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On the strong regularity of degenerate additive noise driven stochastic differential equations with respect to their initial values

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On the strong regularity of degenerate additive noise driven stochastic differential equations with respect to their initial values

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Abstract

Recently in [M. Hairer, M. Hutzenthaler, and A. Jentzen, *Ann. Probab. 43*, 2 (2015), 468–527] and [A. Jentzen, T. Müller-Gronbach, and L. Yaroslavtseva, *Commun. Math. Sci. 14*, 6 (2016), 1477–1500] stochastic differential equations (SDEs) with smooth coefficient functions have been constructed which have an arbitrarily slowly converging modulus of continuity in the initial value. In these SDEs it is crucial that some of the first order partial derivatives of the drift coefficient functions grow at least exponentially and, in particular, quicker than any polynomial. However, in applications SDEs do typically have coefficient functions whose first order partial derivatives are polynomially bounded. In this article we study whether arbitrarily bad regularity phenomena in the initial value may also arise in the latter case and we partially answer this question in the negative. More precisely, we show that every additive noise driven SDE which admits a Lyapunov-type condition (which ensures the existence of a unique solution of the SDE) and which has a drift coefficient function whose first order partial derivatives grow at most polynomially is at least logarithmically Hölder continuous in the initial value.

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1 Introduction

The regularity analysis of nonlinear stochastic differential equations (SDEs) with respect to their initial values is an active research topic in stochastic analysis (cf., e.g., [2, 3, 5, 7, 9, 10, 15, 18, 19, 20, 25] and the references mentioned therein). In particular, it has recently been revealed in the literature that there exist SDEs with smooth coefficient functions which have very poor regularity properties in the initial value. More precisely, it has been shown in [7] that there exist additive noise driven SDEs with infinitely often differentiable drift coefficient functions which have a modulus of continuity in the initial value that converges to zero slower than with any polynomial rate. Moreover, in [13] additive noise driven SDEs with infinitely often differentiable drift coefficient functions have been constructed which even have an arbitrarily slowly converging modulus of continuity in the initial value. In these SDEs it is crucial that the first order partial derivatives of the drift coefficient functions grow at least exponentially and, in particular, quicker than any polynomial. However, in applications SDEs do typically have coefficient functions whose first order partial derivatives grow at most polynomially (cf., e.g., [1, 4, 8, 17, 21, 22, 23, 24], [14, Chapter 7], and [11, Chapter 4] for examples). In particular, in many applications the coefficient functions of the SDEs under consideration are polynomials (cf., e.g., [1, 4, 21, 23, 24], [14, Chapter 7], and [11, Chapter 4] for examples). In view of this, the natural question arises whether such arbitrarily bad regularity phenomena in the initial value may also arise in the case of SDEs with coefficient functions whose first order partial derivatives grow at most polynomially. It is the subject of the main result of this article to partially answer this question in the negative. More precisely, the main result of this article, Theorem 1.1 below, shows that every additive noise driven SDE which admits a Lyapunov-type condition (which ensures the existence of a unique solution of the SDE) and which has a drift coefficient function whose first order partial derivatives grow at most polynomially is at least logarithmically Hölder continuous in the initial value.

Theorem 1.1. *Let $d, m \in \mathbb{N}$, $T, \kappa \in [0,\infty)$, $\alpha \in [0,2)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $V \in C^1(\mathbb{R}^d, [0,\infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ and $|||\cdot||| \colon \mathbb{R}^m \to [0,\infty)$ be norms, assume for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x+\sigma z) \leq \kappa(1 + |||z|||^\alpha)V(x)$, and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then*

(i) there exist unique stochastic processes $X^x \colon [0,T]\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega) \tag{1}$$

and

(ii) it holds for all $R, q \in [0,\infty)$ that there exists $c \in (0,\infty)$ such that for all $x, y \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}$ with $0 < \|x-y\| \neq 1$ it holds that

$$\sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] \leq c\,\big|\ln(\|x-y\|)\big|^{-q}. \tag{2}$$

Theorem 1.1 above is an immediate consequence of Theorem 8.4 in Subsection 8.3 below. Inequality (2) proves, roughly speaking, only Hölder continuity in the initial value in a logarithmic sense but does neither prove local Lipschitz continuity nor prove local Hölder continuity in the initial value in the usual sense. In view of this, the question arises whether the statement of Theorem 1.1 can be strengthened to ensure local Hölder continuity in the initial value in the usual sense. In a subsequent article we show that this is not the case and specify a concrete additive noise driven SDE which satisfies the hypotheses of Theorem 1.1 but whose solution fails for every arbitrarily small $\alpha \in (0, 1]$ to be locally α-Hölder continuous in the initial value. Even more, we show that under the hypotheses of Theorem 1.1 the upper bound in (2) can not be substantially improved in general.

In the following we briefly sketch the key ideas of our proof of inequality (2) in Theorem 1.1. A straightforward approach to estimating the expectation of the Euclidean distance between two solutions of the SDE (1) with different initial values (cf. the left hand side of (2)) would be (i) to apply the fundamental theorem of calculus to the difference of the two solutions with the derivative being taken with respect to the initial value, thereafter, (ii) to employ the triangle inequality to get the Euclidean norm inside of the Riemann integral which has appeared due to the application of the fundamental theorem of calculus, and, finally, (iii) to try to provide a finite upper bound for the expectation of the Euclidean operator norm of the derivative processes of solutions of (1) with respect to the initial value. This approach, however, fails to work in general under the hypotheses of Theorem 1.1 as the derivative processes of solutions may have very poor integrability properties and, in particular, may have infinite absolute moments. A key idea in this article for overcoming the latter obstacle is to estimate the expectation of the Euclidean distance between the two solutions in terms of the expectation of a new distance between the two solutions, which is induced from a very slowly growing norm-type function. As in the approach above, we then also apply the fundamental theorem of calculus to the difference of the two solutions. However, in the latter approach the derivative processes of solutions appear only inside of the argument of the very slowly growing norm-type function and the expectation of the resulting random variable is finite. We then estimate the expectation of this random variable by employing properties of the derivative processes of solutions and the assumption that the first order partial derivatives of the drift coefficient function grow at most polynomially and, thereby, finally establish inequality (2).

The remainder of this article is organized as follows. In Section 2 we establish an essentially well-known existence and uniqueness result for perturbed ordinary differential equations. In Section 3 we recall well-known facts on measurability properties of function limits and in Section 4 we establish a well-known measurability result for solutions of additive noise driven SDEs. In Section 5 we prove existence, uniqueness, and pathwise differentiability with respect to the initial value and in Section 6 we present a few elementary integrability properties for solutions of additive noise driven SDEs with a drift coefficient function which admits a Lyapunov-type condition. In Section 7 we establish an abstract regularity result for solutions of certain additive noise driven SDEs with respect to their initial values. This result together with the results of Sections 5 and 6 is then used to prove the main result of this article, Theorem 8.4, in Section 8.

2 Existence of solutions of perturbed ordinary differential equations (ODEs)

In this section we employ suitable Lyapunov-type functions to establish in Lemma 2.2 in Subsection 2.2 below an essentially well-known existence and uniqueness result for a certain class of perturbed ordinary differential equations (ODEs). Our proof of Lemma 2.2 employs the essentially well-known a priori estimate in Lemma 2.1 in Subsection 2.1 below. Our proof of Lemma 2.1 uses a suitable Lyapunov-type function (denoted by $V\colon \mathbb{R}^d \to \mathbb{R}$ in Lemma 2.1 below).

2.1 A priori estimates for solutions of perturbed ODEs

Lemma 2.1. *Let $d, m \in \mathbb{N}$, $T \in [0,\infty)$, $\xi \in \mathbb{R}^d$, $\mu \in C(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m, [0,\infty))$, $V \in C^1(\mathbb{R}^d, [0,\infty))$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm, let $J \subseteq [0,T]$ be an interval, assume that $0 \in J$, and let $y \in C(J, \mathbb{R}^d)$, $w \in C([0,T], \mathbb{R}^m)$ satisfy for all $x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$, $t \in J$ that $V'(x)\mu(x+\sigma u) \leq \varphi(u)V(x)$, $\|x\| \leq V(x)$, and*

$$y(t) = \xi + \int_0^t \mu(y(s))\,\mathrm{d}s + \sigma w(t). \tag{3}$$

Then it holds that $\sup_{t\in J}\big[\varphi(w(t)) + \|\sigma w(t)\|\big] < \infty$ *and*

$$\sup_{t\in J}\|y(t)\| \leq V(\xi)\exp\!\left(T\left[\sup_{s\in J}\varphi(w(s))\right]\right) + \left[\sup_{t\in J}\|\sigma w(t)\|\right]. \tag{4}$$

Proof of Lemma 2.1. Throughout this proof assume w.l.o.g. that $\sup J > 0$, let $I \subseteq [0,T]$ be the set which satisfies $I = (0, \sup J)$, let $K \in [0,\infty]$ satisfy

$$K = \sup_{s\in J}\varphi(w(s)), \tag{5}$$

and let $z\colon J \to \mathbb{R}^d$ be the function which satisfies for all $t \in J$ that

$$z(t) = y(t) - \sigma w(t). \tag{6}$$

Observe that the fact that φ and w are continuous functions ensures that

$$K \leq \sup_{s\in[0,T]}\varphi(w(s)) < \infty. \tag{7}$$

This and the hypothesis that w is a continuous function ensure that

$$\sup_{t\in J}\big[\varphi(w(t)) + \|\sigma w(t)\|\big] \leq \left[\sup_{t\in J}\varphi(w(t))\right] + \left[\sup_{t\in J}\|\sigma w(t)\|\right] \leq K + \sup_{t\in[0,T]}\|\sigma w(t)\| < \infty. \tag{8}$$

Next note that (3) and (6) imply that for all $t \in J$ it holds that

$$z(t) = \xi + \int_0^t \mu(y(s))\,\mathrm{d}s. \tag{9}$$

The hypothesis that μ and y are continuous functions and the fundamental theorem of calculus hence ensure that for all $t \in I$ it holds that $z|_I \in C^1(I, \mathbb{R}^d)$ and

$$(z|_I)'(t) = \mu(y(t)). \tag{10}$$

This, the assumption that $V \in C^1(\mathbb{R}^d, [0, \infty))$, and the chain rule imply that for all $t \in I$ it holds that $V \circ (z|_I) \in C^1(I, [0, \infty))$ and

$$(V \circ (z|_I))'(t) = V'(z(t))(z|_I)'(t) = V'(z(t))\mu(y(t)). \tag{11}$$

Furthermore, note that the hypothesis that $V \in C^1(\mathbb{R}^d, [0, \infty))$ and the hypothesis that y, w, and μ are continuous functions establish that $J \ni t \mapsto V'(z(t))\mu(y(t)) \in \mathbb{R}$ is a continuous function. Combining this and (11) with the fundamental theorem of calculus and the fact that $z(0) = \xi$ shows that for all $t \in I$ it holds that

$$\begin{aligned} V(z(t)) &= \big[V(z(s))\big]_{s=0}^{s=t} + V(z(0)) \\ &= \int_0^t V'(z(s))\mu(y(s))\,\mathrm{d}s + V(\xi) \\ &= \int_0^t V'(z(s))\mu(z(s) + \sigma w(s))\,\mathrm{d}s + V(\xi). \end{aligned} \tag{12}$$

The hypothesis that for all $x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$ it holds that $V'(x)\mu(x + \sigma u) \le \varphi(u)V(x)$ and (7) hence prove that for all $t \in I$ it holds that

$$V(z(t)) \le \int_0^t \varphi(w(s))V(z(s))\,\mathrm{d}s + V(\xi) \le \int_0^t KV(z(s))\,\mathrm{d}s + V(\xi). \tag{13}$$

The assumption that $\sup J > 0$ and the fact that $J \in t \mapsto V(z(t)) \in [0, \infty)$ is a continuous function therefore imply that for all $u \in \{s \in J \colon s = \sup J\}$ it holds that

$$\begin{aligned} V(z(u)) &= \limsup_{t \nearrow u} V(z(t)) \\ &\le \limsup_{t \nearrow u} \left[\int_0^t KV(z(s))\,\mathrm{d}s + V(\xi)\right] \\ &= \int_0^u KV(z(s))\,\mathrm{d}s + V(\xi). \end{aligned} \tag{14}$$

This, (13), and the fact that $V(z(0)) = V(\xi)$ demonstrate that for all $t \in J$ it holds that

$$V(z(t)) \le \int_0^t KV(z(s))\,\mathrm{d}s + V(\xi). \tag{15}$$

Combining this and (7) with Gronwall's integral inequality (see, e.g., Grohs et al. [6, Lemma 2.11] (with $\alpha \leftarrow V(\xi)$, $\beta \leftarrow K$, $T \leftarrow t$, $f \leftarrow ([0,t] \ni s \mapsto V(z(s)) \in [0,\infty))$ for $t \in J$ in the notation of Grohs et al. [6, Lemma 2.11])) proves that for all $t \in J$ it holds that

$$V(z(t)) \leq V(\xi)\exp(tK) \leq V(\xi)\exp(TK). \tag{16}$$

The triangle inequality and the hypothesis that for all $x \in \mathbb{R}^d$ it holds that $\|x\| \leq V(x)$ hence establish that

$$\begin{aligned}
\sup_{t\in J}\|y(t)\| &= \sup_{t\in J}\|z(t) - \sigma w(t)\| \\
&\leq \sup_{t\in J}\big[\|z(t)\| + \|\sigma w(t)\|\big] \\
&\leq \left[\sup_{t\in J}\|z(t)\|\right] + \left[\sup_{t\in J}\|\sigma w(t)\|\right] \\
&\leq \left[\sup_{t\in J}V(z(t))\right] + \left[\sup_{t\in J}\|\sigma w(t)\|\right] \\
&\leq V(\xi)\exp(TK) + \left[\sup_{t\in J}\|\sigma w(t)\|\right].
\end{aligned} \tag{17}$$

The proof of Lemma 2.1 is thus completed. □

2.2 Existence of solutions of perturbed ODEs

Lemma 2.2. *Let $d, m \in \mathbb{N}$, $T \in [0,\infty)$, $\xi \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m, [0,\infty))$, $V \in C^1(\mathbb{R}^d, [0,\infty))$, $w \in C([0,T], \mathbb{R}^m)$, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, let $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz continuous function, and assume for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $V'(x)\mu(x + \sigma z) \leq \varphi(z)V(x)$ and $\|x\| \leq V(x)$. Then there exists a unique $y \in C([0,T], \mathbb{R}^d)$ such that for all $t \in [0,T]$ it holds that*

$$y(t) = \xi + \int_0^t \mu(y(s))\,\mathrm{d}s + \sigma w(t). \tag{18}$$

Proof of Lemma 2.2. Throughout this proof assume w.l.o.g. that $T > 0$. Note that the hypothesis that μ is a locally Lipschitz continuous function, the hypothesis that w is a continuous function, and [12, Theorem 8.3] (with $(V, \|\cdot\|_V) \leftarrow (\mathbb{R}^d, \|\cdot\|)$, $(W, \|\cdot\|_W) \leftarrow (\mathbb{R}^d, \|\cdot\|)$, $T \leftarrow T$, $F \leftarrow \mu$, $S \leftarrow ((0,T) \ni t \mapsto \mathrm{id}_{\mathbb{R}^d} \in L(\mathbb{R}^d, \mathbb{R}^d))$, $\mathcal{S} \leftarrow ([0,T] \ni t \mapsto \mathrm{id}_{\mathbb{R}^d} \in L(\mathbb{R}^d, \mathbb{R}^d))$, $o \leftarrow ([0,T] \ni t \mapsto \xi + \sigma w(t) \in \mathbb{R}^d)$, $\phi \leftarrow ((0,T) \ni t \mapsto t \in (0,\infty))$ in the notation of [12, Theorem 8.3]) ensure that there exists an interval $J \subseteq [0,T]$ with $0 \in J$ and $\sup J > 0$ such that there exists a unique $x \in C(J, \mathbb{R}^d)$ which satisfies for all $t \in J$ that

$$x(t) = \xi + \int_0^t \mu(x(s))\,\mathrm{d}s + \sigma w(t) \qquad \text{and} \qquad \limsup_{s \nearrow \sup J}\big[(T-s)^{-1} + \|x(s)\|\big] = \infty. \tag{19}$$

Lemma 2.1 hence proves that $\sup_{t\in J}[\varphi(w(s)) + \|\sigma w(t)\|] < \infty$ and

$$\sup_{t\in J}\|x(t)\| \leq V(\xi)\exp\left(T\left[\sup_{s\in J}\varphi(w(s))\right]\right) + \left[\sup_{t\in J}\|\sigma w(t)\|\right] < \infty. \tag{20}$$

Combining this with (19) ensures that $\sup J = T$. Therefore, we obtain that $J = [0,T)$ or $J = [0,T]$. This, the hypothesis that μ is a locally Lipschitz continuous function, (20), and [12, Lemma 8.1] (with $(V, \|\cdot\|_V) \leftarrow (\mathbb{R}^d, \|\cdot\|)$, $(W, \|\cdot\|_W) \leftarrow (\mathbb{R}^d, \|\cdot\|)$, $T \leftarrow T$, $\tau \leftarrow T$, $x \leftarrow x|_{[0,T)}$, $o \leftarrow ([0,T] \ni t \mapsto \xi + \sigma w(t) \in \mathbb{R}^d)$, $F \leftarrow \mu$, $S \leftarrow ((0,T) \ni t \mapsto \mathrm{id}_{\mathbb{R}^d} \in L(\mathbb{R}^d))$, $\phi \leftarrow ((0,T) \ni t \mapsto t \in (0,\infty))$ in the notation of [12, Lemma 8.1]) prove that there exists a continuous function $y\colon [0,T] \to \mathbb{R}^d$ such that it holds for all $t \in [0,T]$ that

$$y|_{[0,T)} = x \qquad \text{and} \qquad y(t) = \xi + \int_0^t \mu(y(s))\,\mathrm{d}s + \sigma w(t). \tag{21}$$

In the next step we observe that (19) and the fact that $\sup J = T$ show that for all $z \in \left\{u \in C([0,T], \mathbb{R}^d)\colon \left(\forall\, t \in [0,T]\colon u(t) = \xi + \int_0^t \mu(u(s))\,\mathrm{d}s + \sigma w(t)\right)\right\}$ it holds that $z|_J = x$. The fact that y is a continuous function and the fact that $y|_{[0,T)} = x$ hence demonstrate that for all $z \in \left\{u \in C([0,T], \mathbb{R}^d)\colon \left(\forall\, t \in [0,T]\colon u(t) = \xi + \int_0^t \mu(u(s))\,\mathrm{d}s + \sigma w(t)\right)\right\}$ it holds that $z = y$. Combining this with (21) establishes (18). The proof of Lemma 2.2 is thus completed. □

3 Measurability properties

In this section we recall in Lemmas 3.1–3.4 in Subsection 3.1 and in Lemmas 3.5 and 3.6 in Subsection 3.2 below a few well-known facts on measurability properties of suitable function limits. For completeness we also include in this section proofs for Lemmas 3.1–3.6.

3.1 Measurability properties for functions

Lemma 3.1. *Let $(\Omega, \mathcal{F})$ be a measurable space, let I be a non-empty and at most countable set, let $Y\colon \Omega \to [-\infty,\infty]$ be a function, and let $X_i\colon \Omega \to [-\infty,\infty]$, $i \in I$, be $\mathcal{F}/\mathcal{B}([-\infty,\infty])$-measurable functions which satisfy for all $\omega \in \Omega$ that*

$$Y(\omega) = \sup_{i\in I} X_i(\omega). \tag{22}$$

Then it holds that Y is an $\mathcal{F}/\mathcal{B}([-\infty,\infty])$-measurable function.

