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Abstract

One of the most challenging problems in applied mathematics is the approximate solution of nonlinear partial differential equations (PDEs) in high dimensions. Standard deterministic approximation methods like finite differences or finite elements suffer from the curse of dimensionality in the sense that the computational effort grows exponentially in the dimension. In this work we overcome this difficulty in the case of reaction-diffusion type PDEs with a locally Lipschitz continuous coervice nonlinearity (such as Allen–Cahn PDEs) by introducing and analyzing truncated variants of the recently introduced full-history recursive multilevel Picard approximation schemes.

Contents

1 Introduction

One of the most challenging problems in applied mathematics is the approximate solution of nonlinear partial differential equations (PDEs) in high dimensions. Standard deterministic approximation methods like finite differences or finite elements suffer from the curse of dimensionality in the sense that the computational effort grows exponentially in the dimension. Linear parabolic PDEs of second order can be solved approximately without the curse of dimensionality by means of Monte Carlo averages. In the last few years, several probabilistic approximation methods, which seem in certain situations to be capable of efficiently approximating high-dimensional nonlinear PDEs, have been proposed. For instance, the articles [6, 18, 20, 21] propose and study approximation methods based on stochastic representations of solutions of PDEs by means of branching diffusion processes (cf., for example, [32, 35, 37] for theoretical relations and cf., for example, [36] for a related method), the articles [1, 2, 3, 4, 5, 8, 9, 12, 13, 14, 15, 16, 17, 19, 22, 27, 29, 30, 31, 33, 34] propose and study approximation methods based on the reformulation of PDEs as stochastic learning problems involving deep artificial neural networks, and the articles [10, 11, 23, 24, 25, 26] propose and study full-history recursive multilevel Picard (MLP) approximation methods. In particular, the articles [24, 25] prove that MLP approximation schemes do indeed overcome the curse of dimensionality in the numerical approximation of semilinear parabolic PDEs. More formally, Theorem 3.8 in [24] shows that MLP approximation schemes are able to approximate the solutions of semilinear parabolic PDEs with a root mean square error of size $\varepsilon \in (0, \infty)$ and a computational effort which grows at most polynomially both in the dimension as well as in the reciprocal $1/\varepsilon$ of the desired approximation accuracy. However, the articles [24, 25] are only applicable in the case where the nonlinearity is globally Lipschitz continuous and, to the best of our knowledge, there exists no result in the scientific literature which shows for every $T \in (0, \infty)$ that the solution of a semilinear parabolic PDE with a non-globally Lipschitz continuous nonlinearity can be efficiently approximated at time T without the curse of dimensionality.

In this work we overcome this difficulty by introducing a truncated variant of the MLP approximation schemes introduced in [10, 24] and by proving that this truncated MLP approximation scheme succeeds in approximately solving reaction-diffusion type PDEs with a locally Lipschitz continuous coercive nonlinearity (such as Allen–Cahn type PDEs) without the curse of dimensionality. More specifically, Theorem 4.5 in Section 3 below, which is the main result of this article, proves under suitable assumptions that for every $\delta \in (0, \infty)$, $\varepsilon \in (0, 1]$ it holds that the proposed truncated MLP approximations can achieve a root mean square error of size at most ε with a computational effort of order $d\varepsilon^{-(2+\delta)}$. To illustrate the findings of this article in more detail, we now present in Theorem 1.1 below a special case of Theorem 4.5.

Theorem 1.1. *Let $\delta, \kappa, T \in (0, \infty)$, $\Theta = \cup_{n \in \mathbb{N}} \mathbb{Z}^n$, $f \in C^1(\mathbb{R}, \mathbb{R})$, $(\mathbf{f}_d)_{d \in \mathbb{N}} \subseteq C(\mathbb{R}, \mathbb{R})$, $(u_d)_{d \in \mathbb{N}} \subseteq C([0, T] \times \mathbb{R}^d, \mathbb{R})$, assume that f' is at most polynomially growing, assume for every $d \in \mathbb{N}$,*

$t \in (0,T]$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ that $vf(v) \leq \kappa(1+v^2)$, $|u_d(0,x)| \leq \kappa$, $u_d|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T] \times \mathbb{R}^d, \mathbb{R})$, $\inf_{c\in\mathbb{R}}(\sup_{s\in[0,T]} \sup_{y=(y_1,\ldots,y_d)\in\mathbb{R}^d}(e^{c(|y_1|^2+\ldots+|y_d|^2)}|u_d(s,y)|)) < \infty$, $\mathbf{f}_d(v) = f(\min\{\ln(1+\ln(d)), \max\{-\ln(1+\ln(d)), v\}\})$, and

$$(\tfrac{\partial}{\partial t}u_d)(t,x) = (\Delta_x u_d)(t,x) + f(u_d(t,x)), \tag{1}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{R}^\theta \colon \Omega \to [0,1]$, $\theta \in \Theta$, be independent $\mathcal{U}_{[0,1]}$-distributed random variables, let $W^{d,\theta} \colon [0,T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, be independent standard Brownian motions, assume that $(\mathcal{R}^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ are independent, let $R^\theta \colon [0,T] \times \Omega \to [0,T]$, $\theta \in \Theta$, satisfy for every $\theta \in \Theta$, $t \in [0,T]$ that $R^\theta_t = t\mathcal{R}^\theta$, for every $d \in \mathbb{N}$, $s \in [0,T]$, $t \in [s,T]$, $x \in \mathbb{R}^d$, $\theta \in \Theta$ let $X^{d,\theta}_{s,t,x} \colon \Omega \to \mathbb{R}^d$ satisfy $X^{d,\theta}_{s,t,x} = x + \sqrt{2}(W^{d,\theta}_t - W^{d,\theta}_s)$, let $U^{d,\theta}_{n,M} \colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d, M \in \mathbb{N}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, satisfy for every $d, n, M \in \mathbb{N}$, $\theta \in \Theta$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $U^{d,\theta}_{0,M}(t,x) = 0$ and

$$\begin{aligned} U^{d,\theta}_{n,M}(t,x) = \sum_{k=1}^{n-1} \frac{t}{M^{n-k}} \Bigg[\sum_{m=1}^{M^{n-k}} \bigg(\mathbf{f}_M\Big(U^{d,(\theta,k,m)}_{k,M}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x} \Big)\Big) \\ - \mathbf{f}_M\Big(U^{d,(\theta,-k,m)}_{k-1,M}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x} \Big)\Big)\bigg)\Bigg] + \frac{1}{M^n}\Bigg[\sum_{m=1}^{M^n} \Big(u_d(0, X^{d,(\theta,0,-m)}_{0,t,x}) + t\, f(0) \Big)\Bigg], \end{aligned} \tag{2}$$

and for every $d, M \in \mathbb{N}$, $n \in \mathbb{N}_0$ let $\mathfrak{C}_{d,n,M} \in \mathbb{N}_0$ be the number of realizations of scalar standard normal random variables which are used to compute one realization of $U^{d,0}_{n,M}(T,0) \colon \Omega \to \mathbb{R}$ (cf. Corollary 5.2 for a precise definition). Then there exist $\mathfrak{N} \colon (0,1] \to \mathbb{N}$ and $c \in \mathbb{R}$ such that for every $d \in \mathbb{N}$, $\varepsilon \in (0,1]$ it holds that $\mathfrak{C}_{d,\mathfrak{N}_\varepsilon,\mathfrak{N}_\varepsilon} \leq cd\varepsilon^{-(2+\delta)}$ and

$$\sup_{x\in\mathbb{R}^d} \Big(\mathbb{E}\Big[\big| U^{d,0}_{\mathfrak{N}_\varepsilon,\mathfrak{N}_\varepsilon}(T,x) - u_d(T,x) \big|^2 \Big] \Big)^{1/2} \leq \varepsilon. \tag{3}$$

Theorem 1.1 above is an immediate consequence of Corollary 5.2 in Section 5 below. Corollary 5.2 follows from Corollary 5.1 which, in turn, is deduced from Theorem 4.5, the main result of this article. Theorem 1.1 establishes under suitable assumptions that for every $\delta \in (0,\infty)$ there exists $c \in (0,\infty)$ such that for every $d \in \mathbb{N}$ the solution $u_d \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ of the reaction-diffusion type partial differential equation in (1) can be approximated by the MLP approximation scheme in (2) with a root mean square error of size $\varepsilon \in (0,\infty)$ while the computational effort is bounded by $cd\varepsilon^{-(2+\delta)}$. The numbers $\mathfrak{C}_{d,n,M}$, $d, M \in \mathbb{N}$, $n \in \mathbb{N}_0$, in Theorem 1.1 model the computational effort. The nonlinearity $f \colon \mathbb{R} \to \mathbb{R}$ in Theorem 1.1 is required to be locally Lipschitz continuous (which follows from the hypothesis in Theorem 1.1 that f' is continuous) and to satisfy a coercivity type condition in the sense that there exists $\kappa \in \mathbb{R}$ such that for all $v \in \mathbb{R}$ it holds that $vf(v) \leq \kappa(1+v^2)$. This coercivity type condition together with the growth assumption on the solutions $u_d \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, allows us to deduce in Section 2 that the solutions $u_d \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, are uniformly bounded. In particular, Corollary 2.4 in Section 2 yields that there exists $\mathfrak{M} \in \mathbb{N}$ such that for every $M \in [\mathfrak{M}, \infty) \cap \mathbb{N}$, $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $(\frac{\partial}{\partial t}u_d)(t,x) = (\Delta_x u_d)(t,x) + \mathbf{f}_M(u_d(t,x))$. The fact that for every $d, M \in \mathbb{N}$ it holds that $(\frac{\partial}{\partial t}u_d)(t,x) = (\Delta_x u_d)(t,x) + \mathbf{f}_M(u_d(t,x))$, $(t,x) \in [0,T] \times \mathbb{R}^d$, is a parabolic PDE with a globally Lipschitz continuous nonlinearity then permits us to bring the machinery from [24] into play. This will finally allow us to prove Theorem 1.1 (see Sections 2 and 3 for details). We note that although Theorem 1.1 uses the assumption that the nonlinearity $f \colon \mathbb{R} \to \mathbb{R}$ satisfies the coercivity type condition that there exists $\kappa \in \mathbb{R}$ such that for all $v \in \mathbb{R}$ it holds that $vf(v) \leq \kappa(1+v^2)$, explicit knowledge of the coercivity constant κ is not required for the implementation of the MLP approximation scheme.

The remainder of this article is organized as follows. In Section 2 we present elementary a priori bounds for classical solutions of reaction-diffusion type PDEs with coercive nonlinearities. In Section 3 we introduce truncated MLP approximation schemes and we provide upper bounds for the root mean square distance between the truncated MLP approximations and the exact solution of the PDE under consideration. In Section 4 we combine the error estimates from Section 3 with estimates for the computational effort for truncated MLP approximations to show under suitable assumptions that for every $\delta \in (0,\infty)$ a root mean square error of size $\varepsilon \in (0,1]$ can be achieved by truncated MLP approximations with a computational effort of order $d\varepsilon^{-(2+\delta)}$. In Section 5 we specialize our findings to Allen–Cahn type PDEs.

2 A priori bounds for reaction-diffusion equations with coercive nonlinearity

For convenience of the reader, we recall the following well-known maximum principle for subsolutions of the heat equation (cf., e.g., John [28, Pages 216–217 in Section 1 in Chapter 7]).

Lemma 2.1. *Let $d \in \mathbb{N}$, $T \in (0,\infty)$, $v \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$, assume that $v|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T] \times \mathbb{R}^d, \mathbb{R})$, assume for every $t \in (0,T]$, $x \in \mathbb{R}^d$ that*

$$(\tfrac{\partial}{\partial t} v)(t,x) \leq (\Delta_x v)(t,x), \tag{4}$$

let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be the d-dimensional Euclidean norm, and assume that

$$\inf_{a\in\mathbb{R}} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{a\|x\|^2} v(t,x)) < \infty. \tag{5}$$

Then it holds that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x). \tag{6}$$

Proof of Lemma 2.1. Throughout this proof assume w.l.o.g. that

$$\sup_{x\in\mathbb{R}^d} v(0,x) < \infty, \tag{7}$$

let $\Phi_\varepsilon \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $\varepsilon \in (0,\infty)$, be the functions which satisfy for every $\varepsilon \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\Phi_\varepsilon(t,x) = [4\pi(T+\varepsilon-t)]^{-d/2} \exp\!\left(\frac{\|x\|^2}{4(T+\varepsilon-t)}\right), \tag{8}$$

and let $w_{\varepsilon,M} \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$, $\varepsilon, M \in (0,\infty)$, be the functions which satisfy for every $\varepsilon, M \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$w_{\varepsilon,M}(t,x) = v(t,x) - M\Phi_\varepsilon(t,x) - \varepsilon t. \tag{9}$$

Observe that for every $\varepsilon \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$(\nabla_x \Phi_\varepsilon)(t,x) = \Phi_\varepsilon(t,x) \left[\frac{x}{2(T+\varepsilon-t)}\right]. \tag{10}$$

This implies that for every $\varepsilon \in (0,\infty)$, $t \in [0,T]$, $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
(\tfrac{\partial^2}{\partial {x_k}^2}\Phi_\varepsilon)(t,x) &= (\tfrac{\partial}{\partial x_k}\Phi_\varepsilon)(t,x) \left[\frac{x_k}{2(T+\varepsilon-t)}\right] + \Phi_\varepsilon(t,x)\left[\frac{1}{2(T+\varepsilon-t)}\right] \\
&= \Phi_\varepsilon(t,x)\left[\frac{x_k}{2(T+\varepsilon-t)}\right]^2 + \Phi_\varepsilon(t,x)\left[\frac{1}{2(T+\varepsilon-t)}\right] \\
&= \Phi_\varepsilon(t,x)\left(\frac{|x_k|^2}{4(T+\varepsilon-t)^2} + \frac{1}{2(T+\varepsilon-t)}\right).
\end{aligned} \tag{11}$$

Therefore, we obtain that for every $\varepsilon \in (0,\infty)$, $t \in [0,T]$, $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$ it holds that

$$\begin{aligned}(\Delta_x \Phi_\varepsilon)(t,x) &= \sum_{k=1}^d (\tfrac{\partial^2}{\partial {x_k}^2}\Phi_\varepsilon)(t,x)\\ &= \sum_{k=1}^d \left[\Phi_\varepsilon(t,x)\left(\frac{|x_k|^2}{4(T+\varepsilon-t)^2} + \frac{1}{2(T+\varepsilon-t)}\right)\right]\\ &= \Phi_\varepsilon(t,x)\left(\frac{\|x\|^2}{4(T+\varepsilon-t)^2} + \frac{d}{2(T+\varepsilon-t)}\right).\end{aligned} \tag{12}$$

Moreover, observe that for every $\varepsilon \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&(\tfrac{\partial}{\partial t}\Phi_\varepsilon)(t,x) = -\frac{d}{2}[4\pi(T+\varepsilon-t)]^{-d/2-1}[-4\pi]\exp\!\left(\frac{\|x\|^2}{4(T+\varepsilon-t)}\right)\\ &+ [4\pi(T+\varepsilon-t)]^{-d/2}\exp\!\left(\frac{\|x\|^2}{4(T+\varepsilon-t)}\right)\left[\frac{\|x\|^2}{4(T+\varepsilon-t)^2}\right]\\ &= [4\pi(T+\varepsilon-t)]^{-d/2}\exp\!\left(\frac{\|x\|^2}{4(T+\varepsilon-t)}\right)\left[-\frac{d}{2}\left(\frac{-4\pi}{4\pi(T+\varepsilon-t)}\right) + \frac{\|x\|^2}{4(T+\varepsilon-t)^2}\right]\\ &= \left(\frac{d}{2(T+\varepsilon-t)} + \frac{\|x\|^2}{4(T+\varepsilon-t)^2}\right)\Phi_\varepsilon(t,x).\end{aligned} \tag{13}$$

Combining this with (12) ensures that for every $\varepsilon \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$(\tfrac{\partial}{\partial t}\Phi_\varepsilon)(t,x) = (\Delta_x \Phi_\varepsilon)(t,x). \tag{14}$$

This, (4), and (9) imply that for every $\varepsilon, M \in (0,\infty)$, $t \in (0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}(\tfrac{\partial}{\partial t}w_{\varepsilon,M})(t,x) &= (\tfrac{\partial}{\partial t}v)(t,x) - M(\tfrac{\partial}{\partial t}\Phi_\varepsilon)(t,x) - \varepsilon\\ &= (\tfrac{\partial}{\partial t}v)(t,x) - M(\Delta_x\Phi_\varepsilon)(t,x) - \varepsilon\\ &\le (\Delta_x v)(t,x) - M(\Delta_x\Phi_\varepsilon)(t,x) - \varepsilon = (\Delta_x w_{\varepsilon,M})(t,x) - \varepsilon.\end{aligned} \tag{15}$$

In addition, observe that (5) ensures that there exist $C \in [0,\infty)$ and $a \in (0,\infty)$ such that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$v(t,x) \le Ce^{a\|x\|^2}. \tag{16}$$

To prove (6) we distinguish between the case $T < \frac{1}{4a}$ and the case $T \ge \frac{1}{4a}$. We first prove (6) in the case $T < \frac{1}{4a}$. Observe that (8), (9) and (16) imply that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}w_{\varepsilon,M}(t,x) &\le v(t,x) - M\Phi_\varepsilon(t,x)\\ &= v(t,x) - \frac{M}{[4\pi(T+\varepsilon-t)]^{d/2}}\exp\!\left(\frac{\|x\|^2}{4(T+\varepsilon-t)}\right)\\ &\le v(t,x) - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(\frac{\|x\|^2}{4(T+\varepsilon)}\right)\\ &\le Ce^{a\|x\|^2} - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(\frac{\|x\|^2}{4(T+\varepsilon)}\right)\\ &= e^{a\|x\|^2}\left[C - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(\|x\|^2\left[\frac{1}{4(T+\varepsilon)} - a\right]\right)\right].\end{aligned} \tag{17}$$

Furthermore, observe that the hypothesis that $v \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ and the fact that the interval $[0,T]$ is compact ensure that $\inf_{s\in[0,T]} v(s,0) \in \mathbb{R}$. Hence, we obtain that for every $\varepsilon, M \in (0,\infty)$ it holds that

$$\min\left\{0, \left[\inf_{s\in[0,T]} v(s,0)\right] - \varepsilon T - \frac{M}{(4\pi\varepsilon)^{d/2}}\right\} \in \mathbb{R}. \tag{18}$$

This and the fact that for every $\varepsilon \in (0, \frac{1}{4a} - T)$ it holds that $a < \frac{1}{4(T+\varepsilon)}$ imply that there exists a function $R = (R_{\varepsilon,M})_{(\varepsilon,M)\in(0,\infty)^2} \colon (0,\infty)^2 \to (0,\infty)$ such that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0,\infty)$ it holds that

$$\begin{aligned} &e^{a|R_{\varepsilon,M}|^2}\left[C - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(|R_{\varepsilon,M}|^2\left[\frac{1}{4(T+\varepsilon)} - a\right]\right)\right] \\ &< \min\left\{0, \left[\inf_{s\in[0,T]} v(s,0)\right] - \varepsilon T - \frac{M}{(4\pi\varepsilon)^{d/2}}\right\}. \end{aligned} \tag{19}$$

