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Abstract

The explicit Euler scheme and similar explicit approximation schemes (such as the Milstein scheme) are known to diverge strongly and numerically weakly in the case of one-dimensional stochastic ordinary differential equations with superlinearly growing nonlinearities. It remained an open question whether such a divergence phenomenon also holds in the case of stochastic partial differential equations with superlinearly growing nonlinearities such as stochastic Allen-Cahn equations. In this work we solve this problem by proving that full-discrete exponential Euler and full-discrete linear-implicit Euler approximations diverge strongly and numerically weakly in the case of stochastic Allen-Cahn equations. This article also contains a short literature

overview on existing numerical approximation results for stochastic differential equations with superlinearly growing nonlinearities.

Contents

1 Introduction

Stochastic differential equations (SDEs), by which we mean both stochastic ordinary differential equations (SODEs) and stochastic partial differential equations (SPDEs), appear in many real-world models in engineering and applied sciences. In particular, SDEs are intensively employed in financial engineering to model prices of financial derivatives (cf., e.g., Filipović et al. [37, (1.3)] and Harms et al. [57, Theorem 3.5]), in molecular dynamics to describe a system of particles immersed in a fluid bath (cf., e.g., Leimkuhler & Matthews [104, (6.32) and (6.33)]), in nonlinear filtering problems in engineering to describe the density of the state variable (cf., e.g., Zakai [163, (18) and (30)] and Kushner [102, (1)]), as well as in quantum mechanics to model the temporal dynamics associated to Euclidean quantum field theories (cf., e.g., Mourrat & Weber [129, (1.1)]). The vast majority of SDEs appearing in these models contain superlinearly growing nonlinearities in their coefficient functions. Such SDEs can usually not be solved explicitly and it is a quite active area of research to design and analyze approximation algorithms which are able to solve SDEs with superlinearly growing nonlinearities approximatively. In particular, we refer, e.g., to [1, 2, 5, 11, 12, 16, 20, 22, 25, 26, 29–31, 34–36, 41, 45–48, 50, 55, 56, 58, 58, 60, 63, 64, 66, 68, 70, 73, 81, 82, 87–90, 92, 93, 97–101, 103, 105, 108–112, 115–117, 120, 122, 127, 128, 132–138, 140, 142–144, 147–151, 153, 154, 156, 164–166, 168, 170, 171, 175–177] for convergence and simulation results for explicit numerical approximation schemes for SODEs with superlinearly growing nonlinearities, we refer, e.g., to [6–8, 13–15, 17–19, 23, 32, 43, 51, 53, 67, 72, 74, 75, 77, 79, 80, 83–85, 91, 121, 155, 158, 167] for convergence and simulation results for explicit numerical approximation schemes for SPDEs with superlinearly growing nonlinearities, we refer, e.g., to [4, 11, 22, 38, 41, 60–62, 65, 73, 90, 107, 118, 119, 123–125, 131, 145, 152, 157, 161, 162, 169, 172–174] for convergence and simulation results for implicit Euler-type numerical approximation schemes for SODEs with superlinearly growing nonlinearities, and we refer, e.g., to [9,10,21,24,27,28,33,39,40,44,49,51,52,86,94–96,106,114,139,167] for convergence and simulation results for implicit Euler-type numerical approximation schemes for SPDEs with superlinearly growing nonlinearities.

The most basic numerical scheme for SODEs, the Euler-Maruyama scheme, and similar explicit approximation schemes for SODEs (such as the Milstein scheme) have been shown to diverge strongly and numerically weakly in the case of one-dimensional SODEs with superlinearly growing nonlinearities; see [69, Theorem 2.1] and [71, Theorem 2.1]. More specifically, Theorem 2.1 in [71] immediately implies the following result.

Theorem 1.1. *Let $\alpha, \beta, c \in (1, \infty)$, $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, let $W \colon [0, T] \times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\xi \colon \Omega \to \mathbb{R}$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable function, let $\mu, \sigma \colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable functions, let $Y_n^N \colon \Omega \to \mathbb{R}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$, satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \ldots, N - 1\}$ that $Y_0^N = \xi$ and*

$$Y_{n+1}^N = Y_n^N + \tfrac{T}{N}\,\mu(Y_n^N) + \sigma(Y_n^N)\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right), \tag{1.1}$$

assume for all $x \in (-\infty, -c] \cup [c, \infty)$ that $|\mu(x)| + |\sigma(x)| \geq \frac{|x|^\alpha}{c}$, and assume for all $x \in [1, \infty)$ that $\big(\big[\mathbb{P}(\sigma(\xi) \neq 0) > 0\big]$ or $\big[\mathbb{P}(|\xi| \geq x) \geq \beta^{(-x^\beta)}\big]\big)$. Then it holds for all $p \in (0, \infty)$ that $\lim_{N\to\infty} \mathbb{E}[|Y_N^N|^p] = \infty$.