Proof of Lemma 3.1. Note that the hypothesis that for all $i \in I$ it holds that X_i is an $\mathcal{F}/\mathcal{B}([-\infty,\infty])$-measurable function and the hypothesis that I is at most countable establish that for all $c \in \mathbb{R}$ it holds that

$$\{Y \leq c\} = \left\{\sup_{i\in I} X_i \leq c\right\} = \bigcap_{i\in I} \underbrace{\{X_i \leq c\}}_{\in\mathcal{F}} \in \mathcal{F}. \tag{23}$$

The proof of Lemma 3.1 is thus completed. ☐

Lemma 3.2. *Let* $(\Omega, \mathcal{F})$ *be a measurable space, let* $Y\colon \Omega \to \mathbb{R}$ *be a function, and let* $X_n\colon \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, *be a sequence of* $\mathcal{F}/\mathcal{B}(\mathbb{R})$*-measurable functions which satisfies for all* $\omega \in \Omega$ *that* $\limsup_{n\to\infty} |X_n(\omega) - Y(\omega)| = 0$. *Then it holds that* Y *is an* $\mathcal{F}/\mathcal{B}(\mathbb{R})$*-measurable function.*

Proof of Lemma 3.2. First, observe that the assumption that for all $\omega \in \Omega$ it holds that $\limsup_{n\to\infty}|X_n(\omega) - Y(\omega)| = 0$ implies that for all $\omega \in \Omega$ it holds that $\mathbb{N} \ni n \mapsto X_n(\omega) \in \mathbb{R}$ is a convergent sequence and

$$\lim_{n\to\infty} X_n(\omega) = Y(\omega). \tag{24}$$

Moreover, note that Lemma 3.1 ensures that for all $n \in \mathbb{N}$ it holds that $\Omega \ni \omega \mapsto \sup_{m\in\{n,n+1,\dots\}} X_m(\omega) \in [-\infty, \infty]$ is an $\mathcal{F}/\mathcal{B}([-\infty,\infty])$-measurable function. This and (24) show that for all $c \in \mathbb{R}$ it holds that

$$\begin{split} \{Y \geq c\} &= \left\{\lim_{n\to\infty} X_n \geq c\right\} = \left\{\limsup_{n\to\infty} X_n \geq c\right\} \\ &= \left\{\lim_{n\to\infty}\left[\sup_{m\in\{n,n+1,\dots\}} X_m\right] \geq c\right\} = \bigcap_{n\in\mathbb{N}} \underbrace{\left\{\left[\sup_{m\in\{n,n+1,\dots\}} X_m\right] \geq c\right\}}_{\in\mathcal{F}} \in \mathcal{F}. \end{split} \tag{25}$$

The proof of Lemma 3.2 is thus completed. ☐

Lemma 3.3. *Let* $(\Omega, \mathcal{F})$ *be a measurable space, let* $d \in \mathbb{N}$, *let* $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ *be a norm, let* $Y\colon \Omega \to \mathbb{R}^d$ *be a function, and let* $X_n\colon \Omega \to \mathbb{R}^d$, $n \in \mathbb{N}$, *be a sequence of* $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$*-measurable functions which satisfies for all* $\omega \in \Omega$ *that* $\limsup_{n\to\infty}\|X_n(\omega) - Y(\omega)\| = 0$. *Then it holds that* Y *is an* $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$*-measurable function.*

Proof of Lemma 3.3. Throughout this proof let $K \in [0,\infty]$ satisfy

$$K = \sup_{v=(v_1,v_2,\dots,v_d)\in\mathbb{R}^d\setminus\{0\}} \left(\frac{\left(\sum_{j=1}^d |v_j|\right)}{\|v\|}\right), \tag{26}$$

let $X_{n,i}\colon \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, $i \in \{1,2,\dots,d\}$, be the functions which satisfy for all $n \in \mathbb{N}$, $\omega \in \Omega$ that

$$X_n(\omega) = (X_{n,1}(\omega), X_{n,2}(\omega), \dots, X_{n,d}(\omega)), \tag{27}$$

and let $Y_i\colon \Omega \to \mathbb{R}$, $i \in \{1,2,\dots,d\}$, be the functions which satisfy for all $\omega \in \Omega$ that

$$Y(\omega) = (Y_1(\omega), Y_2(\omega), \dots, Y_d(\omega)). \tag{28}$$

Observe that the fact that all norms on $\mathbb{R}^d$ are equivalent ensures that $K < \infty$. This implies that for all $n \in \mathbb{N}$, $i \in \{1, 2, \dots, d\}$, $\omega \in \Omega$ it holds that

$$|X_{n,i}(\omega) - Y_i(\omega)| \leq \sum_{j=1}^{d} |X_{n,j}(\omega) - Y_j(\omega)| \leq K \, \|X_n(\omega) - Y(\omega)\|. \tag{29}$$

The assumption that for all $\omega \in \Omega$ it holds that $\limsup_{n \to \infty} \|X_n(\omega) - Y(\omega)\| = 0$ and the fact that $K < \infty$ hence show that for all $i \in \{1, 2, \dots, d\}$, $\omega \in \Omega$ it holds that

$$\limsup_{n \to \infty} |X_{n,i}(\omega) - Y_i(\omega)| = 0. \tag{30}$$

Furthermore, observe that the assumption that for all $n \in \mathbb{N}$ it holds that X_n is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function implies that for all $n \in \mathbb{N}$, $i \in \{1, 2, \dots, d\}$ it holds that $X_{n,i}$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function. Combining this and (30) with Lemma 3.2 establishes that for all $i \in \{1, 2, \dots, d\}$ it holds that Y_i is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function. The fact that $\mathcal{B}(\mathbb{R}^d) = [\mathcal{B}(\mathbb{R})]^{\otimes d}$ hence shows that Y is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. The proof of Lemma 3.3 is thus completed. ☐

Lemma 3.4. *Let $d \in \mathbb{N}$, $x, h \in \mathbb{R}^d$, let $(\Omega, \mathcal{F})$ be a measurable space, and let $y^z \colon \Omega \to \mathbb{R}^d$, $z \in \mathbb{R}^d$, be $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable functions which satisfy for all $\omega \in \Omega$ that $(\mathbb{R}^d \ni z \mapsto y^z(\omega) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$. Then it holds that $\Omega \ni \omega \mapsto \big(\tfrac{\partial}{\partial x} y^x(\omega)\big)(h) \in \mathbb{R}^d$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function.*

Proof of Lemma 3.4. Throughout this proof let $D_n \colon \Omega \to \mathbb{R}^d$, $n \in \mathbb{N}$, be the sequence of functions which satisfies for all $n \in \mathbb{N}$, $\omega \in \Omega$ that

$$D_n(\omega) = \frac{y^{x+n^{-1}h}(\omega) - y^x(\omega)}{n^{-1}} \tag{31}$$

and let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be the d-dimensional Euclidean norm. Note that for all $\omega \in \Omega$ it holds that

$$\limsup_{n \to \infty} \big\| D_n(\omega) - \big(\tfrac{\partial}{\partial x} y^x(\omega)\big)(h) \big\| = 0. \tag{32}$$

Furthermore, observe that the assumption that for all $z \in \mathbb{R}^d$ it holds that y^z is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function ensures that for all $n \in \mathbb{N}$ it holds that D_n is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. Combining this and (32) with Lemma 3.3 (with $(\Omega, \mathcal{F}) \leftarrow (\Omega, \mathcal{F})$, $d \leftarrow d$, $\|\cdot\| \leftarrow \|\cdot\|$, $Y \leftarrow \big(\Omega \ni \omega \mapsto \big(\tfrac{\partial}{\partial x} y^x(\omega)\big)(h) \in \mathbb{R}^d\big)$, $(X_n)_{n \in \mathbb{N}} \leftarrow (D_n)_{n \in \mathbb{N}}$ in the notation of Lemma 3.3) implies that $\Omega \ni \omega \mapsto \big(\tfrac{\partial}{\partial x} y^x(\omega)\big)(h) \in \mathbb{R}^d$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. The proof of Lemma 3.4 is thus completed. ☐

3.2 Measurability properties for stochastic processes

Lemma 3.5. *Let $T \in [0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $Y\colon [0,T]\times\Omega \to \mathbb{R}$ be a stochastic process with continuous sample paths. Then*

(i) it holds for all $\omega\in\Omega$ that $\sup_{t\in[0,T]} Y(t,\omega) = \sup_{t\in[0,T]\cap\mathbb{Q}} Y(t,\omega)$ and

(ii) it holds that $\Omega \ni \omega \mapsto \sup_{t\in[0,T]} Y(t,\omega) \in \mathbb{R}$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function.

Proof of Lemma 3.5. Observe that the hypothesis that for all $\omega\in\Omega$ it holds that $[0,T]\ni t\mapsto Y(t,\omega)\in\mathbb{R}$ is a continuous function and the fact that $[0,T]\cap\mathbb{Q}$ is dense in $[0,T]$ imply that for all $\omega\in\Omega$ it holds that

$$\sup_{t\in[0,T]} Y(t,\omega) = \sup_{t\in[0,T]\cap\mathbb{Q}} Y(t,\omega). \tag{33}$$

Combining this with Lemma 3.1 (with $(\Omega,\mathcal{F}) \leftarrow (\Omega,\mathcal{F})$, $I \leftarrow [0,T]\cap\mathbb{Q}$, $Y \leftarrow \big(\Omega\ni\omega\mapsto \sup_{t\in[0,T]} Y(t,\omega)\in\mathbb{R}\big)$, $(X_t)_{t\in[0,T]\cap\mathbb{Q}} \leftarrow (\Omega\ni\omega\mapsto Y(t,\omega)\in\mathbb{R})_{t\in[0,T]\cap\mathbb{Q}}$ in the notation of Lemma 3.1) shows that $\Omega\ni\omega\mapsto\sup_{t\in[0,T]} Y(t,\omega)\in\mathbb{R}$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function. The proof of Lemma 3.5 is thus completed. ☐

Lemma 3.6. *Let $d\in\mathbb{N}$, $T,R\in[0,\infty)$, let $\|\cdot\|\colon\mathbb{R}^d\to[0,\infty)$ be a norm, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $Y^x\colon[0,T]\times\Omega\to[0,\infty)$, $x\in\mathbb{R}^d$, be stochastic processes with continuous sample paths which satisfy for all $t\in[0,T]$, $\omega\in\Omega$ that $(\mathbb{R}^d\ni x\mapsto Y^x(t,\omega)\in[0,\infty))\in C(\mathbb{R}^d,[0,\infty))$. Then*

(i) it holds for all $\omega\in\Omega$ that

$$\left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}}\ \sup_{t\in[0,T]} Y^x(t,\omega)\right] = \left[\sup_{x\in\{z\in\mathbb{Q}^d\colon \|z\|\le R\}}\ \sup_{t\in[0,T]\cap\mathbb{Q}} Y^x(t,\omega)\right] \tag{34}$$

and

(ii) it holds that

$$\Omega\ni\omega\mapsto\left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}}\ \sup_{t\in[0,T]} Y^x(t,\omega)\right]\in[0,\infty] \tag{35}$$

is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function.

Proof of Lemma 3.6. Throughout this proof let $I\subseteq\mathbb{Q}^d\times([0,T]\cap\mathbb{Q})$ be the set which satisfies

$$I = \{(x,t)\in\mathbb{Q}^d\times([0,T]\cap\mathbb{Q})\colon \|x\|\le R\}. \tag{36}$$

Observe that Lemma 3.5 implies that for all $x\in\mathbb{R}^d$, $\omega\in\Omega$ it holds that

$$\sup_{t\in[0,T]} Y^x(t,\omega) = \sup_{t\in[0,T]\cap\mathbb{Q}} Y^x(t,\omega). \tag{37}$$

The assumption that for all $t \in [0,T]$, $\omega \in \Omega$ it holds that $\mathbb{R}^d \ni x \mapsto Y^x(t,\omega) \in [0,\infty)$ is a continuous function and the fact that $\{z \in \mathbb{Q}^d \colon \|z\| \le R\}$ is dense in $\{z \in \mathbb{R}^d \colon \|z\| \le R\}$ therefore show that for all $\omega \in \Omega$ it holds that

$$\begin{aligned}
\left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]} Y^x(t,\omega)\right] &= \left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]\cap\mathbb{Q}} Y^x(t,\omega)\right] \\
&= \left[\sup_{(x,t)\in\{(z,s)\in\mathbb{R}^d\times([0,T]\cap\mathbb{Q})\colon \|z\|\le R\}} Y^x(t,\omega)\right] \\
&= \left[\sup_{t\in[0,T]\cap\mathbb{Q}} \sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} Y^x(t,\omega)\right] \\
&= \left[\sup_{t\in[0,T]\cap\mathbb{Q}} \sup_{x\in\{z\in\mathbb{Q}^d\colon \|z\|\le R\}} Y^x(t,\omega)\right] \\
&= \left[\sup_{(x,t)\in\{(z,s)\in\mathbb{Q}^d\times([0,T]\cap\mathbb{Q})\colon \|z\|\le R\}} Y^x(t,\omega)\right].
\end{aligned} \tag{38}$$

Hence, we obtain that for all $\omega \in \Omega$ it holds that

$$\left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]} Y^x(t,\omega)\right] = \left[\sup_{x\in\{z\in\mathbb{Q}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]\cap\mathbb{Q}} Y^x(t,\omega)\right]. \tag{39}$$

This establishes (i). In the next step we combine (38) and the fact that I is an at most countable set with Lemma 3.1 (with $(\Omega,\mathcal{F}) \leftarrow (\Omega,\mathcal{F})$, $I \leftarrow I$, $Y \leftarrow \big(\Omega \ni \omega \mapsto \sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]} Y^x(t,\omega) \in [0,\infty]\big)$, $(X_{(x,t)})_{(x,t)\in I} \leftarrow (\Omega \ni \omega \mapsto Y^x(t,\omega) \in [0,\infty))_{(x,t)\in I}$ in the notation of Lemma 3.1) to obtain (ii). The proof of Lemma 3.6 is thus completed. □

4 Measurability properties for solutions of SDEs

In this section we establish in Lemma 4.5 in Subsection 4.3 below the well-known fact that pathwise solutions of certain additive noise driven SDEs are stochastic processes. Our proof of Lemma 4.5 exploits the fact that Euler approximations converge pathwise to solutions of such SDEs (cf. Lemmas 4.2 and 4.3 in Subsection 4.1 and Lemma 4.4 in Subsection 4.2 below) as well as the elementary fact that the Euler approximations, in turn, are indeed stochastic processes. Our proof of the convergence statement for the Euler approximations in Lemma 4.2 exploits the familiar time-discrete Gronwall inequality in Lemma 4.1 in Subsection 4.1 below. For completeness we also include here detailed proofs for Lemmas 4.1–4.5.

4.1 Time-discrete approximations for deterministic differential equations (DEs)

Lemma 4.1. *Let $N \in \mathbb{N}$, $\beta \in [0,\infty)$, $\alpha \in \mathbb{R}$, $f_0, f_1, \dots, f_N \in \mathbb{R} \cup \{\infty\}$ satisfy for all $n \in \{0,1,\dots,N\}$ that*

$$f_n \leq \alpha + \beta \left(\sum_{k=0}^{n-1} f_k \right). \tag{40}$$

Then it holds for all $n \in \{0,1,\dots,N\}$ that

$$f_n \leq \alpha\,(1+\beta)^n \leq |\alpha|\, e^{\beta n} < \infty. \tag{41}$$

Proof of Lemma 4.1. Throughout this proof let $u_0, u_1, \dots, u_N \in \mathbb{R}$ be the real numbers which satisfy for all $n \in \{0,1,2,\dots,N\}$ that

$$u_n = \alpha + \beta \left(\sum_{k=0}^{n-1} u_k \right). \tag{42}$$

Hence, we obtain that for all $n \in \{0,1,\dots,N-1\}$ it holds that

$$u_{n+1} = \alpha + \beta \left(\sum_{k=0}^{n} u_k \right) = \underbrace{\alpha + \beta \left(\sum_{k=0}^{n-1} u_k \right)}_{=u_n} + \beta u_n = (1+\beta)\, u_n. \tag{43}$$

This implies that for all $n \in \{0,1,\dots,N\}$ it holds that

$$u_n = \alpha\,(1+\beta)^n\,. \tag{44}$$

Moreover, observe that induction shows that for all $n \in \{0,1,\dots,N\}$ it holds that

$$f_n \leq u_n. \tag{45}$$

Combining this with (44) establishes (41). This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $d \in \mathbb{N}$, $T \in [0,\infty)$, $f \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $r \in (0,\infty)$ that*

$$\sup_{t\in[0,T]} \sup_{\substack{x,y\in\mathbb{R}^d, x\neq y,\\ \|x\|+\|y\|\leq r}} \frac{\|f(t,x) - f(t,y)\|}{\|x-y\|} < \infty, \tag{46}$$

let $Y \in C([0,T], \mathbb{R}^d)$ satisfy for all $t \in [0,T]$ that $Y(t) = Y(0) + \int_0^t f(s, Y(s))\,\mathrm{d}s$, and let $\mathcal{Y}^N \colon \{0,1,\dots,N\} \to \mathbb{R}^d$, $N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$ that

$$\mathcal{Y}^N(0) = Y(0) \qquad \text{and} \qquad \mathcal{Y}^N(n+1) = \mathcal{Y}^N(n) + \tfrac{T}{N} f\big(\tfrac{nT}{N}, \mathcal{Y}^N(n)\big). \tag{47}$$

Then it holds that

$$\limsup_{N\to\infty} \left[\sup_{n\in\{0,1,\dots,N\}} \big\| \mathcal{Y}^N(n) - Y\big(\tfrac{nT}{N}\big) \big\| \right] = 0. \tag{48}$$

Proof of Lemma 4.2. Throughout this proof let $R \in [0, \infty)$ be the real number which satisfies

$$R = 2\Big[\sup_{t\in[0,T]} \|Y(t)\|\Big] + 1, \tag{49}$$

let $L \in [0, \infty)$ be the real number which satisfies

$$L = \left[\sup_{t\in[0,T]} \sup_{\substack{x,y\in\mathbb{R}^d,\, x\neq y,\\ \|x\|+\|y\|\leq R}} \frac{\|f(t,x) - f(t,y)\|}{\|x-y\|}\right] + \left[\sup_{s,t\in[0,T]} f(s, Y(t))\right], \tag{50}$$

let $\tau_N \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, be the numbers which satisfy for all $N \in \mathbb{N}$ that

$$\tau_N = \min\big(\{N\} \cup \big\{n \in \{0, 1, \ldots, N\} \colon \big\|\mathcal{Y}^N(n) - Y\big(\tfrac{nT}{N}\big)\big\| > 1\big\}\big), \tag{51}$$

and let $\alpha_N \in [0, \infty)$, $N \in \mathbb{N}$, be the real numbers which satisfy for all $N \in \mathbb{N}$ that

$$\alpha_N = \sum_{k=0}^{N-1} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} \big\|f(s, Y(s)) - f\big(\tfrac{kT}{N}, Y(s)\big)\big\| \,\mathrm{d}s. \tag{52}$$

Note that for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N\}$ it holds that

$$\begin{aligned}
&\big\|Y\big(\tfrac{nT}{N}\big) - \mathcal{Y}^N(n)\big\| \\
&= \left\|\left[Y(0) + \int_0^{\frac{nT}{N}} f(s, Y(s)) \,\mathrm{d}s\right] - \left[Y(0) + \frac{T}{N}\sum_{k=0}^{n-1} f\big(\tfrac{kT}{N}, \mathcal{Y}^N(k)\big)\right]\right\| \\
&= \left\|\int_0^{\frac{nT}{N}} f(s, Y(s)) \,\mathrm{d}s - \frac{T}{N}\sum_{k=0}^{n-1} f\big(\tfrac{kT}{N}, \mathcal{Y}^N(k)\big)\right\|.
\end{aligned} \tag{53}$$

Hence, we obtain that for all $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N\}$ it holds that

$$\begin{aligned}
&\|Y\big(\tfrac{nT}{N}\big) - \mathcal{Y}^N(n)\| \\
&= \left\| \sum_{k=0}^{n-1} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} f(s, Y(s))\,\mathrm{d}s - \sum_{k=0}^{n-1} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} f\big(\tfrac{kT}{N}, \mathcal{Y}^N(k)\big)\,\mathrm{d}s \right\| \\
&= \left\| \sum_{k=0}^{n-1} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} f(s, Y(s)) - f\big(\tfrac{kT}{N}, \mathcal{Y}^N(k)\big)\,\mathrm{d}s \right\| \\
&\le \sum_{k=0}^{n-1} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} \big\| f(s, Y(s)) - f\big(\tfrac{kT}{N}, \mathcal{Y}^N(k)\big) \big\|\,\mathrm{d}s \\
&\le \sum_{k=0}^{n-1} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} \big\| f(s, Y(s)) - f\big(\tfrac{kT}{N}, Y(s)\big) \big\| + \big\| f\big(\tfrac{kT}{N}, Y(s)\big) - f\big(\tfrac{kT}{N}, \mathcal{Y}^N(k)\big) \big\|\,\mathrm{d}s \\
&\le \alpha_N + \sum_{k=0}^{n-1} \int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} \big\| f\big(\tfrac{kT}{N}, Y(s)\big) - f\big(\tfrac{kT}{N}, \mathcal{Y}^N(k)\big) \big\|\,\mathrm{d}s.
\end{aligned} \tag{54}$$

Moreover, note that for all $N \in \mathbb{N}$, $k \in \{0, 1, \dots, N-1\} \cap [0, \tau_N)$, $s \in \big[\tfrac{kT}{N}, \tfrac{(k+1)T}{N}\big]$ it holds that

$$\begin{aligned}
\|Y(s) - \mathcal{Y}^N(k)\| &\le \big\|Y(s) - Y\big(\tfrac{kT}{N}\big)\big\| + \big\|Y\big(\tfrac{kT}{N}\big) - \mathcal{Y}^N(k)\big\| \\
&\le \|Y(s)\| + \big\|Y\big(\tfrac{kT}{N}\big)\big\| + \big\|Y\big(\tfrac{kT}{N}\big) - \mathcal{Y}^N(k)\big\| \\
&\le R.
\end{aligned} \tag{55}$$

Furthermore, note that for all $N \in \mathbb{N}$, $k \in \{0, 1, \dots, N-1\}$, $s \in \big[\tfrac{kT}{N}, \tfrac{(k+1)T}{N}\big]$ it holds that

$$\begin{aligned}
\big\|Y(s) - Y\big(\tfrac{kT}{N}\big)\big\| &= \left\| \left[Y(0) + \int_0^s f(u, Y(u))\,\mathrm{d}u \right] - \left[Y(0) + \int_0^{\frac{kT}{N}} f(u, Y(u))\,\mathrm{d}u \right] \right\| \\
&= \left\| \int_{\frac{kT}{N}}^s f(u, Y(u))\,\mathrm{d}u \right\| \\
&\le \int_{\frac{kT}{N}}^s \|f(u, Y(u))\|\,\mathrm{d}u \\
&\le L\big(s - \tfrac{kT}{N}\big) \\
&\le \frac{LT}{N}.
\end{aligned} \tag{56}$$

This and (55) imply that for all $N \in \mathbb{N}$, $k \in \{0, 1, \dots, N-1\} \cap [0, \tau_N)$, $s \in \big[\tfrac{kT}{N}, \tfrac{(k+1)T}{N}\big]$ it holds that

$$
\begin{aligned}
\big\|f\big(\tfrac{kT}{N},Y(s)\big)-f\big(\tfrac{kT}{N},\mathcal{Y}^N(k)\big)\big\| &\le L\|Y(s)-\mathcal{Y}^N(k)\| \\
&\le L\big\|Y(s)-Y\big(\tfrac{kT}{N}\big)\big\|+L\big\|Y\big(\tfrac{kT}{N}\big)-\mathcal{Y}^N(k)\big\| \\
&\le \frac{L^2T}{N}+L\big\|Y\big(\tfrac{kT}{N}\big)-\mathcal{Y}^N(k)\big\|.
\end{aligned}
\tag{57}
$$

Combining this with (54) shows that $N\in\mathbb{N}$, $n\in\{0,1,\dots,\tau_N\}$ it holds that

$$
\begin{aligned}
\big\|Y\big(\tfrac{nT}{N}\big)-\mathcal{Y}^N(n)\big\| &\le \alpha_N+\sum_{k=0}^{n-1}\left[\int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}}\frac{L^2T}{N}+L\big\|Y\big(\tfrac{kT}{N}\big)-\mathcal{Y}^N(k)\big\|\,\mathrm{d}s\right] \\
&= \alpha_N+\sum_{k=0}^{n-1}\left[\frac{L^2T^2}{N^2}+\frac{LT}{N}\big\|Y\big(\tfrac{kT}{N}\big)-\mathcal{Y}^N(k)\big\|\right] \\
&= \alpha_N+\frac{L^2T^2n}{N^2}+\frac{LT}{N}\left(\sum_{k=0}^{n-1}\big\|Y\big(\tfrac{kT}{N}\big)-\mathcal{Y}^N(k)\big\|\right) \\
&\le \alpha_N+\frac{L^2T^2}{N}+\frac{LT}{N}\left(\sum_{k=0}^{n-1}\big\|Y\big(\tfrac{kT}{N}\big)-\mathcal{Y}^N(k)\big\|\right).
\end{aligned}
\tag{58}
$$

Lemma 4.1 (with $N\leftarrow\tau_N$, $\beta\leftarrow\frac{LT}{N}$, $\alpha\leftarrow\alpha_N+\frac{L^2T^2}{N}$, $(f_n)_{n\in\{0,1,\dots,N\}}\leftarrow\big(\big\|Y\big(\frac{nT}{N}\big)-\mathcal{Y}^N(n)\big\|\big)_{n\in\{0,1,\dots,\tau_N\}}$ for $N\in\mathbb{N}$ in the notation of Lemma 4.1) hence establishes that for all $N\in\mathbb{N}$, $n\in\{0,1,\dots,\tau_N\}$ it holds that

$$
\big\|Y\big(\tfrac{nT}{N}\big)-\mathcal{Y}^N(n)\big\|\le\left|\alpha_N+\frac{L^2T^2}{N}\right|\exp\!\left(\frac{LTn}{N}\right)\le\left|\alpha_N+\frac{L^2T^2}{N}\right|\exp(LT). \tag{59}
$$

