a point $\mathbf{x} \notin \Gamma$ as follows

$$\omega(\Gamma, r, \mathbf{x}) \coloneqq \frac{1}{2\pi} \arg(\check{r}(t) - \check{\mathbf{x}}) \Big|_{t=0}^{t=1},$$

where $\arg(z)$ denotes the argument of $z \in \mathbb{C}$ and $\check{\mathbf{x}} = x_1 + \imath x_2 \in \mathbb{C}$, for $\mathbf{x} = (x_1, x_2)$. Observe that if $\mathbf{x} \in \mathrm{D}$, then $\omega(\Gamma, r, \mathbf{x}) = \pm 1$. On the other hand, if $\mathbf{x} \in \mathrm{D}^{\mathrm{c}}$ we have $\omega(\Gamma, r, \mathbf{x}) = 0$.

Definition 3.3. *Let* Γ *be a Jordan curve with a regular boundary representation* $r : \mathrm{I} \to \mathbb{R}^2$*. If* $\omega(\Gamma, r, \mathbf{x}) = 1$ *for* $\mathbf{x} \in \mathrm{D}$*, then we say that* Γ *is positively oriented under the boundary representation* $r : \mathrm{I} \to \mathbb{R}^2$*. Otherwise, we say that* Γ *is negatively oriented under the boundary representation* $r : \mathrm{I} \to \mathbb{R}^2$*.*

Let $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ be orthonormal vectors in $\mathbb{R}^2$. For $\varepsilon > 0$, we define a neighbourhood of $\mathbf{x} \in \mathbb{R}^2$ as

$$U(\mathbf{x}, \varepsilon, \boldsymbol{\tau}, \boldsymbol{\nu}) \coloneqq \left\{ \mathbf{y} \in \mathbb{R}^2 : \; \mathbf{y} = \mathbf{x} + \zeta \boldsymbol{\tau} + \eta \boldsymbol{\nu} : \; \eta, \zeta \in \mathbb{R}, \quad |\zeta| \leq \varepsilon, \; |\eta| \leq \varepsilon \right\}.$$

Proposition 3.4 ([45, Theorem 2.4.3]). *Let* $k \in \mathbb{N}$*. Then, the following conditions are equivalent for a subset* $\Gamma \subset \mathbb{R}^2$*:*

(i) Γ *is a* $\mathscr{C}^k$*–smooth Jordan curve.*

(ii) Γ *is compact, connected, non-empty and for each* $\mathbf{x} \in \Gamma$ *there exist orthonormal vectors* $\boldsymbol{\tau}(\mathbf{x})$ *and* $\boldsymbol{\nu}(\mathbf{x})$ *together with* $\varepsilon_{\mathbf{x}} > 0$*, depending on* $\mathbf{x}$ *such that there holds*

$$U(\mathbf{x}, \varepsilon_{\mathbf{x}}, \boldsymbol{\tau}, \boldsymbol{\nu}) \cap \Gamma = \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x} + \zeta \boldsymbol{\tau}(\mathbf{x}) + f_{\mathbf{x}}(\zeta)\boldsymbol{\nu}(\mathbf{x}), \quad |\zeta| \leq \varepsilon_{\mathbf{x}} \right\},$$

where $f_{\mathbf{x}} : [-\varepsilon_{\mathbf{x}}, \varepsilon_{\mathbf{x}}] \to \mathbb{R}$ *belongs to* $\mathscr{C}^k([-\varepsilon_{\mathbf{x}}, \varepsilon_{\mathbf{x}}])$ *and* $f_{\mathbf{x}}(0) = f'_{\mathbf{x}}(0) = 0$.

Remark 2. When Γ is a positively oriented Jordan curve under the regular boundary representation $r : \mathrm{I} \to \mathbb{R}^2$ (in the sense of Definition 3.3), the vectors $\boldsymbol{\tau}(\mathbf{x})$ and $\boldsymbol{\nu}(\mathbf{x})$ in Proposition 3.4 are given by

$$\boldsymbol{\tau}(r(t)) = \frac{r'(t)}{\|r'(t)\|} \quad \text{and} \quad \boldsymbol{\nu}(r(t)) = \frac{[r'(t)]^{\perp}}{\|r'(t)\|}.$$

Remark 3. Let Γ be a Jordan curve with a boundary representation $r : \mathrm{I} \to \mathbb{R}^2$. If Γ is positively oriented (in the sense of Definition 3.3) the outer normal vector $\boldsymbol{\nu}_{\Gamma}(\mathbf{x})$ from Section 2.3 coincides with $\boldsymbol{\nu}(r(t))$ from Remark 2 at $\mathbf{x} = r(t)$, with $t \in \mathrm{I}$.

Lemma 3.5. *Let* $k \in \mathbb{N}$ *and let* $\Gamma \subset \mathbb{R}^2$ *be a compact, connected boundary of class* $\mathscr{C}^k$*. Then,* Γ *is a* $\mathscr{C}^k$*–smooth, regular Jordan curve.*

Proof. By [32, Definition 3.3.1], being $\Gamma \subset \mathbb{R}^2$ a compact, connected boundary of class $\mathscr{C}^2$, there exists a finite number $J \in \mathbb{N}$ of orthogonal linear transformations $\{\mathcal{B}_\ell\}_{\ell=1}^J \subset \mathbb{R}^{2\times 2}$ (i.e. 2×2 orthogonal matrices), the same number of points $\{\mathbf{x}_\ell\}_{\ell=1}^J \subset \Gamma$ and functions $\chi_\ell : [-\varepsilon, \varepsilon] \to \mathbb{R}$ belonging to $\mathscr{C}^k([-\varepsilon, \varepsilon])$, for $\ell = 1, \ldots, J$, where $\varepsilon > 0$ is a fixed constant such that for each $\mathbf{x} \in \Gamma$ there exists at least one $\ell \in \{1, \ldots, J\}$ providing the following representation of $\mathbf{x} \in \Gamma$

$$\mathbf{x} = \mathbf{x}_\ell + \mathcal{B}_\ell(\zeta, \chi_\ell(\zeta))^\top, \quad |\zeta| < \varepsilon.$$

Furthermore, there exists a $\epsilon > 0$ such that for each $\ell \in \{1, \cdots, J\}$ the open set

$$B_\ell \coloneqq \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x}_\ell + \mathcal{B}_\ell(\zeta, \eta)^\top, \quad |\zeta| < \varepsilon, \quad |\eta| < \epsilon \right\}$$

is the union of the sets

$$B_\ell^- \coloneqq B_\ell \cap \mathrm{D} = \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x}_\ell + \mathcal{B}_\ell\, \mathbf{y}, \quad \begin{array}{l} \mathbf{y} = (\zeta, \eta)^\top \in \mathbb{R}^2, \ |\zeta| < \varepsilon \quad \text{and} \\ \chi_\ell(\zeta) - \epsilon < \eta < \chi_\ell(\zeta) \end{array} \right\},$$

$$B_\ell^+ \coloneqq B_\ell \cap \mathrm{D}^{\mathrm{c}} = \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x}_\ell + \mathcal{B}_\ell\, \mathbf{y}, \quad \begin{array}{l} \mathbf{y} = (\zeta, \eta)^\top \in \mathbb{R}^2, \ |\zeta| < \varepsilon \quad \text{and} \\ \chi_\ell(\zeta) < \eta < \chi_\ell(\zeta) + \epsilon \end{array} \right\},$$

and

$$\Gamma_\ell \coloneqq B_\ell \cap \Gamma = \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x}_\ell + \mathcal{B}_\ell \left(\zeta, \chi_\ell(\zeta)\right)^\top, \quad |\zeta| < \varepsilon \right\}.$$

For each $\mathbf{x} \in \Gamma$ there exists a $\ell \in \{1, \ldots, J\}$ such that $\mathbf{x} \in \Gamma_\ell$ and

$$\mathbf{x} = \mathbf{x}_\ell + \mathcal{B}_\ell(\zeta_{\mathbf{x}}, \chi_\ell(\zeta_{\mathbf{x}}))^\top, \tag{3.1}$$

for a $\zeta_{\mathbf{x}} \in (-\varepsilon, \varepsilon)$, depending on $\mathbf{x}$. Then, using (3.1) we get

$$\begin{aligned} \Gamma_\ell &= \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x}_\ell + \mathcal{B}_\ell \left(\zeta, \chi_\ell(\zeta)\right)^\top, \quad |\zeta| < \varepsilon \right\} \\ &= \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x} + \mathcal{B}_\ell \left(\zeta - \zeta_{\mathbf{x}}, \chi_\ell(\zeta) - \chi_\ell(\zeta_{\mathbf{x}})\right)^\top, \quad |\zeta| < \varepsilon \right\} \\ &= \left\{ \mathbf{y} \in \mathbb{R}^2 : \ \mathbf{y} = \mathbf{x} + \mathcal{B}_\ell \left(u, f_{\mathbf{x}}(u)\right)^\top, \quad -\varepsilon - \zeta_{\mathbf{x}} < u < \varepsilon - \zeta_{\mathbf{x}} \right\}, \end{aligned}$$

where $u \coloneqq \zeta - \zeta_{\mathbf{x}}$, $f_{\mathbf{x}}(u) \coloneqq \chi_\ell(u + \zeta_{\mathbf{x}}) - \chi_\ell(\zeta_{\mathbf{x}})$ and $\varepsilon_{\mathbf{x}} \coloneqq \varepsilon - \zeta_{\mathbf{x}} > 0$. Observe that $f_{\mathbf{x}}(0) = f'_{\mathbf{x}}(0) = 0$. Let $\boldsymbol{\tau}_\ell$, $\boldsymbol{\nu}_\ell$ be the first and second column of $\mathcal{B}_\ell$, respectively. Observe that these vectors depend on $\mathbf{x} \in \Gamma$ through the index ℓ. Then, item (ii) in Proposition 3.4 holds with these quantities. Then, it follows from Proposition 3.4 that Γ is a $\mathscr{C}^k$–smooth, regular Jordan curve. □

3.2. **Admissible Boundary Representations.** Recall that we aim to prove the holomorphic dependence of the Calderón projector on a collection of boundaries of class $\mathscr{C}^2$. In view of Lemma 3.5 and as a way to represent the said collection of boundaries of class $\mathscr{C}^2$, throughout what follows we consider a set $\{\Gamma_r\}_{r \in \mathfrak{T}}$ of $\mathscr{C}^2$–smooth Jordan curves (in the sense of Definition 3.1), where $r : \mathrm{I} \to \mathbb{R}^2$ is a boundary representation of $\Gamma_r \coloneqq \left\{ \mathbf{x} \in \mathbb{R}^2 : \ \mathbf{x} = r(t), \ t \in \mathrm{I} \right\}$ and $\mathfrak{T} \subset \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{R}^2)$ is a set of $\mathscr{C}^2$–smooth, regular boundary representations, whose properties we specify below. We encounter the following issues in the description of the set $\mathfrak{T}$:

- The boundary representation of a Jordan curve is not unique. Given a Jordan curve Γ_r characterized by a boundary representation $r : \mathrm{I} \to \mathbb{R}^2$ and provided a 1-periodic, twice continuously differentiable, bijective function $\chi : \mathrm{I} \to \mathrm{I}$, such that $\chi'(t) > 0$, we have that $r \circ \chi : \mathrm{I} \to \mathbb{R}^2$ is also a regular boundary representation of the Jordan curve Γ_r. Therefore $\Gamma_r = \Gamma_{r\circ\chi}$, i.e. both curves are composed of the exact same set of points of $\mathbb{R}^2$.
- In the ensuing analysis, we need to work with regular boundary representations (in the sense of Definition 3.1, item (iv)). This property allows the construction of an isomorphism between $H^s_{\text{per}}(\mathrm{I})$ and $H^s(\Gamma_r)$, for $|s| \leq 1$, as explained in Section 3.3 that allows us to transport the BIOs from the boundary Γ_r to the reference interval I bijectively.

We address these issues by introducing the concept of *admissible boundary representations* of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves.

Definition 3.6. *We say that $\mathfrak{T}$ is a set of* admissible boundary representations *of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves if:*

(i) *Each $r : \mathrm{I} \to \mathbb{R}^2$ belonging to $\mathfrak{T}$ is a $\mathscr{C}^2$–smooth, regular boundary representation (in the sense of Definition 3.1) of the Jordan curve Γ_r.*

(ii) *Given two Jordan curves Γ_{r_1} and Γ_{r_2} belonging to $\{\Gamma_r\}_{r\in\mathfrak{T}}$ with boundary representations r_1, $r_2 : \mathrm{I} \to \mathbb{R}^2$, respectively, and satisfying $\Gamma_{r_1} = \Gamma_{r_2}$ (i.e., both curves are composed of the exact same point set in $\mathbb{R}^2$), it holds that $r_1(t) = r_2(t)$, for all $t \in \mathrm{I}$.*

Remark 4. The set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of regular Jordan curves is not unique. For each boundary representation $r \in \mathfrak{T}$ one may consider a 1-periodic, twice continuously differentiable and a bijective function $\chi_r : \mathrm{I} \to \mathrm{I}$ (depending on r) such that $\chi_r'(t) > 0$, for $t \in \mathrm{I}$. Then, we have that $r \circ \chi_r : \mathrm{I} \to \mathbb{R}^2$ is a regular boundary representation of the Jordan curve Γ_r. Hence, the set $\widetilde{\mathfrak{T}} := \{r \circ \chi_r\}_{r\in\mathfrak{T}}$ is also a set of admissible boundary representations of the collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of regular Jordan curves.

Remark 5. We do not enforce the boundary representations belonging to the set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ to be parametrized by the arc-length, as we would like all the boundary representations belonging to the set $\mathfrak{T}$ to be functions mapping the (fixed) interval $\mathrm{I} = [0, 1]$ to $\mathbb{R}^2$.

In what follows we work under the following assumption.

Assumption 3.7. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves (in the sense of Definition 3.6). The set $\mathfrak{T}$ has the following properties:*

(i) *For each $r \in \mathfrak{T}$, the Jordan curve Γ_r is positively oriented under the boundary representation $r : \mathrm{I} \to \mathbb{R}^2$ (in the sense of Definition 3.3).*

(ii) *The set $\mathfrak{T}$ is a compact subset of $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$.*

Example 3.8. *Consider the collection of curves $\{\Gamma_r\}_{r\in\mathfrak{T}} \subset \mathbb{R}^2$ defined by the set of boundary representations*

$$\mathfrak{T} = \left\{ r \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2) : \; r(t) = r_0(t) + y\, \psi_\sigma(t) \quad t \in \mathrm{I} \quad \text{and} \quad y \in [-1, 1] \right\}, \tag{3.2}$$

where $r_0 : \mathrm{I} \to \mathbb{R}^2$ is a boundary representation of a $\mathscr{C}^2$–smooth, regular nominal *Jordan curve Γ_{r_0}, $\psi_\sigma(t) = \sigma \cos(2\pi t)$ and $\sigma \in \mathbb{R}_+$. Selecting $\sigma \in \mathbb{R}_+$ sufficiently small, $\mathfrak{T}$ in* (3.2) *is actually a set of admissible boundary representations of the collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves. Assumption 3.7, item (i), is satisfied by the set $\mathfrak{T}$ in* (3.2) *of admissible boundary representations of the collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves. The compactness of $\mathfrak{T}$ in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ follows from* [14, Lemma 2.7]*. Thus, item (ii) in Assumption 3.7 holds.*

Given $r \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$, we define $\mathsf{q}_r : \mathrm{I} \times \mathrm{I} \to \mathbb{R}$ as follows

$$\mathsf{q}_r(t, s) := \begin{cases} \left\| \frac{r(t)-r(s)}{\sin(\pi(t-s))} \right\| & \text{for} \quad t - s \notin \mathbb{Z}, \\ \frac{\left\| r'(s) \right\|}{\pi} & \text{for} \quad t - s \in \mathbb{Z}. \end{cases}$$

The following result establishes a crucial property of a set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves.

Proposition 3.9. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7. Then, for each $r \in \mathfrak{T}$ we have $\mathsf{q}_r \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$ and there exists* *a constant $\alpha(\mathfrak{T}) > 0$ (depending on $\mathfrak{T}$ only) such that*

$$\inf_{r\in\mathfrak{T}} \left(\inf_{(t,s)\in \mathrm{I}\times\mathrm{I}} \mathsf{q}_r(t,s) \right) \geq \alpha(\mathfrak{T}) > 0. \tag{3.3}$$

Proof. We observe that for $(t,s) \in \mathrm{I}\times\mathrm{I}$ such that $t-s \notin \mathbb{Z}$, the function $\mathsf{q}_r : \mathrm{I}\times\mathrm{I} \to \mathbb{R}$ is continuous. We proceed to study q_r when t approaches $s \in \mathrm{I}$. For $s \in \mathrm{I}$, let us compute

$$\lim_{t\to s} \mathsf{q}_r(t,s) = \lim_{t\to s} \left\| \frac{r(t)-r(s)}{\sin(\pi(t-s))} \right\| = \lim_{t\to s} \frac{\left\| \displaystyle\int_0^1 r'(s+\zeta(t-s)d\zeta \right\|}{\left| \frac{\sin(\pi(t-s))}{t-s} \right|} = \frac{\|r'(s)\|}{\pi} = \mathsf{q}_r(s,s).$$

The continuity of q_r for t approaching $s+\mathbb{Z}$ can be obtained by using the 1-biperiodicity of $\mathsf{q}_r(t,s)$. Hence, we conclude that $\mathsf{q}_r \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$.

Let $r \in \mathfrak{T}$. We have that $r : \mathrm{I} \to \Gamma_r$ is injective in $[0,1)$ and is also 1-periodic, i.e. $r(t) = r(t+1)$ for all $t \in \mathbb{R}$. Furthermore, since the boundary representation $r : \mathrm{I} \to \Gamma_r$ is regular, we have $r'(t) \neq \mathbf{0}$, for $t \in \mathrm{I}$. It follows that q_r is strictly positive in the compact domain $\mathrm{I}\times\mathrm{I}$. Hence, $\mathsf{q}_r \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$ and $\mathsf{q}_r(t,s) > 0$ for all $(t,s) \in \mathrm{I}\times\mathrm{I}$. Considering that $\mathrm{I}\times\mathrm{I}$ is a compact subset of $\mathbb{R}^2$, it follows that q_r attains a strictly positive minimum. Therefore, there exists a positive constant α_r, solely depending on the boundary transformation $r \in \mathfrak{T}$, such that

$$\inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_r(t,s) \geq \alpha_r > 0. \tag{3.4}$$

The map

$$r \in \mathfrak{T} \mapsto \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_r. \tag{3.5}$$

is continuous. Indeed, for $r \in \mathfrak{T}$ and $\xi \in \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{R}^2)$ it holds

$$|\mathsf{q}_{r+\xi}(t,s) - \mathsf{q}_r(t,s)| \leq \mathsf{q}_\xi(t,s), \quad \text{for } (t,s) \in \mathrm{I}\times\mathrm{I},$$

and it follows that

$$\|\mathsf{q}_{r+\xi} - \mathsf{q}_r\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})} \leq \|\mathsf{q}_\xi\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})}. \tag{3.6}$$

Observe that

$$\|\mathsf{q}_\xi\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})} \leq \frac{\|\xi\|_{\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{R}^2)}}{\pi}.$$

Furthermore, it holds that

$$\left| \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_{r+\xi} - \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_r \right| \leq \sup_{(t,s)\in\mathrm{I}\times\mathrm{I}} |\mathsf{q}_{r+\xi}(t,s) - \mathsf{q}_\xi(t,s)|.$$

Using (3.6), we obtain

$$\left| \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_{r+\xi} - \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_r \right| \leq \|\mathsf{q}_\xi\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})}.$$

Hence, for all $\epsilon > 0$ we have that

$$\|\xi\|_{\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{R}^2)} < \pi\epsilon \quad \text{implies} \quad \|\mathsf{q}_{r+\xi} - \mathsf{q}_r\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})} < \epsilon.$$

We conclude that the map (3.5) is continuous and, furthermore, strictly positive according to (3.4). Consequently, recalling that $\mathfrak{T}$ is a compact subset of $\mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{R}^2)$ according to Assumption 3.7, the map (3.5) attains a strictly positive minimum. Hence, there exists a constant $\alpha(\mathfrak{T}) > 0$, depending only on $\mathfrak{T}$, such that (3.3) holds. □

3.3. **The Pullback Operator.** Let $\mathfrak{T}$ be a collection of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves (in the sense of Definition 3.6). For $r \in \mathfrak{T}$ and $\varphi \in \mathscr{C}^0(\Gamma_r)$ define $(\tau_r\varphi)(t) := (\varphi \circ r)(t)$, for $t \in \mathrm{I}$. Being the composition of continuous functions, we have that $\tau_r\varphi \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I})$. The operator τ_r admits an inverse τ_r^{-1} given by $\left(\tau_r^{-1}\hat{\varphi}\right)(\mathbf{x}) := \left(\hat{\varphi} \circ r^{-1}\right)(\mathbf{x})$, for $\mathbf{x} \in \Gamma_r$ and $\hat{\varphi} \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I})$. In the following we refer to τ_r as the *pullback operator*.

The pullback operator allows to transform the BIOs introduced in Section 2.3 into 1-periodic integral operators, with mapping properties between appropriate Sobolev spaces of 1-periodic functions that do not depend on the chosen boundary representation. The dependence on the boundary representation will be completely isolated in the integrand of the arising 1-periodic integral operator.

Proposition 3.10. *Let $\mathfrak{T}$ be a collection of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves (in the sense of Definition 3.6) satisfying Assumption 3.7. The pullback operator admits a unique extension, still denoted by τ_r, such that $\tau_r \in \mathscr{L}_{\text{iso}}\left(H^\sigma(\Gamma_r), H^\sigma_{\text{per}}(\mathrm{I})\right)$, for $|\sigma| \le 1$ and $r \in \mathfrak{T}$.*

Proof. The result follows from [32, Lemmas 4.2.4 and 4.2.5]. □

Remark 6. It can be proved that the pullback operator admits a unique extension τ_r having the mapping properties stated in Proposition 3.10, but with $|\sigma| \le 2$ for boundaries of class $\mathscr{C}^2$. However, for our purposes, it is enough to have the mapping properties of the pullback operator for $|\sigma| \le 1$.

4. Holomorphic 1-Periodic Integral Operators

In this section, we consider 1-periodic integral operators depending on a collection of $\mathscr{C}^2$–smooth, regular Jordan curves in $\mathbb{R}^2$ and establish sufficient conditions to obtain shape holomorphy of the corresponding domain-to-operator maps. As we will see in Section 5 ahead, the main result of the present section (Theorem 4.13) provides a common framework to prove shape holomorphy of the BIOs for the Laplace operator appearing in the Calderón projector (2.6). The results presented in this section are developed in slightly greater generality than required in the subsequent development for the Calderón projector for the Laplacean, and can be employed to establish shape holomorphy of more general BIOs in two spatial dimensions.

In order to establish shape holomorphy, we verify *complex* Fréchet differentiability of the corresponding domain-to-operator map. Shape differentiability of the BIOs in acoustic and electromagnetic scattering and the numerical computation of shape gradients play an essential role in the implementation of iterative optimization algorithms (for which a direction of maximum descent is required at each step) and shape sensitivity analyses. Existing results in this regard aim to establish the *real* (Fréchet or Gâteaux) differentiability of the domain-to-operator map and obtain explicit expressions for first and higher-order derivatives. However, to our knowledge, none of the currently available results address the holomorphic dependence of the Calderón projector in (2.6) on the boundary.

Recall that the computation and analysis of complex Fréchet derivatives necessarily requires a map that is well-defined from one complex Banach space to another. However, the Calderón projector (2.6) is only defined on boundaries that are contained in $\mathbb{R}^2$. After the application of the pullback operator introduced in Section 3.3, one can solely construct a domain-to-operator map that is properly defined for a collection of Jordan curves $\{\Gamma_r\}_{r\in\mathfrak{T}}$ in $\mathbb{R}^2$, where $\mathfrak{T} \subset \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ is a set of admissible boundary representations. Hence, up to this point in our analysis, the domain-to-operator map can only be understood for real-valued boundary representations.

To compute the complex Fréchet derivative of the domain-to-operator map one must firstly *extend* the set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves to include *complex-valued* boundary representations, i.e. vector-valued functions of the form $r : \mathrm{I} \to \mathbb{C}^2$ belonging to $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$. This extension is performed by considering an open neighborhood of boundary representations with values in $\mathbb{C}^2$ in the topology induced by the metric $d : \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2) \times \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2) \to \mathbb{R}$ in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$ defined as

$$d(r_1, r_2) := \|r_1 - r_2\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)}, \quad r_1, r_2 \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2). \tag{4.1}$$

The size $\delta > 0$ of the complex open δ-neighborhood of $\mathfrak{T}$ must be judiciously chosen so that the Calderón projector admits a well-defined extension to a set of complex-valued boundary representations contained in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$. In particular, this extension must preserve the mapping properties of the original BIOs. Once this extension is constructed, the complex Fréchet differentiability of the domain-to-operator map can be performed.

This section is divided in two parts. Firstly, in Section 4.1 we introduce notation and results regarding holomorphic maps in Banach spaces of relevance for the subsequent developments. Then, in Section 4.2 we consider a class of 1-periodic integral operators with an integrand defined as the product of a continuous function that depends holomorphically on a set of complex-valued boundary representations and a (possibly singular) function that is independent of the chosen boundary representation. The main result of this section corresponds to Theorem 4.13. Therein, we establish sufficient conditions to prove the holomorphic dependence of 1-periodic integral operators having the structure previously described on a suitable collection of *complex-valued*, $\mathscr{C}^2$–smooth boundary representations.

We remark that the abstract exercise of considering an open neighborhood of boundary representations with values in $\mathbb{C}^2$ and the subsequent extension of the Calderón projector is only required for theoretical purposes. In concrete numerical applications, we neither numerically construct the BIOs for complex-valued boundary representations nor solve BIEs set on complex-valued boundary curves. As will be discussed in Section 6, the presently obtained shape holomorphy result enables us to obtain parametric regularity estimates for the parametric version of the Calderón projector, which are required for the analysis of several techniques commonly used in forward and inverse UQ.

4.1. **Holomorphic Maps in Banach Spaces.** Let E and F be complex Banach spaces equipped with the norms $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively. For $m \in \mathbb{N}$, we denote by $\mathscr{L}(E^{(m)}, F)$ the set of continuous m-linear maps ([40, Definition 1.1])

$$\mathcal{M}_{r_1,\ldots,r_m} : (r_1, \cdots, r_m) \in \underbrace{E \times \cdots \times E}_{m \text{ times}} \to F.$$

A mapping $\mathcal{P}_\xi : \xi \in E \to F$ is said to be an m-homogeneous polynomial if there exists $\mathcal{M} \in \mathscr{L}(E^{(m)}, F)$ such that

$$\mathcal{P}_\xi = \mathcal{M}_{\underbrace{\xi, \ldots, \xi}_{m \text{ times}}}.$$

for all $\xi \in E$. Furthermore, we denote by $\mathscr{P}(E^{(m)}, F)$ the set of all continuous m-homogeneous polynomials from E into F [40, Definition 2.1].

Definition 4.1 ([40, Definition 5.1])**.** *Let U be an open, nonempty subset of E. A map $\mathcal{F}_r : r \in U \to F$ is said to be* holomorphic *if for each $r \in U$ there exists a $\sigma > 0$, a ball $B(\sigma) \coloneqq \{r \in E : \|r\|_E \leq \sigma\}$ and a sequence of polynomials $\mathcal{P}^{(m)}_\xi : \xi \in E \to F$ belonging to $\mathscr{P}(E^{(m)}, F)$ such that for all $r \in U$ and $\xi \in B(\sigma)$ satisfying $r + \xi \in U$*

$$\mathcal{F}_{r+\xi} = \sum_{m=0}^{\infty} \mathcal{P}^{(m)}_\xi$$

holds uniformly

Definition 4.2 ([40, Definition 13.1])**.** *Let U be an open, nonempty subset of E. A map $\mathcal{F}_r : r \in U \to F$ is said to be complex Fréchet differentiable if for each point $r \in U$ there exists a map $(\frac{d}{dr}\mathcal{F}_r)[r, \cdot] \in \mathscr{L}(E, F)$ such that*

$$\left\| \mathcal{F}_{r+\xi} - \mathcal{F}_r - \left(\frac{d}{dr}\mathcal{F}_r\right)[r, \xi] \right\|_F = o\left(\|\xi\|_E\right).$$

We say that $\left(\frac{d}{dr}\mathcal{F}_r\right)[r, \xi]$ is the Fréchet derivative *of $\mathcal{F}_r : r \in U \to F$ at $r \in U$ in the direction $\xi \in E$.*

In the case that the map $r \in U \to (\frac{d}{dr}\mathcal{F}_r)[r, \xi] \in \mathscr{L}(E, F)$ is again complex Fréchet differentiable with continuous derivative we say that the map $\mathcal{F}_r : r \in U \to F$ is *twice complex continuously Fréchet differentiable*. Then, we have that

$$\left(\frac{d^2}{dr^2}\mathcal{F}_r\right)[r, \cdot, \cdot] \coloneqq \frac{d}{dr}\left(\left(\frac{d}{dr}\mathcal{F}_r\right)[r, \cdot]\right)[r, \cdot] \in \mathscr{L}(E^{(2)}, F).$$

Recursively, one may define higher-order Fréchet derivatives.

Definition 4.3. *Let U be an open, nonempty subset of E and let $m \in \mathbb{N}$. We say that the map $\mathcal{F}_r : r \in U \to F$ is m-times Fréchet differentiable if it is $(m-1)$-times Fréchet differentiable and the map*

$$r \in U \to \left(\frac{d^{m-1}}{dr^{m-1}}\mathcal{F}_r\right)[r, \underbrace{\cdot, \ldots, \cdot}_{m-1 \text{ times}}] \in \mathscr{L}\left(E^{(m-1)}, F\right)$$

is Fréchet differentiable as well. We say that $\mathcal{F}_r : r \in U \to F$ is infinitely complex Fréchet differentiable if it is m-times Fréchet differentiable for all $m \in \mathbb{N}$.

