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QUASI-MONTE CARLO BAYESIAN ESTIMATION UNDER BESOV PRIORS IN ELLIPTIC INVERSE PROBLEMS

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Abstract. We analyze rates of convergence for quasi-Monte Carlo (QMC) integration for Bayesian inversion of linear, elliptic PDEs with uncertain input from function spaces. Adopting a Riesz or Schauder basis representation of the uncertain inputs, function space priors are constructed as product measures on spaces of (sequences of) coefficients in the basis representations. The numerical approximation of the posterior expectation, given data, then amounts to a high- or infinite-dimensional numerical integration problem. We consider in particular so-called *Besov priors* on the admissible uncertain inputs. We extend the QMC convergence theory from the Gaussian case, and in particular establish sufficient conditions on the uncertain inputs for achieving dimension-independent convergence rates $> 1/2$ of QMC integration with randomly shifted lattice rules. We apply the theory to a concrete class of linear, 2nd order elliptic boundary value problems with log-Besov uncertain diffusion coefficient.

1. Introduction

The efficient computational Bayesian estimation of response functions of partial differential equations (PDEs for short) with uncertain, and not or partially observable inputs and subject to observation data corrupted by observation noise has received substantial attention recently. One approach towards this problem is the Bayesian approach. In the setting where the so-called forward model is a PDE with uncertain input, sufficient abstract conditions for well-posedness of PDE constrained Bayesian inverse problems (BIPs for short) have been recently given in [35, 8, 9] and the references there. Within this analytic framework, new numerical approaches to Bayesian PDE inversion subject to noisy data have evolved in recent years. Traditionally Bayesian PDE inversion has been approached by sampling and filtering methods, in particular Markov chain Monte Carlo (MCMC for short) and ensemble Kalman filters and their variants. We refer to [13, 32] and the references there. Being essentially (variations of) the Monte Carlo methods, these methods afford only the root mean-square convergence rate 1/2 in terms of the number of proposals.

In the context of PDE inversion, this rate is prohibitive due to the need of one (numerical) PDE solve per sample. Therefore, in recent years higher order

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numerical methods for the computation of Bayesian estimates subject to PDE constrained forward problems with distributed uncertain inputs, i.e., inputs from function spaces, have been developed. Starting with [33] for so-called "uniform priors" subsequent extensions addressed also Gaussian priors (see, e.g., [6] for Smolyak based quadratures and [31] for multilevel Monte Carlo and quasi-Monte Carlo (QMC for short) quadratures).

Higher order QMC quadratures adapted to smoothness and sparsity classes of uncertain inputs from function spaces, again with a uniform prior, were developed in [11]. The choice of a uniform prior on the coefficient sequences of the uncertain inputs implies a strong regularizing effect, and facilitates significant simplifications in the error analysis, as was noted in [11, 10].

The mathematical setting from [35, 9] accomodates more general prior measures than uniform or Gaussian, however. Indeed, so-called *Besov prior measures* on function space inputs have been proposed in [27]. MCMC methods for their efficient numerical evaluation have been analyzed in [8, 9].

1.1. **Our contributions.** We generalize high-dimensional QMC integration for unbounded integration domains by randomly shifted lattice rules from Gaussian densities as considered in [29, 16] to the class of so-called p-exponential densities for $p \in [1, 2]$. The case $p = 2$ is the density of the Gaussian distribution and the case $p = 1$ is the density of the Laplace distribution. They were proposed recently in the context of so-called function space priors of Besov type in Bayesian PDE inversion in, e.g., [27, 8].

We obtain sufficient conditions on the prior models to ensure first order convergence with dimension-independent rates of suitable QMC rules, which are adapted to the structure of the Besov prior, thereby generalizing the error analysis in [11] from the uniform prior and from the Gaussian priors in [31]. A key step in our proofs is the parametric regularity of the integrands arising in the Bayesian estimates. We prove bounds on the derivatives of the parametric posteriors which are explicit in the parameter dimension, the differentiation order and in the sparsity class of the function space prior. In doing so, we also analyse the regularity of parametric PDE solution map. In particular, the presently developed regularity theory for the parameter to solution map for the forward PDE (being a 2nd order, linear elliptic PDE in divergence form) is novel and also covers forward UQ for the considered elliptic PDEs under Besov priors.

To keep technicalities and notation at bay, we confine the present error analysis to so-called single-level algorithms, where the forward PDE is discretized with one, common Galerkin projection for all input samples. We hasten to add, however, that the presently developed tools will cover also multilevel extensions of the present algorithms in certain cases along the lines considered in [10, 23, 21] and the references there. We also confine the regularity analysis to globally supported basis elements in the parametrization of the uncertain function space inputs, as arise, e.g., in the Karhunen–Loève expansion parametrization. Again, we add that locally supported representation systems for the function space input may facilitate more efficient QMC integration procedures (see, e.g., the analysis in [21, 24, 22]).

Nevertheless, the unbounded parameter domain in Besov- and Gaussian priors does entail a number of additional technical issues as compared to the case of uniform priors. Among them is, for example, nonuniform (w.r. to the input samples) ellipticity of the forward PDE w.r. to the function space input. This, in turn, entails numerical ill-conditioning of the discretized forward problem which cannot be resolved by standard preconditioning techniques, and requires to resort to a probabilistic error vs. work analysis [18]. The analysis in [18], is suitably extended in [19, Chapter 5] to cover optimal preconditioning within multilevel QMC algorithms in the forward PDE under Besov priors.

1.2. **Outline.**

The structure of this article is as follows. In Section 2, we review the mathematical framework on Bayesian PDE inversion, in the function space setting as developed e.g. in [35, 8, 9] and the references there. Particular attention is paid on the construction of function space priors of Besov type. Section 3 presents the forward problem with uncertain input, its variational formulation, and also results on the integrability w.r. to the p-exponential probability measure on the data. It also presents the Finite Element (FE for short) discretization of the problem and FE error bounds. Section 4 recapitulates Bayesian estimation accounting in particular for the effects of s-term truncation of the (parametric) input data, and of spatial approximation of the forward problem e.g. due to FE discretization, on the Bayesian estimate. Section 5 extends the theory of QMC integration for the presently considered Besov priors. Section 6 applies the QMC theory from Section 5 to the integrals appearing in the Bayesian estimate. Section 7 verifies the abstract regularity requirements for the QMC integration for the parametric Bayesian posterior density. Section 8 presents several numerical experiments for linear, elliptic model problems which exhibit QMC convergence rates that are in agreement with the theory developed in the previous sections. They also indicate that some of the assumptions in our theory could be weakened. Section 9 gives a summary of the results and outlines several directions for generalizations.

We use standard notation for function spaces on a Euclidean domain $D \subset \mathbb{R}^d$, $d \in \mathbb{N}$. For every $r \in [1, \infty)$, the space of r-integrable functions with respect to the Lebesgue measure is denoted by $L^r(D)$. The space of essentially bounded functions on D is denoted by $L^\infty(D)$. For every $k \in \mathbb{N}$, the space of functions such that their k-th order weak derivatives belong to $L^\infty(D)$ is denoted by $W^{k,\infty}(D)$. The Sobolev–Slobodeckij spaces are denoted by $H^t(D)$, $t \in [0, \infty)$.

2. Bayesian inverse problems

In the present section, we briefly review the Bayesian approach to the inversion of linear, divergence form elliptic PDEs with uncertain inputs from a function space X as laid out in [9, 8]. We detail a particular construction of prior measures on the spaces of admissible inputs, the so-called Besov priors. They are built on a Schauder basis $\boldsymbol{\Psi} = (\psi_j)_{j \geq 1}$ of (a separable subspace of) the input space X with p-exponential distributions of random coordinates for some $p \in [1, 2]$.

2.1. **Abstract formulation.**

We review the formulation of BIP in function spaces, as developed e.g. in [8, 9, Sec. 3.2].

To this end, we denote by X and Y separable Banach spaces of uncertain inputs u and of observation data δ, respectively. In the examples considered below, Y will be finite dimensional and w.l.o.g. we shall then assume $Y = \mathbb{R}^K$ for some $K \in \mathbb{N}$. We endow both X and Y with the corresponding Borel σ-algebras $\mathcal{B}(X)$ and $\mathcal{B}(Y)$. Generally, the inputs u will belong to some separable subspace $X_+ \subset X$

of admissible data from which it can be sampled as an input for a PDE model (referred to as the “forward model” in the following).

To develop the Bayesian approach, we assume given a *prior probability measure* μ on $(X, \mathcal{B}(X))$. The observation data $\delta \in Y$ results from a bounded linear (observation) functional $\mathcal{O}(\cdot)$ on a response $q \in \mathcal{X}$ of the forward PDE with input $u \in X_+$. Denoting the input-to-solution map of the PDE by $\mathcal{S}(\cdot) : X \to \mathcal{X}$, we suppose the observation data $\delta = (\mathcal{O} \circ \mathcal{S})(u)$. The observation data δ is further assumed to be corrupted by *additive, centered Gaussian observation noise* $\eta \sim \mathbb{Q}_0 = \mathcal{N}(0, \Sigma)$ on Y, i.e.

$$\delta = (\mathcal{O} \circ \mathcal{S})(u) + \eta \ , \quad \eta \sim \mathbb{Q}_0 \ . \tag{2.1}$$

We assume that η and u are independent. Then the random variable $\delta | u \sim \mathbb{Q}_u$, the translate of $\mathbb{Q}_0$ by $(\mathcal{O} \circ \mathcal{S})(u)$. We assume that $\mathbb{Q}_u \ll \mathbb{Q}_0$ for μ-a.e. $u \in X_+$. Then, for some (Bayesian) potential $\Phi : X \times Y \to \mathbb{R}$, it holds

$$\frac{d\mathbb{Q}_u}{d\mathbb{Q}_0} = \exp(-\Phi(u, \delta)) \ .$$

This implies that for fixed $u \in X_+$, the map $\Phi(u, \cdot) : Y \to \mathbb{R}$ is measurable with $\mathbb{E}^{\mathbb{Q}_0} [\exp(-\Phi(u, \cdot))] = 1$. For every given data $\delta \in Y$, the map $-\Phi(\cdot; \delta)$ is the *log-likelihood*. Define the product measure $\nu_0 = \mu \otimes \mathbb{Q}_0$. Then, under the assumption that Φ is ν_0-measurable, $(u, \delta) \in X \times Y$ is a random variable with $(u, \delta) \sim \nu$ where $\nu = \mu \otimes \mathbb{Q}_u$, and

$$\nu \ll \nu_0 \ , \quad \frac{d\nu}{d\nu_0} = \exp\left(-\Phi(u; \delta)\right) \ , \quad u \in X_+ \ .$$

In this setting holds the following abstract version of Bayes’ theorem. A proof can be found in [9, Section 3.2].

Theorem 2.1. *Suppose $\mu(X) = 1$. For $\Phi : X \times Y \to \mathbb{R}$ that is ν_0-measurable and such that $Z := \exp(-\Phi(u; \delta))$ satisfies*

$$\mathbb{E}^{\mu}(Z) = \int_X \exp\left(-\Phi(u; \delta)\right) \mu(du) > 0 \tag{2.2}$$

the distribution μ^δ of the random variable $u|\delta$ under ν exists. Furthermore, $\mu^\delta \ll \mu$ and for $\delta \in Y$ it holds ν-a.s.

$$\frac{d\mu^\delta}{d\mu} = \frac{1}{\mathbb{E}^{\mu}(Z)} \exp\left(-\Phi(u; \delta)\right) \ .$$

The Bayesian potential is, for additive, centered Gaussian observation noise η in (2.1), given by

$$(u, \delta) \mapsto \Phi(u; \delta) := \frac{1}{2}((\mathcal{O} \circ \mathcal{S})(u) - \delta)^{\top} \Sigma^{-1} ((\mathcal{O} \circ \mathcal{S})(u) - \delta).$$

For goal functionals $\mathcal{G}$ that are sometimes also referred to as *quantities of interest* (QoI for short), we are interested in the expectation of $(\mathcal{G} \circ \mathcal{S})(u)$ conditioned on the data δ. Specifically, in the approximation of the following integral

$$\mathbb{E}^{\mu^\delta}((\mathcal{G} \circ \mathcal{S})(u)) = \frac{1}{\mathbb{E}^{\mu}(Z)} \mathbb{E}^{\mu}((\mathcal{G} \circ \mathcal{S})(u) \exp(-\Phi(u; \delta)) =: \frac{\mathbb{E}^{\mu}(Z')}{\mathbb{E}^{\mu}(Z)},$$

where $Z' := (\mathcal{G} \circ \mathcal{S})(u) \exp(-\Phi(u; \delta))$.

2.2. **Besov priors.** To consider uncertain input data $u \in X$ where X is a Banach space, a prior probability measure on X needs to be constructed. In view of the ensuing QMC integration considered below, we outline a prior construction in separable Banach spaces X which admit a Schauder basis $\boldsymbol{\Psi} = \{\tilde{\psi}_j\}_{j\geq 1}$. Then, priors μ take the form of countable product measures on coordinate sequences of uncertain inputs $u \in X$ which are represented in $\boldsymbol{\Psi}$. We describe next a) the uncertainty parametrization and b) the construction of the Besov prior probability measure which charges a (separably valued subspace of) the input data space.

2.2.1. *Representation Systems* $\boldsymbol{\Psi}$. We *assume* that there exists a countable representation system $\boldsymbol{\Psi} = \{\tilde{\psi}_j\}_{j\geq 1} \subset X$ such that its X-norm closure $X_+ \subseteq X$. We assume $\boldsymbol{\Psi}$ to be an unconditional Schauder basis of X (or of a norm-closed, separable subspace of X) or, in case that X is a Hilbert space, a Riesz basis. We illustrate this by two prototypical examples, see also [3, Section 6] for the first example. Let for $\rho \in [0,\infty)$ and a Euclidean domain D, the Hölder spaces be denoted by $C^\rho(\overline{D})$.