In the next step we observe that the fact that f is a continuous function ensures that there exist $\delta_\varepsilon\in(0,\infty)$, $\varepsilon\in(0,\infty)$, such that for all $\varepsilon\in(0,\infty)$, $s,t\in[0,T]$, $x\in\{z\in\mathbb{R}^d\colon\|z\|\le R\}$ with $|s-t|\le\delta_\varepsilon$ it holds that

$$
\|f(s,x)-f(t,x)\|<\varepsilon. \tag{60}
$$

Hence, it holds for all $\varepsilon\in(0,\infty)$, $N\in\mathbb{N}\cap[T/\delta_\varepsilon,\infty)$ that

$$
\alpha_N\le\sum_{k=0}^{N-1}\int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}}\varepsilon\,\mathrm{d}s=T\varepsilon. \tag{61}
$$

Therefore, we obtain that

$$
\limsup_{N\to\infty}\alpha_N=0. \tag{62}
$$

This and (59) prove that there exists $M \in \mathbb{N}$ such that for all $N \in \mathbb{N} \cap [M, \infty)$ it holds that

$$\left\| Y\left(\tfrac{\tau_N T}{N}\right) - \mathcal{Y}^N(\tau_N) \right\| < 1. \tag{63}$$

Combining this with (51) shows that for all $N \in \{M, M+1, \dots\}$ it holds that

$$\tau_N = N. \tag{64}$$

This and (59) show that for all $N \in \{M, M+1, \dots\}$ it holds that

$$\sup_{n \in \{0,1,\dots,N\}} \left\| Y\left(\tfrac{nT}{N}\right) - \mathcal{Y}^N(n) \right\| \leq \left| \alpha_N + \frac{L^2 T^2}{N} \right| \exp(LT). \tag{65}$$

Combining this with (62) proves (48). The proof of Lemma 4.2 is thus completed. □

Lemma 4.3. *Let $d \in \mathbb{N}$, $T \in [0, \infty)$, $f \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $w \in C([0,T], \mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, assume for all $r \in (0, \infty)$ that*

$$\sup_{t \in [0,T]} \sup_{\substack{x,y \in \mathbb{R}^d, x \neq y, \\ \|x\| + \|y\| \leq r}} \frac{\|f(t,x) - f(t,y)\|}{\|x - y\|} < \infty, \tag{66}$$

let $Y \in C([0,T], \mathbb{R}^d)$ satisfy for all $t \in [0,T]$ that $Y(t) = \xi + \int_0^t f(s, Y(s))\,\mathrm{d}s + w(t)$, and let $\mathcal{Y}^N \colon \{0, 1, \dots, N\} \to \mathbb{R}^d$, $N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N-1\}$ that $\mathcal{Y}^N(0) = Y(0)$ and

$$\mathcal{Y}^N(n+1) = \mathcal{Y}^N(n) + \tfrac{T}{N} f\left(\tfrac{nT}{N}, \mathcal{Y}^N(n)\right) - w\left(\tfrac{nT}{N}\right) + w\left(\tfrac{(n+1)T}{N}\right). \tag{67}$$

Then it holds that

$$\limsup_{N \to \infty} \left[\sup_{n \in \{0,1,\dots,N\}} \left\| \mathcal{Y}^N(n) - Y\left(\tfrac{nT}{N}\right) \right\| \right] = 0. \tag{68}$$

Proof of Lemma 4.3. Throughout this proof let $Z \colon [0,T] \to \mathbb{R}^d$ be the function which satisfies for all $t \in [0,T]$ that

$$Z(t) = Y(t) - w(t), \tag{69}$$

let $g \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be the function which satisfies for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$g(t,x) = f(t, x + w(t)), \tag{70}$$

let $\mathcal{Z}^N \colon \{0, 1, \dots, N\} \to \mathbb{R}^d$, $N \in \mathbb{N}$, be the functions which satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N\}$ that

$$\mathcal{Z}^N(n) = \mathcal{Y}^N(n) - w\left(\tfrac{nT}{N}\right), \tag{71}$$

let $C_{t,r} \in [0,\infty]$, $t \in [0,T]$, $r \in (0,\infty)$, satisfy for all $t \in [0,T]$, $r \in (0,\infty)$ that

$$C_{t,r} = \sup_{\substack{x,y\in\mathbb{R}^d,\, x\neq y,\\ \|x\|+\|y\|\leq r}} \frac{\|f(t,x) - f(t,y)\|}{\|x-y\|}, \tag{72}$$

and let $K \in [0,\infty]$ satisfy

$$K = \sup_{t\in[0,T]} \|w(t)\|. \tag{73}$$

Note that (66) and the hypothesis that w is a continuous function imply that for all $r \in (0,\infty)$ it holds that

$$\sup_{t\in[0,T]} C_{t,r} < \infty \qquad \text{and} \qquad K < \infty. \tag{74}$$

In addition, observe that the hypothesis that Y, w, and f are continuous functions shows that

$$Z \in C([0,T],\mathbb{R}^d) \qquad \text{and} \qquad g \in C([0,T]\times\mathbb{R}^d,\mathbb{R}^d). \tag{75}$$

Moreover, note that the triangle inequality ensures that for all $t \in [0,T]$, $r \in (0,\infty)$, $x,y \in \mathbb{R}^d$ with $\|x\|+\|y\| \leq r$ it holds that

$$\|x + w(t)\| + \|y + w(t)\| \leq \|x\| + \|w(t)\| + \|y\| + \|w(t)\| \leq r + 2K. \tag{76}$$

This demonstrates that for all $t \in [0,T]$, $r \in (0,\infty)$, $x,y \in \mathbb{R}^d$ with $x \neq y$ and $\|x\|+\|y\| \leq r$ it holds that

$$\begin{aligned} \frac{\|g(t,x) - g(t,y)\|}{\|x-y\|} &= \frac{\|f(t,x+w(t)) - f(t,y+w(t))\|}{\|(x+w(t)) - (y+w(t))\|} \\ &\leq \sup_{\substack{u,v\in\mathbb{R}^d,\, u\neq v,\\ \|u\|+\|v\|\leq r+2K}} \frac{\|f(t,u) - f(t,v)\|}{\|u-v\|} \\ &= C_{t,r+2K}. \end{aligned} \tag{77}$$

Combining this with (74) ensures that for all $r \in (0,\infty)$ it holds that

$$\sup_{t\in[0,T]} \sup_{\substack{x,y\in\mathbb{R}^d,\, x\neq y,\\ \|x\|+\|y\|\leq r}} \frac{\|g(t,x) - g(t,y)\|}{\|x-y\|} \leq \sup_{t\in[0,T]} C_{t,r+2K} < \infty. \tag{78}$$

Moreover, observe that the hypothesis that for all $t \in [0,T]$ it holds that $Y(t) = \xi + \int_0^t f(s,Y(s))\,\mathrm{d}s + w(t)$, (69), (70), and (75) show that for all $t \in [0,T]$ it holds that

$$Z(t) = \xi + \int_0^t f(s,Y(s))\,\mathrm{d}s = \xi + \int_0^t f(s,Z(s)+w(s))\,\mathrm{d}s = Z(0) + \int_0^t g(s,Z(s))\,\mathrm{d}s. \tag{79}$$

Next we combine (67), (70), and (71) to obtain that for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$ it holds that

$$\begin{aligned} \mathcal{Z}^N(n+1) &= \mathcal{Y}^N(n+1) - w\big(\tfrac{(n+1)T}{N}\big) \\ &= \Big[\mathcal{Y}^N(n) + \tfrac{T}{N} f\big(\tfrac{nT}{N}, \mathcal{Y}^N(n)\big) - w\big(\tfrac{nT}{N}\big) + w\big(\tfrac{(n+1)T}{N}\big)\Big] - w\big(\tfrac{(n+1)T}{N}\big) \\ &= \mathcal{Y}^N(n) - w\big(\tfrac{nT}{N}\big) + \tfrac{T}{N} f\big(\tfrac{nT}{N}, \mathcal{Y}^N(n)\big) \\ &= \mathcal{Z}^N(n) + \tfrac{T}{N} f\big(\tfrac{nT}{N}, \mathcal{Z}^N(n) + w\big(\tfrac{nT}{N}\big)\big) \\ &= \mathcal{Z}^N(n) + \tfrac{T}{N} g\big(\tfrac{nT}{N}, \mathcal{Z}^N(n)\big). \end{aligned} \tag{80}$$

Furthermore, observe that the assumption that for all $N \in \mathbb{N}$ it holds that $\mathcal{Y}^N(0) = Y(0)$ ensures that for all $N \in \mathbb{N}$ it holds that

$$\mathcal{Z}^N(0) = \mathcal{Y}^N(0) - w(0) = Y(0) - w(0) = Z(0). \tag{81}$$

Combining this, (75), (78), (79), and (80) with Lemma 4.2 (with $d \leftarrow d$, $T \leftarrow T$, $f \leftarrow g$, $\|\cdot\| \leftarrow \|\cdot\|$, $Y \leftarrow Z$, $(\mathcal{Y}^N)_{N\in\mathbb{N}} \leftarrow (\mathcal{Z}^N)_{N\in\mathbb{N}}$ in the notation of Lemma 4.2) shows that

$$\limsup_{N\to\infty} \left[\sup_{n\in\{0,1,\ldots,N\}} \big\|\mathcal{Z}^N(n) - Z\big(\tfrac{nT}{N}\big)\big\| \right] = 0. \tag{82}$$

Moreover, note that for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N\}$ it holds that

$$\big\|\mathcal{Y}^N(n) - Y\big(\tfrac{nT}{N}\big)\big\| = \big\|\big[\mathcal{Z}^N(n) + w\big(\tfrac{nT}{N}\big)\big] - \big[Z\big(\tfrac{nT}{N}\big) + w\big(\tfrac{nT}{N}\big)\big]\big\| = \big\|\mathcal{Z}^N(n) - Z\big(\tfrac{nT}{N}\big)\big\|. \tag{83}$$

Combining this with (82) establishes that

$$\limsup_{N\to\infty} \left[\sup_{n\in\{0,1,\ldots,N\}} \big\|\mathcal{Y}^N(n) - Y\big(\tfrac{nT}{N}\big)\big\| \right] = 0. \tag{84}$$

This completes the proof of Lemma 4.3. ◻

4.2 Time-continuous approximations for deterministic differential equations

Lemma 4.4. *Let $d \in \mathbb{N}$, $T \in (0, \infty)$, $Y \in C([0,T], \mathbb{R}^d)$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, let $\mathcal{Y}^N \colon [0,T] \to \mathbb{R}^d$, $N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N-1\}$, $t \in \big[\tfrac{nT}{N}, \tfrac{(n+1)T}{N}\big]$ that*

$$\mathcal{Y}^N(t) = \big(1 - \big(\tfrac{tN}{T} - n\big)\big)\mathcal{Y}^N\big(\tfrac{nT}{N}\big) + \big(\tfrac{tN}{T} - n\big)\mathcal{Y}^N\big(\tfrac{(n+1)T}{N}\big), \tag{85}$$

and assume that

$$\limsup_{N\to\infty} \left[\sup_{n\in\{0,1,\ldots,N\}} \big\|\mathcal{Y}^N\big(\tfrac{nT}{N}\big) - Y\big(\tfrac{nT}{N}\big)\big\| \right] = 0. \tag{86}$$

Then it holds that

$$\limsup_{N\to\infty} \left[\sup_{t\in[0,T]} \|\mathcal{Y}^N(t) - Y(t)\| \right] = 0. \tag{87}$$

Proof of Lemma 4.4. First, note that (86) ensures that there exist $M_\varepsilon \in \mathbb{N}$, $\varepsilon \in (0,\infty)$, such that for all $N \in \mathbb{N} \cap [M_\varepsilon, \infty)$ it holds that

$$\sup_{n \in \{0,1,\dots,N\}} \big\| \mathcal{Y}^N\big(\tfrac{nT}{N}\big) - Y\big(\tfrac{nT}{N}\big) \big\| < \varepsilon. \tag{88}$$

Next observe that the hypothesis that $Y\colon [0,T] \to \mathbb{R}^d$ is a continuous function implies that Y is a uniformly continuous function. This ensures that there exist $\delta_\varepsilon \in (0,\infty)$, $\varepsilon \in (0,\infty)$, such that for all $\varepsilon \in (0,\infty)$, $s,t \in [0,T]$ with $|s-t| \le \delta_\varepsilon$ it holds that

$$\|Y(s) - Y(t)\| < \varepsilon. \tag{89}$$

In the next step we observe that for all $\varepsilon \in (0,\infty)$, $N \in \mathbb{N} \cap [T/\delta_{\varepsilon/2}, \infty)$, $n \in \{0,1\dots,N-1\}$, $t \in \big[\tfrac{nT}{N}, \tfrac{(n+1)T}{N}\big]$, $k \in \{n, n+1\}$ it holds that

$$\big|t - \tfrac{kT}{N}\big| \le \tfrac{T}{N} \le \delta_{\varepsilon/2}. \tag{90}$$

This, (88), (89), and the triangle inequality show that for all $\varepsilon \in (0,\infty)$, $N \in \mathbb{N} \cap [\max\{T/\delta_{\varepsilon/2}, M_{\varepsilon/2}\}, \infty)$, $n \in \{0,1,\dots,N-1\}$, $t \in \big[\tfrac{nT}{N}, \tfrac{(n+1)T}{N}\big]$, $k \in \{n, n+1\}$ it holds that

$$\big\|\mathcal{Y}^N\big(\tfrac{kT}{N}\big) - Y(t)\big\| \le \big\|\mathcal{Y}^N\big(\tfrac{kT}{N}\big) - Y\big(\tfrac{kT}{N}\big)\big\| + \big\|Y\big(\tfrac{kT}{N}\big) - Y(t)\big\| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \tag{91}$$

The triangle inequality and (85) hence demonstrate that for all $\varepsilon \in (0,\infty)$, $N \in \mathbb{N} \cap [\max\{T/\delta_{\varepsilon/2}, M_{\varepsilon/2}\}, \infty)$, $n \in \{0,1,\dots,N-1\}$, $t \in \big[\tfrac{nT}{N}, \tfrac{(n+1)T}{N}\big]$ it holds that

$$\begin{aligned}
&\|\mathcal{Y}^N(t) - Y(t)\| \\
&= \big\|\big[\big(1 - \big(\tfrac{tN}{T} - n\big)\big)\mathcal{Y}^N\big(\tfrac{nT}{N}\big) + \big(\tfrac{tN}{T} - n\big)\mathcal{Y}^N\big(\tfrac{(n+1)T}{N}\big)\big] - Y(t)\big\| \\
&= \big\|\big(1 - \big(\tfrac{tN}{T} - n\big)\big)\big(\mathcal{Y}^N\big(\tfrac{nT}{N}\big) - Y(t)\big) + \big(\tfrac{tN}{T} - n\big)\big(\mathcal{Y}^N\big(\tfrac{(n+1)T}{N}\big) - Y(t)\big)\big\| \\
&\le \big\|\big(1 - \big(\tfrac{tN}{T} - n\big)\big)\big(\mathcal{Y}^N\big(\tfrac{nT}{N}\big) - Y(t)\big)\big\| + \big\|\big(\tfrac{tN}{T} - n\big)\big(\mathcal{Y}^N\big(\tfrac{(n+1)T}{N}\big) - Y(t)\big)\big\| \\
&= \big(1 - \big(\tfrac{tN}{T} - n\big)\big)\big\|\mathcal{Y}^N\big(\tfrac{nT}{N}\big) - Y(t)\big\| + \big(\tfrac{tN}{T} - n\big)\big\|\mathcal{Y}^N\big(\tfrac{(n+1)T}{N}\big) - Y(t)\big\| \\
&< \big(1 - \big(\tfrac{tN}{T} - n\big)\big)\varepsilon + \big(\tfrac{tN}{T} - n\big)\varepsilon \\
&= \varepsilon.
\end{aligned} \tag{92}$$

Therefore, we obtain that for all $\varepsilon \in (0,\infty)$, $N \in \mathbb{N} \cap [\max\{T/\delta_{\varepsilon/2}, M_{\varepsilon/2}\}, \infty)$ it holds that

$$\sup_{t \in [0,T]} \|\mathcal{Y}^N(t) - Y(t)\| \le \varepsilon. \tag{93}$$

This establishes (87). The proof of Lemma 4.4 is thus completed. □

4.3 Measurability properties for solutions of SDEs

Lemma 4.5. *Let $d \in \mathbb{N}$, $T \in [0,\infty)$, $f \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}^d$ be a stochastic process with continuous sample paths, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $r \in (0,\infty)$ that*

$$\sup_{t\in[0,T]} \sup_{\substack{x,y\in\mathbb{R}^d, x\neq y,\\ \|x\|+\|y\|\leq r}} \frac{\|f(t,x)-f(t,y)\|}{\|x-y\|} < \infty, \tag{94}$$

and let $Y \colon [0,T] \times \Omega \to \mathbb{R}^d$ satisfy for all $t \in [0,T]$, $\omega \in \Omega$ that $\big([0,T] \ni s \mapsto Y(s,\omega) \in \mathbb{R}^d\big) \in C([0,T], \mathbb{R}^d)$ and

$$Y(t,\omega) = \xi + \int_0^t f(s, Y(s,\omega))\,\mathrm{d}s + W(t,\omega). \tag{95}$$

Then it holds that Y is a stochastic process.

Proof of Lemma 4.5. Throughout this proof assume w.l.o.g. that $T > 0$, let $\mathcal{Z}^N \colon \{0,1,\dots,N\} \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be the sequence of functions which satisfies for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $\omega \in \Omega$ that $\mathcal{Z}^N(0,\omega) = Y(0,\omega)$ and

$$\mathcal{Z}^N(n+1,\omega) = \mathcal{Z}^N(n,\omega) + \tfrac{T}{N} f\big(\tfrac{nT}{N}, \mathcal{Z}^N(n,\omega)\big) - W\big(\tfrac{nT}{N},\omega\big) + W\big(\tfrac{(n+1)T}{N},\omega\big), \tag{96}$$

and let $\mathcal{Y}^N \colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be the sequence of functions which satisfies for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $\omega \in \Omega$ that $\mathcal{Y}^N(0,\omega) = \mathcal{Z}^N(0,\omega)$ and

$$\mathcal{Y}^N(t,\omega) = \big(1 - \big(\tfrac{tN}{T} - n\big)\big)\mathcal{Z}^N(n,\omega) + \big(\tfrac{tN}{T} - n\big)\mathcal{Z}^N(n+1,\omega). \tag{97}$$

Note that (94), (95), (96), and Lemma 4.3 (with $d \leftarrow d$, $T \leftarrow T$, $f \leftarrow f$, $w \leftarrow \big([0,T] \ni t \mapsto W(t,\omega) \in \mathbb{R}^d\big)$, $\xi \leftarrow \xi$, $\|\cdot\| \leftarrow \|\cdot\|$, $Y \leftarrow \big([0,T] \ni t \mapsto Y(t,\omega) \in \mathbb{R}^d\big)$, $(\mathcal{Y}^N)_{N\in\mathbb{N}} \leftarrow \big(\{0,1,\dots,N\} \ni n \mapsto \mathcal{Z}^N(n,\omega) \in \mathbb{R}^d\big)_{N\in\mathbb{N}}$ for $\omega \in \Omega$ in the notation of Lemma 4.3) show that for all $\omega \in \Omega$ it holds that

$$\limsup_{N\to\infty} \Big[\sup_{n\in\{0,1,\dots,N\}} \big\|\mathcal{Z}^N(n,\omega) - Y\big(\tfrac{nT}{N},\omega\big)\big\|\Big] = 0. \tag{98}$$

In the next step we observe that (97) implies that for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$, $\omega \in \Omega$ it holds that

$$\begin{aligned} \mathcal{Y}^N\big(\tfrac{(n+1)T}{N},\omega\big) &= \big(1 - \big((n+1) - n\big)\big)\mathcal{Z}^N(n,\omega) + \big((n+1) - n\big)\mathcal{Z}^N(n+1,\omega) \\ &= \mathcal{Z}^N(n+1,\omega). \end{aligned} \tag{99}$$

Combining this with the fact that for all $N \in \mathbb{N}$, $\omega \in \Omega$ it holds that $\mathcal{Y}^N(0,\omega) = \mathcal{Z}^N(0,\omega)$ shows that for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N\}$, $\omega \in \Omega$ it holds that

$$\mathcal{Y}^N\big(\tfrac{nT}{N},\omega\big) = \mathcal{Z}^N(n,\omega). \tag{100}$$

This and (97) prove that for all $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N-1\}$, $t \in \big[\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $\omega \in \Omega$ it holds that $\mathcal{Y}^N(0, \omega) = \mathcal{Z}^N(0, \omega)$ and

$$\mathcal{Y}^N(t, \omega) = \big(1 - \big(\tfrac{tN}{T} - n\big)\big)\mathcal{Y}^N\big(\tfrac{nT}{N}, \omega\big) + \big(\tfrac{tN}{T} - n\big)\mathcal{Y}^N\big(\tfrac{(n+1)T}{N}, \omega\big). \tag{101}$$

Combining this, (98), and (100), with Lemma 4.4 (with $d \leftarrow d$, $T \leftarrow T$, $Y \leftarrow ([0, T] \ni t \mapsto Y(t, \omega) \in \mathbb{R}^d)$, $\|\cdot\| \leftarrow \|\cdot\|$, $(\mathcal{Y}^N)_{N \in \mathbb{N}} \leftarrow ([0, T] \ni t \mapsto \mathcal{Y}^N(t, \omega) \in \mathbb{R}^d)_{N \in \mathbb{N}}$ for $\omega \in \Omega$ in the notation of Lemma 4.4) establishes that for all $\omega \in \Omega$ it holds that

$$\limsup_{N \to \infty} \left[\sup_{t \in [0, T]} \|\mathcal{Y}^N(t, \omega) - Y(t, \omega)\| \right] = 0. \tag{102}$$

Next observe that the assumption that $\Omega \ni \omega \mapsto W(0, \omega) \in \mathbb{R}^d$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function and the fact that for all $N \in \mathbb{N}$, $\omega \in \Omega$ it holds that $\mathcal{Z}^N(0, \omega) = \xi + W(0, \omega)$ imply that for all $N \in \mathbb{N}$ it holds that

$$\Omega \ni \omega \mapsto \mathcal{Z}^N(0, \omega) \in \mathbb{R}^d \tag{103}$$

is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. Furthermore, observe that (96), the hypothesis that f is a continuous function, and the hypothesis that for all $t \in [0, T]$ it holds that $\Omega \ni \omega \mapsto W(t, \omega) \in \mathbb{R}^d$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function show that for all $N \in \mathbb{N}$, $n \in \{m \in \{0, 1, \dots, N-1\} \colon (\Omega \ni \omega \mapsto \mathcal{Z}^N(m, \omega) \in \mathbb{R}^d)$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function$\}$ it holds that

$$\Omega \ni \omega \mapsto \mathcal{Z}^N(n + 1, \omega) \in \mathbb{R}^d \tag{104}$$

is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. Combining this and (103) with the induction principle proves that for all $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N\}$ it holds that

$$\Omega \ni \omega \mapsto \mathcal{Z}^N(n, \omega) \in \mathbb{R}^d \tag{105}$$

is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. This and (97) demonstrate that for all $N \in \mathbb{N}$, $t \in [0, T]$ it holds that $\Omega \ni \omega \mapsto \mathcal{Y}^N(t, \omega) \in \mathbb{R}^d$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. Combining this and (102) with Lemma 3.3 (with $(\Omega, \mathcal{F}) \leftarrow (\Omega, \mathcal{F})$, $d \leftarrow d$, $\|\cdot\| \leftarrow \|\cdot\|$, $Y \leftarrow (\Omega \ni \omega \mapsto Y(t, \omega) \in \mathbb{R}^d)$, $(X_n)_{n \in \mathbb{N}} \leftarrow (\Omega \ni \omega \mapsto \mathcal{Y}^N(t, \omega) \in \mathbb{R}^d)_{N \in \mathbb{N}}$ for $t \in [0, T]$ in the notation of Lemma 3.3) ensures that for all $t \in [0, T]$ it holds that

$$\Omega \ni \omega \mapsto Y(t, \omega) \in \mathbb{R}^d \tag{106}$$

is an $\mathcal{F}/\mathcal{B}(\mathbb{R}^d)$-measurable function. The proof of Lemma 4.5 is thus completed. □

5 Differentiability with respect to the initial value for SDEs

In this section we establish in Lemma 5.4 in Subsection 5.3 below an existence, uniqueness, and regularity result for solutions of certain additive noise driven SDEs. Our proof of Lemma 5.4 exploits the related regularity results for solutions of certain ODEs in Lemmas 5.1–5.3 below. For the reader's convenience we include in this section also detailed proofs for Lemmas 5.1–5.4.