Theorem 1.1 above proves strong and numerically weak divergence for the Euler-Maruyama scheme and similar approximation schemes (such as the Milstein scheme) in the case of one-dimensional SODEs with superlinearly growing nonlinearities. However, it remained an open question whether the divergence phenomenon in Theorem 1.1 also holds in the case of SPDEs with superlinearly growing nonlinearities. In particular, it remained an open question whether such a divergence phenomenon also holds in the case of reaction-diffusion-type SPDEs with polynomial coefficients such as stochastic Allen-Cahn equations. We answer this question by proving that standard Euler-type approximation schemes for SPDEs (such as exponential Euler and linear-implicit Euler schemes) diverge strongly and numerically weakly in the case of reaction-diffusion-type SPDEs with polynomial coefficients such as stochastic Allen-Cahn equations. To be more precise, the main result of this paper, Theorem 3.7 in Section 3 below, establishes strong and numerically weak divergence for both full-discrete exponential Euler and full-discrete linear-implicit Euler approximations in the case of reaction-diffusion-type SPDEs with polynomial coefficients (including stochastic Allen-Cahn equations as special cases). To illustrate the findings of the main result of this article we now present in the following theorem a special case of Theorem 3.7.

Theorem 1.2. *Let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0, 1)$ to $\mathbb{R}$, let $A \colon D(A) \subseteq H \to H$ be the Laplacian with periodic boundary conditions on H, let $e_n \in H$, $n \in \mathbb{Z}$, satisfy for all $n \in \mathbb{N}$ that $e_0(\cdot) = 1$, $e_n(\cdot) = \sqrt{2}\cos(2n\pi(\cdot))$, and $e_{-n}(\cdot) = \sqrt{2}\sin(2n\pi(\cdot))$, let $T, \eta \in (0, \infty)$, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces* *associated to $\eta - A$, let $P_N\colon H \to H$, $N \in \mathbb{N}$, be the linear operators which satisfy for all $N \in \mathbb{N}$, $v \in H$ that $P_N(v) = \sum_{n=-N}^{N} \langle e_n, v\rangle_H \, e_n$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $q \in \{2, 3, \dots\}$, $a_0, a_1, \dots, a_{q-1} \in \mathbb{R}$, $a_q \in \mathbb{R}\backslash\{0\}$, $\nu \in (1/4, 3/4)$, $\xi \in H_{1/3}$, let $W\colon [0,T] \times \Omega \to H_{-\nu}$ be an Id_H-cylindrical Wiener process, let $S_N\colon H_{-\nu} \to H$, $N \in \mathbb{N}$, be linear operators which satisfy for all $N \in \mathbb{N}$ that $S_N \in \{e^{T/NA}, (I - T/NA)^{-1}\}$, and let $Y^N\colon \{0, 1, \dots, N\} \times \Omega \to H$, $N \in \mathbb{N}$, be the stochastic processes which satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N-1\}$ that $Y_0^N = P_N(\xi)$ and*

$$Y_{n+1}^N = P_N S_N \Big(Y_n^N + \tfrac{T}{N} \Big(\textstyle\sum_{k=0}^{q} a_k \big[Y_n^N\big]^k \Big) + \big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}} \big) \Big). \tag{1.2}$$

Then it holds for all $p \in (0, \infty)$ that $\liminf_{N\to\infty} \mathbb{E}\big[\|Y_N^N\|_H^p\big] = \infty$.

Theorem 1.2 above is an immediate consequence of Corollary 3.8 in Section 3 below. Corollary 3.8, in turn, follows from Theorem 3.7, which is the main result of this article. Note that the assumption in Theorem 1.2 that $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, is a family of interpolation spaces associated to $\eta - A$ ensures that $H_0 = H$, $H_1 = D(A)$, $H_2 = D(A^2)$, $H_3 = D(A^3)$, ... (cf., e.g., Sell & You [146, Section 3.7]). Moreover, observe that in the case where for all $N \in \mathbb{N}$ it holds that $q = 3$, $a_0 = 0$, $a_1 \in (0, \infty)$, $a_2 = 0$, $a_3 \in (-\infty, 0)$, and $S_N = e^{T/NA}$ we have that Theorem 1.2 proves strong and numerically weak divergence for the full-discrete explicit exponential Euler scheme for stochastic Allen-Cahn equations. Furthermore, note that in the case where for all $N \in \mathbb{N}$ it holds that $q = 3$, $a_0 = 0$, $a_1 \in (0, \infty)$, $a_2 = 0$, $a_3 \in (-\infty, 0)$, and $S_N = (I - T/NA)^{-1}$ we have that Theorem 1.2 proves strong and numerically weak divergence for the full-discrete linear-implicit Euler scheme for stochastic Allen-Cahn equations. We prove Theorem 1.2 and Theorem 3.7, respectively, through an application of an abstract divergence theory which we have developed in Section 2 of this paper. We also refer, e.g., to [42, 54, 59, 69, 71, 76, 126, 130, 159, 160] for lower bounds for strong and weak approximation errors for numerical approximation schemes for SDEs with non-globally Lipschitz continuous nonlinearities.

The remainder of this article is organized as follows. In Section 2 we employ reverse Lyapunov-type functions to establish suitable lower bounds for a class of general stochastic processes; cf., e.g., Corollary 2.8. In particular, we establish in Lemma 2.4 in Section 2 lower bounds for the probabilities of certain rare events. Lemma 2.4 is used in our proof of Proposition 2.6, which is the main result of Section 2. Proposition 2.6, in turn, is employed in our proof of Corollary 2.8. In Section 3 we employ the general lower bounds which we have proved in Section 2 to establish Theorem 3.7, which is the main result of this article.