Example 2.2. Consider $X = C^0(\overline{D})$ with $D = (0,1)^d$ and with $\boldsymbol{\Psi}$ denoting an unconditional Schauder basis $\boldsymbol{\Psi} = \{\tilde{\psi}_\lambda\}_{\lambda\in\mathcal{J}}$ of X which is indexed by a countable set $\mathcal{J}$ of indices and which is dense w.r. to the uniform norm. Here, $\lambda = (l,j)$ where $l := |\lambda|$ denotes a scaling factor and where $j \in \mathcal{J}_l$ denotes $\#(\mathcal{J}_l) = O(2^{|\lambda| d}) = O(2^{ld})$ indices which describe localization of $\mathrm{supp}(\tilde{\psi}_\lambda)$ in D. The system $\boldsymbol{\Psi}$ is assumed to satisfy the following hypotheses:

- $\boldsymbol{\Psi}$ is a Schauder basis of $C^0(\overline{D})$, i.e. for every $u \in C^0(\overline{D})$ there exists a unique sequence $\boldsymbol{y} = \{y_\lambda\}_{\lambda\in\mathcal{J}} \in \mathbb{R}^{\mathbb{N}}$ of coefficients such that

$$u(x) = \sum_{l\geq 0}\sum_{j\in\mathcal{J}_l} y_{lj}\tilde{\psi}_{lj}(x) \quad \text{in} \quad C^0(\overline{D}) \,, \tag{2.3}$$

  where $\mathrm{diam}\,\mathrm{supp}(\tilde{\psi}_{lj}) \leq C_0 2^{-l}$,
- there exists $C_1 > 0$ such that for all $\lambda \in \mathcal{J}$ and for all $x, x' \in D$

$$|\tilde{\psi}_\lambda(x) - \tilde{\psi}_\lambda(x')| \leq C_1 2^{|\lambda|(\rho+d/2)}|x-x'|^\rho \,, \quad 0 \leq \rho \leq \bar{\rho}\wedge 1 \,.$$

- there exists $C_2 > 0$ such that $\left\|\sum_{|\lambda|=l} y_\lambda\tilde{\psi}_\lambda\right\|_{L^\infty(D)} \leq C_2 2^{dl/2}\sup_{|\lambda|=l}|y_\lambda|$.
- for $0 < \rho < \bar{\rho}$, $\rho \notin \mathbb{N}$, $u \in C^\rho(\overline{D})$ if and only if $\sup_{l\geq 0,|\lambda|=l} 2^{l(d/2+\rho)}|y_\lambda| < \infty$.

Example 2.3. Consider the torus $D = \mathbb{T}^d$, and the Hilbert space $X = L^2(\mathbb{T}^d)$. A Riesz basis of X is then given by the collection $\boldsymbol{\Psi} = \{\exp(i\boldsymbol{m}\cdot x) : \boldsymbol{m} \in \mathbb{Z}^d\}$. It induces an isometry between X and $\ell^2(\mathbb{N})$. Subspaces $X_+ \subset X$ of higher regularity in $\mathbb{T}^d$ are obtained by imposing stronger coefficient decay than mere 2-summability.

2.2.2. *Prior Construction.* Having at hand the Schauder basis $\boldsymbol{\Psi}$, we construct probability measures charging the space X_+ through product measures on the space $\mathbb{R}^{\mathbb{N}}$ of (sequences of) expansion coefficients of $u \in X$. To this end, we endow $\mathbb{R}^{\mathbb{N}}$ with the product sigma algebra $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$, which results in a measurable space $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$. On $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$, we introduce so-called *p-exponential probability measures.*

Definition 2.4. [weighted p-exponential measure $\mu_{\boldsymbol{\beta}}$] Let $\boldsymbol{\beta} = (\beta_j)_{j\geq 1} \in (0,\infty)^{\mathbb{N}}$ denote a decreasing sequence of positive real numbers and let ξ_j, $j \in \mathbb{N}$, denote i.i.d. random variables in $\mathbb{R}$ with p-exponential probability density function $\phi_p(y) \simeq \exp(-|y|^p/p)$, $y \in \mathbb{R}$, where $p \in [1,2]$ denotes a parameter. On $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$, we define the probability measure $\mu_{\boldsymbol{\beta}}$ to be the law of $(\beta_j \xi_j)_{j\geq 1}$ and refer to this measure as *p-exponential measure* with *scaling sequence* $\boldsymbol{\beta} = (\beta_j)_{j\geq 1}$.

Having at hand $\mu_{\boldsymbol{\beta}}$, a probability measure on $X = C^0(\overline{D})$ is obtained by randomizing the coefficients $\{y_\lambda\}_{\lambda\in\mathcal{J}}$ in (2.3). To this end, we select

$$y_\lambda = \beta_\lambda \xi_\lambda,\ \ \xi_\lambda \overset{\text{i.i.d.}}{\sim} \phi_p,\ p\in[1,2],\ \ \beta_\lambda = 2^{-(d/2+\alpha)|\lambda|},\ \alpha > 0\ . \tag{2.4}$$

For α and p as in (2.4), we refer to $\mu_{\boldsymbol{\beta}}$ as in Definition 2.4 as an α-regular, p-exponential probability measure $\mu_{\boldsymbol{\beta}}$ on $C^0(\overline{D})$. The Schauder basis representation (2.3) induces a prior probability measure on the space X of uncertain PDE inputs, as pushforward of $\mu_{\boldsymbol{\beta}}$ on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$. Specifically, the law of u as in (2.3) with respect to $\mu_{\boldsymbol{\beta}}$ is a probability measure on $C^0(\overline{D})$.

We shall be interested in the case that X_+ is a subspace of X which is norm closed in X and which identifies elements of X with additional spatial regularity. The weight sequence $\boldsymbol{\beta}$ concentrates i.i.d. samples under $\mu_{\boldsymbol{\beta}}$ in subspaces of $\mathbb{R}^{\mathbb{N}}$.

Proposition 2.5 ([3, Proposition 6.1])**.** *For the p-exponential measure $\mu_{\boldsymbol{\beta}}$ as specified in* (2.4) *with weight sequence $\boldsymbol{\beta}$ and for any $p \in [1,2]$ and $\alpha > 0$, there holds $(\mu_{\boldsymbol{\beta}} \circ u^{-1})(C^\rho(\overline{D})) = 1$ for $0 \leq \rho < \min\{\alpha, 1\}$.*

The case $p = 2$ is, therefore, set in a Hilbertian sequence space and for $p = 2$ a random element in a separable Hilbert space X can be identified with a Gaussian product measure, where the basis elements $\tilde{\psi}_j$ in the expansion [34] are (suitably scaled) eigenfunctions of the covariance operator.

We illustrated the abstract concepts with several concrete constructions of Besov priors; some of these shall be employed in the numerical experiments ahead. For a more comprehensive presentation on Schauder bases and corresponding Banach spaces we refer to [38]. In [38, Chapter 5], abstract constructions of representation systems with stability properties in scales of Besov spaces, in general Lipschitz domains, are provided. Wavelet-type Riesz bases $\boldsymbol{\Psi}$ with locally supported elements in bounded, polygonal and polyhedral domains D with stability in scales of Hilbertian Sobolev spaces have been constructed in [28, 30]. Function systems of the type considered in Example 2.2 are also available in certain Lipschitz domains. Specifically, Ψ may be chosen as a system of biorthogonal, piecewise polynomial wavelet functions in the case that D is a polygon in $\mathbb{R}^2$, cf. [30]. More generally for Lipschitz domains satisfying [38, Definition 3.4(iii)], by [38, Theorem 4.23] wavelet systems exist that satisfy the conditions specified in Example 2.2.

3. Elliptic forward problem

For the ensuing presentation and analysis of QMC quadrature in UQ, we use the notation for the p-exponential measure that

$$\mu_p(\,\mathrm{d}\boldsymbol{y}) := \bigotimes_{j\geq 1} \phi_p(y_j)\,\mathrm{d}y_j,$$

where we recall the p-exponential probability density $\phi_p(y) \simeq \exp(-|y|^p/p)$, $y \in \mathbb{R}$, $p \in [1,2]$. Here, we have replaced the subscript of the weight sequence $\boldsymbol{\beta}$ by the parameter p. It is costumary in QMC literature to study unweighted product measures. The weight sequence $\boldsymbol{\beta}$ from Definition 2.4 will appear as a certain decay condition on the function system Ψ (see ahead (3.5)). On the countable product of real lines

$$\Omega := \mathbb{R}^{\mathbb{N}},$$

endowed with the sigma-algebra $\mathcal{A}$ generated by cylinders of Borel sets on $\mathbb{R}$, the product measure μ_p is a probability measure. Let us denote Lebesgue–Bochner spaces by $L^r(\Omega, \mu_p; E)$ for Banach spaces E and any $r \in [1, \infty)$. Elements of $L^r(\Omega, \mu_p; E)$ are strongly measurable maps from $(\Omega, \mathcal{A})$ such that the r-th power of their E-norm is μ_p-integrable, see for example [39, Sections V.4 and V.5].

In the bounded Lipschitz domain $D \subset \mathbb{R}^d$ of dimension $d \geq 2$, we consider formally the elliptic PDE with uncertain coefficient u and given, deterministic right hand side f:

$$-\nabla \cdot ((a_0 + \exp(u))\nabla q) = f, \quad q\Big|_{\partial D} = 0. \tag{3.1}$$

We shall work with a standard, primal variational formulation of (3.1). Consider a random coefficient u in (3.1) such that $u : \Omega \to L^\infty(D)$ is strongly measurable and there holds, μ_p-a.s., $\|u\|_{L^\infty(D)} < \infty$. Assume furthermore given a deterministic $a_0 \in L^\infty(D)$ such that $\operatorname{ess\,inf}_{x\in D}\{a_0(x)\} \geq 0$. Then for $a = a_0 + \exp(u)$ it holds μ_p-a.s. that $\operatorname{ess\,inf}_{x\in D} a(x, \omega) \geq \exp(-\|u\|_{L^\infty(D)}) > 0$. By the Lax–Milgram lemma, with μ_p probability one, for every deterministic, bounded linear functional f on $H_0^1(D)$, the variational form of (3.1): find $q : \Omega \to H_0^1(D)$ such that

$$\int_D (a_0 + \exp(u))\nabla q \cdot \nabla v \,\mathrm{d}x = \langle f, v\rangle \quad \forall v \in H_0^1(D) \tag{3.2}$$

admits a unique solution $q \in H_0^1(D)$.

Here $X = L^\infty(D)$, and $\mathcal{X} = H_0^1(D) = \{v \in H^1(D) : v|_{\partial D} = 0\}$ with dual space $\mathcal{X}^*$. The dual pairing on $\mathcal{X}^* \times \mathcal{X}$ is denoted by $\langle \cdot, \cdot \rangle$. For given, fixed $f \in \mathcal{X}^*$, the data-to-solution map $\mathcal{S}$, associates to each coefficient realization $u \in X$ in (3.1) the unique solution $q \in \mathcal{X}$ to (3.2). This allows to define sample-wise the random solution $[q : \Omega \to \mathcal{X} : \boldsymbol{y} \mapsto \mathcal{S}(u(\boldsymbol{y}))]$. The coefficient-to-solution map $\mathcal{S} : X \to \mathcal{X}$ is locally Lipschitz continuous: for fixed $f \in \mathcal{X}^*$ and for any $u_1, u_2 \in L^\infty(D)$ holds

$$\begin{aligned} \|\mathcal{S}(u_1) - \mathcal{S}(u_2)\|_{\mathcal{X}} &\leq \|f\|_{\mathcal{X}^*} \exp(\|u_1\|_X + \|u_2\|_X)\| \exp(u_1) - \exp(u_2)\|_X \\ &\leq \|f\|_{\mathcal{X}^*} \exp(2(\|u_1\|_X + \|u_2\|_X))\|u_1 - u_2\|_X. \end{aligned} \tag{3.3}$$

This implies $q = \mathcal{S}(u) : \Omega \to \mathcal{X}$ is strongly μ_p-measurable for $1 \leq p \leq 2$ as a composition of a strongly measurable and a continuous map, see [20, Lemma A.5]. The exponential growth of the map $u \mapsto \exp(u)$ in the diffusion coefficient in (3.2) naturally motivates distinguishing the case $p = 1$ and $p \in (1, 2]$. Analogous results to the Gaussian case ($p = 2$) can be expected and will be proved for $p \in (1, 2]$. The more delicate case $p = 1$ will require a smallness condition on u.

3.1. **Uncertainty parametrization.** The uncertain input u is assumed to be given by a series expansion

$$u(\boldsymbol{y}) = \sum_{j\geq 1} y_j \psi_j, \tag{3.4}$$

where the elements of the sequence $(\psi_j)_{j\geq 1}$ comprising the representation system $\boldsymbol{\Psi}$ are assumed bounded measurable on D and such that

$$\sum_{j\geq 1} \|\psi_j\|_X^t < \infty \tag{3.5}$$

for some $t \in (0,2]$. The coefficients y_j are assumed random, and i.i.d. with $y_j \sim \phi_p$. For parametric u as in (3.4), the space of admissible uncertain inputs X_+ is the closure of $\mathrm{span}\{\psi_j : j \geq 1\}$ with respect to the X-norm. With these assumptions, the uncertain PDE input a is for μ_p-a.e. $\boldsymbol{y} \in \Omega$ given by

$$a(\boldsymbol{y}) = a_0 + \exp(u(\boldsymbol{y})), \tag{3.6}$$

where $a_0 \in X$ is such that $\operatorname{ess\,inf}_{x\in D}\{a_0(x)\} \geq 0$. For every $s \in \mathbb{N}$, define the s-term truncations u^s and a^s of $[\boldsymbol{y} \mapsto u(\boldsymbol{y})]$ and of $[\boldsymbol{y} \mapsto a(\boldsymbol{y})]$ in (3.6) by

$$u^s := \sum_{j=1}^{s} y_j \psi_j \quad \text{and} \quad a^s := a_0 + \exp(u^s). \tag{3.7}$$

For $t \in (0,1]$ in (3.5), for every $r \in [1,\infty)$ and every $s \in \mathbb{N}$ holds, with $C_r = (\int_{\mathbb{R}} |y|^r \phi_p(y)\,\mathrm{d}y)^{1/r}$,

$$\|u - u^s\|_{L^r(\Omega,\mu_p;X)} \leq C_r \sum_{j>s} \|\psi_j\|_X \to 0 \ \text{ as } \ s \to \infty. \tag{3.8}$$

This can be seen by considering the partial sum of u, i.e., for every integer $s' > s$, by the triangle inequality, $\|u^{s'} - u^s\|_{L^r(\Omega,\mu_p;X)} \leq C_r \sum_{j=s+1}^{s'} \|\psi_j\|_X$. The limit $s' \to \infty$ exists in $L^r(\Omega,\mu_p;X)$, since this is a Banach space. It does not depend on $r \in [1,\infty)$, because for every $1 \leq r_1 < r_2$, $L^{r_2}(\Omega,\mu_p;L^{r_2}(D))$ is continuously embedded in $L^{r_1}(\Omega,\mu_p;L^{r_1}(D))$ by Hölder's inequality. As a consequence of (3.8), $u : \Omega \to X$ is strongly measurable.