5.1 Local Lipschitz continuity for deterministic DEs

Lemma 5.1. *Let $d \in \mathbb{N}$, $w \in \mathbb{R}^d$, $T \in [0,\infty)$, $f \in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm, and let $y^x \in C([0,T],\mathbb{R}^d)$, $x \in \mathbb{R}^d$, be functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$ that*

$$y^x(t) = x + \int_0^t f(s, y^x(s))\,\mathrm{d}s. \tag{107}$$

Then there exist $r, L \in (0,\infty)$ such that for all $v \in \{u \in \mathbb{R}^d \colon \|u - w\| \le r\}$, $t \in [0,T]$ it holds that

$$\|y^v(t) - y^w(t)\| \le L\|v - w\|. \tag{108}$$

Proof of Lemma 5.1. Throughout this proof assume w.l.o.g. that $T > 0$, let $D\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be the function which satisfies for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$D(t,x) = \tfrac{\partial}{\partial x} f(t,x), \tag{109}$$

let $R \in (0,\infty)$ be the real number which satisfies

$$R = 1 + \left[\sup_{t\in[0,T]} \|y^w(t) - w\|\right], \tag{110}$$

let $K \in [0,\infty]$ satisfy

$$K = \left[\sup_{x\in\{u\in\mathbb{R}^d\colon \|u-w\|\le R\}} \sup_{t\in[0,T]} \sup_{h\in\mathbb{R}^d\setminus\{0\}} \left(\frac{\|D(t,x)h\|}{\|h\|}\right)\right], \tag{111}$$

and let $\tau_x \in [0,T]$, $x \in \mathbb{R}^d$, be the real numbers which satisfy for all $x \in \mathbb{R}^d$ that

$$\tau_x = \inf(\{T\} \cup \{t \in [0,T] \colon \|y^x(t) - w\| \ge R\}). \tag{112}$$

Note that

$$\begin{aligned} K &= \left[\sup_{x\in\{u\in\mathbb{R}^d\colon \|u-w\|\le R\}} \sup_{t\in[0,T]} \sup_{h\in\mathbb{R}^d\setminus\{0\}} \left\| D(t,x)\left(\frac{h}{\|h\|}\right)\right\|\right] \\ &= \sup_{\substack{(x,t,v)\in\mathbb{R}^d\times[0,T]\times\mathbb{R}^d,\\ \|x-w\|\le R,\, \|v\|=1}} \|D(t,x)v\|. \end{aligned} \tag{113}$$

In addition, observe that the hypothesis that $f \in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ implies that $\mathbb{R}^d \times [0,T] \times \mathbb{R}^d \ni (x,t,v) \mapsto D(t,x)v \in \mathbb{R}^d$ is a continuous function. Combining this with (113) establishes that

$$K < \infty. \tag{114}$$

The fundamental theorem of calculus hence implies that for all $t \in [0,T]$, $u, v \in \{z \in \mathbb{R}^d \colon \|z - w\| \leq R\}$ it holds that

$$\begin{aligned}
\|f(t,u) - f(t,v)\| &= \big\| \big[f(t, v + s(u-v)) \big]_{s=0}^{s=1} \big\| \\
&= \left\| \int_0^1 D(t, v + s(u-v))(u-v) \, \mathrm{d}s \right\| \\
&\leq \int_0^1 \|D(t, v + s(u-v))(u-v)\| \, \mathrm{d}s \\
&\leq \int_0^1 K\|u-v\| \, \mathrm{d}s \\
&= K\|u-v\|.
\end{aligned} \tag{115}$$

Moreover, note that (112) ensures that for all $v \in \mathbb{R}^d$, $t \in [0,T] \cap (-\infty, \tau_v)$ it holds that

$$\|y^v(t) - w\| \leq R. \tag{116}$$

The triangle inequality, (107), and (115) hence show that for all $v \in \mathbb{R}^d$, $t \in [0,T] \cap (-\infty, \tau_v)$ it holds that

$$\begin{aligned}
\|y^v(t) - y^w(t)\| &= \left\| \left[v + \int_0^t f(s, y^v(s)) \, \mathrm{d}s \right] - \left[w + \int_0^t f(s, y^w(s)) \, \mathrm{d}s \right] \right\| \\
&= \left\| v - w + \int_0^t f(s, y^v(s)) - f(s, y^w(s)) \, \mathrm{d}s \right\| \\
&\leq \|v - w\| + \left\| \int_0^t f(s, y^v(s)) - f(s, y^w(s)) \, \mathrm{d}s \right\| \\
&\leq \|v - w\| + \int_0^t \|f(s, y^v(s)) - f(s, y^w(s))\| \, \mathrm{d}s \\
&\leq \|v - w\| + \int_0^t K\|y^v(s) - y^w(s)\| \, \mathrm{d}s.
\end{aligned} \tag{117}$$

Gronwall's integral inequality (see, e.g., Grohs et al. [6, Lemma 2.11] (with $\alpha \leftarrow \|v - w\|$, $\beta \leftarrow K$, $T \leftarrow t$, $f \leftarrow ([0,t] \ni s \mapsto \|y^v(s) - y^w(s)\| \in [0,\infty))$ for $v \in \mathbb{R}^d$, $t \in [0,T] \cap (-\infty, \tau_v)$ in the notation of Grohs et al. [6, Lemma 2.11])) and (114) hence establish that for all $v \in \mathbb{R}^d$, $t \in [0,T] \cap (-\infty, \tau_v)$ it holds that

$$\|y^v(t) - y^w(t)\| \leq \|v - w\| \exp(Kt) \leq \|v - w\| \exp(KT). \tag{118}$$

The triangle inequality therefore proves that for all $v \in \mathbb{R}^d$, $t \in [0,T] \cap (-\infty, \tau_v)$ with $\|v - w\| < (2\exp(KT))^{-1}$ it holds that

$$\begin{aligned}
\|y^v(t) - w\| &\leq \|y^v(t) - y^w(t)\| + \|y^w(t) - w\| \\
&\leq \|v - w\| \exp(KT) + R - 1 \\
&\leq R - \frac{1}{2}.
\end{aligned} \tag{119}$$

In addition, observe that for all $v \in \mathbb{R}^d$ with $\|v - w\| < (2\exp(KT))^{-1}$ it holds that

$$\|y^v(0) - w\| = \|v - w\| < 1 \leq R. \tag{120}$$

This, the assumption that $T > 0$, and the hypothesis that for all $x \in \mathbb{R}^d$ it holds that y^x is a continuous function ensure that for all $v \in \mathbb{R}^d$ with $\|v - w\| < (2\exp(KT))^{-1}$ it holds that

$$\tau_v > 0. \tag{121}$$

Combining this and the hypothesis that for all $x \in \mathbb{R}^d$ it holds that y^x is a continuous function with (119) implies that for all $v \in \mathbb{R}^d$ with $\|v - w\| < (2\exp(KT))^{-1}$ it holds that

$$\tau_v = T. \tag{122}$$

The fact that for all $x \in \mathbb{R}^d$ it holds that y^x is a continuous function and (118) therefore ensure that for all $t \in [0, T]$, $v \in \mathbb{R}^d$ with $\|v - w\| < (2\exp(KT))^{-1}$ it holds that

$$\|y^v(t) - y^w(t)\| \leq \|v - w\| \exp(KT). \tag{123}$$

The proof of Lemma 5.1 is thus completed. □

5.2 Differentiability with respect to the initial value for deterministic DEs

Lemma 5.2. *Let $d \in \mathbb{N}$, $T \in [0, \infty)$, $f \in C^{0,1}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$, let $D\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ be a function, and let $y^x \in C([0, T], \mathbb{R}^d)$, $x \in \mathbb{R}^d$, be functions which satisfy for all $t \in [0, T]$, $x \in \mathbb{R}^d$ that $D(t, x) = \frac{\partial}{\partial x} f(t, x)$ and*

$$y^x(t) = x + \int_0^t f(s, y^x(s))\, \mathrm{d}s. \tag{124}$$

Then

(i) it holds that $([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto y^x(t) \in \mathbb{R}^d) \in C^{0,1}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ and

(ii) it holds for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$ that

$$\big(\tfrac{\partial}{\partial x} y^x(t)\big)(h) = h + \int_0^t D(s, y^x(s))\big(\big(\tfrac{\partial}{\partial x} y^x(s)\big)(h)\big)\, \mathrm{d}s. \tag{125}$$

Proof of Lemma 5.2. Throughout this proof let $\|\cdot\|\colon \mathbb{R}^d \to [0, \infty)$ be the d-dimensional Euclidean norm, let $C_r \in [0, \infty)$, $r \in [0, \infty)$, be the real numbers which satisfy for all $r \in [0, \infty)$ that

$$C_r = \left[\sup_{t \in [0,T]} \sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq r\}} \|f(t, x)\|\right], \tag{126}$$

let $R_x \in [0,\infty)$, $x \in \mathbb{R}^d$, be the real numbers which satisfy for all $x \in \mathbb{R}^d$ that

$$R_x = \sup_{t\in[0,T]} \|y^x(t)\|, \tag{127}$$

and let $K_r \in [0,\infty]$, $r \in [0,\infty)$, satisfy for all $r \in [0,\infty)$ that

$$K_r = \left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le r\}} \sup_{t\in[0,T]} \sup_{h\in\mathbb{R}^d\setminus\{0\}} \left(\frac{\|D(t,x)h\|}{\|h\|} \right) \right]. \tag{128}$$

Observe that for all $r \in [0,\infty)$ it holds that

$$\begin{aligned} K_r &= \left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le r\}} \sup_{t\in[0,T]} \sup_{h\in\mathbb{R}^d\setminus\{0\}} \left\| D(t,x)\left(\frac{h}{\|h\|}\right) \right\| \right] \\ &= \sup_{\substack{(x,t,w)\in\mathbb{R}^d\times[0,T]\times\mathbb{R}^d,\\ \|x\|\le r,\, \|w\|=1}} \|D(t,x)w\|. \end{aligned} \tag{129}$$

In addition, observe that the fact that $f \in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ implies that $\mathbb{R}^d\times[0,T]\times\mathbb{R}^d \ni (x,t,w) \mapsto D(t,x)w \in \mathbb{R}^d$ is a continuous function. Combining this with (129) establishes that for all $r \in [0,\infty)$ it holds that

$$K_r < \infty. \tag{130}$$

Note that Lemma 5.1 proves that there exist $L_w, r_w \in (0,\infty)$, $w \in \mathbb{R}^d$, such that for all $v, w \in \mathbb{R}^d$, $t \in [0,T]$ with $\|v - w\| < r_w$ it holds that

$$\|y^v(t) - y^w(t)\| \le L_w\|v - w\|. \tag{131}$$

In the next step we observe that (124) implies that for all $w \in \mathbb{R}^d$, $t \in [0,T]$, $u \in [0,t]$ it holds that

$$\begin{aligned} \|y^w(t) - y^w(u)\| &= \left\| \left[w + \int_0^t f(s, y^w(s))\,\mathrm{d}s \right] - \left[w + \int_0^u f(s, y^w(s))\,\mathrm{d}s \right] \right\| \\ &= \left\| \int_u^t f(s, y^w(s))\,\mathrm{d}s \right\| \\ &\le \int_u^t \|f(s, y^w(s))\|\,\mathrm{d}s \\ &\le \int_u^t C_{R_w}\,\mathrm{d}s \\ &= (t-u)C_{R_w}. \end{aligned} \tag{132}$$

This, (131), and the triangle inequality hence prove that for all $v, w \in \mathbb{R}^d$, $t, u \in [0,T]$ with $\|v - w\| < r_w$ it holds that

$$\|y^v(t) - y^w(u)\| \le \|y^v(t) - y^w(t)\| + \|y^w(t) - y^w(u)\| \le L_w\|v - w\| + C_{R_w}|t - u|. \tag{133}$$

Therefore, we obtain that for all $v, w \in \mathbb{R}^d$, $t, u \in [0, T]$, $\varepsilon \in (0, \infty)$ with $\|v - w\| < \min\{r_w, (2L_w)^{-1}\varepsilon\}$ and $|t - u| < (2C_{R_w} + 1)^{-1}\varepsilon$ it holds that

$$\|y^v(t) - y^w(u)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{134}$$

This establishes that

$$[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto y^x(t) \in \mathbb{R}^d \tag{135}$$

is a continuous function. Next note that there exist unique $v^{x,h} \in C([0, T], \mathbb{R}^d)$, $x, h \in \mathbb{R}^d$, such that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$v^{x,h}(t) = h + \int_0^t D(s, y^x(s))(v^{x,h}(s)) \, \mathrm{d}s. \tag{136}$$

This implies that for all $x, h, k \in \mathbb{R}^d$, $\lambda, \mu \in \mathbb{R}$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\lambda v^{x,h}(t) + \mu v^{x,k}(t) \\
&= \lambda \left[h + \int_0^t D(s, y^x(s))(v^{x,h}(s)) \, \mathrm{d}s \right] + \mu \left[k + \int_0^t D(s, y^x(s))(v^{x,k}(s)) \, \mathrm{d}s \right] \\
&= \lambda h + \mu k + \left[\int_0^t \lambda D(s, y^x(s))(v^{x,h}(s)) + \mu D(s, y^x(s))(v^{x,k}(s)) \, \mathrm{d}s \right] \\
&= \lambda h + \mu k + \left[\int_0^t D(s, y^x(s)) \big(\lambda v^{x,h}(s) + \mu v^{x,k}(s) \big) \, \mathrm{d}s \right].
\end{aligned} \tag{137}$$

Combining this with (136) proves that for all $x, h, k \in \mathbb{R}^d$, $\lambda, \mu \in \mathbb{R}$, $t \in [0, T]$ it holds that

$$v^{x, \lambda h + \mu k}(t) = \lambda v^{x,h}(t) + \mu v^{x,k}(t). \tag{138}$$

This shows that for all $x \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\mathbb{R}^d \ni h \mapsto v^{x,h}(t) \in \mathbb{R}^d \tag{139}$$

is a linear function. Next observe that the fact that $[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto D(t, x) \in \mathbb{R}^{d \times d}$ is a continuous function implies that there exist $\delta_\varepsilon^\rho \in (0, \infty)$, $\rho, \varepsilon \in (0, \infty)$, such that for all $\rho, \varepsilon \in (0, \infty)$, $t \in [0, T]$, $\theta \in \{x \in \mathbb{R}^d \colon \|x\| \leq \rho\}$, $\vartheta \in \{x \in \mathbb{R}^d \colon \|x - \theta\| \leq \delta_\varepsilon^\rho\}$, $h \in \mathbb{R}^d$ it holds that

$$\|D(t, \vartheta)h - D(t, \theta)h\| \leq \varepsilon \|h\|. \tag{140}$$

In addition, note that (131) implies that for all $\rho, \varepsilon \in (0, \infty)$, $z, x \in \mathbb{R}^d$, $t \in [0, T]$, $u \in [0, 1]$ with $\|x - z\| < \min\{r_z, (L_z)^{-1}\delta_\varepsilon^\rho\}$ it holds that

$$\big\| \big[y^z(t) + u(y^x(t) - y^z(t)) \big] - y^z(t) \big\| = u \|y^x(t) - y^z(t)\| \leq u L_z \|x - z\| \leq \delta_\varepsilon^\rho. \tag{141}$$

Combining this with (140) shows that for all $\varepsilon \in (0, \infty)$, $z, x, h \in \mathbb{R}^d$, $t \in [0, T]$, $u \in [0, 1]$ with $\|x - z\| < \min\{r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}$ it holds that

$$\big\|D\big(t, y^z(t) + u(y^x(t) - y^z(t))\big)h - D(t, y^z(t))h\big\| \leq \varepsilon\|h\|. \tag{142}$$

The triangle inequality therefore implies that for all $\varepsilon \in (0, \infty)$, $z, x, h, \mathfrak{h} \in \mathbb{R}^d$, $t \in [0, T]$, $u \in [0, 1]$ with $\|x - z\| < \min\{r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}$ it holds that

$$\begin{aligned}
&\big\|D\big(t, y^z(t) + u(y^x(t) - y^z(t))\big)h - D(t, y^z(t))\mathfrak{h}\big\| \\
&\leq \big\|D\big(t, y^z(t) + u(y^x(t) - y^z(t))\big)h - D(t, y^z(t))h\big\| + \|D(t, y^z(t))h - D(t, y^z(t))\mathfrak{h}\| \\
&\leq \varepsilon\|h\| + \|D(t, y^z(t))(h - \mathfrak{h})\| \\
&\leq \varepsilon\|h\| + K_{R_z}\|h - \mathfrak{h}\|.
\end{aligned} \tag{143}$$

The fundamental theorem of calculus and (131) hence prove that for all $\varepsilon \in (0, \infty)$, $z, k \in \mathbb{R}^d$, $t \in [0, T]$ with $\|k\| < \min\{r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}$ it holds that

$$\begin{aligned}
&\big\|f(t, y^{z+k}(t)) - f(t, y^z(t)) - D(t, y^z(t))(v^{z,k}(t))\big\| \\
&= \big\|\big[f\big(t, y^z(t) + u(y^{z+k}(t) - y^z(t))\big)\big]_{u=0}^{u=1} - D(t, y^z(t))(v^{z,k}(t))\big\| \\
&= \bigg\|\bigg[\int_0^1 D\big(t, y^z(t) + u(y^{z+k}(t) - y^z(t))\big)(y^{z+k}(t) - y^z(t))\,\mathrm{d}u\bigg] \\
&\qquad - D(t, y^z(t))(v^{z,k}(t))\bigg\| \\
&= \bigg\|\int_0^1 D\big(t, y^z(t) + u(y^{z+k}(t) - y^z(t))\big)(y^{z+k}(t) - y^z(t)) - D(t, y^z(t))(v^{z,k}(t))\,\mathrm{d}u\bigg\| \\
&\leq \int_0^1 \big\|D\big(t, y^z(t) + u(y^{z+k}(t) - y^z(t))\big)(y^{z+k}(t) - y^z(t)) - D(t, y^z(t))(v^{z,k}(t))\big\|\,\mathrm{d}u \\
&\leq \int_0^1 \varepsilon\|y^{z+k}(t) - y^z(t)\| + K_{R_z}\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\|\,\mathrm{d}u \\
&= \varepsilon\|y^{z+k}(t) - y^z(t)\| + K_{R_z}\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\| \\
&\leq \varepsilon L_z\|k\| + K_{R_z}\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\|.
\end{aligned} \tag{144}$$

Combining this with (124) and (136) shows that for all $\varepsilon \in (0, \infty)$, $z, k \in \mathbb{R}^d$, $t \in [0, T]$

with $\|k\| < \min\{r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}$ it holds that

$$
\begin{aligned}
&\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\| \\
&= \left\| \left[z + k + \int_0^t f(s, y^{z+k}(s))\,\mathrm{d}s\right] - \left[z + \int_0^t f(s, y^z(s))\,\mathrm{d}s\right] \right. \\
&\qquad \left. - \left[k + \int_0^t D(s, y^z(s))(v^{z,k}(s))\,\mathrm{d}s\right] \right\| \\
&= \left\| \int_0^t f(s, y^{z+k}(s)) - f(s, y^z(s)) - D(s, y^z(s))(v^{z,k}(s))\,\mathrm{d}s \right\| \\
&\leq \int_0^t \left\| f(s, y^{z+k}(s)) - f(s, y^z(s)) - D(s, y^z(s))(v^{z,k}(s)) \right\| \mathrm{d}s \\
&\leq \int_0^t \varepsilon L_z \|k\| + K_{R_z} \|y^{z+k}(s) - y^z(s) - v^{z,k}(s)\|\,\mathrm{d}s \\
&= t\varepsilon L_z \|k\| + \int_0^t K_{R_z} \|y^{z+k}(s) - y^z(s) - v^{z,k}(s)\|\,\mathrm{d}s \\
&\leq T\varepsilon L_z \|k\| + \int_0^t K_{R_z} \|y^{z+k}(s) - y^z(s) - v^{z,k}(s)\|\,\mathrm{d}s.
\end{aligned}
\tag{145}
$$

Gronwall's integral inequality (see, e.g., Grohs et al. [6, Lemma 2.11] (with $\alpha \leftarrow T\varepsilon L_z\|k\|$, $\beta \leftarrow K_{R_z}$, $T \leftarrow T$, $f \leftarrow ([0,T] \ni s \mapsto \|y^{z+k}(s) - y^z(s) - v^{z,k}(s)\| \in [0,\infty))$ for $\varepsilon \in (0,\infty)$, $z \in \mathbb{R}^d$, $k \in \{h \in \mathbb{R}^d \colon \|h\| < \min\{r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}\}$ in the notation of Grohs et al. [6, Lemma 2.11])) and (130) therefore show that for all $\varepsilon \in (0,\infty)$, $z, k \in \mathbb{R}^d$, $t \in [0,T]$ with $\|k\| < \min\{r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}$ it holds that

$$
\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\| \leq T\varepsilon L_z \|k\| \exp(K_{R_z} t). \tag{146}
$$

This establishes that for all $\varepsilon \in (0,\infty)$, $z \in \mathbb{R}^d$, $k \in \mathbb{R}^d \setminus \{0\}$, $t \in [0,T]$ with $\|k\| < \min\bigl\{r_z, (L_z)^{-1}\delta^{R_z}_{(TL_z \exp(K_{R_z}t))^{-1}\varepsilon}\bigr\}$ it holds that

$$
\frac{\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\|}{\|k\|} \leq \varepsilon. \tag{147}
$$