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2 Reverse a priori bounds based on Lyapunov-type functions

Throughout this section the following setting is frequently used.

Setting 2.1. *For every two measurable spaces* $(\Omega_1, \mathcal{F}_1)$ *and* $(\Omega_2, \mathcal{F}_2)$ *let* $\mathcal{M}(\mathcal{F}_1, \mathcal{F}_2)$ *be the set of all* $\mathcal{F}_1/\mathcal{F}_2$*-measurable functions, let* $(H, \mathcal{H})$ *and* $(U, \mathcal{U})$ *be measurable spaces, let* $\Phi\colon H \times U \to H$ *be an* $(\mathcal{H} \otimes \mathcal{U})/\mathcal{H}$*-measurable function, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, for every set* $R \subseteq [-\infty, \infty]$ *and every function* $f\colon \Omega \to R$ *let* $[\![f]\!]$ *be the set given by* $[\![f]\!] = \{g \in \mathcal{M}(\mathcal{F}, \mathcal{B}([0,\infty)))\colon (\exists\, A \in \{B \in \mathcal{F}\colon \mathbb{P}(B) = 1\}\colon (\forall\, \omega \in A\colon f(\omega) = g(\omega)))\}$*, let* $N \in \mathbb{N}$*,* $c \in (0,1]$*,* $\alpha, \theta \in (1,\infty)$*,* $\mathbb{H}_0, \mathbb{H}_1, \ldots, \mathbb{H}_N \in \mathcal{H}$*, let* $Z_1, Z_2, \ldots, Z_N\colon \Omega \to U$ *be i.i.d. random variables, let* $Y_0, Y_1, \ldots, Y_N\colon \Omega \to H$ *be random variables which satisfy for all* $n \in \{1, 2, \ldots, N\}$ *that* $Y_n = \Phi(Y_{n-1}, Z_n)$*, assume that* $\sigma(Y_0)$ *and* $\sigma(Z_1, Z_2, \ldots, Z_N)$ *are independent on* $(\Omega, \mathcal{F}, \mathbb{P})$*, and let* $\mathcal{V}\colon H \to [0,\infty)$ *be an* $\mathcal{H}/\mathcal{B}([0,\infty))$*-measurable function.*

2.1 A reverse Gronwall-type inequality

In the next elementary result, Lemma 2.2 below, we present a reverse Gronwall-type inequality. We employ this reverse Gronwall-type inequality to establish lower bounds for the probabilities of certain rare events in Lemma 2.4 below.

Lemma 2.2. *Let* $c, \alpha \in (0,\infty)$*,* $N \in \mathbb{N}$*,* $e_0, e_1, \ldots, e_N \in [0,\infty)$ *satisfy for all* $n \in \{0, 1, \ldots, N-1\}$ *that*

$$e_{n+1} \geq c\,[e_n]^{\alpha}. \tag{2.1}$$

Then it holds for all $n \in \{0, 1, \ldots, N\}$ *that*

$$e_n \geq c^{\left(\sum_{k=0}^{n-1} \alpha^k\right)} \cdot [e_0]^{(\alpha^n)}. \tag{2.2}$$

Proof of Lemma 2.2. We prove (2.2) by induction on $n \in \{0, 1, \ldots, N\}$. For the base $n = 0$ we note that

$$e_0 = c^0 \cdot e_0 = c^{(\sum_{k=0}^{-1} \alpha^k)} \cdot [e_0]^{(\alpha^0)}. \tag{2.3}$$

This proves (2.2) in the base case $n = 0$. For the induction step $\{0, 1, \ldots, N-1\} \ni n \to n+1 \in \{1, 2, \ldots, N\}$ assume that (2.2) is fulfilled for an $n \in \{0, 1, \ldots, N-1\}$. The induction hypothesis and (2.1) ensure that

$$\begin{aligned} e_{n+1} &\geq c\,[e_n]^{\alpha} \geq c\left[c^{(\sum_{k=0}^{n-1} \alpha^k)} \cdot [e_0]^{(\alpha^n)}\right]^{\alpha} \\ &= c\left[c^{(\alpha \cdot \sum_{k=0}^{n-1} \alpha^k)} \cdot [e_0]^{(\alpha \cdot \alpha^n)}\right] = c\left[c^{(\sum_{k=0}^{n-1} \alpha^{k+1})} \cdot [e_0]^{(\alpha^{n+1})}\right] \\ &= c^{(1+\sum_{k=0}^{n-1} \alpha^{k+1})} \cdot [e_0]^{(\alpha^{n+1})} = c^{(1+\sum_{k=1}^{n} \alpha^k)} \cdot [e_0]^{(\alpha^{n+1})} \\ &= c^{(\sum_{k=0}^{n} \alpha^k)} \cdot [e_0]^{(\alpha^{n+1})}. \end{aligned} \tag{2.4}$$

This proves (2.2) in the case $n + 1$. Induction thus completes the proof of Lemma 2.2. ∎