Combining this with (8) and (9) proves that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0,\infty)$, $t \in [0,T]$ it holds that

$$\begin{aligned} &e^{a|R_{\varepsilon,M}|^2}\left[C - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(|R_{\varepsilon,M}|^2\left[\frac{1}{4(T+\varepsilon)} - a\right]\right)\right] \\ &< \left[\inf_{s\in[0,T]} v(s,0)\right] - \varepsilon T - \frac{M}{(4\pi\varepsilon)^{d/2}} \le \left[\inf_{s\in[0,T]} v(s,0)\right] - \varepsilon t - \frac{M}{(4\pi\varepsilon)^{d/2}} \\ &\le \left[\inf_{s\in[0,T]} v(s,0)\right] - \varepsilon t - \frac{M}{[4\pi(T+\varepsilon-t)]^{d/2}} \le v(t,0) - \varepsilon t - \frac{M}{[4\pi(T+\varepsilon-t)]^{d/2}} \\ &= v(t,0) - \varepsilon t - M\Phi_\varepsilon(t,0) = w_{\varepsilon,M}(t,0) \le \sup_{\substack{(s,x)\in[0,T]\times\mathbb{R}^d,\\ \|x\|\le R_{\varepsilon,M}}} w_{\varepsilon,M}(s,x). \end{aligned} \tag{20}$$

This, (17), and (19) ensure that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ with $\|x\| > R_{\varepsilon,M}$ it holds that

$$\begin{aligned} w_{\varepsilon,M}(t,x) &\le e^{a\|x\|^2}\left[C - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(\|x\|^2\left[\frac{1}{4(T+\varepsilon)} - a\right]\right)\right] \\ &\le e^{a\|x\|^2}\left[C - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(|R_{\varepsilon,M}|^2\left[\frac{1}{4(T+\varepsilon)} - a\right]\right)\right] \\ &\le e^{a|R_{\varepsilon,M}|^2}\left[C - \frac{M}{[4\pi(T+\varepsilon)]^{d/2}}\exp\!\left(|R_{\varepsilon,M}|^2\left[\frac{1}{4(T+\varepsilon)} - a\right]\right)\right] \\ &\le \sup_{\substack{(s,y)\in[0,T]\times\mathbb{R}^d,\\ \|y\|\le R_{\varepsilon,M}}} w_{\varepsilon,M}(s,y). \end{aligned} \tag{21}$$

Therefore, we obtain that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0,\infty)$ it holds that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} w_{\varepsilon,M}(t,x) = \sup_{\substack{(t,x)\in[0,T]\times\mathbb{R}^d,\\ \|x\|\le R_{\varepsilon,M}}} w_{\varepsilon,M}(t,x). \tag{22}$$

The fact that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0,\infty)$ it holds that the function $w_{\varepsilon,M} \colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is continuous hence demonstrates that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0,\infty)$ there exists $(t_{\varepsilon,M}, x_{\varepsilon,M}) \in [0,T]\times\mathbb{R}^d$ such that it holds that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} w_{\varepsilon,M}(t,x) = w_{\varepsilon,M}(t_{\varepsilon,M}, x_{\varepsilon,M}). \tag{23}$$

The fact that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0, \infty)$ it holds that $w_{\varepsilon,M}|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T]\times\mathbb{R}^d, \mathbb{R})$ therefore ensures that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0, \infty)$, $v \in \mathbb{R}^d$ with $t_{\varepsilon,M} > 0$ it holds that

$$(\tfrac{\partial}{\partial t} w_{\varepsilon,M})(t_{\varepsilon,M}, x_{\varepsilon,M}) = 0 \qquad \text{and} \qquad \Big((\tfrac{\partial^2}{\partial x^2} w_{\varepsilon,M})(t_{\varepsilon,M}, x_{\varepsilon,M})\Big)(v,v) \leq 0. \tag{24}$$

Hence, we obtain that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0, \infty)$ with $t_{\varepsilon,M} > 0$ it holds that

$$(\tfrac{\partial}{\partial t} w_{\varepsilon,M})(t_{\varepsilon,M}, x_{\varepsilon,M}) \geq 0 \tag{25}$$

and

$$(\Delta_x w_{\varepsilon,M})(t_{\varepsilon,M}, x_{\varepsilon,M}) = \sum_{k=1}^{d} (\tfrac{\partial^2}{\partial {x_k}^2} w_{\varepsilon,M})(t_{\varepsilon,M}, x_{\varepsilon,M}) \leq 0. \tag{26}$$

This and (15) imply that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0, \infty)$ with $t_{\varepsilon,M} > 0$ it holds that

$$0 \leq (\tfrac{\partial}{\partial t} w_{\varepsilon,M})(t_{\varepsilon,M}, x_{\varepsilon,M}) \leq (\Delta_x w_{\varepsilon,M})(t_{\varepsilon,M}, x_{\varepsilon,M}) - \varepsilon \leq -\varepsilon < 0. \tag{27}$$

Hence, we obtain for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0, \infty)$ that $t_{\varepsilon,M} = 0$. Combining this with (23) proves that for every $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0, \infty)$ it holds that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} w_{\varepsilon,M}(t,x) = w_{\varepsilon,M}(t_{\varepsilon,M}, x_{\varepsilon,M}) = w_{\varepsilon,M}(0, x_{\varepsilon,M}) \leq \sup_{x\in\mathbb{R}^d} w_{\varepsilon,M}(0,x). \tag{28}$$

This and (9) imply that for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $\varepsilon \in (0, \frac{1}{4a} - T)$, $M \in (0, \infty)$ it holds that

$$\begin{aligned} v(t,x) &= w_{\varepsilon,M}(t,x) + M\Phi_\varepsilon(t,x) + \varepsilon t \leq \left[\sup_{y\in\mathbb{R}^d} w_{\varepsilon,M}(0,y)\right] + M\Phi_\varepsilon(t,x) + \varepsilon t \\ &\leq \left[\sup_{y\in\mathbb{R}^d} v(0,y)\right] + M\Phi_\varepsilon(t,x) + \varepsilon t. \end{aligned} \tag{29}$$

Therefore, we obtain that for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $\varepsilon \in (0, \frac{1}{4a} - T)$ it holds that

$$v(t,x) \leq \liminf_{M\searrow 0} \left(\left[\sup_{y\in\mathbb{R}^d} v(0,y)\right] + M\Phi_\varepsilon(t,x) + \varepsilon t\right) = \left[\sup_{y\in\mathbb{R}^d} v(0,y)\right] + \varepsilon t. \tag{30}$$

Hence, we obtain that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$v(t,x) \leq \liminf_{\varepsilon\searrow 0} \left(\left[\sup_{y\in\mathbb{R}^d} v(0,y)\right] + \varepsilon t\right) = \sup_{y\in\mathbb{R}^d} v(0,y). \tag{31}$$

This establishes (6) in the case $T < \frac{1}{4a}$. We now prove (6) in the case $T \geq \frac{1}{4a}$. For this let $k \in \mathbb{N}$ and $\mathcal{T} \in (0, \frac{1}{8a}]$ be the real numbers which satisfy that

$$T = \frac{k}{8a} + \mathcal{T}, \tag{32}$$

let $\tau_l \in \mathbb{R}$, $l \in \{0, 1, \ldots, k+1\}$, be the real numbers which satisfy for all $l \in \{0, 1, \ldots, k\}$ that

$$\tau_l = \frac{l}{8a} \qquad \text{and} \qquad \tau_{k+1} = T, \tag{33}$$

and let $\mathfrak{v}_l \colon [0, \tau_{l+1} - \tau_l] \times \mathbb{R}^d \to \mathbb{R}$, $l \in \{0, 1, \ldots, k\}$, be the functions which satisfy for all $l \in \{0, 1, \ldots, k\}$, $t \in [0, \tau_{l+1} - \tau_l]$, $x \in \mathbb{R}^d$ that

$$\mathfrak{v}_l(t,x) = v(t + \tau_l, x). \tag{34}$$

Next we claim that for every $l \in \{0, 1, \dots, k+1\}$ it holds that

$$\sup_{(t,x)\in[0,\tau_l]\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x). \tag{35}$$

We now prove (35) by induction on $l \in \{0, 1, \dots, k+1\}$. Observe that the fact that

$$\sup_{(t,x)\in[0,\tau_0]\times\mathbb{R}^d} v(t,x) = \sup_{(t,x)\in\{0\}\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x) \tag{36}$$

establishes (35) in the base case $l = 0$. For the induction step $\{0, 1, \dots, k\} \ni l \to l+1 \in \{1, 2, \dots, k+1\}$ assume that there exists $l \in \{0, 1, \dots, k\}$ such that

$$\sup_{(t,x)\in[0,\tau_l]\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x). \tag{37}$$

In addition, note that (4), (16), and (34) ensure that for every $t \in (0, \tau_{l+1} - \tau_l]$, $x \in \mathbb{R}^d$ it holds that

$$(\tfrac{\partial}{\partial t}\mathfrak{v}_l)(t,x) = (\tfrac{\partial}{\partial t}v)(t+\tau_l,x) \le (\Delta_x v)(t+\tau_l,x) = (\Delta_x \mathfrak{v}_l)(t,x) \tag{38}$$

and

$$\sup_{(t,x)\in[0,\tau_{l+1}-\tau_l]\times\mathbb{R}^d} (e^{-a\|x\|^2}\mathfrak{v}_l(t,x)) = \sup_{(t,x)\in[\tau_l,\tau_{l+1}]\times\mathbb{R}^d} (e^{-a\|x\|^2}v(t,x)) \le C < \infty. \tag{39}$$

This, (37), and (6) in the case $T < \frac{1}{4a}$ show that

$$\begin{aligned}\sup_{(t,x)\in[\tau_l,\tau_{l+1}]\times\mathbb{R}^d} v(t,x) &= \sup_{(t,x)\in[0,\tau_{l+1}-\tau_l]\times\mathbb{R}^d} \mathfrak{v}_l(t,x) = \sup_{x\in\mathbb{R}^d} \mathfrak{v}_l(0,x) = \sup_{x\in\mathbb{R}^d} v(\tau_l,x)\\ &\le \sup_{(t,x)\in[0,\tau_l]\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x).\end{aligned} \tag{40}$$

Therefore, we obtain that

$$\begin{aligned}\sup_{(t,x)\in[0,\tau_{l+1}]\times\mathbb{R}^d} v(t,x) &= \max\left\{\sup_{(t,x)\in[0,\tau_l]\times\mathbb{R}^d} v(t,x), \sup_{(t,x)\in[\tau_l,\tau_{l+1}]\times\mathbb{R}^d} v(t,x)\right\}\\ &= \max\left\{\sup_{x\in\mathbb{R}^d} v(0,x), \sup_{(t,x)\in[\tau_l,\tau_{l+1}]\times\mathbb{R}^d} v(t,x)\right\}\\ &\le \sup_{x\in\mathbb{R}^d} v(0,x).\end{aligned} \tag{41}$$

Induction hence proves (35). Furthermore, note that (35) and the fact that $T = \tau_{k+1}$ imply that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} v(t,x) = \sup_{(t,x)\in[0,\tau_{k+1}]\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x). \tag{42}$$

This establishes (6) in the case $T \ge \frac{1}{4a}$. The proof of Lemma 2.1 is thus completed. □

Corollary 2.2. *Let $d \in \mathbb{N}$, $T \in (0,\infty)$, $v \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, assume that $v|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$, assume for every $t \in (0,T]$, $x \in \mathbb{R}^d$ that*

$$(\tfrac{\partial}{\partial t}v)(t,x) \le (\Delta_x v)(t,x), \tag{43}$$

let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm, and assume that

$$\inf_{a\in\mathbb{R}} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{a\|x\|^2}v(t,x)) < \infty. \tag{44}$$

Then it holds that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x). \tag{45}$$

Proof of Corollary 2.2. Throughout this proof let $|||\cdot|||\colon \mathbb{R}^d \to [0,\infty)$ be the d-dimensional Euclidean norm and let $c \in (0,\infty)$ be the real number which satisfies that

$$c = \sup_{x\in\mathbb{R}^d\setminus\{0\}} \left(\frac{\|x\|}{|||x|||}\right). \tag{46}$$

Note that (44) ensures that there exists $a \in (-\infty, 0]$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{a\|x\|^2} v(t,x)) < \infty. \tag{47}$$

In addition, observe that (46) implies that for all $x \in \mathbb{R}^d \setminus \{0\}$ it holds that

$$a\|x\|^2 = a\left[\frac{\|x\|}{|||x|||}\right]^2 |||x|||^2 \geq ac^2|||x|||^2. \tag{48}$$

Combining this with (47) demonstrates that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} e^{ac^2|||x|||^2} v(t,x) &\leq e^{ac^2|||x|||^2} \max\{0, v(t,x)\} \\ &\leq e^{a\|x\|^2} \max\{0, v(t,x)\} \\ &= \max\Big\{0, e^{a\|x\|^2} v(t,x)\Big\} \\ &\leq \max\left\{0, \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} (e^{a\|y\|^2} v(s,y))\right\} < \infty. \end{aligned} \tag{49}$$

This ensures that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{ac^2|||x|||^2} v(t,x)) \leq \max\left\{0, \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{a\|x\|^2} v(t,x))\right\} < \infty. \tag{50}$$

Hence, we obtain that

$$\inf_{\alpha\in\mathbb{R}} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{\alpha|||x|||^2} v(t,x)) < \infty. \tag{51}$$

Combining this with (43) enables us to apply Lemma 2.1 to obtain that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} v(t,x) = \sup_{x\in\mathbb{R}^d} v(0,x). \tag{52}$$

The proof of Corollary 2.2 is thus completed. □

Theorem 2.3. *Let $d \in \mathbb{N}$, $T \in (0,\infty)$, $c \in \mathbb{R}$, let $f\colon [0,T]\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ be a function, assume for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ that*

$$yf(t,x,y) \leq c(1+y^2), \tag{53}$$

let $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, assume that $u|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm, assume that

$$\inf_{a\in\mathbb{R}} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{a\|x\|^2}|u(t,x)|) < \infty, \tag{54}$$

and assume for every $t \in (0,T]$, $x \in \mathbb{R}^d$ that

$$(\tfrac{\partial u}{\partial t})(t,x) = (\Delta_x u)(t,x) + f(t,x,u(t,x)). \tag{55}$$

Then it holds for every $t \in [0,T]$ that

$$\sup_{x\in\mathbb{R}^d} |u(t,x)| \leq \left[\sup_{x\in\mathbb{R}^d}(1+|u(t,x)|^2)\right]^{1/2} \leq e^{ct}\left[1+\sup_{x\in\mathbb{R}^d}|u(0,x)|^2\right]^{1/2}. \tag{56}$$

Proof of Theorem 2.3. Throughout this proof let $v\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be the function which satisfies for every $t\in[0,T]$, $x\in\mathbb{R}^d$ that

$$v(t,x) = e^{-2ct}(1+|u(t,x)|^2). \tag{57}$$

Note that (54) ensures that there exists $a\in(-\infty,0]$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(e^{a\|x\|^2}|u(t,x)|\right) < \infty. \tag{58}$$

Moreover, observe that (53) demonstrates that $c\geq 0$. Combining this with (57) and (58) implies that

$$\begin{aligned}
&\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(e^{2a\|x\|^2} v(t,x)\right)\\
&= \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(e^{2a\|x\|^2} e^{-2ct}\left(1+|u(t,x)|^2\right)\right)\\
&\leq \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(e^{2a\|x\|^2}\left(1+|u(t,x)|^2\right)\right)\\
&\leq \left[\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(e^{2a\|x\|^2}\right)\right] + \left[\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(e^{2a\|x\|^2}|u(t,x)|^2\right)\right]\\
&= 1 + \left[\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(e^{a\|x\|^2}|u(t,x)|\right)\right]^2 < \infty.
\end{aligned} \tag{59}$$

Hence, we obtain that

$$\inf_{\alpha\in\mathbb{R}} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} (e^{\alpha\|x\|^2} v(t,x)) < \infty. \tag{60}$$

Next observe that (57), the hypothesis that $u\in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, and the hypothesis that $u|_{(0,T]\times\mathbb{R}^d}\in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$ ensure that

$$v\in C([0,T]\times\mathbb{R}^d,\mathbb{R}) \qquad \text{and} \qquad v|_{(0,T]\times\mathbb{R}^d}\in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R}). \tag{61}$$

Furthermore, note that (57) demonstrates that for every $t\in(0,T]$, $x\in\mathbb{R}^d$ it holds that $v(0,x) = 1+|u(0,x)|^2$ and

$$(\tfrac{\partial}{\partial t}v)(t,x) = -2ce^{-2ct}(1+|u(t,x)|^2) + 2e^{-2ct}u(t,x)(\tfrac{\partial u}{\partial t})(t,x). \tag{62}$$

This, (55), and (53) imply that for every $t\in(0,T]$, $x\in\mathbb{R}^d$ it holds that

$$\begin{aligned}
(\tfrac{\partial}{\partial t}v)(t,x) &= -2ce^{-2ct}(1+|u(t,x)|^2) + 2e^{-2ct}u(t,x)\left((\Delta_x u)(t,x) + f(t,x,u(t,x))\right)\\
&\leq -2ce^{-2ct}(1+|u(t,x)|^2) + 2e^{-2ct}u(t,x)(\Delta_x u)(t,x) + 2ce^{-2ct}(1+|u(t,x)|^2)\\
&= 2e^{-2ct}u(t,x)(\Delta_x u)(t,x).
\end{aligned} \tag{63}$$

The fact that for every twice differentiable function $w\colon\mathbb{R}^d\to\mathbb{R}$ and every $x=(x_1,\dots,x_d)\in\mathbb{R}^d$ it holds that

$$\begin{aligned}
(\Delta(|w|^2))(x) &= \sum_{k=1}^d \left[\tfrac{\partial^2}{\partial {x_k}^2}\left(|w(x)|^2\right)\right] = \sum_{k=1}^d \left[\tfrac{\partial}{\partial x_k}\left(2w(x)(\tfrac{\partial}{\partial x_k}w)(x)\right)\right]\\
&= \sum_{k=1}^d \left[2|(\tfrac{\partial}{\partial x_k}w)(x)|^2 + 2w(x)(\tfrac{\partial^2}{\partial {x_k}^2}w)(x)\right]\\
&= 2\left[\sum_{k=1}^d |(\tfrac{\partial}{\partial x_k}w)(x)|^2\right] + 2w(x)\left[\sum_{k=1}^d (\tfrac{\partial^2}{\partial {x_k}^2}w)(x)\right]\\
&= 2w(x)(\Delta w)(x) + 2\left[\sum_{k=1}^d |(\tfrac{\partial}{\partial x_k}w)(x)|^2\right]
\end{aligned} \tag{64}$$

therefore implies that for every $t \in (0, T]$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ it holds that

$$
\begin{aligned}
(\tfrac{\partial}{\partial t} v)(t,x) &\leq e^{-2ct}\Big(2u(t,x)(\Delta_x u)(t,x)\Big) \\
&= e^{-2ct}\left(\Big(\Delta_x(|u|^2)\Big)(t,x) - 2\left[\sum_{k=1}^d |(\tfrac{\partial}{\partial x_k} u)(t,x)|^2\right]\right) \\
&= (\Delta_x v)(t,x) - 2e^{-2ct}\left[\sum_{k=1}^d |(\tfrac{\partial}{\partial x_k} u)(t,x)|^2\right] \leq (\Delta_x v)(t,x).
\end{aligned}
\tag{65}
$$