Therefore, we obtain that for all $z \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$
\limsup_{\substack{k \to 0, \\ k \in \mathbb{R}^d \setminus \{0\}}} \left[\frac{\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\|}{\|k\|}\right] = 0. \tag{148}
$$

Combining this with (139) shows that for all $t \in [0,T]$, $x, h \in \mathbb{R}^d$ it holds that $\mathbb{R}^d \ni z \mapsto y^z(t) \in \mathbb{R}^d$ is a differentiable function and

$$
\bigl(\tfrac{\partial}{\partial x} y^x(t)\bigr)(h) = v^{x,h}(t). \tag{149}
$$

This and (136) establish (ii). Next note that the triangle inequality and (136) imply that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\begin{aligned} \|v^{x,h}(t)\| &\leq \|h\| + \left\| \int_0^t D(s, y^x(s))(v^{x,h}(s)) \, \mathrm{d}s \right\| \\ &\leq \|h\| + \int_0^t \big\| D(s, y^x(s))(v^{x,h}(s)) \big\| \, \mathrm{d}s \\ &\leq \|h\| + \int_0^t K_{R_x} \|v^{x,h}(s)\| \, \mathrm{d}s. \end{aligned} \tag{150}$$

Gronwall's integral inequality (see, e.g., Grohs et al. [6, Lemma 2.11] (with $\alpha \leftarrow \|h\|$, $\beta \leftarrow K_{R_x}$, $T \leftarrow T$, $f \leftarrow ([0, T] \ni s \mapsto \|v^{x,h}(s)\| \in [0, \infty])$ for $x, h \in \mathbb{R}^d$ in the notation of Grohs et al. [6, Lemma 2.11])) and (130) hence ensure that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\|v^{x,h}(t)\| \leq \|h\| \exp(K_{R_x} t) \leq \|h\| \exp(K_{R_x} T). \tag{151}$$

In addition, observe that (131) and the triangle inequality imply that for all $x, z \in \mathbb{R}^d$, $t \in [0, T]$ with $\|x - z\| < \min\{1, r_z\}$ it holds that

$$\|y^x(t)\| \leq \|y^x(t) - y^z(t)\| + \|y^z(t)\| \leq L_z \|x - z\| + R_z \leq L_z + R_z. \tag{152}$$

This ensures that for all $x, z \in \mathbb{R}^d$ with $\|x - z\| < \min\{1, r_z\}$ it holds that

$$R_x \leq R_z + L_z. \tag{153}$$

Combining this with (151) proves that for all $x, z, h \in \mathbb{R}^d$, $t \in [0, T]$ with $\|x - z\| < \min\{1, r_z\}$ it holds that

$$\|v^{x,h}(t)\| \leq \|h\| \exp(K_{R_z + L_z} T). \tag{154}$$

Next note that (143) implies that for all $\varepsilon \in (0, \infty)$, $x, z, h \in \mathbb{R}^d$, $t \in [0, T]$ with $\|x - z\| <$

$\min\{1, r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}$ it holds that

$$\begin{aligned}\|v^{x,h}(t) - v^{z,h}(t)\| &= \left\| \left[h + \int_0^t D(s, y^x(s))(v^{x,h}(s)) \, \mathrm{d}s \right] \right. \\ &\qquad \left. - \left[h + \int_0^t D(s, y^z(s))(v^{z,h}(s)) \, \mathrm{d}s \right] \right\| \\ &= \left\| \int_0^t D(s, y^x(s))(v^{x,h}(s)) - D(s, y^z(s))(v^{z,h}(s)) \, \mathrm{d}s \right\| \\ &\leq \int_0^t \left\| D(s, y^x(s))(v^{x,h}(s)) - D(s, y^z(s))(v^{z,h}(s)) \right\| \mathrm{d}s \\ &\leq \int_0^t \varepsilon \|v^{x,h}(s)\| + K_{R_z} \|v^{x,h}(s) - v^{z,h}(s)\| \, \mathrm{d}s \\ &\leq \int_0^t \varepsilon \|h\| \exp(K_{R_z+L_z} T) + K_{R_z} \|v^{x,h}(s) - v^{z,h}(s)\| \, \mathrm{d}s \\ &= \varepsilon \|h\| \exp(K_{R_z+L_z} T) + \int_0^t K_{R_z} \|v^{x,h}(s) - v^{z,h}(s)\| \, \mathrm{d}s.\end{aligned} \tag{155}$$

This, Gronwall's integral inequality (see, e.g., Grohs et al. [6, Lemma 2.11] (with $\alpha \leftarrow \varepsilon\|h\| \exp(K_{R_z+L_z}T)$, $\beta \leftarrow K_{R_z}$, $T \leftarrow T$, $f \leftarrow ([0,T] \ni t \mapsto \|v^{x,h}(t) - v^{z,h}(t)\| \in [0,\infty))$ for $\varepsilon \in (0,\infty)$, $z, h \in \mathbb{R}^d$, $x \in \{w \in \mathbb{R}^d \colon \|w - z\| < \min\{1, r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}\}$ in the notation of Grohs et al. [6, Lemma 2.11])), and (130) show that for all $\varepsilon \in (0,\infty)$, $z, x, h \in \mathbb{R}^d$, $t \in [0,T]$ with $\|x - z\| < \min\{1, r_z, (L_z)^{-1}\delta_\varepsilon^{R_z}\}$ it holds that

$$\begin{aligned}\|v^{x,h}(t) - v^{z,h}(t)\| &\leq \varepsilon \|h\| \exp(K_{R_z+L_z} T) \exp(K_{R_z} t) \\ &\leq \varepsilon \|h\| \exp(K_{R_z+L_z} T) \exp(K_{R_z} T) \\ &\leq \varepsilon \|h\| \exp(2K_{R_z+L_z} T).\end{aligned} \tag{156}$$

Moreover, (136) and (151) show that for all $z, h \in \mathbb{R}^d$, $s \in [0,T]$, $t \in [0,s]$ it holds that

$$\begin{aligned} \|v^{z,h}(s) - v^{z,h}(t)\| &= \bigg\| \bigg[h + \int_0^s D(u, y^z(u))(v^{z,h}(u)) \, \mathrm{d}u \bigg] \\ &\qquad - \bigg[h + \int_0^t D(u, y^z(u))(v^{z,h}(u)) \, \mathrm{d}u \bigg] \bigg\| \\ &= \bigg\| \int_t^s D(u, y^z(u))(v^{z,h}(u)) \, \mathrm{d}u \bigg\| \\ &\le \int_t^s \big\| D(u, y^z(u))(v^{z,h}(u)) \big\| \, \mathrm{d}u \\ &\le \int_t^s K_{R_z} \|v^{z,h}(u)\| \, \mathrm{d}u \\ &\le \int_t^s K_{R_z} \|h\| \exp(K_{R_z} T) \, \mathrm{d}u \\ &= (s-t) K_{R_z} \|h\| \exp(K_{R_z} T). \end{aligned} \tag{157}$$

Combining this with (156) proves that for all $\varepsilon \in (0,\infty)$, $z, x, h \in \mathbb{R}^d$, $s, t \in [0,T]$ with $\|x - z\| < \min\{1, r_z, (L_z)^{-1} \delta^{R_z}_{\exp(-2K_{R_z+L_z}T)2^{-1}\varepsilon}\}$ and $|s - t| < (2K_{R_z} \exp(K_{R_z} T) + 1)^{-1} \varepsilon$ it holds that

$$\begin{aligned} \|v^{x,h}(s) - v^{z,h}(t)\| &\le \|v^{x,h}(s) - v^{z,h}(s)\| + \|v^{z,h}(s) - v^{z,h}(t)\| \\ &\le \frac{\varepsilon\|h\|}{2} + \frac{\varepsilon\|h\|}{2} \\ &= \varepsilon\|h\|. \end{aligned} \tag{158}$$

Combining this with (149) establishes that

$$[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto \big(\tfrac{\partial}{\partial x} y^x(t) \big) \in \mathbb{R}^{d \times d} \tag{159}$$

is a continuous function. This and (135) prove (i). The proof of Lemma 5.2 is thus completed. ∎

Lemma 5.3. *Let $d, m \in \mathbb{N}$, $T \in [0,\infty)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d \times m}$, $w \in C([0,T], \mathbb{R}^m)$ and let $y^x \in C([0,T], \mathbb{R}^d)$, $x \in \mathbb{R}^d$, satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$ that*

$$y^x(t) = x + \int_0^t \mu(y^x(s)) \, \mathrm{d}s + \sigma w(t). \tag{160}$$

Then

(i) it holds that $([0,T] \times \mathbb{R}^d \ni (t,x) \mapsto y^x(t) \in \mathbb{R}^d) \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and

(ii) it holds for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$ that

$$\big(\tfrac{\partial}{\partial x} y^x(t)\big)(h) = h + \int_0^t \mu'(y^x(s))\big(\big(\tfrac{\partial}{\partial x} y^x(s)\big)(h)\big)\,\mathrm{d}s. \tag{161}$$

Proof of Lemma 5.3. Throughout this proof let $f\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ be the function which satisfies for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that

$$f(t,x) = \mu(x + \sigma w(t)) \tag{162}$$

and let $z^x\colon [0,T] \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be the functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$ that

$$z^x(t) = y^x(t) - \sigma w(t). \tag{163}$$

Observe that the hypothesis that $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and the hypothesis that w is a continuous function show that for all $t \in [0,T]$, $x \in \mathbb{R}^d$ it holds that

$$\begin{gathered} f \in C([0,T]\times\mathbb{R}^d, \mathbb{R}^d), \qquad (\mathbb{R}^d \ni v \mapsto f(t,v) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d, \mathbb{R}^d), \\ \text{and} \qquad \tfrac{\partial}{\partial x} f(t,x) = \mu'(x + \sigma w(t)). \end{gathered} \tag{164}$$

The hypothesis that $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and the hypothesis that w is a continuous function hence imply that $[0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \frac{\partial}{\partial x} f(t,x) \in \mathbb{R}^{d\times d}$ is a continuous function. This and (164) ensure that

$$f \in C^{0,1}([0,T]\times\mathbb{R}^d, \mathbb{R}^d). \tag{165}$$

Next we combine (160) and (163) to obtain that for all $x \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$z^x(t) = x + \int_0^t \mu(y^x(s))\,\mathrm{d}s = x + \int_0^t \mu(z^x(s) + \sigma w(s))\,\mathrm{d}s = x + \int_0^t f(s, z^x(s))\,\mathrm{d}s. \tag{166}$$

In addition, note that the assumption that for all $x \in \mathbb{R}^d$ it holds that y^x is a continuous function and the assumption that w is a continuous function imply that for all $x \in \mathbb{R}^d$ it holds that z^x is a continuous function. Combining this, (164), (165), and (166) with Lemma 5.2 (with $d \leftarrow d$, $T \leftarrow T$, $f \leftarrow f$, $D \leftarrow \big([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto \mu'(x+\sigma w(t)) \in \mathbb{R}^{d\times d}\big)$, $(y^x)_{x\in\mathbb{R}^d} \leftarrow (z^x)_{x\in\mathbb{R}^d}$ in the notation of Lemma 5.2) shows

(a) that

$$([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto z^x(t) \in \mathbb{R}^d) \in C^{0,1}([0,T]\times\mathbb{R}^d, \mathbb{R}^d) \tag{167}$$

and

(b) that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\big(\tfrac{\partial}{\partial x} z^x(t)\big)(h) = h + \int_0^t \mu'(z^x(s) + \sigma w(s))\big(\big(\tfrac{\partial}{\partial x} z^x(s)\big)(h)\big)\,\mathrm{d}s. \tag{168}$$

Observe that (163) and (167) imply that $([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto y^x(t) \in \mathbb{R}^d) \in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$. This and (163) establish that for all $x \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\tfrac{\partial}{\partial x} y^x(t) = \tfrac{\partial}{\partial x} z^x(t). \tag{169}$$

Combining this with (168) proves that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\big(\tfrac{\partial}{\partial x} y^x(t)\big)(h) = h + \int_0^t \mu'(y^x(s))\big(\big(\tfrac{\partial}{\partial x} y^x(s)\big)(h)\big)\,\mathrm{d}s. \tag{170}$$

The proof of Lemma 5.3 is thus completed. □

5.3 Differentiability with respect to the initial value for SDEs

Lemma 5.4. *Let $d,m \in \mathbb{N}$, $T \in [0,\infty)$, $\mu \in C^1(\mathbb{R}^d,\mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m,[0,\infty))$, $V \in C^1(\mathbb{R}^d,[0,\infty))$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $V'(x)\mu(x+\sigma z) \le \varphi(z)V(x)$ and $\|x\| \le V(x)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a stochastic process with continuous sample paths. Then*

(i) there exist unique stochastic processes $X^x\colon [0,T]\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{171}$$

(ii) it holds for all $\omega \in \Omega$ that $([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto X^x(t,\omega) \in \mathbb{R}^d) \in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$, and

(iii) it holds for all $x,h \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$\big(\tfrac{\partial}{\partial x} X^x(t,\omega)\big)(h) = h + \int_0^t \mu'(X^x(s,\omega))\big(\big(\tfrac{\partial}{\partial x} X^x(s,\omega)\big)(h)\big)\,\mathrm{d}s. \tag{172}$$

Proof of Lemma 5.4. First, observe that Lemma 2.2 (with $d \leftarrow d$, $m \leftarrow m$, $T \leftarrow T$, $\xi \leftarrow x$, $\mu \leftarrow \mu$, $\sigma \leftarrow \sigma$, $\varphi \leftarrow \varphi$, $V \leftarrow V$, $w \leftarrow ([0,T] \ni t \mapsto W(t,\omega) \in \mathbb{R}^m)$, $\|\cdot\| \leftarrow \|\cdot\|$ for $x \in \mathbb{R}^d$, $\omega \in \Omega$ in the notation of Lemma 2.2) proves that there exist unique $y^x_\omega \in C([0,T],\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, such that for all $x \in \mathbb{R}^d$, $\omega \in \Omega$, $t \in [0,T]$ it holds that

$$y^x_\omega(t) = x + \int_0^t \mu(y^x_\omega(s))\,\mathrm{d}s + \sigma W(t,\omega). \tag{173}$$

In addition, note that the hypothesis that $\mu \in C^1(\mathbb{R}^d,\mathbb{R}^d)$ ensures that for all $r \in (0,\infty)$ it holds that

$$\sup_{\substack{x,y\in\mathbb{R}^d,\, x\neq y,\\ \|x\|+\|y\|\le r}} \frac{\|\mu(x)-\mu(y)\|}{\|x-y\|} < \infty. \tag{174}$$

Combining this and (173) with Lemma 4.5 (with $d \leftarrow d$, $T \leftarrow T$, $f \leftarrow ([0,T]\times\mathbb{R}^d \ni (t,y) \mapsto \mu(y) \in \mathbb{R}^d)$, $\xi \leftarrow x$, $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $W \leftarrow ([0,T]\times\Omega \ni (t,\omega) \mapsto \sigma W(t,\omega) \in \mathbb{R}^d)$, $\|\cdot\| \leftarrow \|\cdot\|$, $Y \leftarrow ([0,T]\times\Omega \ni (t,\omega) \mapsto y^x_\omega(t) \in \mathbb{R}^d)$ for $x \in \mathbb{R}^d$ in the notation of Lemma 4.5) shows that for all $x \in \mathbb{R}^d$ it holds that $[0,T]\times\Omega \ni (t,\omega) \mapsto y^x_\omega(t) \in \mathbb{R}^d$ is a stochastic process. This and (173) establish (i). Next note that (173) and Lemma 5.3 (with $d \leftarrow d$, $m \leftarrow m$, $T \leftarrow T$, $\mu \leftarrow \mu$, $\sigma \leftarrow \sigma$, $w \leftarrow ([0,T] \ni t \mapsto W(t,\omega) \in \mathbb{R}^m)$, $(y^x)_{x\in\mathbb{R}^d} = (y^x_\omega)_{x\in\mathbb{R}^d}$ for $\omega \in \Omega$ in the notation of Lemma 5.3) establishes (ii) and (iii). The proof of Lemma 5.4 is thus completed. □

6 Integrability properties for stochastic differential equations (SDEs)

In this section we present in Lemma 6.1 in Subsection 6.1 below, in Lemmas 6.3–6.5 in Subsection 6.2 below, and in Lemma 6.6 in Subsection 6.3 below a few elementary integrability properties for standard Brownian motions (see Lemmas 6.1 and Lemmas 6.3–6.5) and solutions of certain additive noise driven stochastic differential equations (see Lemma 6.6). Lemma 6.1 establishes exponential integrability properties for one-dimensional standard Brownian motions and is a straightforward consequence of Ledoux-Talagrand [16, Corollary 3.2]. Lemmas 6.3 and 6.4 establish exponential integrability properties for multi-dimensional standard Brownian motions. Our proof of Lemma 6.3 uses Lemma 6.1 and an application of the well-known inequality for real numbers in Lemma 6.2 below. Lemma 6.4, in turn, is an immediate consequence of Lemma 6.3. Lemma 6.5 establishes polynomial integrability properties for multi-dimensional standard Brownian motions and is a direct consequence of Lemma 6.4. Lemmas 6.1 –6.5 are essentially well-known and for the reader's convenience, we include in this section full proofs for these lemmas.

6.1 Integrability properties for scalar Brownian motions

Lemma 6.1. *Let $T, c \in [0,\infty)$, $\alpha \in [0,2)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, and let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a Brownian motion with continuous sample paths. Then*

(i) it holds that $\Omega \ni \omega \mapsto \sup_{t\in[0,T]}(|W(t,\omega)|^\alpha) \in \mathbb{R}$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function and

(ii) it holds that $\mathbb{E}\big[\exp\big(c\big[\sup_{t\in[0,T]}(|W(t)|^\alpha)\big]\big)\big] < \infty$.

Proof of Lemma 6.1. Throughout this proof assume w.l.o.g. that $T > 0$ and $\alpha > 0$ and let $K \in [0,\infty)$ satisfy

$$K = \Big(\frac{2-\alpha}{2}\Big)\Big(\frac{2}{4\alpha T}\Big)^{\frac{\alpha}{\alpha-2}} c^{\frac{2}{2-\alpha}}. \tag{175}$$

Note that the fact that $\frac{2-\alpha}{2} + \frac{\alpha}{2} = 1$ and the fact that for all $a, b \in [0,\infty)$, $p, q \in (0,\infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ it holds that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ (Young inequality) implies that for all $\omega \in \Omega$ it holds that

$$
\begin{aligned}
&c\Big[\sup_{t\in[0,T]}\big(|W(t,\omega)|^{\alpha}\big)\Big]\\
&= c\Big(\frac{2}{4\alpha T}\Big)^{-\frac{\alpha}{2}}\bigg(\Big(\frac{2}{4\alpha T}\Big)^{\frac{\alpha}{2}}\Big[\sup_{t\in[0,T]}\big(|W(t,\omega)|^{\alpha}\big)\Big]\bigg)\\
&\leq \frac{2-\alpha}{2}\bigg[c\Big(\frac{2}{4\alpha T}\Big)^{-\frac{\alpha}{2}}\bigg]^{\frac{2}{2-\alpha}} + \frac{\alpha}{2}\bigg(\Big(\frac{2}{4\alpha T}\Big)^{\frac{\alpha}{2}}\Big[\sup_{t\in[0,T]}\big(|W(t,\omega)|^{\alpha}\big)\Big]\bigg)^{\frac{2}{\alpha}}\\
&= \frac{2-\alpha}{2}\Big(\frac{2}{4\alpha T}\Big)^{\frac{\alpha}{\alpha-2}} c^{\frac{2}{2-\alpha}} + \frac{\alpha}{2}\Big(\frac{2}{4\alpha T}\Big)\Big[\sup_{t\in[0,T]}\big(|W(t,\omega)|^{2}\big)\Big]\\
&= K + (4T)^{-1}\Big[\sup_{t\in[0,T]}\big(|W(t,\omega)|^{2}\big)\Big].
\end{aligned}
\tag{176}
$$

Furthermore, observe that Lemma 3.5 ensures that for all $\beta \in [0,\infty)$ it holds that

$$\Omega \ni \omega \mapsto \sup_{t\in[0,T]}(|W(t,\omega)|^{\beta}) \in \mathbb{R} \tag{177}$$

is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function. In addition, note that for all $\kappa \in [0, \frac{1}{2T})$ it holds that

$$\mathbb{E}\big[\exp\big(\kappa\big[\sup_{t\in[0,T]}\big(|W(t)|^2\big)\big]\big)\big] < \infty \tag{178}$$

(cf., e.g., Ledoux-Talagrand [16, Corollary 3.2]). Combining this and (177) with (176) establishes that

$$
\begin{aligned}
\mathbb{E}\big[\exp\big(c\big[\sup_{t\in[0,T]}\big(|W(t)|^{\alpha}\big)\big]\big)\big] &\leq \mathbb{E}\big[\exp\big(K + (4T)^{-1}\sup_{t\in[0,T]}\big(|W(t)|^2\big)\big)\big]\\
&= \exp(K)\,\mathbb{E}\big[\exp\big((4T)^{-1}\sup_{t\in[0,T]}\big(|W(t)|^2\big)\big)\big] < \infty.
\end{aligned}
\tag{179}
$$

This completes the proof of Lemma 6.1. □

6.2 Integrability properties for multi-dimensional Brownian motions

Lemma 6.2. *It holds for all $\beta \in [0,\infty)$, $m \in \mathbb{N}$, $a_1, a_2, \ldots, a_m \in \mathbb{R}$ that*

$$\Big|\sum\nolimits_{i=1}^{m} a_i\Big|^{\beta} \leq m^{\max\{0,\beta-1\}}\Big[\sum\nolimits_{i=1}^{m}\big(|a_i|^{\beta}\big)\Big]. \tag{180}$$

Proof of Lemma 6.2. Throughout this proof let $\varphi_\beta \colon \mathbb{R} \to \mathbb{R}$, $\beta \in [1,\infty)$, be the functions which satisfy for all $\beta \in [1,\infty)$, $x \in \mathbb{R}$ that

$$\varphi_\beta(x) = |x|^{\beta}. \tag{181}$$

Note that for all $\beta \in [0,1]$, $m \in \mathbb{N}$, $a_1, a_2, \ldots, a_m \in \mathbb{R}$ it holds that

$$\Big[\sum\nolimits_{i=1}^{m} |a_i|\Big]^{\beta} \leq 2^0\Big[\sum\nolimits_{i=1}^{m}\big(|a_i|^{\beta}\big)\Big]. \tag{182}$$

Next observe that for all $\beta \in [1,\infty)$ it holds that φ_β is a convex function. Jensen's inequality hence establishes that for all $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{R}$ it holds that

$$\frac{\left|\sum_{i=1}^m a_i\right|^\beta}{m^\beta} = \varphi_\beta\left(\frac{\sum_{i=1}^m a_i}{m}\right) \leq \frac{\sum_{i=1}^m \varphi_\beta(a_i)}{m} = \frac{\sum_{i=1}^m (|a_i|^\beta)}{m}. \tag{183}$$