2.2 Lower bounds for the probabilities of certain rare events

Lemma 2.3. *Assume Setting 2.1, let $A_n \subseteq \Omega$, $n \in \{0, 1, \dots, N\}$, be the sets which satisfy for all $n \in \{1, 2, \dots, N\}$ that $A_0 = \{Y_0 \in \mathbb{H}_0\}$ and*

$$A_n = \{\mathcal{V}(Y_n) \geq c[\mathcal{V}(Y_{n-1})]^\alpha\} \cap \{Y_n \in \mathbb{H}_n\}, \tag{2.5}$$

and let $p_n \colon H \to [0, 1]$, $n \in \{1, 2, \dots, N\}$, be the functions which satisfy for all $n \in \{1, 2, \dots, N\}$, $v \in H$ that

$$p_n(v) = \mathbb{P}\Big(\{\mathcal{V}(\Phi(v, Z_n)) \geq c[\mathcal{V}(v)]^\alpha\} \cap \{\Phi(v, Z_n) \in \mathbb{H}_n\}\Big). \tag{2.6}$$

Then

(i) it holds for all $n \in \{1, 2, \dots, N\}$ that $p_n \in \mathcal{M}(\mathcal{H}, \mathcal{B}([0, 1]))$,

(ii) it holds for all $n \in \{1, 2, \dots, N\}$ that $A_0 \in \sigma(Y_0)$ and $A_n \in \sigma(Y_0, Z_1, Z_2, \dots, Z_n)$,

(iii) it holds for all $n \in \{1, 2, \dots, N\}$ that

$$\begin{aligned} &\mathbb{P}\Big((\cap_{k=0}^n A_k) \,\Big|\, \sigma(Y_0)\Big) \\ &= \mathbb{E}\bigg[p_n(Y_{n-1})\, \mathbb{1}^\Omega_{\{\mathcal{V}(Y_{n-1}) \geq c^{(\sum_{l=0}^{n-2} \alpha^l)}\, [\mathcal{V}(Y_0)]^{(\alpha^{(n-1)})}\}}\, \mathbb{1}^\Omega_{(\cap_{k=0}^{n-1} A_k)} \,\bigg|\, \sigma(Y_0)\bigg], \end{aligned} \tag{2.7}$$

and

(iv) it holds for all $n \in \{1, 2, \dots, N\}$ that

$$\begin{aligned} &\mathbb{P}\Big((\cap_{k=0}^n A_k) \,\Big|\, \sigma(Y_0)\Big) \Big[\!\!\Big[\mathbb{1}^\Omega_{\{\mathcal{V}(Y_0) \geq c^{(1/(1-\alpha))}\theta\}}\Big]\!\!\Big] \\ &\geq \inf\Big(\{p_n(v) \colon (v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})})\} \cup \{1\}\Big) \\ &\quad \cdot \mathbb{P}\Big((\cap_{k=0}^{n-1} A_k) \,\Big|\, \sigma(Y_0)\Big) \Big[\!\!\Big[\mathbb{1}^\Omega_{\{\mathcal{V}(Y_0) \geq c^{(1/(1-\alpha))}\theta\}}\Big]\!\!\Big]. \end{aligned} \tag{2.8}$$

Proof of Lemma 2.3 . Throughout this proof let $\mathcal{G}_n \subseteq \mathcal{P}(\Omega)$, $n \in \{0, 1, \dots, N\}$, be the sigma-algebras on Ω which satisfy for all $n \in \{1, 2, \dots, N\}$ that $\mathcal{G}_0 = \sigma(Y_0)$ and

$$\mathcal{G}_n = \sigma(Y_0, Z_1, Z_2, \dots, Z_n), \tag{2.9}$$

let $C_k \subseteq H \times \Omega$, $k \in \{1, 2, \dots, N\}$, and $D_k \subseteq H \times \Omega$, $k \in \{1, 2, \dots, N\}$, be the sets which satisfy for all $k \in \{1, 2, \dots, N\}$ that

$$C_k = \{(x, \omega) \in H \times \Omega \colon \mathcal{V}(\Phi(x, Z_k(\omega))) - c[\mathcal{V}(x)]^\alpha \geq 0\} \tag{2.10}$$

and

$$D_k = \{(x, \omega) \in H \times \Omega \colon \Phi(x, Z_k(\omega)) \in \mathbb{H}_k\}, \tag{2.11}$$

let $\pi \colon H \times \Omega \to H$ and $f_k \colon H \times \Omega \to \mathbb{R}$, $k \in \{1, 2, \dots, N\}$, be the functions which satisfy for all $k \in \{1, 2, \dots, N\}$, $(v, \omega) \in H \times \Omega$ that

$$\pi(v, \omega) = v \qquad \textit{and} \qquad f_k(v, \omega) = \mathbb{1}^{H \times \Omega}_{C_k \cap D_k}(v, \omega), \tag{2.12}$$

let $\Psi_k\colon H \times \Omega \to H \times U$, $k \in \{1, 2, \dots, N\}$, be the functions which satisfy for all $k \in \{1, 2, \dots, N\}$, $(x, \omega) \in H \times \Omega$ that

$$\Psi_k(x, \omega) = (x, Z_k(\omega)), \tag{2.13}$$

and let $\Upsilon\colon H \times \Omega \to [0, \infty)$ be the function which satisfies for all $(x, \omega) \in H \times \Omega$ that

$$\Upsilon(x, \omega) = c[(\mathcal{V} \circ \pi)(x, \omega)]^\alpha. \tag{2.14}$$