Combining this with (60) and (61) enables us to apply Corollary 2.2 to obtain that

$$
0 \leq \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} v(t,x) \leq \sup_{x\in\mathbb{R}^d} v(0,x) = 1 + \sup_{x\in\mathbb{R}^d} |u(0,x)|^2. \tag{66}
$$

Therefore, we obtain that for every $t \in [0, T]$ it holds that

$$
\begin{aligned}
\sup_{x\in\mathbb{R}^d} |u(t,x)| &= \left[\sup_{x\in\mathbb{R}^d} |u(t,x)|^2\right]^{1/2} \leq \left[\sup_{x\in\mathbb{R}^d} \Big(1 + |u(t,x)|^2\Big)\right]^{1/2} \\
&= e^{ct}\left[\sup_{x\in\mathbb{R}^d} \Big(e^{-2ct}\Big(1 + |u(t,x)|^2\Big)\Big)\right]^{1/2} = e^{ct}\left[\sup_{x\in\mathbb{R}^d} v(t,x)\right]^{1/2} \\
&\leq e^{ct}\left[\sup_{(s,x)\in[0,T]\times\mathbb{R}^d} v(s,x)\right]^{1/2} \leq e^{ct}\left[1 + \sup_{x\in\mathbb{R}^d} |u(0,x)|^2\right]^{1/2}.
\end{aligned}
\tag{67}
$$

The proof of Theorem 2.3 is thus completed. ∎

Corollary 2.4. *Let $d \in \mathbb{N}$, $T \in (0,\infty)$, $c \in \mathbb{R}$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm, let $f\colon [0,T]\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ be a function which satisfies for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ that $yf(t,x,y) \leq c(1+y^2)$, and let $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ satisfy for every $t \in [0,T)$, $x \in \mathbb{R}^d$ that $u|_{[0,T)\times\mathbb{R}^d} \in C^{1,2}([0,T)\times\mathbb{R}^d,\mathbb{R})$, $\inf_{a\in\mathbb{R}} \sup_{(s,y)\in[0,T]\times\mathbb{R}^d}(e^{a\|y\|^2}|u(s,y)|) < \infty$, and*

$$
(\tfrac{\partial}{\partial t} u)(t,x) + \tfrac{1}{2}(\Delta_x u)(t,x) + f(t,x,u(t,x)) = 0. \tag{68}
$$

Then it holds for every $t \in [0,T]$ that

$$
\sup_{x\in\mathbb{R}^d} |u(t,x)| \leq \left[\sup_{x\in\mathbb{R}^d} (1 + |u(t,x)|^2)\right]^{1/2} \leq e^{c(T-t)}\left[1 + \sup_{x\in\mathbb{R}^d} |u(T,x)|^2\right]^{1/2}. \tag{69}
$$

Proof of Corollary 2.4. Throughout this proof let $U\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ and $F\colon [0,T]\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$ be the functions which satisfy for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ that $U(t,x) = u(T-t, \frac{x}{\sqrt{2}})$ and $F(t,x,y) = f(T-t, \frac{x}{\sqrt{2}}, y)$. Observe that the assumption that for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ it holds that $vf(t,x,y) \leq c(1+y^2)$ implies for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ that

$$
yF(t,x,y) = yf(T-t, \tfrac{x}{\sqrt{2}}, y) \leq c(1+y^2). \tag{70}
$$

Moreover, observe that the hypothesis that $\inf_{a\in\mathbb{R}} \sup_{(s,y)\in[0,T]\times\mathbb{R}^d}(e^{a\|y\|^2}|u(s,y)|) < \infty$ ensures that there exists $\alpha \in \mathbb{R}$ which satisfies that

$$
\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \Big(e^{\alpha\|x\|^2}|u(t,x)|\Big) < \infty. \tag{71}
$$

This implies that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\Big(e^{\frac{\alpha}{2}\|x\|^2}|U(t,x)|\Big)=\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\Big(e^{\alpha\|x/\sqrt{2}\|^2}|u(T-t,\tfrac{x}{\sqrt{2}})|\Big)<\infty. \tag{72}$$

Hence, we obtain that

$$\inf_{a\in\mathbb{R}}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\Big(e^{a\|x\|^2}|U(t,x)|\Big)<\infty. \tag{73}$$

In addition, note that the hypothesis that $u\in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, the hypothesis that $u|_{[0,T)\times\mathbb{R}^d}\in C^{1,2}([0,T)\times\mathbb{R}^d,\mathbb{R})$, the chain rule, and (68) ensure that for every $t\in(0,T]$, $x\in\mathbb{R}^d$ it holds that $U\in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, that $U|_{(0,T]\times\mathbb{R}^d}\in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$, and that

$$(\tfrac{\partial}{\partial t}U)(t,x)=(\Delta_x U)(t,x)+F(t,x,U(t,x)). \tag{74}$$

Combining this, (70), and (73) with Theorem 2.3 (with $f=F$, $u=U$ in the notation of Theorem 2.3) demonstrates for every $t\in[0,T]$ that

$$\begin{aligned}\sup_{x\in\mathbb{R}^d}|u(t,x)|&=\Big[1+\sup_{x\in\mathbb{R}^d}|u(t,x)|^2\Big]^{1/2}=\Big[1+\sup_{x\in\mathbb{R}^d}|U(T-t,x)|^2\Big]^{1/2}\\&\le e^{c(T-t)}\Big[1+\sup_{x\in\mathbb{R}^d}|U(0,x)|^2\Big]^{1/2}=e^{c(T-t)}\Big[1+\sup_{x\in\mathbb{R}^d}|u(T,x)|^2\Big]^{1/2}.\end{aligned} \tag{75}$$

This completes the proof of Corollary 2.4. □

3 Truncated full-history recursive multilevel Picard (MLP) approximations

In this section we present and analyze a (truncated) MLP approximation scheme for reaction-diffusion type PDEs with coercive nonlinearity (see Setting 3.1 below for details). The error analysis relies on results in [24, Section 3] (cf. also Proposition 3.4 below) in combination with a Feynman–Kac representation (cf. Lemma 3.3) and the a priori estimates in Section 2 above.

Setting 3.1 (Setting and algorithm). *Let $d\in\mathbb{N}$, $T\in(0,\infty)$, $\Theta=\cup_{n\in\mathbb{N}}\mathbb{Z}^n$, $f\in C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $g\in C(\mathbb{R}^d,\mathbb{R})$, let $\mathbf{f}_r\colon[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$, $r\in(0,\infty)$, satisfy for every $r\in(0,\infty)$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $u\in\mathbb{R}$ that*

$$\mathbf{f}_r(t,x,u)=f(t,x,\min\{r,\max\{-r,u\}\}), \tag{76}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mathcal{R}^\theta\colon\Omega\to[0,1]$, $\theta\in\Theta$, be independent $\mathcal{U}_{[0,1]}$-distributed random variables, let $W^\theta\colon[0,T]\times\Omega\to\mathbb{R}^d$, $\theta\in\Theta$, be independent standard Brownian motions, assume that $(\mathcal{R}^\theta)_{\theta\in\Theta}$ and $(W^\theta)_{\theta\in\Theta}$ are independent, let $R^\theta\colon[0,T]\times\Omega\to[0,T]$, $\theta\in\Theta$, satisfy for every $\theta\in\Theta$, $t\in[0,T]$ that $R^\theta_t=t+(T-t)\mathcal{R}^\theta$, for every $\theta\in\Theta$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$ let $X^\theta_{t,s,x}\colon\Omega\to\mathbb{R}^d$ satisfy $X^\theta_{t,s,x}=x+W^\theta_s-W^\theta_t$, and let $U^\theta_{n,M,r}\colon[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $\theta\in\Theta$, $n\in\mathbb{N}_0$, $M\in\mathbb{N}$, $r\in(0,\infty)$, satisfy for every $\theta\in\Theta$, $n,M\in\mathbb{N}$, $r\in(0,\infty)$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $U^\theta_{0,M,r}(t,x)=0$ and

$$\begin{aligned}U^\theta_{n,M,r}(t,x)&=\frac{1}{M^n}\Bigg[\sum_{m=1}^{M^n}\Big(g(X^{(\theta,0,-m)}_{t,T,x})+(T-t)\,f\Big(R^{(\theta,0,m)}_t,X^{(\theta,0,m)}_{t,R^{(\theta,0,m)}_t,x},0\Big)\Big)\Bigg]\\&+\sum_{k=1}^{n-1}\frac{(T-t)}{M^{n-k}}\Bigg[\sum_{m=1}^{M^{n-k}}\Big(\mathbf{f}_r\Big(R^{(\theta,k,m)}_t,X^{(\theta,k,m)}_{t,R^{(\theta,k,m)}_t,x},U^{(\theta,k,m)}_{k,M,r}\Big(R^{(\theta,k,m)}_t,X^{(\theta,k,m)}_{t,R^{(\theta,k,m)}_t,x}\Big)\Big)\\&\qquad-\mathbf{f}_r\Big(R^{(\theta,k,m)}_t,X^{(\theta,k,m)}_{t,R^{(\theta,k,m)}_t,x},U^{(\theta,-k,m)}_{k-1,M,r}\Big(R^{(\theta,k,m)}_t,X^{(\theta,k,m)}_{t,R^{(\theta,k,m)}_t,x}\Big)\Big)\Big)\Bigg].\end{aligned} \tag{77}$$

The next result, Lemma 3.2 below, is an adaptation of [24, Theorem 3.5] to Setting 3.1.

Lemma 3.2 (Convergence rate for stochastic fixed point equations)**.** *Assume Setting 3.1, let $\rho \in (0,\infty)$, let $L\colon (0,\infty) \to [0,\infty)$ satisfy for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $v,w \in [-r,r]$ that*

$$|f(t,x,v) - f(t,x,w)| \leq L(r)|v-w|, \tag{78}$$

let $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ satisfy for every $r \in [\rho,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\mathbb{E}\Big[|g(X^0_{0,T,x})|\Big] + \int_0^T \Big(\mathbb{E}\Big[|u(s,X^0_{0,s,x})|^2\Big]\Big)^{1/2}\,ds \\ &\quad + \int_t^T \mathbb{E}\Big[\Big|\mathbf{f}_r\Big(s,X^0_{t,s,x},u(s,X^0_{t,s,x})\Big)\Big| + \Big|f(s,X^0_{t,s,x},0)\Big|\Big]\,ds < \infty \end{aligned} \tag{79}$$

$$\text{and} \qquad u(t,x) = \mathbb{E}\bigg[g(X^0_{t,T,x}) + \int_t^T \mathbf{f}_r\Big(s,X^0_{t,s,x},u(s,X^0_{t,s,x})\Big)\,ds\bigg]. \tag{80}$$

Then it holds for every $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $r \in [\rho,\infty)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\Big(\mathbb{E}\Big[|U^0_{n,M,r}(0,x) - u(0,x)|^2\Big]\Big)^{1/2} \\ &\leq e^{L(r)T}\left[\Big(\mathbb{E}\Big[|g(X^0_{0,T,x})|^2\Big]\Big)^{1/2} + \sqrt{T}\left|\int_0^T \mathbb{E}\Big[|f(s,X^0_{0,s,x},0)|^2\Big]\,ds\right|^{1/2}\right]\left[\frac{e^{M/2}(1+2L(r)T)^n}{M^{n/2}}\right]. \end{aligned} \tag{81}$$

Proof of Lemma 3.2. Throughout this proof let $P_r\colon \mathbb{R}\to\mathbb{R}$, $r \in (0,\infty)$, be the functions which satisfy for every $v \in \mathbb{R}$ that $P_r(v) = \min\{r,\max\{-r,v\}\}$ and assume w.l.o.g. that there exists a standard Brownian motion $\mathbf{W}\colon [0,T]\times\Omega\to\mathbb{R}^d$ which satisfies that $(\mathcal{R}^\theta)_{\theta\in\Theta}$, $(W^\theta)_{\theta\in\Theta}$, and $\mathbf{W}$ are independent. Observe that for every $r \in (0,\infty)$ it holds that $P_r\colon \mathbb{R}\to\mathbb{R}$ is the projection onto the closed convex interval $[-r,r]$. Therefore, we obtain for every $r \in (0,\infty)$, $v,w \in \mathbb{R}$ that

$$|P_r(v) - P_r(w)| \leq |v-w| \tag{82}$$

(cf., e.g., Brézis [7, Proposition 5.3]). This, (76), and (78) imply for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $v,w \in \mathbb{R}$ that

$$\begin{aligned} |\mathbf{f}_r(t,x,v) - \mathbf{f}_r(t,x,w)| &= |f(t,x,P_r(v)) - f(t,x,P_r(w))| \\ &\leq L(r)|P_r(v) - P_r(w)| \leq L(r)|v-w|. \end{aligned} \tag{83}$$

This and [24, Theorem 3.5] (with $d = d$, $T = T$, $L = L(r)$, $\xi = x$, $F = (C([0,T]\times\mathbb{R}^d,\mathbb{R}) \ni v \mapsto ([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \mathbf{f}_r(t,x,v(t,x)) \in \mathbb{R}) \in C([0,T]\times\mathbb{R}^d,\mathbb{R}))$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $g = g$, $u = u$, $\Theta = \Theta$, $W^\theta = W^\theta$, $\mathfrak{r}^\theta = \mathcal{R}^\theta$, $\mathcal{R}^\theta = R^\theta$, $U^\theta_{n,M} = U^\theta_{n,M,r}$ for $\theta\in\Theta$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ in the notation of [24, Theorem 3.5]) ensure for every $n,M \in \mathbb{N}$, $r \in [\rho,\infty)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\Big(\mathbb{E}\Big[|U^0_{n,M,r}(0,x) - u(0,x)|^2\Big]\Big)^{1/2} \\ &\leq e^{L(r)T}\left[\Big(\mathbb{E}\Big[|g(X^0_{0,T,x})|^2\Big]\Big)^{1/2} + \sqrt{T}\left|\int_0^T \mathbb{E}\Big[|f(s,X^0_{0,s,x},0)|^2\Big]\,ds\right|^{1/2}\right]\left[\frac{e^{M/2}(1+2L(r)T)^n}{M^{n/2}}\right]. \end{aligned} \tag{84}$$

Moreover, note that (83) and [24, Lemma 3.4] (with $d = d$, $T = T$, $L = L(r)$, $\xi = x$, $F = (C([0,T]\times\mathbb{R}^d,\mathbb{R}) \ni v \mapsto ([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \mathbf{f}_r(t,x,v(t,x)) \in \mathbb{R}) \in C([0,T]\times\mathbb{R}^d,\mathbb{R}))$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $g = g$, $u = u$, $\Theta = \Theta$, $W^\theta = W^\theta$, $\mathfrak{r}^\theta = \mathcal{R}^\theta$, $\mathcal{R}^\theta = R^\theta$, $U^\theta_{n,M} = U^\theta_{n,M,r}$ for

$\theta \in \Theta$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ in the notation of [24, Lemma 3.4]) yield that for every $M \in \mathbb{N}$, $r \in [\rho, \infty)$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} &\Big(\mathbb{E}\Big[\big|U^0_{0,M,r}(0,x) - u(0,x)\big|^2\Big]\Big)^{1/2} \\ &\leq e^{L(r)T}\left[\Big(\mathbb{E}\big[|g(X^0_{0,T,x})|^2\big]\Big)^{1/2} + \sqrt{T}\left|\int_0^T \mathbb{E}\big[|f(s, X^0_{0,s,x}, 0)|^2\big]\, ds\right|^{1/2}\right]. \end{aligned} \tag{85}$$

Combining this with (84) establishes (81). The proof of Lemma 3.2 is thus completed. □

Lemma 3.3 (Feyman–Kac formula)**.** *Let $d \in \mathbb{N}$, $T \in (0,\infty)$, $u, h \in C([0,T]\times\mathbb{R}^d, \mathbb{R})$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^d$ be a standard Brownian motion, for every $t \in [0,T]$, $s \in [t,T]$, $x \in \mathbb{R}^d$ let $X_{t,s,x}\colon \Omega \to \mathbb{R}^d$ satisfy $X_{t,s,x} = x + W_s - W_t$, and assume for every $t \in [0,T)$, $x \in \mathbb{R}^d$ that $\sup_{s\in[0,T], y\in\mathbb{R}^d} |u(s,y)| < \infty$, $\mathbb{E}[\int_t^T |h(s, X_{t,s,x})|\, ds] < \infty$, $u|_{[0,T)\times\mathbb{R}^d} \in C^{1,2}([0,T)\times\mathbb{R}^d, \mathbb{R})$, and*

$$(\tfrac{\partial}{\partial t}u)(t,x) + \tfrac{1}{2}(\Delta_x u)(t,x) + h(t,x) = 0. \tag{86}$$

Then it holds for every $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$u(t,x) = \mathbb{E}\left[u(T, X_{t,T,x}) + \int_t^T h(s, X_{t,s,x})\, ds\right]. \tag{87}$$

Proof of Lemma 3.3. Throughout this proof let $\langle\cdot,\cdot\rangle\colon \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ be the Euclidean scalar product on $\mathbb{R}^d$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be the Euclidean norm on $\mathbb{R}^d$, and for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ with $t < T - {}^1\!/\!_r$ let the function $\tau_r^{t,x}\colon \Omega \to [t, T - {}^1\!/\!_r]$ satisfy that $\tau_r^{t,x} = \inf(\{s \in [t,T]\colon \|X_{t,s,x} - x\| > r\} \cup \{T - {}^1\!/\!_r\})$. Observe that Itô's formula and the hypothesis that $u|_{[0,T)\times\mathbb{R}^d} \in C^{1,2}([0,T)\times\mathbb{R}^d, \mathbb{R})$ ensure that for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ with $t < T - {}^1\!/\!_r$ it holds $\mathbb{P}$-a.s. that

$$u(\tau_r^{t,x}, X_{t,\tau_r^{t,x},x}) = u(t,x) + \int_t^{\tau_r^{t,x}} \langle(\nabla_x u)(s, X_{t,s,x}), dW_s\rangle - \int_t^{\tau_r^{t,x}} h(s, X_{t,s,x})\, ds. \tag{88}$$

This implies for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ with $t < T - {}^1\!/\!_r$ that

$$u(t,x) = \mathbb{E}\left[u(\tau_r^{t,x}, X_{t,\tau_r^{t,x},x}) + \int_t^{\tau_r^{t,x}} h(s, X_{t,s,x})\, ds\right]. \tag{89}$$

Combining the fact that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds $\mathbb{P}$-a.s. that $\limsup_{r\to\infty} |\tau_r^{t,x} - T| = 0$ and the hypothesis that $u\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a bounded continuous function with Lebesgue's dominated convergence theorem hence implies that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\limsup_{r\to\infty} \mathbb{E}\Big[\big|u(\tau_r^{t,x}, X_{t,\tau_r^{t,x},x}) - u(T, X_{t,T,x})\big|\Big] = 0. \tag{90}$$