A more refined argument presented in the following proposition yields an approximation rate bound in the range $t \in (0,2]$ in (3.5).

Proposition 3.1. *Let assumption* (3.5) *be satisfied for some* $t \in (0,2]$. *Then, for every* $r \in [1,\infty)$, $u \in L^r(\Omega,\mu_p;L^r(D))$. *Furthermore, there exists a positive constant* C_r *such that for every* $s \in \mathbb{N}$

$$\|u - u^s\|_{L^r(\Omega,\mu_p;L^r(D))} \leq C \left(\sum_{j>s} \|\psi_j\|_X^2 \right)^{1/2} .$$

Proof. For every $1 \leq r_1 < r_2$, $L^{r_2}(\Omega,\mu_p;L^{r_2}(D))$ is continuously embedded in $L^{r_1}(\Omega,\mu_p;L^{r_1}(D))$ by Hölder's inequality. It is therefore sufficient to consider the case $r \in 2\mathbb{N}$. Let thus $r = 2k$ for some $k \in \mathbb{N}$. Then, Fubini's theorem gives that for any $1 \leq s_1 \leq s_2 < \infty$ holds

$$\left\| \sum_{j=s_1}^{s_2} y_j \psi_j \right\|_{L^{2k}(\Omega,\mu_p;L^{2k}(D))}^{2k} = \int_D \int_\Omega \left(\sum_{j=s_1}^{s_2} y_j \psi_j \right)^{2k} \mu_{\mathrm{p}}(\mathrm{d}\boldsymbol{y})\,\mathrm{d}x,$$

where $|D|$ denotes the Lebesgue measure of D. Since the univariate density ϕ_p is an even function with respect to $y = 0$, all odd order moments of ϕ_p vanish. Thus, the twofold application of the multinomial theorem implies with the Jensen inequality that there exists a constant C_k, which only depends on k, such that

(3.9)
$$\begin{aligned}\int_D \int_\Omega \left(\sum_{j=s_1}^{s_2} y_j \psi_j \right)^{2k} \mu_{\mathrm{p}}(\mathrm{d}\boldsymbol{y})\, \mathrm{d}x &\leq |D| \int_{\mathbb{R}} |y|^{2k} \phi_p(y)\, \mathrm{d}y \sum_{|\boldsymbol{\tau}|=2k, \tau_j \text{ is even}} \binom{2k}{\boldsymbol{\tau}} \|\psi_j\|_X^{\tau_j} \\ &\leq |D| \int_{\mathbb{R}} |y|^{2k} \phi_p(y)\, \mathrm{d}y C_k \left(\sum_{j=s_1}^{s_2} \|\psi_j\|_X^2 \right)^k .\end{aligned}$$

Taking the $2k$-th root, implies that $(u^s)_{s\geq 1}$ is a Cauchy sequence in the Banach space $L^{2k}(\Omega, \mu_p; L^{2k}(D))$. The completeness of $L^{2k}(\Omega, \mu_p; L^{2k}(D))$, the sequence of partial sums admits a unique limit in $L^{2k}(\Omega, \mu_p; L^{2k}(D))$ which we denote by u. Since $L^{r_2}(\Omega, \mu_p; L^{r_2}(D))$ is continuously embedded in $L^{r_1}(\Omega, \mu_p; L^{r_1}(D))$ for every $1 \leq r_1 < r_2$, the limit u does not depend on k. The asserted estimate follows by choosing in (3.9) $s_1 = s+1$ and by taking the limit $s_2 \to \infty$. □

Proposition 3.2. *Suppose* (3.5) *holds with some* $t \in (0,1]$. *Then, for every* $(p,r) \in (1,2] \times [1,\infty)$ *or every* $(p,r) \in \{1\} \times [1, \inf_{j\geq 1} \|\psi_j\|_X^{-1})$, $a = a_0 + \exp(u) \in L^r(\Omega, \mu_p; X)$. *Furthermore, there exists a constant* C_r *such that for every* $s \in \mathbb{N}$

$$\|a^s\|_{L^r(\Omega,\mu_p;X)} \leq C_r$$

and

$$\|a - a^s\|_{L^r(\Omega,\mu_p;L^r(D))} \leq C_r \left(\sum_{j>s} \|\psi_j\|_X^2 \right)^{1/2} . \tag{3.10}$$

Proof. We begin with the case $p \in (1,2]$. The following elementary considerations will be of use in the ensuing proof. For any $x, y > 0$ and $\theta \in (0,1)$, Young's inequality implies that

$$xy = x^\theta x^{1-\theta} y \leq \frac{1}{p'} x^{\theta p'} + \frac{1}{p} x^{(1-\theta)p} y^p,$$

where $p' = p/(p-1)$ is the conjugate of p. The choice $\theta = p/(p'+p)$ implies

$$\forall x, y \geq 0 : \qquad xy \leq \frac{(p-1)x}{p} + \frac{xy^p}{p}. \tag{3.11}$$

Since $(\|\psi_j\|_X)_{j\geq 1} \in \ell^1(\mathbb{N})$, there exists $j_0 \in \mathbb{N}$ such that for every $j > j_0$, $r\|\psi_j\|_X < 1$. Thus,

$$\begin{aligned}\|\exp(u)\|_{L^r(\Omega,\mu_p;X)}^r &\leq \prod_{j\geq 1} \int_{\mathbb{R}} e^{r\|\psi_j\|_X |y_j|} \phi_p(y_j)\, \mathrm{d}y_j \\ &= C_{j_0} \prod_{j>j_0} \int_{\mathbb{R}} e^{r\|\psi_j\|_X |y_j|} \phi_p(y_j)\, \mathrm{d}y_j,\end{aligned} \tag{3.12}$$

where $C_{j_0} = \prod_{j=1}^{j_0} \int_{\mathbb{R}} e^{r\|\psi_j\|_X |y_j|} \phi_p(y_j)\, \mathrm{d}y_j$ is finite. By (3.11) for $j > j_0$,

$$\begin{aligned}\int_{\mathbb{R}} e^{r\|\psi_j\|_X |y_j|} \phi_p(y_j)\, \mathrm{d}y_j &\leq e^{(p-1)r\|\psi_j\|_X/p} \int_{\mathbb{R}} e^{r\|\psi_j\|_X |y_j|^p/p} \phi_p(y_j)\, \mathrm{d}y_j \\ &= \frac{e^{(p-1)r\|\psi_j\|_X/p}}{(1-r\|\psi_j\|_X)^{1/p}},\end{aligned} \tag{3.13}$$

where we used the fact that for $\phi_p(y) = c_p e^{-|y|^p/p}$ and any $\theta > 0$, $c_p \int_{\mathbb{R}} e^{-\theta|y|^p/p}\,\mathrm{d}y = \theta^{-1/p}$, which follows readily by the coordinate transformation $y \mapsto y\theta^{1/p}$. The fact that for any $b \in [0,1)$, $1/(1-b) = 1 + b/(1-b) \le \exp(b/(1-b))$ implies that

$$\prod_{j>j_0} \frac{e^{(p-1)r\|\psi_j\|_X/p}}{(1-r\|\psi_j\|_X)^{1/p}} \le \exp\left(\sum_{j>j_0} \frac{(p-1)r\|\psi_j\|_X}{p} + \frac{1}{p}\frac{r\|\psi_j\|_X}{1-r\|\psi_j\|_X}\right), \tag{3.14}$$

where we use that the sequence $(\|\psi_j\|_X)_{j\ge1}$ is summable. Thus, $\exp(u) \in L^r(\Omega,\mu_p;X)$. Since j_0 does not depend on s, the same proof also yields the second assertion.

In the case $p = 1$ and $r\sup_{j\ge1}\|\psi_j\|_X < 1$ for $r \ge 1$, by (3.12) (and as a special case of (3.14) with $j_0 = 0$)

$$\|\exp(u)\|^r_{L^r(\Omega,\mu_p;X)} \le \exp\left(\sum_{j\ge1} \frac{r\|\psi_j\|_X}{1-r\|\psi_j\|_X}\right) < \infty.$$

By the same argument, there exists a constant $C_r > 0$ such that for every $s \in \mathbb{N}$ holds $\|\exp(u^s)\|_{L^r(\Omega,\mu_p;X)} \le C_r$.

The fundamental theorem of calculus implies that for every $t_1, t_2 \in \mathbb{R}$, $|e^{t_2} - e^{t_1}| \le (e^{t_2} + e^{t_1})|t_2 - t_1|$. By the first assertion applied with $\bar r > r$ (for $p = 1$, $1 \le r < \bar r < \inf_{j\ge1}\|\psi_j\|_X^{-1}$), Proposition 3.1, and by the Hölder inequality,

$$\begin{aligned}\|a - a^s\|_{L^r(\Omega,\mu_p;L^r(D))} &\le \|a + a^s\|_{L^{\bar r}(\Omega,\mu_p;L^{\bar r}(D))}\|u - u^s\|_{L^{r\bar r/(\bar r - r)}(\Omega,\mu_p;L^{r\bar r/(\bar r - r)}(D))} \\ &\le C_{\bar r}\left(\sum_{j>s}\|\psi_j\|_X^2\right)^{1/2},\end{aligned}$$

where we use that the domain D is bounded. □

By Proposition 3.2, $\operatorname{ess\,inf}_{x\in D}\{a(x)\} > 0$ μ_p-a.s. Therefore, the bilinear form $\mathcal{X} \times \mathcal{X} \ni (w,v) \mapsto \int_{\mathbb{R}} a\nabla w \cdot \nabla v\,\mathrm{d}x \in \mathbb{R}$ is μ_p-a.s. continuous and coercive with random continuity "constant" $\|a\|_X$ and random coercivity "constant"

$$a_{\min} := \operatorname{ess\,inf}_{x\in D}\{a(x)\}.$$

Since $[X \ni v \mapsto \|v\|_X]$ and $[X \ni v \mapsto \operatorname{ess\,inf}_{x\in D}\{v(x)\}]$ are continuous, $\|a\|_X$ and $a_{\min}$ are μ_p-measurable and thus random variables. By the Lax–Milgram lemma, the solution to (3.2) $q : \Omega \to \mathcal{X}$ exists and is unique. Moreover, by the Lax–Milgram lemma and Proposition 3.2 for every $r \in [1,\infty)$ and every $p \in (1,2]$ holds

$$\|q\|_{L^r(\Omega,\mu_p;\mathcal{X})} \le \left\|\frac{1}{a_{\min}}\right\|_{L^r(\Omega,\mu_p)} \|f\|_{\mathcal{X}^*} < \infty. \tag{3.15}$$

For $p = 1$ and for any $r \in [1, \inf_{j\ge1}\|\psi_j\|_X^{-1})$, the estimate (3.15) holds also by Proposition 3.2.

Remark 3.3. Results corresponding to those in this section in the case of locally supported function systems $(\psi_j)_{j\ge1}$ have been derived in [19, Chapter 1] and generalize the analysis in [24, 22] to the case of Besov priors.

3.2. **Dimension truncation.** Let $q^s : \Omega \to \mathcal{X}$ be the solution to (3.2) with respect to the s-parametric random coefficient a^s for some $s \in \mathbb{N}$, i.e., $q^s = \mathcal{S}(u^s)$. For $s \in \mathbb{N}$, we define

$$a^s_{\min} := \operatorname{ess\,inf}_{x\in D}\{a^s(x)\}\ .$$

Since the mapping $[v \mapsto \operatorname{ess\,inf}_{x\in D}\{v(x)\}]$ is continuous from X to $\mathbb{R}$, $a^s_{\min} : \Omega \to (0,\infty)$ is μ_p-measurable, hence a random variable. Furthermore, for every $s \in \mathbb{N}$ holds μ_p-a.s. the lower bound $a^s_{\min} \geq \exp(-\|u^s\|_X)$.

Remark 3.4. The forward PDE solution $q^s : \Omega \to \mathcal{X}$ for s-parametric input with finite $s \in \mathbb{N}$, may also be viewed as a conditional expectation

$$q^s = \mathbb{E}^{\mu_p}(q|\mathcal{A}_s) \quad \text{with} \quad \mathcal{A}_s := \mathcal{B}(\mathbb{R}^s) \otimes \left(\bigotimes_{j>s} \sigma(\{y_j = 0\}) \right) . \tag{3.16}$$

The relation (3.16) states that the s-term truncation of the affine-parametric representation (3.4) of the uncertain input u can be interpreted statistically as inclusion of a-priori information on u into the prior expectation. Apart from facilitating QMC integration (which is based on finite-dimensional integration domains, see Section 5 ahead), s-term truncation also regularizes the Bayesian inverse problem.

Similarly to (3.15), as a consequence of Proposition 3.2, for every $r \in [1,\infty)$ and $p \in (1,2]$ there exists a constant C_r such that for every $s \in \mathbb{N}$ holds

$$\|q^s\|_{L^r(\Omega,\mu_p;\mathcal{X})} \leq \left\| \frac{1}{a^s_{\min}} \right\|_{L^r(\Omega,\mu_p)} \|f\|_{\mathcal{X}^*} \leq C_r \|f\|_{\mathcal{X}^*}. \tag{3.17}$$

For $p = 1$ and for any $r \in [1, \inf_{j\geq 1} \|\psi_j\|_X^{-1})$, the estimate (3.17) holds also due to Proposition 3.2.

Proposition 3.5. *Suppose* (3.5) *holds for some* $t \in (0,1]$ *and let* $p \in (1,2]$*. Assume that there exists* $\varepsilon > 0$ *such that* $q \in L^{\hat{r}}(\Omega,\mu_p;W^{1,2+\varepsilon}(D))$ *for every* $\hat{r} \in [1,\infty)$*. Then, for every* $r \in [1,\infty)$ *there exists a constant* C_r *such that for every* $s \in \mathbb{N}$

$$\|q - q^s\|_{L^r(\Omega,\mu_p;\mathcal{X})} \leq C_r \left(\sum_{j>s} \|\psi_j\|_X^2 \right)^{1/2} .$$

Proof. We observe that for every $r' \in [2,\infty)$ and $r'' = 2r'/(r'-2)$

$$\|q - q^s\|_{\mathcal{X}} \leq \frac{\||\nabla q|\|_{L^{r'}(D)}}{a^s_{\min}} \|a - a^s\|_{L^{r''}(D)}. \tag{3.18}$$

The assertion follows by (3.10) in Proposition 3.2 and by the choice $r' = 2 + \varepsilon$. □

If the assumption $q \in L^{\hat{r}}(\Omega,\mu_p;W^{1,2+\varepsilon}(D))$ in Proposition 3.5 only holds with $\varepsilon = 0$ as implied by (3.15), still an error estimate holds.