Observe that for all $k \in \{1, 2, \dots, N\}$ it holds that $f_k \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{B}(\mathbb{R}))$ if and only if it holds that

$$(C_k \cap D_k) \in \mathcal{H} \otimes \mathcal{F}. \tag{2.15}$$

Next note that for all $k \in \{1, 2, \dots, N\}$ it holds that

$$\Psi_k \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{H} \otimes \mathcal{U}). \tag{2.16}$$

Moreover, observe that

$$\Phi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{U}, \mathcal{H}). \tag{2.17}$$

Combining this with (2.16) implies for all $k \in \{1, 2, \dots, N\}$ that

$$\Phi \circ \Psi_k \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{H}). \tag{2.18}$$

The fact that $\forall\, k \in \{1, 2, \dots, N\}\colon \mathbb{H}_k \in \mathcal{H}$ therefore proves for all $k \in \{1, 2, \dots, N\}$ that

$$D_k = (\Phi \circ \Psi_k)^{-1}(\mathbb{H}_k) \in \mathcal{H} \otimes \mathcal{F}. \tag{2.19}$$

In addition, note that

$$\pi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{H}) \qquad \text{and} \qquad \mathcal{V} \in \mathcal{M}(\mathcal{H}, \mathcal{B}([0, \infty))). \tag{2.20}$$

This and (2.18) imply for all $k \in \{1, 2, \dots, N\}$ that

$$\Upsilon \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{B}([0, \infty))) \tag{2.21}$$

and

$$\mathcal{V} \circ \Phi \circ \Psi_k \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{B}([0, \infty))). \tag{2.22}$$

This ensures for all $k \in \{1, 2, \dots, N\}$ that

$$[\mathcal{V} \circ \Phi \circ \Psi_k - \Upsilon] \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{B}(\mathbb{R})). \tag{2.23}$$

Hence, we obtain for all $k \in \{1, 2, \dots, N\}$ that

$$C_k = (\mathcal{V} \circ \Phi \circ \Psi_k - \Upsilon)^{-1}([0, \infty)) \in \mathcal{H} \otimes \mathcal{F}. \tag{2.24}$$

Combining this with (2.19) establishes for all $k \in \{1, 2, \dots, N\}$ that

$$C_k \cap D_k \in \mathcal{H} \otimes \mathcal{F}. \tag{2.25}$$

This and (2.15) demonstrate that for all $k \in \{1, 2, \dots, N\}$ it holds that

$$f_k \in \mathcal{M}(\mathcal{H} \otimes \mathcal{F}, \mathcal{B}(\mathbb{R})). \tag{2.26}$$

Furthermore, note that it holds for all $k \in \{1, 2, \ldots, N\}$, $v \in H$ that

$$p_k(v) = \int_\Omega f_k(v, \omega)\, \mathbb{P}(d\omega). \tag{2.27}$$

Fubini's theorem and (2.26) therefore ensure that for all $k \in \{1, 2, \ldots, N\}$ it holds that

$$p_k \in \mathcal{M}(\mathcal{H}, \mathcal{B}([0, 1])). \tag{2.28}$$

This proves Item (i). Next note that

$$A_0 = \{Y_0 \in \mathbb{H}_0\} = Y_0^{-1}(\mathbb{H}_0) \in \sigma(Y_0). \tag{2.29}$$

Furthermore, observe that it holds for all $n \in \{1, 2, \ldots, N\}$ that

$$\{\mathcal{V}(Y_n) \geq c[\mathcal{V}(Y_{n-1})]^\alpha\} \in \sigma(Y_n, Y_{n-1}) \qquad \text{and} \qquad \{Y_n \in \mathbb{H}_n\} \in \sigma(Y_n). \tag{2.30}$$

In the next step we demonstrate that for all $n \in \{1, 2, \ldots, N\}$ it holds that

$$\sigma(Y_n) \subseteq \sigma(Y_0, Z_1, Z_2, \ldots, Z_n). \tag{2.31}$$

We prove (2.31) by induction on $n \in \{1, 2, \ldots, N\}$. Observe that the assumption that $Y_1 = \Phi(Y_0, Z_1)$ and the assumption that $\Phi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{U}, \mathcal{H})$ ensure that

$$\sigma(Y_1) \subseteq \sigma(Y_0, Z_1). \tag{2.32}$$

This establishes (2.31) in the base case $n = 1$. For the induction step $\{1, 2, \ldots, N-1\} \ni n \to n+1 \in \{2, 3, \ldots, N\}$ assume that (2.31) is fulfilled for an $n \in \{1, 2, \ldots, N-1\}$. The assumption that $\forall\, m \in \{1, 2, \ldots, N\}\colon Y_m = \Phi(Y_{m-1}, Z_m)$ and the assumption that $\Phi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{U}, \mathcal{H})$ assure that

$$\sigma(Y_{n+1}) \subseteq \sigma(Y_n, Z_{n+1}). \tag{2.33}$$

Moreover, note that the induction hypothesis implies that

$$\sigma(Y_n, Z_{n+1}) \subseteq \sigma(Y_0, Z_1, Z_2, \ldots, Z_{n+1}). \tag{2.34}$$