In addition, note that the fact that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds $\mathbb{P}$-a.s. that $\limsup_{r\to\infty} |\tau_r^{t,x,\mathfrak{t}} - \mathfrak{t}| = 0$, the hypothesis that $h\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is a continuous function, the hypothesis that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $\int_t^T \mathbb{E}[|h(s, X_{t,s,x})|]\, ds < \infty$, and Lebesgue's dominated convergence theorem ensure for every $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\limsup_{r\to\infty} \left|\mathbb{E}\left[\int_t^{\tau_r^{t,x}} h(s, X_{t,s,x})\, ds\right] - \mathbb{E}\left[\int_t^T h(s, X_{t,s,x})\, ds\right]\right| = 0. \tag{91}$$

This, (89), and (90) imply for every $t \in [0,T)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} u(t,x) &= \lim_{r\to\infty} \left(\mathbb{E}\left[u(\tau_r^{t,x}, X_{t,\tau_r^{t,x},x}) + \int_t^{\tau_r^{t,x}} h(s, X_{t,s,x})\, ds \right] \right) \\ &= \mathbb{E}\left[u(T, X_{t,T,x}) + \int_t^T h(s, X_{t,s,x})\, ds \right]. \end{aligned} \tag{92}$$

This establishes (87). The proof of Lemma 3.3 is thus completed. ☐

Proposition 3.4 (Convergence rate for Allen–Cahn PDEs)**.** *Assume Setting 3.1, let $\rho \in (0,\infty)$, $c \in [0,\infty)$, let $\|\cdot\| : \mathbb{R}^d \to [0,\infty)$ be a norm, let $L\colon (0,\infty) \to [0,\infty)$ satisfy for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $v,w \in [-r,r]$ that*

$$|f(t,x,v) - f(t,x,w)| \le L(r)|v-w|, \tag{93}$$

let $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$ satisfy that $\inf_{a\in\mathbb{R}}[\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d}(e^{a\|x\|^2}|u(t,x)|)] < \infty$ and $u|_{[0,T)\times\mathbb{R}^d} \in C^{1,2}([0,T)\times\mathbb{R}^d,\mathbb{R})$, and assume for every $t \in [0,T)$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}$ that $\rho \ge e^{cT}(1+|g(x)|^2)^{1/2}$, $vf(t,x,v) \le c(1+v^2)$, $\int_t^T \mathbb{E}[|f(s,X^0_{t,s,x},0)|]\, ds < \infty$, $u(T,x) = g(x)$, and

$$(\tfrac{\partial}{\partial t}u)(t,x) + \tfrac{1}{2}(\Delta_x u)(t,x) + f(t,x,u(t,x)) = 0. \tag{94}$$

Then it holds for every $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $r \in [\rho,\infty)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} &\left(\mathbb{E}\left[|U^0_{n,M,r}(0,x) - u(0,x)|^2\right]\right)^{1/2} \\ &\le e^{L(r)T}\left[\left(\mathbb{E}\left[|g(X^0_{0,T,x})|^2\right]\right)^{1/2} + \sqrt{T}\left|\int_0^T \mathbb{E}\left[|f(s,X^0_{0,s,x},0)|^2\right] ds\right|^{1/2}\right]\left[\frac{e^{M/2}(1+2L(r)T)^n}{M^{n/2}}\right]. \end{aligned} \tag{95}$$

Proof of Proposition 3.4. First, observe that the hypothesis that $\sup_{x\in\mathbb{R}^d}|g(x)| < \infty$ implies that for every $x \in \mathbb{R}^d$ it holds that

$$\mathbb{E}\left[|g(X^0_{0,T,x})|\right] < \infty. \tag{96}$$

Next note that Corollary 2.4 (with $d = d$, $T = T$, $c = c$, $\|\cdot\| = \|\cdot\|$, $f = f$, $u = u$ in the notation of Corollary 2.4) ensures for every $t \in [0,T]$ that

$$\sup_{x\in\mathbb{R}^d}|u(t,x)| \le e^{c(T-t)}\left[1 + \sup_{x\in\mathbb{R}^d}|u(T,x)|^2\right]^{1/2} \le e^{cT}\left[1 + \sup_{x\in\mathbb{R}^d}|g(x)|^2\right]^{1/2} \le \rho. \tag{97}$$

Combining this with (76) yields for every $r \in [\rho,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\mathbf{f}_r(t,x,u(t,x)) = f(t,x,\min\{r,\max\{-r,u(t,x)\}\}) = f(t,x,u(t,x)). \tag{98}$$

This and (94) demonstrate that for every $r \in [\rho,\infty)$, $t \in [0,T)$, $x \in \mathbb{R}^d$ it holds that

$$(\tfrac{\partial}{\partial t}u)(t,x) + \tfrac{1}{2}(\Delta_x u)(t,x) + \mathbf{f}_r(t,x,u(t,x)) = 0. \tag{99}$$

Next observe that the fact that $\sup_{t\in[0,T],x\in\mathbb{R}^d}|u(t,x)| \le \rho$ and (93) ensure that for every $r \in [\rho,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $(\mathbb{E}[|u(s,X^0_{0,s,x})|^2])^{1/2} \le \rho < \infty$ and

$$\begin{aligned} &\mathbb{E}\left[\int_t^T \left|\mathbf{f}_r\left(s,X^0_{t,s,x},u(s,X^0_{t,s,x})\right)\right| ds\right] \\ &\le \mathbb{E}\left[\int_t^T \left|\mathbf{f}_r\left(s,X^0_{t,s,x},0\right)\right| ds\right] + \int_t^T L(r)\,\mathbb{E}\left[|u(s,X^0_{t,s,x})|\right] ds \\ &\le \mathbb{E}\left[\int_t^T |f(s,X^0_{t,s,x},0)|\, ds\right] + L(r)T\rho < \infty. \end{aligned} \tag{100}$$

Hence, we obtain that (99) and Lemma 3.3 (with $d = d$, $T = T$, $u = u$, $h = ([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \mathbf{f}_r(t,x,u(t,x)) \in \mathbb{R})$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $W = W^0$, $X_{t,s,x} = X^0_{t,s,x}$ for $t \in [0,T]$, $s \in [t,T]$, $x \in \mathbb{R}^d$ in the notation of Lemma 3.3) demonstrate that for every $r \in [\rho,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$u(t,x) = \mathbb{E}\bigg[g(X^0_{t,T,x}) + \int_t^T \mathbf{f}_r\Big(s, X^0_{t,s,x}, u(s,X^0_{t,s,x})\Big)\,ds\bigg]. \tag{101}$$

Lemma 3.2 (with $\rho = \rho$, $L = L$, $u = u$ in the notation of Lemma 3.2), (96), and (100) hence establish (95). The proof of Proposition 3.4 is thus completed. ☐

Proposition 3.5. *Let $d \in \mathbb{N}$, $\rho, T \in (0,\infty)$, $c \in [0,\infty)$, $\Theta = \cup_{n\in\mathbb{N}}\mathbb{Z}^n$, $f \in C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $(\mathbf{f}_r)_{r\in(0,\infty)} \subseteq C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, $u \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $L\colon (0,\infty)\to[0,\infty)$ be a function, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mathcal{R}^\theta\colon \Omega\to[0,1]$, $\theta\in\Theta$, be independent $\mathcal{U}_{[0,1]}$-distributed random variables, let $W^\theta\colon [0,T]\times\Omega\to\mathbb{R}^d$, $\theta\in\Theta$, be independent standard Brownian motions, assume that $(\mathcal{R}^\theta)_{\theta\in\Theta}$ and $(W^\theta)_{\theta\in\Theta}$ are independent, let $R^\theta\colon [0,T]\times\Omega\to[0,T]$, $\theta\in\Theta$, satisfy for every $\theta\in\Theta$, $t\in[0,T]$ that $R^\theta_t = t\mathcal{R}^\theta$, for every $\theta\in\Theta$, $s\in[0,T]$, $t\in[s,T]$, $x\in\mathbb{R}^d$ let $X^\theta_{s,t,x}\colon\Omega\to\mathbb{R}^d$ satisfy $X^\theta_{s,t,x} = x + \sqrt{2}(W^\theta_t - W^\theta_s)$, assume for every $r\in(0,\infty)$, $t\in(0,T]$, $x\in\mathbb{R}^d$, $v\in\mathbb{R}$, $w,\mathfrak{w}\in[-r,r]$ that $vf(t,x,v)\le c(1+v^2)$, $|f(t,x,w)-f(t,x,\mathfrak{w})|\le L(r)|w-\mathfrak{w}|$, $\int_0^t\mathbb{E}[|f(s,X^0_{s,t,x},0)|]\,ds<\infty$, $\mathbf{f}_r(t,x,v) = f(t,x,\min\{r,\max\{-r,v\}\})$, $e^{cT}(1+|u(0,x)|^2)^{1/2}\le\rho$, $u|_{(0,T]\times\mathbb{R}^d}\in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$, $\inf_{a\in\mathbb{R}}[\sup_{s\in[0,T]}\sup_{y\in\mathbb{R}^d}(e^{a\|y\|^2}|u(s,y)|)]<\infty$, and*

$$(\tfrac{\partial}{\partial t}u)(t,x) = (\Delta_x u)(t,x) + f(t,x,u(t,x)), \tag{102}$$

and let $U^\theta_{n,M,r}\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $\theta\in\Theta$, $n\in\mathbb{N}_0$, $M\in\mathbb{N}$, $r\in(0,\infty)$, satisfy for every $\theta\in\Theta$, $n,M\in\mathbb{N}$, $r\in(0,\infty)$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $U^\theta_{0,M,r}(t,x) = 0$ and

$$\begin{aligned} U^\theta_{n,M,r}(t,x) &= \frac{1}{M^n}\left[\sum_{m=1}^{M^n}\Big(u(0,X^{(\theta,0,-m)}_{0,t,x}) + t\,f\Big(R^{(\theta,0,m)}_t, X^{(\theta,0,m)}_{R^{(\theta,0,m)}_t,t,x},0\Big)\Big)\right] \\ &+ \sum_{k=1}^{n-1}\frac{t}{M^{n-k}}\left[\sum_{m=1}^{M^{n-k}}\bigg(\mathbf{f}_r\Big(R^{(\theta,k,m)}_t, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}, U^{(\theta,k,m)}_{k,M,r}\Big(R^{(\theta,k,m)}_t, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\right. \\ &\left.\qquad - \mathbf{f}_r\Big(R^{(\theta,k,m)}_t, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}, U^{(\theta,-k,m)}_{k-1,M,r}\Big(R^{(\theta,k,m)}_t, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\bigg)\right]. \end{aligned} \tag{103}$$

Then it holds for every $n\in\mathbb{N}_0$, $M\in\mathbb{N}$, $r\in[\rho,\infty)$, $x\in\mathbb{R}^d$ that

$$\begin{aligned} \Big(\mathbb{E}\Big[|U^0_{n,M,r}(T,x)-u(T,x)|^2\Big]\Big)^{1/2} &\le e^{L(r)T}\left[\frac{e^{M/2}(1+2L(r)T)^n}{M^{n/2}}\right] \\ &\cdot\left[\Big(\mathbb{E}\Big[|u(0,X^0_{0,T,x})|^2\Big]\Big)^{1/2} + \sqrt{T}\left|\int_0^T\mathbb{E}\Big[|f(s,X^0_{s,T,x},0)|^2\Big]\,ds\right|^{1/2}\right]. \end{aligned} \tag{104}$$

Proof of Proposition 3.5. Throughout this proof let $v\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}$ be the function which satisfies for every $t\in[0,T]$, $x\in\mathbb{R}^d$ that $v(t,x) = u(T-t,x\sqrt{2})$, let $F\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ be the function which satisfies for every $t\in[0,T]$, $x\in\mathbb{R}^d$, $w\in\mathbb{R}$ that $F(t,x,w) = f(T-t,x\sqrt{2},w)$, let $\mathbf{F}_r\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$, $r\in(0,\infty)$, be the functions which satisfy for every $r\in(0,\infty)$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $w\in\mathbb{R}$ that $\mathbf{F}_r(t,x,w) = F(t,x,\min\{r,\max\{-r,w\}\})$, let $\mathcal{S}^\theta\colon\Omega\to[0,1]$, $\theta\in\Theta$, satisfy for every $\theta\in\Theta$ that $\mathcal{S}^\theta = 1-\mathcal{R}^\theta$, let $S^\theta\colon [0,T]\times\Omega\to[0,T]$, $\theta\in\Theta$, satisfy for every $\theta\in\Theta$, $t\in[0,T]$ that $S^\theta_t = t+(T-t)\mathcal{S}^\theta$, for every $\theta\in\Theta$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$ let $Y^\theta_{t,s,x}\colon\Omega\to\mathbb{R}^d$ satisfy that $Y^\theta_{t,s,x} = \frac{1}{\sqrt{2}}X^\theta_{T-s,T-t,x\sqrt{2}} = x + W^\theta_{T-t} - W^\theta_{T-s} = x + (W^\theta_T - W^\theta_{T-s}) - (W^\theta_T - W^\theta_{T-t})$, and let $V^\theta_{n,M,r}\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $\theta\in\Theta$, $n\in\mathbb{N}_0$, $M\in\mathbb{N}$, $r\in(0,\infty)$ satisfy for every $n,M\in\mathbb{N}$,

$\theta \in \Theta$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $V^\theta_{n,M,r}(t,x) = U^\theta_{n,M,r}(T-t, x\sqrt{2})$. Note that (102) hence ensures for every $t \in [0,T)$, $x \in \mathbb{R}^d$ that $v \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$, $v|_{[0,T)\times\mathbb{R}^d} \in C^{1,2}([0,T) \times \mathbb{R}^d, \mathbb{R})$, $\inf_{a\in\mathbb{R}}[\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}(e^{a\|y\|^2}|v(s,y)|)] < \infty$, and

$$\begin{aligned}&(\tfrac{\partial}{\partial t}v)(t,x) + \tfrac{1}{2}(\Delta_x v)(t,x) + F(t,x,v(t,x))\\ &= -(\tfrac{\partial u}{\partial t})(T-t, x\sqrt{2}) + (\Delta_x u)(T-t,x\sqrt{2}) + f(T-t, x\sqrt{2}, u(T-t,x\sqrt{2})) = 0.\end{aligned} \tag{105}$$

In addition, note that the hypothesis that for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}$ it holds that $wf(t,x,w) \le c(1+w^2)$ guarantees that for every $t \in [0,T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}$ it holds that

$$wF(t,x,w) = wf(T-t, x\sqrt{2}, w) \le c(1+w^2). \tag{106}$$

Moreover, observe that it holds for every $t \in [0,T]$, $\theta \in \Theta$ that

$$S^\theta_t = t + (T-t)\mathcal{S}^\theta = t + (T-t)(1-\mathcal{R}^\theta) = T - (T-t)\mathcal{R}^\theta = T - R^\theta_{T-t}. \tag{107}$$

Next observe that the assumption that for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $w, \mathfrak{w} \in [-r,r]$ it holds that $|f(t,x,w) - f(t,x,\mathfrak{w})| \le L(r)|w-\mathfrak{w}|$ implies that for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $w, \mathfrak{w} \in [-r,r]$ it holds that

$$|F(t,x,w) - F(t,x,\mathfrak{w})| = |f(T-t,x\sqrt{2},w) - f(T-t,x\sqrt{2},\mathfrak{w})| \le L(r)|w-\mathfrak{w}|. \tag{108}$$

In addition, note that

$$e^{cT}\left[1 + \sup_{x\in\mathbb{R}^d}|v(T,x)|^2\right]^{1/2} = e^{cT}\left[1 + \sup_{x\in\mathbb{R}^d}|u(0,x\sqrt{2})|^2\right]^{1/2} \le \rho. \tag{109}$$

Furthermore, note that for every $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\int_t^T \mathbb{E}\Big[|F(s, Y^0_{t,s,x}, 0)|\Big]\,ds = \int_0^{T-t} \mathbb{E}\Big[|F(T-s, Y^0_{t,T-s,x}, 0)|\Big]\,ds\\ &= \int_0^{T-t} \mathbb{E}\Big[|f(s, \sqrt{2}Y^0_{t,T-s,x}, 0)|\Big]\,ds = \int_0^{T-t} \mathbb{E}\Big[|f(s, X^0_{s,T-t,x\sqrt{2}}, 0)|\Big]\,ds < \infty\end{aligned} \tag{110}$$

and

$$\begin{aligned}&\int_0^T \mathbb{E}\Big[|F(s, Y^0_{0,s,x/\sqrt{2}}, 0)|^2\Big]\,ds = \int_0^T \mathbb{E}\Big[|f(T-s, \sqrt{2}\,Y^0_{0,s,x/\sqrt{2}}, 0)|^2\Big]\,ds\\ &= \int_0^T \mathbb{E}\Big[|f(T-s, X^0_{T-s,T,x}, 0)|^2\Big]\,ds = \int_0^T \mathbb{E}\Big[|f(s, X^0_{s,T,x}, 0)|^2\Big]\,ds.\end{aligned} \tag{111}$$

Moreover, observe that (103) guarantees for every $n, M \in \mathbb{N}$, $\theta \in \Theta$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$\begin{aligned}&U^\theta_{n,M,r}(T-t, x\sqrt{2})\\ &= \frac{1}{M^n}\left[\sum_{m=1}^{M^n}\left(u\Big(0, X^{(\theta,0,-m)}_{0,T-t,x\sqrt{2}}\Big) + (T-t)\,f\Big(R^{(\theta,0,m)}_{T-t}, X^{(\theta,0,m)}_{R^{(\theta,0,m)}_{T-t},T-t,x\sqrt{2}}, 0\Big)\right)\right]\\ &+ \sum_{k=1}^{n-1}\frac{T-t}{M^{n-k}}\left[\sum_{m=1}^{M^{n-k}}\left(\mathbf{f}_r\Big(R^{(\theta,k,m)}_{T-t}, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_{T-t},T-t,x\sqrt{2}}, U^{(\theta,k,m)}_{k,M,r}\Big(R^{(\theta,k,m)}_{T-t}, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_{T-t},T-t,x\sqrt{2}}\Big)\Big)\right.\right.\\ &\qquad\left.\left. - \mathbf{f}_r\Big(R^{(\theta,k,m)}_{T-t}, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_{T-t},T-t,x\sqrt{2}}, U^{(\theta,-k,m)}_{k-1,M,r}\Big(R^{(\theta,k,m)}_{T-t}, X^{(\theta,k,m)}_{R^{(\theta,k,m)}_{T-t},T-t,x\sqrt{2}}\Big)\Big)\right)\right].\end{aligned} \tag{112}$$