This implies that for all $m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{R}$ it holds that

$$\left|\sum\nolimits_{i=1}^m a_i\right|^\beta \leq m^{\beta-1} \sum\nolimits_{i=1}^m (|a_i|^\beta). \tag{184}$$

The proof of Lemma 6.2 is thus completed. □

Lemma 6.3. *Let $m \in \mathbb{N}$, $T, c \in [0,\infty)$, $\alpha \in [0,2)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths, and let $\|\cdot\| \colon \mathbb{R}^m \to [0,\infty)$ be a norm. Then it holds that*

$$\mathbb{E}\left[\sup_{t \in [0,T] \cap \mathbb{Q}} \exp\!\big(c \, \|W(t)\|^\alpha\big)\right] < \infty. \tag{185}$$

Proof of Lemma 6.3. Throughout this proof let $W_i \colon [0,T] \times \Omega \to \mathbb{R}$, $i \in \{1, 2, \ldots, m\}$, be the functions which satisfy for all $t \in [0,T]$, $\omega \in \Omega$ that

$$W(t,\omega) = \big(W_1(t,\omega), W_2(t,\omega), \ldots, W_m(t,\omega)\big) \tag{186}$$

and let $K \in [0,\infty]$ satisfy

$$K = \sup_{z=(z_1,z_2,\ldots,z_m) \in \mathbb{R}^m \setminus \{0\}} \left(\frac{\|z\|}{\left(\sum_{i=1}^m |z_i|\right)}\right). \tag{187}$$

Note that the fact that all norms on $\mathbb{R}^m$ are equivalent ensures that $K < \infty$. Hence, we obtain that for all $\omega \in \Omega$ it holds that

$$\begin{aligned}\sup_{t \in [0,T]} (\|W(t,\omega)\|^\alpha) &\leq \sup_{t \in [0,T]} \left(\left[K\left[\sum\nolimits_{i=1}^m |W_i(t,\omega)|\right]\right]^\alpha\right) \\ &= K^\alpha \left[\sup_{t \in [0,T]} \left(\left[\sum\nolimits_{i=1}^m |W_i(t,\omega)|\right]^\alpha\right)\right].\end{aligned} \tag{188}$$

This, the fact that $\alpha < 2$, and Lemma 6.2 show that for all $\omega \in \Omega$ it holds that

$$\begin{aligned}c\left[\sup_{t \in [0,T]} (\|W(t,\omega)\|^\alpha)\right] &\leq cK^\alpha \left[\sup_{t \in [0,T]} \left(m\left[\sum\nolimits_{i=1}^m (|W_i(t,\omega)|^\alpha)\right]\right)\right] \\ &= mcK^\alpha \left[\sup_{t \in [0,T]} \left(\sum\nolimits_{i=1}^m (|W_i(t,\omega)|^\alpha)\right)\right] \\ &\leq mcK^\alpha \left[\sum_{i=1}^m \left(\sup_{t \in [0,T]} \left(|W_i(t,\omega)|^\alpha\right)\right)\right] \\ &= \sum_{i=1}^m \left(mcK^\alpha \left[\sup_{t \in [0,T]} \left(|W_i(t,\omega)|^\alpha\right)\right]\right).\end{aligned} \tag{189}$$

Hence, we obtain that for all $\omega \in \Omega$ it holds that

$$\exp\left(c\left[\sup_{t\in[0,T]}(\|W(t,\omega)\|^\alpha)\right]\right) \leq \prod_{i=1}^m \exp\left(mcK^\alpha\left[\sup_{t\in[0,T]}\big(|W_i(t,\omega)|^\alpha\big)\right]\right). \tag{190}$$

In the next step we note that Lemma 3.5 (with $T \leftarrow T$, $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $Y \leftarrow ([0,T]\times\Omega \ni (t,\omega) \mapsto \|W(t,\omega)\|^\alpha \in [0,\infty))$ in the notation of Lemma 3.5) ensures that for all $\omega \in \Omega$ it holds that

$$\sup_{t\in[0,T]}(\|W(t,\omega)\|^\alpha) = \sup_{t\in[0,T]\cap\mathbb{Q}}(\|W(t,\omega)\|^\alpha). \tag{191}$$

Combining this with (190) shows that for all $\omega \in \Omega$ it holds that

$$\begin{aligned}\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(c\,\|W(t,\omega)\|^\alpha\big) &= \exp\left(c\left[\sup_{t\in[0,T]\cap\mathbb{Q}}(\|W(t,\omega)\|^\alpha)\right]\right)\\ &\leq \prod_{i=1}^m \exp\left(mcK^\alpha\left[\sup_{t\in[0,T]}\big(|W_i(t,\omega)|^\alpha\big)\right]\right).\end{aligned} \tag{192}$$

In addition, observe that Lemma 6.1 (with $T \leftarrow T$, $c \leftarrow 2^m cK^\alpha$, $\alpha \leftarrow \alpha$, $(\Omega,\mathcal{F},\mathbb{P}) \leftarrow (\Omega,\mathcal{F},\mathbb{P})$, $W \leftarrow W_i$ for $i \in \{1,2,\ldots,m\}$ in the notation of Lemma 6.1) proves

(A) that for all $i \in \{1,2,\ldots,m\}$ it holds that $\Omega \ni \omega \mapsto \sup_{t\in[0,T]}\big(|W_i(t,\omega)|^\alpha\big) \in \mathbb{R}$ is an $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable function and

(B) that for all $i \in \{1,2,\ldots,m\}$ it holds that

$$\mathbb{E}\left[\exp\left(mcK^\alpha\left[\sup_{t\in[0,T]}\big(|W_i(t)|^\alpha\big)\right]\right)\right] < \infty. \tag{193}$$

Note that the fact that $W_1, W_2, \ldots, W_m$ are independent stochastic processes, (192), and (193) establish that

$$\mathbb{E}\left[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(c\,\|W(t)\|^\alpha\big)\right] \leq \prod_{i=1}^m \mathbb{E}\left[\exp\left(mcK^\alpha\left[\sup_{t\in[0,T]}\big(|W_i(t)|^\alpha\big)\right]\right)\right] < \infty. \tag{194}$$

The proof of Lemma 6.3 is thus completed. □

Lemma 6.4. *Let $m \in \mathbb{N}$, $T, C \in [0,\infty)$, $\alpha \in [0,2)$, let $\|\cdot\|\colon \mathbb{R}^m \to [0,\infty)$ be a norm, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths, and let $\varphi\colon \mathbb{R}^m \to [0,\infty)$ satisfy for all $z \in \mathbb{R}^m$ that $\varphi(z) \leq C(1+\|z\|^\alpha)$. Then it holds for all $c \in [0,\infty)$ that*

$$\mathbb{E}\left[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(c\,\varphi(W(t))\big)\right] < \infty. \tag{195}$$

Proof of Lemma 6.4. Note that the assumption that for all $z \in \mathbb{R}^m$ it holds that $\varphi(z) \leq C(1 + \|z\|^\alpha)$ implies that for all $c \in [0, \infty)$, $\omega \in \Omega$ it holds that

$$\begin{aligned}\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\bigl(c\,\varphi(W(t,\omega))\bigr) &\leq \sup_{t\in[0,T]\cap\mathbb{Q}} \exp\bigl(cC(1 + \|W(t,\omega)\|^\alpha)\bigr) \\ &= \sup_{t\in[0,T]\cap\mathbb{Q}} \exp\bigl(cC + cC\,\|W(t,\omega)\|^\alpha\bigr) \\ &= \exp(cC)\biggl[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\bigl(cC\,\|W(t,\omega)\|^\alpha\bigr)\biggr].\end{aligned} \tag{196}$$

Combining this with Lemma 6.3 establishes that for all $c \in [0, \infty)$ it holds that

$$\mathbb{E}\biggl[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\bigl(c\,\varphi(W(t))\bigr)\biggr] \leq \exp(cC)\,\mathbb{E}\biggl[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\bigl(cC\,\|W(t)\|^\alpha\bigr)\biggr] < \infty. \tag{197}$$

This completes the proof of Lemma 6.4. □

Lemma 6.5. *Let $d, m \in \mathbb{N}$, $T, r \in [0, \infty)$, $\sigma \in \mathbb{R}^{d\times m}$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then it holds that*

$$\mathbb{E}\biggl[\sup_{t\in[0,T]\cap\mathbb{Q}} (\|\sigma W(t)\|^r)\biggr] < \infty. \tag{198}$$

Proof of Lemma 6.5. Throughout this proof let $|||\cdot||| \colon \mathbb{R}^m \to [0, \infty)$ be the m-dimensional Euclidean norm and let $C \in [0, \infty]$ satisfy

$$C = \sup_{x\in\mathbb{R}^m\setminus\{0\}} \left(\frac{\|\sigma x\|}{|||x|||}\right). \tag{199}$$

Note that

$$C = \sup_{x\in\mathbb{R}^m\setminus\{0\}} \left\|\sigma\left(\frac{x}{|||x|||}\right)\right\| \leq \sup_{y\in\{v\in\mathbb{R}^m \colon |||v|||=1\}} \|\sigma y\|. \tag{200}$$

The fact that $\mathbb{R}^m \ni y \mapsto \|\sigma y\| \in [0, \infty)$ is a continuous function and the fact that $\{v \in \mathbb{R}^m \colon |||v||| = 1\}$ is a compact set hence prove that

$$C < \infty. \tag{201}$$

In the next step we observe that for all $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$\|\sigma W(t,\omega)\|^r \leq [1 + \|\sigma W(t,\omega)\|]^r = \exp\bigl(r\ln(1 + \|\sigma W(t,\omega)\|)\bigr). \tag{202}$$

Furthermore, note that (201) and the fact that for all $y \in [0, \infty)$ it holds that $\ln(1+y) \leq y$ ensure that for all $z \in \mathbb{R}^m$ it holds that

$$\ln(1 + \|\sigma z\|) \leq \|\sigma z\| \leq C|||z||| \leq C(1 + |||z|||). \tag{203}$$

This, (201), and Lemma 6.4 (with $m \leftarrow m$, $T \leftarrow T$, $C \leftarrow C$, $\alpha \leftarrow 1$, $\|\cdot\| \leftarrow |||\cdot|||$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$, $\varphi \leftarrow (\mathbb{R}^m \ni z \mapsto \ln(1 + \|\sigma z\|) \in [0, \infty))$ in the notation of Lemma 6.4) show that

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(r \ln(1 + \|\sigma W(t)\|)\big)\Bigg] < \infty. \tag{204}$$

Combining this with (202) establishes that

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]\cap\mathbb{Q}} (\|\sigma W(t)\|^r)\Bigg] \leq \mathbb{E}\Bigg[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(r \ln(1 + \|\sigma W(t)\|)\big)\Bigg] < \infty. \tag{205}$$

This completes the proof of Lemma 6.5. □

6.3 Integrability properties for solutions of SDEs

Lemma 6.6. *Let $d, m \in \mathbb{N}$, $T \in [0, \infty)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m, [0, \infty))$, $V \in C^1(\mathbb{R}^d, [0, \infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, assume for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $V'(x)\mu(x + \sigma z) \leq \varphi(z)V(x)$ and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a stochastic process with continuous sample paths, let $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes with continuous sample paths, assume for all $c \in [0, \infty)$ that $\mathbb{E}\big[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(c\,\varphi(W(t))\big)\big] + \mathbb{E}\big[\sup_{t\in[0,T]\cap\mathbb{Q}} (\|\sigma W(t)\|^c)\big] < \infty$, and assume for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ that*

$$X^x(t, \omega) = x + \int_0^t \mu(X^x(s, \omega))\, ds + \sigma W(t, \omega). \tag{206}$$

Then

(i) it holds for all $R, r \in [0, \infty)$ that

$$\Omega \ni \omega \mapsto \Bigg[\sup_{x\in\{z\in\mathbb{R}^d \colon \|z\|\leq R\}} \sup_{t\in[0,T]} \big(\|X^x(t, \omega)\|^r\big)\Bigg] \in [0, \infty] \tag{207}$$

is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function and

(ii) it holds for all $R, r \in [0, \infty)$ that

$$\mathbb{E}\Bigg[\sup_{x\in\{z\in\mathbb{R}^d \colon \|z\|\leq R\}} \sup_{t\in[0,T]} \big(\|X^x(t)\|^r\big)\Bigg] < \infty. \tag{208}$$

Proof of Lemma 6.6. Throughout this proof let $Y, Z \colon \Omega \to [0, \infty)$ be the functions which satisfy for all $\omega \in \Omega$ that

$$Y(\omega) = \sup_{t\in[0,T]} \exp\big(\varphi(W(t, \omega))\big) \qquad \text{and} \qquad Z(\omega) = \sup_{t\in[0,T]} \|\sigma W(t, \omega)\|. \tag{209}$$

Note that Lemma 2.1 (with $d \leftarrow d$, $m \leftarrow m$, $T \leftarrow T$, $\xi \leftarrow x$, $\mu \leftarrow \mu$, $\sigma \leftarrow \sigma$, $\varphi \leftarrow \varphi$, $V \leftarrow V$, $\|\cdot\| \leftarrow \|\cdot\|$, $J \leftarrow [0,T]$, $y \leftarrow ([0,T] \ni t \mapsto X^x(t,\omega) \in \mathbb{R}^d)$, $w \leftarrow ([0,T] \ni t \mapsto W(t,\omega) \in \mathbb{R}^m)$ for $x \in \mathbb{R}^d$, $\omega \in \Omega$ in the notation of Lemma 2.1) ensures that for all $x \in \mathbb{R}^d$, $\omega \in \Omega$ it holds that

$$\begin{aligned} \sup_{t\in[0,T]} \|X^x(t,\omega)\| &\leq V(x)\exp\!\left(T\left[\sup_{t\in[0,T]} \varphi(W(t,\omega))\right]\right) + \left[\sup_{t\in[0,T]} \|\sigma W(t,\omega)\|\right] \\ &= V(x)\left[\sup_{t\in[0,T]} \exp\big(\varphi(W(t,\omega))\big)\right]^T + \left[\sup_{t\in[0,T]} \|\sigma W(t,\omega)\|\right] \\ &= V(x)[Y(\omega)]^T + Z(\omega). \end{aligned} \tag{210}$$

The hypothesis that for all $x \in \mathbb{R}^d$, $\omega \in \Omega$ it holds that $[0,T] \ni t \mapsto X^x(t,\omega) \in \mathbb{R}^d$ is a continuous function and the fact that for all $a,b \in \mathbb{R}$, $r \in [0,\infty)$ it holds that $|a+b|^r \leq 2^r(|a|^r + |b|^r)$ hence ensure that for all $\omega \in \Omega$, $R, r \in [0,\infty)$ it holds that

$$\begin{aligned} &\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R\}} \sup_{t\in[0,T]} \big(\|X^x(t,\omega)\|^r\big) \\ &= \sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R\}} \left(\left[\sup_{t\in[0,T]} \|X^x(t,\omega)\|\right]^r\right) \\ &\leq \sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R\}} \big(2^r\big([V(x)]^r[Y(\omega)]^{Tr} + [Z(\omega)]^r\big)\big) \\ &= 2^r\big(\big[\sup\nolimits_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R\}} V(x)\big]^r[Y(\omega)]^{Tr} + [Z(\omega)]^r\big). \end{aligned} \tag{211}$$

Next we combine the assumption that for all $\omega \in \Omega$ it holds that $[0,T] \ni t \mapsto W(t,\omega) \in \mathbb{R}^m$ is a continuous function and the assumption that φ is a continuous function with Lemma 3.5 to obtain that

(a) for all $\omega \in \Omega$ it holds that

$$Y(\omega) = \sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(\varphi(W(t,\omega))\big) \qquad \text{and} \qquad Z(\omega) = \sup_{t\in[0,T]\cap\mathbb{Q}} \|\sigma W(t,\omega)\| \tag{212}$$

and

(b) it holds that Y and Z are $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable functions.

Moreover, note that Lemma 5.3 (with $d \leftarrow d$, $m \leftarrow m$, $T \leftarrow T$, $\mu \leftarrow \mu$, $\sigma \leftarrow \sigma$, $w \leftarrow ([0,T] \ni t \mapsto W(t,\omega) \in \mathbb{R}^m)$, $(y^x)_{x\in\mathbb{R}^d} \leftarrow ([0,T] \ni t \mapsto X^x(t,\omega) \in \mathbb{R}^d)_{x\in\mathbb{R}^d}$ for $\omega \in \Omega$ in the notation of Lemma 5.3) ensures that for all $\omega \in \Omega$ it holds that

$$([0,T] \times \mathbb{R}^d \ni (t,x) \mapsto X^x(t,\omega) \in \mathbb{R}^d) \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d). \tag{213}$$

Combining this with Lemma 3.6 shows that for all $R, r \in [0,\infty)$ it holds that

$$\Omega \ni \omega \mapsto \left[\sup_{x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}} \sup_{t \in [0,T]} \left(\|X^x(t,\omega)\|^r \right) \right] \in [0,\infty] \tag{214}$$

is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function. In the next step we observe that the assumption that for all $c \in [0,\infty)$ it holds that $\mathbb{E}\big[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(c\,\varphi(W(t))\big)\big] < \infty$, (212), and the fact that Y is an $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable function ensure that for all $r \in [0,\infty)$ it holds that

$$\mathbb{E}\big[Y^{Tr}\big] = \mathbb{E}\left[\sup_{t\in[0,T]\cap\mathbb{Q}} \exp\big(Tr\,\varphi(W(t))\big) \right] < \infty. \tag{215}$$

In addition, note that the hypothesis that V is a continuous function implies that for all $R \in [0,\infty)$ it holds that

$$\sup_{x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}} V(x) < \infty. \tag{216}$$

Furthermore, observe that (212), the fact that Z is an $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable function, and the hypothesis that for all $c \in [0,\infty)$ it holds that $\mathbb{E}\big[\sup_{t\in[0,T]\cap\mathbb{Q}}(\|\sigma W(t)\|^c)\big] < \infty$ show that for all $r \in [0,\infty)$ it holds that

$$\mathbb{E}[Z^r] = \mathbb{E}\left[\sup_{t\in[0,T]\cap\mathbb{Q}} \big(\|\sigma W(t)\|^r\big) \right] < \infty. \tag{217}$$

Combining this, (214), (215), and (216) with (211) implies that for all $R, r \in [0,\infty)$ it holds that

$$\begin{aligned} &\mathbb{E}\left[\sup_{x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}} \sup_{t \in [0,T]} \big(\|X^x(t)\|^r\big) \right] \\ &\leq 2^r \Big(\big[\sup\nolimits_{x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}} V(x)\big]^r \mathbb{E}\big[Y^{Tr}\big] + \mathbb{E}\big[Z^r\big] \Big) < \infty. \end{aligned} \tag{218}$$

This completes the proof of Lemma 6.6. □

7 Conditional regularity with respect to the initial value for SDEs

In this section we study in Lemmas 7.4 and 7.5 in Subsection 7.2 below regularity properties of solutions of certain additive noise driven SDEs with respect to their initial values. In particular, in Lemma 7.5 we establish in inequality (269) a quantitative estimate for the difference of two solutions of certain additive noise driven SDEs. Our proof of Lemma 7.5 is based on an application of Lemma 7.4 which establishes a similar statement in wider generality. Our proof of Lemma 7.4, in turn, uses, besides other arguments, the auxiliary results in Lemma 7.1 in Subsection 7.1 below and in Lemmas 7.2 and 7.3 in Subsection 7.2 below. For completeness we include in this section also the detailed proofs for the two elementary results in Lemmas 7.2 and 7.3.