Combining this with (2.33) ensures that

$$\sigma(Y_{n+1}) \subseteq \sigma(Y_0, Z_1, Z_2, \ldots, Z_{n+1}). \tag{2.35}$$

This proves (2.31) in the case $n + 1$. Induction thus completes the proof of (2.31). Combining (2.31) with (2.30) and (2.5) proves that it holds for all $n \in \{1, 2, \ldots, N\}$ that

$$A_n \in \sigma(Y_0, Z_1, \ldots, Z_n). \tag{2.36}$$

This establishes Item (ii). Next note that the tower property for conditional expectations implies for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}
\mathbb{P}\Big((\cap_{n=0}^k A_n) \,\big|\, \sigma(Y_0)\Big) &= \mathbb{E}\Big[\mathbb{1}^\Omega_{(\cap_{n=0}^k A_n)} \,\big|\, \mathcal{G}_0\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{1}^\Omega_{(\cap_{n=0}^k A_n)} \,\big|\, \mathcal{G}_{k-1}\Big] \,\Big|\, \mathcal{G}_0\Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{1}^\Omega_{A_k} \mathbb{1}^\Omega_{(\cap_{n=0}^{k-1} A_n)} \,\big|\, \mathcal{G}_{k-1}\Big] \,\Big|\, \mathcal{G}_0\Big].
\end{aligned} \tag{2.37}$$

This and Item (ii) assure for all $k \in \{1, 2, \ldots, N\}$ that

$$\mathbb{P}\Big((\cap_{n=0}^{k} A_n) \,\big|\, \sigma(Y_0)\Big) = \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{1}_{A_k}^{\Omega} \,\big|\, \mathcal{G}_{k-1}\Big] \mathbb{1}_{(\cap_{n=0}^{k-1} A_n)}^{\Omega} \,\Big|\, \mathcal{G}_0\Big]. \tag{2.38}$$

Combining this with (2.5) ensures for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}
\mathbb{P}\Big((\cap_{n=0}^{k} A_n) \,\big|\, \sigma(Y_0)\Big) &= \mathbb{E}\Big[\mathbb{E}\Big[\mathbb{1}_{\{\mathcal{V}(Y_k) \geq c[\mathcal{V}(Y_{k-1})]^{\alpha}\} \cap \{Y_k \in \mathbb{H}_k\}}^{\Omega} \,\big|\, \mathcal{G}_{k-1}\Big] \mathbb{1}_{(\cap_{n=0}^{k-1} A_n)}^{\Omega} \,\Big|\, \mathcal{G}_0\Big] \\
&= \mathbb{E}\Big[\mathbb{P}\Big(\big\{\mathcal{V}(Y_k) \geq c[\mathcal{V}(Y_{k-1})]^{\alpha}\big\} \cap \big\{Y_k \in \mathbb{H}_k\big\} \,\big|\, \mathcal{G}_{k-1}\Big) \mathbb{1}_{(\cap_{n=0}^{k-1} A_n)}^{\Omega} \,\Big|\, \mathcal{G}_0\Big] \\
&= \mathbb{E}\Big[\mathbb{P}\Big(\big\{\mathcal{V}(\Phi(Y_{k-1}, Z_k)) \geq c[\mathcal{V}(Y_{k-1})]^{\alpha}\big\} \cap \big\{\Phi(Y_{k-1}, Z_k) \in \mathbb{H}_k\big\} \,\big|\, \mathcal{G}_{k-1}\Big) \\
&\qquad \cdot \mathbb{1}_{(\cap_{n=0}^{k-1} A_n)}^{\Omega} \,\Big|\, \mathcal{G}_0\Big].
\end{aligned} \tag{2.39}$$

Moreover, observe that [78, Lemma 2.9] (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $(D, \mathcal{D}) = (H, \mathcal{H})$, $(E, \mathcal{E}) = (U, \mathcal{U})$, $\mathcal{X} = \mathcal{G}_{k-1}$, $\mathcal{Y} = \sigma(Z_k)$, $X = Y_{k-1}$, $Y = Z_k$, $\Phi = \mathbb{1}_{\{(v,u) \in H \times U \colon \mathcal{V}(\Phi(v,u)) \geq c[\mathcal{V}(v)]^{\alpha}\} \cap \{(v,u) \in H \times U \colon \Phi(v,u) \in \mathbb{H}_k\}}^{H \times U}$ for $k \in \{1, 2, \ldots, N\}$ in the notation of [78, Lemma 2.9]) establishes for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{P}\Big(\big\{\mathcal{V}(\Phi(Y_{k-1}, Z_k)) \geq c[\mathcal{V}(Y_{k-1})]^{\alpha}\big\} \cap \big\{\Phi(Y_{k-1}, Z_k) \in \mathbb{H}_k\big\} \,\big|\, \mathcal{G}_{k-1}\Big) \\
&\quad \cdot \mathbb{1}_{(\cap_{n=0}^{k-1} A_n)}^{\Omega} \,\Big|\, \mathcal{G}_0\Big] = \mathbb{E}\Big[p_k(Y_{k-1}) \, \mathbb{1}_{(\cap_{n=0}^{k-1} A_n)}^{\Omega} \,\big|\, \mathcal{G}_0\Big].
\end{aligned} \tag{2.40}$$

This and (2.39) prove for all $k \in \{1, 2, \ldots, N\}$ that

$$\mathbb{P}\Big((\cap_{n=0}^{k} A_n) \,\big|\, \sigma(Y_0)\Big) = \mathbb{E}\Big[p_k(Y_{k-1}) \, \mathbb{1}_{(\cap_{n=0}^{k-1} A_n)}^{\Omega} \,\big|\, \mathcal{G}_0\Big]. \tag{2.41}$$