The fact that for every $\theta \in \Theta$, $t \in [0,T]$, $s \in [t,T]$, $x \in \mathbb{R}^d$ it holds that $X^\theta_{t,s,x\sqrt{2}} = \sqrt{2}Y^\theta_{T-s,T-t,x}$ and (107) therefore imply that for every $n, M \in \mathbb{N}$, $\theta \in \Theta$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} & U^\theta_{n,M,r}(T-t, x\sqrt{2}) \\ &= \frac{1}{M^n}\left[\sum_{m=1}^{M^n} \left(u\Big(0, \sqrt{2}\, Y^{(\theta,0,-m)}_{t,T,x}\Big) + (T-t)\, f\Big(T - S^{(\theta,0,m)}_t, \sqrt{2}\, Y^{(\theta,0,m)}_{t,S^{(\theta,0,m)}_t,x}, 0\Big)\right)\right] \\ &+ \sum_{k=1}^{n-1} \tfrac{T-t}{M^{n-k}} \Bigg[\sum_{m=1}^{M^{n-k}} \bigg(\mathbf{f}_r\Big(T - S^{(\theta,k,m)}_t, \sqrt{2}\, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}, U^{(\theta,k,m)}_{k,M,r}\Big(T - S^{(\theta,k,m)}_t, \sqrt{2}\, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}\Big)\Big) \\ &\qquad - \mathbf{f}_r\Big(T - S^{(\theta,k,m)}_t, \sqrt{2}\, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}, U^{(\theta,-k,m)}_{k-1,M,r}\Big(T - S^{(\theta,k,m)}_t, \sqrt{2}\, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}\Big)\Big)\bigg)\Bigg]. \end{aligned} \tag{113}$$

Combining this with (103) and the fact that for every $M \in \mathbb{N}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $V^\theta_{n,M,r}(t,x) = U^\theta_{n,M,r}(T-t, x\sqrt{2})$ yields hat for every $\theta \in \Theta$, $n, M \in \mathbb{N}$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $V^\theta_{0,M,r}(t,x) = 0$ and

$$\begin{aligned} & V^\theta_{n,M,r}(t,x) \\ &= \frac{1}{M^n}\left[\sum_{m=1}^{M^n} \left(u\Big(0, \sqrt{2}\, Y^{(\theta,0,-m)}_{t,T,x}\Big) + (T-t)\, f\Big(T - S^{(\theta,0,m)}_t, \sqrt{2}\, Y^{(\theta,0,m)}_{t,S^{(\theta,0,m)}_t,x}, 0\Big)\right)\right] \\ &+ \sum_{k=1}^{n-1} \tfrac{T-t}{M^{n-k}} \Bigg[\sum_{m=1}^{M^{n-k}} \bigg(\mathbf{f}_r\Big(T - S^{(\theta,k,m)}_t, \sqrt{2}\, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}, V^{(\theta,k,m)}_{k,M,r}\Big(S^{(\theta,k,m)}_t, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}\Big)\Big) \\ &\qquad - \mathbf{f}_r\Big(T - S^{(\theta,k,m)}_t, \sqrt{2}\, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}, V^{(\theta,-k,m)}_{k-1,M,r}\Big(S^{(\theta,k,m)}_t, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}\Big)\Big)\bigg)\Bigg]. \end{aligned} \tag{114}$$

This and the fact that for every $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}$ it holds that $u(0, x\sqrt{2}) = v(T,x)$ and $\mathbf{F}_r(t,x,w) = F(t,x,\min\{r,\max\{-r,w\}\}) = f(T-t, x\sqrt{2}, \min\{r,\max\{-r,w\}\}) = \mathbf{f}_r(T-t, x\sqrt{2}, w)$ demonstrate that for every $\theta \in \Theta$, $n, M \in \mathbb{N}$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $V^\theta_{0,M,r}(t,x) = 0$ and

$$\begin{aligned} V^\theta_{n,M,r}(t,x) &= \frac{1}{M^n}\left[\sum_{m=1}^{M^n} \left(v(T, Y^{(\theta,0,-m)}_{t,T,x}) + (T-t)\, F(S^{(\theta,0,m)}_t, Y^{(\theta,0,m)}_{t,S^{(\theta,0,m)}_t,x}, 0)\right)\right] \\ &+ \sum_{k=1}^{n-1} \frac{(T-t)}{M^{n-k}} \Bigg[\sum_{m=1}^{M^{n-k}} \bigg(\mathbf{F}_r\Big(S^{(\theta,k,m)}_t, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}, V^{(\theta,k,m)}_{k,M,r}\Big(S^{(\theta,k,m)}_t, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}\Big)\Big) \\ &\qquad - \mathbf{F}_r\Big(S^{(\theta,k,m)}_t, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}, V^{(\theta,-k,m)}_{k-1,M,r}\Big(S^{(\theta,k,m)}_t, Y^{(\theta,k,m)}_{t,S^{(\theta,k,m)}_t,x}\Big)\Big)\bigg)\Bigg]. \end{aligned} \tag{115}$$

This, (105)–(111), and Proposition 3.4 (with $d = d$, $T = T$, $\Theta = \Theta$, $f = F$, $g = (\mathbb{R}^d \ni x \mapsto v(T,x) = u(0, x\sqrt{2}) \in \mathbb{R})$, $\mathbf{f}_r = \mathbf{F}_r$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{R}^\theta = \mathcal{S}^\theta$, $W^\theta = ([0,T] \times \Omega \ni (t,\omega) \mapsto W^\theta_T(\omega) - W^\theta_{T-t}(\omega) \in \mathbb{R}^d)$, $X^\theta_{t,s,x} = Y^\theta_{t,s,x}$, $R^\theta = S^\theta$, $U^\theta_{n,M,r} = V^\theta_{n,M,r}$, $\rho = \rho$, $c = c$, $\|\cdot\| = \|\cdot\|$, $L = L$, $u = v$ for $\theta \in \Theta$, $r \in (0,\infty)$, $t \in [0,T]$, $s \in [t,T]$ $x \in \mathbb{R}^d$ in the notation of Proposition 3.4) demonstrate that for every $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $r \in [\rho,\infty)$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned} \Big(\mathbb{E}\Big[|V^0_{n,M,r}(0, {}^{x}\!/\!\sqrt{2}) - v(0, {}^{x}\!/\!\sqrt{2})|^2\Big]\Big)^{1/2} &\le \frac{e^{M/2}(1 + 2L(r)T)^n}{M^{n/2}} \\ \cdot\, e^{L(r)T} &\left[\Big(\mathbb{E}\Big[|v(T, Y^0_{0,T,x/\sqrt{2}})|^2\Big]\Big)^{1/2} + \sqrt{T}\left|\int_0^T \mathbb{E}\Big[|F(s, Y^0_{0,s,x/\sqrt{2}}, 0)|^2\Big]\, ds\right|^{1/2}\right]. \end{aligned} \tag{116}$$

Combining this with (111) and the fact that for every $\theta \in \Theta$, $n \in \mathbb{N}_0$, $M \in \mathbb{N}$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that $u(t,x) = v(T-t, x/\sqrt{2})$, $U^\theta_{n,M,r}(t,x) = V^\theta_{n,M,r}(T-t, x/\sqrt{2})$, and $\sqrt{2}\, Y^0_{0,T,x/\sqrt{2}} = X^0_{0,T,x}$ establishes (104). The proof of Proposition 3.5 is thus completed. □

4 Computational cost analysis for truncated MLP approximations

Our next goal is to estimate the overall complexity of the MLP approximation scheme. This is achieved in Theorem 4.5 below. We first quote an elementary result (see [24, Lemma 3.6]) which provides a bound for the computational cost. Lemma 4.2–Lemma 4.4 are technical statements needed for the proof of Theorem 4.5.

Lemma 4.1 (Computational cost)**.** *Let $d \in \mathbb{N}$, $(\mathfrak{C}_{n,M})_{n\in\mathbb{N}_0, M\in\mathbb{N}} \subseteq \mathbb{N}_0$ satisfy for every $n, M \in \mathbb{N}$ that $\mathfrak{C}_{0,M} = 0$ and*

$$\mathfrak{C}_{n,M} \leq (2d+1)M^n + \sum_{l=1}^{n-1} M^{n-l}\left(d + 1 + \mathfrak{C}_{l,M} + \mathfrak{C}_{l-1,M}\right). \tag{117}$$

Then it holds for every $n, M \in \mathbb{N}$ that $\mathfrak{C}_{n,M} \leq d(5M)^n$.

Proof of Lemma 4.1. This is an immediate consequence of [24, Lemma 3.6] (with $d = d$, $RV_{n,M} = \mathfrak{C}_{n,M}$ for $n \in \mathbb{N}_0$, $M \in \mathbb{N}$ in the notation of [24, Lemma 3.6]). The proof of Lemma 4.1 is thus completed. □

Lemma 4.2. *Let $\alpha, \beta, c, \kappa, \rho \in (0,\infty)$, $K \in \mathbb{N}_0$, $(\gamma_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$, $(\epsilon_{n,r})_{n\in\mathbb{N}, r\in[\rho,\infty)} \subseteq [0,\infty)$, let $L\colon (0,\infty) \to [0,\infty)$ be a function, assume for every $n \in \mathbb{N}$, $r \in [\rho,\infty)$ that $\gamma_n \leq (\alpha n)^n$ and $\epsilon_{n,r} \leq c e^{L(r)} \kappa^n (1 + \beta L(r))^n n^{-n/2}$, and let $\varrho\colon \mathbb{N} \to (0,\infty)$ satisfy that*

$$\limsup_{n\to\infty} \left[\frac{L(\varrho_n)}{\ln(n)} + \frac{1}{\varrho_n}\right] = 0. \tag{118}$$

Then there exist $\mathfrak{N}\colon (0,1] \to \mathbb{N}$ and $\mathfrak{c}\colon (0,\infty) \to [0,\infty)$ such that for every $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$ it holds that $\sup_{n\in[1,\mathfrak{N}_\varepsilon+K]\cap\mathbb{N}} \gamma_n \leq \mathfrak{c}_\delta \varepsilon^{-(2+2\delta)}$ and $\sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n} \leq \varepsilon$.

Proof of Lemma 4.2. Throughout this proof let $\mathfrak{a}_\delta \in [0,\infty]$, $\delta \in (0,\infty)$, and $\mathfrak{b} \in [0,\infty)$ satisfy for every $\delta \in (0,\infty)$ that

$$\mathfrak{a}_\delta = c^{2+2\delta} \sup_{n\in\mathbb{N}} \left[\frac{[\max\{\alpha,1\}(n+1)]^{(n+1)}}{n^{n(1+\delta)}} e^{L(\varrho_n)(2+2\delta)} [\kappa(1+\beta L(\varrho_n))]^{n(2+2\delta)}\right] \tag{119}$$

and

$$\mathfrak{b} = [\max\{\alpha,1\}(K+1)]^{(K+1)}. \tag{120}$$

First, observe that the fact that for every $t \in (0,\infty)$ it holds that $\ln(t) \leq t-1$ and (118) ensure that

$$\begin{aligned}
&\limsup_{n\to\infty} \left[\ln\!\left(c e^{L(\varrho_n)} \kappa^n (1+\beta L(\varrho_n))^n n^{-n/2}\right)\right] \\
&= \limsup_{n\to\infty} \left[\ln(c) + L(\varrho_n) + n\ln(\kappa) + n\ln(1+\beta L(\varrho_n)) - \frac{n}{2}\ln(n)\right] \\
&\leq \limsup_{n\to\infty} \left[\ln(c) + L(\varrho_n) + n\ln(\kappa) + n\beta L(\varrho_n) - \frac{n}{2}\ln(n)\right] \\
&= \limsup_{n\to\infty} \left[n\ln(n)\left(\frac{\ln(c)}{n\ln(n)} + \frac{L(\varrho_n)}{n\ln(n)} + \frac{\ln(\kappa)}{\ln(n)} + \frac{\beta L(\varrho_n)}{\ln(n)} - \frac{1}{2}\right)\right] = -\infty.
\end{aligned} \tag{121}$$

This and the fact that $\lim_{s\to-\infty} e^s = 0$ imply that

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} \Big[c e^{L(\varrho_n)} \kappa^n (1+\beta L(\varrho_n))^n n^{-n/2} \Big] \\ &= \limsup_{n\to\infty} \Big[\exp\Big(\ln\Big(c e^{L(\varrho_n)} \kappa^n (1+\beta L(\varrho_n))^n n^{-n/2} \Big)\Big) \Big] = 0. \end{aligned} \tag{122}$$

Hence, we obtain that there exist $N_\varepsilon \in \mathbb{N}$, $\varepsilon \in (0,\infty)$, which satisfy for every $\varepsilon \in (0,\infty)$ that

$$N_\varepsilon = \min\bigg\{ n \in \mathbb{N} \colon \sup_{m \in [n,\infty)\cap\mathbb{N}} \Big[c e^{L(\varrho_m)} \kappa^m (1+\beta L(\varrho_m))^m m^{-m/2} \Big] \le \varepsilon \bigg\}. \tag{123}$$

Moreover, the assumption that $\liminf_{n\to\infty} \varrho_n = \infty$ implies that there exists $\mathfrak{n} \in \mathbb{N}$ which satisfies that $\inf_{n\in[\mathfrak{n},\infty)\cap\mathbb{N}} \varrho_n \ge \rho$. Next let $\eta \in (0,\infty)$ satisfy that $\eta < c e^{L(\varrho_\mathfrak{n})} \kappa^\mathfrak{n} (1+\beta L(\varrho_\mathfrak{n}))^\mathfrak{n} \mathfrak{n}^{-\mathfrak{n}/2}$. This implies for every $\varepsilon \in (0,\eta]$ that $N_\varepsilon > \mathfrak{n}$. Hence, we obtain that for every $\varepsilon \in (0,\eta]$ it holds that $\inf_{n\in[N_\varepsilon,\infty)\cap\mathbb{N}} \varrho_n \ge \inf_{n\in[\mathfrak{n},\infty)\cap\mathbb{N}} \varrho_n \ge \rho$. This, the assumption that for every $n \in \mathbb{N}$, $r \in [\rho,\infty)$ it holds that $\epsilon_{n,r} \le c e^{L(r)} \kappa^n (1+\beta L(r))^n n^{-n/2}$, and (123) ensure that for every $\varepsilon \in (0,\eta]$ it holds that

$$\sup_{n\in[N_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n} \le \sup_{n\in[N_\varepsilon,\infty)\cap\mathbb{N}} [c e^{L(\varrho_n)} \kappa^n (1+\beta L(\varrho_n))^n n^{-n/2}] \le \varepsilon. \tag{124}$$

Next let $E = \{\varepsilon \in (0,\infty) \colon N_\varepsilon > 1\}$. Observe that (123) yields for every $\varepsilon \in E$ that

$$(N_\varepsilon - 1)^{(N_\varepsilon-1)/2} < \frac{c}{\varepsilon} e^{L(\varrho_{N_\varepsilon-1})} [\kappa(1+\beta L(\varrho_{N_\varepsilon-1}))]^{(N_\varepsilon-1)}. \tag{125}$$

This and the assumption that for every $n \in \mathbb{N}$ it holds that $\gamma_n \le (\alpha n)^n$ imply that for every $\varepsilon \in E$, $\delta \in (0,\infty)$ it holds that

$$\begin{aligned} &\sup_{n\in[1,N_\varepsilon+K]\cap\mathbb{N}} \gamma_n \le \sup_{n\in[1,N_\varepsilon+K]\cap\mathbb{N}} (\alpha n)^n \le \sup_{n\in[1,N_\varepsilon+K]\cap\mathbb{N}} (\max\{\alpha,1\} n)^n \\ &= [\max\{\alpha,1\}(N_\varepsilon+K)]^{N_\varepsilon+K} = \frac{[\max\{\alpha,1\}(N_\varepsilon+K)]^{N_\varepsilon+K}}{(N_\varepsilon-1)^{(N_\varepsilon-1)(1+\delta)}} (N_\varepsilon-1)^{(N_\varepsilon-1)(1+\delta)} \\ &\le \frac{[\max\{\alpha,1\}(N_\varepsilon+K)]^{N_\varepsilon+K}}{(N_\varepsilon-1)^{(N_\varepsilon-1)(1+\delta)}} \frac{c^{2+2\delta}}{\varepsilon^{2+2\delta}} e^{L(\varrho_{N_\varepsilon-1})(2+2\delta)} [\kappa(1+\beta L(\varrho_{N_\varepsilon-1}))]^{(N_\varepsilon-1)(2+2\delta)} \\ &\le c^{2+2\delta} \varepsilon^{-(2+2\delta)} \sup_{n\in\mathbb{N}} \left[\frac{[\max\{\alpha,1\}(n+K+1)]^{(n+K+1)}}{n^{n(1+\delta)}} e^{L(\varrho_n)(2+2\delta)} [\kappa(1+\beta L(\varrho_n))]^{n(2+2\delta)} \right] \\ &= \mathfrak{a}_\delta \varepsilon^{-(2+2\delta)}. \end{aligned} \tag{126}$$

Next observe that the fact that for every $t \in (0,\infty)$ it holds that $\ln(t) \le t-1$ and (118) ensure once again that for every $\delta \in (0,\infty)$ it holds that

$$\begin{aligned} &\limsup_{n\to\infty} \left[\ln\left(\frac{[\max\{\alpha,1\}(n+K+1)]^{(n+K+1)}}{n^{n(1+\delta)}} e^{L(\varrho_n)(2+2\delta)} [\kappa(1+\beta L(\varrho_n))]^{n(2+2\delta)} \right) \right] \\ &= \limsup_{n\to\infty} \bigg[(n+K+1)\ln(\max\{\alpha,1\}) + (n+K+1)\ln(n+K+1) - n(1+\delta)\ln(n) \\ &\qquad + L(\varrho_n)(2+2\delta) + n(2+2\delta)\ln(\kappa) + n(2+2\delta)\ln(1+\beta L(\varrho_n)) \bigg] \\ &\le \limsup_{n\to\infty} \bigg[n\ln(n) \Big(\frac{(n+K+1)\ln(\max\{\alpha,1\})}{n\ln(n)} + \frac{n+K+1}{n}\frac{\ln(n+K+1)}{\ln(n)} - (1+\delta) \\ &\qquad + \frac{L(\varrho_n)}{n\ln(n)}(2+2\delta) + (2+2\delta)\frac{\ln(\kappa)}{\ln(n)} + (2+2\delta)\beta\frac{L(\varrho_n)}{\ln(n)} \Big) \bigg] = -\infty. \end{aligned} \tag{127}$$

This, (118), and (119) imply for every $\delta \in (0,\infty)$ that

$$a_\delta = c^{2+2\delta} \sup_{n\in\mathbb{N}} \left[\frac{[\max\{\alpha,1\}(n+K+1)]^{(n+K+1)}}{n^{n(1+\delta)}} e^{L(\varrho_n)(2+2\delta)} [\kappa(1+\beta L(\varrho_n))]^{n(2+2\delta)} \right] < \infty. \quad (128)$$

Next observe that the assumption that for every $n \in \mathbb{N}$ it holds that $\gamma_n \le (\alpha n)^n$ and (120) ensure that for every $\varepsilon \in (0,\eta] \setminus E$, $\delta \in (0,\infty)$ it holds that

$$\sup_{n\in[1,N_\varepsilon+K]\cap\mathbb{N}} \gamma_n = \sup_{n\in[1,K+1]\cap\mathbb{N}} \gamma_n \le [\max\{\alpha,1\}(K+1)]^{(K+1)} \left\lceil \frac{\eta}{\varepsilon} \right\rceil^{(2+2\delta)} = \mathfrak{b}\eta^{(2+2\delta)}\varepsilon^{-(2+2\delta)}. \quad (129)$$