In the following, we adopt the notation

$$\left(\frac{d^m}{dr^m}\mathcal{F}_r\right)[r, \xi] = \left(\frac{d^m}{dr^m}\mathcal{F}_r\right)[r, \underbrace{\xi, \cdots, \xi}_{m \text{ times}}], \quad m \in \mathbb{N}.$$

We recall results regarding the extension of Taylor's formula to maps between Banach spaces.

Lemma 4.4 ([23, Lemma 5.40]). *Let U be an open, nonempty subset of E and let $\mathcal{F}_r : r \in U \to F$ be complex Fréchet differentiable. If the segment joining $r \in U$ and $r + \xi \in U$ is contained in U, then it holds*

$$\mathcal{F}_{r+\xi} - \mathcal{F}_r = \int_0^1 \left(\frac{d}{dr} \mathcal{F}_r \right) [r + \eta\xi, \xi] d\eta.$$

Lemma 4.5 ([23, Theorem 5.42]). *Let U be an open, nonempty subset of E and let $\mathcal{F}_r : r \in U \to F$ be m-times continuously complex Fréchet differentiable, for some $m \in \mathbb{N}$. In case that the segment joining $r \in U$ and $r + \xi \in U$ is contained in U it holds*

$$\mathcal{F}_{r+\xi} = \sum_{\ell=0}^{m} \frac{1}{\ell!} \left(\frac{d^\ell}{dr^\ell} \mathcal{F}_r \right) [r, \xi] + \int_0^1 \left(\left(\frac{d^m}{dr^m} \mathcal{F}_r \right) [r + \eta\xi, \xi] - \left(\frac{d^m}{dr^m} \mathcal{F}_r \right) [r, \xi] \right) d\vartheta_m(\eta),$$

where $\vartheta_m(\eta) = -(1 - \eta)^m/m!$, for $\eta \in [0, 1]$.

Theorem 4.6 ([40, Theorem 14.7]). *Let U be an open, nonempty subset of E. For the map $\mathcal{F}_r : r \in U \to F$ the following conditions are equivalent:*

(i) *$\mathcal{F}_r$ is holomorphic.*
(ii) *$\mathcal{F}_r$ is complex Fréchet differentiable.*
(iii) *$\mathcal{F}_r$ is infinitely complex Fréchet differentiable.*

Furthermore, for all $r \in U$ and $\xi \in E$ such that $r + \xi \in U$ it holds

$$\left(\frac{d^m}{dr^m} \mathcal{F}_r \right) [r, \xi] = m!\, \mathcal{P}_\xi^{(m)}, \quad m \in \mathbb{N},$$

for all $r \in U$ and $\xi \in E$ such that $r + \xi \in U$, where $\mathcal{P}_\xi^{(m)}$ is as in Definition 4.1, for $m \in \mathbb{N}$.

Remark 7. As we aim to prove the holomorphic dependence of the BIOs on a collection of planar curves of class $\mathscr{C}^2$, we emphasize the complex nature of the Banach spaces E and F. Due to Theorem 4.6 it is necessary and sufficient to prove *complex* Fréchet differentiability of the BIOs with respect to a suitable collection of *complex-valued* planar curves of class $\mathscr{C}^2$. We elaborate on this issue in Section 4.2.

We proceed to present further properties of holomorphic maps in complex Banach spaces, to be used in the ensuing analysis.

Theorem 4.7. *Let U be an open, nonempty subset of E and let $\mathcal{F}_r : r \in U \to F$ be holomorphic. Then, if the segment joining $r \in U$ and $r + \xi \in U$ is contained in U, for all $m \in \mathbb{N}_0$ it holds*

$$\mathcal{F}_{r+\xi} = \sum_{\ell=0}^{m} \frac{1}{\ell!} \left(\frac{d^\ell}{dr^\ell} \mathcal{F}_r \right) [r, \xi] + \int_0^1 \frac{(1-\eta)^m}{m!} \left(\frac{d^{m+1}}{dr^{m+1}} \mathcal{F}_r \right) [r + \eta\xi, \xi] d\eta.$$

Proof. According to Lemma 4.4 and since $\mathcal{F}_r : r \in U \to F$ is holomorphic,

$$\begin{aligned}
\left(\frac{d^m}{dr^m} \mathcal{F}_r \right) [r + \eta\xi, \xi] - \left(\frac{d^m}{dr^m} \mathcal{F}_r \right) [r, \xi] &= \int_0^1 \left(\frac{d^{m+1}}{dr^{m+1}} \mathcal{F}_r \right) [r + t\eta\xi, \xi] dt \\
&= \int_0^\eta \left(\frac{d^{m+1}}{dr^{m+1}} \mathcal{F}_r \right) [r + t\xi, \xi] dt,
\end{aligned}$$

for $\eta \in [0, 1]$. Applying integration by parts

$$\begin{aligned}
\int_0^1 \left(\left(\frac{d^m}{dr^m} \mathcal{F}_r \right) [r + \eta\xi, \xi] - \left(\frac{d^m}{dr^m} \mathcal{F}_r \right) [r, \xi] \right) d\vartheta_m(\eta) &= \underbrace{\left(\vartheta_m(\eta) \int_0^\eta \left(\frac{d^{m+1}}{dr^{m+1}} \mathcal{F}_r \right) [r + t\xi, \xi] dt \right) \Bigg|_{\eta=0}^{1}}_{=0} \\
&\quad - \int_0^1 \left(\frac{d^{m+1}}{dr^{m+1}} \mathcal{F}_r \right) [r + \eta\xi, \xi] \vartheta_m(\eta) d\eta \\
&= \int_0^1 \frac{(1-\eta)^m}{m!} \left(\frac{d^{m+1}}{dr^{m+1}} \mathcal{F}_r \right) [r + \eta\xi, \xi] d\eta,
\end{aligned}$$

and the result follows from these computations. $\square$

Let $\mathscr{D}(\vartheta) := \{z \in \mathbb{C} : \ |z| < \vartheta\}$ be the complex open disc of radius $\vartheta > 0$ centered in the origin of the complex plane and let $\overline{\mathscr{D}}(\vartheta)$ be its closure. The following result corresponds to the version of Cauchy's integral formula for *holomorphic* maps in complex Banach spaces.

Proposition 4.8 ([40, Corollary 7.3])**.** *Let U be an open and nonempty subset of E and let $\mathcal{F}_r : r \in U \to F$ be Fréchet differentiable. Let $r \in U$, $\xi \in E$ and $\vartheta > 0$ be such that $r + \sigma\xi \in U$, for all $\sigma \in \overline{\mathscr{D}}(\vartheta)$. Then, for each $m \in \mathbb{N}_0$ we have the Cauchy's integral formula*

$$\left(\frac{d^m}{d\, r^m}\mathcal{F}_r\right)[r,\xi] = \frac{m!}{2\pi\imath} \int\limits_{|\lambda|=\vartheta} \frac{\mathcal{F}_{r+\lambda\xi}}{\lambda^{m+1}} d\lambda.$$

We conclude this section by recalling the following result that asserts the uniqueness of holomorphic extensions.

Theorem 4.9 ([23, Theorem 5.34])**.** *Let R be a real vector subspace of E such that the complex subspace $R_c := R + \imath R$ is dense in E. If $\mathcal{F}_r, \mathcal{G}_r : r \in U \to F$ are holomophic on a connected open set U in E, and $\mathcal{F}_r = \mathcal{G}_r$ on some nonempty set $V \subset U \cap R$, relatively open in R, then $\mathcal{F}_r = \mathcal{G}_r$ on U.*

4.2. **Shape Holomorphy of 1-periodic Integral Operators.** Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ (in the sense of Definition 3.6). Provided a function $\mathsf{f} : \mathbb{R}\backslash\mathbb{Z} \to \mathbb{C}$ and, for each $r \in \mathfrak{T}$, a function $\mathsf{p}_r : \mathrm{I} \times \mathrm{I} \to \mathbb{C}$, we define for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ the following 1-periodic integral operator

$$\left(\mathsf{P}_r\, \hat{\varphi}\right)(t) := \int\limits_0^1 \mathsf{f}(t-s)\, \mathsf{p}_r(t,s)\hat{\varphi}(s) ds, \quad t \in \mathrm{I}. \tag{4.2}$$

Observe that P_r depends on the boundary representation $r \in \mathfrak{T}$ only through $\mathsf{p}_r : \mathrm{I} \times \mathrm{I} \to \mathbb{C}$. In the following and throughout this section we work under the following assumption.

Assumption 4.10. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$.*

(i) *For each $r \in \mathfrak{T}$, we have that $\mathsf{p}_r \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$.*

(ii) *The function f is a 1-periodic, weakly singular kernel, i.e. there exists a $\upsilon \in (0,1)$ and a finite constant $C(\mathsf{f},\upsilon) > 0$ (depending on f and υ only) such that*

$$|\mathsf{f}(t)| \leq C(\mathsf{f},\upsilon)\, |\sin(\pi t)|^{-\upsilon}, \quad t \in \mathbb{R}\backslash\mathbb{Z}.$$

The function f is continuous in $\mathbb{R}\backslash\mathbb{Z}$ and does not depend on the boundary representation $r \in \mathfrak{T}$.

As we shall prove in Section 5, the BIOs appearing in the Calderón projector set on a Jordan curve Γ_r with a boundary representation $r : \mathrm{I} \to \mathbb{R}^2$ may be a cast as in (4.2). After the application of pullback operator τ_r introduced for each for $r \in \mathfrak{T}$ in Section 3.3 to the BIOs defined in Section 2.3, we obtain 1-periodic integral operators defined in the reference domain I with the structure of P_r. Hence the importance of 1-periodic integral operators satisfying this framework. The singular component of the BIOs is contained in the function f, while the dependence on the boundary representation $r : \mathrm{I} \to \mathbb{R}^2$ of Γ_r is isolated in the continuous function $\mathsf{p}_r \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$. A detailed description of this procedure for the components of the Calderón projector is provided in Section 5.

Under Assumption 4.10, the integral in (4.2) exists in the Lebesgue sense. The proof of the following result may be found in Appendix A.

Lemma 4.11. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ and let Assumption 4.10 be satisfied. For $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$, the integral in (4.2), i.e. in the definition of the 1-periodic operator P_r, exists in the Lebesgue sense. Furthermore, for each $r \in \mathfrak{T}$ and for all $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$, we have that $\mathsf{P}_r\, \hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$.*

The main goal of this subsection is to introduce sufficient conditions that allow us to establish the holomorphic dependence of the 1-periodic integral operator P_r in (4.2) (as an element of the complex Banach space of bounded linear operators satisfying suitable mapping properties) on a set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$.

Due to Theorem 4.6 and as explained in Remark 7, we aim to prove *complex Fréchet differentiability* of the 1-periodic integral operator P_r with respect to the boundary representation $r \in \mathfrak{T}$. In so doing, first we have to extend the set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves to include complex-valued boundary representations, as described at the beginning of this section. Given $\delta > 0$, we define the complex open δ-neighborhood of $\mathfrak{T}$ as

$$\mathfrak{T}_\delta := \left\{ r \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2) : \exists \widetilde{r} \in \mathfrak{T} \text{ such that } d(\widetilde{r}, r) < \delta \right\},$$

where $d(\cdot,\cdot)$ has been introduced in (4.1). For some $\delta > 0$ to be specified, we define an extension of the 1-periodic integral operator P_r to the set $\mathfrak{T}_\delta$ and for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ as

$$(\mathsf{P}_{r,\mathbb{C}}\, \hat{\varphi})\,(t) := \int_0^1 \mathsf{f}(t-s)\, \mathsf{p}_{r,\mathbb{C}}(t,s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I} \quad \text{and} \quad r \in \mathfrak{T}_\delta, \tag{4.3}$$

where $\mathsf{p}_{r,\mathbb{C}}$ is a suitable extension of $\mathsf{p}_r \in \mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})$ to the set $\mathfrak{T}_\delta$. In the following, we work under the assumption stated below.

Assumption 4.12. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ (in the sense of Definition 3.6). There exists $\delta > 0$ such that:*

(i) *for each $r \in \mathfrak{T}_\delta$, the extension $\mathsf{p}_{r,\mathbb{C}}$ of $\mathsf{p}_r \in \mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})$ to $\mathfrak{T}_\delta$ belongs to $\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})$,*

(ii) *the 1-periodic integral operator $\mathsf{P}_{r,\mathbb{C}}$ defined in (4.3) admits a unique extension to 1-periodic Sobolev spaces, still denoted by $\mathsf{P}_{r,\mathbb{C}}$, such that for some $\kappa, \rho \in \mathbb{R}$ and for all $r \in \mathfrak{T}_\delta$ the 1-periodic integral operator $\mathsf{P}_{r,\mathbb{C}} : H^\kappa_{\text{per}}(\mathrm{I}) \to H^\rho_{\text{per}}(\mathrm{I})$ is linear and bounded.*

(iii) *the map*

$$r \in \mathfrak{T}_\delta \mapsto \mathsf{P}_{r,\mathbb{C}} \in \mathscr{L}\left(H^\kappa_{\text{per}}(\mathrm{I}), H^\rho_{\text{per}}(\mathrm{I})\right)$$

is uniformly bounded on the set $\mathfrak{T}_\delta$, i.e. there exists a finite constant $C_{\mathsf{P}}(\mathfrak{T},\delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that

$$\sup_{r\in\mathfrak{T}_\delta} \|\mathsf{P}_{r,\mathbb{C}}\|_{\mathscr{L}\left(H^\kappa_{\text{per}}(\mathrm{I}), H^\rho_{\text{per}}(\mathrm{I})\right)} \leq C_{\mathsf{P}}\left(\mathfrak{T},\delta\right), \tag{4.4}$$

(iv) *the map*

$$r \in \mathfrak{T}_\delta \mapsto \mathsf{p}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}\left(\mathrm{I}\times\mathrm{I}\right)$$

is holomorphic.

Under Assumptions 4.10 and 4.12, the next theorem establishes the holomorphic dependence of $\mathsf{P}_{r,\mathbb{C}}$ on the set $\mathfrak{T}_\varepsilon$ for some $\varepsilon > 0$ to be specified.

Remark 8. When item (i) in Assumption 4.12 is satisfied, Assumptions 4.10 holds for $\mathsf{P}_{r,\mathbb{C}}$. Hence, the statements of Lemma 4.11 hold for $\mathsf{P}_{r,\mathbb{C}}$ as well.

Theorem 4.13. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$. Let Assumptions 4.10 and 4.12 hold with some $\delta > 0$. Then, for any $\varepsilon \in (0,\delta)$ the map*

$$r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{P}_{r,\mathbb{C}} \in \mathscr{L}\big(H^\kappa_{\text{per}}(\mathrm{I}), H^\rho_{\text{per}}(\mathrm{I})\big) \tag{4.5}$$

is holomorphic. Its Fréchet derivative at $r \in \mathfrak{T}_\varepsilon$, for $\varepsilon \in (0,\delta)$, in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ and for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ reads

$$\left(\frac{d}{dr}\mathsf{P}_{r,\mathbb{C}}\,\hat{\varphi}\right)[r,\xi](t) = \int_0^1 \mathsf{f}(t-s)\left(\frac{d}{dr}\mathsf{p}_{r,\mathbb{C}}\right)[r,\xi](t,s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I}.$$

Proof. Let $\varepsilon \in (0,\delta]$ where $\delta > 0$ is as in Assumption 4.12. The statements of Assumption 4.12 hold as well for the set $\mathfrak{T}_\varepsilon$.

The proof is divided into two steps:

(a) For $m \in \mathbb{N}$, $r \in \mathfrak{T}_\varepsilon$ and $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$, we define

$$\left(\mathsf{P}^{(m)}_{r,\xi}\,\hat{\varphi}\right)(t) := \int_0^1 \mathsf{f}(t-s)\left(\frac{d^m}{d\,r^m}\mathsf{p}_{r,\mathbb{C}}\right)[r,\xi](t,s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I},$$

where $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$. Firstly, we prove that $\mathsf{P}^{(m)}_{r,\xi} : H^\kappa_{\text{per}}(\mathrm{I}) \to H^\rho_{\text{per}}(\mathrm{I})$ is linear and bounded for all $r \in \mathfrak{T}_\varepsilon$ and $m \in \mathbb{N}$. More precisely, that the following estimate holds for all $m \in \mathbb{N}$

$$\left\|\mathsf{P}^{(m)}_{r,\xi}\right\|_{\mathscr{L}\left(H^\kappa_{\text{per}}(\mathrm{I}), H^\rho_{\text{per}}(\mathrm{I})\right)} \leq m!\frac{C_{\mathsf{P}}\left(\mathfrak{T},\delta\right)}{\vartheta^m_\xi},$$

where $C_{\mathsf{P}}\left(\mathfrak{T},\delta\right)$ is as in Assumption 4.12 and $\vartheta_\xi > 0$ (depending upon ξ) together with $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ are chosen so that $r + \lambda\xi \in \mathfrak{T}_\delta$, for all $\lambda \in \overline{\mathscr{D}}(\vartheta_\xi)$.

(b) Using Taylor's expansion we prove that

$$\left\|\mathsf{P}_{r+\xi,\mathbb{C}}-\mathsf{P}_{r,\mathbb{C}}-\mathsf{P}^{(1)}_{r,\xi}\right\|_{\mathscr{L}\left(H^{\kappa}_{\mathrm{per}}(\mathrm{I}),H^{\rho}_{\mathrm{per}}(\mathrm{I})\right)}=o\left(\|\xi\|_{\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\right),$$

for $r\in\mathfrak{T}_\varepsilon$. This implies *complex* Fréchet differentiability of the map (4.5) and that $\mathsf{P}^{(1)}_{r,\xi}\in\mathscr{L}\big(H^{\kappa}_{\mathrm{per}}(\mathrm{I}),H^{\rho}_{\mathrm{per}}(\mathrm{I})\big)$ is actually the Fréchet derivative of $\mathsf{P}_{r,\mathbb{C}}$ at $r\in\mathfrak{T}_\varepsilon$ in the direction $\xi\in\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$.

We proceed with step (a). Let $r\in\mathfrak{T}_\varepsilon$ and $0\neq\xi\in\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$. Let us assume, for the moment, that there exists $\vartheta_\xi>0$ (depending on ξ) such that $r+\lambda\xi\in\mathfrak{T}_\delta$ for all $\lambda\in\overline{\mathscr{D}}(\vartheta_\xi)$. We will specify these quantities in step (b) of the proof. Therefore, since $r\in\mathfrak{T}_\delta\mapsto\mathsf{p}_{r,\mathbb{C}}\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$ is holomorphic according to Assumption 4.12, item (iv), Cauchy's integral formula (Proposition 4.8) delivers

$$\left(\frac{d^m}{dr^m}\mathsf{p}_{r,\mathbb{C}}\right)[r,\xi](t,s)=\frac{m!}{2\pi\imath}\int\limits_{|\lambda|=\vartheta_\xi}\frac{\mathsf{p}_{r+\lambda\xi,\mathbb{C}}(t,s)}{\lambda^{m+1}}d\lambda,\quad (t,s)\in\mathrm{I}\times\mathrm{I}.$$

Hence, for $t\in\mathrm{I}$ and $\hat{\varphi}\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I})$ we have

$$\begin{aligned}\left(\mathsf{P}^{(m)}_{r,\xi}\hat{\varphi}\right)(t)&=\frac{m!}{2\pi\imath}\int_0^1\mathsf{f}(t-s)\left(\int\limits_{|\lambda|=\vartheta_\xi}\frac{\mathsf{p}_{r+\lambda\xi,\mathbb{C}}(t,s)}{\lambda^{m+1}}d\lambda\right)\hat{\varphi}(s)ds\\&=\frac{m!}{2\pi\imath}\int\limits_{|\lambda|=\vartheta_\xi}\frac{1}{\lambda^{m+1}}\left(\int_0^1\mathsf{f}(t-s)\mathsf{p}_{r+\lambda\xi,\mathbb{C}}(t,s)\hat{\varphi}(s)ds\right)d\lambda\\&=\frac{m!}{2\pi\imath}\int\limits_{|\lambda|=\vartheta_\xi}\frac{1}{\lambda^{m+1}}\left(\mathsf{P}_{r+\lambda\xi}\hat{\varphi}\right)(t)d\lambda.\end{aligned}$$

Using the uniform boundedness of P_r over $r\in\mathfrak{T}_\delta$, i.e. item (iii) in Assumption 4.12, and recalling that $r+\lambda\xi\in\mathfrak{T}_\delta$ for $\lambda\in\overline{\mathscr{D}}(\vartheta_\xi)$, for $\hat{\varphi}\in H^{\kappa}_{\mathrm{per}}(\mathrm{I})$, we obtain

$$\left\|\mathsf{P}^{(m)}_{r,\xi}\hat{\varphi}\right\|_{H^{\rho}_{\mathrm{per}}(\mathrm{I})}\leq\frac{m!}{2\pi}\int\limits_{|\lambda|=\vartheta_\xi}\frac{1}{|\lambda|^{m+1}}\left\|\mathsf{P}_{r+\lambda\xi}\hat{\varphi}\right\|_{H^{\rho}_{\mathrm{per}}(\mathrm{I})}d\lambda\leq C_{\mathsf{P}}(\mathfrak{T},\delta)\frac{m!}{\vartheta_\xi^m}\left\|\hat{\varphi}\right\|_{H^{\kappa}_{\mathrm{per}}(\mathrm{I})},$$

where the constant $C_{\mathsf{P}}(\mathfrak{T},\delta)>0$ is that of estimate (4.4). Then, using item (ii) in Assumption 4.12, for $\hat{\varphi}\in H^{\kappa}_{\mathrm{per}}(\mathrm{I})$, we obtain

$$\left\|\mathsf{P}^{(m)}_{r,\xi}\right\|_{\mathscr{L}\left(H^{\kappa}_{\mathrm{per}}(\mathrm{I}),H^{\rho}_{\mathrm{per}}(\mathrm{I})\right)}\leq C_{\mathsf{P}}(\mathfrak{T},\delta)\frac{m!}{\vartheta_\xi^m},$$

for $r\in\mathfrak{T}_\varepsilon$ and $\xi\in\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$ such that $r+\lambda\xi\in\mathfrak{T}_\delta$. This concludes step (a) of the proof.

We continue with step (b) of the proof. Recalling the holomorphic dependence of $\mathsf{p}_{r,\mathbb{C}}$ on the set $\mathfrak{T}_\delta$, item (iv) in Assumption 4.12 and an application of Taylor's theorem (Theorem 4.7) yields

$$\mathsf{p}_{r+\xi,\mathbb{C}}(t,s)=\mathsf{p}_{r,\mathbb{C}}(t,s)+\left(\frac{d}{dr}\mathsf{p}_{r,\mathbb{C}}\right)[r,\xi](t,s)+\int_0^1(1-\eta)\left(\frac{d^2}{dr^2}\mathsf{p}_{r,\mathbb{C}}\right)[r+\eta\xi,\xi](t,s)d\eta,$$

Let $\hat{\varphi}\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I})$, then using Lemma 4.11 and Remark 8 we obtain

$$\left(\mathsf{P}_{r+\xi}\hat{\varphi}\right)(t)=\left(\mathsf{P}_r\hat{\varphi}\right)(t)+\left(\mathsf{P}^{(1)}_{r,\xi}\hat{\varphi}\right)(t)+\int_0^1(1-\eta)\left(\mathsf{P}^{(2)}_{r+\eta\xi,\xi}\hat{\varphi}\right)(t)d\eta,\quad t\in\mathrm{I},$$

and it holds that

$$\left\|\left(\mathsf{P}_{r+\xi}-\mathsf{P}_r-\mathsf{P}^{(1)}_{r,\xi}\right)\hat{\varphi}\right\|_{H^{\rho}_{\mathrm{per}}(\mathrm{I})}\leq\sup_{\eta\in[0,1]}\left\|\mathsf{P}^{(2)}_{r+\eta\xi,\xi}\hat{\varphi}\right\|_{H^{\rho}_{\mathrm{per}}(\mathrm{I})}.\tag{4.6}$$

Let $r\in\mathfrak{T}_\varepsilon$ and consider $0\neq\xi\in\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$ such that $\|\xi\|_{\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}<\delta-\varepsilon$. We claim that by choosing ϑ_ξ as follows

$$0<\vartheta_\xi:=\frac{\delta-\varepsilon}{\|\xi\|_{\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}}-1,$$

it holds that $r + \eta\xi + \lambda\xi \in \mathfrak{T}_\delta$ for all $\lambda \in \overline{\mathscr{D}}(\vartheta_\xi)$ and for all $\eta \in [0,1]$. Indeed, we have that

$$\|r + \eta\xi + \lambda\xi - \widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} < \varepsilon + (1 + \vartheta_\xi)\, \|\xi\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} < \delta$$

where $\widetilde{r} \in \mathfrak{T}$ is such that $\|r - \widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} < \varepsilon$ (recall that $r \in \mathfrak{T}_\varepsilon$). Then, according to step (a) of the proof, one obtains for $r \in \mathfrak{T}_\varepsilon$ and $0 \neq \xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ the bound

$$\sup_{\eta\in[0,1]} \left\| \mathsf{P}^{(2)}_{r+\eta\xi,\xi} \right\|_{\mathscr{L}\left(H^\kappa_{\text{per}}(\mathrm{I}),H^\rho_{\text{per}}(\mathrm{I})\right)} \leq 2\frac{C_{\mathsf{P}}\left(\mathfrak{T},\delta\right)}{\vartheta^2_\xi}.$$

Observe that

$$\frac{1}{\vartheta^2_\xi} = o\left(\|\xi\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)}\right).$$

Then, from (4.6) one may conclude that

$$\left\| \mathsf{P}_{r+\xi} - \mathsf{P}_r - \mathsf{P}^{(1)}_{r,\xi} \right\|_{\mathscr{L}\left(H^\kappa_{\text{per}}(\mathrm{I}),H^\rho_{\text{per}}(\mathrm{I})\right)} = o\left(\|\xi\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)}\right),$$

for $r \in \mathfrak{T}_\varepsilon$ and $0 \neq \xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ such that $\|\xi\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} < \delta - \varepsilon$. Hence, for any $\varepsilon \in (0,\delta)$, the map

$$r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{P}_r \in \mathscr{L}\big(H^\kappa_{\text{per}}(\mathrm{I}), H^\rho_{\text{per}}(\mathrm{I})\big)$$

is complex Fréchet differentiable and, furthermore, its Fréchet derivative at $r \in \mathfrak{T}_\varepsilon$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ is $\mathsf{P}^{(1)}_{r,\xi} \in \mathscr{L}\big(H^\kappa_{\text{per}}(\mathrm{I}), H^\rho_{\text{per}}(\mathrm{I})\big)$. This concludes step (b) of the proof and shows that the map (4.5) is holomorphic by invoking Theorem 4.6. □

5. Shape Holomorphy of the Calderón Projector

In this section, we prove shape holomorphy of the Calderón projector introduced in (2.6). We proceed as follows. Firstly, we consider a set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$ (in the sense of Definition 3.6). Then, using the pullback operator defined in Section 3.3, we transform the BIOs originally posed on the boundary Γ_r into 1-periodic integral operators defined on the reference domain I. For $r \in \mathfrak{T}$, we define $\hat{\mathsf{C}}_r := \tau_r \circ \mathsf{C}_r \circ \tau_r^{-1}$ (the application of the pullback operator to C_r is understood component-wise), where $\mathsf{C}_r := \mathsf{C}_{\Gamma_r}$ is the Calderón projector, introduced in Section 2.3. Recalling that for $|\sigma| \leq 1$ and for all $r \in \mathfrak{T}$ we have $\tau_r \in \mathscr{L}_{\text{iso}}\left(H^\sigma(\Gamma_r) H^\sigma_{\text{per}}(\mathrm{I})\right)$ (Proposition 3.10) and together with the mapping properties of the BIOs (Theorem 2.8), we analyze the smoothness of the map

$$r \in \mathfrak{T} \mapsto \hat{\mathsf{C}}_r \in \mathscr{L}\left(\mathbf{V}_{\text{per}}, \mathbf{V}_{\text{per}}\right), \tag{5.1}$$

where $\mathbf{V}_{\text{per}} := H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \times H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$. More precisely, we study the holomorphic dependence of $\hat{\mathsf{C}}_r$ (as an element of the complex Banach space of bounded linear operators) on a set $\mathfrak{T}_\delta$, for some $\delta > 0$ to be specified. In so doing, we first need to construct a well-defined extension of $\hat{\mathsf{C}}_r$ to $r \in \mathfrak{T}_\delta$, denoted by $\hat{\mathsf{C}}_{r,\mathbb{C}}$, satisfying the appropriate mapping properties between 1-periodic Sobolev spaces, as indicated in (5.1). Then, in view of Theorem 4.6, we must study the *complex*[1] Fréchet differentiability of the map

$$r \in \mathfrak{T}_\delta \mapsto \hat{\mathsf{C}}_{r,\mathbb{C}} \in \mathscr{L}\left(\mathbf{V}_{\text{per}}, \mathbf{V}_{\text{per}}\right). \tag{5.2}$$

The complex Fréchet differentiability of the map in (5.2) is equivalent to that of the four 1-periodic integral operators contained in the Calderón projector.

This section is structured as follows. In Section 5.1 we investigate holomorphic functions of relevance for the subsequent analysis. In Sections 5.2 and 5.3 we establish shape holomorphy of the single and double layer BIOs, respectively. These results are based on the framework developed in Section 4.2, hence the main task here is to verify that the extension of these 1–periodic integral operators to complex-valued boundary representations satisfy Assumptions 4.10 and 4.12. Then, Theorem 4.13 provides the sought shape holomorphy result.

In Sections 5.4 and 5.5 we establish shape holomorphy of the adjoint double layer and hypersingular BIOs, respectively. Therein, the arguments to obtain shape holomorphy read differently. The technique used in Section 5.4 for the adjoint double layer operator hinges on the result obtained for double layer operator in Section 5.3. For the hypersingular operator, we construct an extension to complex-valued boundary representations using Maue's formula (Lemma 2.7) and the result for the single layer operator.