Furthermore, note that (2.5) implies for all $k \in \{2, 3, \ldots, N\}$, $n \in \{0, 1, \ldots, k-2\}$, $\omega \in (\cap_{l=0}^{k-1} A_l)$ that

$$\mathcal{V}(Y_{n+1}(\omega)) \geq c[\mathcal{V}(Y_n(\omega))]^{\alpha}. \tag{2.42}$$

This and Lemma 2.2 (with $c = c$, $\alpha = \alpha$, $N = k-1$, $e_n = \mathcal{V}(Y_n(\omega))$ for $k \in \{2, 3, \ldots, N\}$, $n \in \{0, 1, \ldots, k-1\}$, $\omega \in (\cap_{l=0}^{k-1} A_l)$ in the notation of Lemma 2.2) assure that it holds for all $k \in \{2, 3, \ldots, N\}$, $n \in \{0, 1, \ldots, k-1\}$, $\omega \in (\cap_{l=0}^{k-1} A_l)$ that

$$\mathcal{V}(Y_n(\omega)) \geq c^{(\sum_{l=0}^{n-1} \alpha^l)} \, [\mathcal{V}(Y_0(\omega))]^{(\alpha^n)}. \tag{2.43}$$

Hence, we obtain for all $k \in \{2, 3, \ldots, N\}$, $\omega \in (\cap_{l=0}^{k-1} A_l)$ that

$$\mathcal{V}(Y_{k-1}(\omega)) \geq c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0(\omega))]^{(\alpha^{(k-1)})}. \tag{2.44}$$

Next observe that

$$\Big\{\mathcal{V}(Y_0) \geq c^{(\sum_{l=0}^{-1} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^0)}\Big\} = \Big\{\mathcal{V}(Y_0) \geq \mathcal{V}(Y_0)\Big\} = \Omega. \tag{2.45}$$

The fact that $A_0 \subseteq \Omega$ therefore proves that

$$A_0 = (\cap_{n=0}^{0} A_n) \subseteq \Big\{\mathcal{V}(Y_0) \geq c^{(\sum_{l=0}^{-1} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^0)}\Big\}. \tag{2.46}$$

Combining this with (2.44) implies for all $k \in \{1, 2, \dots, N\}$ that

$$(\cap_{n=0}^{k-1} A_n) \subseteq \Big\{ \mathcal{V}(Y_{k-1}) \ge c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})} \Big\}. \tag{2.47}$$

Therefore, we obtain for all $k \in \{1, 2, \dots, N\}$ that

$$\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1}) \ge c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}} \, \mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1} A_n)} = \mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1} A_n)}. \tag{2.48}$$

Combining this with (2.41) assures for all $k \in \{1, 2, \dots, N\}$ that

$$\begin{split} \mathbb{P}\Big((\cap_{n=0}^{k} A_n) \,\big|\, \sigma(Y_0) \Big) &= \mathbb{E}\Big[p_k(Y_{k-1}) \, \mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1} A_n)} \,\big|\, \mathcal{G}_0 \Big] \\ &= \mathbb{E}\bigg[p_k(Y_{k-1}) \, \mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1}) \ge c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}} \, \mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1} A_n)} \,\bigg|\, \mathcal{G}_0 \bigg]. \end{split} \tag{2.49}$$

This establishes Item (iii). It thus remains to prove Item (iv). For this we note that (2.7) implies that for all $k \in \{1, 2, \dots, N\}$ it holds that

$$\begin{split} &\mathbb{P}\Big((\cap_{n=0}^{k} A_n) \,\big|\, \sigma(Y_0) \Big) \, \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \ge c^{(1/(1-\alpha))} \theta\}} \Big]\!\Big] \\ &= \mathbb{E}\bigg[p_k(Y_{k-1}) \, \mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1}) \ge c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}} \, \mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1} A_n)} \,\bigg|\, \mathcal{G}_0 \bigg] \, \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \ge c^{(1/(1-\alpha))} \theta\}} \Big]\!\Big] \\ &\ge \mathbb{E}\bigg[p_k(Y_{k-1}) \mathbb{1}^{\Omega}_{\{Y_{k-1} \in \{w \in \mathbb{H}_{k-1} \colon \mathcal{V}(w) \ge \theta^{(\alpha^{(k-1)})}\}\}} \, \mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1}) \ge c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}} \\ &\qquad \cdot \mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1} A_n)} \, \mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \ge c^{(1/(1-\alpha))} \theta\}} \,\bigg|\, \mathcal{G}_0 \bigg]. \end{split} \tag{2.50}$$