Combining this with (119), (120), (124), and (126) we obtain that for every $\delta \in (0,\infty)$, $\varepsilon \in (0,\eta]$ it holds that $\sup_{n\in[N_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n} \le \varepsilon$ and

$$\sup_{n\in[1,N_\varepsilon+K]\cap\mathbb{N}} \gamma_n \le \varepsilon^{-(2+2\delta)} \max\Big\{\mathfrak{a}_\delta, \mathfrak{b}\eta^{(2+2\delta)}\Big\}. \quad (130)$$

Next let $\mathfrak{N}_\varepsilon \in \mathbb{N}_0$, $\varepsilon \in (0,1]$, satisfy for every $\varepsilon \in (0,1]$ that

$$\mathfrak{N}_\varepsilon = \begin{cases} N_\varepsilon & : 0 < \varepsilon \le \eta \\ N_\eta & : \eta < \varepsilon \le 1. \end{cases} \quad (131)$$

This and (130) ensure that for every $\delta \in (0,\infty)$, $\varepsilon \in (\eta,1]$ it holds that $\sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\rho_n} = \sup_{n\in[N_\eta,\infty)\cap\mathbb{N}} \epsilon_{n,\rho_n} \le \eta \le \varepsilon$ and

$$\begin{aligned} \sup_{n\in[1,\mathfrak{N}_\varepsilon+K]\cap\mathbb{N}} \gamma_n = \sup_{n\in[1,N_\eta+K]\cap\mathbb{N}} \gamma_n &\le \max\left\{\mathfrak{a}_\delta, \mathfrak{b}\eta^{2+2\delta}\right\} \eta^{-(2+2\delta)} = \max\left\{\mathfrak{a}_\delta\eta^{-(2+2\delta)}, \mathfrak{b}\right\} \\ &\le \max\left\{\mathfrak{a}_\delta\eta^{-(2+2\delta)}, \mathfrak{b}\right\} \varepsilon^{-(2+2\delta)}. \end{aligned} \quad (132)$$

Combining this with (130) and (131) establishes that for every $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$ it holds that

$$\sup_{n\in[1,\mathfrak{N}_\varepsilon+K]\cap\mathbb{N}} \gamma_n \le \Big(\max\Big\{1,\eta^{2+2\delta}\Big\}\max\Big\{\mathfrak{a}_\delta\eta^{-(2+2\delta)}, \mathfrak{b}\Big\}\Big)\varepsilon^{-(2+2\delta)} \text{ and } \sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n} \le \varepsilon. \quad (133)$$

The proof of Lemma 4.2 is thus completed. ☐

Lemma 4.3. *Let $\alpha \in [1,\infty)$. Then it holds for every $n \in \mathbb{N}$ that $\sum_{m=1}^n (\alpha m)^m \le 2(\alpha n)^n$.*

Proof of Lemma 4.3. First, note that the claim is clear in the case $n = 1$. Next observe that for all $n \in \mathbb{N} \cap [2,\infty)$ it holds that $\alpha n \ge 2$. This implies that for all $n \in \mathbb{N} \cap [2,\infty)$ it holds that

$$\sum_{m=1}^n \frac{(\alpha m)^m}{(\alpha n)^n} \le \sum_{m=1}^n \frac{(\alpha n)^m}{(\alpha n)^n} = \sum_{m=1}^n \frac{1}{(\alpha n)^{n-m}} = \sum_{k=0}^{n-1} \frac{1}{(\alpha n)^k} \le \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^k \le 2. \quad (134)$$

The proof of Lemma 4.3 is thus completed. ☐

Lemma 4.4. *Let $\alpha, \beta, c, \kappa, \rho \in (0,\infty)$, $K \in \mathbb{N}_0$, $(\gamma_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$, $(\epsilon_{n,r})_{n\in\mathbb{N},r\in[\rho,\infty)} \subseteq [0,\infty)$, let $L\colon (0,\infty) \to [0,\infty)$ be a function, assume for every $n \in \mathbb{N}$, $r \in [\rho,\infty)$ that $\gamma_n \le (\alpha n)^n$ and $\epsilon_{n,r} \le ce^{L(r)}\kappa^n(1+\beta L(r))^n n^{-n/2}$, and let $\varrho\colon \mathbb{N} \to (0,\infty)$ satisfy $\limsup_{n\to\infty}(\frac{L(\varrho_n)}{\ln(n)} + \frac{1}{\varrho_n}) = 0$. Then there exist $\mathfrak{N}\colon (0,1] \to \mathbb{N}$ and $\mathfrak{c}\colon (0,\infty) \to [0,\infty)$ such that for every $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$ it holds that $\sum_{n=1}^{\mathfrak{N}_\varepsilon+K} \gamma_n \le \mathfrak{c}_\delta\varepsilon^{-(2+2\delta)}$ and $\sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n} \le \varepsilon$.*

Proof of Lemma 4.4. First, observe that for every $n \in \mathbb{N}$ it holds that $\gamma_n \leq (\max\{\alpha,1\}n)^n$. Lemma 4.2 (with $\alpha = \max\{\alpha,1\}$, $\beta = \beta$, $c = c$, $\kappa = \kappa$, $\rho = \rho$, $K = K$, $L = L$, $\varrho_n = \varrho_n$, $\gamma_n = (\max\{\alpha,1\}n)^n$, $\epsilon_{n,r} = \epsilon_{n,r}$ for $r \in [\rho,\infty)$, $n \in \mathbb{N}$ in the notation of Lemma 4.2) therefore guarantees that there exist $\mathfrak{N}_\varepsilon \in \mathbb{N}$, $\varepsilon \in (0,1]$, and $\mathfrak{c}_\delta \in [0,\infty)$, $\delta \in (0,\infty)$, such that for every $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$ it holds that $\sup_{n\in[1,\mathfrak{N}_\varepsilon+K]\cap\mathbb{N}}(\max\{\alpha,1\}n)^n \leq \mathfrak{c}_\delta\varepsilon^{-(2+2\delta)}$ and $\sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n} \leq \varepsilon$. The fact that for every $n \in \mathbb{N}$ it holds that $\gamma_n \leq (\max\{\alpha,1\}n)^n$, the fact that for every $N \in \mathbb{N}$ it holds that $\sup_{n\in[1,N]\cap\mathbb{N}}(\max\{\alpha,1\}n)^n = (\max\{\alpha,1\}N)^N$, and Lemma 4.3 hence imply that for every $\varepsilon \in (0,1]$ it holds that $\sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n} \leq \varepsilon$ and

$$\sum_{n=1}^{\mathfrak{N}_\varepsilon+K} \gamma_n \leq \sum_{n=1}^{\mathfrak{N}_\varepsilon+K} (\max\{\alpha,1\}n)^n \leq 2(\max\{\alpha,1\}(\mathfrak{N}_\varepsilon+K)^{(\mathfrak{N}_\varepsilon+K)} \leq 2\mathfrak{c}_\delta\varepsilon^{-(2+2\delta)}. \tag{135}$$

The proof of Lemma 4.4 is thus completed. □

Theorem 4.5. *Let $\rho, T \in (0,\infty)$, $c, \gamma, p \in [0,\infty)$, $K \in \mathbb{N}_0$, $\Theta = \cup_{n\in\mathbb{N}}\mathbb{Z}^n$, $(f_d)_{d\in\mathbb{N}}, (\mathbf{f}_{d,r})_{d\in\mathbb{N},r\in(0,\infty)} \subseteq C([0,T]\times\mathbb{R}^d\times\mathbb{R},\mathbb{R})$, let $L\colon (0,\infty) \to [0,\infty)$ be a function, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mathcal{R}^\theta\colon \Omega \to [0,1]$, $\theta \in \Theta$, be independent $\mathcal{U}_{[0,1]}$-distributed random variables, let $W^{d,\theta}\colon [0,T]\times\Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, be independent standard Brownian motions, assume that $(\mathcal{R}^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ are independent, let $R^\theta\colon [0,T]\times\Omega \to [0,T]$, $\theta \in \Theta$, satisfy for every $\theta \in \Theta$, $t \in [0,T]$ that $R^\theta_t = t\mathcal{R}^\theta$, for every $d \in \mathbb{N}$, $\theta \in \Theta$, $s \in [0,T]$, $t \in [s,T]$, $x \in \mathbb{R}^d$ let $X^{d,\theta}_{s,t,x}\colon \Omega \to \mathbb{R}^d$ satisfy $X^{d,\theta}_{s,t,x} = x + \sqrt{2}(W^{d,\theta}_t - W^{d,\theta}_s)$, assume for every $d \in \mathbb{N}$, $r \in (0,\infty)$, $t \in (0,T]$, $x \in \mathbb{R}^d$, $u, v \in [-r,r]$, $w \in \mathbb{R}$ that $wf_d(t,x,w) \leq c(1+w^2)$, $\mathbf{f}_{d,r}(t,x,w) = f_d(t,x,\min\{r,\max\{-r,w\}\})$, $\mathbb{E}[\int_0^t |f_d(s,X^{d,0}_{s,t,x},0)|\,ds] < \infty$, and $|f_d(t,x,u) - f_d(t,x,v)| \leq L(r)|u-v|$, let $u_d \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d \in \mathbb{N}$, satisfy for every $d \in \mathbb{N}$, $t \in (0,T]$, $x \in \mathbb{R}^d$ that $e^{cT}(1+|u_d(0,x)|^2)^{1/2} \leq \rho$, $\inf_{a\in\mathbb{R}}[\sup_{s\in[0,T]}\sup_{y=(y_1,\dots,y_d)\in\mathbb{R}^d}(e^{a(|y_1|^2+\ldots+|y_d|^2)}|u_d(s,y)|)] < \infty$, $u_d|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$, and*

$$(\tfrac{\partial}{\partial t}u_d)(t,x) = (\Delta_x u_d)(t,x) + f_d(t,x,u_d(t,x)), \tag{136}$$

let $U^{d,\theta}_{n,M,r}\colon [0,T]\times\mathbb{R}^d\times\Omega \to \mathbb{R}$, $d, M \in \mathbb{N}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, $r \in (0,\infty)$, satisfy for every $d, n, M \in \mathbb{N}$, $\theta \in \Theta$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $U^{d,\theta}_{0,M,r}(t,x) = 0$ and

$$\begin{aligned} U^{d,\theta}_{n,M,r}(t,x) &= \frac{1}{M^n}\left[\sum_{m=1}^{M^n}\left(u_d(0,X^{d,(\theta,0,-m)}_{0,t,x}) + t\,f_d\Big(R^{(\theta,0,m)}_t, X^{d,(\theta,0,m)}_{R^{(\theta,0,m)}_t,t,x},0\Big)\right)\right] \\ &+ \sum_{k=1}^{n-1}\frac{t}{M^{n-k}}\left[\sum_{m=1}^{M^{n-k}}\left(\mathbf{f}_{d,r}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}, U^{d,(\theta,k,m)}_{k,M,r}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\right.\right. \\ &\qquad \left.\left. - \mathbf{f}_{d,r}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}, U^{d,(\theta,-k,m)}_{k-1,M,r}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\right)\right], \end{aligned} \tag{137}$$

let $\varrho\colon \mathbb{N} \to (0,\infty)$ satisfy $\limsup_{n\to\infty}(\frac{L(\varrho_n)}{\ln(n)} + \frac{1}{\varrho_n}) = 0$, and let $\mathfrak{C}_{d,n,M} \in \mathbb{N}_0$, $d, M \in \mathbb{N}$, $n \in \mathbb{N}_0$, satisfy for every $d, n, M \in \mathbb{N}$ that $\mathfrak{C}_{d,0,M} = 0$ and

$$\mathfrak{C}_{d,n,M} \leq (2d+1)M^n + \sum_{l=1}^{n-1} M^{n-l}\,(d+1+\mathfrak{C}_{d,l,M} + \mathfrak{C}_{d,l-1,M}). \tag{138}$$

Then there exist $\mathfrak{N}\colon (0,1] \to \mathbb{N}$ and $\mathfrak{c}\colon (0,\infty) \to [0,\infty)$ such that for every $d \in \mathbb{N}$, $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$, $x \in \mathbb{R}^d$ with $(\int_0^T \mathbb{E}[|f_d(s,X^{d,0}_{s,T,x},0)|^2]\,ds)^{1/2} \leq \gamma d^p$ it holds that $\sum_{n=1}^{\mathfrak{N}_{(\varepsilon/d^p)}+K} \mathfrak{C}_{d,n,n} \leq \mathfrak{c}_\delta d^{1+p(2+\delta)}\varepsilon^{-(2+\delta)}$ and

$$\left[\sup_{n\in[\mathfrak{N}_{(\varepsilon/d^p)},\infty)\cap\mathbb{N}}\Big(\mathbb{E}\Big[|U^{d,0}_{n,n,\varrho_n}(T,x) - u_d(T,x)|^2\Big]\Big)^{1/2}\right] \leq \varepsilon. \tag{139}$$

Proof of Theorem 4.5. Throughout this proof let $\mathfrak{X}_d \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, satisfy for every $d \in \mathbb{N}$ that

$$\mathfrak{X}_d = \left\{ x \in \mathbb{R}^d \colon \left(\int_0^T \mathbb{E}\Big[\big| f_d(s, X^{d,0}_{s,T,x}, 0) \big|^2 \Big] \, ds \right)^{1/2} \leq \gamma d^p \right\}, \tag{140}$$

let $\epsilon_{n,r} \in [0,\infty]$, $n \in \mathbb{N}$, $r \in (0,\infty)$, satisfy for every $n \in \mathbb{N}$, $r \in (0,\infty)$ that

$$\epsilon_{n,r} = \sup\!\left(\left\{ \frac{1}{d^p} \left(\mathbb{E}\Big[\big| U^{d,0}_{n,n,r}(T,x) - u_d(T,x) \big|^2 \Big] \right)^{1/2} : x \in \mathfrak{X}_d, d \in \mathbb{N} \right\} \cup \{0\} \right), \tag{141}$$

and let $\gamma_n \in [0,\infty]$, $n \in \mathbb{N}$, satisfy for every $n \in \mathbb{N}$ that

$$\gamma_n = \sup_{d \in \mathbb{N}} \left(\frac{\mathfrak{C}_{d,n,n}}{d} \right). \tag{142}$$

Note that Lemma 4.1 demonstrates that for every $d, n, M \in \mathbb{N}$ it holds that $\mathfrak{C}_{d,n,M} \leq d(5M)^n$. This implies for every $n \in \mathbb{N}$ that

$$\gamma_n = \sup_{d \in \mathbb{N}} \left(\frac{\mathfrak{C}_{d,n,n}}{d} \right) \leq \sup_{d \in \mathbb{N}} \left(\frac{d(5n)^n}{d} \right) = (5n)^n < \infty. \tag{143}$$

Next observe that Proposition 3.5 (with $d = d$, $T = T$, $\Theta = \Theta$, $f = f_d$, $\mathbf{f}_r = \mathbf{f}_{d,r}$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{R}^\theta = \mathcal{R}^\theta$, $W^\theta = W^{d,\theta}$, $R^\theta = R^\theta$, $X^\theta_{t,s,x} = X^{d,\theta}_{t,s,x}$, $U^\theta_{n,M,r}(t,x) = U^{d,\theta}_{n,M,r}(t,x)$, $\rho = \rho$, $c = c$, $\|\cdot\| = (\mathbb{R}^d \ni y = (y_1, \ldots, y_d) \mapsto |y_1|^2 + \ldots + |y_d|^2 \in \mathbb{R})$, $L = L$, $u = u_d$ for $d, M \in \mathbb{N}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, $r \in (0,\infty)$, $t \in [0,T]$, $s \in [t,T]$, $x \in \mathbb{R}^d$ in the notation of Proposition 3.5) ensures that it holds for every $d, M \in \mathbb{N}$, $n \in \mathbb{N}_0$, $r \in [\rho,\infty)$, $x \in \mathbb{R}^d$ that

$$\begin{aligned} \left(\mathbb{E}\Big[|U^{d,0}_{n,M,r}(T,x) - u_d(T,x)|^2 \Big] \right)^{1/2} &\leq \frac{e^{M/2}(1 + 2L(r)T)^n}{M^{n/2}} \\ &\quad \cdot e^{L(r)T} \left[\left(\mathbb{E}\Big[|u_d(0, X^{d,0}_{0,T,x})|^2 \Big] \right)^{1/2} + \sqrt{T} \left| \int_0^T \mathbb{E}\Big[|f_d(s, X^{d,0}_{s,T,x}, 0)|^2 \Big] \, ds \right|^{1/2} \right]. \end{aligned} \tag{144}$$

This implies that for every $n \in \mathbb{N}$, $r \in [\rho,\infty)$ it holds that

$$\begin{aligned} \epsilon_{n,r} &= \sup\!\left(\left\{ \frac{1}{d^p} \left(\mathbb{E}\Big[\big| U^{d,0}_{n,n,r}(T,x) - u_d(T,x) \big|^2 \Big] \right)^{1/2} : x \in \mathfrak{X}_d, d \in \mathbb{N} \right\} \cup \{0\} \right) \\ &\leq \sup_{d \in \mathbb{N}} \left(\frac{e^{n/2}(1 + 2L(r)T)^n}{n^{n/2}} \frac{e^{L(r)T}}{d^p} \left[\sup_{m \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} |u_m(0,x)| + \gamma \sqrt{T} d^p \right] \right) \\ &\leq \left[\left(\sup_{d \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} |u_d(0,x)| \right) + \gamma \sqrt{T} \right] e^{L(r)T} \frac{e^{n/2}(1 + 2L(r)T)^n}{n^{n/2}} < \infty. \end{aligned} \tag{145}$$

This, (143), and Lemma 4.4 (with $\alpha = 5$, $\beta = 2$, $c = 1 + \sup_{d \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} |u_d(0,x)| + \gamma\sqrt{T}$, $\kappa = \sqrt{e}$, $\rho = \rho$, $K = K$, $L(s) = L(s)T$, $\gamma_n = \gamma_n$, $\epsilon_{n,r} = \epsilon_{n,r}$, $\varrho_n = \varrho_n$ for $n \in \mathbb{N}$, $r \in [\rho,\infty)$, $s \in (0,\infty)$ in the notation of Lemma 4.2) guarantee that there exist $\mathfrak{N} \colon (0,1] \to \mathbb{N}$ and $\mathfrak{c} \colon (0,\infty) \to [0,\infty)$ which satisfy that for every $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$ it holds that

$$\left[\sum_{n=1}^{\mathfrak{N}_\varepsilon + K} \gamma_n \right] \leq \mathfrak{c}_\delta \varepsilon^{-(2+2\delta)} \qquad \text{and} \qquad \sup_{n \in [\mathfrak{N}_\varepsilon, \infty) \cap \mathbb{N}} \epsilon_{n,\varrho_n} \leq \varepsilon \tag{146}$$

This implies that for every $d \in \mathbb{N}$, $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$, $x \in \mathfrak{X}_d$ it holds that

$$\left[\sum_{n=1}^{\mathfrak{N}_{(\varepsilon/d^p)} + K} \mathfrak{C}_{d,n,n} \right] \leq \mathfrak{c}_\delta d \left[\frac{\varepsilon}{d^p} \right]^{-(2+2\delta)} = \mathfrak{c}_\delta d^{1+p(2+2\delta)} \varepsilon^{-(2+2\delta)} \tag{147}$$

and

$$\sup_{n\in[\mathfrak{N}_{(\varepsilon/d^p)},\infty)\cap\mathbb{N}} \Big(\mathbb{E}\Big[|U^{d,0}_{n,n,\varrho_n}(T,x)-u_d(T,x)|^2\Big]\Big)^{1/2} \le d^p \Bigg[\sup_{n\in[\mathfrak{N}_{(\varepsilon/d^p)},\infty)\cap\mathbb{N}} \epsilon_{n,\varrho_n}\Bigg] \le d^p\frac{\varepsilon}{d^p} = \varepsilon. \tag{148}$$

This establishes (139). The proof of Theorem 4.5 is thus completed. □

5 MLP approximations for Allen–Cahn type partial differential equations

In this section we consider sample applications of Theorem 4.5. This provides us with examples of Allen–Cahn PDEs for which the curse of dimensionality can be broken in numerical approximations.