[1]In this section we study the complex Fréchet differentiability of maps between complex Banach spaces. For the sake of readability, we drop the word "complex" as it is already implied that Fréchet differentiability only in this sense is established here, as we work only with complex Banach spaces.

Shape holomorphy is obtained by observing that this extension can be written as composition of holomophic maps. In Section 5.6 we put together the results for each of the 1-periodic integrals operators analyzed here and establish shape holomorphy of the Calderón projector. Finally, in Section 5.7, we introduce an abstract framework to obtain the holomorphic dependence of the solution of a well-posed BIE on a family of boundary curves in $\mathbb{R}^2$. As an example we consider the boundary integral formulations used to convert the Laplace problem equipped with Dirichlet boundary conditions into a BIE.

5.1. **Holomorphic Boundary Representations.** Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ (in the sense of Definition 3.6). For some $\delta > 0$ to be specified and $r \in \mathfrak{T}_\delta$ we define

$$\mathsf{m}_{r,\mathbb{C}}(t,s) := \begin{cases} \frac{(r(t)-r(s))\cdot(r(t)-r(s))}{\sin^2(\pi(t-s))} & t-s \notin \mathbb{Z}, \\ \frac{(r'\cdot r')(s)}{\pi^2} & t-s\in\mathbb{Z}. \end{cases} \tag{5.3}$$

We present a technical result that will be used extensively throughout this work. Its proof may be found in Appendix B.

Proposition 5.1. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$ (in the sense of Definition 3.6) satisfying Assumption 3.7. Then, for $\delta = \delta(\mathfrak{T}) > 0$ given by*

$$\delta = \frac{1}{2}\inf_{r\in\mathfrak{T}}\left(-\,|r|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \sqrt{|r|^2_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + (\alpha(\mathfrak{T}))^2}\right),$$

where $\alpha(\mathfrak{T}) > 0$ is as in Proposition 3.9, there exists a finite constant $\widetilde{\alpha}(\mathfrak{T},\delta) > 0$ (depending only on $\mathfrak{T}$ and δ) such that

$$\inf_{r\in\mathfrak{T}_\delta}\inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \Re\left\{\mathsf{m}_{r,\mathbb{C}}(t,s)\right\} \geq \widetilde{\alpha}(\mathfrak{T},\delta). \tag{5.4}$$

As explained in Remark 7, we must provide a well-defined holomorphic extension of the BIOs to the set $\mathfrak{T}_\delta$, for some $\delta > 0$. This implies that we have to construct a well-defined holomorphic extension of the map $\mathbf{x}\in\mathbb{R}^2 \mapsto \|\mathbf{x}\| \in \mathbb{R}$. Let us define for $\mathbf{x}\in\mathbb{C}^2$

$$\|\mathbf{x}\|_{\mathbb{C}} := \sqrt{\mathbf{x}\cdot\mathbf{x}}, \quad \text{where} \quad \mathbf{x}\cdot\widetilde{\mathbf{x}} := \sum_{i=1}^{2} x_i\,\widetilde{x}_i$$

for $\mathbf{x} = (x_1,x_2)$ and $\widetilde{\mathbf{x}} = (\widetilde{x}_1,\widetilde{x}_2)$. Here, considering $\mathfrak{U} := \mathbb{C}\backslash(-\infty,0]$, we denote by $\sqrt{\cdot}:\mathfrak{U}\to\mathbb{C}$ the *principal branch* of the square root defined as $\sqrt{z} = \sqrt{r}\exp(\imath\frac{\theta}{2})$, where $z = r\exp(\imath\theta)$ is such that $\theta\in(-\pi,\pi)$. This branch is holomorphic and its complex derivative is $\frac{d}{dz}\sqrt{z} = \frac{1}{2\sqrt{z}}$, for $z\in\mathfrak{U}$.

Lemma 5.2. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1. The maps*

$$r\in\mathfrak{T}_\delta \to r,\, r',\, r'' \in \mathscr{C}^0_{\text{per}}(\mathrm{I},\mathbb{C}^2)$$

are holomorphic and uniformly bounded on the set $\mathfrak{T}_\delta$, i.e. there exists a finite constant $C(\mathfrak{T},\delta) > 0$ (depending on $\mathfrak{T}$ and δ only) such that

$$\sup_{r\in\mathfrak{T}_\delta}\|r\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} \leq C(\mathfrak{T},\delta). \tag{5.5}$$

Proof. For each $r\in\mathfrak{T}_\delta$, there exists a $\widetilde{r}\in\mathfrak{T}$ such that $\|r-\widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} < \delta$. The map $\widetilde{r}\in\mathfrak{T}\mapsto \|\widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)}\in\mathbb{R}$ is continuous and considering that $\mathfrak{T}$ is a compact subset of $\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)$, according to Assumption 3.7, it attains its maximum in $\mathfrak{T}$. Therefore,

$$\|r\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} \leq \|r-\widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} + \|\widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} \leq \underbrace{\delta + \sup_{\widetilde{r}\in\mathfrak{T}}\|\widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)}}_{C(\mathfrak{T},\delta)} < \infty.$$

Consequently, (5.5) holds with a finite constant $C(\mathfrak{T},\delta) > 0$ that depends only on $\mathfrak{T}$ and δ. $\square$

Lemma 5.3. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For all $r\in\mathfrak{T}_\delta$, we have that $\mathsf{m}_{r,\mathbb{C}}\in\mathscr{C}^1_{\text{per}}(\mathrm{I}\times\mathrm{I})$.*
(ii) *The map*

$$r\in\mathfrak{T}_\delta\mapsto \mathsf{m}_{r,\mathbb{C}}\in\mathscr{C}^1_{\text{per}}\left(\mathrm{I}\times\mathrm{I}\right) \tag{5.6}$$

is holomorphic.

(iii) *There exists a finite constant $C_{\mathsf{m}}(\mathfrak{T},\delta)>0$, depending only on $\mathfrak{T}$ and δ only, such that*

$$\sup_{r\in\mathfrak{T}_\delta}\|\mathsf{m}_{r,\mathbb{C}}\|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})}\le C_{\mathsf{m}}(\mathfrak{T},\delta). \tag{5.7}$$

(iv) *The Fréchet derivative of the map in* (5.6) *at $r\in\mathfrak{T}_\delta$ in the direction $\xi\in\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$ reads*

$$\left(\frac{d}{d\,r}\mathsf{m}_{r,\mathbb{C}}\right)[r,\xi](t,s)=2\frac{(r(t)-r(s))\cdot(\xi(t)-\xi(s))}{\sin^2(\pi(t-s))},\quad (t,s)\in\mathrm{I}\times\mathrm{I}\quad\text{and}\quad t-s\notin\mathbb{Z}.$$

Proof. Let $\delta>0$ be as in Proposition 5.1. Firstly, we prove that for all $r\in\mathfrak{T}_\delta$ we have that $\mathsf{m}_{r,\mathbb{C}}(t,s)\in\mathscr{C}^1_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$. Due to Proposition 5.1 $\mathsf{m}_{r,\mathbb{C}}$ is continuously differentiable for $t-s\notin\mathbb{Z}$. The only possible discontinuity is located at $t-s\in\mathbb{Z}$. We analyze the behaviour for $t\in\mathrm{I}$ approaching $s\in\mathrm{I}$. For a fixed $s\in\mathrm{I}$, let us compute

$$\begin{aligned}\lim_{t\to s}\mathsf{m}_{r,\mathbb{C}}(t,s)&=\lim_{t\to s}\frac{(r(t)-r(s))\cdot(r(t)-r(s))}{\sin^2(\pi(t-s))}\\&=\lim_{t\to s}\frac{(t-s)^2}{\sin^2(\pi(t-s))}\left(\int_0^1 r'(s+\zeta(t-s))d\zeta\right)\cdot\left(\int_0^1 r'(s+\zeta(t-s))d\zeta\right)\\&=\frac{(r'\cdot r')(s)}{\pi^2}=\mathsf{m}_{r,\mathbb{C}}(s,s).\end{aligned}$$

The extension to $t-s\in\mathbb{Z}$ is based on the 1-biperiodicity of $\mathsf{m}_{r,\mathbb{C}}(t,s)$. Therefore, $\mathsf{m}_{r,\mathbb{C}}(t,s)\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$. We prove that $\mathsf{m}_{r,\mathbb{C}}(t,s)$ belongs to $\mathscr{C}^1_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$. We calculate

$$\frac{\partial}{\partial t}\mathsf{m}_{r,\mathbb{C}}(t,s)=\begin{cases}2\frac{r'(t)\cdot(r(t)-r(s))}{\sin^2(\pi(t-s))}-2\pi\frac{(r(t)-r(s))\cdot(r(t)-r(s))}{\sin^2(\pi(t-s))}\operatorname{cotg}(\pi(t-s)), & t-s\notin\mathbb{Z},\\ \frac{(r'\cdot r'')(s)}{\pi^2}, & t-s\in\mathbb{Z}.\end{cases}$$

To analyze the behaviour when t approaches $s\in\mathrm{I}$, with a second order Taylor expansion of $r(s)$ at t we get

$$r(s)=r(t)+(s-t)r'(t)+(s-t)^2\int_0^1(1-\zeta)r''(t+\zeta(s-t))d\zeta. \tag{5.8}$$

For $s\in\mathrm{I}$, we compute

$$\begin{aligned}\lim_{t\to s}\frac{\partial}{\partial t}\mathsf{m}_{r,\mathbb{C}}(t,s)=&\lim_{t\to s}2\frac{(r(t)-r(s))}{\sin^2(\pi(t-s))}\cdot\left(r'(t)-\frac{r(t)-r(s)}{t-s}\right)\\&+\lim_{t\to s}2\pi\frac{(r(t)-r(s))\cdot(r(t)-r(s))}{\sin^2(\pi(t-s))}\left(\frac{1}{\pi(t-s)}-\operatorname{cotg}(\pi(t-s))\right).\end{aligned}$$

Using (5.8),

$$\begin{aligned}&\lim_{t\to s}\frac{2(r(t)-r(s))}{\sin^2(\pi(t-s))}\cdot\left(r'(t)-\frac{r(t)-r(s)}{t-s}\right)\\&\qquad=\lim_{t\to s}\frac{2(t-s)}{\sin^2(\pi(t-s))}(r(t)-r(s))\cdot\int_0^1(1-\zeta)r''(t+\zeta(s-t))d\zeta\\&\qquad=\lim_{t\to s}\frac{2(t-s)^2}{\sin^2(\pi(t-s))}\int_0^1 r'(t+\zeta(t-s))d\zeta\cdot\int_0^1(1-\zeta)r''(t+\zeta(s-t))d\zeta=\frac{(r'\cdot r'')(s)}{\pi^2}.\end{aligned}$$

Furthermore,

$$\lim_{t\to s}2\pi\frac{(r(t)-r(s))\cdot(r(t)-r(s))}{2\sin^2(\pi(t-s))}\left(\frac{1}{\pi(t-s)}-\operatorname{cotg}(\pi(t-s))\right)=0,$$

and

$$\lim_{t\to s}\frac{\partial}{\partial t}\mathsf{m}_{r,\mathbb{C}}(t,s)=\frac{(r'\cdot r'')(s)}{\pi^2}=\left(\frac{\partial}{\partial t}\mathsf{m}_{r,\mathbb{C}}(t,s)\right)\Big|_{t=s}.$$

Again, the analysis for $t\in\mathrm{I}$ approaching $s+\mathbb{Z}$ can be approached via the 1-biperiodicity of $\frac{\partial}{\partial t}\mathsf{m}_{r,\mathbb{C}}(t,s)$. Therefore, $\frac{\partial}{\partial t}\mathsf{m}_{r,\mathbb{C}}\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$. Similarly, one can prove that $\frac{\partial}{\partial s}\mathsf{m}_{r,\mathbb{C}}\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$ and we have that $\mathsf{m}_{r,\mathbb{C}}\in\mathscr{C}^1_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})$, for all $r\in\mathfrak{T}_\delta$.

We claim that the map $r \in \mathfrak{T}_\delta \mapsto \mathsf{m}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$ is holomorphic and its Fréchet derivative at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$ reads

$$\left(\frac{d}{d\,r}\mathsf{m}_{r,\mathbb{C}}\right)[r,\xi](t,s) := \begin{cases} 2\frac{(r(t)-r(s))\cdot(\xi(t)-\xi(s))}{\sin^2(\pi(t-s))}, & t-s \notin \mathbb{Z}, \\ 2\frac{\left(r'\cdot\xi'\right)(s)}{\pi^2}, & t-s \in \mathbb{Z}. \end{cases}$$

Indeed, we have

$$\left|\mathsf{m}_{r+\xi,\mathbb{C}}(t,s) - \mathsf{m}_{r,\mathbb{C}}(t,s) - \left(\frac{d}{d\,r}\mathsf{m}_{r,\mathbb{C}}\right)[r,\xi](t,s)\right| \leq \begin{cases} \frac{|(\xi(t)-\xi(s))\cdot(\xi(t)-\xi(s))|}{\sin^2(\pi(t-s))}, & t-s \notin \mathbb{Z}, \\ \frac{\left|\left(\xi'\cdot\xi'\right)(s)\right|}{\pi^2}, & t-s \in \mathbb{Z}, \end{cases}$$

and, using Lemma B.1, we obtain

$$\left\|\mathsf{m}_{r+\xi,\mathbb{C}} - \mathsf{m}_{r,\mathbb{C}} - \left(\frac{d}{d\,r}\mathsf{m}_{r,\mathbb{C}}\right)[r,\xi]\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} \leq \frac{1}{\pi^2}\left\|\xi\right\|^2_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)}.$$

Hence, it follows that $\mathsf{m}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$ depends holomorphically on $r \in \mathfrak{T}_\delta$. Using Lemmas B.1 and 5.2 one obtains the uniform boundedness claimed in (5.7). Using the exact same arguments, one can prove that the derivatives of $\mathsf{m}_{r,\mathbb{C}}$ (as elements of $\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})$) depend holomorphically on $\mathfrak{T}_\delta$ and conclude that $r \in \mathfrak{T}_\delta \mapsto \mathsf{m}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I} \times \mathrm{I})$ is holomorphic. □

Let $\mathcal{J}_r(t) := \|r'(t)\|$ for $t \in \mathrm{I}$ denote the Jacobian of the boundary representation $r \in \mathfrak{T}$ of the curve Γ_r. For $r \in \mathfrak{T}_\delta$, with $\delta > 0$ as in Proposition 5.1, we consider an extension of $\mathcal{J}_r$, denoted by $\mathcal{J}_{r,\mathbb{C}}$, to the set $\mathfrak{T}_\delta$ defined as $\mathcal{J}_{r,\mathbb{C}}(t) := \|r'(t)\|_{\mathbb{C}}$, for $t \in \mathrm{I}$. Observe that due to Proposition 5.1 the function $\mathcal{J}_{r,\mathbb{C}}$ is well-defined for each $r \in \mathfrak{T}_\delta$. We study the holomorphic dependence of this function on $r \in \mathfrak{T}_\delta$.

Lemma 5.4. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For all $r \in \mathfrak{T}_\delta$, we have that $\mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$.*

(ii) *The map*

$$r \in \mathfrak{T}_\delta \mapsto \mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I}) \tag{5.9}$$

is holomorphic.

(iii) *There exists a finite constant $C_{\mathcal{J}}(\mathfrak{T},\delta) > 0$ (depending on $\mathfrak{T}$ and δ only) such that*

$$\sup_{r\in\mathfrak{T}_\delta} \|\mathcal{J}_{r,\mathbb{C}}\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \leq C_{\mathcal{J}}(\mathfrak{T},\delta). \tag{5.10}$$

(iv) *The Fréchet derivative of the map in (5.9) at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ reads*

$$\left(\frac{d}{d\,r}\mathcal{J}_{r,\mathbb{C}}\right)[r,\xi](t) = \frac{r'(t)\cdot\xi'(t)}{\|r'(t)\|_{\mathbb{C}}}, \quad t \in \mathrm{I}. \tag{5.11}$$

Proof. Let $\delta > 0$ be as in Proposition 5.1. It follows from Proposition 5.1 that $\|r'(t)\|_{\mathbb{C}} \neq 0$ for $t \in \mathrm{I}$ and $r \in \mathfrak{T}_\delta$. Furthermore, we have that

$$\mathcal{J}'_{r,\mathbb{C}}(t) = \frac{(r'\cdot r'')(t)}{\|r'(t)\|_{\mathbb{C}}}, \quad t \in \mathrm{I}.$$

Since $r \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$, it follows that $\mathcal{J}'_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ and, therefore $\mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$. The map

$$r \in \mathfrak{T}_\delta \mapsto r'\cdot r' \in \mathscr{C}^1_{\text{per}}(\mathrm{I}) \tag{5.12}$$

is holomorphic and its Fréchet derivative at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ is

$$\left(\frac{d}{dr}\left(r'\cdot r'\right)\right)[r,\xi](t) = 2\left(r'\cdot\xi'\right)(t), \quad t \in \mathrm{I}.$$

Due to Proposition 5.1, the holomorphy of the map (5.12) and that of the principal branch of the square root, we may conclude that the map (5.9) is holomorphic. Using the chain rule for Fréchet differentiable maps, we obtain the expression for the Fréchet derivative of the map (5.9), namely (5.11). Finally, recalling Proposition 5.1 and the uniform boundedness of the maps $r \in \mathfrak{T}_\delta \mapsto r',\, r'' \in \mathscr{C}^0_{\text{per}}(\mathrm{I},\mathbb{C}^2)$, established in Lemma 5.2, it follows that (5.10) holds with a constant $C_{\mathcal{J}}(\mathfrak{T},\delta)$ that depends on $\mathfrak{T}$ and δ only. □

As a consequence of Proposition 5.1 and Lemma 5.4, one may obtain the following result.

Corollary 5.5. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For $r \in \mathfrak{T}_\delta$, we have that* $\frac{1}{\mathcal{J}_{r,\mathbb{C}}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$.

(ii) *The map*

$$r \in \mathfrak{T}_\delta \mapsto \frac{1}{\mathcal{J}_{r,\mathbb{C}}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I}) \tag{5.13}$$

is holomorphic.

(iii) *There exists a finite constant* $C_{\frac{1}{\mathcal{J}}}(\mathfrak{T},\delta) > 0$ *(depending on* $\mathfrak{T}$ *and* δ *only) such that*

$$\sup_{r \in \mathfrak{T}_\delta} \left\| \frac{1}{\mathcal{J}_{r,\mathbb{C}}} \right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \leq C_{\frac{1}{\mathcal{J}}}(\mathfrak{T},\delta).$$

(iv) *The Fréchet derivative of the map in* (5.13) *at* $r \in \mathfrak{T}_\delta$ *in the direction* $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ *reads*

$$\left(\frac{d}{dr}\frac{1}{\mathcal{J}_{r,\mathbb{C}}}\right)[r,\xi] = -\frac{1}{\mathcal{J}^2_{r,\mathbb{C}}}\left(\frac{d}{dr}\mathcal{J}_{r,\mathbb{C}}\right)[r,\xi].$$

5.2. **Shape Holomorphy of the Single Layer Operator.** We analyze the map

$$r \in \mathfrak{T} \mapsto \hat{\mathsf{V}}_r \in \mathscr{L}\big(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\big),$$

where $\hat{\mathsf{V}}_r \coloneqq \tau_r \circ \mathsf{V}_r \circ \tau_r^{-1}$ yields a representation of V_r in the reference domain $\mathrm{I} = [0,1]$ with the notation $\mathsf{V}_r \coloneqq \mathsf{V}_{\Gamma_r}$, for $r \in \mathfrak{T}$. The explicit representation of the operator V_r given in Lemma 2.6 provides for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$

$$\left(\hat{\mathsf{V}}_r\,\hat{\varphi}\right)(t) = \int_0^1 \mathsf{v}_r(t,s)\hat{\varphi}(s)\mathcal{J}_r(s)ds, \quad t \in \mathrm{I},$$

where

$$\mathsf{v}_r(t,s) \coloneqq -\frac{1}{2\pi}\log\|r(t)-r(s)\|, \quad (t,s) \in \mathrm{I}\times\mathrm{I} \quad \text{and} \quad t-s \notin \mathbb{Z}.$$

We decompose $\hat{\mathsf{V}}_r$ as follows

$$\hat{\mathsf{V}}_r = \mathsf{S}_r + \mathsf{G}_r,$$

where, for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$, the operator

$$(\mathsf{S}_r\,\hat{\varphi})(t) \coloneqq -\frac{1}{4\pi}\int_0^1 \log\left(4\sin^2(\pi(t-s))\right)\mathcal{J}_r(s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I},$$

contains the logarithmic singularity of $\hat{\mathsf{V}}_r$ and for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$

$$(\mathsf{G}_r\,\hat{\varphi})(t) \coloneqq \int_0^1 \mathsf{g}_r(t,s)\mathcal{J}_r(s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I},$$

is a 1-periodic integral operator with $\mathsf{g}_r : \mathrm{I}\times\mathrm{I} \to \mathbb{R}$ defined as

$$\mathsf{g}_r(t,s) \coloneqq -\frac{1}{4\pi}\log\left(\frac{\|r(t)-r(s)\|^2}{4\sin^2(\pi(t-s))}\right), \quad (t,s) \in \mathrm{I}\times\mathrm{I} \quad \text{and} \quad t-s \notin \mathbb{Z}.$$

Let $\mathrm{Log}\ z : \mathfrak{U} \to \mathbb{C}$ be the *principal branch of the logarithm* (recall that $\mathfrak{U} \coloneqq \mathbb{C}\backslash(-\infty,0]$). For a complex number $z = r\exp(\imath\theta)$ with $-\pi < \theta < \pi$ we have that $\mathrm{Log}\ z = \log\ r + \imath\theta$. The principal branch of the logarithm is holomorphic and its complex derivative is $\frac{d}{dz}\mathrm{Log}(z) = z^{-1}$ [16, Corollary 2.21]. For $\delta > 0$ as in Proposition 5.1, we consider an extension $\mathsf{g}_{r,\mathbb{C}} : \mathrm{I}\times\mathrm{I} \to \mathbb{C}$ of g_r to the set $\mathfrak{T}_\delta$ given by

$$\mathsf{g}_{r,\mathbb{C}}(t,s) \coloneqq -\frac{1}{4\pi}\mathrm{Log}\left(\frac{(r(t)-r(s))\cdot(r(t)-r(s))}{4\sin^2(\pi(t-s))}\right), \quad (t,s) \in \mathrm{I}\times\mathrm{I} \quad \text{and} \quad t-s \notin \mathbb{Z}.$$

Observe that $\mathsf{g}_{r,\mathbb{C}}$ is actually the composition of the principal branch of the logarithm and $\mathsf{m}_{r,\mathbb{C}} : \mathrm{I}\times\mathrm{I} \to \mathbb{C}$ defined in (5.3). It follows from Proposition 5.1 that $\mathsf{g}_{r,\mathbb{C}}$ is well-defined for each $r \in \mathfrak{T}_\delta$. This is due to the the fact that the argument of the logarithm has a real part that is bounded from below away from zero, according to Proposition 5.1.

We define an extension of the 1-periodic integral operator $\hat{\mathsf{V}}_r$ to the set $\mathfrak{T}_\delta$ as follows

$$\hat{\mathsf{V}}_{r,\mathbb{C}} \coloneqq \mathsf{S}_{r,\mathbb{C}} + \mathsf{G}_{r,\mathbb{C}}, \tag{5.14}$$

where for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$

$$\left(\mathsf{S}_{r,\mathbb{C}}\, \hat{\varphi}\right)(t) \coloneqq -\frac{1}{4\pi} \int_0^1 \log\left(4\sin^2\left(\pi(t-s)\right)\right) \mathcal{J}_{r,\mathbb{C}}(s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I},$$

and

$$\left(\mathsf{G}_{r,\mathbb{C}}\, \hat{\varphi}\right)(t) \coloneqq \int_0^1 \mathsf{g}_{r,\mathbb{C}}(t,s)\mathcal{J}_{r,\mathbb{C}}(s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I}.$$

The following result provides the regularity of $\mathsf{g}_{r,\mathbb{C}}$ and establishes its holomorphic dependence on the set $\mathfrak{T}_\delta$, with $\delta > 0$ as in Proposition 5.1

Lemma 5.6. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, we have $\mathsf{g}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I}\times\mathrm{I})$.*

(ii) *The map*

$$r \in \mathfrak{T}_\delta \mapsto \mathsf{g}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}\left(\mathrm{I}\times\mathrm{I}\right) \tag{5.15}$$

is holomorphic and uniformly bounded on the set $\mathfrak{T}_\delta$,

(iii) *There exists a finite constant $C_{\mathsf{g}}(\mathfrak{T},\delta) > 0$ (depending on $\mathfrak{T}$ and δ only) such that*

$$\sup_{r\in\mathfrak{T}_\delta} \|\mathsf{g}_{r,\mathbb{C}}\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I}\times\mathrm{I})} \leq C_{\mathsf{g}}(\mathfrak{T},\delta). \tag{5.16}$$

(iv) *The Fréchet derivative of the map in (5.15) at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ reads*

$$\left(\frac{d}{d\,r}\mathsf{g}_{r,\mathbb{C}}\right)[r,\xi](t,s) = -\frac{1}{2\pi}\frac{(r(t)-r(s))\cdot(\xi(t)-\xi(s))}{(r(t)-r(s))\cdot(r(t)-r(s))}, \quad (t,s)\in\mathrm{I}\times\mathrm{I} \quad \text{and} \quad t-s\notin\mathbb{Z}. \tag{5.17}$$

Proof. From Proposition 5.1, we have

$$\Re\left\{\mathsf{m}_{r,\mathbb{C}}(t,s)\right\} \geq \widetilde{\alpha}(\mathfrak{T}) > 0 \quad \text{for all } (t,s)\in\mathrm{I}\times\mathrm{I}, \tag{5.18}$$

where $\delta > 0$ is as in Proposition 5.1. The function $\mathsf{g}_{r,\mathbb{C}} : \mathrm{I}\times\mathrm{I}\to\mathbb{C}$ corresponds to the composition of the principal branch of the logarithm and $\mathsf{m}_{r,\mathbb{C}} : \mathrm{I}\times\mathrm{I}\to\mathbb{C}$. Due to (5.18), we have that $\mathsf{g}_{r,\mathbb{C}}$ is holomorphic and its Fréchet derivative at $r\in\mathfrak{T}_\delta$ in the direction $\xi\in\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ is given in (5.17). From Proposition 5.1 and Lemma 5.3 it follows that the map $r\in\mathfrak{T}_\delta\mapsto\mathsf{m}_{r,\mathbb{C}}\in\mathscr{C}^1_{\text{per}}(\mathrm{I}\times\mathrm{I})$ is uniformly bounded from below away from zero and, considering that the principal branch of the logarithm is holomorphic in $\mathfrak{U}$, we conclude that (5.16) holds. ☐

We proceed to prove that $\mathsf{G}_{r,\mathbb{C}}$ and $\mathsf{S}_{r,\mathbb{C}}$ are bounded linear operator for each $r:\mathrm{I}\to\mathbb{C}^2$ belonging to the set $\mathfrak{T}_\delta$. Furthermore, we show their holomorphic dependence upon the set $\mathfrak{T}_\varepsilon$, for some $\varepsilon > 0$ that depends on $\delta > 0$ from Proposition 5.1.

Remark 9. In the following, we adopt the notation $\lesssim$ to denote boundedness up to a multiplicative constant that is completely independent of any complex-valued boundary representation $r\in\mathfrak{T}_\delta$ and of the set $\mathfrak{T}_\delta$ itself.

Lemma 5.7. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r\in\mathfrak{T}_\delta$, the 1-periodic integral operator $\mathsf{S}_{r,\mathbb{C}} : H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\to H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded.*

(ii) *For any $\varepsilon\in(0,\delta)$, the map*

$$r\in\mathfrak{T}_\varepsilon\mapsto\mathsf{S}_{r,\mathbb{C}}\in\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right) \tag{5.19}$$

is holomorphic.

(iii) *There exists a constant $C_{\mathsf{S}}(\mathfrak{T},\delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that for any $\varepsilon\in(0,\delta)$*

$$\sup_{r\in\mathfrak{T}_\varepsilon}\|\mathsf{S}_{r,\mathbb{C}}\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}),H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \leq C_{\mathsf{S}}(\mathfrak{T},\delta). \tag{5.20}$$

(iii) *The Fréchet derivative of the map in (5.19) at $r\in\mathfrak{T}_\varepsilon$ in the direction $\xi\in\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ and for $\hat{\varphi}\in\mathscr{C}^0_{\text{per}}(\mathrm{I})$ reads*

$$\left(\frac{d}{dr}\mathsf{S}_{r,\mathbb{C}}\,\hat{\varphi}\right)[r,\xi](t) \coloneqq -\frac{1}{4\pi}\int_0^1\log\left(4\sin^2\left(\pi(t-s)\right)\right)\left(\frac{d}{d\,r}\mathcal{J}_{r,\mathbb{C}}\right)[r,\xi](s)\hat{\varphi}(s)ds, \quad t\in\mathrm{I}.$$

Proof. The 1-periodic integral operator $\mathsf{S}_{r,\mathbb{C}}$ fits the framework of Section 4.2 and satisfies Assumption 4.10 with $\mathsf{f}(t) = \log(4\sin^2(\pi(t))$ and $\mathsf{p}_{r,\mathbb{C}}(t,s) = \mathscr{J}_{r,\mathbb{C}}(s)$. It remains to prove that $\mathsf{S}_{r,\mathbb{C}}$ fulfils Assumption 4.12 with $\delta > 0$ as in Proposition 5.1.

(i) It follows from Lemma 5.4 that for each $r \in \mathfrak{T}_\delta$ we have $\mathscr{J}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$.