7.1 Conditional local Lipschitz continuity for deterministic DEs

Lemma 7.1. *Let $d \in \mathbb{N}$, $T \in [0,\infty)$, $\varphi \in C(\mathbb{R}^d, [0,\infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, let $z^x \in C([0,T], \mathbb{R}^d)$, $x \in \mathbb{R}^d$, be functions which satisfy for all $t \in [0,T]$ that $(\mathbb{R}^d \ni x \mapsto z^x(t) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, and assume for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $h \in \big\{v \in \mathbb{R}^d \colon \big([0,T] \ni s \mapsto \big(\tfrac{\partial}{\partial x} z^x(s)\big)(v) \in \mathbb{R}^d\big)$ is a $\mathcal{B}([0,T])/\mathcal{B}(\mathbb{R}^d)$-measurable function$\big\}$ that $\int_0^T \big\|\big(\tfrac{\partial}{\partial x} z^x(s)\big)(h)\big\| \, \mathrm{d}s < \infty$ and*

$$\big\|\big(\tfrac{\partial}{\partial x} z^x(t)\big)(h)\big\| \leq \|h\| + \int_0^t \varphi(z^x(s)) \, \big\|\big(\tfrac{\partial}{\partial x} z^x(s)\big)(h)\big\| \, \mathrm{d}s. \tag{219}$$

Then it holds for all $x, y \in \mathbb{R}^d$, $t \in [0,T]$ that

$$\|z^x(t) - z^y(t)\| \leq \sup_{u \in [0,1]} \left[\|x-y\| \exp\!\left(T \left[\sup_{s \in [0,T]} \varphi(z^{(1-u)y+ux}(s)) \right] \right) \right]. \tag{220}$$

Proof of Lemma 7.1. Throughout this proof let $D^x \colon [0,T] \to \mathbb{R}^{d \times d}$, $x \in \mathbb{R}^d$, be the functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$ that

$$D^x(t) = \tfrac{\partial}{\partial x} z^x(t). \tag{221}$$

Note that the assumption that for all $t \in [0,T]$ it holds that $(\mathbb{R}^d \ni x \mapsto z^x(t) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ ensures that for all $t \in [0,T]$, $h \in \mathbb{R}^d$ it holds that $\mathbb{R}^d \ni x \mapsto D^x(t)h \in \mathbb{R}^d$ is a continuous function. The fundamental theorem of calculus hence implies that for all $x, y \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\begin{aligned} \|z^x(t) - z^y(t)\| &= \big\| \big[z^{(1-u)y+ux}(t) \big]_{u=0}^{u=1} \big\| \\ &= \left\| \int_0^1 D^{(1-u)y+ux}(t)(x-y) \, \mathrm{d}u \right\| \\ &\leq \int_0^1 \|D^{(1-u)y+ux}(t)(x-y)\| \, \mathrm{d}u \\ &\leq \sup_{u \in [0,1]} \|D^{(1-u)y+ux}(t)(x-y)\|. \end{aligned} \tag{222}$$

Moreover, observe that the fact that for all $x \in \mathbb{R}^d$ it holds that z^x is a continuous function, the fact that for all $t \in [0,T]$ it holds that $(\mathbb{R}^d \ni x \mapsto z^x(t) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, and Lemma 3.4 (with $d \leftarrow d$, $x \leftarrow x$, $h \leftarrow h$, $(\Omega, \mathcal{F}) \leftarrow ([0,T], \mathcal{B}([0,T]))$, $(y^u)_{u \in \mathbb{R}^d} \leftarrow (z^u)_{u \in \mathbb{R}^d}$ for $x, h \in \mathbb{R}^d$ in the notation of Lemma 3.4) show that for all $x, h \in \mathbb{R}^d$ it holds that

$$[0,T] \ni t \mapsto \big(\tfrac{\partial}{\partial x} z^x(t)\big)(h) \in \mathbb{R}^d \tag{223}$$

is a $\mathcal{B}([0,T])/\mathcal{B}(\mathbb{R}^d)$-measurable function. This and the hypothesis that for all $x \in \mathbb{R}^d$, $h \in \big\{v \in \mathbb{R}^d \colon \big([0,T] \ni t \mapsto \big(\tfrac{\partial}{\partial x} z^x(t)\big)(v) \in \mathbb{R}^d\big)$ is a $\mathcal{B}([0,T])/\mathcal{B}(\mathbb{R}^d)$-measurable function$\big\}$

it holds that $\int_0^T \big\| \big(\frac{\partial}{\partial x} z^x(s)\big)(h) \big\| \, \mathrm{d}s < \infty$ implies that for all $x, h \in \mathbb{R}^d$ it holds that

$$\int_0^T \|D^x(s)h\| \, \mathrm{d}s < \infty. \tag{224}$$

In addition, observe that the hypothesis that for all $w \in \mathbb{R}^d$ it holds that z^w is a continuous function and the hypothesis that φ is a continuous function ensure that for all $w \in \mathbb{R}^d$ it holds that

$$\sup_{s \in [0,T]} \varphi(z^w(s)) < \infty. \tag{225}$$

This, (219), and (223) ensure that for all $w, h \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\begin{aligned} \|D^w(t)(h)\| &\leq \|h\| + \int_0^t \varphi(z^w(s)) \, \|D^w(s)h\| \, \mathrm{d}s \\ &\leq \|h\| + \left[\sup_{s \in [0,T]} \varphi(z^w(s)) \right] \int_0^t \|D^w(s)h\| \, \mathrm{d}s. \end{aligned} \tag{226}$$

Combining this, (224), and (225) with Gronwall's integral inequality (see, e.g., Grohs et al. [6, Lemma 2.11] (with $\alpha \leftarrow \|h\|$, $\beta \leftarrow \sup_{s \in [0,T]} \varphi(z^w(s))$, $T \leftarrow T$, $f \leftarrow ([0, T] \ni s \mapsto \|D^w(s)h\| \in [0, \infty))$ for $w, h \in \mathbb{R}^d$ in the notation of Grohs et al. [6, Lemma 2.11])) shows that for all $w, h \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\|D^w(t)h\| \leq \|h\| \exp\!\left(\left[\sup_{s \in [0,T]} \varphi(z^w(s)) \right] t \right) \leq \|h\| \exp\!\left(T \left[\sup_{s \in [0,T]} \varphi(z^w(s)) \right] \right). \tag{227}$$

Combining this with (222) shows that for all $x, y \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\|z^x(t) - z^y(t)\| \leq \sup_{u \in [0,1]} \left[\|x - y\| \exp\!\left(T \left[\sup_{s \in [0,T]} \varphi(z^{(1-u)y+ux}(s)) \right] \right) \right]. \tag{228}$$

This completes the proof of Lemma 7.1. □

7.2 Conditional sub-Hoelder continuity for SDEs

Lemma 7.2. *Let $q \in (0, \infty)$. Then it holds for all $a, b \in [e^q, \infty)$ with $a \leq b$ that*

$$\frac{a^2}{|\ln(a)|^{2q}} \leq \frac{b^2}{|\ln(b)|^{2q}}. \tag{229}$$

Proof of Lemma 7.2. Throughout this proof let $f \colon (1, \infty) \to [0, \infty)$ be the function which satisfies for all $z \in (1, \infty)$ that

$$f(z) = \frac{z^2}{|\ln(z)|^{2q}}. \tag{230}$$

Note that f is a continuously differentiable function and for all $z \in [e^q, \infty)$ it holds that

$$f'(z) = \frac{2z[\ln(z)]^{2q} - 2qz^2[\ln(z)]^{2q-1}z^{-1}}{[\ln(z)]^{4q}} = \frac{2z\ln(z) - 2qz}{[\ln(z)]^{2q+1}} \geq 0. \tag{231}$$

Hence, we obtain that $f|_{[e^q,\infty)}$ is an increasing function. This establishes (229). The proof of Lemma 7.2 is thus completed. □

Lemma 7.3. *Let $q \in [0, \infty)$. Then it holds for all $a, b \in [1, \infty)$ with $a \leq b$ that*

$$\frac{(e^q a)^2}{|\ln(e^q a)|^{2q}} \leq \frac{(e^q b)^2}{|\ln(e^q b)|^{2q}}. \tag{232}$$

Proof of Lemma 7.3. Throughout this proof assume w.l.o.g. that $q > 0$. Observe that Lemma 7.2 (with $q \leftarrow q$, $a \leftarrow e^q a$, $b \leftarrow e^q b$ in the notation of Lemma 7.2) implies (232). The proof of Lemma 7.3 is thus completed. □

Lemma 7.4. *Let $d \in \mathbb{N}$, $T, R, q, K, \mathcal{K} \in [0, \infty)$, $\varphi \in C(\mathbb{R}^d, [0, \infty))$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, let $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes with continuous sample paths which satisfy for all $t \in [0, T]$, $\omega \in \Omega$ that $(\mathbb{R}^d \ni x \mapsto X^x(t, \omega) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, assume for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$, $h \in \big\{v \in \mathbb{R}^d \colon \big([0, T] \ni s \mapsto \big(\frac{\partial}{\partial x} X^x(s, \omega)\big)(v) \in \mathbb{R}^d\big)$ is a $\mathcal{B}([0, T])/\mathcal{B}(\mathbb{R}^d)$-measurable function$\big\}$ that $\int_0^T \big\|\big(\frac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big\| \, \mathrm{d}s < \infty$ and*

$$\big\|\big(\tfrac{\partial}{\partial x} X^x(t, \omega)\big)(h)\big\| \leq \|h\| + \int_0^t \varphi(X^x(s, \omega)) \, \big\|\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big\| \, \mathrm{d}s, \tag{233}$$

assume that $\mathbb{E}\big[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R+1\}} \sup_{t \in [0,T] \cap \mathbb{Q}} \big([\varphi(X^x(t))]^{4q+4}\big)\big] \leq K$, and assume for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R + 1\}$, $t \in [0, T]$ that $\mathbb{E}\big[\|X^x(t)\|^2\big] \leq K$ and $\mathcal{K} = 1 + 2^{4q+4}\big(|\ln(2 + e^q)|^{4q+4} + T^{4q+4}K\big)$. Then it holds for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$, $t \in [0, T]$ that

$$\mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] \leq 2\sqrt{(1 + 4K)\mathcal{K}} \, \big|\ln(\|h\|)\big|^{-q}. \tag{234}$$

Proof of Lemma 7.4. Throughout this proof let $F, G \colon [0, \infty) \to [0, \infty)$ be the functions which satisfy for all $y \in [0, \infty)$ that

$$F(y) = \ln(1 + y), \qquad G(y) = \begin{cases} 0 & : y = 0 \\ [\ln(1 + y)]^{-1} y & : y \neq 0, \end{cases} \tag{235}$$

let $D^x \colon [0, T] \times \Omega \to \mathbb{R}^{d \times d}$, $x \in \mathbb{R}^d$, be the functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ that

$$D^x(t, \omega) = \tfrac{\partial}{\partial x} X^x(t, \omega), \tag{236}$$

let $Y\colon \Omega \to [0,\infty]$ be the function which satisfies for all $\omega \in \Omega$ that

$$Y(\omega) = \left[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R+1\}} \sup_{t\in[0,T]} \varphi(X^x(t,\omega))\right], \tag{237}$$

let $A \subseteq \Omega$ be the set which satisfies

$$A = \{\omega \in \Omega\colon Y(\omega) < \infty\}, \tag{238}$$

and let $Z\colon \Omega \to [0,\infty)$ be the function which satisfies for all $\omega \in \Omega$ that

$$Z(\omega) = \begin{cases} \exp(T\,Y(\omega)) & \colon \omega \in A \\ 1 & \colon \omega \in \Omega \setminus A. \end{cases} \tag{239}$$

Note that (235) implies that for all $y \in [0,\infty)$ it holds that

$$y = G(y)F(y). \tag{240}$$

Hence, we obtain that for all $x,h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] = \mathbb{E}\big[G(\|X^{x+h}(t) - X^x(t)\|)F(\|X^{x+h}(t) - X^x(t)\|)\big]. \tag{241}$$

Next observe that the fundamental theorem of calculus ensures that for all $y \in [0,\infty)$ it holds that

$$\ln(1+y) = \big[\ln(1+z)\big]_{z=0}^{z=y} = \int_0^y \frac{1}{1+z}\,\mathrm{d}z \geq y\left[\inf_{z\in[0,y]} \frac{1}{1+z}\right] = \frac{y}{1+y}. \tag{242}$$

This and (235) show that for all $y \in [0,\infty)$ it holds that

$$G(y) \leq 1 + y. \tag{243}$$

This, the fact that G is a $\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty))$-measurable function, the fact that for all $a,b \in \mathbb{R}$ it holds that $|a+b|^2 \leq 2(|a|^2 + |b|^2)$, and the triangle inequality show that for all $x,h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\begin{aligned} \mathbb{E}\big[\big|G(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] &\leq \mathbb{E}\big[\big(1 + \|X^{x+h}(t) - X^x(t)\|\big)^2\big] \\ &\leq \mathbb{E}\big[2\big(1 + \|X^{x+h}(t) - X^x(t)\|^2\big)\big] \\ &= 2\big(1 + \mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|^2\big]\big) \\ &\leq 2\big(1 + \mathbb{E}\big[(\|X^{x+h}(t)\| + \|X^x(t)\|)^2\big]\big) \\ &\leq 2\big(1 + \mathbb{E}\big[2\big(\|X^{x+h}(t)\|^2 + \|X^x(t)\|^2\big)\big]\big) \\ &= 2\big(1 + 2\big(\mathbb{E}\big[\|X^{x+h}(t)\|^2\big] + \mathbb{E}\big[\|X^x(t)\|^2\big]\big)\big). \end{aligned} \tag{244}$$

The hypothesis that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R+1\}$, $t \in [0,T]$ it holds that $\mathbb{E}[\|X^x(t)\|^2] \leq K$ hence implies that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \colon \|v\| < 1\}$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\big|G(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] \leq 2(1 + 2(K+K)) = 2 + 8K. \tag{245}$$

In the next step we note that (233), the hypothesis that for all $x \in \mathbb{R}^d$, $\omega \in \Omega$, $h \in \big\{v \in \mathbb{R}^d \colon \big([0,T] \ni s \mapsto \big(\tfrac{\partial}{\partial x} X^x(s,\omega)\big)(v) \in \mathbb{R}^d\big) \text{ is a } \mathcal{B}([0,T])/\mathcal{B}(\mathbb{R}^d)\text{-measurable function}\big\}$ it holds that $\int_0^T \big\|\big(\tfrac{\partial}{\partial x} X^x(s,\omega)\big)(h)\big\| \, \mathrm{d}s < \infty$, and Lemma 7.1 (with $d \leftarrow d$, $T \leftarrow T$, $\varphi \leftarrow \varphi$, $\|\cdot\| \leftarrow \|\cdot\|$, $(z^x)_{x \in \mathbb{R}^d} \leftarrow \big([0,T] \ni t \mapsto X^x(t,\omega) \in \mathbb{R}^d\big)_{x \in \mathbb{R}^d}$ for $\omega \in \Omega$ in the notation of Lemma 7.1) show that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\|X^{x+h}(t,\omega) - X^x(t,\omega)\| \leq \sup_{u \in [0,1]} \left[\|h\| \exp\!\left(T\left[\sup_{s \in [0,T]} \varphi(X^{x+uh}(s,\omega))\right]\right)\right]. \tag{246}$$

Therefore, we obtain that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \colon \|v\| < 1\}$, $t \in [0,T]$, $\omega \in A$ it holds that

$$\begin{aligned} \|X^{x+h}(t,\omega) - X^x(t,\omega)\| &\leq \sup_{y \in \{z \in \mathbb{R}^d \colon \|z\| \leq R+1\}} \left[\|h\| \exp\!\left(T\left[\sup_{s \in [0,T]} \varphi(X^y(s,\omega)\right]\right)\right] \\ &= \|h\| \exp(T\,Y(\omega)) \\ &= \|h\|\, Z(\omega). \end{aligned} \tag{247}$$

Furthermore, observe that Lemma 3.6 (with $d \leftarrow d$, $T \leftarrow T$, $R \leftarrow R+1$, $\|\cdot\| \leftarrow \|\cdot\|$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(Y^x)_{x \in \mathbb{R}^d} \leftarrow \big([0,T] \times \Omega \ni (t,\omega) \mapsto \varphi(X^x(t,\omega)) \in [0,\infty)\big)_{x \in \mathbb{R}^d}$ in the notation of Lemma 3.6) ensures

(a) that for all $\omega \in \Omega$ it holds that

$$Y(\omega) = \left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R+1\}} \sup_{t \in [0,T] \cap \mathbb{Q}} \varphi(X^x(t,\omega))\right] \tag{248}$$

and

(b) that Y is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function.

Observe that (238) and the fact that Y is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function imply that

$$A \in \mathcal{F}. \tag{249}$$

Next note that the hypothesis that $\mathbb{E}\big[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R+1\}} \sup_{t \in [0,T] \cap \mathbb{Q}} \big([\varphi(X^x(t))]^{4q+4}\big)\big] \leq K$, the fact that Y is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function, and (248) ensure that

$$\mathbb{E}[Y] \leq \mathbb{E}\left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R+1\}} \sup_{t \in [0,T] \cap \mathbb{Q}} \big(1 + [\varphi(X^x(t))]^{4q+4}\big)\right] \leq 1 + K < \infty. \tag{250}$$

Combining this with (249) implies that

$$\mathbb{P}(A) = 1. \tag{251}$$

Furthermore, note that (239), (249), and the fact that Y is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function show that Z is an $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable function. Combining this, (247), and (251) with (235) and the fact that F is a $\mathcal{B}([0,\infty))/\mathcal{B}([0,\infty))$-measurable function demonstrates that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \colon \|v\| < 1\}$, $t \in [0,T]$ it holds that

$$\begin{aligned} \mathbb{E}\big[\big|F(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] &= \mathbb{E}\big[\big|\ln(1 + \|X^{x+h}(t) - X^x(t)\|)\big|^2\big] \\ &\leq \mathbb{E}\big[\big|\ln(1 + \|h\| Z)\big|^2\big]. \end{aligned} \tag{252}$$

In the next step we observe that for all $\omega \in A$ it holds that

$$[Y(\omega)]^{4q+4} = \left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R+1\}} \sup_{t \in [0,T] \cap \mathbb{Q}} \big([\varphi(X^x(t,\omega))]^{4q+4}\big)\right]. \tag{253}$$

The hypothesis that $\mathbb{E}\big[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R+1\}} \sup_{t \in [0,T] \cap \mathbb{Q}} \big([\varphi(X^x(t))]^{4q+4}\big)\big] \leq K$ and (251) hence show that

$$\mathbb{E}\big[Y^{4q+4}\big] \leq K. \tag{254}$$

Moreover, note that the fact that for all $a, b \in \mathbb{R}$, $r \in [0,\infty)$ it holds that $|a+b|^r \leq 2^r(|a|^r + |b|^r)$ ensures that for all $C \in [1,\infty)$, $r \in [0, 4q+4]$, $\omega \in A$ it holds that

$$\begin{aligned} |\ln(CZ(\omega))|^r &\leq 1 + |\ln(CZ(\omega))|^{4q+4} \\ &= 1 + |\ln(C) + \ln(Z(\omega))|^{4q+4} \\ &= 1 + |\ln(C) + T\,Y(\omega)|^{4q+4} \\ &\leq 1 + 2^{4q+4}\big(|\ln(C)|^{4q+4} + T^{4q+4}[Y(\omega)]^{4q+4}\big). \end{aligned} \tag{255}$$

Combining this and the fact that Z is an $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable function with (251) and (254) proves that for all $C \in [1, 2+e^q]$, $r \in [0, 4q+4]$ it holds that

$$\mathbb{E}\big[|\ln(CZ)|^r\big] \leq 1 + 2^{4q+4}\big(|\ln(2+e^q)|^{4q+4} + T^{4q+4}K\big) = \mathcal{K}. \tag{256}$$

Next note that the fact that Z is an $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable function, the fact that for all $\omega \in \Omega$ it holds that $Z(\omega) \geq 1$, and the fact that for all $y \in [0,\infty)$ it holds that $\ln(1+y) \leq y$ show that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\begin{aligned} \mathbb{E}\big[|\ln(1 + \|h\|Z)|^2 \mathbb{1}_{\{Z \leq 1/\|h\|\}}\big] &\leq \mathbb{E}\big[\|h\|^2 Z^2 \mathbb{1}_{\{Z \leq 1/\|h\|\}}\big] \\ &= \|h\|^2 e^{-2q}\, \mathbb{E}\big[(e^q Z)^2 \mathbb{1}_{\{Z \leq 1/\|h\|\}}\big] \\ &= \|h\|^2 e^{-2q}\, \mathbb{E}\left[\frac{(e^q Z)^2}{|\ln(e^q Z)|^{2q}} |\ln(e^q Z)|^{2q} \mathbb{1}_{\{Z \leq 1/\|h\|\}}\right]. \end{aligned} \tag{257}$$

The fact that for all $\omega \in \Omega$ it holds that $Z(\omega) \geq 1$ and Lemma 7.3 hence prove that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\begin{aligned}
\mathbb{E}\big[|\ln(1 + \|h\|Z)|^2 \mathbb{1}_{\{Z \leq 1/\|h\|\}}\big] &\leq \|h\|^2 e^{-2q}\, \mathbb{E}\left[\frac{\big(\frac{e^q}{\|h\|}\big)^2}{\big|\ln\big(\frac{e^q}{\|h\|}\big)\big|^{2q}} |\ln(e^q Z)|^{2q} \mathbb{1}_{\{Z \leq 1/\|h\|\}}\right] \\
&= \big|\ln\big(\tfrac{e^q}{\|h\|}\big)\big|^{-2q}\, \mathbb{E}\big[|\ln(e^q Z)|^{2q} \mathbb{1}_{\{Z \leq 1/\|h\|\}}\big] \\
&\leq \big|\ln\big(\tfrac{e^q}{\|h\|}\big)\big|^{-2q}\, \mathbb{E}\big[|\ln(e^q Z)|^{2q}\big] \\
&= \big|q - \ln(\|h\|)\big|^{-2q}\, \mathbb{E}\big[|\ln(e^q Z)|^{2q}\big] \\
&= \big(q + \big|\ln(\|h\|)\big|\big)^{-2q}\, \mathbb{E}\big[|\ln(e^q Z)|^{2q}\big] \\
&\leq \big|\ln(\|h\|)\big|^{-2q}\, \mathbb{E}\big[|\ln(e^q Z)|^{2q}\big].
\end{aligned} \tag{258}$$

This and (256) (with $C \leftarrow e^q$, $r \leftarrow 2q$ in the notation of (256)) establish that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\mathbb{E}\big[|\ln(1 + \|h\|Z)|^2 \mathbb{1}_{\{Z \leq 1/\|h\|\}}\big] \leq \mathcal{K}\, \big|\ln(\|h\|)\big|^{-2q}. \tag{259}$$

In the next step we observe that (256) (with $C \leftarrow 1$, $r \leftarrow 4q$ in the notation of (256)) and the fact that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$, $\omega \in \big\{Z > \frac{1}{\|h\|}\big\}$ it holds that $|\ln(Z(\omega))|^{4q} |\ln(1/\|h\|)|^{-4q} \geq 1$ imply that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\begin{aligned}
\mathbb{E}\big[\mathbb{1}_{\{Z > 1/\|h\|\}}\big] &\leq \mathbb{E}\big[|\ln(Z)|^{4q} |\ln(1/\|h\|)|^{-4q} \mathbb{1}_{\{Z > 1/\|h\|\}}\big] \\
&\leq \mathbb{E}\big[|\ln(Z)|^{4q} |\ln(1/\|h\|)|^{-4q}\big] \\
&= \mathbb{E}\big[|\ln(Z)|^{4q}\big] \big|\ln(\|h\|)\big|^{-4q} \\
&\leq \mathcal{K}\, \big|\ln(\|h\|)\big|^{-4q}.
\end{aligned} \tag{260}$$

Furthermore, observe that (256) (with $C \leftarrow 2$, $r \leftarrow 4$ in the notation of (256)) shows that

$$\mathbb{E}\big[|\ln(2Z)|^4\big] \leq \mathcal{K}. \tag{261}$$

This and (260) ensure that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\mathbb{E}\big[\mathbb{1}_{\{Z > 1/\|h\|\}}\big] < \infty \qquad \text{and} \qquad \mathbb{E}\big[|\ln(2Z)|^4\big] < \infty. \tag{262}$$

Combining this, (260), (261), and the fact that for all $\omega \in \Omega$ it holds that $Z(\omega) \geq 1$ with the Cauchy-Schwarz inequality establishes that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\begin{aligned}
\mathbb{E}\big[|\ln(1 + \|h\|Z)|^2 \mathbb{1}_{\{Z > 1/\|h\|\}}\big] &\leq \mathbb{E}\big[|\ln(Z + Z)|^2 \mathbb{1}_{\{Z > 1/\|h\|\}}\big] \\
&\leq \Big(\mathbb{E}\big[|\ln(2Z)|^4\big] \mathbb{E}\big[\mathbb{1}_{\{Z > 1/\|h\|\}}\big]\Big)^{1/2} \\
&\leq \big(\mathcal{K}^2\, \big|\ln(\|h\|)\big|^{-4q}\big)^{1/2} \\
&= \mathcal{K}\, \big|\ln(\|h\|)\big|^{-2q}.
\end{aligned} \tag{263}$$

Combining this, (252), and (259) proves that for all $x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} : \|v\| < 1\}$, $t \in [0, T]$ it holds that

$$\begin{aligned}
&\mathbb{E}\big[\big|F(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] \\
&\leq \mathbb{E}\big[|\ln(1 + \|h\|Z)|^2(\mathbb{1}_{\{Z \leq 1/\|h\|\}} + \mathbb{1}_{\{Z > 1/\|h\|\}})\big] \\
&= \mathbb{E}\big[|\ln(1 + \|h\|Z)|^2\mathbb{1}_{\{Z \leq 1/\|h\|\}}\big] + \mathbb{E}\big[|\ln(1 + \|h\|Z)|^2\mathbb{1}_{\{Z > 1/\|h\|\}}\big] \\
&\leq \mathcal{K}\,\big|\ln(\|h\|)\big|^{-2q} + \mathcal{K}\,\big|\ln(\|h\|)\big|^{-2q} \\
&= 2\mathcal{K}\,\big|\ln(\|h\|)\big|^{-2q}.
\end{aligned} \tag{264}$$