Corollary 5.1. *Let $T\in(0,\infty)$, $c,\rho\in[0,\infty)$, $K\in\mathbb{N}_0$, $\Theta=\cup_{n\in\mathbb{N}}\mathbb{Z}^n$, $f\in C(\mathbb{R},\mathbb{R})$, $(\mathbf{f}_r)_{r\in(0,\infty)}\subseteq C(\mathbb{R},\mathbb{R})$, let $L\colon(0,\infty)\to[0,\infty)$ be a function, assume for every $r\in(0,\infty)$, $u,v\in[-r,r]$, $w\in\mathbb{R}$ that $wf(w)\le c(1+w^2)$, $\mathbf{f}_r(w)=f(\min\{r,\max\{-r,w\}\})$, and $|f(u)-f(v)|\le L(r)|u-v|$, let $\varrho\colon\mathbb{N}\to(0,\infty)$ satisfy $\limsup_{n\to\infty}(\frac{L(\varrho_n)}{\ln(n)}+\frac{1}{\varrho_n})=0$, let $u_d\in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d\in\mathbb{N}$, satisfy for every $d\in\mathbb{N}$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $e^{cT}(1+|u_d(0,x)|^2)^{1/2}\le\rho$, $u_d|_{(0,T]\times\mathbb{R}^d}\in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$, $\inf_{a\in\mathbb{R}}[\sup_{s\in[0,T]}\sup_{y=(y_1,\ldots,y_d)\in\mathbb{R}^d}(e^{a(|y_1|^2+\ldots+|y_d|^2)}|u_d(s,y)|)]<\infty$, and*

$$(\tfrac{\partial}{\partial t}u_d)(t,x)=(\Delta_x u_d)(t,x)+f(u_d(t,x)), \tag{149}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $\mathcal{R}^\theta\colon\Omega\to[0,1]$, $\theta\in\Theta$, be independent $\mathcal{U}_{[0,1]}$-distributed random variables, let $W^{d,\theta}\colon[0,T]\times\Omega\to\mathbb{R}^d$, $d\in\mathbb{N}$, $\theta\in\Theta$, be independent standard Brownian motions, assume that $(\mathcal{R}^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ are independent, let $R^\theta_t\colon\Omega\to[0,t]$, $\theta\in\Theta$, $t\in[0,T]$, satisfy for every $t\in[0,T]$ that $R^\theta_t=t\mathcal{R}^\theta$, for every $d\in\mathbb{N}$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$, $\theta\in\Theta$ let $X^{d,\theta}_{t,s,x}\colon\Omega\to\mathbb{R}^d$ satisfy $X^{d,\theta}_{t,s,x}=x+\sqrt{2}(W^{d,\theta}_s-W^{d,\theta}_t)$, let $U^{d,\theta}_{n,M,r}\colon[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $\theta\in\Theta$, $d,M\in\mathbb{N}$, $n\in\mathbb{N}_0$, $r\in(0,\infty)$, satisfy for every $d,n,M\in\mathbb{N}$, $\theta\in\Theta$, $r\in(0,\infty)$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $U^{d,\theta}_{0,M,r}(t,x)=0$ and

$$\begin{aligned} U^{d,\theta}_{n,M,r}(t,x) &= \sum_{k=1}^{n-1}\frac{t}{M^{n-k}}\Bigg[\sum_{m=1}^{M^{n-k}}\Big(\mathbf{f}_r\Big(U^{d,(\theta,k,m)}_{k,M,r}(R^{(\theta,k,m)}_t,X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x})\Big) \\ &\quad -\mathbf{f}_r\Big(U^{d,(\theta,-k,m)}_{k-1,M,r}(R^{(\theta,k,m)}_t,X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x})\Big)\Big)\Bigg]+\frac{1}{M^n}\Bigg[\sum_{m=1}^{M^n}\Big(u_d(0,X^{d,(\theta,0,-m)}_{0,t,x})+t\,f(0)\Big)\Bigg], \end{aligned} \tag{150}$$

and let $\mathfrak{C}_{d,n,M}\in\mathbb{N}_0$, $d,M\in\mathbb{N}$, $n\in\mathbb{N}_0$, satisfy for every $d,n,M\in\mathbb{N}$ that $\mathfrak{C}_{d,0,M}=0$ and

$$\mathfrak{C}_{d,n,M}\le(2d+1)M^n+\sum_{l=1}^{n-1}M^{n-l}\,(d+1+\mathfrak{C}_{d,l,M}+\mathfrak{C}_{d,l-1,M}). \tag{151}$$

Then

(i) it holds for every $d,M\in\mathbb{N}$, $n\in\mathbb{N}_0$, $r\in[\rho,\infty)$ that

$$\begin{aligned} &\sup_{x\in\mathbb{R}^d}\Big(\mathbb{E}\Big[|U^{d,0}_{n,M,r}(T,x)-u_d(T,x)|^2\Big]\Big)^{1/2} \\ &\le e^{L(r)T}\Bigg[\sup_{x\in\mathbb{R}^d}|u_d(0,x)|+T|f(0)|\Bigg]\Bigg[\frac{e^{M/2}(1+2L(r)T)^n}{M^{n/2}}\Bigg] \end{aligned} \tag{152}$$

and

(ii) there exist $\mathfrak{N}\colon (0,1] \to \mathbb{N}$ *and* $\mathfrak{c}\colon (0,\infty) \to [0,\infty)$ *such that for every* $d \in \mathbb{N}$, $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$ *it holds that* $\sum_{n=1}^{\mathfrak{N}_\varepsilon+K} \mathfrak{C}_{d,n,n} \le d\mathfrak{c}_\delta \varepsilon^{-(2+\delta)}$ *and*

$$\sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \left[\sup_{x\in\mathbb{R}^d} \left(\mathbb{E}\Big[|U^{d,0}_{n,n,\varrho_n}(T,x) - u_d(T,x)|^2\Big]\right)^{1/2}\right] \le \varepsilon. \tag{153}$$

Proof of Corollary 5.1. Throughout this proof let $F_d\colon [0,T]\times\mathbb{R}^d\times\mathbb{R} \to \mathbb{R}$, $d \in \mathbb{N}$, be the functions which satisfy for every $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}$ that

$$F_d(t,x,w) = f(w). \tag{154}$$

Observe that the fact that for every $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ it holds that $|u_d(0,x)| \le \rho < \infty$ and (154) ensure that for every $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$, $w \in \mathbb{R}$ it holds that $\mathbf{f}_r(w) = F_d(t,x,\min\{r,\max\{-r,w\}\})$, $\mathbb{E}[\int_0^t |F_d(s,X^{d,0}_{s,t,x},0)|\,ds] = t|f(0)| < \infty$, $(\mathbb{E}[\int_0^T |F_d(s,X^{d,0}_{s,T,x},0)|^2\,ds])^{1/2} = \sqrt{T}|f(0)|$, and

$$\left(\mathbb{E}\Big[|u_d(0,X^{d,0}_{0,T,x})|^2\Big]\right)^{1/2} + \sqrt{T}\left|\int_0^T \mathbb{E}\Big[|F_d(s,X^{d,0}_{s,T,x},0)|^2\Big]\,ds\right|^{1/2} \le \sup_{\xi\in\mathbb{R}^d}|u_d(0,\xi)| + T|f(0)|. \tag{155}$$

This and Theorem 4.5 (with $\rho = \rho$, $T = T$, $c = c$, $\gamma = \sqrt{T}|f(0)|$, $p = 0$, $K = K$, $\Theta = \Theta$, $L = L$, $f_d = F_d$, $\mathbf{f}_{d,r} = ([0,T]\times\mathbb{R}^d\times\mathbb{R} \ni (t,x,y) \mapsto \mathbf{f}_r(y) \in \mathbb{R})$, $u_d = u_d$, $\varrho = \varrho$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{R}^\theta = \mathcal{R}^\theta$, $W^{d,\theta} = W^{d,\theta}$, $R^\theta = R^\theta$, $X^{d,\theta}_{t,s,x} = X^{d,\theta}_{t,s,x}$, $U^{d,\theta}_{n,M,r}(t,x) = U^{d,\theta}_{n,M,r}(t,x)$, $\mathfrak{C}_{d,n,M} = \mathfrak{C}_{d,n,M}$ for $d, M \in \mathbb{N}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, $r \in (0,\infty)$, $t \in [0,T]$, $s \in [t,T]$, $x \in \mathbb{R}^d$ in the notation of Theorem 4.5) ensure that

(I) for every $d, M \in \mathbb{N}$, $n \in \mathbb{N}_0$, $r \in [\rho,\infty)$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}&\left(\mathbb{E}\Big[|U^{d,0}_{n,M,r}(T,x) - u_d(T,x)|^2\Big]\right)^{1/2}\\ &\le e^{L(r)T}\left[\sup_{\xi\in\mathbb{R}^d}|u_d(0,\xi)| + T|f(0)|\right]\left[\frac{e^{M/2}(1+2L(r)T)^n}{M^{n/2}}\right]\end{aligned} \tag{156}$$

and

(II) there exist $\mathfrak{N}\colon (0,1] \to \mathbb{N}$ and $\mathfrak{c}\colon (0,\infty) \to [0,\infty)$ such that for every $d \in \mathbb{N}$, $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$, $x \in \mathbb{R}^d$ it holds that

$$\left[\sum_{n=1}^{\mathfrak{N}_\varepsilon+K} \mathfrak{C}_{d,n,n}\right] \le \mathfrak{c}_\delta d\varepsilon^{-(2+2\delta)} \quad \text{and} \quad \sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \left(\mathbb{E}\Big[|U^{d,0}_{n,n,\varrho_n}(T,x) - u_d(T,x)|^2\Big]\right)^{1/2} \le \varepsilon. \tag{157}$$

This establishes Items (i) and (ii). The proof of Corollary 5.1 is thus completed. □

Corollary 5.2. *Let* $T \in (0,\infty)$, $c \in [0,\infty)$, $K \in \mathbb{N}_0$, $\Theta = \cup_{n\in\mathbb{N}}\mathbb{Z}^n$, $f \in C(\mathbb{R},\mathbb{R})$, $(\mathbf{f}_n)_{n\in\mathbb{N}} \subseteq C(\mathbb{R},\mathbb{R})$, *let* $\varrho\colon \mathbb{N} \to (0,\infty)$ *satisfy* $\limsup_{n\to\infty}(\frac{\varrho_n}{\ln(\ln(n))}) < \infty = \liminf_{n\to\infty}\varrho_n$, *assume for every* $n \in \mathbb{N}$, $u, v \in \mathbb{R}$ *that* $|f(u) - f(v)| \le c(1 + |u|^c + |v|^c)|u - v|$, $vf(v) \le c(1 + v^2)$, *and* $\mathbf{f}_n(v) = f(\min\{\varrho_n,\max\{-\varrho_n,v\}\})$, *let* $u_d \in C([0,T]\times\mathbb{R}^d,\mathbb{R})$, $d \in \mathbb{N}$, *satisfy for every* $d \in \mathbb{N}$, $t \in (0,T]$, $x \in \mathbb{R}^d$ *that* $\inf_{a\in\mathbb{R}}[\sup_{s\in[0,T]}\sup_{y=(y_1,\dots,y_d)\in\mathbb{R}^d}(e^{a(|y_1|^2+\dots+|y_d|^2)}|u_d(s,y)|)] < \infty$, $|u_d(0,x)| \le c$, $u_d|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T]\times\mathbb{R}^d,\mathbb{R})$, *and*

$$(\tfrac{\partial}{\partial t}u_d)(t,x) = (\Delta_x u_d)(t,x) + f(u_d(t,x)), \tag{158}$$

let $(\Omega,\mathcal{F},\mathbb{P})$ *be a probability space, let* $\mathcal{R}^\theta\colon \Omega \to [0,1]$, $\theta \in \Theta$, *be independent* $\mathcal{U}_{[0,1]}$*-distributed random variables, let* $W^{d,\theta}\colon [0,T]\times\Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, *be independent standard Brownian* *motions, assume that* $(\mathcal{R}^\theta)_{\theta\in\Theta}$ *and* $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ *are independent, let* $R^\theta\colon [0,T]\times\Omega\to[0,T]$, $\theta\in\Theta$, *satisfy for every* $\theta\in\Theta$, $t\in[0,T]$ *that* $R^\theta_t = t\mathcal{R}^\theta$, *for every* $d\in\mathbb{N}$, $t\in[0,T]$, $s\in[t,T]$, $x\in\mathbb{R}^d$, $\theta\in\Theta$ *let* $X^{d,\theta}_{t,s,x}\colon\Omega\to\mathbb{R}^d$ *satisfy* $X^{d,\theta}_{t,s,x} = x+\sqrt{2}(W^{d,\theta}_s - W^{d,\theta}_t)$, *let* $U^{d,\theta}_{n,M}\colon[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $d,M\in\mathbb{N}$, $\theta\in\Theta$, $n\in\mathbb{N}_0$, *satisfy for every* $d,n,M\in\mathbb{N}$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$ *that* $U^{d,\theta}_{0,M}(t,x)=0$ *and*

$$U^{d,\theta}_{n,M}(t,x) = \sum_{k=1}^{n-1}\frac{t}{M^{n-k}}\left[\sum_{m=1}^{M^{n-k}}\left(\mathbf{f}_M\Big(U^{d,(\theta,k,m)}_{k,M}(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x})\Big) - \mathbf{f}_M\Big(U^{d,(\theta,-k,m)}_{k-1,M}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\right)\right] + \frac{1}{M^n}\left[\sum_{m=1}^{M^n}\Big(u_d(0,X^{d,(\theta,0,-m)}_{0,t,x}) + t\,f(0)\Big)\right], \tag{159}$$

and let $\mathfrak{C}_{d,n,M}\in\mathbb{N}_0$, $d,M\in\mathbb{N}$, $n\in\mathbb{N}_0$, *satisfy for every* $d,n,M\in\mathbb{N}$ *that* $\mathfrak{C}_{d,0,M}=0$ *and*

$$\mathfrak{C}_{d,n,M} \le (2d+1)M^n + \sum_{l=1}^{n-1} M^{n-l}\,(d+1+\mathfrak{C}_{d,l,M}+\mathfrak{C}_{d,l-1,M}). \tag{160}$$

Then there exist $\mathfrak{N}\colon(0,1]\to\mathbb{N}$ *and* $\mathfrak{c}\colon(0,\infty)\to[0,\infty)$ *such that for every* $d\in\mathbb{N}$, $\delta\in(0,\infty)$, $\varepsilon\in(0,1]$ *it holds that*

$$\left[\sum_{n=1}^{\mathfrak{N}_\varepsilon+K}\mathfrak{C}_{d,n,n}\right]\le cd\varepsilon^{-(2+\delta)} \quad \text{and} \quad \sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}}\left[\sup_{x\in\mathbb{R}^d}\left(\mathbb{E}\Big[\big|U^{d,0}_{n,n}(T,x)-u_d(T,x)\big|^2\Big]\right)^{1/2}\right]\le\varepsilon. \tag{161}$$

Proof of Corollary 5.2. Throughout this proof let $L\colon(0,\infty)\to[0,\infty)$ satisfy for every $r\in(0,\infty)$ that $L(r)=c(1+2r^c)$, let $F_r\colon\mathbb{R}\to\mathbb{R}$, $r\in(0,\infty)$, be the functions which satisfy for every $r\in(0,\infty)$, $v\in\mathbb{R}$ that $F_r(v)=f(\min\{r,\max\{-r,v\}\})$, and let $V^{d,\theta}_{n,M,r}\colon[0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $d,M\in\mathbb{N}$, $\theta\in\Theta$, $n\in\mathbb{N}_0$, satisfy for every $d,n,M\in\mathbb{N}$, $\theta\in\Theta$, $r\in(0,\infty)$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $V^{d,\theta}_{n,M,r}(0,x)=0$ and

$$V^{d,\theta}_{n,M,r}(t,x) = \sum_{k=1}^{n-1}\frac{t}{M^{n-k}}\left[\sum_{m=1}^{M^{n-k}}\left(F_r\Big(V^{d,(\theta,k,m)}_{k,M,r}(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x})\Big) - F_r\Big(V^{d,(\theta,-k,m)}_{k-1,M,r}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\right)\right] + \frac{1}{M^n}\left[\sum_{m=1}^{M^n}\Big(u_d(0,X^{d,(\theta,0,-m)}_{0,t,x}) + t\,f(0)\Big)\right]. \tag{162}$$

Next observe that the hypothesis that $\limsup_{n\to\infty}(\frac{\varrho_n}{\ln(\ln(n))})<\infty$ implies that there exists $\gamma\in(0,\infty)$ which satisfies that for every $n\in[3,\infty)\cap\mathbb{N}$ it holds that $\varrho_n\le\gamma\ln(\ln(n))$. This yields that

$$\begin{aligned}&\limsup_{n\to\infty}\left[\frac{L(\varrho_n)}{\ln(n)}\right]\le\limsup_{n\to\infty}\left[\frac{1+2(\gamma\ln(\ln(n)))^c}{\ln(n)}\right]\\ &\le\limsup_{n\to\infty}\left[\frac{1}{\ln(n)}\right]+\limsup_{n\to\infty}\left[\frac{2(\gamma\ln(\ln(n)))^c}{\ln(n)}\right] = 2\gamma^c\limsup_{n\to\infty}\left[\frac{(\ln(\ln(n)))^c}{\ln(n)}\right]=0.\end{aligned} \tag{163}$$

Next let $A\subseteq\mathbb{N}_0$ be the set given by

$$A=\left\{n\in\mathbb{N}\colon \begin{array}{c}\text{For all } d,M\in\mathbb{N},\theta\in\Theta,k\in\mathbb{N}_0\cap[0,n-1],t\in[0,T],x\in\mathbb{R}^d\\ \text{it holds that } U^{d,\theta}_{k,M}(t,x)=V^{d,\theta}_{k,M,\varrho_M}(t,x)\end{array}\right\}. \tag{164}$$