Proposition 3.6. *Suppose* (3.5) *holds for some* $t \in (0,1]$*. Let either* $p \in (1,2]$ *be arbitrary and* $r \in [1,\infty)$ *or let* $p = 1$ *and* $r \in [1, \inf_{j\geq 1} \|\psi_j\|_X^{-1}/4)$*.*

Then, there exists a constant C_r *such that for every* $s \in \mathbb{N}$

$$\|q - q^s\|_{L^r(\Omega,\mu_p;\mathcal{X})} \leq C_r \sum_{j>s} \|\psi_j\|_X.$$

Proof. A twofold application of Hölder's inequality implies with (3.18) and (3.3) and Proposition 3.2

$$\|q - q^s\|_{L^r(\Omega,\mu_p;\mathcal{X})} \leq \|f\|_{\mathcal{X}^*} \|\exp(2\|u\|_X)\|_{L^{2r(1+\varepsilon)}(\Omega,\mu_p)} \|\exp(2\|u^s\|_X)\|_{L^{2r(1+\varepsilon)}(\Omega,\mu_p)} \times \|u - u^s\|_{L^{r(1+\varepsilon)/\varepsilon}(\Omega,\mu_p;X)},$$

where $\varepsilon > 0$ satisfies $r(1+\varepsilon) < \inf_{j\geq 1} \|\psi_j\|_X^{-1}/4$. The assertion follows by Proposition 3.2 and (3.8). ☐

3.3. **Spatial discretization.** We discuss sufficent conditions on the function system $(\psi_j)_{j\geq 1}$ that imply error estimates of the spatial discretization.

3.3.1. *Solution regularity.* For the convergence rate analysis of the spatial approximation by the Finite Element Method (FEM), we study the regularity of the solution q with respect to the spatial coordinate $x \in D$. For conciseness of presentation, we suppose that D is either a bounded polygon in $\mathbb{R}^2$ with straight sides or D is a bounded polyhedron in $\mathbb{R}^3$ with plane faces.

Suppose that for some $\bar{t} \in \{1, 2\}$,

$$a_0 \in W^{1,\infty}(D) \quad \text{and} \quad \sum_{j>1} \||\nabla\psi_j|\|_X^{\bar{t}} < \infty. \tag{3.19}$$

Lemma 3.7. *In space dimension $d = 2$, let D be a bounded polygon with straight sides. Let $\omega \in (0, 2\pi)$ denote the largest interior angle at a corner of D. Suppose $(\psi_j)_{j\geq 1}$ satisfies (3.5) and (3.19) with some $t \in (0, 1]$ and some $\bar{t} \in \{1, 2\}$, respectively. Let $p \in (1, 2]$. Assume furthermore that $f \in L^2(D)$ in (3.1). Then, for any $0 < \tau < \min\{1, \pi/\omega\}$ and for every $r \in [1,\infty)$, there holds $q \in L^r(\Omega, \mu_p; H^{1+\tau}(D))$. Furthermore, for every $r \in [1,\infty)$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$*

$$\|q^s\|_{L^r(\Omega,\mu_p;H^{1+\tau}(D))} \leq C.$$

Proof. The proof of Proposition 3.1 is applicable and yields $u \in L^r(\Omega, \mu_p; W^{1,r}(D))$ for every $r \in [1,\infty)$. Thus, by the Sobolev embedding (see for example [37, Theorem 1.107]) $u \in L^r(\Omega, \mu_p; C^{1-\varepsilon}(\overline{D}))$ for every $\varepsilon \in (0, 1)$ and every $r \in [1,\infty)$. Since for every $v \in C^\rho(\overline{D})$ and every $\rho \in [0, 1]$, $\|\exp(v)\|_{C^\rho(\overline{D})} \leq \|\exp(v)\|_{C^0(\overline{D})}(1 + \|v\|_{C^\rho(\overline{D})})$, we conclude with Proposition 3.2 by the Cauchy–Schwarz inequality that $[u \mapsto \exp(u)] \in L^r(\Omega, \mu_p; C^{1-\varepsilon}(\overline{D}))$ for every $\varepsilon \in (0, 1)$ and for every $r \in [1,\infty)$. The regularity estimate [36, Lemma 5.2] implies the first assertion with the Cauchy–Schwarz inequality.

The uniformity with respect to s in the second assertion follows, since it is provided by Propositions 3.1 and 3.2 that were used in this proof previously. ☐

Lemma 3.8. *Let the assumptions of Lemma 3.7 be satisfied, but assume that $p = 1$, $\bar{t} = 1$. Then, for any $0 < \tau < \min\{1, \pi/\omega\}$ and for every $r \in [1, \inf_{j\geq 1} \|\psi_j\|_X^{-1})$, there holds $q \in L^r(\Omega, \mu_p; H^{1+\tau}(D))$. Furthermore, for every $r \in [1, \inf_{j\geq 1} \|\psi_j\|_X^{-1})$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$*

$$\|q^s\|_{L^r(\Omega,\mu_p;H^{1+\tau}(D))} \leq C.$$

Proof. The same argument that results in (3.8) implies that $u \in L^{r'}(\Omega, \mu_1; W^{1,\infty}(D))$ for every $r' \in [1,\infty)$. Moreover, for every $r' \in [1,\infty)$, there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$, $\|u^s\|_{L^{r'}(\Omega,\mu_1;W^{1,\infty}(D))} \leq C$.

The forward problem (3.1) may be rewritten as $-\Delta q = (\nabla a \cdot \nabla q + f)/a$, where Δ denotes the Dirichlet Laplacian. Using that $a = a_0 + \exp(u)$,

$$\|\Delta q\|_{L^2(D)} \leq \|\nabla u \cdot \nabla q\|_{L^2(D)} + \frac{\|\nabla a_0 \cdot \nabla q\|_{L^2(D)} + \|f\|_{L^2(D)}}{a_{\min}}.$$

Note that by the Hölder inequality and by (3.15), for every $\varepsilon > 0$ such that $r(1+\varepsilon) < \inf_{j\geq 1} \|\psi_j\|_X^{-1}$,

$$\|\nabla u \cdot \nabla q\|_{L^r(\Omega,\mu_1;L^2(D))} \leq \|u\|_{L^{r(1+\varepsilon)/\varepsilon}(\Omega,\mu_1;L^2(D))} \|f\|_{\mathcal{X}^*} \|a_{\min}^{-1}\|_{L^{r(1+\varepsilon)}(\Omega,\mu_1;L^2(D))}.$$

Thus, $\Delta q \in L^r(\Omega, \mu_1; L^2(D))$. The first assertion of the lemma follows, since $\Delta : H^{1+\tau}(D) \to L^2(D)$ is boundedly invertible. The uniform estimate with respect to the truncation s follows by (3.17). □

3.3.2. *Finite element discretization.* In the bounded Lipschitz polytope $D \subset \mathbb{R}^d$, $d = 2, 3$, consider families of quasiuniform, simplicial, shape regular, conforming triangulations of D. These triangulations can be obtained, for example, by uniform (so-called "red") refinement of a regular, coarse simplicial partition $\mathcal{T}_0$ of the domain D. As usual, the term "regular" indicates that any two (closed) distinct n-simplices $T, T' \in \mathcal{T}_0$ either have empty intersection, or intersect in an *entire* $n - k$ simplex for some $1 \leq k \leq n$.

On the sequence $\{\mathcal{T}_\ell\}_{\ell \geq 0}$ of such regular, simplicial triangulations, we consider continuous, piecewise linear Finite Element (FE) spaces

$$\mathcal{X}_\ell := \{v \in \mathcal{X} : v|_K \in \mathcal{P}^1(K)\ \forall K \in \mathcal{T}_\ell\}, \quad \ell \geq 0, \tag{3.20}$$

where $\mathcal{P}^1(K)$ denotes the polynomials of degree 1. Let $h_\ell = \max\{\mathrm{diam}(T) : T \in \mathcal{T}_\ell\}$ denote the meshwidth of $\mathcal{T}_\ell$. For every $s \in \mathbb{N}$, the random FE solution $q^{s,h_\ell} : \Omega \to \mathcal{X}_\ell$ is the sample-wise unique solution which solves, μ_p-a.s., the variational form: find $q^{s,h_\ell} \in \mathcal{X}_\ell$ such that

$$\int_D a^s \nabla q^{s,h_\ell} \cdot \nabla v = \langle f, v\rangle \quad \forall v \in \mathcal{X}_\ell. \tag{3.21}$$

The corresponding input-to-solution map is denoted by $\mathcal{S}_{h_\ell}$, i.e., $q^{s,h_\ell} = \mathcal{S}_{h_\ell}(u^s)$. As in (3.17), by Proposition 3.2 for every $r \in [1, \infty)$ and $p \in (1, 2]$ or $r \in [1, \inf_{j\geq 1} \|\psi_j\|_X^{-1})$ and $p = 1$, there exists a constant $C_r > 0$ such that for every $s \in \mathbb{N}$ and $h_\ell > 0$

$$\|q^{s,h_\ell}\|_{L^r(\Omega,\mu_p;\mathcal{X})} \leq \left\| \frac{1}{a^s_{\min}} \right\|_{L^r(\Omega,\mu_p)} \|f\|_{\mathcal{X}^*} \leq C_r \|f\|_{\mathcal{X}^*}. \tag{3.22}$$

By [24, Proposition 15] and its proof and Lemma 3.7 (and Lemma 3.8 in the case $p = 1$), we obtain a version of [24, Proposition 15] in the case of p-exponential densities.

Proposition 3.9. *Let $d = 2$ and let $\omega \in (0, 2\pi)$ denote the largest interior angle at a corner of the polygon D. Suppose that* (3.5) *holds for some $t \in (0, 1]$ and* (3.19) *holds for some $\bar{t} \in \{1, 2\}$. Assume either $p \in (1, 2]$ and $r \in [1, \infty)$ or assume $p = 1$, $r \in [1, \inf_{j\geq 1} \|\psi_j\|_X^{-1})$, and additionally $\bar{t} = 1$. For $\tau, \tau' \in [0, \min\{1, \pi/\omega\})$ and for every $f \in H^{-1+\tau}(D)$, there exists a constant $C > 0$ independent of $h_\ell > 0$ and $s \in \mathbb{N}$, such that*

$$\|q^s - q^{s,h_\ell}\|_{L^r(\Omega,\mu_p;H^{1-\tau'}(D))} \leq C h_\ell^{\tau+\tau'}. \tag{3.23}$$

3.4. **Remarks on higher order FEM in D.** The convergence bound (3.23) pertained to first order FEM on quasiuniform families of regular, simplicial triangulations of D. There, the FE convergence rate is limited by the maximal regularity of the solution in standard Sobolev spaces in D. As it is well-known, *local regularity* of solutions in compact subsets of the polygon D is maximal, i.e., it is only limited by the values of τ and τ' in Proposition 3.9, regardless of the values of the corner angle π/ω. Near corners of D, loss of Sobolev regularity is quantified by so-called *corner-weighted spaces* of Kondrat'ev type (see, e.g., [4, 17, 5]).

Consider again a bounded polygon $D \subset \mathbb{R}^2$ with straight edges and assume that $\tau \in \mathbb{N}_0$. For $\bar{t} = 1$ in (3.19), one finds as in (3.8) and in the proof of Lemma 3.7 that $u \in L^r(\Omega, \mu_p; W^{1,\infty}(D))$ for every $r \in [1, \infty)$. This allows us conclude regularity in Kondrat'ev spaces of the solution by using the regularity shifts of the Dirichlet Laplacian from [4] and [5], see [22, Section 2] for details. For parametric solutions taking values in weighted Kondrat'ev spaces, *graded local mesh refinement towards the vertices of D* will restore maximal FE convergence rates. We refer to [2] and the references there.

Specifically, for FE order $k \in \mathbb{N}$, the spaces $\mathcal{X}_\ell$ of continuous, Lagrangean Finite Elements of order k on regular, simplicial triangulations $\mathcal{T}_\ell$ of D in (3.20) are

$$\mathcal{X}_\ell := \{v \in \mathcal{X} : v|_K \in \mathcal{P}^k(K)\ \forall K \in \mathcal{T}_\ell\}, \quad \ell \geq 0, \tag{3.24}$$

where $\mathcal{P}^k(K)$ denotes the space of polynomials of total degree k. For optimal convergence rates of FEM of order $k > 1$, we assume in (3.6) that a_0 is $W^{k,\infty}$-regular and positive[1], i.e.

$$a_0 \in W^{k,\infty}(D), \quad \operatorname*{ess\,inf}_{x \in D} a_0(x) \geq 0\ . \tag{3.25}$$

For integer $k > 1$, the summability condition (3.19) is strengthened. Let us assume there exists $\bar{t} \in \{1, 2\}$ such that

$$\sum_{j>1} \|\psi_j\|^{\bar{t}}_{W^{k,\infty}(D)} < \infty. \tag{3.26}$$

Under conditions (3.5) with $t \in (0, 1]$, (3.25) and (3.26) with $\bar{t} = 1$, for every $\tau, \tau' > 0$, $1 < p \leq 2$ and for every $f \in H^{-1+\tau}(D)$, and for families $\{\mathcal{T}_\ell\}_{\ell \geq 0}$ of properly graded, regular triangular meshes in D (with mesh-grading depending on k and on $\tau, \tau' \geq 0$, see [2]), one obtains in place of (3.23) that for every $r \in [1, \infty)$ there exists a constant $C > 0$ independent of s and of ℓ such that there holds the bound

$$\|q^s - q^{s,h_\ell}\|_{L^r(\Omega,\mu_p;H^{1-\tau'}(D))} \leq C h_\ell^{\min\{k,\tau\}+\min\{k,\tau'\}}\ . \tag{3.27}$$

Here, for $\tau' > 1$, $H^{1-\tau'}(D)$ is the dual space of $\mathcal{X} \cap H^{-1+\tau'}(D)$.