This and (245) ensure that for all $x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} : \|v\| < 1\}$, $t \in [0, T]$ it holds that

$$\mathbb{E}\big[\big|G(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] < \infty \qquad \text{and} \qquad \mathbb{E}\big[\big|F(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] < \infty. \tag{265}$$

Combining this and the Cauchy-Schwarz inequality with (241) proves that for all $x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} : \|v\| < 1\}$, $t \in [0, T]$ it holds that

$$\mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] \leq \Big(\mathbb{E}\big[\big|G(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big]\mathbb{E}\big[\big|F(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big]\Big)^{1/2}. \tag{266}$$

This, (245), and (264) prove that for all $x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} : \|v\| < 1\}$, $t \in [0, T]$ it holds that

$$\mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] \leq \big((4 + 16K)\mathcal{K}\,\big|\ln(\|h\|)\big|^{-2q}\big)^{1/2} = 2\sqrt{(1 + 4K)\mathcal{K}}\,\big|\ln(\|h\|)\big|^{-q}. \tag{267}$$

This completes the proof of Lemma 7.4. □

Lemma 7.5. *Let $d \in \mathbb{N}$, $T, \kappa \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\|\cdot\| : \mathbb{R}^d \to [0, \infty)$ be a norm, let $X^x : [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes which satisfy for all $\omega \in \Omega$ that $([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto X^x(t, \omega) \in \mathbb{R}^d) \in C^{0,1}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$, assume for all $R, r \in [0, \infty)$ that $\mathbb{E}\big[\sup_{x \in \{z \in \mathbb{Q}^d : \|z\| \leq R\}} \sup_{t \in [0,T] \cap \mathbb{Q}}\big(\|X^x(t)\|^r\big)\big] < \infty$, and assume for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ that*

$$\big\|\big(\tfrac{\partial}{\partial x} X^x(t, \omega)\big)(h)\big\| \leq \|h\| + \kappa \int_0^t (1 + \|X^x(s, \omega)\|^\kappa)\,\big\|\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big\|\,\mathrm{d}s. \tag{268}$$

Then it holds for all $R, q \in [0, \infty)$ that there exists $c \in [0, \infty)$ such that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} : \|v\| < 1\}$ it holds that

$$\Bigg[\sup_{x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}} \sup_{t \in [0,T]} \mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big]\Bigg] \leq c\,\big|\ln(\|h\|)\big|^{-q}. \tag{269}$$

Proof of Lemma 7.5. Throughout this proof let $\varphi\colon \mathbb{R}^d \to [0,\infty)$ be the function which satisfies for all $x \in \mathbb{R}^d$ that

$$\varphi(x) = \kappa(1 + \|x\|^\kappa) \tag{270}$$

and let $K_{R,q} \in [0,\infty]$, $R, q \in [0,\infty)$, satisfy for all $R, q \in [0,\infty)$ that

$$K_{R,q} = \max\left\{ \mathbb{E}\left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \le R+1\}} \sup_{t \in [0,T]\cap\mathbb{Q}} \left([\varphi(X^x(t))]^{4q+4} \right) \right], \mathbb{E}\left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \le R+1\}} \sup_{t \in [0,T]\cap\mathbb{Q}} \left(\|X^x(t)\|^2 \right) \right] \right\}. \tag{271}$$

Note that (270) and Lemma 6.2 show that for all $q \in [0,\infty)$, $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} [\varphi(X^x(t,\omega))]^{4q+4} &= \left[\kappa + \kappa\|X^x(t,\omega)\|^\kappa\right]^{4q+4} \\ &\le 2^{4q+3}\left(\kappa^{4q+4} + \kappa^{4q+4}\|X^x(t,\omega)\|^{\kappa(4q+4)}\right). \end{aligned} \tag{272}$$

Furthermore, observe that the hypothesis that for all $\omega \in \Omega$ it holds that $[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto X^x(t,\omega) \in \mathbb{R}^d$ is a continuous function ensures that for all $R, q \in [0,\infty)$, $\omega \in \Omega$ it holds that

$$\left[\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \le R+1\}} \sup_{t \in [0,T]} \|X^x(t,\omega)\|^{\kappa(4q+4)} \right] < \infty. \tag{273}$$

Combining this with (272) and the hypothesis that for all $R, r \in [0,\infty)$ it holds that $\mathbb{E}\big[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \le R\}} \sup_{t \in [0,T]\cap\mathbb{Q}}\big(\|X^x(t)\|^r\big)\big] < \infty$ demonstrates that for all $R, q \in [0,\infty)$ it holds that

$$\begin{aligned} &\mathbb{E}\left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \le R+1\}} \sup_{t \in [0,T]\cap\mathbb{Q}} \left([\varphi(X^x(t))]^{4q+4} \right) \right] \\ &\le 2^{4q+3}\left(\kappa^{4q+4} + \kappa^{4q+4}\, \mathbb{E}\left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \le R+1\}} \sup_{t \in [0,T]\cap\mathbb{Q}} \left(\|X^x(t)\|^{4\kappa(q+1)} \right) \right] \right) < \infty. \end{aligned} \tag{274}$$

This and the hypothesis that for all $R, r \in [0,\infty)$ it holds that $\mathbb{E}\big[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \le R\}} \sup_{t \in [0,T]\cap\mathbb{Q}}\big(\|X^x(t)\|^r\big)\big] < \infty$ prove that for all $R, q \in [0,\infty)$ it holds that

$$K_{R,q} < \infty. \tag{275}$$

Furthermore, note that Lemma 3.6 (with $d \leftarrow d$, $T \leftarrow T$, $R \leftarrow R+1$, $\|\cdot\| \leftarrow \|\cdot\|$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $(Y^x)_{x \in \mathbb{R}^d} \leftarrow \big([0,T] \times \Omega \ni (t,\omega) \mapsto \|X^x(t,\omega)\|^2 \in [0,\infty)\big)_{x \in \mathbb{R}^d}$ for $R \in [0,\infty)$ in the notation of Lemma 3.6) shows that for all $R \in [0,\infty)$, $\omega \in \Omega$ it holds that

$$\left[\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \le R+1\}} \sup_{t \in [0,T]} \left(\|X^x(t,\omega)\|^2 \right) \right] = \left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \le R+1\}} \sup_{t \in [0,T]\cap\mathbb{Q}} \left(\|X^x(t,\omega)\|^2 \right) \right]. \tag{276}$$

Hence, we obtain that for all $R, q \in [0, \infty)$, $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R + 1\}$, $t \in [0, T]$ it holds that

$$\mathbb{E}\big[\|X^x(t)\|^2\big] \leq \mathbb{E}\Bigg[\sup_{y \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R+1\}} \sup_{s \in [0,T] \cap \mathbb{Q}} \big(\|X^y(s)\|^2\big)\Bigg] \leq K_{R,q}. \tag{277}$$

Moreover, observe that the hypothesis that for all $\omega \in \Omega$ it holds that $([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto X^x(t, \omega) \in \mathbb{R}^d) \in C^{0,1}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ implies that for all $x, h \in \mathbb{R}^d$, $\omega \in \Omega$ it holds that $[0, T] \ni t \mapsto \big(\tfrac{\partial}{\partial x} X^x(t, \omega)\big)(h) \in \mathbb{R}^d$ is a continuous function. This implies that for all $x, h \in \mathbb{R}^d$, $\omega \in \Omega$ it holds that

$$\int_0^T \big\|\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big\| \, \mathrm{d}s < \infty. \tag{278}$$

Furthermore, note that (268) and (270) demonstrate that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$\big\|\big(\tfrac{\partial}{\partial x} X^x(t, \omega)\big)(h)\big\| \leq \|h\| + \int_0^t \varphi(X^x(s, \omega)) \, \big\|\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big\| \, \mathrm{d}s. \tag{279}$$

Combining this, (275), (277), and (278) with Lemma 7.4 (with $d \leftarrow d$, $T \leftarrow T$, $R \leftarrow R$, $q \leftarrow q$, $K \leftarrow K_{R,q}$, $\mathcal{K} \leftarrow 1 + 2^{4q+4}(|\ln(2 + e^q)|^{4q+4} + T^{4q+4} K_{R,q})$, $\varphi \leftarrow \varphi$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $\|\cdot\| \leftarrow \|\cdot\|$, $(X^x)_{x \in \mathbb{R}^d} \leftarrow (X^x)_{x \in \mathbb{R}^d}$ for $R, q \in [0, \infty)$ in the notation of Lemma 7.4) hence proves that for all $R, q \in [0, \infty)$, $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$, $t \in [0, T]$ it holds that

$$\begin{aligned} &\mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] \\ &\leq 2\sqrt{(1 + 4K_{R,q})\big(1 + 2^{4q+4}\big(|\ln(2 + e^q)|^{4q+4} + T^{4q+4} K_{R,q}\big)\big)} \, \big|\ln(\|h\|)\big|^{-q}. \end{aligned} \tag{280}$$

This completes the proof of Lemma 7.5. □

8 Regularity with respect to the initial value for SDEs

In this section we establish in Theorem 8.4 in Subsection 8.3 below the main result of this article. Theorem 8.4 proves that every additive noise driven SDE with a drift coefficient function whose derivatives grows at most polynomially and which also admits a Lyapunov-type condition (which ensures the existence of a unique solution) is at least logarithmically Hoelder continuous in the initial value (see (303) in Theorem 8.4 below for the precise statement). Our proof of Theorem 8.4 exploits Corollary 8.2 in Subsection 8.2 below and the auxiliary continuity-regularity result in Lemma 8.3 in Subsection 8.3 below. Our proof of Corollary 8.2 is based on an application of Proposition 8.1 below. Our proof of Proposition 8.1, in turn, uses the regularity result in Lemma 7.5 in Subsection 7.2 above.

8.1 Regularity with respect to the initial value for SDEs with general noise

Proposition 8.1. *Let $d, m \in \mathbb{N}$, $T, \kappa \in [0, \infty)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d \times m}$, $\varphi \in C(\mathbb{R}^m, [0, \infty))$, $V \in C^1(\mathbb{R}^d, [0, \infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, assume for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \varphi(z)V(x)$, and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a stochastic process with continuous sample paths, and assume for all $c \in [0, \infty)$ that $\mathbb{E}\big[\sup_{t \in [0,T] \cap \mathbb{Q}} \exp\big(c\,\varphi(W(t))\big)\big] + \mathbb{E}\big[\sup_{t \in [0,T] \cap \mathbb{Q}} (\|\sigma W(t)\|^c)\big] < \infty$. Then*

(i) there exist unique stochastic processes $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths which satisfy for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ that

$$X^x(t, \omega) = x + \int_0^t \mu(X^x(s, \omega))\,\mathrm{d}s + \sigma W(t, \omega), \tag{281}$$

(ii) it holds for all $R, r \in [0, \infty)$ that $\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T]} \mathbb{E}\big[\|X^x(t)\|^r\big] < \infty$, and

(iii) it holds for all $R, q \in [0, \infty)$ that there exists $c \in (0, \infty)$ such that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\left[\sup_{x \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}} \sup_{t \in [0,T]} \mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big]\right] \leq c\,\big|\ln(\|h\|)\big|^{-q}. \tag{282}$$

Proof of Proposition 8.1. First, observe that Lemma 5.4 shows

(a) that there exist unique stochastic processes $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths which satisfy for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ that

$$X^x(t, \omega) = x + \int_0^t \mu(X^x(s, \omega))\,\mathrm{d}s + \sigma W(t, \omega), \tag{283}$$

(b) that for all $\omega \in \Omega$ it holds that

$$\big([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto X^x(t, \omega) \in \mathbb{R}^d\big) \in C^{0,1}([0, T] \times \mathbb{R}^d, \mathbb{R}^d), \tag{284}$$

and

(c) that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$\big(\tfrac{\partial}{\partial x} X^x(t, \omega)\big)(h) = h + \int_0^t \mu'(X^x(s, \omega))\big(\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big)\,\mathrm{d}s. \tag{285}$$

In the next step we note that Lemma 6.6 ensures

(A) that for all $R, r \in [0, \infty)$ it holds that

$$\Omega \ni \omega \mapsto \left[\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T]} (\|X^x(t, \omega)\|^r) \right] \in [0, \infty] \tag{286}$$

is an $\mathcal{F}/\mathcal{B}([0, \infty])$-measurable function and

(B) that for all $R, r \in [0, \infty)$ it holds that

$$\mathbb{E}\left[\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T]} \big(\|X^x(t)\|^r\big) \right] < \infty. \tag{287}$$

Furthermore, observe that for all $R, r \in [0, \infty)$, $y \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $s \in [0, T]$ it holds that

$$\mathbb{E}\big[\|X^y(s)\|^r\big] \leq \mathbb{E}\left[\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T]} \big(\|X^x(t)\|^r\big) \right]. \tag{288}$$

Combining this with (287) establishes (ii). Moreover, note that (287) and Lemma 3.6 prove that for all $R, r \in [0, \infty)$ it holds that

$$\mathbb{E}\left[\sup_{x \in \{z \in \mathbb{Q}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T] \cap \mathbb{Q}} \big(\|X^x(t)\|^r\big) \right] < \infty. \tag{289}$$

In the next step we combine (285), the triangle inequality, and the hypothesis that for all $x, h \in \mathbb{R}^d$ it holds that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$ to obtain that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned} \big\|\big(\tfrac{\partial}{\partial x} X^x(t, \omega)\big)(h)\big\| &\leq \|h\| + \left\| \int_0^t \mu'(X^x(s, \omega))\big(\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big) \, \mathrm{d}s \right\| \\ &\leq \|h\| + \int_0^t \big\|\mu'(X^x(s, \omega))\big(\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big)\big\| \, \mathrm{d}s \\ &\leq \|h\| + \int_0^t \kappa(1 + \|X^x(s, \omega)\|^\kappa) \, \big\|\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big\| \, \mathrm{d}s \\ &= \|h\| + \kappa \int_0^t (1 + \|X^x(s, \omega)\|^\kappa) \, \big\|\big(\tfrac{\partial}{\partial x} X^x(s, \omega)\big)(h)\big\| \, \mathrm{d}s. \end{aligned} \tag{290}$$

Combining this, (284), and (289) with Lemma 7.5 (with $d \leftarrow d$, $T \leftarrow T$, $\kappa \leftarrow \kappa$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $\|\cdot\| \leftarrow \|\cdot\|$, $(X^x)_{x \in \mathbb{R}^d} \leftarrow (X^x)_{x \in \mathbb{R}^d}$ in the notation of Lemma 7.5) establishes (iii). The proof of Proposition 8.1 is thus completed. □

8.2 Regularity with respect to the initial value for SDEs with Wiener noise

Corollary 8.2. *Let $d, m \in \mathbb{N}$, $T, \kappa \in [0, \infty)$, $\alpha \in [0, 2)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d \times m}$, $V \in C^1(\mathbb{R}^d, [0, \infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ and $|||\cdot||| \colon \mathbb{R}^m \to [0, \infty)$ be norms, assume for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |||z|||^\alpha)V(x)$, and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then*

(i) there exist unique stochastic processes $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$X^x(t, \omega) = x + \int_0^t \mu(X^x(s, \omega))\, \mathrm{d}s + \sigma W(t, \omega), \tag{291}$$

(ii) it holds for all $R, r \in [0, \infty)$ that $\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0, T]} \mathbb{E}\big[\|X^x(t)\|^r\big] < \infty$, and

(iii) it holds for all $R, q \in [0, \infty)$ that there exists $c \in (0, \infty)$ such that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\left[\sup_{x \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}} \sup_{t \in [0, T]} \mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big]\right] \leq c\, \big|\ln(\|h\|)\big|^{-q}. \tag{292}$$

Proof of Corollary 8.2. Throughout this proof let $\varphi \colon \mathbb{R}^m \to [0, \infty)$ be the function which satisfies for all $z \in \mathbb{R}^m$ that

$$\varphi(z) = \kappa(1 + |||z|||^\alpha). \tag{293}$$

Note that Lemma 6.4 (with $m \leftarrow m$, $T \leftarrow T$, $C \leftarrow \kappa$, $\alpha \leftarrow \alpha$, $\|\cdot\| \leftarrow |||\cdot|||$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$, $\varphi \leftarrow \varphi$ in the notation of Lemma 6.4) shows that for all $c \in [0, \infty)$ it holds that

$$\mathbb{E}\left[\sup_{t \in [0, T] \cap \mathbb{Q}} \exp\big(c\, \varphi(W(t))\big)\right] < \infty. \tag{294}$$

In addition, observe that Lemma 6.5 (with $d \leftarrow d$, $m \leftarrow m$, $T \leftarrow T$, $r \leftarrow c$, $\sigma \leftarrow \sigma$, $\|\cdot\| \leftarrow \|\cdot\|$, $(\Omega, \mathcal{F}, \mathbb{P}) \leftarrow (\Omega, \mathcal{F}, \mathbb{P})$, $W \leftarrow W$ for $c \in [0, \infty)$ in the notation of Lemma 6.5) ensures that for all $c \in [0, \infty)$ it holds that

$$\mathbb{E}\left[\sup_{t \in [0, T] \cap \mathbb{Q}} (\|\sigma W(t)\|^c)\right] < \infty. \tag{295}$$

Combining this and (294) with Proposition 8.1 establishes (i), (ii), and (iii). The proof of Corollary 8.2 is thus completed. □

8.3 Sub-Hoelder continuity with respect to the initial value for SDEs

Lemma 8.3. *Let $d \in \mathbb{N}$, $T, R, q, c, C \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X^x \colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ that*

$$\left[\sup_{x \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}} \sup_{t \in [0,T]} \mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big]\right] \leq c\,\big|\ln(\|h\|)\big|^{-q}, \tag{296}$$

and assume that $C = \sup_{x \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}} \sup_{t \in [0,T]} \mathbb{E}\big[\|X^x(t)\|\big]$. Then it holds for all $x, y \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}$ with $0 < \|x - y\| \neq 1$ that

$$\sup_{t \in [0,T]} \mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] \leq \max\{c, 2C\,|\ln(2R+1)|^q\}\,\big|\ln(\|x-y\|)\big|^{-q}. \tag{297}$$

Proof of Lemma 8.3. First, note that (296) implies that for all $x, y \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}$ with $x \neq y$ and $\|x - y\| < 1$ it holds that

$$\sup_{t \in [0,T]} \mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] = \sup_{t \in [0,T]} \mathbb{E}\big[\|X^{y+(x-y)}(t) - X^y(t)\|\big] \leq \frac{c}{|\ln(\|x-y\|)|^q}. \tag{298}$$

Furthermore, observe that the triangle inequality and the hypothesis that $C < \infty$ show that for all $x, y \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] \leq \mathbb{E}\big[\|X^x(t)\| + \|X^y(t)\|\big] = \mathbb{E}\big[\|X^x(t)\|\big] + \mathbb{E}\big[\|X^y(t)\|\big] \leq 2C. \tag{299}$$

The fact that for all $q \in [0,\infty)$ it holds that $[1,\infty) \ni z \mapsto |\ln(z)|^q \in \mathbb{R}$ is an increasing function and the fact that for all $x, y \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}$ it holds that $\|x - y\| \leq 2R$ hence show that for all $t \in [0,T]$, $x, y \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}$ with $\|x - y\| > 1$ it holds that

$$\sup_{t \in [0,T]} \mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] \leq \frac{2C\,|\ln(\|x-y\|)|^q}{|\ln(\|x-y\|)|^q} \leq \frac{2C\,|\ln(2R+1)|^q}{|\ln(\|x-y\|)|^q}. \tag{300}$$

Combining this with (298) demonstrates that for all $x, y \in \{v \in \mathbb{R}^d \colon \|v\| \leq R\}$ with $0 < \|x - y\| \neq 1$ it holds that

$$\begin{aligned}\sup_{t \in [0,T]} \mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] &\leq \max\left\{\frac{c}{|\ln(\|x-y\|)|^q}, \frac{2C\,|\ln(2R+1)|^q}{|\ln(\|x-y\|)|^q}\right\} \\ &= \max\{c, 2C\,|\ln(2R+1)|^q\}\,\big|\ln(\|x-y\|)\big|^{-q}.\end{aligned} \tag{301}$$

The proof of Lemma 8.3 is thus completed. □

Theorem 8.4. *Let $d, m \in \mathbb{N}$, $T, \kappa \in [0,\infty)$, $\alpha \in [0,2)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d \times m}$, $V \in C^1(\mathbb{R}^d, [0,\infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ and $|||\cdot||| \colon \mathbb{R}^m \to [0,\infty)$ be norms, assume for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |||z|||^\alpha)V(x)$, and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then* *(i) there exist unique stochastic processes $X^x\colon [0,T]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, with continuous sample paths such that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ it holds that*

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega) \tag{302}$$

and

(ii) it holds for all $R,q\in[0,\infty)$ that there exists $c\in(0,\infty)$ such that for all $x,y\in\{v\in\mathbb{R}^d\colon \|v\|\le R\}$ with $0<\|x-y\|\neq 1$ it holds that

$$\sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t)-X^y(t)\|\big] \le c\,\big|\ln(\|x-y\|)\big|^{-q}. \tag{303}$$

Proof of Theorem 8.4. Throughout this proof let $\varphi\colon\mathbb{R}^m\to[0,\infty)$ be the function which satisfies for all $z\in\mathbb{R}^m$ that

$$\varphi(z) = \kappa(1+|||z|||^\alpha). \tag{304}$$

Note that Corollary 8.2 establishes

(a) that there exist unique stochastic processes $X^x\colon[0,T]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, with continuous sample paths such that for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ it holds that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{305}$$

(b) that for all $R,r\in[0,\infty)$ it holds that

$$\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}}\ \sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t)\|^r\big] < \infty, \tag{306}$$

and

(c) that there exist $c_{R,q}\in(0,\infty)$, $R,q\in[0,\infty)$, such that for all $R,q\in[0,\infty)$, $h\in\{v\in\mathbb{R}^d\setminus\{0\}\colon \|v\|<1\}$ it holds that

$$\sup_{x\in\{v\in\mathbb{R}^d\colon \|v\|\le R\}}\ \sup_{t\in[0,T]} \mathbb{E}\big[\|X^{x+h}(t)-X^x(t)\|\big] \le c_{R,q}\,\big|\ln(\|h\|)\big|^{-q}. \tag{307}$$

Combining (306) and (307) with Lemma 8.3 (with $d\leftarrow d$, $T\leftarrow T$, $R\leftarrow R$, $q\leftarrow q$, $c\leftarrow c_{R,q}$, $C\leftarrow \sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}}\sup_{t\in[0,T]}\mathbb{E}\big[\|X^x(t)\|\big]$, $(\Omega,\mathcal{F},\mathbb{P})\leftarrow(\Omega,\mathcal{F},\mathbb{P})$, $(X^x)_{x\in\mathbb{R}^d}\leftarrow(X^x)_{x\in\mathbb{R}^d}$, $\|\cdot\|\leftarrow\|\cdot\|$ for $R,q\in[0,\infty)$ in the notation of Lemma 8.3) establishes (ii). The proof of Theorem 8.4 is thus completed. ∎

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