Note that the fact that for every $d,M\in\mathbb{N}$, $\theta\in\Theta$, $t\in[0,T]$, $x\in\mathbb{R}^d$, $r\in(0,\infty)$ it holds that $V^{d,\theta}_{0,M,r}(t,x)=0=U^{d,\theta}_{0,M}(t,x)$ ensures that $1\in A$. Moreover, note that (159), (162), and (164) ensure that for every $d, M \in \mathbb{N}$, $\theta \in \Theta$, $n \in A$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}
U^{d,\theta}_{n,M}(t,x) &= \sum_{k=1}^{n-1} \frac{t}{M^{n-k}} \left[\sum_{m=1}^{M^{n-k}} \left(\mathbf{f}_M\Big(U^{d,(\theta,k,m)}_{k,M}(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x})\Big) \right.\right. \\
&\quad \left.\left. - \mathbf{f}_M\Big(U^{d,(\theta,-k,m)}_{k-1,M}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\right)\right] + \frac{1}{M^n}\left[\sum_{m=1}^{M^n} \Big(u_d(0, X^{d,(\theta,0,-m)}_{0,t,x}) + t\, f(0)\Big)\right] \\
&= \sum_{k=1}^{n-1} \frac{t}{M^{n-k}} \left[\sum_{m=1}^{M^{n-k}} \left(F_{\varrho_M}\Big(V^{d,(\theta,k,m)}_{k,M,\varrho_M}(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x})\Big) \right.\right. \\
&\quad \left.\left. - F_{\varrho_M}\Big(V^{d,(\theta,-k,m)}_{k-1,M,\varrho_M}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big)\right)\right] + \frac{1}{M^n}\left[\sum_{m=1}^{M^n} \Big(u_d(0, X^{d,(\theta,0,-m)}_{0,t,x}) + t\, f(0)\Big)\right] \\
&= V^{d,\theta}_{n,M,\varrho_M}(t,x).
\end{aligned} \tag{165}$$

Hence, we obtain that for every $n \in A$ it holds that $n+1 \in A$. Combining this with the fact that $1 \in A$ and induction ensures that $A = \mathbb{N}$. This yields that for every $d, M \in \mathbb{N}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$V^{d,\theta}_{n,M,\varrho_M}(t,x) = U^{d,\theta}_{n,M}(t,x). \tag{166}$$

The fact that for all $r \in (0,\infty)$, $w, \mathfrak{w} \in [-r,r]$ it holds that $|f(w) - f(\mathfrak{w})| \le L(r)|w - \mathfrak{w}|$, (163), and Corollary 5.1 (with $\rho = \exp(T \sup_{v\in\mathbb{R}}(\frac{vf(v)}{1+v^2}))[1 + \sup_{d\in\mathbb{N}} \sup_{x\in\mathbb{R}^d} |u_d(0,x)|^2]^{1/2}$, $c = \sup_{v\in\mathbb{R}}(\frac{vf(v)}{1+v^2})$, $T = T$, $K = K$, $\Theta = \Theta$, $f = f$, $\mathbf{f}_r = \mathbf{f}_r$, $L = L$, $\|\cdot\|_d = \|\cdot\|_d$, $\varrho = \varrho$, $u_d = u_d$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{R}^\Theta = \mathcal{R}^\theta$, $W^{d,\theta} = W^{d,\theta}$, $X^{d,\theta}_{t,s,x} = X^{d,\theta}_{t,s,x}$, $U^{d,\theta}_{n,M,r}(t,x) = V^{d,\theta}_{n,M,r}(t,x)$, $\mathfrak{C}_{d,n,M} = \mathfrak{C}_{d,n,M}$ for $d, M \in \mathbb{N}$, $\theta \in \Theta$, $n \in \mathbb{N}_0$, $r \in (0,\infty)$, $t \in [0,T]$, $x \in \mathbb{R}^d$ in the notation of Corollary 5.1) therefore guarantee that there exist $\mathfrak{N}\colon (0,1] \to \mathbb{N}$ and $\mathfrak{c}\colon (0,\infty) \to [0,\infty)$ such that for every $d \in \mathbb{N}$, $\delta \in (0,\infty)$, $\varepsilon \in (0,1]$ it holds that $\sum_{n=1}^{\mathfrak{N}_\varepsilon + K} \mathfrak{C}_{d,n,n} \le \mathfrak{c}_\delta d \varepsilon^{-(2+2\delta)}$ and

$$\begin{aligned}
&\sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \left[\sup_{x\in\mathbb{R}^d} \Big(\mathbb{E}\Big[|U^{d,0}_{n,n}(T,x) - u_d(T,x)|^2\Big]\Big)^{1/2}\right] \\
&= \sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}} \left[\sup_{x\in\mathbb{R}^d} \Big(\mathbb{E}\Big[|V^{d,0}_{n,n,\varrho_n}(T,x) - u_d(T,x)|^2\Big]\Big)^{1/2}\right] \le \varepsilon.
\end{aligned} \tag{167}$$

The proof of Corollary 5.2 is thus completed. □

Corollary 5.3. *Let $c, T \in (0,\infty)$, $K \in \mathbb{N}_0$, $\Theta = \cup_{n\in\mathbb{N}} \mathbb{Z}^n$, for every $d \in \mathbb{N}$ let $\|\cdot\|_d : \mathbb{R}^d \to [0,\infty)$ be a norm on $\mathbb{R}^d$, let $u_d \in C([0,T]\times\mathbb{R}^d, \mathbb{R})$, $d \in \mathbb{N}$, satisfy for every $d \in \mathbb{N}$, $t \in [0,T]$, $x \in \mathbb{R}^d$ that $|u_d(0,x)| \le c$, $\inf_{a\in\mathbb{R}}[\sup_{s\in[0,T]} \sup_{y=(y_1,\ldots,y_d)\in\mathbb{R}^d}(e^{a(|y_1|^2+\ldots+|y_d|^2)}|u_d(s,y)|)] < \infty$, $u_d|_{(0,T]\times\mathbb{R}^d} \in C^{1,2}((0,T]\times\mathbb{R}^d, \mathbb{R})$, and*

$$(\tfrac{\partial}{\partial t} u_d)(t,x) = (\Delta_x u_d)(t,x) + u_d(t,x) - (u_d(t,x))^3, \tag{168}$$

let $\varrho\colon \mathbb{N} \to (0,\infty)$ be a function which satisfies that $\limsup_{n\to\infty}(\frac{\varrho_n}{\ln(\ln(n))}) < \infty = \liminf_{n\to\infty} \varrho_n$, let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be the functions which satisfy for every $n \in \mathbb{N}$, $v \in \mathbb{R}$ that $f_n(v) = (\min\{\varrho_n, \max\{-\varrho_n, v\}\}) - (\min\{\varrho_n, \max\{-\varrho_n, v\}\})^3$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{R}^\theta\colon \Omega \to [0,1]$, $\theta \in \Theta$, be independent $\mathcal{U}_{[0,1]}$-distributed random variables, let $W^{d,\theta}\colon [0,T]\times\Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, be independent standard Brownian motions, assume that $(\mathcal{R}^\theta)_{\theta\in\Theta}$ and $(W^{d,\theta})_{(d,\theta)\in\mathbb{N}\times\Theta}$ are independent, let $R^\theta\colon \Omega\times[0,T] \to [0,T]$, $\theta \in \Theta$, satisfy for every $\theta \in \Theta$, $t \in [0,T]$ that $R^\theta_t = t\mathcal{R}^\theta$, for every $d \in \mathbb{N}$, $s \in [0,T]$, $t \in [s,T]$, $x \in \mathbb{R}^d$, $\theta \in \Theta$ let $X^{d,\theta}_{s,t}\colon \Omega \to \mathbb{R}^d$ satisfy

$X^{d,\theta}_{s,t,x} = x + \sqrt{2}\,(W^{d,\theta}_t - W^{d,\theta}_s)$, let $U^{d,\theta}_{n,M}\colon [0,T]\times\mathbb{R}^d\times\Omega\to\mathbb{R}$, $d,M\in\mathbb{N}$, $\theta\in\Theta$, $n\in\mathbb{N}_0$, satisfy for every $d,n,M\in\mathbb{N}$, $\theta\in\Theta$, $r\in(0,\infty)$, $t\in[0,T]$, $x\in\mathbb{R}^d$ that $U^{d,\theta}_{0,M}(t,x)=0$ and

$$\begin{aligned} U^{d,\theta}_{n,M}(t,x) = \sum_{k=1}^{n-1}\frac{t}{M^{n-k}}\Bigg[\sum_{m=1}^{M^{n-k}}\bigg(f_M\Big(U^{d,(\theta,k,m)}_{k,M,r}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)}_t,t,x}\Big)\Big) \\ - f_M\Big(U^{d,(\theta,-k,m)}_{k-1,M}\Big(R^{(\theta,k,m)}_t, X^{d,(\theta,k,m)}_{R^{(\theta,k,m)},t,x}\Big)\Big)\bigg)\Bigg] + \frac{1}{M^n}\left[\sum_{m=1}^{M^n} u_d(0, X^{d,(\theta,0,-m)}_{0,t,x})\right], \end{aligned} \tag{169}$$

and let $\mathfrak{C}_{d,n,M}\in\mathbb{N}_0$, $d,M\in\mathbb{N}$, $n\in\mathbb{N}_0$, satisfy for every $d,n,M\in\mathbb{N}$ that $\mathfrak{C}_{d,0,M}=0$ and

$$\mathfrak{C}_{d,n,M} \le (2d+1)M^n + \sum_{l=1}^{n-1} M^{n-l}\,(d+1+\mathfrak{C}_{d,l,M}+\mathfrak{C}_{d,l-1,M}). \tag{170}$$

Then there exist $\mathfrak{N}\colon (0,1]\to\mathbb{N}$ and $\mathfrak{c}\colon (0,\infty)\to[0,\infty)$ such that for every $d\in\mathbb{N}$, $\delta\in(0,\infty)$, $\varepsilon\in(0,1)$ it holds that

$$\left[\sum_{n=1}^{\mathfrak{N}_\varepsilon+K}\mathfrak{C}_{d,n,n}\right] \le d\mathfrak{c}_\delta\varepsilon^{-(2+\delta)} \quad \textit{and} \quad \sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}}\left[\sup_{x\in\mathbb{R}^d}\Big(\mathbb{E}\Big[|U^{d,0}_{n,n}(T,x)-u_d(T,x)|^2\Big]\Big)^{1/2}\right]\le\varepsilon. \tag{171}$$

Proof of Corollary 5.3. First, observe that for every $r\in(0,\infty)$, $v,w\in[-r,r]$ it holds that

$$\begin{aligned} |(v-v^3)-(w-w^3)| = |(v-w)(1-w^2-wv-v^2)| &\le (1+|v|^2+|w|^2+|wv|)|v-w| \\ &\le 2(1+|v|^2+|w|^2)|v-w|. \end{aligned} \tag{172}$$

Moreover, note that for every $v\in\mathbb{R}$ it holds that $v(v-v^3) = v^2 - v^4 \le 1+v^2$. This, the hypothesis that for every $d\in\mathbb{N}$, $x\in\mathbb{R}^d$ it holds that $|u_d(0,x)|\le c$, the fact that for every $d\in\mathbb{N}$ it holds that $\|\cdot\|_d$ and the Euclidean norm on $\mathbb{R}^d$ are equivalent, (172), and Corollary 5.2 (with $T=T$, $c=\max\{c,2\}$, $K=K$, $\Theta=\Theta$, $f=(\mathbb{R}\ni u\mapsto u-u^3\in\mathbb{R})$, $\mathbf{f}_M=f_M$, $\varrho=\varrho$, $u_d=u_d$, $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $\mathcal{R}^\theta=\mathcal{R}^\theta$, $W^{d,\theta}=W^{d,\theta}$, $R^\theta=R^\theta$, $X^{d,\theta}_{s,t,x}=X^{d,\theta}_{s,t,x}$, $U^{d,\theta}_{n,M}(t,x)=U^{d,\theta}_{n,M}(t,x)$, $\mathfrak{C}_{d,n,M}=\mathfrak{C}_{d,n,M}$ for $d,M\in\mathbb{N}$, $\theta\in\Theta$, $n\in\mathbb{N}_0$, $r\in(0,\infty)$, $t\in[0,T]$, $s\in[0,t]$, $x\in\mathbb{R}^d$ in the notation of Corollary 5.1) ensure that there exist $\mathfrak{N}\colon(0,1]\to\mathbb{N}$ and $\mathfrak{c}\colon(0,\infty)\to[0,\infty)$ such that for every $d\in\mathbb{N}$, $\delta\in(0,\infty)$, $\varepsilon\in(0,1]$ it holds that

$$\left[\sum_{n=1}^{\mathfrak{N}_\varepsilon+K}\mathfrak{C}_{d,n,n}\right] \le \mathfrak{c}_\delta d\varepsilon^{-(2+2\delta)} \quad \text{and} \quad \sup_{n\in[\mathfrak{N}_\varepsilon,\infty)\cap\mathbb{N}}\left[\sup_{x\in\mathbb{R}^d}\Big(\mathbb{E}\Big[|V^{d,0}_{n,n}(T,x)-v_d(T,x)|^2\Big]\Big)^{1/2}\right]\le\varepsilon. \tag{173}$$

The proof of Corollary 5.3 is thus completed. ◻

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References

[1] Andersson, A., Beck, C., Becker, S., Cheridito, P., Jentzen, A., and Neufeld, A. Deep splitting method for parabolic PDEs. *arXiv:1907.03452* (2019), 40 pages.

[2] Beck, C., Becker, S., Grohs, P., Jaafari, N., and Jentzen, A. Solving stochastic differential equations and Kolmogorov equations by means of deep learning. *arXiv:1806.00421* (2018), 56 pages.

[3] Beck, C., E, W., and Jentzen, A. Machine learning approximation algorithms for high-dimensional fully nonlinear partial differential equations and second-order backward stochastic differential equations. *Journal of Nonlinear Science* (2017), 1–57.

[4] Becker, S., Cheridito, P., and Jentzen, A. Deep optimal stopping. *To appear in The Journal of Machine Learning Research, arXiv:1804.05394* (2018), 26 pages.

[5] Berg, J., and Nyström, K. A unified deep artificial neural network approach to partial differential equations in complex geometries. *Neurocomputing 317* (2018), 28–41.

[6] Bouchard, B., Tan, X., Warin, X., and Zou, Y. Numerical approximation of BSDEs using local polynomial drivers and branching processes. *Monte Carlo Methods Appl. 23*, 4 (2017), 241–263.

[7] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Springer Science & Business Media, 2010.

[8] Chan-Wai-Nam, Q., Mikael, J., and Warin, X. Machine learning for semi linear PDEs. *arXiv:1809.07609* (2018), 38 pages.

[9] E, W., Han, J., and Jentzen, A. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Communications in Mathematics and Statistics 5* (2017), 349–380.

[10] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. Multilevel Picard iterations for solving smooth semilinear parabolic heat equations. *arXiv:1607.03295* (2016), 19 pages.

[11] E, W., Hutzenthaler, M., Jentzen, A., and Kruse, T. On multilevel Picard numerical approximations for high-dimensional nonlinear parabolic partial differential equations and high-dimensional nonlinear backward stochastic differential equations. *J. Sci. Comput. 79*, 3 (2019), 1534–1571.

[12] E, W., and Yu, B. The deep Ritz method: A deep learning-based numerical algorithm for solving variational problems. *arXiv:1710.00211* (2017), 14 pages.

[13] Farahmand, A.-m., Nabi, S., and Nikovski, D. N. Deep reinforcement learning for partial differential equation control. *2017 American Control Conference (ACC)* (2017), 3120–3127.

[14] Fujii, M., Takahashi, A., and Takahashi, M. Asymptotic expansion as prior knowledge in deep learning method for high dimensional BSDEs. *arXiv:1710.07030* (2017), 16 pages.

[15] Goudenege, L., Molent, A., and Zanette, A. Machine learning for pricing american options in high dimension. *arXiv:1903.11275* (2019), 11 pages.

[16] Han, J., Jentzen, A., and E, W. Solving high-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences 115*, 34 (2018), 8505–8510.

[17] Han, J., and Long, J. Convergence of the deep BSDE method for coupled FBSDEs. *arXiv:1811.01165* (2018), 26 pages.

[18] Henry-Labordere, P. Counterparty risk valuation: A marked branching diffusion approach. *SSRN:1995503* (2012), 17 pages.

[19] Henry-Labordere, P. Deep primal-dual algorithm for BSDEs: Applications of machine learning to CVA and IM. *Preprint, SSRN–id3071506* (2017), 16 pages.

[20] Henry-Labordère, P., Oudjane, N., Tan, X., Touzi, N., and Warin, X. Branching diffusion representation of semilinear PDEs and Monte Carlo approximation. *Ann. Inst. Henri Poincaré Probab. Stat. 55*, 1 (2019), 184–210.

[21] Henry-Labordère, P., Tan, X., and Touzi, N. A numerical algorithm for a class of BSDEs via the branching process. *Stochastic Process. Appl. 124*, 2 (2014), 1112–1140.

[22] Huré, C., Pham, H., and Warin, X. Some machine learning schemes for high-dimensional nonlinear PDEs. *arXiv:1902.01599* (2019), 33 pages.

[23] Hutzenthaler, M., Jentzen, A., Kruse, T., and Nguyen, T. A. A proof that rectified deep neural networks overcome the curse of dimensionality in the numerical approximation of semilinear heat equations. *arXiv:1901.10854* (2019), 24 pages.

[24] Hutzenthaler, M., Jentzen, A., Kruse, T., Nguyen, T. A., and von Wurstemberger, P. Overcoming the curse of dimensionality in the numerical approximation of semilinear parabolic partial differential equations. *arXiv:1807.01212* (2018), 30 pages.

[25] Hutzenthaler, M., Jentzen, A., and von Wurstemberger, P. Overcoming the curse of dimensionality in the approximative pricing of financial derivatives with default risks. *arXiv:1903.05985* (2019), 71 pages.

[26] Hutzenthaler, M., and Kruse, T. Multi-level Picard approximations of high-dimensional semilinear parabolic differential equations with gradient-dependent nonlinearities. *arXiv:1711.01080* (2017), 19 pages.

[27] Jacquier, A., and Oumgari, M. Deep PPDEs for rough local stochastic volatility. *arXiv:1906.02551* (2019), 21 pages.

[28] John, F. *Partial differential equations.* Applied mathematical sciences. Springer, 1982.

[29] Long, Z., Lu, Y., Ma, X., and Dong, B. PDE-Net: Learning PDEs from Data. *arXiv:1710.09668* (2017), 15 pages.

[30] Lye, K. O., Mishra, S., and Ray, D. Deep learning observables in computational fluid dynamics. *arXiv:1903.03040* (2019), 57 pages.

[31] Magill, M., Qureshi, F., and de Haan, H. Neural networks trained to solve differential equations learn general representations. *Advances in Neural Information Processing Systems* (2018), 4075–4085.

[32] McKean, H. P. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Comm. Pure Appl. Math. 28*, 3 (1975), 323–331.

[33] Raissi, M. Deep hidden physics models: Deep learning of nonlinear partial differential equations. *arXiv:1801.06637* (2018), 26 pages.

[34] Sirignano, J., and Spiliopoulos, K. DGM: A deep learning algorithm for solving partial differential equations. *Journal of Computational Physics 375* (2018), 1339–1364.

[35] Skorohod, A. V. Branching diffusion processes. *Teor. Verojatnost. i Primenen. 9* (1964), 492–497.

[36] Warin, X. Nesting Monte Carlo for high-dimensional non-linear PDEs. *Monte Carlo Methods Appl. 24*, 4 (2018), 225–247.

[37] Watanabe, S. On the branching process for Brownian particles with an absorbing boundary. *J. Math. Kyoto Univ. 4* (1965), 385–398.