(ii) The operator $\widetilde{\mathsf{S}} : H^p_{\text{per}}(\mathrm{I}) \to H^{p+1}_{\text{per}}(\mathrm{I})$ defined for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ as

$$\left(\widetilde{\mathsf{S}}\,\hat{\varphi}\right)(t) := -\frac{1}{4\pi}\int_0^1 \log\left(4\sin^2\left(\pi(t-s)\right)\right)\hat{\varphi}(s)ds, \quad t \in \mathrm{I},$$

is linear and bounded for all $p \in \mathbb{R}$ [37, Theorem 8.29]. According to Lemma 2.5 we have

$$\left\|\mathscr{J}_{r,\mathbb{C}}\,\hat{\varphi}\right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})} \lesssim \left\|\mathscr{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \left\|\hat{\varphi}\right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})}, \tag{5.21}$$

for $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$. Observe that $\mathsf{S}_{r,\mathbb{C}}\,\hat{\varphi} = \widetilde{\mathsf{S}}\left(\mathscr{J}_{r,\mathbb{C}}\,\hat{\varphi}\right)$. Using the mapping properties of $\widetilde{\mathsf{S}}$ and (5.21), we obtain for $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$

$$\begin{aligned}\left\|\widetilde{\mathsf{S}}\left(\mathscr{J}_{r,\mathbb{C}}\,\hat{\varphi}\right)\right\|_{H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})} &\leq \left\|\widetilde{\mathsf{S}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \left\|\mathscr{J}_{r,\mathbb{C}}\,\hat{\varphi}\right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})} \\ &\lesssim \left\|\mathscr{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \left\|\widetilde{\mathsf{S}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \left\|\hat{\varphi}\right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})}.\end{aligned} \tag{5.22}$$

Recalling that $\mathscr{C}^1_{\text{per}}(\mathrm{I})$ is dense in $H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$, we conclude that $\mathsf{S}_{r,\mathbb{C}} : H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded.

(iii) Using (5.22) and the uniform boundedness of the map $r \in \mathfrak{T}_\delta \mapsto \mathscr{J}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ established in Lemma 5.4 we get

$$\begin{aligned}\left\|\mathsf{S}_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} &\lesssim \left\|\mathscr{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \left\|\widetilde{\mathsf{S}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \\ &\lesssim \left\|\widetilde{\mathsf{S}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} C_{\mathscr{J}}(\mathfrak{T}, \delta).\end{aligned}$$

Hence, the map $r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{S}_{r,\mathbb{C}} \in \mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)$ is uniformly bounded, for any $\varepsilon \in (0, \delta]$. Therefore, (5.20) holds for a finite constant $C_{\mathsf{S}}(\mathfrak{T}, \delta) > 0$, depending upon $\mathfrak{T}$ and δ only.

(iv) The map $r \in \mathfrak{T}_\delta \mapsto \mathscr{J}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ is holomorphic according to Lemma 5.4.

Therefore, Theorem 4.13 asserts that the map in (5.19) is holomorphic for any $\varepsilon \in (0, \delta)$ and provides the expression of the corresponding Fréchet derivative. $\square$

The following lemma establishes the holomorphic dependence of the 1-periodic integral operator $\mathsf{G}_{r,\mathbb{C}}$ on $r \in \mathfrak{T}_\varepsilon$, for some $\varepsilon > 0$ to be specified.

Lemma 5.8. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, the 1-periodic integral operator $\mathsf{G}_{r,\mathbb{C}} : H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded.*

(ii) *For any $\varepsilon \in (0, \delta)$, the map*

$$r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{G}_{r,\mathbb{C}} \in \mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right) \tag{5.23}$$

is holomorphic and uniformly bounded on the set $\mathfrak{T}_\varepsilon$,

(iii) *There exists a constant $C_{\mathsf{G}}(\mathfrak{T}, \delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that for any $\varepsilon \in (0, \delta)$*

$$\sup_{r\in\mathfrak{T}_\varepsilon} \left\|\mathsf{G}_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \leq C_{\mathsf{G}}(\mathfrak{T}, \delta). \tag{5.24}$$

(iv) *The Fréchet derivative of the map in (5.23) at $r \in \mathfrak{T}_\varepsilon$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$ and for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ reads*

$$\left(\frac{d}{dr}\,\mathsf{G}_{r,\mathbb{C}}\,\hat{\varphi}\right)[r, \xi](t) := \int_0^1 \left(\frac{d}{dr}\mathsf{g}_{r,\mathbb{C}}\,\mathscr{J}_{r,\mathbb{C}}\right)[r, \xi](t,s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I}.$$

Proof. The 1-periodic integral operator $\mathsf{G}_{r,\mathbb{C}}$ fits the framework of Section 4.2 and satisfies Assumption 4.10 with $\mathsf{f}(t) = 1$ and $\mathsf{p}_{r,\mathbb{C}}(t,s) = \mathsf{g}_{r,\mathbb{C}}(t,s)\mathscr{J}_{r,\mathbb{C}}(s)$. We show that Assumption 4.12 with $\delta > 0$ as in Proposition 5.1 is fulfilled by $\mathsf{G}_{r,\mathbb{C}}$.

(i) According to Lemmas 5.4 and 5.6 for each $r \in \mathfrak{T}_\delta$ we have $\mathsf{g}_{r,\mathbb{C}}\mathscr{J}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$.

(ii) For $r \in \mathfrak{T}_\delta$ and $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$, we obtain

$$\left(\mathsf{G}_{r,\mathbb{C}}\hat{\varphi}\right)'(t) = \int_0^1 \frac{\partial}{\partial t}\mathsf{g}_{r,\mathbb{C}}(t,s)\,\mathcal{J}_{r,\mathbb{C}}(s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I}.$$

Therefore,

$$\left\|\mathsf{G}_{r,\mathbb{C}}\,\hat{\varphi}\right\|_{L^2(\mathrm{I})} \leq \left\|\mathsf{g}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})}\left\|\mathcal{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I})}\left\|\hat{\varphi}\right\|_{L^2(\mathrm{I})}$$

and

$$\left\|\left(\mathsf{G}_{r,\mathbb{C}}\,\hat{\varphi}\right)'\right\|_{L^2(\mathrm{I})} \leq \left\|\frac{\partial}{\partial t}\mathsf{g}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})}\left\|\mathcal{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I})}\left\|\hat{\varphi}\right\|_{L^2(\mathrm{I})}.$$

Recalling that in $\mathscr{C}^0_{\text{per}}(\mathrm{I})$ is dense in $L^2(\mathrm{I})$, that $L^2(\mathrm{I})$ can be identified with $H^0_{\text{per}}(\mathrm{I})$ and the norm equivalence stated in Lemma 2.3, we conclude that $\mathsf{G}_{r,\mathbb{C}} : H^0_{\text{per}}(\mathrm{I}) \to H^1_{\text{per}}(\mathrm{I})$ is linear and bounded. Furthermore, we obtain

$$\left\|\mathsf{G}_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^0_{\text{per}}(\mathrm{I}),H^1_{\text{per}}(\mathrm{I})\right)} \lesssim \left(\left\|\mathsf{g}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} + \left\|\frac{\partial}{\partial t}\mathsf{g}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})}\right)\left\|\mathcal{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})}.$$

The $\dagger$-adjoint operator of $\mathsf{G}_{r,\mathbb{C}}$ is given by

$$\left(\mathsf{G}^\dagger_{r,\mathbb{C}}\,\hat{\varphi}\right)(s) = \int_0^1 \mathcal{J}_{r,\mathbb{C}}(s)\mathsf{g}_{r,\mathbb{C}}(t,s)\hat{\varphi}(t)dt, \quad s \in \mathrm{I}.$$

It follows that $\mathsf{G}^\dagger_{r,\mathbb{C}} : H^0_{\text{per}}(\mathrm{I}) \to H^1_{\text{per}}(\mathrm{I})$ is linear and bounded and that

$$\left\|\mathsf{G}^\dagger_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^0_{\text{per}}(\mathrm{I}),H^1_{\text{per}}(\mathrm{I})\right)} \lesssim \left(\left\|\mathsf{g}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} + \left\|\frac{\partial}{\partial s}\mathsf{g}_{r,\mathbb{C}}\right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})}\right)\left\|\mathcal{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})},$$

For $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$, we have

$$\begin{aligned}\left\|\mathsf{G}_{r,\mathbb{C}}\,\hat{\varphi}\right\|_{H^0_{\text{per}}(\mathrm{I})} &= \sup_{0\neq\hat{\psi}\in H^0_{\text{per}}(\mathrm{I})}\frac{\left|\left\langle\mathsf{G}_{r,\mathbb{C}}\,\hat{\varphi},\hat{\psi}\right\rangle_{\text{per}}\right|}{\left\|\hat{\psi}\right\|_{H^0_{\text{per}}(\mathrm{I})}} \\ &= \sup_{0\neq\hat{\psi}\in H^0_{\text{per}}(\mathrm{I})}\frac{\left|\left\langle\hat{\varphi},\mathsf{G}^\dagger_{r,\mathbb{C}}\,\hat{\psi}\right\rangle_{\text{per}}\right|}{\left\|\hat{\psi}\right\|_{H^0_{\text{per}}(\mathrm{I})}} \leq \left\|\mathsf{G}^\dagger_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^0_{\text{per}}(\mathrm{I}),H^1_{\text{per}}(\mathrm{I})\right)}\left\|\hat{\varphi}\right\|_{H^{-1}_{\text{per}}(\mathrm{I})}.\end{aligned}$$

Again, considering that in $\mathscr{C}^1_{\text{per}}(\mathrm{I})$ is dense in $H^{-1}_{\text{per}}(\mathrm{I})$, we conclude that $\mathsf{G}_{r,\mathbb{C}} : H^{-1}_{\text{per}}(\mathrm{I}) \to H^0_{\text{per}}(\mathrm{I})$ is linear and bounded. Recalling Lemma 2.4, we conclude that for each $r \in \mathfrak{T}_\delta$ we have that $\mathsf{G}_{r,\mathbb{C}} : H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \mapsto H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded. Moreover, the following bounds hold

$$\left\|\mathsf{G}_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}),H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \lesssim \left\|\mathsf{g}_{r,\mathbb{C}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I}\times\mathrm{I})}\left\|\mathcal{J}_{r,\mathbb{C}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})}. \tag{5.25}$$

(iii) Using (5.25), the uniform boundedness of $r \in \mathfrak{T}_\delta \mapsto \mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ (Lemma 5.4) and that of $r \in \mathfrak{T}_\delta \mapsto \mathsf{g}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ (Lemma 5.6) we have that the map $r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{G}_{r,\mathbb{C}} \in \mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)$ is uniformly bounded, for any $\varepsilon \in (0,\delta]$. Hence, (5.24) holds for a finite constant $C_{\mathsf{G}}(\mathfrak{T},\delta) > 0$ that depends on $\mathfrak{T}$ and δ only.

(iv) The map $r \in \mathfrak{T}_\delta \mapsto \mathsf{g}_{r,\mathbb{C}}\,\mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})$ is holomorphic according to Lemmas 5.4 and 5.6.

Theorem 4.13 asserts that the map in (5.19) is holomorphic and provides the expression for the corresponding Fréchet derivative. □

Lemmas 5.7 and 5.8 allow us to establish the uniform boundedness of the operator $\hat{\mathsf{V}}_{r,\mathbb{C}}$ and its holomorphic dependence on the set $\mathfrak{T}_\varepsilon$, for some $\varepsilon > 0$ that depends on $\delta > 0$ from Proposition 5.1.

Theorem 5.9. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, the 1-periodic integral operator $\hat{\mathsf{V}}_{r,\mathbb{C}} : H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded.*

(ii) *For any $\varepsilon \in (0,\delta)$, the map*

$$r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{V}}_{r,\mathbb{C}} \in \mathscr{L}\big(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\big) \tag{5.26}$$

is holomorphic.

(iii) *there exists a constant $C_{\hat{\mathsf{V}}}(\mathfrak{T},\delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that for any $\varepsilon \in (0,\delta)$*

$$\sup_{r\in\mathfrak{T}_\varepsilon} \left\|\hat{\mathsf{V}}_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \leq C_{\hat{\mathsf{V}}}(\mathfrak{T},\delta).$$

(iv) *The Fréchet derivative of the map in* (5.26) *at $r \in \mathfrak{T}_\varepsilon$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ reads*

$$\left(\frac{d}{dr}\hat{\mathsf{V}}_{r,\mathbb{C}}\right)[r,\xi] = \left(\frac{d}{dr}\mathsf{S}_{r,\mathbb{C}}\right)[r,\xi] + \left(\frac{d}{dr}\mathsf{G}_{r,\mathbb{C}}\right)[r,\xi].$$

Proof. The result follows directly from Lemmas 5.7 and 5.8 together with (5.14). □

5.3. **Shape Holomorphy of the Double Layer Operator.** We analyze the map

$$r \in \mathfrak{T} \mapsto \hat{\mathsf{K}}_r \in \mathscr{L}\big(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\big),$$

where $\hat{\mathsf{K}}_r := \tau_r \circ \mathsf{K}_r \circ \tau_r^{-1}$ yields a representation of K_r in the reference domain $\mathrm{I} = [0,1]$ with the notation $\mathsf{K}_r := \mathsf{K}_{\Gamma_r}$, for $r \in \mathfrak{T}$. The explicit representation of the double layer operator K_r on a boundary curve of class $\mathscr{C}^2$ (*cf.* Lemma 2.6), yields the following expression of the operator $\hat{\mathsf{K}}_r$ for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$

$$\left(\hat{\mathsf{K}}_r\,\hat{\varphi}\right)(t) = \int_0^1 \mathsf{k}_r(t,s)\mathcal{J}_r(s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I},$$

Here, $\mathsf{k}_r : \mathrm{I}\times\mathrm{I} \to \mathbb{R}$ is given by

$$\mathsf{k}_r(t,s) := \frac{1}{2\pi}\frac{(r(t)-r(s))\cdot\hat{\boldsymbol{\nu}}_r(s)}{\|r(t)-r(s)\|^2}, \quad (t,s) \in \mathrm{I}\times\mathrm{I}, \quad \text{and} \quad t-s \notin \mathbb{Z},$$

where $\hat{\boldsymbol{\nu}}_r := \tau_r \boldsymbol{\nu}_{\Gamma_r}$ is given (for a positively oriented Jordan curves) by

$$\hat{\boldsymbol{\nu}}_r(s) = \frac{[r'(s)]^\perp}{\|r'(s)\|}, \quad s \in \mathrm{I}.$$

In the previous expression, for a vector $\boldsymbol{v} = (v_1,v_2) \in \mathbb{R}^2$ we use the notation $[\boldsymbol{v}]^\perp := (v_2,-v_1)$. We proceed to define an extension of the double layer operator $\hat{\mathsf{K}}_r$ to the set $\mathfrak{T}_\delta$. In so doing, we define first the corresponding extension for k_r. For $r \in \mathfrak{T}_\delta$, with $\delta > 0$ as in Proposition 5.1, we set

$$\mathsf{k}_{r,\mathbb{C}}(t,s) := \frac{1}{2\pi}\frac{(r(t)-r(s))\cdot\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s)}{(r(t)-r(s))\cdot(r(t)-r(s))}, \quad (t,s) \in \mathrm{I}\times\mathrm{I} \quad \text{and} \quad t-s \notin \mathbb{Z}, \tag{5.27}$$

where

$$\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) := \frac{[r'(s)]^\perp}{\|r'(s)\|_{\mathbb{C}}}, \quad s \in \mathrm{I},$$

is an extension of $\hat{\boldsymbol{\nu}}_r$ to the set $\mathfrak{T}_\delta$. We define an extension of the operator $\hat{\mathsf{K}}_r$ to the set $\mathfrak{T}_\delta$ and for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ as follows

$$\left(\hat{\mathsf{K}}_{r,\mathbb{C}}\,\hat{\varphi}\right)(t) = \int_0^1 \mathsf{k}_{r,\mathbb{C}}(t,s)\mathcal{J}_{r,\mathbb{C}}(s)\hat{\varphi}(s)ds, \quad t \in \mathrm{I}.$$

To facilitate the forthcoming analysis, we define for $r \in \mathfrak{T}_\delta$

$$\mathsf{n}_{r,\mathbb{C}}(t,s) := -\mathsf{k}_{r,\mathbb{C}}(t,s)\sin(\pi(t-s)), \quad (t,s) \in \mathrm{I}\times\mathrm{I}, \quad \text{and} \quad t-s \notin \mathbb{Z}.$$

Lemma 5.10. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, we have $\hat{\boldsymbol{\nu}}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{C}^2)$.*

(ii) *The map*

$$r \in \mathfrak{T}_\delta \mapsto \hat{\boldsymbol{\nu}}_{r,\mathbb{C}} \in \mathscr{C}^1_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2) \tag{5.28}$$

is holomorphic and uniformly bounded on the set of complex-valued, i.e. there exists a finite constant $C_{\hat{\boldsymbol{\nu}}}(\mathfrak{T}, \delta) > 0$ (depending only on $\mathfrak{T}$ and δ only) such that

$$\sup_{r \in \mathfrak{T}_\delta} \|\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}\|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)} \leq C_{\hat{\boldsymbol{\nu}}}(\mathfrak{T}, \delta). \tag{5.29}$$

(iii) *The Fréchet derivative of the map in* (5.28) *at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2)$ is*

$$\left(\frac{d}{dr}\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}\right)[r, \xi](s) = \frac{[\xi'(s)]^\perp}{\|r'(s)\|_\mathbb{C}} - [r'(s)]^\perp \frac{r'(s) \cdot \xi'(s)}{\|r'(s)\|^3_\mathbb{C}}, \quad s \in \mathrm{I}.$$

Proof. Let $\delta > 0$ be as in Proposition 5.1. Observe that for all $r \in \mathfrak{T}_\delta$ the extension of the normal derivative $\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}$ is well-defined due to Proposition 5.1. Furthermore, being the quotient of 1-periodic continuously differentiable functions with a denominator that does not vanish, it follows that $\hat{\boldsymbol{\nu}}_{r,\mathbb{C}} \in \mathscr{C}^1_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2)$. The holomorphy of the map in (5.28) is a consequence of that of the map in (5.9), the fact that $\|r'(s)\|_\mathbb{C} \neq 0$ for $s \in \mathrm{I}$ and the linearity of the operator $[\cdot]^\perp$. The statement for the Fréchet derivative of the map (5.28) can be deduced using the Fréchet derivative of the map (5.13) and the product rule for Fréchet differentiable maps. The uniform boundedness claimed in (5.29) follows from Proposition 5.1 and Lemma 5.2. $\square$

Lemma 5.11. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, we have $\mathsf{k}_{r,\mathbb{C}} \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I} \times \mathrm{I})$ and $\mathsf{n}_{r,\mathbb{C}} \in \mathscr{C}^1_{\mathrm{per}}(\mathrm{I} \times \mathrm{I})$.*

(ii) *The map*

$$r \in \mathfrak{T}_\delta \mapsto \mathsf{k}_{r,\mathbb{C}} \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I} \times \mathrm{I}) \tag{5.30}$$

with $\mathsf{k}_{r,\mathbb{C}}$ as defined in (5.27) *is holomorphic.*

(iii) *There exists a finite constant $C_\mathsf{k}(\mathfrak{T}, \delta) > 0$ (depending on $\mathfrak{T}$ and δ only) such that*

$$\sup_{r \in \mathfrak{T}_\delta} \|\mathsf{k}_{r,\mathbb{C}}\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})} \leq C_\mathsf{k}(\mathfrak{T}, \delta).$$

(iv) *The Fréchet derivative of the map in* (5.30) *at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2)$ and for $(t, s) \in \mathrm{I} \times \mathrm{I}$ such that $t - s \notin \mathbb{Z}$ reads*

$$\begin{aligned}\left(\frac{d}{dr}\mathsf{k}_{r,\mathbb{C}}\right)[r, \xi](t, s) =& \frac{1}{2\pi} \frac{(\xi(t) - \xi(s)) \cdot \hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) + (r(t) - r(s)) \cdot \left(\frac{d}{dr}\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}\right)[r, \xi](s)}{(r(t) - r(s)) \cdot (r(t) - r(s))} \\ &- \frac{1}{\pi} \frac{(r(t) - r(s)) \cdot \hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s)\,(\xi(t) - \xi(s)) \cdot (r(t) - r(s))}{[(r(t) - r(s)) \cdot (r(t) - r(s))]^2}.\end{aligned}$$

(v) *There exists a finite constant $C_\mathsf{n}(\mathfrak{T}, \delta) > 0$ (depending on $\mathfrak{T}$ and δ only) such that*

$$\sup_{r \in \mathfrak{T}_\delta} \|\mathsf{n}_{r,\mathbb{C}}\|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})} \leq C_\mathsf{n}(\mathfrak{T}, \delta).$$

Proof. Let $\delta > 0$ be as in Proposition 5.1. Observe that $\mathsf{k}_{r,\mathbb{C}}$ is well-defined for $r \in \mathfrak{T}_\delta$, due to Proposition 5.1. We claim that $\mathsf{k}_{r,\mathbb{C}} : \mathrm{I} \times \mathrm{I} \to \mathbb{C}$ is continuous. Indeed, $\mathsf{k}_{r,\mathbb{C}}$ can be continuously extended as follows

$$\mathsf{k}_{r,\mathbb{C}}(t, s) := \begin{cases} \frac{1}{2\pi} \frac{\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) \cdot (r(t) - r(s))}{(r(t) - r(s)) \cdot (r(t) - r(s))}, & t - s \notin \mathbb{Z}, \\ \frac{1}{4\pi} \frac{\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) \cdot r''(s)}{(r' \cdot r')(s)}, & t - s \in \mathbb{Z}. \end{cases}$$

We now study the behaviour when t tends to s. Since $r \in \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2)$, for $(t, s) \in \mathrm{I} \times \mathrm{I}$ such that $|t - s| < \frac{1}{2}$ Taylor's expansion delivers

$$r(t) = r(s) + (t - s)r'(s) + (t - s)^2 \int_0^1 (1 - \zeta)r''(s + \zeta(t - s))d\zeta.$$

Observe that $r'(s) \cdot \hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) = 0$, for $s \in \mathrm{I}$. Let us calculate

$$\begin{aligned}
\lim_{t\to s} \mathsf{k}_{r,\mathbb{C}}(t,s) &= \frac{1}{2\pi} \lim_{t\to s} \frac{\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) \cdot (r(t) - r(s))}{(r(t) - r(s)) \cdot (r(t) - r(s))} \\
&= \frac{1}{2\pi} \lim_{t\to s} \frac{\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) \cdot \int_0^1 (1-\zeta) r''(s + \zeta(t-s)) d\zeta}{\left(\int_0^1 r'(s+\zeta(t-s))d\zeta\right) \cdot \left(\int_0^1 r'(s+\zeta(t-s))d\zeta\right)} \\
&= \frac{1}{4\pi} \frac{\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) \cdot r''(s)}{(r' \cdot r')(s)} \\
&= \mathsf{k}_{r,\mathbb{C}}(s,s).
\end{aligned}$$

If follows straightforwardly that $\mathsf{n}_{r,\mathbb{C}} \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I} \times \mathrm{I})$. Moreover, we have

$$\mathsf{n}_{r,\mathbb{C}}(t,s) = -\frac{1}{8\pi} \frac{\frac{(r(t)-r(s))\cdot \hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s)}{\sin(\pi(t-s))}}{\mathsf{m}_{r,\mathbb{C}}(t,s)}, \quad (t,s) \in (\mathrm{I} \times \mathrm{I}).$$

Hence, for each $r \in \mathfrak{T}_\delta$ the function $\mathsf{n}_{r,\mathbb{C}}$ can be expressed as the quotient of 1–biperiodic, continuously differentiably function with a nonvanishing denominator, according to Proposition 5.1. Observe that for each for each $r \in \mathfrak{T}_\delta$

$$\mathsf{k}_{r,\mathbb{C}}(t,s) = -\frac{1}{8\pi} \frac{\frac{\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s)\cdot(r(t)-r(s))}{\sin^2(\pi(t-s))}}{\mathsf{m}_{r,\mathbb{C}}(t,s)}, \quad (t,s) \in (\mathrm{I} \times \mathrm{I}).$$

It follows from Proposition 5.1 and Lemma 5.10 together with the holomorphy of the map

$$r \in \mathfrak{T}_\delta \mapsto \frac{\hat{\boldsymbol{\nu}}_{r,\mathbb{C}}(s) \cdot (r(t) - r(s))}{\sin^2(\pi(t-s))} \in \mathscr{C}^0_{\mathrm{per}}(\mathrm{I} \times \mathrm{I})$$

that the map in (5.30) is holomorphic as well. □

Proposition 5.12. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1. Then, for each $r \in \mathfrak{T}_\delta$ and $\hat{\varphi} \in \mathscr{C}^1_{\mathrm{per}}(\mathrm{I})$ we have that $\hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi} \in \mathscr{C}^1_{\mathrm{per}}(\mathrm{I})$ and*

$$\left(\hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi}\right)'(t) = \int_0^1 \left(\frac{\partial}{\partial t}\mathsf{n}_{r,\mathbb{C}}(t,s) - \pi\, \mathsf{k}_{r,\mathbb{C}}(t,s)\cos(\pi(t-s))\right) \mathcal{J}_{r,\mathbb{C}}(s) \frac{\hat{\varphi}(t) - \hat{\varphi}(s)}{\sin(\pi(t-s))} ds, \quad t \in \mathrm{I}. \tag{5.31}$$

Proof. Let $\delta > 0$ be as in Proposition 5.1. For $\hat{\varphi} \in \mathscr{C}^1_{\mathrm{per}}(\mathrm{I})$ and $r \in \mathfrak{T}_\delta$, we have

$$\left(\hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi}\right)(t) = \int_0^1 \mathsf{n}_{r,\mathbb{C}}(t,s)\mathcal{J}_{r,\mathbb{C}}(s) \frac{\hat{\varphi}(t) - \hat{\varphi}(s)}{\sin(\pi(t-s))} ds + \hat{\varphi}(t) \int_0^1 \mathsf{k}_{r,\mathbb{C}}(t,s)\mathcal{J}_{r,\mathbb{C}}(s) ds,$$

Let us prove that for all $r \in \mathfrak{T}_\delta$ and for all $t \in \mathrm{I}$ it holds

$$\int_0^1 \mathsf{k}_{r,\mathbb{C}}(t,s)\mathcal{J}_{r,\mathbb{C}}(s) ds = -\frac{1}{2}. \tag{5.32}$$

For a given $r_0 \in \mathfrak{T}$, we define

$$B(r_0,\delta) \coloneqq \left\{ r \in \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{R}^2) : \ d(r_0, r) < \delta \right\},$$

where $d(\cdot,\cdot) : \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2) \times \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2) \to \mathbb{R}$ is defined in (4.1). Throughout this proof we set $B_{\Re}(r_0,\delta) \coloneqq B(r_0,\delta) \cap \mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{R}^2)$. Observe that for each $r_0 \in \mathfrak{T}_\delta$ we have $B_{\Re}(r_0,\delta) \subset B(r_0,\delta)$ and

$$\mathfrak{T}_\delta = \bigcup_{r_0 \in \mathfrak{T}} B(r_0,\delta). \tag{5.34}$$

Define $\mathbb{1}(\mathbf{x}) = 1$, for $\mathbf{x} \in \Gamma_r$, where $r \in B_{\Re}(r_0, \delta)$ and $r_0 \in \mathfrak{T}$. According to Lemma 2.6 and [37, Example 6.14], for all $r \in B_{\Re}(r_0, \delta)$ and $r_0 \in \mathfrak{T}$ it holds that

$$\left(\mathsf{K}_r\, \mathbb{1}\right)(\mathbf{x}) = \int\limits_{\Gamma_r} \boldsymbol{\nu}_{\Gamma_r}(\mathbf{y}) \cdot \mathbf{grad}_{\mathbf{y}} \mathrm{G}(\mathbf{x}, \mathbf{y}) ds_{\mathbf{y}} = -\frac{1}{2}, \quad \mathbf{x} \in \Gamma_r.$$

Using the pullback operator, we obtain $\tau_r \circ (\mathsf{K}_r\, \mathbb{1}) \circ \tau_r^{-1} = -\frac{1}{2}$. For $r_0 \in \mathfrak{T}$, let us consider the map

$$r \in B_{\Re}(r_0, \delta) \mapsto \tau_r \circ (\mathsf{K}_r\, \mathbb{1}) \circ \tau_r^{-1} = \int\limits_0^1 \mathsf{k}_r(t, s) \mathcal{J}_r(s) ds \in \mathscr{C}^0_{\text{per}}(\mathrm{I}).$$

Observe that the map

$$r \in B(r_0, \delta) \mapsto \int\limits_0^1 \mathsf{k}_{r,\mathbb{C}}(t, s) \mathcal{J}_{r,\mathbb{C}}(s) ds \in \mathscr{C}^0_{\text{per}}(\mathrm{I}), \tag{5.35}$$

is a well-defined extension to $B(r_0, \delta)$ of the map $r \in B_{\Re}(r_0, \delta) \mapsto \tau_r \circ (\mathsf{K}_r\, \mathbb{1}) \circ \tau_r^{-1}$. This map also admits the extension $r \in B(r_0, \delta) \mapsto -\frac{1}{2}$. These two extensions coincide for real-valued boundary transformations $r \in B_{\Re}(r_0, \delta)$, with $r_0 \in \mathfrak{T}$. The set $B(r_0, \delta)$ is open in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$. Hence, $B_{\Re}(r_0, \delta)$ is relatively open in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$. For $r_0 \in \mathfrak{T}$, due to Lemma 5.4 and Lemma 5.11, the map (5.35) is holomorphic on the connected open set $B(r_0, \delta)$. Clearly, $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ is a real vector subspace of $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$. Since each $r \in \mathscr{C}^2(\mathrm{I}, \mathbb{C}^2)$ admits the unique decomposition $r(t) = \Re\{r(t)\} + \imath \Im\{r(t)\}$, where $\Re\{r(t)\}, \Im\{r(t)\} \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$, we have $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2) \cong \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2) + \imath\, \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$. Recalling Theorem 4.9, we conclude that the extension to complex-valued boundary representations of the map $r \in B_{\Re}(r_0, \delta) \mapsto \tau_r \circ (\mathsf{K}_r\, \mathbb{1}) \circ \tau_r^{-1} = -\frac{1}{2}$ is unique. It follows that the map (5.35) is constant and equal to $-\frac{1}{2}$, for all $r \in B(r_0, \delta)$. Recalling (5.34), (5.32) holds for all $r \in \mathfrak{T}_\delta$ together with

$$\left(\hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi}\right)(t) = \int\limits_0^1 \mathsf{n}_{r,\mathbb{C}}(t, s) \mathcal{J}_{r,\mathbb{C}}(s) \frac{\hat{\varphi}(t) - \hat{\varphi}(s)}{\sin(\pi(t - s))} ds - \frac{1}{2} \hat{\varphi}(t), \quad t \in \mathrm{I}.$$

For $t \in \mathrm{I}$, let us compute

$$\begin{aligned}
\left(\hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi}\right)'(t) &= \int\limits_0^1 \frac{\partial}{\partial t} \mathsf{n}_{r,\mathbb{C}}(t, s) \mathcal{J}_{r,\mathbb{C}}(s) \frac{\hat{\varphi}(t) - \hat{\varphi}(s)}{\sin(\pi(t - s))} ds - \hat{\varphi}'(t) \int\limits_0^1 \mathsf{k}_{r,\mathbb{C}}(t, s) \mathcal{J}_{r,\mathbb{C}}(s) ds \\
&\quad - \pi \int\limits_0^1 \mathsf{k}_{r,\mathbb{C}}(t, s) \cos(\pi(t - s)) \mathcal{J}_{r,\mathbb{C}}(s) \frac{\hat{\varphi}(t) - \hat{\varphi}(s)}{\sin(\pi(t - s))} ds - \frac{1}{2} \hat{\varphi}'(t) \\
&= \int\limits_0^1 \left(\frac{\partial}{\partial t} \mathsf{n}_{r,\mathbb{C}}(t, s) - \pi\, \mathsf{k}_{r,\mathbb{C}}(t, s) \cos(\pi(t - s)) \right) \mathcal{J}_{r,\mathbb{C}}(s) \frac{\hat{\varphi}(t) - \hat{\varphi}(s)}{\sin(\pi(t - s))} ds.
\end{aligned}$$

It follows that for all $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ and $r \in \mathfrak{T}_\delta$ we have $\hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ and that (5.31) holds. □

Theorem 5.13. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r \in \mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, the 1-periodic integral operator $\hat{\mathsf{K}}_{r,\mathbb{C}} : H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear, bounded and compact.*

(ii) *For any $\varepsilon \in (0, \delta)$, the map*

$$r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{K}}_{r,\mathbb{C}} \in \mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right), \tag{5.36}$$

is holomorphic.