We outline the argument for the proof of (3.27). Due to Céa's lemma and suitable approximation properties of $\mathcal{X}_\ell$ from, e.g., [2], it is sufficient that the $(k+1)$-th order Kondrat'ev space norm of q^s may be bounded uniformly with respect to truncation dimension s. For $k = 1$, regularity of the solution in 2nd order Kondrat'ev spaces follows by [22, Equation (15)] and the regularity shift of the Dirichlet Laplacian. Higher order regularity in Kondrat'ev spaces follows by the bootstrap argument as in the proof of [24, Proposition 15]. For $\bar{t} = 2$ in the condition (3.26), Proposition 3.9 (here with uniform refinement) holds also with the bound (3.27) provided that

[1] This can be weakened by introducing the weighted spaces $\mathcal{W}^{k,\infty}(D)$ of [5]; see also [19]

$\tau, \tau' \in [0, \min\{k, \pi/\omega\})$. These assertions are proved in a slightly more general setting in [19, Proposition 1.1.12 and Theorem 1.4.3].

4. Dimension truncation and spatial approximation in Bayesian estimation

The approximation properties by truncating the parameter dimension to s and the spatial approximation error by FEM in the forward map translate directly to corresponding approximation errors of the BIP. For a finite truncation dimension $s \in \mathbb{N}$ of the parametric expansions (3.7) in the forward models, define random variables in the corresponding Bayesian inverse estimate according to Section 2.1

$$Z'_s := (\mathcal{G} \circ \mathcal{S})(u^s) \exp(-\Phi(u^s; \delta)), \quad Z_s := \exp(-\Phi(u^s; \delta))$$

and

$$Z'_{s,h_\ell} := (\mathcal{G} \circ \mathcal{S}_{h_\ell})(u^s) \exp(-\Phi_h(u^s; \delta)), \quad Z_{s,h_\ell} := \exp(-\Phi_h(u^s; \delta)), \tag{4.1}$$

where

$$u \mapsto \Phi_h(u; \delta) := \frac{1}{2}((\mathcal{O} \circ \mathcal{S}_{h_\ell})(u) - \delta)^\top \Sigma^{-1}((\mathcal{O} \circ \mathcal{S}_{h_\ell})(u) - \delta).$$

Note that under the assumptions of Proposition 3.2, it follows by Proposition 3.2 that $\mu_p(X_+) = 1$. Since X_+ is separable by construction, Theorem 2.1 is applicable.

Proposition 4.1. *Let the assumptions of Propositions 3.5 and 3.9 be satisfied. If $p = 1$, assume additionally $\sup_{j\geq 1} \|\psi_j\|_X < 1/4$. Then, there exists a constant $C > 0$ (which does not depend on s and h_ℓ), such that*

$$\left| \frac{\mathbb{E}^{\mu_p}(Z')}{\mathbb{E}^{\mu_p}(Z)} - \frac{\mathbb{E}^{\mu_p}(Z'_{s,h_\ell})}{\mathbb{E}^{\mu_p}(Z_{s,h_\ell})} \right| \leq C \left(\sum_{j>s} \|\psi_j\|_X^\iota \right)^{1/\iota} + C h^{\tau+\tau'},$$

where $\iota = 2$ for $p \in (1, 2]$ and $\iota = 1$ for $p = 1$.

Proof. By elementary manipulations, it holds μ_p-a.s.

$$\begin{aligned} |Z - Z_{s,h_\ell}| &= |\exp(-\|\mathcal{O}(q) - \delta\|^2_{\Sigma^{-1}}/2) - \exp(-\|\mathcal{O}(q_{s,h_\ell}) - \delta\|^2_{\Sigma^{-1}}/2)| \\ &\leq |\|\mathcal{O}(q) - \delta\|^2_{\Sigma^{-1}} - \|\mathcal{O}(q_{s,h_\ell}) - \delta\|^2_{\Sigma^{-1}}| \\ &\leq (|\|\mathcal{O}(q) - \delta\|_{\Sigma^{-1}}| + \|\mathcal{O}(q_{s,h_\ell}) - \delta\|_{\Sigma^{-1}}|)|\mathcal{O}(q - q_{s,h_\ell})|, \end{aligned}$$

where $\|\xi\|^2_{\Sigma^{-1}} := \xi^\top \Sigma^{-1} \xi$, $\xi \in \mathbb{R}^K$. Thus, by the Cauchy–Schwarz inequality, Proposition 3.5 (and Remark 3.6 in the case $p = 1$) and Proposition 3.9,

$$|\mathbb{E}^{\mu_p}(Z - Z_{s,h_\ell})| \leq \mathbb{E}^{\mu_p}(|Z - Z_{s,h_\ell}|) \leq C \left(\sum_{j>s} \|\psi_j\|_X^\iota \right)^{1/\iota} + C h^{\tau+\tau'}, \tag{4.2}$$

where $\iota = 2$ for $p \in (1, 2]$ and $\iota = 1$ for $p = 1$. By a similar argument $|\mathbb{E}^{\mu_p}(Z' - Z'_{s,h_\ell})|$ is also upper bounded by the right hand side of (4.2) with possibly a different constant. Since $\mathbb{E}^{\mu_p}(Z) > 0$ by Theorem 2.1, we conclude that there exist $C > 0$, $\ell_0 \geq 0$, $s_0 \in \mathbb{N}$ such that for any $s \geq s_0$ and $h_\ell < h_{\ell_0}$,

$$\mathbb{E}^{\mu_p}(Z_{s,h_\ell}) \geq C > 0. \tag{4.3}$$

The assertion is then implied by the following observation

$$\left| \frac{\mathbb{E}^{\mu_p}(Z')}{\mathbb{E}^{\mu_p}(Z)} - \frac{\mathbb{E}^{\mu_p}(Z'_{s,h_\ell})}{\mathbb{E}^{\mu_p}(Z_{s,h_\ell})} \right| \leq \frac{|\mathbb{E}^{\mu_p}(Z')||\mathbb{E}^{\mu_p}(Z' - Z'_{s,h_\ell})|}{\mathbb{E}^{\mu_p}(Z)\mathbb{E}^{\mu_p}(Z_{s,h_\ell})} + \frac{|\mathbb{E}^{\mu_p}(Z - Z_{s,h_\ell})|}{\mathbb{E}^{\mu_p}(Z_{s,h_\ell})},$$

where $|\mathbb{E}^{\mu_p}(Z')| < \infty$ by (3.15). □

We remark that the constant $C > 0$ in the error bound strongly depends on the observation noise covariance $\Sigma > 0$ in the model (2.1). In general, $C \sim \exp(b\widetilde{\sigma}^{-1})$ for some $b > 0$, where $\widetilde{\sigma} > 0$ is the smallest eigenvalue of the matrix Σ.

5. Quasi-Monte Carlo integration with Besov priors

QMC integration by randomly shifted lattice rules is applicable to high dimensional integrals with respect to certain measures with unbounded support in $\mathbb{R}^s$. The dimension $s \in \mathbb{N}$ is an explicit discretization parameter. QMC error estimates and convergence rates will be uniform with respect to the integration dimension s, which is achieved by a suitable weighting of the coordinates in the basis representation (3.4) of the uncertain inputs. Specifically, this is encoded in the following norm of parametric integrands $F : \mathbb{R}^s \to \mathbb{R}$. For arbitrary, finite integration dimension $s \in \mathbb{N}$, it is given by

(5.1)
$$\|F\|_{\boldsymbol{\gamma},s} = \left(\sum_{\mathfrak{u} \subset \{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left| \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \partial_{\boldsymbol{y}}^{\mathfrak{u}} F(\boldsymbol{y}) \prod_{j \notin \mathfrak{u}} \phi_p(y_j) \,\mathrm{d}y_j \right|^2 \prod_{j \in \mathfrak{u}} \boldsymbol{w}^2(y_j) \,\mathrm{d}y_j \right)^{1/2},$$

where $\partial_{\boldsymbol{y}}^{\mathfrak{u}} = \frac{\partial^{|\mathfrak{u}|}}{\prod_{j \in \mathfrak{u}} \partial_{y_j}}$. Let $\mathcal{W}_{\boldsymbol{\gamma},s}$ be the closure of $\{F \in C^\infty(\mathbb{R}^s) : \|F\|_{\boldsymbol{\gamma},s} < \infty\}$ with respect to the norm in (5.1). Here, the *QMC weights* $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \{1:s\}}$ are a collection of positive numbers and $\{1 : s\} := \{1, \dots, s\}$. In (5.1), the weight function $\boldsymbol{w}$ is yet to be specified. In the present work, we choose, for $1 \le p \le 2$, the p-exponential probability density given by

$$\phi_p(y) = \frac{1}{2p^{1/2}\Gamma(1/p)} e^{-\frac{|y|^p}{p}}, \quad y \in \mathbb{R},$$

where $\Gamma(\cdot)$ denotes the gamma function. We select the QMC weight function $\boldsymbol{w}$ in (5.1) as exponentially decaying with parameter $\alpha > 0$

$$\boldsymbol{w}^2(y) := e^{-\alpha|y|}, \quad y \in \mathbb{R}. \tag{5.2}$$

We consider QMC integration by so-called *randomly shifted lattice rules*, where the QMC points are given by [12, 25, 26]

$$\boldsymbol{y}^{(i)} = \Phi_s^{-1}\left(\left\{ \frac{(i+1)\boldsymbol{z}}{N} + \boldsymbol{\Delta} \right\} \right), \quad i = 0, \dots, N-1, \tag{5.3}$$

where the generating vector $\boldsymbol{z} \in \{1, \dots, N-1\}^s$ is such that every entry z_j and N are coprime. The random shift $\boldsymbol{\Delta}$ has independent and uniformly distributed entries in $[0, 1]$. The expectation with respect to $\boldsymbol{\Delta}$ will be denoted by $\mathbb{E}^{\boldsymbol{\Delta}}$. In (5.3), Φ_s denote the coordinate marginal distribution function corresponding to the density ϕ_p. For any truncation dimension $s \in \mathbb{N}$, denote $\mu_p^s(\mathrm{d}\boldsymbol{y}) := \bigotimes_{j=1}^s \phi_p(y_j)\,\mathrm{d}y_j$. Concrete choices of QMC weights $\boldsymbol{\gamma}$ in (5.1) and of the weight function $\boldsymbol{w}^2$ in (5.2) are an input for the so-called *fast component-by-component (CBC) algorithm* for the effiicent computation of the QMC generating vector $\boldsymbol{z}$, cf. [29, Section 5.2]. The equal weight QMC quadrature with random shift $\boldsymbol{\Delta}$ and generating vector $\boldsymbol{z}$ is then given by

$$Q^{\mathbf{\Delta}}_{s,N}(F) = \frac{1}{N}\sum_{i=0}^{N-1} F(\boldsymbol{y}^{(i)}) \approx \int_{\mathbb{R}^s} F(\boldsymbol{y})\mu_p^s(\,\mathrm{d}\boldsymbol{y}) = \mathbb{E}^{\mu_p^s}(F).$$

Theorem 5.1. *Let $p \in [1,2]$. Suppose that the generating vector $\boldsymbol{z}$ in (5.3) is obtained by the CBC algorithm with respect to QMC weights $\boldsymbol{\gamma}$. For every $\lambda \in (1/(2r),1]$ and for integrands $F \in \mathcal{W}_{\boldsymbol{\gamma},s}$,*

$$\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|\mathbb{E}^{\mu_p^s}(F) - Q^{\mathbf{\Delta}}_{s,N}(F)|^2)} \le \left(\frac{1}{\varphi(N)}\sum_{\emptyset\neq\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} C^{|\mathfrak{u}|}\right)^{1/(2\lambda)} \|F\|_{\boldsymbol{\gamma},s},$$

where $\varphi(\cdot)$ denotes Euler's totient function. For $p\in(1,2]$

$$C = \left[\frac{2p^{1/p}\Gamma(1/p)}{\pi^{2r}r(1-r)}\exp\left(\left(\frac{\alpha^p}{2(1-r)}\right)^{1/(p-1)}(1-p^{-1})\right)\right]^{\lambda}\zeta(2r\lambda),$$

where $r\in(1/2,1)$ is arbitrary and where $\zeta(\cdot)$ denotes Riemann's zeta function. For $p=1$ additionally suppose $\alpha\in(0,1)$. Then

$$C = \left(\frac{2\pi^{\alpha}}{(2-\alpha)\alpha}\right)^{\lambda}\zeta(2r\lambda) \quad \text{and} \quad r = 1-\frac{\alpha}{2}.$$

Proof. The assertion will follow by [29, Theorem 8], once we estimated a certain function $\widehat{\theta}$, which is defined below in (5.5). We shall verify that

$$\exp\left(\frac{|\Phi^{-1}(t)|^p}{p}\right) \le \frac{1}{t}, \quad \forall t\in(0,1/2). \tag{5.4}$$

Along the lines of the equivalences in [26, Equation (26)], for every $t\in(0,1/2)$,

$$\exp\left(\frac{|\Phi^{-1}(t)|^p}{p}\right) \le \frac{1}{t} \Leftrightarrow \Phi^{-1}(t) \ge -(-p\log(t))^{1/p} \Leftrightarrow t \ge \Phi(-(-p\log(t))^{1/p}).$$

The first derivative of $f(t) := t - \Phi(-(-p\log(t))^{1/p})$, $t\in(0,1/2)$, is given by $f'(t) := 1 - 1/(2\Gamma(1/p)(-\log(t))^{1-1/p})$. It holds that $f'(t)\ge 0$, $t\in(0,1/2)$ if $t \le \exp(-(2\Gamma(1/p))^{p/(1-p)})$, $t\in(0,1/2)$. The latter estimate follows, since

$$0.5685 \le \exp(-(2\Gamma_{\min})^{-1}) \le \exp(-(2\Gamma(1/p))^{p/(1-p)}),$$

where $\Gamma_{\min} = \min_{x\in(0,\infty)}\{\Gamma(x)\} \ge 0.8856$. We thus conlude the claim in (5.4).