(iii) *There exists a finite constant $C_{\hat{\mathsf{K}}}(\mathfrak{T}, \delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that for any $\varepsilon \in (0, \delta)$*

$$\sup_{r \in \mathfrak{T}_\varepsilon} \left\| \hat{\mathsf{K}}_{r,\mathbb{C}} \right\|_{\mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \leq C_{\hat{\mathsf{K}}}(\mathfrak{T}, \delta). \tag{5.37}$$

(iv) *The Fréchet derivative of the map in* (5.36) *at* $r \in \mathfrak{T}_\varepsilon$ *in the direction* $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)$ *and for* $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ *reads*

$$\left(\frac{d}{dr} \hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi} \right) [r, \xi](t) = \int_0^1 \left(\frac{d}{d\,r} \mathsf{k}_{r,\mathbb{C}}\, \mathcal{J}_{r,\mathbb{C}} \right) [r, \xi](t, s) \hat{\varphi}(s) ds, \quad t \in \mathrm{I}.$$

Proof. The 1-periodic integral operator $\mathsf{K}_{r,\mathbb{C}}$ fits the framework of Section 4.2 and satisfies Assumption 4.10 with $\mathsf{f}(t) = 1$ and $\mathsf{p}_{r,\mathbb{C}}(t, s) = \mathsf{k}_{r,\mathbb{C}}(t, s)\mathcal{J}_{r,\mathbb{C}}(s)$. We proceed to prove that the operator $\hat{\mathsf{K}}_{r,\mathbb{C}}$ fulfils Assumption 4.12, with $\delta > 0$ as in Proposition 5.1.

(i) According to Lemma 5.4 and Lemma 5.11, we have that $\mathsf{k}_{r,\mathbb{C}}\, \mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$, for all $r \in \mathfrak{T}_\delta$.

(ii) The 1-periodic integral operator $\hat{\mathsf{K}}_{r,\mathbb{C}} : H^0_{\text{per}}(\mathrm{I}) \to H^0_{\text{per}}(\mathrm{I})$ is linear and bounded, for all $r \in \mathfrak{T}_\delta$. Furthermore, it holds for all $r \in \mathfrak{T}_\delta$

$$\left\| \hat{\mathsf{K}}_{r,\mathbb{C}} \right\|_{\mathscr{L}\left(H^0_{\text{per}}(\mathrm{I}), H^0_{\text{per}}(\mathrm{I})\right)} \leq \left\| \mathsf{k}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} \left\| \mathcal{J}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I})},$$

Recalling Proposition 5.12, we have that

$$\left\| \left(\hat{\mathsf{K}}_{r,\mathbb{C}}\, \hat{\varphi} \right)' \right\|_{L^2(\mathrm{I})} \lesssim \left(\left\| \frac{\partial}{\partial t} \mathsf{n}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} + \pi \left\| \mathsf{k}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} \right) \left\| \mathcal{J}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I})} \left\| \hat{\varphi} \right\|_{\frac{1}{2},\text{per}}.$$

It follows from the norm equivalences stated in Lemma 2.3 that

$$\left\| \hat{\mathsf{K}}_{r,\mathbb{C}} \right\|_{\mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^1_{\text{per}}(\mathrm{I})\right)} \lesssim \left(\left\| \frac{\partial}{\partial t} \mathsf{n}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} + \left\| \mathsf{k}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I}\times\mathrm{I})} \right) \left\| \mathcal{J}_{r,\mathbb{C}} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I})}. \tag{5.38}$$

Since $\mathscr{C}^1_{\text{per}}(\mathrm{I})$ is dense in $H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$, the 1-periodic integral operator $\hat{\mathsf{K}}_{r,\mathbb{C}} : H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^1_{\text{per}}(\mathrm{I}) \subset H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear, bounded and compact due to the compactness of the embedding $H^1_{\text{per}}(\mathrm{I}) \subset H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$, see e.g. [37, Theorem 8.3].

(iii) Due to the uniform boundedness of the maps $r \in \mathfrak{T}_\delta \mapsto \mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$, $r \in \mathfrak{T}_\delta \mapsto \mathsf{k}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ and $r \in \mathfrak{T}_\delta \mapsto \mathsf{n}_{r,\mathbb{C}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ established in Lemmas 5.4 and 5.11, respectively, together with (5.38), the map $r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{K}}_{r,\mathbb{C}} \in \mathscr{L}\big(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\big)$ is uniformly bounded for any $\varepsilon \in (0, \delta]$. Therefore, (5.37) holds for a finite constant $C_{\hat{\mathsf{K}}}(\mathfrak{T}, \delta) > 0$ that depends upon $\mathfrak{T}$ and δ only.

(iv) Finally, the map $r \in \mathfrak{T}_\delta \mapsto \mathsf{k}_{r,\mathbb{C}}\mathcal{J}_{r,\mathbb{C}} \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$ is holomorphic according to Lemmas 5.4 and 5.11.

Assumption 4.12 is fulfilled by the operator $\hat{\mathsf{K}}_{r,\mathbb{C}}$. Consequently, Theorem 4.13 implies that the map (5.36) is holomorphic and provides the expression for the corresponding Fréchet derivative. □

5.4. **Shape Holomorphy of the Adjoint Double Layer Operator.** We analyze the map

$$r \in \mathfrak{T} \mapsto \hat{\mathsf{K}}'_r \in \mathscr{L}\big(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\big),$$

where $\hat{\mathsf{K}}'_r := \tau_r \circ \mathsf{K}'_r \circ \tau_r^{-1}$ yields a representation of K'_r in the reference domain I with the notation $\mathsf{K}'_r := \mathsf{K}'_{\Gamma_r}$, for $r \in \mathfrak{T}$. For each $r \in \mathfrak{T}$ and for all $\hat{\varphi} \in H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$ and $\hat{\phi} \in H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ it holds

$$\left\langle \hat{\mathsf{K}}'_r\, \hat{\varphi}, \hat{\phi} \right\rangle_{\text{per}} = \left\langle \hat{\varphi}, \mathcal{J}_r\, \hat{\mathsf{K}}_r \left(\mathcal{J}_r^{-1} \hat{\phi} \right) \right\rangle_{\text{per}}. \tag{5.39}$$

It follows from (5.39) that

$$\hat{\mathsf{K}}'_r = \left(\mathsf{M}_r \circ \hat{\mathsf{K}}_r \circ \mathsf{M}_r^{-1} \right)^\dagger,$$

where for each $r \in \mathfrak{T}_\delta$ the map $\mathsf{M}_r : H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is defined as $\mathsf{M}_r\hat{\varphi} := \mathcal{J}_r\hat{\varphi}$ with inverse $\mathsf{M}_r^{-1} : H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ given by $\mathsf{M}_r^{-1}\hat{\varphi} := \mathcal{J}_r^{-1}\hat{\varphi}$, for $\hat{\varphi} \in H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$. Observe that for a set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ and due to Lemma 2.5 these maps are linear and bounded. We define an extension of the adjoint double layer operator to the set $\mathfrak{T}_\delta$, with $\delta > 0$ as in Proposition 5.1, as

$$\hat{\mathsf{K}}'_{r,\mathbb{C}} := \left(\mathsf{M}_{r,\mathbb{C}} \circ \hat{\mathsf{K}}_{r,\mathbb{C}} \circ \mathsf{M}_{r,\mathbb{C}}^{-1} \right)^\dagger,$$

where $\mathsf{M}_{r,\mathbb{C}}$ and $\mathsf{M}^{-1}_{r,\mathbb{C}}$ are extended to $\mathfrak{T}_\delta$ by using $\mathcal{J}_{r,\mathbb{C}}$, i.e. $\mathsf{M}_{r,\mathbb{C}}\,\hat{\varphi} \coloneqq \mathcal{J}_{r,\mathbb{C}}\hat{\varphi}$ and $\mathsf{M}^{-1}_{r,\mathbb{C}}\,\hat{\varphi} \coloneqq \mathcal{J}^{-1}_{r,\mathbb{C}}\hat{\varphi}$, respectively, for $\hat{\varphi} \in H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$. Again, due to Lemmas 2.5 and 5.4, these maps are linear and bounded. Furthermore, it follows from Lemma 5.4 and Corollary 5.5 that the maps

$$r \in \mathfrak{T}_\delta \mapsto \mathsf{M}_{r,\mathbb{C}} \in \mathscr{L}\big(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\big) \quad \text{and} \quad r \in \mathfrak{T}_\delta \mapsto \mathsf{M}^{-1}_{r,\mathbb{C}} \in \mathscr{L}\big(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\big) \tag{5.40}$$

are holomorphic and uniformly bounded on $\mathfrak{T}_\delta$. We now proceed to the main result regarding shape holomorphy of the adjoint double layer operator. Its proof relies on Theorem 5.13.

Theorem 5.14. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each for $r \in \mathfrak{T}_\delta$, the 1-periodic integral operator $\hat{\mathsf{K}}'_{r,\mathbb{C}} : H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear, bounded and compact.*

(ii) *For any $\varepsilon \in (0,\delta)$, the map*

$$r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{K}}'_{r,\mathbb{C}} \in \mathscr{L}\big(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\big), \tag{5.41}$$

is holomorphic and uniformly bounded on the set $\mathfrak{T}_\varepsilon$,

(iii) *There exists a finite constant $C_{\hat{\mathsf{K}}'}(\mathfrak{T},\delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that for any $\varepsilon \in (0,\delta)$*

$$\sup_{r\in\mathfrak{T}_\varepsilon} \left\| \hat{\mathsf{K}}'_{r,\mathbb{C}} \right\|_{\mathscr{L}\left(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \leq C_{\hat{\mathsf{K}}'}(\mathfrak{T},\delta). \tag{5.42}$$

(iv) *The Fréchet derivative of the map in (5.41) at $r \in \mathfrak{T}_\varepsilon$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ reads*

$$\left(\frac{d}{dr}\hat{\mathsf{K}}'_{r,\mathbb{C}}\right)[r,\xi](t) = \left(\frac{d}{dr}\mathsf{M}_{r,\mathbb{C}}[r,\xi] \circ \hat{\mathsf{K}}_{r,\mathbb{C}} \circ \mathsf{M}^{-1}_{r,\mathbb{C}}\right)^\dagger + \left(\mathsf{M}_{r,\mathbb{C}} \circ \frac{d}{dr}\hat{\mathsf{K}}_{r,\mathbb{C}}[r,\xi] \circ \mathsf{M}^{-1}_{r,\mathbb{C}}\right)^\dagger + \left(\mathsf{M}_{r,\mathbb{C}} \circ \hat{\mathsf{K}}_{r,\mathbb{C}} \circ \frac{d}{dr}\mathsf{M}^{-1}_{r,\mathbb{C}}[r,\xi]\right)^\dagger.$$

Proof. Let $\delta > 0$ be as in Proposition 5.1. The 1–periodic integral operator $\hat{\mathsf{K}}'_{r,\mathbb{C}}$ corresponds to the composition of bounded linear operators that are uniformly bounded on the set $\mathfrak{T}_\delta$, therefore it is linear, bounded and (5.42) holds with a finite constant $C_{\hat{\mathsf{K}}'}(\mathfrak{T},\delta) > 0$, depending upon $\mathfrak{T}$ and δ only. The application of $\dagger$ to a bounded linear operator is linear, thus holomorphic. Recalling the holomorphy of the maps in (5.40) and Theorem 5.13, the map in (5.41) can be expressed as the successive application of holomorphic maps. Therefore, it is holomorphic itself. $\square$

5.5. **Shape Holomorphy of the Hypersingular Operator.** We consider the map

$$r \in \mathfrak{T} \mapsto \hat{\mathsf{W}}_r \in \mathscr{L}\big(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\big),$$

where $\hat{\mathsf{W}}_r \coloneqq \tau_r \circ \mathsf{W}_r \circ \tau_r^{-1}$ yields a representation of W_r and where $\mathsf{W}_r \coloneqq \mathsf{W}_{\Gamma_r}$, for $r \in \mathfrak{T}$. According to Lemma 2.7 (the so-called Maue's formula), for each $r \in \mathfrak{T}$ and $\varphi \in \mathscr{C}^1(\Gamma_r)$ it holds

$$\mathsf{W}_{\Gamma_r}\varphi = -\operatorname{curl}_{\Gamma_r} \circ \mathsf{V}_{\Gamma_r} \circ \operatorname{curl}_{\Gamma_r}\varphi.$$

For each $r \in \mathfrak{T}$ we have

$$\hat{\mathsf{W}}_r\,\hat{\varphi} = -(\tau_r \circ \operatorname{curl}_{\Gamma_r} \circ \tau_r^{-1}) \circ \hat{\mathsf{V}}_r \circ (\tau_r \circ \operatorname{curl}_{\Gamma_r} \circ \tau_r^{-1})\,\hat{\varphi}, \tag{5.43}$$

where $\hat{\varphi} = \tau_r\varphi \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$. Let us define $\widehat{\operatorname{curl}}_r \coloneqq \tau_r \circ \operatorname{curl}_{\Gamma_r} \circ \tau_r^{-1}$. For each $r \in \mathfrak{T}$ and $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ it holds

$$\left(\widehat{\operatorname{curl}}_r\hat{\varphi}\right)(t) = \frac{\hat{\varphi}'(t)}{\mathcal{J}_r(t)}, \quad t \in \mathrm{I}.$$

The strategy to establish the shape holomorphy result for the hypersingular operator differs from that of the the single layer operator (Section 5.2) and that of the double layer operator (Section 5.3). The proof hinges on the following ingredients. Firstly, we use the representation of $\hat{\mathsf{W}}_r$ in (5.43) to construct an extension of the hypersingular operator to $\mathfrak{T}_\delta$, with $\delta > 0$ as in Proposition 5.1, by using the extension of the single layer operator $\hat{\mathsf{V}}_r$ studied in Section 5.2. This entails the construction of a well-defined extension of the operator $\widehat{\operatorname{curl}}_r$ to the set $\mathfrak{T}_\delta$ that depends holomorphically on the set $\mathfrak{T}_\delta$. Secondly, recalling the shape holomorphy result for the single layer operator established in Theorem 5.2 and by writing the extension of the hypersingular operator to $\mathfrak{T}_\delta$ as the successive application of holomorphic maps, one may obtain the desired result.

For $r \in \mathfrak{T}_\delta$, with $\delta > 0$ as in Proposition 5.1, we define the extension to $\mathfrak{T}_\delta$ of the 1-periodic integral operator $\hat{\mathsf{W}}_r$ as

$$\hat{\mathsf{W}}_{r,\mathbb{C}} := -\widehat{\operatorname{curl}}_{r,\mathbb{C}} \circ \hat{\mathsf{V}}_{r,\mathbb{C}} \circ \widehat{\operatorname{curl}}_{r,\mathbb{C}}, \tag{5.44}$$

where, for $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$, we define

$$\left(\widehat{\operatorname{curl}}_{r,\mathbb{C}}\, \hat{\varphi}\right)(t) := \frac{\hat{\varphi}'(t)}{\mathcal{J}_{r,\mathbb{C}}(t)}, \quad t \in \mathrm{I}.$$

Due to Proposition 5.1 the operator $\widehat{\operatorname{curl}}_{r,\mathbb{C}}$ is well-defined for all $r \in \mathfrak{T}_\delta$.

Lemma 5.15. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, the operator $\widehat{\operatorname{curl}}_{r,\mathbb{C}} : H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$, is linear and bounded.*

(ii) *The map*

$$r \in \mathfrak{T}_\delta \mapsto \widehat{\operatorname{curl}}_{r,\mathbb{C}} \in \mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})\right) \tag{5.45}$$

is holomorphic.

(iii) *There exists a finite constant $C_{\widehat{\operatorname{curl}}}(\mathfrak{T},\delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that*

$$\sup_{r\in\mathfrak{T}_\delta} \left\|\widehat{\operatorname{curl}}_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \leq C_{\widehat{\operatorname{curl}}}(\mathfrak{T},\delta).$$

(iv) *The Fréchet derivative of the map in (5.45) at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ reads*

$$\left(\frac{d}{dr}\widehat{\operatorname{curl}}_{r,\mathbb{C}}\right)[r,\xi] = -\frac{1}{\mathcal{J}_{r,\mathbb{C}}}\left(\frac{d}{dr}\mathcal{J}_{r,\mathbb{C}}\right)[r,\xi]\, \widehat{\operatorname{curl}}_{r,\mathbb{C}}. \tag{5.46}$$

Proof. Let $\delta > 0$ be as in Proposition 5.1. For $r \in \mathfrak{T}_\delta$ and $\hat{\varphi}$, $\hat{\psi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$, we have

$$\int_0^1 \left(\widehat{\operatorname{curl}}_{r,\mathbb{C}}\, \hat{\varphi}\right)(t)\hat{\psi}(t)dt = -\int_0^1 \hat{\varphi}(t)\left(\frac{\hat{\psi}}{\mathcal{J}_{r,\mathbb{C}}}\right)'(t)dt.$$

Then, we obtain

$$\begin{aligned}
\left|\left\langle \widehat{\operatorname{curl}}_{r,\mathbb{C}}\, \hat{\varphi}, \hat{\psi}\right\rangle_{\text{per}}\right| = \left|\left\langle \hat{\varphi}, \left(\frac{\hat{\psi}}{\mathcal{J}_{r,\mathbb{C}}}\right)'\right\rangle_{\text{per}}\right| &\leq \|\hat{\varphi}\|_{H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})} \left\|\left(\frac{\hat{\psi}}{\mathcal{J}_{r,\mathbb{C}}}\right)'\right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})} \\
&\lesssim \|\hat{\varphi}\|_{H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})} \left\|\frac{\hat{\psi}}{\mathcal{J}_{r,\mathbb{C}}}\right\|_{H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})} \\
&\lesssim \left\|\frac{1}{\mathcal{J}_{r,\mathbb{C}}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \|\hat{\varphi}\|_{H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})} \left\|\hat{\psi}\right\|_{H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})}.
\end{aligned}$$

Recalling that $\mathscr{C}^1_{\text{per}}(\mathrm{I})$ is dense in $H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$, we get

$$\left\|\widehat{\operatorname{curl}}_{r,\mathbb{C}}\right\|_{\mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\right)} \lesssim \left\|\frac{1}{\mathcal{J}_{r,\mathbb{C}}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{C})}.$$

Corollary 5.5 provides the uniform boundedness of $\left\|\frac{1}{\mathcal{J}_{r,\mathbb{C}}}\right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})}$ on the set $\mathfrak{T}_\delta$. Therefore $\widehat{\operatorname{curl}}_{r,\mathbb{C}}$: $H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded for all $r \in \mathfrak{T}_\delta$ and the map (5.45) is uniformly bounded. According to Corollary 5.5 the map

$$r \in \mathfrak{T}_\delta \mapsto \frac{1}{\mathcal{J}_{r,\mathbb{C}}} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$$

is holomorphic and its Fréchet derivative at $r \in \mathfrak{T}_\delta$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ is

$$\left(\frac{d}{dr}\frac{1}{\mathcal{J}_{r,\mathbb{C}}}\right)[r,\xi] = -\frac{1}{\mathcal{J}^2_{r,\mathbb{C}}}\left(\frac{d}{dr}\mathcal{J}_{r,\mathbb{C}}\right)[r,\xi].$$

Recalling Lemma 2.5, we have that for $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$ it holds

$$\begin{aligned}
&\left\| \left(\widehat{\operatorname{curl}}_{r+\xi,\mathbb{C}} - \widehat{\operatorname{curl}}_{r,\mathbb{C}} - \left(\frac{d}{dr} \widehat{\operatorname{curl}}_{r,\mathbb{C}} \right) [r,\xi] \right) \hat{\varphi}' \right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})} \\
&\qquad \leq \left\| \left(\frac{1}{\mathcal{J}_{r+\xi,\mathbb{C}}} - \frac{1}{\mathcal{J}_{r,\mathbb{C}}} - \left(\frac{d}{dr} \frac{1}{\mathcal{J}_{r,\mathbb{C}}} \right) [r,\xi] \right) \hat{\varphi}' \right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})} \\
&\qquad \lesssim \left\| \left(\frac{1}{\mathcal{J}_{r+\xi,\mathbb{C}}} - \frac{1}{\mathcal{J}_{r,\mathbb{C}}} - \left(\frac{d}{dr} \frac{1}{\mathcal{J}_{r,\mathbb{C}}} \right) [r,\xi] \right) \right\|_{\mathscr{C}^1_{\text{per}}(\mathrm{I})} \left\| \hat{\varphi}' \right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})} \\
&\qquad = o\left(\|\xi\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} \right) \left\| \hat{\varphi}' \right\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})}.
\end{aligned}$$

Therefore, recalling that $\mathscr{C}^1_{\text{per}}(\mathrm{I})$ is dense in $H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ and that $\|\hat{\varphi}'\|_{H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})} \lesssim \|\hat{\varphi}\|_{H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})}$ for $\hat{\varphi} \in \mathscr{C}^1_{\text{per}}(\mathrm{I})$, we obtain

$$\left\| \left(\widehat{\operatorname{curl}}_{r+\xi,\mathbb{C}} - \widehat{\operatorname{curl}}_{r,\mathbb{C}} - \left(\frac{d}{dr} \widehat{\operatorname{curl}}_{r,\mathbb{C}} \right) [r,\xi] \right) \right\|_{\mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \right)} = o\left(\|\xi\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} \right).$$

It follows that the map (5.45) is holomorphic and its Fréchet derivative is given by (5.46). $\square$

Lemma 5.15 together with Theorem 5.9 allows us to establish the boundedness of the 1-periodic integral operator $\hat{\mathsf{W}}_{r,\mathbb{C}}$ and its holomorphic dependence on the set $\mathfrak{T}_\varepsilon$, for some $\varepsilon > 0$ to be specified.

Theorem 5.16. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, the* 1*-periodic integral operator $\hat{\mathsf{W}}_{r,\mathbb{C}} : H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded.*

(ii) *For any $\varepsilon \in (0,\delta)$, the map*

$$r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{W}}_{r,\mathbb{C}} \in \mathscr{L}\big(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})\big) \tag{5.47}$$

is holomorphic.

(iii) *There exists a finite constant $C_{\hat{\mathsf{W}}}(\mathfrak{T},\delta) > 0$ (depending upon $\mathfrak{T}$ and δ only) such that*

$$\sup_{r\in\mathfrak{T}_\varepsilon} \left\| \hat{\mathsf{W}}_{r,\mathbb{C}} \right\|_{\mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \right)} \leq C_{\hat{\mathsf{W}}}(\mathfrak{T},\delta).$$

(iv) *The Fréchet derivative of the map in* (5.47) *at $r \in \mathfrak{T}_\varepsilon$ in the direction $\xi \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)$ reads*

$$\begin{aligned}
\left(\frac{d}{dr} \hat{\mathsf{W}}_{r,\mathbb{C}} \right) [r,\xi] = &- \left(\frac{d}{dr} \widehat{\operatorname{curl}}_{r,\mathbb{C}} \right) [r,\xi] \circ \hat{\mathsf{V}}_{r,\mathbb{C}} \circ \widehat{\operatorname{curl}}_{r,\mathbb{C}} - \widehat{\operatorname{curl}}_{r,\mathbb{C}} \circ \left(\frac{d}{dr} \hat{\mathsf{V}}_{r,\mathbb{C}} \right) [r,\xi] \circ \widehat{\operatorname{curl}}_{r,\mathbb{C}} \\
&- \widehat{\operatorname{curl}}_{r,\mathbb{C}} \circ \hat{\mathsf{V}}_{r,\mathbb{C}} \circ \left(\frac{d}{dr} \widehat{\operatorname{curl}}_{r,\mathbb{C}} \right) [r,\xi].
\end{aligned}$$

Proof. Let $\delta > 0$ be as in Proposition 5.1. Recalling (5.44) and using Lemma 5.15 together with Theorem 5.9, we conclude that the 1-periodic integral operator $\hat{\mathsf{W}}_{r,\mathbb{C}} : H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \to H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is linear and bounded, for all $r \in \mathfrak{T}_\delta$. Furthermore, for any $\varepsilon \in (0,\delta)$

$$\sup_{r\in\mathfrak{T}_\varepsilon} \left\| \hat{\mathsf{W}}_{r,\mathbb{C}} \right\|_{\mathscr{L}\left(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \right)} \leq \left(C_{\widehat{\operatorname{curl}}}(\mathfrak{T},\delta) \right)^2 C_{\hat{\mathsf{V}}}(\mathfrak{T},\delta),$$

therefore the map (5.47) is uniformly bounded on $\mathfrak{T}_\varepsilon$. We observe that $\hat{\mathsf{W}}_{r,\mathbb{C}}$ is defined as the successive application of holomorphic maps. Therefore, the map in (5.47) is holomorphic itself. Indeed, the composition of bounded linear operators being linear on each component is holomorphic. Hence, the map in (5.47) can be written as the composition of the maps in (5.45) and (5.26). Then, it follows from Theorem 5.9 and Lemma 5.15 that the map in (5.47) is holomorphic. $\square$

5.6. **Shape Holomorphy of the Calderón Projector.** Let $\delta > 0$ be as in Proposition 5.1. We define the extension of the Calderón operator $\hat{\mathsf{C}}_r$ (recall that $\hat{\mathsf{C}}_r \coloneqq \tau_r \circ \mathsf{C}_r \circ \tau_r^{-1}$ and that C_r is the Calderón projector defined on Γ_r) to the set $\mathfrak{T}_\delta$ as follows

$$\hat{\mathsf{C}}_{r,\mathbb{C}} \coloneqq \begin{pmatrix} \frac{1}{2}\mathsf{I} - \hat{\mathsf{K}}_{r,\mathbb{C}} & \hat{\mathsf{V}}_{r,\mathbb{C}} \\ \hat{\mathsf{W}}_{r,\mathbb{C}} & \frac{1}{2}\mathsf{I} + \hat{\mathsf{K}}'_{r,\mathbb{C}} \end{pmatrix}, \quad r \in \mathfrak{T}_\delta.$$

As a consequence of Theorems 5.9, 5.13, 5.14 and 5.16, we may establish the following result. Recall that $\mathbf{V}_{\text{per}} = H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}) \times H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I})$.

Theorem 5.17. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7 and let $\delta > 0$ be as in Proposition 5.1.*

(i) *For each $r \in \mathfrak{T}_\delta$, the 1-periodic integral operator $\hat{\mathsf{C}}_{r,\mathbb{C}} : \mathbf{V}_{\text{per}} \to \mathbf{V}_{\text{per}}$ is linear and bounded.*

(ii) *For any $\varepsilon \in (0, \delta)$, the map*

$$r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{C}}_{r,\mathbb{C}} \in \mathscr{L}(\mathbf{V}_{\text{per}}, \mathbf{V}_{\text{per}})$$

is holomorphic.

(iii) *There exists a finite constant $C_{\hat{\mathsf{C}}}(\mathfrak{T}, \delta)$, depending on $\mathfrak{T}$ and $\delta > 0$ only, such that*

$$\sup_{r\in\mathfrak{T}_\varepsilon} \left\| \hat{\mathsf{C}}_{r,\mathbb{C}} \right\|_{\mathscr{L}(\mathbf{V}_{\text{per}},\mathbf{V}_{\text{per}})} \le C_{\hat{\mathsf{C}}}(\mathfrak{T}, \delta).$$

5.7. **Shape Holomorphy of the Domain-to-Solution Map.** Theorem 5.17 in Section 5.6 establishes the holomorphic dependence of the Calderón projector on a family of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$. However, one is also interested in the shape holomorphy of the domain-to-solution map associated to a BIE, which in turn is obtained by means of a boundary reduction of the original boundary value problem using the BIOs contained in the Calderón projector.

Different approaches may be used to derive a boundary integral formulation for a particular boundary value problem. As an example, we consider the Laplace problem in an open bounded domain equipped with Dirichlet boundary conditions. We proceed to summarize the commonly available approaches to convert this problem into an equivalent BIE.

Example 5.18 (Interior Laplace problem with Dirichlet boundary conditions). *Let $\mathfrak{T}$ be a set of admissible boundary representation of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$. Let $\mathrm{D}_r \subset \mathbb{R}^2$ be the open bounded domain enclosed by Γ_r (the existence of this open bounded domain is guaranteed by Proposition 3.2), i.e. $\Gamma_r = \partial\mathrm{D}_r$. We assume that*

$$\sup_{r\in\mathfrak{T}} \operatorname{diam}\{\mathrm{D}_r\} < 1, \tag{5.48}$$

where, for each $r \in \mathfrak{T}$, $\operatorname{diam}\{\mathrm{D}_r\}$ signifies the diameter of the bounded domain D_r. For each $r \in \mathfrak{T}$, let us consider the Dirichlet problem of finding $u_r \in H^1(\mathrm{D}_r)$ such that

$$-\Delta u_r = 0 \quad \text{in} \quad \mathrm{D}_r \quad \text{and} \quad u_r = \mathsf{g}_r \quad \text{on} \quad \Gamma_r, \tag{5.49}$$

where $\mathsf{g}_r \in H^{\frac{1}{2}}(\Gamma_r)$ is the boundary data on Γ_r. Set $\mathcal{S}_r := \mathcal{S}_{\Gamma_r}$ and $\mathcal{D}_r := \mathcal{D}_{\Gamma_r}$, where $\mathcal{S}_{\Gamma_r}$ and $\mathcal{D}_{\Gamma_r}$ correspond to the single and double layer potentials on Γ_r, respectively, as introduced in (2.2) *and* (2.3)*. We review the approaches to obtain a boundary integral formulation for* (5.49)*, i.e. to cast* (5.49) *as an equivalent BIE.*

✓ **Direct method**. *We express $u_r \in H^1(\mathrm{D}_r)$ by using Green's representation formula, i.e.*

$$u_r = \mathcal{S}_r\left(\partial_{\boldsymbol{\nu}_r} u_r\right) - \mathcal{D}_r\left(u_r|_{\Gamma_r}\right) \quad \textit{in} \quad \mathrm{D}_r,$$

where $\partial_{\boldsymbol{\nu}_r} : H^1(\mathrm{D}_r, \Delta) \to H^{-\frac{1}{2}}(\Gamma_r)$ stands for the Neumann trace operator and $u_r|_{\Gamma_r}$ corresponds to the Dirichlet trace of $u_r \in H^1(\mathrm{D}_r)$ on Γ_r.

◇ **First kind BIE**. *Find $\phi_r := \partial_{\boldsymbol{\nu}_r} u_r \in H^{-\frac{1}{2}}(\Gamma_r)$ such that*

$$\mathsf{V}_r\phi_r = \left(\frac{1}{2}\mathsf{I} + \mathsf{K}_r\right)\mathsf{g}_r \quad \textit{in} \quad H^{\frac{1}{2}}(\Gamma_r).$$

◇ **Second kind BIE** *Find $\chi_r := \partial_{\boldsymbol{\nu}_r} u_r \in H^{-\frac{1}{2}}(\Gamma_r)$ such that*

$$\left(\frac{1}{2}\mathsf{I} - \mathsf{K}'_r\right)\chi_r = \mathsf{W}_r\mathsf{g}_r \quad \textit{in} \quad H^{-\frac{1}{2}}(\Gamma_r).$$

✓ **Indirect method**. *Recalling that both the single and double layer potentials are solutions to the Laplace equation in $\mathbb{R}^2$* [50, Lemmas 6.6 & 6.10]*, we express $u_r \in H^1(\mathrm{D}_r)$ in terms of one of them only.*

◇ **First kind BIE**. *We use the single layer ansatz for $u_r \in H^1(\mathrm{D}_r)$*

$$u_r = \mathcal{S}_r(\vartheta_r) \quad \textit{in} \quad \mathrm{D}_r,$$

where $\vartheta_r \in H^{-\frac{1}{2}}(\Gamma_r)$ is the unknown. The BIE reads: find $\vartheta_r \in H^{-\frac{1}{2}}(\Gamma_r)$ such that

$$\mathsf{V}_r\vartheta_r = \mathsf{g}_r \quad \textit{in} \quad H^{\frac{1}{2}}(\Gamma_r)$$

$\diamond$ **Second kind BIE**. *We use the double layer ansatz for $u_r \in H^1(\mathrm{D}_r)$*

$$u_r = \mathcal{D}_r\left(\psi_r\right) \quad \textit{in} \quad \mathrm{D}_r,$$

where $\psi_r \in H^{\frac{1}{2}}(\Gamma_r)$ is the unknown. The BIE reads: find $\psi_r \in H^{\frac{1}{2}}(\Gamma_r)$ such that

$$\left(\frac{1}{2}\mathsf{I} - \mathsf{K}_r\right)\psi_r = -\mathsf{g}_r \quad \textit{in} \quad H^{\frac{1}{2}}(\Gamma_r).$$

Remark 10 (Solvability of the first kind BIEs in Example 5.18). For each $r \in \mathfrak{T}$, it follows from the Lax-Milgram Lemma and the $H^{-\frac{1}{2}}(\Gamma_r)$-ellipticity property of the integral operator V_r stated in [50, Theorem 6.23] (in two dimensions, it suffices (5.48) to hold in order to have this property) that $\mathsf{V}_r \in \mathscr{L}_{\text{iso}}(H^{-\frac{1}{2}}(\Gamma_r), H^{\frac{1}{2}}(\Gamma_r))$. Hence, the first kind formulations introduced in Example 5.18 are uniquely solvable. In turn, due to Proposition 3.10, for all $r \in \mathfrak{T}$ we have that

$$\hat{\mathsf{V}}_r \in \mathscr{L}_{\text{iso}}(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})), \tag{5.50}$$

provided that (5.48) holds. Therefore, the domain-to-solution maps

$$r \in \mathfrak{T} \mapsto \hat{\phi}_r := \hat{\mathsf{V}}_r^{-1}\left(\frac{1}{2}\mathsf{I} + \hat{\mathsf{K}}_r\right)\hat{\mathsf{g}}_r \in H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \tag{5.51}$$

and

$$r \in \mathfrak{T} \mapsto \hat{\vartheta}_r := \hat{\mathsf{V}}_r^{-1}\hat{\mathsf{g}}_r \in H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \tag{5.52}$$

are well-defined, where for each $r \in \mathfrak{T}$ we have set $\hat{\mathsf{g}}_r := \tau_r \mathsf{g}_r$.

Remark 11 (Solvability of the second kind Fredholm BIEs in Example 5.18). For each $r \in \mathfrak{T}$, the single layer operator $\mathsf{V}_r : H^{-\frac{1}{2}}(\Gamma_r) \to H^{\frac{1}{2}}(\Gamma_r)$ is $H^{-\frac{1}{2}}(\Gamma_r)$-elliptic, i.e. there exists a constant $\alpha_r > 0$, depending on $r \in \mathfrak{T}$, such that

$$\Re\left\{\langle \mathsf{V}_r\varphi, \overline{\varphi}\rangle_{\Gamma_r}\right\} \geq \alpha_r \left\|\varphi\right\|^2_{H^{-\frac{1}{2}}(\Gamma_r)} \quad \text{for all} \quad \varphi \in H^{-\frac{1}{2}}(\Gamma_r),$$

and self-adjoint in the $\langle\cdot,\cdot\rangle_{\Gamma_r}$ duality pairing, i.e.

$$\langle \mathsf{V}_r\varphi, \psi\rangle_{\Gamma_r} = \langle \varphi, \mathsf{V}_r\psi\rangle_{\Gamma_r} \quad \text{for all} \quad \varphi, \psi \in H^{-\frac{1}{2}}(\Gamma_r).$$

Then, for each $r \in \mathfrak{T}$ and for all $\varphi \in H^{-\frac{1}{2}}(\Gamma_r)$ and $\phi \in H^{\frac{1}{2}}(\Gamma_r)$

$$\|\varphi\|_{\mathsf{V}_r} := \sqrt{\langle \mathsf{V}_r\varphi, \overline{\varphi}\rangle_{\Gamma_r}} \quad \text{and} \quad \|\phi\|_{\mathsf{V}_r^{-1}} := \sqrt{\langle \mathsf{V}_r^{-1}\phi, \overline{\phi}\rangle_{\Gamma_r}}$$

are norms equivalent to $\|\varphi\|_{H^{-\frac{1}{2}}(\Gamma_r)}$ and to $\|\phi\|_{H^{\frac{1}{2}}(\Gamma_r)}$, respectively. According to [50, Corollaries 6.27 & 6.30] for each $r \in \mathfrak{T}$ there exists $c_r \in (0,1)$, (depending on $r \in \mathfrak{T}$) such that for all $\varphi \in H^{-\frac{1}{2}}(\Gamma_r)$ and $\phi \in H^{\frac{1}{2}}(\Gamma_r)$

$$\left\|\left(\frac{1}{2}\mathsf{I} + \mathsf{K}'_r\right)\varphi\right\|_{\mathsf{V}_r} \leq c_r \left\|\varphi\right\|_{\mathsf{V}_r} \quad \text{and} \quad \left\|\left(\frac{1}{2}\mathsf{I} + \mathsf{K}_r\right)\phi\right\|_{\mathsf{V}_r^{-1}} \leq c_r \left\|\phi\right\|_{\mathsf{V}_r^{-1}}. \tag{5.53}$$

The solution to the second kind formulations introduced in Example 5.18 are formally given by the Neumann series

$$\vartheta_r = \sum_{\ell=0}^{\infty}\left(\frac{1}{2}\mathsf{I} + \mathsf{K}'_r\right)^{\ell}\mathsf{W}_r\mathsf{g}_r \quad \text{and} \quad \psi_r = -\sum_{\ell=0}^{\infty}\left(\frac{1}{2}\mathsf{I} + \mathsf{K}_r\right)^{\ell}\mathsf{g}_r. \tag{5.54}$$

Due to (5.53), the Neumann series in (5.54) converge in the norms $\|\cdot\|_{\mathsf{V}_r}$ and $\|\cdot\|_{\mathsf{V}_r^{-1}}$, respectively. Recalling the equivalence with the norms $\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_r)}$ and $\|\cdot\|_{H^{\frac{1}{2}}(\Gamma_r)}$ and that $\mathsf{K}'_r : H^{-\frac{1}{2}}(\Gamma_r) \to H^{-\frac{1}{2}}(\Gamma_r)$ together with $\mathsf{K}'_r : H^{\frac{1}{2}}(\Gamma_r) \to H^{\frac{1}{2}}(\Gamma_r)$ are linear and bounded, we conclude that

$$\frac{1}{2}\mathsf{I} - \mathsf{K}'_r \in \mathscr{L}_{\text{iso}}(H^{-\frac{1}{2}}(\Gamma_r), H^{-\frac{1}{2}}(\Gamma_r)) \quad \text{and} \quad \frac{1}{2}\mathsf{I} - \mathsf{K}_r \in \mathscr{L}_{\text{iso}}(H^{\frac{1}{2}}(\Gamma_r), H^{\frac{1}{2}}(\Gamma_r)).$$

In turn

$$\frac{1}{2}\mathsf{I} - \hat{\mathsf{K}}'_r \in \mathscr{L}_{\text{iso}}(H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}))) \quad \text{and} \quad \frac{1}{2}\mathsf{I} - \hat{\mathsf{K}}_r \in \mathscr{L}_{\text{iso}}(H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}), H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}))). \tag{5.55}$$

It follows that the domain-to-solution maps

$$r \in \mathfrak{T} \mapsto \hat{\chi}_r := \left(\frac{1}{2}\mathsf{I} - \mathsf{K}'_r\right)^{-1}\hat{\mathsf{W}}_r\,\hat{\mathsf{g}}_r \in H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}) \tag{5.56}$$

and

$$r \in \mathfrak{T} \mapsto \hat{\psi}_r := -\left(\frac{1}{2}\mathsf{I} - \mathsf{K}_r\right)^{-1} \hat{\mathsf{g}}_r \in H^{\frac{1}{2}}_{\mathrm{per}}(\mathrm{I}) \tag{5.57}$$

are well-defined.

Remark 12. In Remarks 10 and 11 we have established the "pointwise solvability" of the first and second kind boundary integral formulations introduced in Example 5.18, i.e. for each boundary representation $r \in \mathfrak{T}$ we have proved well–posednes of the aforementioned formulations. Under Assumption 3.7, the set $\mathfrak{T}$ of admissible boundary representation of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves is a compact subset of $\mathscr{C}^2_{\mathrm{per}}(\mathrm{I}, \mathbb{C}^2)$. Moreover, the implied constants in the boundedness of the isomorphisms from Remarks 10 and 11 depend continuously on the set $\mathfrak{T}$. It follows that the isomorphisms in (5.50) and in (5.55) from Remarks 10 and 11, respectively, and their inverses are uniformly bounded (in the corresponding operator norm) on the set $\mathfrak{T}$. As a consequence, the domain-to-solution maps (5.51)–(5.52) and (5.56)–(5.57) in Remarks 10 and 11, respectively, are also uniformly bounded on $\mathfrak{T}$, provided that the right-hand sides of the first and second kind BIEs in Example 5.18 possess this property as well.

Let X, Y and Z be complex Banach spaces equipped with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and $\|\cdot\|_Z$, respectively. As usual, let $\mathfrak{T}$ be a set of admissible boundary representations of a family $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$. Consider the following domain-to-operator maps

$$r \in \mathfrak{T} \mapsto \mathsf{A}_r \in \mathscr{L}(X, Y) \quad \text{and} \quad r \in \mathfrak{T} \mapsto \mathsf{B}_r \in \mathscr{L}(Z, Y) \tag{5.58}$$

together with the domain-to-data map

$$r \in \mathfrak{T} \mapsto \mathsf{g}_r \in Z. \tag{5.59}$$

Throughout this section we assume that for each $r \in \mathfrak{T}$ we have that $\mathsf{A}_r \in \mathscr{L}_{\mathrm{iso}}(X, Y)$.

Problem 5.19. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a family $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves. For each $r \in \mathfrak{T}$, we seek $\varphi_r \in X$ such that*

$$\mathsf{A}_r\, \varphi_r = \mathsf{B}_r \mathsf{g}_r.$$

Recalling that $\mathsf{A}_r \in \mathscr{L}_{\mathrm{iso}}(X, Y)$ for each $r \in \mathfrak{T}$, there exists a unique $\varphi_r \in X$ solution to Problem 5.19. Consequently, we may define the domain-to-solution map associated to Problem 5.19 as follows

$$r \in \mathfrak{T} \mapsto \varphi_r := \mathsf{A}_r^{-1}\, \mathsf{B}_r\, \mathsf{g}_r \in X. \tag{5.60}$$

Observe that, after the application of the pullback operator introduced in Subsection 3.3, all four formulations presented in Example 5.18 fit the framework of Problem 5.19. After these preparations, we turn to the main purpose of this section. We proceed to establish shape holomorphy of the domain-to-solution map in (5.60). We work under the assumption stated below.

Assumption 5.20. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a family $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves in $\mathbb{R}^2$ satisfying Assumption 3.7. There exists $\varepsilon > 0$ such that:*

(i) *the domain-to-operator maps in* (5.58) *admit extensions to the set $\mathfrak{T}_\varepsilon$ denoted by*

$$r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{A}_{r,\mathbb{C}} \in \mathscr{L}(X, Y) \quad \textit{and} \quad r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{B}_{r,\mathbb{C}} \in \mathscr{L}(Z, Y), \tag{5.61}$$

(ii) *the maps in* (5.61) *are holomorphic and uniformly bounded on the set $\mathfrak{T}_\varepsilon$, i.e. there exist finite constants $C_{\mathsf{A}}(\mathfrak{T}, \varepsilon) > 0$ and $C_{\mathsf{B}}(\mathfrak{T}, \varepsilon) > 0$ depending upon $\mathfrak{T}$ and ε only) such that*

$$\sup_{r\in\mathfrak{T}_\varepsilon} \|\mathsf{A}_{r,\mathbb{C}}\|_{\mathscr{L}(X,Y)} \le C_{\mathsf{A}}(\mathfrak{T}, \varepsilon) \quad \textit{and} \quad \sup_{r\in\mathfrak{T}_\varepsilon} \|\mathsf{B}_{r,\mathbb{C}}\|_{\mathscr{L}(Z,Y)} \le C_{\mathsf{B}}(\mathfrak{T}, \varepsilon),$$

(iii) *the domain-to-data map in* (5.59) *admits an extension to $\mathfrak{T}_\varepsilon$ denoted by*

$$r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{g}_{r,\mathbb{C}} \in Z$$

that is holomorphic and uniformly bounded on the set $\mathfrak{T}_\varepsilon$,

(iv) *there exist a finite constant $C_{\mathsf{g}}(\mathfrak{T}, \varepsilon) > 0$, depending on $\mathfrak{T}$ and ε only, such that*

$$\sup_{r\in\mathfrak{T}_\varepsilon} \|\mathsf{g}_{r,\mathbb{C}}\|_Z \le C_{\mathsf{g}}(\mathfrak{T}, \varepsilon).$$

Theorem 5.21. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a family $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7 and let Assumption 5.20 hold with $\varepsilon > 0$. Then, there exists $\eta = \eta(\mathfrak{T}, \varepsilon) > 0$, depending only on $\mathfrak{T}$ and ε, such that:*

(i) *for each $r \in \mathfrak{T}_\eta$, we have that $\mathsf{A}_{r,\mathbb{C}} \in \mathscr{L}_{\mathrm{iso}}(X, Y)$,*

(ii) *the map in* (5.60) *admits an extension to the set* $\mathfrak{T}_\eta$ *given by*

$$r \in \mathfrak{T}_\eta \mapsto \varphi_{r,\mathbb{C}} := \mathsf{A}_{r,\mathbb{C}}^{-1} \mathsf{B}_{r,\mathbb{C}} \mathsf{g}_{r,\mathbb{C}} \in X, \tag{5.62}$$

(iii) *the map in* (5.62) *is holomorphic and uniformly bounded on the set* $\mathfrak{T}_\eta$*, i.e. there exists a finite constant* $C(\mathfrak{T}, \eta, \varepsilon) > 0$ *(depending on* $\mathfrak{T}$*,* η *and* ε *only) such that*

$$\sup_{r \in \mathfrak{T}_\eta} \|\varphi_{r,\mathbb{C}}\|_X \leq C(\mathfrak{T}, \eta, \varepsilon). \tag{5.63}$$

Before presenting the proof of this result, we introduce a technical proposition regarding holomorphic maps in complex Banach spaces.

Proposition 5.22. *Let* X*,* Y *be complex Banach spaces.*

(i) *Let* $\mathsf{M} \in \mathscr{L}_{\text{iso}}(X, Y)$*. Then*

$$\mathcal{C}_{\mathsf{M}} := \left\{ \mathsf{T} \in \mathscr{L}(X, Y) : \ \|\mathsf{M} - \mathsf{T}\|_{\mathscr{L}(X,Y)} < \left\|\mathsf{M}^{-1}\right\|_{\mathscr{L}(Y,X)}^{-1} \right\} \subseteq \mathscr{L}_{\text{iso}}(X, Y)$$

and for all $\mathsf{T} \in \mathcal{C}_{\mathsf{M}}$ *it holds*

$$\left\|\mathsf{T}^{-1}\right\|_{\mathscr{L}(Y,X)} \leq \frac{\left\|\mathsf{M}^{-1}\right\|_{\mathscr{L}(Y,X)}}{1 - \|\mathsf{M} - \mathsf{T}\|_{\mathscr{L}(X,Y)} \left\|\mathsf{M}^{-1}\right\|_{\mathscr{L}(Y,X)}}.$$

(ii) *The* inversion *map*

$$\text{inv} : \ \mathscr{L}_{\text{iso}}(X, Y) \mapsto \mathscr{L}_{\text{iso}}(Y, X) : \mathsf{M} \mapsto \mathsf{M}^{-1}$$

is holomorphic.

(iii) *The* application *map*

$$\text{app} : \ (\mathscr{L}(X, Y), X) \to Y : \ (\mathsf{M}, \mathsf{g}) \mapsto \mathsf{M}\,\mathsf{g} \tag{5.64}$$

is holomorphic.

Proof. Items (i) and (ii) have been stated in [2, Proposition 4.2]. For the sake of completeness, we include the proofs.

(i) Let $\mathsf{T} \in \mathcal{C}_{\mathsf{M}}$. Using the Neumann series expansion of $(\mathsf{I} - (\mathsf{M} - \mathsf{T})\mathsf{M}^{-1})^{-1}$ [37, Theorem 2.14] we obtain

$$\mathsf{T}^{-1} = (\mathsf{M} - (\mathsf{M} - \mathsf{T}))^{-1} = \mathsf{M}^{-1}(\mathsf{I} - (\mathsf{M} - \mathsf{T}))^{-1} = \mathsf{M}^{-1} \sum_{\ell=0}^{\infty} \left[(\mathsf{M} - \mathsf{T})\mathsf{M}^{-1}\right]^{\ell}. \tag{5.65}$$

Since $\mathsf{T} \in \mathcal{C}_{\mathsf{M}}$, the above series converges absolutely. We conclude that $\mathsf{T}^{-1} \in \mathscr{L}(Y, X)$ and that $\mathcal{C}_{\mathsf{M}} \subseteq \mathscr{L}_{\text{iso}}(X, Y)$.

(ii) The series in (5.65) corresponds to the power series expansion of the map $\mathsf{T} \in \mathscr{L}(X, Y) \mapsto \mathsf{T}^{-1} \in \mathscr{L}(Y, X)$. According to [4, Section 11.12], maps having this structure are complex Fréchet differentiable, thus holomorphic according to Theorem 4.6.

(iii) The application map introduced in (5.64) is linear on each component, hence is holomorphic.

□

Proof of Theorem 5.21. We proceed to prove the claims of Theorem 5.21.

(i) We divide the proof of item (i) into two parts.

Part A. Throughout this part of the proof, let $\widetilde{r} \in \mathfrak{T}$ be arbitrary but fixed. According to Assumption 5.20, item (ii), the map $r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{A}_{r,\mathbb{C}} \in \mathscr{L}(X, Y)$ is holomorphic. It follows from Theorem 4.7 and Proposition 4.8 that for $r \in \mathfrak{T}_\varepsilon$ it holds

$$\|\mathsf{A}_{r,\mathbb{C}} - \mathsf{A}_{\widetilde{r}}\|_{\mathscr{L}(X,Y)} \leq 2\frac{C_{\mathsf{A}}(\mathfrak{T}, \varepsilon)}{\varepsilon} \|r - \widetilde{r}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)}.$$

For $\widetilde{r} \in \mathfrak{T}$, define

$$\eta_{\max}(\widetilde{r}) := \frac{\varepsilon}{2C_{\mathsf{A}}(\mathfrak{T}, \varepsilon)} \left\|\mathsf{A}_{\widetilde{r}}^{-1}\right\|_{\mathscr{L}(Y,X)}^{-1}. \tag{5.66}$$

Since $\mathsf{A}_{\widetilde{r}} \in \mathscr{L}_{\text{iso}}(X, Y)$, we have that $\eta_{\max}(\widetilde{r})$ is strictly positive. Set $\eta(\widetilde{r}) \in (0, \eta_{\max}(\widetilde{r}))$. Then, for a fixed $\widetilde{r} \in \mathfrak{T}$ and for all $r \in B(\widetilde{r}, \eta(\widetilde{r}))$, it holds

$$\|\mathsf{A}_{r,\mathbb{C}} - \mathsf{A}_{\widetilde{r}}\|_{\mathscr{L}(X,Y)} \left\|\mathsf{A}_{\widetilde{r}}^{-1}\right\|_{\mathscr{L}(Y,X)}^{-1} < 1,$$

where for $\varepsilon > 0$

$$B(\widetilde{r}, \varepsilon) = \left\{ r \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2) : \ d(r, \widetilde{r}) < \varepsilon \right\},$$

with $d(\cdot,\cdot)$ as defined in (4.1). Then, according to Proposition 5.22, item (i), for all $r \in B(\widetilde{r}, \eta(\widetilde{r}))$ we have that $\mathsf{A}_{r,\mathbb{C}} \in \mathscr{L}_{\text{iso}}(X,Y)$.

Part B. The set $\mathfrak{T}$ admits the following covering by open sets

$$\mathfrak{T} \subset \bigcup_{\widetilde{r}\in\mathfrak{T}} B\left(\widetilde{r}, \frac{\eta(\widetilde{r})}{2}\right).$$

According to Assumption 3.7, $\mathfrak{T}$ is a compact subset of $\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)$. Therefore, there exist $J \in \mathbb{N}$ and a set $\{\widetilde{r}_1,\dots,\widetilde{r}_J\} \subset \mathfrak{T}$ such that

$$\mathfrak{T} \subset \bigcup_{j=1}^{J} B\left(\widetilde{r}_j, \frac{\eta(\widetilde{r}_j)}{2}\right), \tag{5.67}$$

where $\eta(\widetilde{r}_j) \in (0, \eta_{\max}(\widetilde{r}_j))$ and $\eta_{\max}(\widetilde{r}_j)$ is as in (5.66), but with $\widetilde{r}_j$ instead of $\widetilde{r}$, for $j = 1,\dots,J$. Set

$$\eta = \frac{1}{2}\min_{j=1,\dots,J} \eta(\widetilde{r}_j) > 0. \tag{5.68}$$

With $J \in \mathbb{N}$ as in (5.67), the following inclusions hold true

$$\mathfrak{T} \subset \mathfrak{T}_\eta \subset \bigcup_{j=1}^{J} B(\widetilde{r}_j, \eta(\widetilde{r}_j)).$$

Hence, for each $r \in \mathfrak{T}_\eta$, with $\eta > 0$ as in (5.68), there exists $\widetilde{r}_j \in \mathfrak{T}$ such that $r \in B(\widetilde{r}_j, \eta(\widetilde{r}_j))$. It follows from **Part A** of the proof that for all $r \in \mathfrak{T}_\eta$ we have that $\mathsf{A}_{r,\mathbb{C}} \in \mathscr{L}_{\text{iso}}(X,Y)$.

(ii) Using the inversion and application map from Proposition 5.22, one may cast the map in (5.62) as follows

$$r \in \mathfrak{T}_\eta \mapsto \varphi_{r,\mathbb{C}} = \operatorname{app}(\operatorname{inv}(\mathsf{A}_{r,\mathbb{C}}), \operatorname{app}(\mathsf{B}_{r,\mathbb{C}}, \mathsf{g}_{r,\mathbb{C}})) \in X. \tag{5.69}$$

On the one hand, according to Assumption 5.20 and Proposition 5.22, item (iii), the map $r \in \mathfrak{T}_\eta \mapsto \operatorname{app}(\mathsf{B}_{r,\mathbb{C}}, \mathsf{g}_{r,\mathbb{C}}) \in Y$ is holomorphic. On the other hand, it follows from Assumption 5.20 and Proposition 5.22, item (ii), that the map $r \in \mathfrak{T}_\eta \mapsto \operatorname{inv}(\mathsf{A}_{r,\mathbb{C}}) \in \mathscr{L}(Y,X)$ is holomorphic. Recalling again Proposition 5.22, item (iii), we have that the map in (5.69) is holomorphic.

(iii) Observe that

$$\sup_{r\in\mathfrak{T}_\eta} \|\varphi_{r,\mathbb{C}}\|_X \leq C_{\mathsf{B}}(\mathfrak{T},\varepsilon) C_{\mathsf{g}}(\mathfrak{T},\varepsilon) \sup_{r\in\mathfrak{T}_\eta} \left\|\mathsf{A}^{-1}_{r,\mathbb{C}}\right\|_{\mathscr{L}(Y,X)}.$$

According to Proposition 5.22, item (i), for all $r \in \mathfrak{T}_\eta$ with $\eta > 0$ as in (5.68) there exists $\widetilde{r}_j \in \mathfrak{T}$ as in the proof of item (i) such that

$$\begin{aligned}\left\|\mathsf{A}^{-1}_{r,\mathbb{C}}\right\|_{\mathscr{L}(Y,X)} &\leq \frac{\left\|\mathsf{A}^{-1}_{\widetilde{r}_j}\right\|_{\mathscr{L}(Y,X)}}{1 - \left\|\mathsf{A}_{\widetilde{r}_j} - \mathsf{A}_{r,\mathbb{C}}\right\|_{\mathscr{L}(X,Y)} \left\|\mathsf{A}^{-1}_{\widetilde{r}_j}\right\|_{\mathscr{L}(Y,X)}} \\ &\leq \frac{\left\|\mathsf{A}^{-1}_{\widetilde{r}_j}\right\|_{\mathscr{L}(Y,X)}}{1 - 2\frac{C_{\mathsf{A}}(\mathfrak{T},\varepsilon)}{\varepsilon}\eta \left\|\mathsf{A}^{-1}_{\widetilde{r}_j}\right\|_{\mathscr{L}(Y,X)}} < \infty.\end{aligned} \tag{5.70}$$

The bound in (5.70) is uniform over $r \in B(\widetilde{r}_j, \eta(\widetilde{r}_j))$. Hence,

$$\begin{aligned}\sup_{r\in\mathfrak{T}_\eta} \left\|\mathsf{A}^{-1}_{r,\mathbb{C}}\right\|_{\mathscr{L}(Y,X)} &\leq \max_{j=1,\dots,J} \left(\sup_{r\in B(\widetilde{r}_j,\eta(\widetilde{r}_j))} \left\|\mathsf{A}^{-1}_{r,\mathbb{C}}\right\|_{\mathscr{L}(Y,X)} \right) \\ &\leq \max_{j=1,\dots,J} \frac{\left\|\mathsf{A}^{-1}_{\widetilde{r}_j}\right\|_{\mathscr{L}(Y,X)}}{1 - 2\frac{C_{\mathsf{A}}(\mathfrak{T},\varepsilon)}{\varepsilon}\eta \left\|\mathsf{A}^{-1}_{\widetilde{r}_j}\right\|_{\mathscr{L}(Y,X)}} < \infty.\end{aligned}$$

Then (5.63) holds with a finite constant $C(\mathfrak{T},\eta,\varepsilon) > 0$ that depends only on $\mathfrak{T}$, η and ε.