Let us consider the case $p\in(1,2]$ first. We observe that (5.4) implies for arbitrary $h\ge 1$

$$\begin{aligned} \widehat{\theta}(h) &:= \frac{2}{\pi^2h^2}\int_0^{1/2}\frac{\sin(\pi h t)}{\boldsymbol{w}^2(\Phi^{-1}(t))\phi(\Phi^{-1}(t))}\,\mathrm{d}t \\ &= \frac{p^{1/p}\Gamma(1/p)}{\pi^2h^2}\int_0^{1/2}\sin(\pi h t)\exp\left(p^{-1}|\Phi^{-1}(t)|^p - \alpha\Phi^{-1}(t)\right)\,\mathrm{d}t \\ &\le \frac{p^{1/p}\Gamma(1/p)}{\pi^2h^2}\int_0^{1/2}\frac{\sin(\pi h t)}{t}\exp\left(\alpha(-p\log(t))^{1/p}\right)\,\mathrm{d}t. \end{aligned} \tag{5.5}$$

For any $\sigma > 0$, the function $[1,\infty) \ni y \mapsto \exp(\nu(\log(y))^{1/p})/y^\sigma$ attains its maximum

$$\exp\left(\nu\left(\frac{\nu}{p\sigma}\right)^{1/(p-1)} - \sigma\left(\frac{\nu}{p\sigma}\right)^{p/(p-1)}\right) \tag{5.6}$$

at $y_0 = \exp((\nu/(p\sigma))^{p/(p-1)})$, which can be verified by elementary manipulations. Thus, choosing $\nu = \alpha p^{1/p}$ and $\sigma = 2(1-r) =: 2\delta$ in (5.6), (5.5) implies with [26, Lemma 3]

$$\widehat{\theta}(h) \le \frac{2p^{1/p}\Gamma(1/p)}{\pi^{2-2\delta}(1-\delta)\delta}\exp\left(\left(\frac{\alpha^p}{2\delta}\right)^{1/(p-1)}(1-p^{-1})\right)h^{-2(1-\delta)}.$$

Thus, the assumption of [29, Theorem 8], i.e., [29, Equation (42)], is satisfied with $r = (1-\delta)$ and the assertion follows for $p \in (1,2]$.

For $p = 1$, we can directly estimate (5.5) and obtain with [26, Lemma 3]

$$\widehat{\theta}(h) = \frac{1}{\pi^2 h^2}\int_0^{1/2}\frac{\sin(\pi h t)}{t^{1+\alpha}}\,\mathrm{d}t \le \frac{2\pi^\alpha}{(2-\alpha)\alpha}h^{-2+\alpha}$$

and conclude the assertion with [29, Theorem 8]. □

Remark 5.2. A convergence rate bound of QMC with randomly shifted lattice rules and Besov-type QMC weight functions has been obtained in [19, Chapter 2].

6. Quasi-Monte Carlo Bayesian estimation

We begin by a novel convergence estimate of QMC with Besov priors for a class of generic integrands F, which will be used in the following discussion of the approximation properties of QMC by randomly shifted lattice rules for BIP under Besov priors.

Theorem 6.1. *Let $p \in [1,2]$. Suppose that integrands F satisfy the bound*

$$|\partial_{\boldsymbol{y}}^{\mathfrak{u}} F(\boldsymbol{y})| \le C_1 \exp\left(\sum_{j\ge1}|y_j|\mathfrak{b}_j\right)(|\mathfrak{u}|!)^\varrho C_2^{|\mathfrak{u}|}\prod_{j\in\mathfrak{u}} b_j \tag{6.1}$$

for some $\varrho \ge 1$ and for some $(b_j)_{j\ge1} \in (0,\infty)^{\mathbb{N}} \cap \ell^t(\mathbb{N})$ for some $t \in (0, 1/\varrho)$ and $(\mathfrak{b}_j)_{j\ge1} \in (0,\infty)^{\mathbb{N}} \cap \ell^1(\mathbb{N})$. Let $\alpha > 0$ in (5.2) satisfy $\alpha > 2\sup_{j\ge1}\{\mathfrak{b}_j\}$. If $p = 1$, additionally assume that $\alpha < 1$. Define product and order dependent QMC weights by $\gamma_\emptyset = 1$ and

$$\gamma_{\mathfrak{u}} = (|\mathfrak{u}|!)^{2\varrho/(1+\lambda)}\prod_{j\in\mathfrak{u}}\left(\frac{2C_2^2 b_j^2}{(\alpha-2\mathfrak{b}_j)C}\right)^{1/(1+\lambda)}, \quad \mathfrak{u} \subset \mathbb{N}, 1 \le |\mathfrak{u}| < \infty, \tag{6.2}$$

where C is specified in Theorem 5.1, and

$$\lambda = \begin{cases} t/(2-t) & \text{if } \varrho < 3/2, t \in (2/3, 1/\varrho), \\ 1/(2-2\varepsilon) & \text{if } t \in (0, 2/3], t < 1/\varrho, \end{cases}$$

and some $\varepsilon \in (0,1/2)$. Furthermore for $p = 1$, $\lambda > 1/(2-\alpha)$. Let $s, N \in \mathbb{N}$. Then, there exist a constant $C' > 0'$ (independent of s, N) and a QMC randomly shifted *lattice rule given by a generating vector constructed by the fast CBC algorithm* [29, Section 5.2] *such that*

$$\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|\mathbb{E}^{\mu_p^s}(F) - Q^{\mathbf{\Delta}}_{s,N}(F)|^2)} \leq C' \begin{cases} \varphi(N)^{-\min\{1/t-1/2,1-\varepsilon\}} & \textit{if } p \in (1,2], \\ \varphi(N)^{-\min\{1/t-1/2,1-\alpha/2-\varepsilon\}} & \textit{if } p = 1. \end{cases}$$

Proof. The proof consists of two steps. Firstly, the $\mathcal{W}_{\boldsymbol{\gamma},s}$-norm of the integrand F is bounded. Secondly, the choice of weights is justified to ensure convergence bounds that do not depend on the integration dimension s.

Since $(\mathfrak{b}_j)_{j\geq 1} \in \ell^1(\mathbb{N})$, there exists $j_0 \in \mathbb{N}_0$ such that $2\mathfrak{b}_j < 1$ for every $j > j_0$. By [16, Equations (4.14) and (4.16)] and by (3.13) and (3.14),

(6.3)

$$\begin{aligned} &\|F\|^2_{\boldsymbol{\gamma},s} \\ &\leq C_{j_0} C_1^2 \sum_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1} (|\mathfrak{u}|!)^{2\varrho} \prod_{j\in\mathfrak{u}} \left(\frac{2C_2^2 b_j^2}{\alpha - 2\mathfrak{b}_j}\right) \prod_{j\in\{1:s\}\setminus(\mathfrak{u}\cup\{1:j_0\})} \left(\frac{e^{2(p-1)\mathfrak{b}_j/p}}{(1-2\mathfrak{b}_j)^{1/p}}\right) \\ &\leq C_{j_0} C_1^2 \exp\left(\sum_{j>j_0} \frac{2(p-1)\mathfrak{b}_j}{p} + \frac{1}{p}\frac{2\mathfrak{b}_j}{1-2\mathfrak{b}_j}\right) \sum_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1} (|\mathfrak{u}|!)^{2\varrho} \prod_{j\in\mathfrak{u}} \left(\frac{2C_2^2 b_j^2}{\alpha - 2\mathfrak{b}_j}\right), \end{aligned}$$

where the constant $C_{j_0} = \prod_{j=1}^{j_0} \int_{\mathbb{R}} e^{2\mathfrak{b}_j|y_j|} \phi_p(y_j)\,\mathrm{d}y_j$ is finite. For $p = 1$, $j_0 = 0$ and $C_{j_0} = 1$ by the convention that empty products are equal to one.

The choice of the QMC weights in (6.2) is a consequence of the optimization procedure of the expression (6.3) and $\sum_{\mathfrak{u}\subset\mathbb{N}} \gamma_{\mathfrak{u}}^{\lambda} C^{|\mathfrak{u}|}$ from Theorem 5.1, see the proof of [16, Theorem 20]. Inserting the QMC weights, we obtain

(6.4)

$$\sum_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{-1} (|\mathfrak{u}|!)^{2\varrho} \prod_{j\in\mathfrak{u}} \left(\frac{2C_2^2 b_j^2}{\alpha - 2\mathfrak{b}_j}\right) = \sum_{\mathfrak{u}\subset\{1:s\}} (|\mathfrak{u}|!)^{2\varrho\lambda/(1+\lambda)} \prod_{j\in\mathfrak{u}} \left(\frac{2C_2^2 b_j^2 C^{1/\lambda}}{\alpha - 2\mathfrak{b}_j}\right)^{\lambda/(1+\lambda)}.$$

The right hand side of (6.4) also equals the expression $\sum_{\mathfrak{u}\subset\mathbb{N}} \gamma_{\mathfrak{u}}^{\lambda} C^{|\mathfrak{u}|}$ from Theorem 5.1. We differentiate two regimes of the value of ϱ. For $\varrho < 3/2$, $\lambda \in (1/2, 1/(2\varrho - 1))$ is admissible and we obtain by Hölder's inequality with conjugate exponents $(1+\lambda)/2\lambda\varrho > 1$ and $(1+\lambda)/(1-\lambda(2\varrho-1)) > 1$,

$$\begin{aligned} &\sum_{\mathfrak{u}\subset\{1:s\}} (|\mathfrak{u}|!)^{2\varrho\lambda/(1+\lambda)} \prod_{j\in\mathfrak{u}} \left(\frac{2C_2^2 b_j^2 C^{1/\lambda}}{\alpha - 2\mathfrak{b}_j}\right)^{\lambda/(1+\lambda)} \\ &\leq \left(\sum_{\mathfrak{u}\subset\{1:s\}} |\mathfrak{u}|! \prod_{j\in\mathfrak{u}} A_j\right)^{2\varrho\lambda/(1+\lambda)} \\ &\quad\times \left(\sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \left(\frac{2C_2^2 b_j^2 C^{1/\lambda}}{(\alpha - 2\mathfrak{b}_j) A_j^{2\varrho}}\right)^{\lambda/(1-\lambda(2\varrho-1))}\right)^{(1-\lambda(2\varrho-1))/(1+\lambda)} \\ &\leq \left(\frac{1}{1-\sum_{j\geq 1} A_j}\right)^{2\varrho\lambda/(1+\lambda)} \exp\left(\widetilde{C} \sum_{j\geq 1} \left(\frac{b_j^2}{A_j^{2\varrho}}\right)^{\lambda/(1-\lambda(2\varrho-1))}\right), \end{aligned} \tag{6.5}$$

where

$$\widetilde{C} = \frac{1-\lambda(2\varrho-1)}{1+\lambda}\left(\frac{2C_2^2 C^{1/\lambda}}{\alpha - 2\sup_{j>1}\{\mathfrak{b}_j\}}\right)^{\lambda/(1-\lambda(2\varrho-1))}$$

and $(A_j)_{j\geq 1} \in (0,\infty)^{\mathbb{N}}$ is such that $\sum_{j\geq 1} A_j < 1$. We choose $(A_j)_{j\geq 1}$ as

$$A_j := \frac{b_j^t}{\sum_{j'\geq 1} b_{j'}^t + \varepsilon'}, \quad j \geq 1,$$

for some $\varepsilon' > 0$. There holds $\sum_{j\geq 1}(b_j^2/A_j^{2\varrho})^{\lambda/(1-\lambda(2\varrho-1))} < \infty$ if

$$(1-\varrho t)\frac{2\lambda}{1-\lambda(2\varrho-1)} \geq t \quad \Leftrightarrow \quad \lambda \geq \frac{t}{2-t},$$

where the latter is satisfied for $\lambda = t/(2-t)$ for $t \in (2/3, 1/\varrho)$ and $\lambda = 1/(2-2\varepsilon)$ for $t \in (0, 2/3]$ and for some $\varepsilon \in (0, 1/2)$. Thus, the assertion follows in the regime $\varrho < 3/2$ by Theorem 5.1.

For $\varrho \geq 3/2$, we prove boundedness of the expression on the left hand side of (6.5) by the simple fact that for any positive sequence $(a_{\mathfrak{u}})_{\mathfrak{u}\subset\{1:s\}}$ and any $\rho < 1$, $\sum_{\mathfrak{u}\subset\{1:s\}} a_{\mathfrak{u}} \leq (\sum_{\mathfrak{u}\subset\{1:s\}} a_{\mathfrak{u}}^{\rho})^{1/\rho}$. Specifically, for some $c \in [1/\varrho, 3/(2\varrho))$, $\widetilde{\varrho} := \varrho c \in [1, 3/2)$, $\widetilde{b}_j := b_j^c$, $j \geq 1$, and

$$\begin{aligned}
&\sum_{\mathfrak{u}\subset\{1:s\}} (|\mathfrak{u}|!)^{2\varrho\lambda/(1+\lambda)} \prod_{j\in\mathfrak{u}} \left(\frac{2C_2^2 b_j^2 C^{1/\lambda}}{\alpha - 2b_j}\right)^{\lambda/(1+\lambda)} \\
&\leq \left(\sum_{\mathfrak{u}\subset\{1:s\}} (|\mathfrak{u}|!)^{2\widetilde{\varrho}\lambda/(1+\lambda)} \left(\frac{2C_2^2 C^{1/\lambda}}{\alpha - 2\mathfrak{b}_j}\right)^{|\mathfrak{u}|c\lambda/(1+\lambda)} \prod_{j\in\mathfrak{u}} \widetilde{b}_j^{2\lambda/(1+\lambda)}\right)^{1/c}.
\end{aligned}$$

Now the same argument as above can be applied with $\widetilde{\varrho} = \varrho c$, $(\widetilde{b}_j = b_j^c)_{j\geq 1} \in \ell^{\widetilde{t}}(\mathbb{N})$ for $\widetilde{t} = t/c$ in the latter case $\widetilde{t} \in (0, 2/3]$. Particularly, the choice $c = (3/2)t$ is admissible, where we may assume w.l.o.g. that $(3/2)t \geq 1/\varrho$. This shows the assertion also in the regime $0 < t < 1/\varrho \leq 2/3$. □

Remark 6.2. For every linear functional $\mathcal{G} \in \mathcal{X}^*$, the assumption (6.1) in Theorem 6.1 is satisfied for $[\boldsymbol{y} \to \mathcal{G}(q(\boldsymbol{y}))]$ with $\mathfrak{b}_j = b_j = \|\psi_j\|_X$, $j \geq 1$, and $\varrho = 1$ by [16, Theorem 14]. This enables QMC with dimension-independent convergence rates to approximate the prior expectation of $\mathcal{G}(q)$.