□

As a consequence of Theorem 5.17, Theorem 5.21 and Remarks 10 and 11, one may establish the following result for the domain-to-solution maps associated to the BIEs from Example 5.18. We remind that the 1–periodic Sobolev space $H^s_{\text{per}}(\mathrm{I})$ for $s \in \mathbb{R}$ are complex-valued ones, according to Section 2.2.

Corollary 5.23. *Let $\mathfrak{T}$ be a set of admissible boundary representations of a family $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7 and let us consider the setting from Example 5.18. Assume that there exists $\varepsilon > 0$ such that the map $r \in \mathfrak{T}_\varepsilon \mapsto \mathsf{g}_{r,\mathbb{C}} \in H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ is holomorphic and uniformly bounded on the $\mathfrak{T}_\varepsilon$ and that $\mathsf{g}_{r,\mathbb{C}} \in H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ extends $\mathsf{g}_r \in H^{\frac{1}{2}}_{\text{per}}(\mathrm{I})$ to $\mathfrak{T}_\varepsilon$. Then, there exists $\eta > 0$ such that:*

(i) *the map in* (5.51) *admits an extension to $\mathfrak{T}_\eta$ given by*

$$r \in \mathfrak{T}_\eta \mapsto \hat{\phi}_{r,\mathbb{C}} := \hat{\mathsf{V}}^{-1}_{r,\mathbb{C}} \left(\frac{1}{2}\mathsf{I} + \hat{\mathsf{K}}_{r,\mathbb{C}}\right) \hat{\mathsf{g}}_{r,\mathbb{C}} \in H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), \tag{5.71}$$

(ii) *the map in* (5.52) *admits an extension to $\mathfrak{T}_\eta$ given by*

$$r \in \mathfrak{T}_\eta \mapsto \hat{\vartheta}_{r,\mathbb{C}} := \hat{\mathsf{V}}^{-1}_{r,\mathbb{C}}\, \hat{\mathsf{g}}_{r,\mathbb{C}} \in H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), \tag{5.72}$$

(iii) *the map in* (5.56) *admits an extension to $\mathfrak{T}_\eta$ given by*

$$r \in \mathfrak{T}_\eta \mapsto \hat{\chi}_{r,\mathbb{C}} := \left(\frac{1}{2}\mathsf{I} - \hat{\mathsf{K}}'_{r,\mathbb{C}}\right)^{-1} \hat{\mathsf{W}}_{r,\mathbb{C}}\, \hat{\mathsf{g}}_{r,\mathbb{C}} \in H^{-\frac{1}{2}}_{\text{per}}(\mathrm{I}), \tag{5.73}$$

(iv) *the map in* (5.57) *admits an extension to $\mathfrak{T}_\eta$ given by*

$$r \in \mathfrak{T}_\eta \mapsto \hat{\psi}_{r,\mathbb{C}} := -\left(\frac{1}{2}\mathsf{I} - \hat{\mathsf{K}}_{r,\mathbb{C}}\right)^{-1} \hat{\mathsf{g}}_{r,\mathbb{C}} \in H^{\frac{1}{2}}_{\text{per}}(\mathrm{I}). \tag{5.74}$$

Moreover, the maps in (5.71)*,* (5.72)*,* (5.73) *and* (5.74) *are holomorphic and uniformly bounded on the set $\mathfrak{T}_\eta$.*

6. Parametric Holomorphy

The presently obtained result establishes the holomorphic dependence of the Calderón projector on a collection of $\mathscr{C}^2$–smooth, regular Jordan curves. However, in practical applications and for computational purposes, one usually deals with a *parametric* representation of the boundary. Namely, each boundary representation belonging to a set $\mathfrak{T}$ of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves is identified by means of a parameter sequence $\boldsymbol{y} \in [-1,1]^s$, where $s \in \mathbb{N}$ corresponds to the parametric dimension. Examples of parametric representation of the boundary may be constructed by means of Fourier polynomials, wavelets bases, B-splines and NURBS (Non-uniform rational B-spline). This parametric representation naturally defines the map $\boldsymbol{y} \in [-1,1]^s \mapsto r_{\boldsymbol{y}} \in \mathfrak{T} \subset \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ and motivates us to consider the following *parametric* version of the Calderón projector

$$\boldsymbol{y} \mapsto \hat{\mathsf{C}}_{\boldsymbol{y}} := \hat{\mathsf{C}}_{r_{\boldsymbol{y}}} \in \mathscr{L}\left(\mathbf{V}_{\text{per}}, \mathbf{V}_{\text{per}}\right). \tag{6.1}$$

It follows from the shape holomorphy result established in Theorem 5.17 that the parametric Calderón projector depends holomorphically on the parameter sequence $\boldsymbol{y}$ provided that the parametric boundary representations $r_{\boldsymbol{y}} \in \mathfrak{T}$ does so as well.

Nevertheless, the efficient approximation of the parametric Calderón projector, and of every parametric map having a structure similar to that of (6.1), is a challenge due to the high dimensionality of the input parameter sequence $\boldsymbol{y}$. In fact, the construction of sparse surrogates of polynomials type for the approximation of these maps is a non-trivial task and suffer generally from the so-called curse of dimensionality.

Recent results regarding the polynomial approximation of parametric maps have identified a precise notion of holomorphy that allows us to obtain dimension-independent convergence rates for the polynomial approximation of these maps: the so-called $(\boldsymbol{b}, \varepsilon)$-holomorphy (Definition 6.3 ahead). This concept, originally introduced in [12], has been recognized as a paramount property to obtain algebraic convergence rate that are independent of the parametric dimension in several techniques used in forward and inverse UQ. As a consequence of the results to be presented in this section, sparse tensor interpolation methods [11, 53, 47, 48, 52, 10], higher-order quasi-Monte Carlo quadratures [29, 30, 22, 21] and model order reduction techniques [9, 7, 8, 3] will be available and mathematically justified for the analysis of forward and inverse shape UQ by means of BIOs and BIEs, with convergence rates that are immune to the growth of the parametric dimension of the underlying problem.

In this section, we analyze the holomorphic dependence of the parametric Calderón projector defined in (6.1) on the parameter sequence $\boldsymbol{y} \in \mathbb{U}$ used to construct a parametric description of the boundary. In Section 6.1, we introduce a collection of affine-parametric boundary representations and establish sufficient conditions to obtain a set of admissible boundary representations of a collection of Jordan curves. Then, in Section 6.2, we establish *parametric holomorphy* of the Calderón projector. More precisely, we prove $(\boldsymbol{b}, \varepsilon)$-holomorphy of the map (6.1) provided that an affine-parametric boundary representation is used. Finally, using the framework established in Section 5.7, in Section 6.3 we establish parametric holomorphy of the domain-to-solution map.

6.1. **Affine-Parametric Boundary Representations.** Recall that $\mathrm{I} = [0,1]$ and define $\mathbb{U} \coloneqq [-1,1]^{\mathbb{N}}$. Let us consider the following class of affine-parametric boundary representations

$$r_{\boldsymbol{y}}(t) \coloneqq r_0(t) + \sum_{j\geq 1} y_j r_j(t), \quad t \in \mathrm{I}, \quad \boldsymbol{y} = (y_j)_{j\geq 1} \in \mathbb{U}, \tag{6.2}$$

where $r_j \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$, for $j \in \mathbb{N}_0$. Let us define

$$\mathsf{q}_j(t,s) \coloneqq \begin{cases} \left\| \frac{r_j(t)-r_j(s)}{\sin(\pi(t-s))} \right\|, & t-s \notin \mathbb{Z}, \\ \frac{\left\| r_j'(t) \right\|}{\pi}, & t-s \in \mathbb{Z}, \end{cases} \qquad j \in \mathbb{N}_0.$$

We work under the assumptions stated below.

Assumption 6.1. *Let $\boldsymbol{b} = \{b_j\}_{j\in\mathbb{N}}$ be defined by $b_j \coloneqq \|r_j\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)}$, for $j \in \mathbb{N}$. We assume that r_0 is a positively oriented and regular boundary representation of a $\mathscr{C}^2$–smooth Jordan curve (in the sense of Definition 3.1), that there exists a $p \in (0,1)$ such that $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and that for some $\eta \in (0,1)$ it holds*

$$\sup_{(t,s)\in \mathrm{I}\times\mathrm{I}} \sum_{j\geq 1} \mathsf{q}_j(t,s) \leq \eta \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_0(t,s). \tag{6.3}$$

Assumption 6.1 enables us to prove the following properties of the map $\boldsymbol{y} \in \mathbb{U} \mapsto r_{\boldsymbol{y}} \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$.

Lemma 6.2. *Let Assumption 6.1 hold. Then, the set*

$$\mathfrak{T} \coloneqq \{r_{\boldsymbol{y}} : \boldsymbol{y} \in \mathbb{U}\} \subset \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2).$$

is a set of admissible boundary representations (in the sense of Definition 3.6) of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7.

Proof. Let us show that the affine-parametric boundary representation (6.2) actually provides a boundary representation of a $\mathscr{C}^2$–smooth, regular Jordan curve (in the sense of Definition 3.1) satisfying the properties listed in Assumption 3.7. First we observe that Assumption 6.1 entails absolute convergence of $r_{\boldsymbol{y}}$ (as defined in (6.2)) in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$, uniformly with respect to $\boldsymbol{y} \in \mathbb{U}$, in the sense that

$$\sup_{\boldsymbol{y}\in\mathbb{U}} \sum_{j\geq 1} \|y_j r_j\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} = \sum_{j\geq 1} \|r_j\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} < \infty,$$

since $\boldsymbol{b} \in \ell^p(\mathbb{N}) \subset \ell^1(\mathbb{N})$ for some $p \in (0,1)$. Hence, $\mathfrak{T}$ is actually contained in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$. For $(t,s) \in \mathrm{I}\times\mathrm{I}$ and for $\boldsymbol{y} \in \mathbb{U}$, we have

$$\left\| \frac{r_{\boldsymbol{y}}(t) - r_{\boldsymbol{y}}(s)}{\sin(\pi(t-s))} \right\| = \left\| \frac{r_0(t) - r_0(s)}{\sin(\pi(t-s))} + \sum_{j\geq 1} y_j \frac{r_j(t) - r_j(s)}{\sin(\pi(t-s))} \right\|.$$

Using the triangle inequality, we get

$$\left\| \frac{r_{\boldsymbol{y}}(t) - r_{\boldsymbol{y}}(s)}{\sin(\pi(t-s))} \right\| \geq \left| \mathsf{q}_0(t,s) - \left\| \sum_{j\geq 1} y_j \frac{r_j(t) - r_j(s)}{\sin(\pi(t-s))} \right\| \right|, \quad \boldsymbol{y} = \{y_j\}_{j\geq 1} \in \mathbb{U}.$$

Due to (6.3) in Assumption 6.1, we have that for $(t,s) \in \mathrm{I}\times\mathrm{I}$ it holds

$$\mathsf{q}_0(t,s) - \sum_{j\geq 1} \mathsf{q}_j(t,s) \geq \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_0(t,s) - \sup_{(t,s)\in\mathrm{I}\times\mathrm{I}} \sum_{j\geq 1} \mathsf{q}_j(t,s) \geq (1-\eta) \inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_0(t,s).$$

By Assumption 6.1, r_0 is the boundary representation of a $\mathscr{C}^2$–smooth, regular Jordan curve, there exists a constant γ_0, depending on $r_0 \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ only, such that

$$\inf_{(t,s)\in\mathrm{I}\times\mathrm{I}} \mathsf{q}_0(t,s) \geq \gamma_0 > 0.$$

Hence, for $(t,s) \in \mathrm{I}\times\mathrm{I}$ and $\boldsymbol{y} \in \mathbb{U}$ we have that

$$\left\| \frac{r_{\boldsymbol{y}}(t) - r_{\boldsymbol{y}}(s)}{\sin(\pi(t-s))} \right\| \geq (1-\eta)\gamma_0 > 0$$

and we conclude that $r_{\boldsymbol{y}} : [0,1) \to \mathbb{R}^2$ is injective and 1-periodic, for all $\boldsymbol{y} \in \mathbb{U}$. Computing the limit of $t \in \mathrm{I}$ tending to $s \in \mathrm{I}$, we obtain $\left\| r'_{\boldsymbol{y}}(s) \right\| \geq \pi(1-\eta)\gamma_0 > 0$, for $s \in \mathrm{I}$, and the boundary representation $r_{\boldsymbol{y}}$

is regular for all $\boldsymbol{y} \in \mathbb{U}$. The parametric boundary representation $r_{\boldsymbol{y}} \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ inherits the orientation of $r_0 \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$. The compactness of $\mathfrak{T}$ in $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ has been proved in [14, Lemma 2.7]. $\square$

6.2. **Parametric Holomorphy of the Calderón Projector.** For $s > 1$, we consider the Bernstein ellipse in the complex plane

$$\mathcal{E}_s \coloneqq \left\{ \frac{z + z^{-1}}{2} : \; 1 \leq |z| \leq s \right\} \subset \mathbb{C}.$$

This ellipse has foci at $z = \pm 1$ and semi-axes of length $a \coloneqq (s + s^{-1})/2$ and $b \coloneqq (s - s^{-1})/2$. For $\boldsymbol{\rho} \coloneqq \{\rho_j\}_{j \geq 1}$ with $\rho_j > 1$, for $j \in \mathbb{N}$, consider the tensorized poly-ellipse

$$\mathcal{E}_{\boldsymbol{\rho}} \coloneqq \bigotimes_{j \geq 1} \mathcal{E}_{\rho_j} \subset \mathbb{C}^{\mathbb{N}}.$$

We adopt the convention $\mathcal{E}_1 \coloneqq [-1, 1]$ to include the case $\rho_j = 1$.

Definition 6.3 ([12, Definition 2.1]). *Let X be a complex Banach space equipped with the norm $\|\cdot\|_X$. For $\varepsilon > 0$ and $p \in (0, 1)$, we say that map $\boldsymbol{y} \in \mathbb{U} \mapsto u_{\boldsymbol{y}} \in X$ is $(\boldsymbol{b}, \varepsilon)$-holomorphic if and only if*

(i) *The map $\boldsymbol{y} \in \mathbb{U} \mapsto u_{\boldsymbol{y}} \in X$ is uniformly bounded, i.e.*

$$\sup_{\boldsymbol{y} \in \mathbb{U}} \|u_{\boldsymbol{y}}\|_X \leq C_0,$$

for some finite constant $C_0 > 0$.

(ii) *There exists a positive sequence $\boldsymbol{b} \coloneqq \{b_j\}_{j \geq 1} \in \ell^p(\mathbb{N})$ and a constant $C_\varepsilon > 0$ such that for any sequence $\boldsymbol{\rho} \coloneqq \{\rho_j\}_{j \geq 1}$ of numbers strictly larger than one that is $(\boldsymbol{b}, \varepsilon)$-admissible, i.e. satisyfing*

$$\sum_{j \geq 1} (\rho_j - 1) b_j \leq \varepsilon,$$

the map $\boldsymbol{y} \mapsto u_{\boldsymbol{y}}$ admits a complex extension $\boldsymbol{z} \mapsto u_{\boldsymbol{z}}$ that is holomorphic with respect to each variable z_j on a set of the form

$$\mathcal{O}_{\boldsymbol{\rho}} \coloneqq \bigotimes_{j \geq 1} \mathcal{O}_{\rho_j},$$

where $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open set containing $\mathcal{E}_{\rho_j}$. This extension is bounded on $\mathcal{E}_{\boldsymbol{\rho}}$ according to

$$\sup_{\boldsymbol{z} \in \mathcal{E}_{\boldsymbol{\rho}}} \|u_{\boldsymbol{z}}\|_X \leq C_\varepsilon. \tag{6.4}$$

Given $s > 1$, let us define

$$\mathcal{T}_s \coloneqq \{z \in \mathbb{C} : \; \operatorname{dist}(z, [-1, 1]) < s - 1\} \quad \text{and} \quad \mathcal{T}_{\boldsymbol{\rho}} \coloneqq \bigotimes_{j \geq 1} \mathcal{T}_{\rho_j},$$

where $\boldsymbol{\rho} \coloneqq \{\rho_j\}_{j \geq 1}$ is such that $\rho_j > 1$, for $j \in \mathbb{N}$. For a $(\boldsymbol{b}, \varepsilon)$-admissible sequence $\boldsymbol{\rho} = \{\rho_j\}_{j \in \mathbb{N}}$, let us consider the following extension of the affine-parametric boundary representation to complex-valued parametric inputs

$$r_{\boldsymbol{z}}(t) \coloneqq r_0(t) + \sum_{j \geq 1} z_j r_j(t), \quad t \in \mathrm{I} \quad \text{and} \quad \boldsymbol{z} = \{z_j\}_{j \in \mathbb{N}} \in \mathcal{T}_{\boldsymbol{\rho}}. \tag{6.5}$$

Lemma 6.4. *Let Assumption 6.1 hold. Then the map $\boldsymbol{y} \in \mathbb{U} \mapsto r_{\boldsymbol{y}} \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ is $(\boldsymbol{b}, \varepsilon)$-holomorphic with the same $\boldsymbol{b}$ and $p \in (0, 1)$ used in Assumption 6.1 and for any $\varepsilon > 0$.*

Proof. Observe that

$$\sup_{\boldsymbol{y} \in \mathbb{U}} \|r_{\boldsymbol{y}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)} \leq \|r_0\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)} + \sum_{j \geq 1} \|r_j\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)}.$$

Hence, item (i) in Definition 6.3 is satisfied with $C_0 = \|r_0\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)} + \|\boldsymbol{b}\|_{\ell^1(\mathbb{N})}$. Being an affine function in each variable, we conclude that the extension (6.5) is holomorphic in each variable z_j in the set $\mathcal{T}_{\boldsymbol{\rho}}$, for any $(\boldsymbol{b}, \varepsilon)$-admissible sequence $\boldsymbol{\rho} \coloneqq \{\rho_j\}_{j \in \mathbb{N}}$ of numbers strictly larger than 1. Recalling that $\mathcal{T}_s$ is an open neighborhood of $\mathcal{E}_s$ [12, Lemma 4.4], we have that item (ii) in Definition 6.3 is satisfied as well. Furthermore, for any $(\boldsymbol{b}, \varepsilon)$-admissible sequence $\boldsymbol{\rho} \coloneqq \{\rho_j\}_{j \in \mathbb{N}}$ of numbers strictly larger than 1 and any $\boldsymbol{z} \coloneqq \{z_j\}_{j \in \mathbb{N}} \in \mathcal{T}_{\boldsymbol{\rho}}$ there exists a $\boldsymbol{y} \in \mathbb{U}$ such that $|z_j - y_j| \leq \rho_j - 1$, for all $j \in \mathbb{N}$. Then, for such $\boldsymbol{y} \in \mathbb{U}$, we have

$$\|r_{\boldsymbol{z}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)} \leq \|r_{\boldsymbol{z}} - r_{\boldsymbol{y}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)} + \|r_{\boldsymbol{y}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{C}^2)},$$

which implies

$$\begin{aligned}\|r_{\boldsymbol{z}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} &\leq \sum_{j=1}^{\infty} |z_j - y_j| \, \|r_j\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \|r_{\boldsymbol{y}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} \\ &\leq \sum_{j=1}^{\infty} (\rho_j - 1) b_j + \sup_{\boldsymbol{y}\in\mathbb{U}} \|r_{\boldsymbol{y}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} \\ &\leq \varepsilon + \|r_0\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \|\boldsymbol{b}\|_{\ell^1(\mathbb{N})}\end{aligned}$$

and

$$\sup_{\boldsymbol{z}\in\mathcal{E}_{\boldsymbol{\rho}}} \|r_{\boldsymbol{z}}\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{C}^2)} \leq \varepsilon + \|r_0\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \|\boldsymbol{b}\|_{\ell^1(\mathbb{N})} \,.$$

Therefore, the estimate (6.4) in Definition 6.3 is satisfied with $C_\varepsilon \coloneqq \varepsilon + \|r_0\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \|\boldsymbol{b}\|_{\ell^1(\mathbb{N})}$. □

Lemma 6.5. *Let Assumption 6.1 hold. Then, the map $\boldsymbol{y} \in \mathbb{U} \mapsto r_{\boldsymbol{y}} \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)$ is continuous provided $\mathbb{U}$ is endowed with the product topology.*

Proof. We take our cue from [15, Lemma 5.7]. According to Assumption 6.1, we have that $\boldsymbol{b} \in \ell^1(\mathbb{N})$. Hence, for all $\epsilon > 0$ there exists a $J_1 = J_1(\epsilon) \in \mathbb{N}$ such that $\sum_{j=J+1}^{\infty} b_j < \epsilon$. Let $\boldsymbol{y}_n \coloneqq \{y_{j,n}\}_{j\in\mathbb{N}} \in \mathbb{U}$ be a sequence converging to $\boldsymbol{y} \coloneqq \{y_j\}_{j\in\mathbb{N}} \in \mathbb{U}$ pointwise. This implies that for all $\epsilon > 0$, there exists $J_2 = J_2(\epsilon) \in \mathbb{N}$ such that $\max_{j\in\mathbb{N}} |y_j - y_{j,n}| < \epsilon$, for all $n > J_2$. Then, for all $\epsilon > 0$ we select $J \coloneqq \max\{J_1, J_2\}$ and we obtain

$$\left\|r_{\boldsymbol{y}} - r_{\boldsymbol{y}_n}\right\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} \leq \sum_{j=1}^{J} |y_j - y_{j,n}| \, \|r_j\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + 2\sum_{j>J} \|r_j\|_{\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)} \leq \epsilon \left(\|\boldsymbol{b}\|_{\ell^1(\mathbb{N})} + 2\right).$$

It follows that $\boldsymbol{y} \in \mathbb{U} \mapsto r_{\boldsymbol{y}} \in \mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)$ is continuous. □

For $\boldsymbol{y} \in \mathbb{U}$, recall that $\hat{\mathsf{C}}_{\boldsymbol{y}} = \hat{\mathsf{C}}_{r_{\boldsymbol{y}}}$. The following result establishes the holomorphic dependence of the Calderón projector on $\mathbb{U}$ in the sense of Definition 6.3.

Theorem 6.6. *Let Assumption 6.1 hold and let $\delta > 0$ be as in Proposition 5.1. Then, for any $\varepsilon \in (0,\delta)$, the map*

$$\boldsymbol{y} \in \mathbb{U} \to \hat{\mathsf{C}}_{\boldsymbol{y}} \in \mathscr{L}\left(\mathbf{V}_{\text{per}}, \mathbf{V}_{\text{per}}\right). \tag{6.6}$$

is $(\boldsymbol{b},\varepsilon)$-holomorphic, with the same $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and $p \in (0,1)$ as in Assumption 6.1. Moreover, the map (6.6) is continuous when $\mathbb{U}$ is equipped with the product topology.

Proof. Let $\delta > 0$ be as in Proposition 5.1. For any $\varepsilon \in (0,\delta)$, we consider a $(\boldsymbol{b},\varepsilon)$-admissible sequence $\boldsymbol{\rho} = \{\rho_j\}_{j\in\mathbb{N}}$ of numbers strictly larger than one. Observe that $\boldsymbol{z} \in \mathcal{T}_{\boldsymbol{\rho}} \Rightarrow r_{\boldsymbol{z}} \in \mathfrak{T}_\varepsilon$, where $r_{\boldsymbol{z}}$ is as in (6.5). Therefore, the chain of compositions

$$\boldsymbol{z} \in \mathcal{T}_{\boldsymbol{\rho}} \mapsto r_{\boldsymbol{z}} \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{C}}_{r_{\boldsymbol{z}}} \in \mathscr{L}\left(\mathbf{V}_{\text{per}}, \mathbf{V}_{\text{per}}\right).$$

is well-defined. The map $\boldsymbol{y} \in \mathcal{T}_{\boldsymbol{\rho}} \mapsto r_{\boldsymbol{y}} \in \mathfrak{T}_\varepsilon$ is $(\boldsymbol{b},\varepsilon)$-holomorphic with the same $\boldsymbol{b}$ and $p \in (0,1)$ as in Assumption 6.1 and any $\varepsilon > 0$. The map

$$r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{C}}_{r,\mathbb{C}} \in \mathscr{L}\left(\mathbf{V}_{\text{per}}, \mathbf{V}_{\text{per}}\right).$$

is holomorphic and uniformly bounded, according to Theorem 5.17. Therefore, the composition is $(\boldsymbol{b},\varepsilon)$-holomorphic, again with the same $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and $p \in (0,1)$ as in Assumption 6.1 and any $\varepsilon \in (0,\delta)$.

The map $\boldsymbol{y} \in \mathbb{U} \to r_{\boldsymbol{y}} \in \mathfrak{T}_\varepsilon$ is continuous for any $\varepsilon \in (0,\delta)$, according to Lemma 6.5. Being holomorphic, the map $r \in \mathfrak{T}_\varepsilon \mapsto \hat{\mathsf{C}}_{r,\mathbb{C}}$ is continuous as well. Therefore, the composition is continuous itself. □

6.3. **Parametric Holomorphy of the Domain-to-Solution Map.** We consider the setting from Section 5.7. For $\boldsymbol{y} \in \mathbb{U}$, we define

$$\mathsf{A}_{\boldsymbol{y}} \coloneqq \mathsf{A}_{r_{\boldsymbol{y}}} \in \mathscr{L}(X,Y), \quad \mathsf{B}_{\boldsymbol{y}} \coloneqq \mathsf{B}_{r_{\boldsymbol{y}}} \in \mathscr{L}(Z,Y) \quad \text{and} \quad \mathsf{g}_{\boldsymbol{y}} \coloneqq \mathsf{g}_{r_{\boldsymbol{y}}} \in Y.$$

Moreover, for $\boldsymbol{y} \in \mathbb{U}$ we set

$$\varphi_{\boldsymbol{y}} \coloneqq \mathsf{A}_{\boldsymbol{y}}^{-1} \, \mathsf{B}_{\boldsymbol{y}} \, \mathsf{g}_{\boldsymbol{y}} \in X.$$

Theorem 6.7. *Let Assumptions 5.20 and 6.1 hold and let $\eta > 0$ be as in Theorem 5.21. Then, for any $\varepsilon \in (0, \eta)$ the map*

$$\boldsymbol{y} \in \mathbb{U} \mapsto \varphi_{\boldsymbol{y}} \in X \tag{6.7}$$

is $(\boldsymbol{b}, \varepsilon)$-holomorphic, with the same $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and $p \in (0,1)$ as in Assumption 6.1. Moreover, the map (6.7) is continuous when $\mathbb{U}$ is equipped with the product topology.

Proof. The proof follows the exact same steps of that of Theorem 6.6. According to Assumptions 5.20 and Theorem 5.21, the map

$$r \in \mathfrak{T}_\eta \mapsto \varphi_{r,\mathbb{C}} = \mathsf{A}_{r,\mathbb{C}}^{-1} \mathsf{B}_{r,\mathbb{C}} \mathsf{g}_{r,\mathbb{C}} \in X$$

is holomorphic and uniformly bounded. As in the proof of Theorem 6.6, it follows that the map $\boldsymbol{y} \in \mathcal{T}_{\boldsymbol{\rho}} \mapsto r_{\boldsymbol{y}} \in \mathfrak{T}_\varepsilon$ is $(\boldsymbol{b}, \varepsilon)$-holomorphic with the same $\boldsymbol{b}$ and $p \in (0,1)$ as in Assumption 6.1 and any $\varepsilon > 0$. It follows that the map in (6.7) is also is $(\boldsymbol{b}, \varepsilon)$-holomorphic with the exact same $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and $p \in (0,1)$ and $p \in (0,1)$ as in Assumption 6.1 and any $\varepsilon \in (0, \eta)$.

Again, the map $\boldsymbol{y} \in \mathbb{U} \mapsto r_{\boldsymbol{y}} \in \mathfrak{T}_\varepsilon$ is continuous for any $\varepsilon \in (0, \eta)$, according to Lemma 6.5. Recalling that $r \in \mathfrak{T}_\eta \mapsto \varphi_{r,\mathbb{C}} \in X$ is holomorphic, we have that this map is also continuous. One concludes that (6.7) is continuous as well. ☐

Remark 13. Since the inversion operation of linear isomorphisms is holomorphic as stated in Proposition 5.22, the $(\boldsymbol{b}, \varepsilon)$-holomorphy property of the Calderón projector obtained in Theorem 6.6 implies that the parametric counterparts of the domain-to-solution maps from Corollary 5.23 inherit the $(\boldsymbol{b}, \varepsilon)$-holomorphy property with the same $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and $p \in (0,1)$, however possibly with a different $\varepsilon > 0$.

7. Concluding Remarks

We consider the Calderón projector for the Laplace equation in two dimensions and prove its holomorphically dependence on a collection of $\mathscr{C}^2$–smooth Jordan curves in $\mathbb{R}^2$. The presently obtained result allows us to establish that the solution of well-posed BIE both of the first or second kind arising from the boundary reduction of the Laplace equation (equipped with suitable boundary conditions) depends holomorphically on the shape of the boundary, provided that the corresponding right-hand side possesses this property as well. Moreover, shape holomorphy of the Calderón projector for the Laplace equation entails the holomorphic dependence of the discrete solution to a well-posed BIE obtained, for instance, by means of Galerkin or collocation discretization methods upon the boundary shape.