Recall from Section 4, the dimensionally truncated and FE discretized integrands that are part of the BIP. Specifically, $Z'(\boldsymbol{y}) = \mathcal{G}(q(\boldsymbol{y}))\exp(-\Phi(u(\boldsymbol{y});\delta)$ and $Z(\boldsymbol{y}) = \exp(-\Phi(u(\boldsymbol{y});\delta)$ and the respective versions after dimension truncation and FE discretization, denoted by Z'_{s,h_ℓ} and Z_{s,h_ℓ}, were introduced in (4.1).

Proposition 6.3. *Let $p \in [1,2]$. Suppose that the observation functional $\mathcal{O}$ and the parametric posterior $[\boldsymbol{y} \mapsto u(\boldsymbol{y})]$ are such that (6.1) is satisfied by $[\boldsymbol{y} \mapsto Z_{s,h_\ell}(\boldsymbol{y})]$ for some $\varrho \geq 1$ and sequences $(b_j)_{j\geq 1}$, $(\mathfrak{b}_j)_{j\geq 1}$ with constants that neither depend on $s \in \mathbb{N}$ nor on $h_\ell > 0$. Let α be as in the assumptions of Theorem 6.1. For $p \in (1,2]$, let $\varepsilon > 0$ be arbitrary and for $p = 1$, let $\varepsilon > 0$ satisfy $\varepsilon > 2\sup_{j\geq 1}\|\psi_j\|_X/(1 - 2\sup_{j\geq 1}\|\psi_j\|_X)$. Then, there exists a constant $C_\varepsilon > 0$ that neither depends on s*

nor on h_ℓ *such that*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}\left(\left|\frac{\mathbb{E}^{\mu_p}(Z'_{s,h_\ell})}{\mathbb{E}^{\mu_p}(Z_{s,h_\ell})}-\frac{Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})}{Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})}\right|^2\right)}\leq C_\varepsilon N^{-\chi+\varepsilon},$$

where, for any $0<t<1/\varrho\leq 1$,

$$\chi=\begin{cases}\min\{1/t-1/2,1\} & \textit{if } p\in(1,2],\\ \min\{1/t-1/2,1-\alpha/2\} & \textit{if } p=1.\end{cases} \tag{6.6}$$

Proof. It holds that

$$\left|\frac{\mathbb{E}^{\mu_p}(Z'_{s,h_\ell})}{\mathbb{E}^{\mu_p}(Z_{s,h_\ell})}-\frac{Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})}{Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})}\right|^2\leq\frac{2}{\mathbb{E}^{\mu_p}(Z_{s,h_\ell})^2}\Bigg[|\mathbb{E}^{\mu_p}(Z'_{s,h_\ell})-Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})|^2 + \frac{|Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})|^2}{Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})^2}|\mathbb{E}^{\mu_p}(Z_{s,h_\ell})-Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})|^2\Bigg]. \tag{6.7}$$

The previous estimate is to be understood in a.s. sense with respect to the random shift $\boldsymbol{\Delta}$. Since $\mathbb{E}^{\mu_p}(Z_{s,h_\ell})\geq C>0$ for some C that does not depend s,h, see (4.3), the QMC convergence rate bound of the first summand $\mathbb{E}^{\mu_p}(Z'_{s,h_\ell})-Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})$ in (6.7) follows by Theorem 6.1. The map $[\boldsymbol{y}\mapsto Z'(\boldsymbol{y})]$ also satisfies (6.3). This follows, for example, by [14, Proposition 3.4].

By the proof of [31, Lemma 5.3], the random quantity $|Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})|/Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})$ satisfies that for every $r,\bar r\in[1,\infty)$ such that $r\leq\bar r$,

$$\left(\mathbb{E}^{\boldsymbol{\Delta}}\left(\left(\frac{|Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})|}{Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})}\right)^r\right)\right)^{1/r}\leq\|\mathcal{O}(q_{s,h_\ell})\|_{L^{\bar r}(\Omega,\mu_p;\mathcal{X})}N^{1/\bar r}.$$

For every $\varepsilon>0$ by Hölder's inequality

$$\begin{aligned}&\mathbb{E}^{\boldsymbol{\Delta}}\left(\frac{|Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})|^2}{Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})^2}|\mathbb{E}^{\mu_p}(Z_{s,h_\ell})-Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})|^2\right)\\ &\leq\mathbb{E}^{\boldsymbol{\Delta}}\left(\frac{|Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})|^{2(1+\varepsilon)/\varepsilon}}{Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})^{2(1+\varepsilon)/\varepsilon}}\right)^{\varepsilon/(1+\varepsilon)}\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}^{\mu_p}(Z_{s,h_\ell})-Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})|^{2(1+\varepsilon)})^{1/(1+\varepsilon)}.\end{aligned}$$

Since $\|\mathcal{O}(q_{s,h_\ell})\|_{L^{\bar r}(\Omega,\mu_p;\mathcal{X})}$ may be bounded uniformly with respect to s,h_ℓ for $\bar r\in[1,\infty)$ by (3.22), the previous two estimates imply

$$\begin{aligned}\mathbb{E}^{\boldsymbol{\Delta}}&\left(\frac{|Q^{\boldsymbol{\Delta}}_{s,N}(Z'_{s,h_\ell})|^2}{Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})^2}|\mathbb{E}^{\mu_p}(Z_{s,h_\ell})-Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})|^2\right)\\ &\leq C_{\bar\varepsilon}N^{\bar\varepsilon}\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}^{\mu_p}(Z_{s,h_\ell})-Q^{\boldsymbol{\Delta}}_{s,N}(Z_{s,h_\ell})|^{2(1+\varepsilon)})^{1/(1+\varepsilon)}\end{aligned}$$

where the constant $C_{\bar\varepsilon}>0$ depends on $0<\bar\varepsilon\leq\varepsilon/(1+\varepsilon)$, but is independent of s and of h_ℓ. For $p\in(1,2]$, ε may be chosen to be arbitrarily small. However, for $p=1$, we require $\sup_{j\geq1}\|\psi_j\|_X<\varepsilon/(2(1+\varepsilon))$ for (3.22) to hold. This is equivalent to

$$\frac{2\sup_{j\geq1}\|\psi_j\|_X}{1-2\sup_{j\geq1}\|\psi_j\|_X}<\varepsilon$$

and results in the particular reduced convergence rate for $p = 1$.

The QMC points in (5.3) are randomized by the random shift $\boldsymbol{\Delta}$ and are distributed according to μ_p^s. By the triangle inequality, the QMC quadrature is stable, i.e., for every $r \in [1, \infty]$

$$\mathbb{E}^{\boldsymbol{\Delta}} \left(|Q_{s,N}^{\boldsymbol{\Delta}}(Z_{s,h_\ell})|^r \right)^{1/r} \leq \|Z_{s,h_\ell}\|_{L^r(\Omega,\mu_p)} \leq 1.$$

Note that in general, $r = \infty$ is excluded. However, here $Z_{s,h_\ell} \leq 1$. By the real method of interpolation of the linear operator $F \mapsto \mathbb{E}^{\mu_p}(F) - Q_{s,N}^{\boldsymbol{\Delta}}(F)$ with interpolation couple $(L^2_{\boldsymbol{\Delta}}([0,1]^s), L^\infty_{\boldsymbol{\Delta}}([0,1]^s))$, Theorem 6.1 implies with the previously derived estimates that for every $1/2\varepsilon > 0$ there exists $C_\varepsilon > 0$ (independent of s, h_ℓ) such that

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}} \left(\frac{|Q_{s,N}^{\boldsymbol{\Delta}}(Z'_{s,h_\ell})|^2}{Q_{s,N}^{\boldsymbol{\Delta}}(Z_{s,h_\ell})^2} |\mathbb{E}^{\mu_p}(Z_{s,h_\ell}) - Q_{s,N}^{\boldsymbol{\Delta}}(Z_{s,h_\ell})|^2 \right)} \leq C_\varepsilon N^{-\chi+\varepsilon}.$$

Note that Euler's totient function satisfies that $\varphi(N)^{-1} < N^{-1}[e^{\widetilde{\gamma}} \log(\log(N)) + 3\log(\log(N))^{-1}]$ for any $N \geq 3$, where $\widetilde{\gamma} \approx 0.5772$ is the Euler–Mascheroni constant. Hence, for every $\widetilde{\varepsilon} > 0$, there exists $C_{\widetilde{\varepsilon}} > 0$ such that $\varphi(N)^{-1} \leq C_{\widetilde{\varepsilon}} N^{-1+\widetilde{\varepsilon}}$. Thus, the proposition is proven. □

We summarize the error bounds of dimension truncation, FEM, and QMC quadrature and state an overall error estimate in the following corollary.

Corollary 6.4. *Let the assumptions of Propositions 6.3 and 4.1 be satisfied. Suppose that the sequence* $(\|\psi_j\|_{L^\infty(D)})_{j\geq 1} \in \ell^t(\mathbb{N})$ *is decreasing. For* $p \in (1,2]$*, let* $\varepsilon > 0$ *be arbitrary. For* $p = 1$*, assume that* $\varepsilon > 2\|\psi_1\|_X/(1 - 2\|\psi_1\|_X)$*. Then, there exists a constant* $C > 0$ *(independent of* s*,* h_ℓ*, and* N*) such that for any* $0 < \varepsilon \ll 1/2$*,*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}} \left(\left| \frac{\mathbb{E}^{\mu_p}(Z')}{\mathbb{E}^{\mu_p}(Z)} - \frac{Q_{s,N}^{\boldsymbol{\Delta}}(Z'_{s,h_\ell})}{Q_{s,N}^{\boldsymbol{\Delta}}(Z_{s,h_\ell})} \right|^2 \right)} \leq C \left(\|\psi_s\|_{L^\infty(D)}^{1-t/\iota} + h_\ell^{\tau+\tau'} + N^{-\chi+\varepsilon} \right),$$

where χ *is specified in* (6.6)*. For* $p \in (1,2]$*,* $\iota = 2$*, and for* $p = 1$*,* $\iota = 1$*.*

Proof. Since the sequence $(\|\psi_j\|_{L^\infty(D)})_{j\geq 1}$ is decreasing, it holds that

$$\sum_{j>s} \|\psi_j\|_{L^\infty(D)}^\iota \leq \|\psi_s\|_{L^\infty(D)}^{\iota-t} \sum_{j>s} \|\psi_j\|_{L^\infty(D)}^t \leq \|\psi_s\|_{L^\infty(D)}^{\iota-t} \sum_{j\geq 1} \|\psi_j\|_{L^\infty(D)}^t < \infty.$$

The assertion follows now by Propositions 6.3 and 4.1. □

7. Parametric regularity of the posterior

In this section, we investigate parametric regularity of the mappings $[\boldsymbol{y} \mapsto Z(\boldsymbol{y})]$ and $[\boldsymbol{y} \mapsto Z'(\boldsymbol{y})]$ in certain cases. Up to this point $a_0 \equiv 0$ in (3.6) was admissible. In the ensuing analysis of parametric regularity of the posterior density, the stronger assumption $\operatorname{ess\,inf}_{x\in D}\{a_0(x)\} > 0$ is required as stated in the following proposition.

Proposition 7.1. *Suppose that $\mathcal{O} \in \mathcal{X}^*$. Assume that $\bar{a} := \operatorname{ess\,inf}_{x\in D}\{a_0(x)\} > 0$. Then, with a constant $C_0 \le 1.1$ for any $\delta \in \mathbb{R}$ and every $\mathfrak{u} \subset \mathbb{N}$ such that $|\mathfrak{u}| < \infty$,*

$$\left|\partial_{\boldsymbol{y}}^{\mathfrak{u}} \exp\left(-\frac{|\delta - \mathcal{O}(q(\boldsymbol{y}))|^2}{2}\right)\right| \le C_0 \frac{(|\mathfrak{u}|!)^{3/2}}{(\log(2))^{|\mathfrak{u}|}} \max\left\{\frac{\|\mathcal{O}\|_{\mathcal{X}^*}\|f\|_{\mathcal{X}^*}}{\bar{a}}, 1\right\}^{|\mathfrak{u}|} \exp\left(\pi\sqrt{\frac{2|\mathfrak{u}|}{3}}\right)\prod_{j\in\mathfrak{u}} b_j.$$

Proof. We use multi-index notation: for every $\mathfrak{u} \subset \mathbb{N}$ with $|\mathfrak{u}| < \infty$, there exists $\boldsymbol{\tau} \in \{0,1\}^{\mathbb{N}}$ with $|\boldsymbol{\tau}| = \sum_{j\ge1}\tau_j < \infty$ such that $j \in \mathfrak{u} \Leftrightarrow \tau_j = 1$. Define the Hermite polynomials by $H_0 \equiv 1$ and

$$H_n(\xi) := (-1)^n e^{\xi^2/2}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}e^{-\xi^2/2}, \quad n \in \mathbb{N}, \xi \in \mathbb{R}.$$

Cramér's inequality (see e.g. [1, Equations (22.5.18) and (22.14.17)]) states

$$|H_n(\xi)| \le C_0 e^{\xi^2/4}\sqrt{n!} \quad \forall n \in \mathbb{N}, \forall \xi \in \mathbb{R},$$

where $C_0 \le 1.0866$. Thus,

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}e^{-\xi^2/2}\right| \le C_0 e^{-\xi^2/4}\sqrt{n!} \quad \forall n \in \mathbb{N}, \forall \xi \in \mathbb{R}. \tag{7.1}$$

By Faa di Bruno's formula (e.g. [7, Corollary 2.10]), for any multi-index $\boldsymbol{\tau} \in \{0,1\}^{\mathbb{N}}$ such that $|\boldsymbol{\tau}| < \infty$,

$$\partial_{\boldsymbol{y}}^{\boldsymbol{\tau}} \exp\left(-\frac{|\delta - \mathcal{O}(q(\boldsymbol{y}))|^2}{2}\right) = \sum_{r=1}^{|\boldsymbol{\tau}|}\frac{\mathrm{d}^r}{\mathrm{d}\xi^r}e^{-\xi^2/2}\Big|_{\xi=\delta-\mathcal{O}(q(\boldsymbol{y}))}\sum_{P(r,\boldsymbol{\tau})}\prod_{i=1}^{r}\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}^{(i)}}\mathcal{O}(q(\boldsymbol{y})),$$

where

$$P(r,\boldsymbol{\tau}) := \left\{\boldsymbol{\nu}^{(1)},\ldots,\boldsymbol{\nu}^{(r)} \in \{0,1\}^{\mathbb{N}} : \mathbf{0} \prec \boldsymbol{\nu}^{(1)} \prec \ldots \prec \boldsymbol{\nu}^{(r)} \text{ and } \sum_{i=1}^{r}\boldsymbol{\nu}^{(i)} = \boldsymbol{\tau}\right\}.$$

The linear order "$\prec$" is defined on [7, p. 505]. By [16, Theorem 14], (7.1), and the fact that $a(\boldsymbol{y})^{-1} \le \bar{a}^{-1} < \infty$ by assumption,

$$\begin{aligned}
&\left|\partial_{\boldsymbol{y}}^{\boldsymbol{\tau}} \exp\left(-\frac{|\delta - \mathcal{O}(q(\boldsymbol{y}))|^2}{2}\right)\right| \\
&\le C_0 e^{-|\delta-\mathcal{O}(q(\boldsymbol{y}))|^2/4}\sum_{r=1}^{|\boldsymbol{\tau}|}\sqrt{r!}\left(\frac{\|\mathcal{O}\|_{\mathcal{X}^*}\|f\|_{\mathcal{X}^*}}{\bar{a}}\right)^r\sum_{P(r,\boldsymbol{\tau})}\prod_{i=1}^{r}\left(\frac{|\boldsymbol{\nu}^{(i)}|!}{(\log(2))^{|\boldsymbol{\nu}^{(i)}|}}\prod_{j\ge1}b_j^{\boldsymbol{\nu}_j^{(i)}}\right) \\
&\le C_0 C_1^{|\boldsymbol{\tau}|}e^{-|\delta-\mathcal{O}(q(\boldsymbol{y}))|^2/4}\frac{\sqrt{|\boldsymbol{\tau}|!}}{(\log(2))^{|\boldsymbol{\tau}|}}\prod_{j\ge1}b_j^{\boldsymbol{\tau}_j}\sum_{r=1}^{|\boldsymbol{\tau}|}\sum_{P(r,\boldsymbol{\tau})}\prod_{i=1}^{r}|\boldsymbol{\nu}^{(i)}|!,
\end{aligned}$$

where $C_1 := \max\{\|\mathcal{O}\|_{\mathcal{X}^*}\|f\|_{\mathcal{X}^*}/\bar{a}, 1\}$. The quantity $\sum_{r=1}^{|\boldsymbol{\tau}|}\sum_{P(r,\boldsymbol{\tau})}\prod_{i=1}^{r}|\boldsymbol{\nu}^{(i)}|!$ is estimated on [31, p. 516]. There, it is shown as the last display equation on [31, p. 516] that

$$\sum_{r=1}^{|\boldsymbol{\tau}|}\sum_{P(r,\boldsymbol{\tau})}\prod_{i=1}^{r}|\boldsymbol{\nu}^{(i)}|! \le \exp\left(\pi\sqrt{\frac{2|\boldsymbol{\tau}|}{3}}\right)|\boldsymbol{\tau}|!,$$

which implies the assertion of this proposition. $\square$

The parametric regularity estimate in Proposition 7.1 is a version of [31, Assumption A4]. The proof follows the exposition in [31, Appendix A]. Detail to a certain extent is provided, since a different bound for Hermite polynomials is used compared to [31, Equation (A.1)].

Lemma 7.2. *Under the assumptions of Proposition 7.1. Suppose that $\mathcal{G} \in \mathcal{X}^*$. Then, there exist constants $C_1, C_2 > 0$ (depending on $\bar{u}$ f, $\mathcal{O}$, and δ) such that for any $\mathfrak{u} \subset \mathbb{N}$ with $|\mathfrak{u}| < \infty$,*

$$\left| \partial_{\boldsymbol{y}}^{\mathfrak{u}} \mathcal{G}(q(\boldsymbol{y})) \exp\left(-\frac{|\delta - \mathcal{O}(q(\boldsymbol{y}))|^2}{2} \right) \right| \leq C_1 (|\mathfrak{u}|!)^{3/2} C_2^{|\mathfrak{u}|} \prod_{j \in \mathfrak{u}} b_j.$$

Proof. This follows by Proposition 7.1 and the proof of [14, Proposition 3.4]. □

Remark 7.3. In the setting of Lemma 7.2, the assumptions of Proposition 6.3 are satisfied with $(\mathfrak{b}_j)_{j \geq 1} \equiv 0$. Then, for $p = 1$ in Proposition 6.3 and in Corollary 6.4, also every $\varepsilon > 0$ is admissible, which results in a dimension-independent QMC convergence rate of essentially χ, where χ is specified in (6.6).

8. Numerical experiments

We study numerically the convergence of QMC under a Besov prior for a BIP. Therefore, we fix a truncation dimension $s \in \mathbb{N}$ and a FE mesh width h and vary the number of QMC points N. In our numerical tests, $D = (0, 1)^2$. We subdivide D into four squares D_i, $i = 1, \ldots, 4$, where $D_1 = (0, 1/2) \times (0, 1/2)$, $D_2 = (1/2, 1) \times (0, 1/2)$, $D_3 = (1/2, 1) \times (1/2, 1)$, and $D_4 = (0, 1/2) \times (1/2, 1)$. Thus, $\overline{D} = \bigcup_{i=1}^4 \overline{D}_i$. The *observation* functional $\mathcal{O}$ is given by a vector $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3)$ such that $\mathcal{O}_i \in \mathcal{X}^*$, $i = 1, 2, 3$, is the average over a subdomain of D, i.e., for $i = 1, 2, 3$,

$$\mathcal{O}_i(v) = \frac{1}{|D_i|} \int_{D_i} v \, \mathrm{d}x, \quad v \in \mathcal{X},$$

where $|D_i|$ denotes the area of D_i. The function system $(\psi_j)_{j \geq 1}$ is chosen to be

$$\psi_{j(k_1,k_2)}(x_1, x_2) = \sigma \frac{\kappa^{\beta-1}}{(k_1^2 + k_2^2 + \kappa^2)^{\beta/2}} \sin(\pi k_1 x_1) \sin(\pi k_2 x_2), \quad k_1, k_2 \in \mathbb{N}, \tag{8.1}$$

for some $\kappa \geq 1$, $\sigma > 0$, $\eta > 1$ (which are specified below), where the index mapping $j : \mathbb{N}^2 \to \mathbb{N}$ is such that $(\|\psi_j\|_{L^\infty(D)})_{j \geq 1}$ is monotonically decreasing. This particular decay in (8.1) is inspired by so-called *Matérn random fields*, see for example [11, Section 6.7]. The QoI $\mathcal{G} \in \mathcal{X}^*$ is given by

$$\mathcal{G}(v) = \frac{1}{|D_4|} \int_{D_4} v \, \mathrm{d}x, \quad v \in \mathcal{X}.$$

We consider the input-to-solution forward map $\mathcal{S}$, which maps the uncertain coefficient u to the solution q to (3.2); here with right hand side function $f(x_1, x_2) = x_1$. These are approximated by $\mathcal{S}_h$ and $q^{s,h} = \mathcal{S}_h(u^s)$ with fixed $s \in \mathbb{N}$ and $h > 0$ as mentioned above. The function a_0 in (3.6) is assumed to be constant. For $a_0 > 0$, Theorem 6.1 with $(\mathfrak{b}_j)_{j \geq 1} \equiv 0$, Propositions 6.3 and 7.1, and Lemma 7.2 imply dimension-independent convergence rates of QMC with QMC weights

$$\gamma_{\mathfrak{u}} = \left((|\mathfrak{u}|!)^{\varrho} \prod_{j \in \mathfrak{u}} c \|\psi_j\|_X \right)^{2/(1+\lambda)}, \quad \mathfrak{u} \subset \{1 : s\}, \tag{8.2}$$

for $\varrho = 3/2$, and some $\lambda \in (1/2, 1]$, with some $c > 0$ to be specified, see also the discussion on [16, p. 359] on the "alternative choice of weights". The idea to scale the weight sequence by a positive constant to obtain numerically more suitable generating vectors was proposed in [15].

Here, $\varrho = 3/2$ is the borderline case in Theorem 6.1. We compute the generating vectors using product and order dependent (POD for short) QMC weights (8.2) by the fast CBC algorithm with parameters $c = 0.1$, $\lambda = 0.55$, and QMC weight function parameter $\alpha = (2.1)^{-1}$. We use first order FEM on a uniform mesh of axiparallel squares with $h = 2^{-7}$ and truncate the the affine-parametric input u in (3.4) to $s = 400$ terms, which subsequently is the integration dimension in the Bayesian inversion. The occurring linear systems are solved by a sparse direct solver. We note that for large scale problems fast iterative solver are admissible also in the cases of unbounded priors, cf. [18]. The error vs. work analysis has been extended to multilevel QMC under Besov priors with fast iterative solvers for the discretized forward PDE in [19, Chapter 5]. There, numerical experiments are reported which are in agreement with the theoretical error bounds.

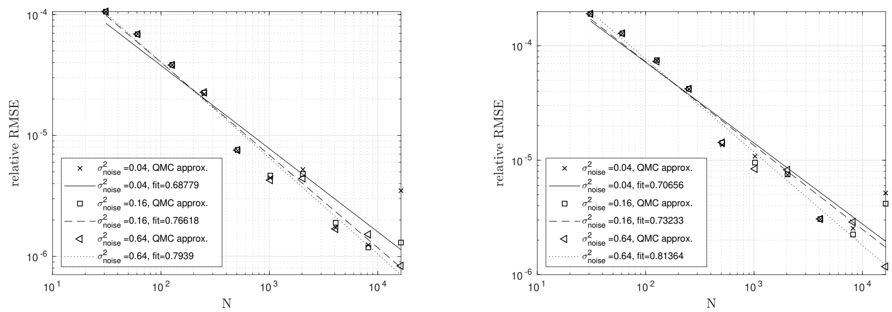
We suppose that the additive Gaussian noise η in (2.1) is uncorrelated with covariance $\Sigma = \sigma^2_{\text{noise}}\text{Id}$ for $\sigma^2_{\text{noise}} > 0$ to be specified. In Figures 1(a) and 1(b), we compare the approximation by QMC under Besov priors for different noise levels $\sigma^2_{\text{noise}} > 0$ and in the cases $\operatorname{ess\,inf}_{x\in D}\{a_0\} > 0$ and $\operatorname{ess\,inf}_{x\in D}\{a_0\} = 0$. The following numbers of QMC points $N \in \{31, 61, 127, 251, 509, 1021, 2039, 4093, 8191, 16381\}$ are used. The root-mean-square error (RMSE for short) has been approximated by 20 independent random shifts, where the reference value has been computed as the average over 20 independent random shifts with $N_{\text{ref}} = 32761$ many QMC points. In the case that $\operatorname{ess\,inf}_{x\in D}\{a_0\} > 0$, Propositions 6.3 and 7.1 imply a convergence rate of essentially first order and with chosen value of $\lambda = 0.55$ a convergence rate independent of the dimension of ≈ 0.91. The data $\delta_i = (\mathcal{O}_i \circ \mathcal{S}_h)(u^s(\boldsymbol{y}_0))$, $i = 1, 2, 3$, has been synthetically generated by a fixed $y_0 \in \mathbb{R}^s$ drawn from the Besov prior defined by (3.4) and (8.1) with $p = 1.5$, $s = 400$, $h = 2^{-7}$. Recall that $q^{s,h} = \mathcal{S}_h(u^s)$ has been introduced in (3.21).

We observe that QMC also seems applicable to this BIP in the case that $a_0 \equiv 0$. However, the dependence on the additive Gaussian noise seems more sensitive for small variances $\sigma^2_{\text{noise}} > 0$ and larger number of QMC points. As indicated at the end of Section 4, the constants in the error estimates depend expontially on $\sigma^{-2}_{\text{noise}}$.

9. Conclusions

We established the first dimension-independent convergence rate bounds for quasi-Monte Carlo integration for Bayesian inverse problems for second order, linear elliptic divergence-form PDEs with uncertain coefficients that have a distribution with unbounded support. We admitted Besov priors μ_p in the regime $p \in [1, 2]$. In particular, in the case $p = 1$, which corresponds to the Laplace distribution, a smallness assumption on the function system $\boldsymbol{\Psi}$ representing the uncertain log-diffusion coefficient was required in the theoretical analysis in order for log-Besov coefficients in the elliptic PDE (3.1) to be admissible.

Forward UQ under the presently considered Besov function space prior μ_p with QMC quadratures has been analyzed in [19]. In particular in [19] it was shown in [19] that representation systems $\boldsymbol{\Psi}$ with locally supported functions allow for QMC integration rules with so-called product weights. This allows fast CBC algorithms



(a) $a_0 \equiv 0.5$ (b) $a_0 \equiv 0$

FIGURE 1. Convergence of QMC: comparison of $a_0 \equiv 0.5$ with $a_0 \equiv 0$ for $p = 1.5$, $\sigma = 0.5$, $\beta = 3$, $\kappa = 5$

with computational cost that scales linear with respect to the integration dimension s. The QMC POD weights used in this work allow generally for fast CBC algorithms with computational cost that is quadratic with respect to s. Due to the appearance of the particular order dependent term while differentiating Hermite polynomials (see (7.1)) in the estimates of parametric regularity of the posterior density, locally supported ψ_j in the affine-parametric representation (3.4) are not straightforwardly giving rise to QMC product weights in the context of Bayesian estimation by QMC. Locally supported representation systems for the function space input may facilitate more efficient QMC integration procedures (see, e.g., the analysis in [21, 24, 22]).

We also explained (Remark 3.4) the connection between s-term truncation of the parametric uncertain function space input $u(\boldsymbol{y})$ in (3.4) and the corresponding stochastic interpretation in connection with Bayesian prior modelling.

The present, single-level QMC-FE algorithms naturally can be generalized in several directions. In particular, multilevel QMC methods could be considered, e.g., for $\operatorname{ess\,inf}_{x\in D}\{a_0(x)\} > 0$, along the lines of the analysis in [10] for uniform priors. Rather than the model linear, elliptic divergence-form PDE (3.1), Bayesian inverse problems for more general forward models, such as anisotropic diffusion or linearized elastostatics, with distributed, parametric inputs (2.2) could be treated in exactly the same fashion.

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