We remark that the framework constructed in Section 4, used in the present work only for the Calderón projector for the Laplace equation, can also be employed to establish shape holomorphy of the BIOs arising in the Helmholtz, Stokes and linear elasticity problems.

After considering a suitable affine-parametric boundary representation, shape holomorphy of the BIOs implies *parametric holomorphy* of the corresponding parametric versions of these operators and of the solution of well-posed BIEs set on a $\mathscr{C}^2$–smooth, regular Jordan curve. This property provides the mathematical justification for the construction of sparse surrogates of the polynomial type for the approximation of the resulting parametric BIEs and their numerical approximations by means of either Galerkin or collocation techniques, with convergence rates that do not suffer from the so-called curse of dimensionality of the parameter space. Moreover, as discussed in Section 6, the theoretical foundations of several algorithms used in forward and in inverse UQ and their capability to afford dimension-independent convergence rates rely on the notion of parametric holomorphy presented in Definition 6.3. Although we have considered an affine-parametric representation of the boundary, we remark that the results obtained in Section 6 remain valid inasmuch as the parametric boundary representation $\boldsymbol{y} \in \mathbb{U} \mapsto r_{\boldsymbol{y}} \in \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$ is $(\boldsymbol{b}, \varepsilon)$-holomorphic (in the sense of Definition 6.3) with $\boldsymbol{b} \in \ell^p(\mathbb{N})$ and for some $p \in (0,1)$. An analysis of the implications of our findings in computational UQ using BIEs and BIOs, with the appropiate Galerkin discretization of the BIEs and for the forward and inverse problems, will be elaborated elsewhere.

References

[1] K. Atkinson and W. Han, *Theoretical Numerical Analysis: A Functional Analysis Framework.*, vol. 39, Springer-Verlag, New York, 3rd ed., 2009.

[2] R. Aylwin, C. Jerez-Hanckes, C. Schwab, and J. Zech, *Domain uncertainty quantification in computational electromagnetics*, Tech. Rep. 2019-04, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2019.

[3] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova, and P. Wojtaszczyk, *Convergence rates for greedy algorithms in reduced basis methods*, SIAM J. Math. Anal., 43 (2011), pp. 1457–1472.

[4] S. B. Chae, *Holomorphy and Calculus in Normed Spaces*, vol. 92, CRC Press, 1985.

[5] R. Chapko, D. Gintides, and L. Mindrinos, *The inverse scattering problem by an elastic inclusion*, Adv. Comput. Math., 44 (2018), pp. 453–476.

[6] A. Charalambopoulos, *On the Fréchet differentiability of boundary integral operators in the inverse elastic scattering problem*, Inverse Problems, 11 (1995), p. 1137.

[7] P. Chen and C. Schwab, *Adaptive sparse grid model order reduction for fast Bayesian estimation and inversion*, in Sparse Grids and Applications-Stuttgart 2014, Springer, 2016, pp. 1–27.

[8] ———, *Model order reduction methods in computational uncertainty quantification*, Handbook of Uncertainty Quantification, (2016), pp. 1–53.

[9] ———, *Sparse-grid, reduced-basis Bayesian inversion: Nonaffine-parametric nonlinear equations*, Journal of Computational Physics, 316 (2016), pp. 470–503.

[10] A. Chkifa, A. Cohen, R. DeVore, and C. Schwab, *Sparse adaptive Taylor approximation algorithms for parametric and stochastic elliptic PDEs*, ESAIM: Mathematical Modelling and Numerical Analysis, 47 (2013), pp. 253–280.

[11] A. Chkifa, A. Cohen, and C. Schwab, *High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs*, Foundations of Computational Mathematics, 14 (2014), pp. 601–633.

[12] ———, *Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs*, Journal de Mathématiques Pures et Appliquées, 103 (2015), pp. 400–428.

[13] J. H. Choi and B. M. Kwak, *Shape design sensitivity analysis of elliptic problems in boundary integral equation formulation*, Mechanics of structures and machines, 16 (1988), pp. 147–165.

[14] A. Cohen and R. DeVore, *Approximation of high-dimensional parametric PDEs*, Acta Numerica, 24 (2015), pp. 1–159.

[15] A. Cohen, C. Schwab, and J. Zech, *Shape holomorphy of the stationary Navier-Stokes equations*, SIAM J. Math. Analysis, 50 (2018), pp. 1720–1752.

[16] J. Conway, *Functions of One Complex Variable I*, vol. 159, Springer Science & Business Media, 2nd ed., 1978.

[17] M. Costabel, *Boundary integral operators on Lipschitz domains: elementary results*, SIAM Journal on Mathematical Analysis, 19 (1988), pp. 613–626.

[18] M. Costabel and F. Le Louër, *Shape derivatives of boundary integral operators in electromagnetic scattering. Part I: Shape differentiability of pseudo-homogeneous boundary integral operators*, Integral Equations and Operator Theory, 72 (2012), pp. 509–535.

[19] ———, *Shape derivatives of boundary integral operators in electromagnetic scattering. Part II: Application to scattering by a homogeneous dielectric obstacle*, Integral Equations and Operator Theory, 73 (2012), pp. 17–48.

[20] M. Delfour and J. P. Zolesio, *Shapes and geometries: metrics, analysis, differential calculus, and optimization*, vol. 22, SIAM, 2011.

[21] J. Dick, R. N. Gantner, Q. T. Le Gia, and C. Schwab, *Higher order Quasi-Monte Carlo integration for Bayesian estimation*, Computers and Mathematics with Applications, 77 (2019), pp. 144–172.

[22] J. Dick, Q. T. Le Gia, and C. Schwab, *Higher order Quasi–Monte Carlo integration for holomorphic, parametric operator equations*, SIAM/ASA Journal on Uncertainty Quantification, 4 (2016), pp. 48–79.

[23] R. M. Dudley and R. Norvaiša, *Concrete functional calculus*, New York: Springer, 2011.

[24] H. Eckel and R. Kress, *Nonlinear integral equations for the inverse electrical impedance problem*, Inverse Problems, 23 (2007), p. 475.

[25] K. Eppler, *Boundary integral representations of second derivatives in shape optimization*, Discussiones Mathematicae, Differential Inclusions, Control and Optimization, 20 (2000), pp. 63–78.

[26] ———, *Optimal shape design for elliptic equations via BIE-methods*, Int. J. Appl. Math. Comput. Sci, 10 (2000), pp. 487–516.

[27] K. Eppler and H. Harbrecht, *A regularized Newton method in electrical impedance tomography using shape hessian information*, Control and Cybernetics, 34 (2005), pp. 203–225.

[28] ———, *Second-order shape optimization using wavelet BEM*, Optimization Methods and Software, 21 (2006), pp. 135–153.

[29] R. Gantner and M. Peters, *Higher order Quasi-Monte Carlo for Bayesian shape inversion*, SIAM/ASA J. Uncertainty Quantification, 6 (2016), pp. 707–736.

[30] R. N. Gantner, *A generic C++ library for multilevel Quasi-Monte Carlo*, in Proceedings of the Platform for Advanced Scientific Computing Conference, ACM, 2016, p. 11.

[31] R. Hiptmair, L. Scarabosio, C. Schillings, and C. Schwab, *Large deformation shape uncertainty quantification in acoustic scattering*, Advances in Computational Mathematics, 44 (2018), pp. 1475–1518.

[32] G. Hsiao and W. Wendland, *Boundary Integral Equations*, vol. 164, Springer Berlin Heidelberg, 2008.

[33] O. Ivanyshyn and R. Kress, *Nonlinear integral equations for solving inverse boundary value problems for inclusions and cracks*, The Journal of Integral Equations and Applications, 18 (2006), pp. 13–38.

[34] C. Jerez-Hanckes and C. Schwab, *Electromagnetic wave scattering by random surfaces: uncertainty quantification via sparse tensor boundary elements*, IMA Journal of Numerical Analysis, 37 (2016), pp. 1175–1210.

[35] C. Jerez-Hanckes, C. Schwab, and J. Zech, *Electromagnetic wave scattering by random surfaces: Shape holomorphy*, Mathematical Models and Methods in Applied Sciences, 27 (2016), pp. 2229–2259.

[36] R. Kress, *Inverse elastic scattering from a crack*, Inverse Problems, 12 (1996), p. 667.

[37] ———, *Linear integral equations*, vol. 82 of Applied Mathematical Sciences, Springer-Verlag, New York, 1999.

[38] R. Kress and W. Rundell, *Nonlinear integral equations and the iterative solution for an inverse boundary value problem*, Inverse problems, 21 (2005), p. 1207.

[39] ———, *A nonlinear integral equation and an iterative algorithm for an inverse source problem*, Journal of Integral Equations and Applications, 27 (2015), pp. 179–197.

[40] J. Mujica, *Complex analysis in Banach spaces,*, vol. 120, North-Holland Publishing Co., Amsterdam, 1986.

[41] R. Potthast, *Fréchet differentiability of boundary integral operators in inverse acoustic scattering*, Inverse Problems, 10 (1994), pp. 431–447.

[42] ———, *Domain derivatives in electromagnetic scattering*, Mathematical Methods in the Applied sciences, 19 (1996), pp. 1157–1175.

[43] ———, *Fréchet differentiability of the solution to the acoustic Neumann scattering problem with respect to the domain*, Journal of Inverse and Ill-Posed Problems, 4 (1996), pp. 67–84.
[44] M. Reed and B. Simon, *Functional Analysis*, vol. I, New York: Academic press, 1980.
[45] J. Saranen and G. Vainikko, *Periodic integral and pseudodifferential equations with numerical approximation*, Springer Science & Business Media, 2013.
[46] S. Sauter and C. Schwab, *Boundary Element Methods*, Springer, Berlin, Heidelberg, 2010.
[47] C. Schillings and C. Schwab, *Sparse, adaptive Smolyak quadratures for Bayesian inverse problems*, Inverse Problems, 29 (2013), p. 065011.
[48] ———, *Sparsity in bayesian inversion of parametric operator equations*, Inverse Problems, 30 (2014), p. 065007.
[49] J. Sokolowski and J. P. Zolesio, *Introduction to shape optimization*, Springer, Berlin, Heidelberg, 1992.
[50] O. Steinbach, *Numerical approximation methods for elliptic boundary value problems: finite and boundary elements*, Springer Science & Business Media, 2007.
[51] O. Yaman and F. Le Louër, *Material derivatives of boundary integral operators in electromagnetism and application to inverse scattering problems*, Inverse Problems, 32 (2016), p. 095003.
[52] J. Zech, D. Dung, and C. Schwab, *Multilevel approximation of parametric and stochastic PDEs*, Tech. Rep. 2018-05, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2018.
[53] J. Zech and C. Schwab, *Convergence rates of high dimensional Smolyak quadrature*, Tech. Rep. 2017-27, Seminar for Applied Mathematics, ETH Zürich, Switzerland, 2017.

Appendix A. Proof of Lemma 4.11

Let $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ and let $\mathrm{B}_t \subset \mathbb{R}$ be an open interval containing the point $t \in \mathrm{I}$. Assume that the length of B_t is strictly smaller than $\frac{1}{2}$. For $r \in \mathfrak{T}$, we split the integral in (4.2) as follows

$$\left(\mathsf{P}_r\, \hat{\varphi}\right)(t) = \int\limits_{s \in \mathrm{I} \setminus \mathrm{B}_t} \mathsf{f}(t-s)\, \mathsf{p}_r(t,s) \hat{\varphi}(s) ds + \int\limits_{s \in \mathrm{I} \cap \mathrm{B}_t} \mathsf{f}(t-s)\, \mathsf{p}_r(t,s) \hat{\varphi}(s) ds, \quad t \in \mathrm{I}.$$

Observe that, for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$, we have

$$\left| \int\limits_{s \in \mathrm{I} \cap \mathrm{B}_t} \mathsf{f}(t-s)\, \mathsf{p}_r(t,s) \hat{\varphi}(s) ds, \right| \leq C(\mathsf{f}, \upsilon) \left\| \mathsf{p}_r\, \hat{\varphi} \right\|_{\mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})} \int\limits_{s \in \mathrm{I} \cap \mathrm{B}_t} \left| \sin(\pi(t-s)) \right|^{-\upsilon} ds, \quad t \in \mathrm{I}. \tag{A.1}$$

Since $s \in \mathrm{I} \cap \mathrm{B}_t$ and $t \in \mathrm{I}$, it follows that $|t-s| < \frac{1}{2}$. Moreover, for $(t,s) \in \mathrm{I} \times \mathrm{I}$ such that $|t-s| < \frac{1}{2}$, it holds $|\sin(\pi(t-s))| = \sin(\pi\,|t-s|) \geq |t-s|$. Using this estimate together with (A.1) and recalling that $\upsilon \in (0,1)$, we obtain

$$\begin{aligned}
\int\limits_{s \in \mathrm{I} \cap \mathrm{B}_t} \left| \sin(\pi(t-s)) \right|^{-\upsilon} ds \leq \int\limits_{s \in \mathrm{I} \cap \mathrm{B}_t} |t-s|^{-\upsilon} ds &= \int\limits_{t - \sup\{\mathrm{I} \cap \mathrm{B}_t\}}^{t - \inf\{\mathrm{I} \cap \mathrm{B}_t\}} |\eta|^{-\upsilon} d\eta \\
&= \int\limits_0^{t - \inf\{\mathrm{I} \cap \mathrm{B}_t\}} \eta^{-\upsilon} d\eta + \int\limits_0^{\sup\{\mathrm{I} \cap \mathrm{B}_t\} - t} \eta^{-\upsilon} d\eta \\
&= \frac{(t - \inf\{\mathrm{I} \cap \mathrm{B}_t\})^{1-\upsilon} + (\sup\{\mathrm{I} \cap \mathrm{B}_t\} - t)^{1-\upsilon}}{1-\upsilon}.
\end{aligned}$$

Hence, for each fixed $t \in \mathrm{I}$ and denoting $|\mathrm{B}_t|$ the length of B_t, we have that

$$\int\limits_{s \in \mathrm{I} \cap \mathrm{B}_t} \mathsf{f}(t-s)\, \mathsf{p}_r(t,s) \hat{\varphi}(s) ds \to 0, \quad \text{as} \quad |\mathrm{B}_t| \to 0.$$

Therefore, the integral in (4.2) exists in the Lebesgue sense.

Let $\chi : [0, \infty) \to \mathbb{R}$ be a continuous function satisfying the following properties: $\chi(t) = 0$ for $t \in [0, \frac{1}{2}]$, $\chi(t) = 1$ for $t \geq 1$ and $\chi(t) \in [0,1]$ for $t \in [0, \infty)$. Let us define

$$\mathsf{p}_r^{(n)}(t,s) \coloneqq \chi\left(n \sin^2(\pi(t-s))\right) \mathsf{p}_r(t,s), \quad n \in \mathbb{N} \quad \text{and} \quad (t,s) \in \mathrm{I} \times \mathrm{I}.$$

Moreover, we set

$$\left(\mathsf{P}_r^{(n)}\, \hat{\varphi}\right)(t) \coloneqq \int\limits_0^1 \mathsf{f}(t-s)\, \mathsf{p}_r^{(n)}(t,s) \hat{\varphi}(s) ds, \quad t \in \mathrm{I}.$$

Observe that $\mathsf{f}(t-s)\mathsf{p}_r^{(n)}(t,s) \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$ and that $\left(\mathsf{P}_r^{(n)}\, \hat{\varphi}\right) \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$, for $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I})$ and $r \in \mathfrak{T}$. For $t \in \mathrm{I}$ and $\hat{\varphi} \in \mathscr{C}^0_{\text{per}}(\mathrm{I} \times \mathrm{I})$, we have

$$\left|\left(\mathsf{P}_r\,\hat\varphi\right)(t)-\left(\mathsf{P}_r^{(n)}\,\hat\varphi\right)(t)\right|\leq C(\mathsf{f},\upsilon)\,\|\mathsf{p}_r\,\hat\varphi\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})}\int_0^1|\sin(\pi(t-s))|^{-\upsilon}\,|1-\chi\left(n\,|\sin(\pi(t-s))|\right)|\,ds \tag{A.2}$$

Let us consider the change of variables $u=\sin(\pi(t-s))$. For $t\in\mathrm{I}$, we have for $n\in\mathbb{N}$, $n\geq 2$

$$\int_0^1|\sin(\pi(t-s))|^{-\upsilon}\,|1-\chi\left(n\,|\sin(\pi(t-s))|\right)|\leq\frac{1}{\pi}\int_{-\frac{1}{n}}^{\frac{1}{n}}\frac{|u|^{-\upsilon}}{\sqrt{1-u^2}}du=\frac{2}{\pi}\int_0^{\frac{1}{n}}\frac{u^{-\upsilon}}{\sqrt{1-u^2}}du.$$

Furthermore, recalling that $\upsilon\in(0,1)$, we find for $n\in\mathbb{N}$, $n\geq 2$,

$$\int_0^{\frac{1}{n}}\frac{u^{-\upsilon}}{\sqrt{1-u^2}}du\leq\frac{n}{\sqrt{n^2-1}}\int_0^{\frac{1}{n}}u^{-\upsilon}du=\frac{n}{\sqrt{n^2-1}}\left.\frac{u^{1-\upsilon}}{1-\upsilon}\right|_0^{\frac{1}{n}}=\frac{1}{1-\upsilon}\frac{n^{\upsilon}}{\sqrt{n^2-1}}.$$

Recalling (A.2), we obtain

$$\left|\left(\mathsf{P}_r\,\hat\varphi\right)(t)-\left(\mathsf{P}_r^{(n)}\,\hat\varphi\right)(t)\right|\leq C(\mathsf{f},\upsilon)\,\|\mathsf{p}_r\,\hat\varphi\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I}\times\mathrm{I})}\frac{n^{\upsilon}}{\sqrt{n^2-1}},\quad n\in\mathbb{N},\quad n\geq 2,\quad t\in\mathrm{I}.$$

Hence, for a fixed $\hat\varphi\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I})$ we have

$$\left\|\left(\mathsf{P}_r\,\hat\varphi\right)-\left(\mathsf{P}_r^{(n)}\,\hat\varphi\right)\right\|_{\mathscr{C}^0_{\mathrm{per}}(\mathrm{I})}\to 0\quad\text{as } n\to\infty.$$

Since the uniform limit of continuous functions is continuous, we conclude that $\mathsf{P}_r\,\hat\varphi\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I})$, for all $r\in\mathfrak{T}$ provided that $\hat\varphi\in\mathscr{C}^0_{\mathrm{per}}(\mathrm{I})$.

Appendix B. Proof of Proposition 5.1

Lemma B.1. *For all $r\in\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$ we have that for all $(t,s)\in\mathrm{I}\times\mathrm{I}$ it holds*

$$\|r(t)-r(s)\|\leq|r|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\,|\sin(\pi(t-s))|\,.$$

Proof. Let t, $s\in\mathrm{I}$ and we assume w.l.o.g. that $t>s$. If $t-s\in\left[0,\frac{1}{2}\right]$, we have that $\sin(\pi(t-s))\geq\frac{\pi(t-s)}{2}$. Then, for $r\in\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$ it holds

$$\begin{aligned}\|r(t)-r(s)\|\leq|r|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\,|t-s|&\leq\frac{2}{\pi}\,|r|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\sin(\pi(t-s))\\&\leq|r|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\sin(\pi(t-s)).\end{aligned}$$

On the other hand, if $t-s\in\left[\frac{1}{2},1\right]$, $1-(t-s)\in\left[0,\frac{1}{2}\right]$. Hence, using the 1-periodicity of $r\in\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$, we get

$$\|r(t)-r(s)\|=\|r(s)-r(t-1)\|\leq|r|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\sin(\pi(t-s)).$$

It follows that $\|r(t)-r(s)\|\leq|r|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\,|\sin(\pi(t-s))|$ holds for all $(t,s)\in\mathrm{I}\times\mathrm{I}$. ☐

Proof of Proposition 5.1. Let $\mathfrak{T}$ be a set of admissible boundary representations of a collection $\{\Gamma_r\}_{r\in\mathfrak{T}}$ of Jordan curves satisfying Assumption 3.7. Given $\widetilde{r}\in\mathfrak{T}\subset\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{R}^2)$, we consider the open ball in $\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)$ centered in $\widetilde{r}$ and of size $\delta\left(\widetilde{r}\right)>0$ (with a dependence on $\widetilde{r}$ to be specified later) i.e.

$$B(\widetilde{r},\delta(\widetilde{r}))\left\{r\in\mathscr{C}^2_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2):\,d(r,\widetilde{r})<\delta\left(\widetilde{r}\right)\right\},$$

where $d(\cdot,\cdot)$ has been defined in (4.1). Let $r\in B(\widetilde{r},\delta(\widetilde{r}))$. For $(t,s)\in\mathrm{I}\times\mathrm{I}$, we have

$$\begin{aligned}(r(t)-r(s))\cdot(r(t)-r(s))=&(\widetilde{r}(t)-\widetilde{r}(s))\cdot(\widetilde{r}(t)-\widetilde{r}(s))\\&+2\,(\widetilde{r}(t)-\widetilde{r}(s))\cdot((r-\widetilde{r})(t)-(r-\widetilde{r})(s))\\&+((r-\widetilde{r})(t)-(r-\widetilde{r})(s))\cdot(r-\widetilde{r})(t)-(r-\widetilde{r})(s).\end{aligned}$$

On the one hand, using Lemma B.1, we obtain

$$\begin{aligned}|(\widetilde{r}(t)-\widetilde{r}(s))\cdot((r-\widetilde{r})(t)-(r-\widetilde{r})(s))|&\leq\|\widetilde{r}(t)-\widetilde{r}(s)\|\,\|(r-\widetilde{r})(t)-(r-\widetilde{r})(s)\|\\&\leq|r-\widetilde{r}|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\,|\widetilde{r}|_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{R}^2)}\sin^2(\pi(t-s)).\end{aligned}$$

On the other hand, again using Lemma B.1, we get

$$|((r-\widetilde{r})(t)-(r-\widetilde{r})(s))\cdot((r-\widetilde{r})(t)-(r-\widetilde{r})(s))|\leq|r-\widetilde{r}|^2_{\mathscr{C}^1_{\mathrm{per}}(\mathrm{I},\mathbb{C}^2)}\sin^2(\pi(t-s)).$$

Thus, we obtain

$$(r(t)-r(s))\cdot(r(t)-r(s)) = \|\widetilde{r}(t)-\widetilde{r}(s)\|^2 + \mathcal{E}(r,\widetilde{r}), \tag{B.1}$$

where $|\mathcal{E}(r,\widetilde{r})| \leq \mathcal{U}(r,\widetilde{r})\sin^2(\pi(t-s))$ and

$$\mathcal{U}(r,\widetilde{r}) \coloneqq 2\,|r-\widetilde{r}|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{C}^2)}\,|\widetilde{r}|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + |r-\widetilde{r}|^2_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{C}^2)}\,.$$

Observe that

$$\mathcal{U}(r,\widetilde{r}) \leq 2\delta(\widetilde{r})\,|\widetilde{r}|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + (\delta(\widetilde{r}))^2. \tag{B.2}$$

Using (B.1), we obtain the following lower bound

$$\Re\left\{\frac{(r(t)-r(s))\cdot r(t)-r(s)}{\sin^2(\pi(t-s))}\right\} \geq \left\|\frac{\widetilde{r}(t)-\widetilde{r}(s)}{\sin(\pi(t-s))}\right\|^2 - \mathcal{U}(r,\widetilde{r}).$$

Recalling Proposition 3.9, we obtain

$$\Re\left\{\frac{(r(t)-r(s))\cdot r(t)-r(s)}{\sin^2(\pi(t-s))}\right\} \geq (\alpha(\mathfrak{T}))^2 - \mathcal{U}(r,\widetilde{r}),$$

where $\alpha(\mathfrak{T}) > 0$ is as in Proposition 3.9 and $(t,s) \in \mathrm{I}\times\mathrm{I}$ are such that $t-s \notin \mathbb{Z}$. We proceed to find $\delta_{\max}(\widetilde{r}) > 0$ (depending on the boundary representation $\widetilde{r} \in \mathfrak{T}$) such that $(\alpha(\mathfrak{T}))^2 - \mathcal{U}(r,\widetilde{r}) > 0$, for all $\delta \in (0,\delta_{\max}(\widetilde{r}))$. Using (B.2), we obtain

$$\delta_{\max}(\widetilde{r}) = -\,|\widetilde{r}|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \sqrt{|\widetilde{r}|^2_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + (\alpha(\mathfrak{T}))^2}. \tag{B.3}$$

By selecting $\delta(\widetilde{r}) \in (0,\delta_{\max}(\widetilde{r}))$, we get the following bound

$$\begin{aligned}\Re\left\{\frac{(r(t)-r(s))\cdot(r(t)-r(s))}{\sin^2(\pi(t-s)}\right\} &\geq (\alpha(\mathfrak{T}))^2 - \mathcal{U}(r,\widetilde{r})\\ &\geq \underbrace{(\alpha(\mathfrak{T}))^2 - 2\delta(\widetilde{r})\,|\widetilde{r}|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} - \delta(\widetilde{r})^2}_{=:\widetilde{\alpha}(\widetilde{r},\delta(\widetilde{r})} > 0,\end{aligned} \tag{B.4}$$

for $(t,s) \in \mathrm{I}\times\mathrm{I}$ such that $t-s \notin \mathbb{Z}$ and for all $r \in B(\widetilde{r},\delta(\widetilde{r}))$. Observe that as s approaches $t+\mathbb{Z}$, we obtain

$$\Re\left\{\frac{(r'\cdot r')(t)}{\pi^2}\right\} \geq \widetilde{\alpha}(\widetilde{r},\delta(\widetilde{r}),\mathfrak{T}), \tag{B.5}$$

for all $r \in B(\widetilde{r},\delta(\widetilde{r}))$. The bounds (B.4) and (B.5) are uniform over $r \in B(\widetilde{r},\delta(\widetilde{r}))$, therefore

$$\inf_{r\in B(\widetilde{r},\delta(\widetilde{r}))}\ \inf_{\substack{(t,s)\in\mathrm{I}\times\mathrm{I}\\ t\neq s}} \Re\left\{\frac{(r(t)-r(s))\cdot(r(t)-r(s))}{\sin^2(\pi(t-s))}\right\} \geq \widetilde{\alpha}(\widetilde{r},\delta(\widetilde{r})), \tag{B.6}$$

and

$$\inf_{r\in B(\widetilde{r},\delta(\widetilde{r}))}\ \inf_{t\in\mathrm{I}} \Re\left\{\frac{(r\cdot r)(t)}{\pi^2}\right\} \geq \widetilde{\alpha}(\widetilde{r},\delta(\widetilde{r})), \tag{B.7}$$

where $\delta(\widetilde{r}) \in (0,\delta_{\max}(\widetilde{r}))$. As in the proof of Theorem 5.21 (**Part B**) and recalling that according to Assumption 3.7 the set $\mathfrak{T}$ is a compact subset of $\mathscr{C}^2_{\text{per}}(\mathrm{I},\mathbb{R}^2)$, there exist $J \in \mathbb{N}$ and a set $\{\widetilde{r}_1,\ldots,\widetilde{r}_J\} \subset \mathfrak{T}$ such that

$$\mathfrak{T} \subset \bigcup_{i=1}^{J} B\left(\widetilde{r}_i, \frac{\delta(\widetilde{r}_i)}{2}\right), \tag{B.8}$$

where $\delta(\widetilde{r}_i) \in (0,\delta_{\max}(\widetilde{r}_i))$ and $\delta_{\max}(\widetilde{r}_i)$ is as in (B.3), but with $\widetilde{r}_i$ instead of $\widetilde{r}$, for $i = 1,\ldots,J$. Let us set

$$\delta(\mathfrak{T}) = \frac{1}{2}\inf_{r\in\mathfrak{T}}\left(-\,|r|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \sqrt{|r|^2_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + (\alpha(\mathfrak{T}))^2}\right).$$

We claim that $\delta(\mathfrak{T})$ is strictly positive. Observe that the map

$$r \in \mathfrak{T} \mapsto -\,|r|_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + \sqrt{|r|^2_{\mathscr{C}^1_{\text{per}}(\mathrm{I},\mathbb{R}^2)} + (\alpha(\mathfrak{T}))^2}\ \in \mathbb{R}$$

is continuous and strictly positive over $\mathfrak{T} \subset \mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$. According to Assumption 3.7, $\mathfrak{T}$ is a compact subset of $\mathscr{C}^2_{\text{per}}(\mathrm{I}, \mathbb{R}^2)$. Consequently, this map attains a strictly positive minimum, thereby providing the strict positiveness of $\delta(\mathfrak{T})$. With $J \in \mathbb{N}$ as in (B.8), the following inclusions hold true

$$\mathfrak{T} \subset \mathfrak{T}_\delta \subset \bigcup_{i=1}^{J} B\left(\widetilde{r}_i, \delta\left(\widetilde{r}_i\right)\right). \tag{B.9}$$

Together with (B.6) and (B.7), (B.9) leads us to

$$\begin{aligned}\inf_{r \in \mathfrak{T}_\delta} \inf_{(t,s) \in \mathrm{I} \times \mathrm{I}} \Re\left\{\mathsf{m}_{r,\mathbb{C}}(t,s)\right\} &\geq \inf_{i=1,\dots,J} \inf_{r \in B(\widetilde{r}_i, \delta(\widetilde{r}_i))} \inf_{(t,s) \in \mathrm{I} \times \mathrm{I}} \Re\left\{\mathsf{m}_{r,\mathbb{C}}(t,s)\right\} \\ &\geq \underbrace{\inf_{i=1,\dots,J} \widetilde{\alpha}(\widetilde{r}_i, \delta(\widetilde{r}_i))}_{=:\widetilde{\alpha}(\mathfrak{T},\delta)} > 0.\end{aligned}$$

We conclude that (5.4) holds with the strictly positive constant $\widetilde{\alpha}(\mathfrak{T}, \delta)$. $\square$

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