Next observe that (2.5) assures for all $k \in \{1, 2, \dots, N\}$, $\omega \in (\cap_{n=0}^{k-1} A_n)$ that

$$Y_{k-1}(\omega) \in \mathbb{H}_{k-1}. \tag{2.51}$$

Moreover, note that for all $k \in \{2, 3, \dots, N\}$, $\omega \in \{\mathcal{V}(Y_0) \ge c^{(1/(1-\alpha))} \theta\} \cap \{\mathcal{V}(Y_{k-1}) \ge c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}$ it holds that

$$\begin{split} \mathcal{V}(Y_{k-1}(\omega)) &\ge c^{(\sum_{l=0}^{k-2} \alpha^l)} \, [\mathcal{V}(Y_0(\omega))]^{(\alpha^{(k-1)})} \\ &= c^{(\alpha^{(k-1)} - 1)/(\alpha - 1)} \, [\mathcal{V}(Y_0(\omega))]^{(\alpha^{(k-1)})} \\ &\ge c^{(\alpha^{(k-1)})/(\alpha-1)} \, c^{(\alpha^{(k-1)})/(1-\alpha)} \, \theta^{(\alpha^{(k-1)})} = \theta^{(\alpha^{(k-1)})}. \end{split} \tag{2.52}$$

Furthermore, observe that the hypothesis that $c \in (0, 1]$ and the hypothesis that $\alpha \in (1, \infty)$ prove for all $\omega \in A_0 \cap \{\mathcal{V}(Y_0) \ge c^{1/(1-\alpha)} \theta\}$ that

$$\mathcal{V}(Y_0(\omega)) \ge c^{1/(1-\alpha)} \theta \ge \theta. \tag{2.53}$$

This demonstrates that

$$A_0 \cap \{\mathcal{V}(Y_0) \ge c^{1/(1-\alpha)} \theta\} \subseteq \Big\{ Y_0 \in \Big\{ w \in \mathbb{H}_0 \colon \mathcal{V}(w) \ge \theta^{(\alpha^0)} \Big\} \Big\}. \tag{2.54}$$

Combining this with (2.51) and (2.52) ensures for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}&(\cap_{n=0}^{k-1} A_n) \cap \Big\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\Big\} \cap \Big\{\mathcal{V}(Y_{k-1}) \geq c^{(\sum_{l=0}^{k-2}\alpha^l)}\,[\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\Big\}\\&\subseteq \Big\{Y_{k-1} \in \Big\{w \in \mathbb{H}_{k-1} \colon \mathcal{V}(w) \geq \theta^{(\alpha^{(k-1)})}\Big\}\Big\}.\end{aligned} \tag{2.55}$$

This implies for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}&\mathbb{1}^{\Omega}_{\{Y_{k-1}\in\{w\in\mathbb{H}_{k-1}\colon \mathcal{V}(w)\geq\theta^{(\alpha^{(k-1)})}\}\}}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1})\geq c^{(\sum_{l=0}^{k-2}\alpha^l)}\,[\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}}\\&\quad\cdot\mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1}A_n)}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\\&=\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1})\geq c^{(\sum_{l=0}^{k-2}\alpha^l)}\,[\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}}\,\mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1}A_n)}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}.\end{aligned} \tag{2.56}$$

Combining this with (2.50) assures for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}&\mathbb{P}\Big((\cap_{n=0}^{k}A_n)\,\big|\,\sigma(Y_0)\Big)\,\Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\Big]\!\Big]\\&\geq\mathbb{E}\bigg[\inf\Big(\Big\{p_k(v)\colon (v\in\mathbb{H}_{k-1}\colon \mathcal{V}(v)\geq\theta^{(\alpha^{(k-1)})})\Big\}\cup\{1\}\Big)\,\mathbb{1}^{\Omega}_{\{Y_{k-1}\in\{w\in\mathbb{H}_{k-1}\,:\,\mathcal{V}(w)\geq\theta^{(\alpha^{(k-1)})}\}\}}\\&\qquad\cdot\mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1}A_n)}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1})\geq c^{(\sum_{l=0}^{k-2}\alpha^l)}\,[\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\,\Big|\,\mathcal{G}_0\bigg]\\&=\mathbb{E}\bigg[\inf\Big(\Big\{p_k(v)\colon (v\in\mathbb{H}_{k-1}\colon \mathcal{V}(v)\geq\theta^{(\alpha^{(k-1)})})\Big\}\cup\{1\}\Big)\,\mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1}A_n)}\\&\qquad\cdot\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_{k-1})\geq c^{(\sum_{l=0}^{k-2}\alpha^l)}\,[\mathcal{V}(Y_0)]^{(\alpha^{(k-1)})}\}}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\,\Big|\,\mathcal{G}_0\bigg].\end{aligned} \tag{2.57}$$

This and (2.48) ensure for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}&\mathbb{P}\Big((\cap_{n=0}^{k}A_n)\,\big|\,\sigma(Y_0)\Big)\,\Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\Big]\!\Big]\\&\geq\inf\Big(\Big\{p_k(v)\colon (v\in\mathbb{H}_{k-1}\colon \mathcal{V}(v)\geq\theta^{(\alpha^{(k-1)})})\Big\}\cup\{1\}\Big)\\&\quad\cdot\mathbb{E}\Big[\mathbb{1}^{\Omega}_{(\cap_{n=0}^{k-1}A_n)}\,\big|\,\mathcal{G}_0\Big]\,\Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\Big]\!\Big]\\&=\inf\Big(\Big\{p_k(v)\colon (v\in\mathbb{H}_{k-1}\colon \mathcal{V}(v)\geq\theta^{(\alpha^{(k-1)})})\Big\}\cup\{1\}\Big)\\&\quad\cdot\mathbb{P}\Big((\cap_{n=0}^{k-1}A_n)\,\big|\,\sigma(Y_0)\Big)\,\Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\Big]\!\Big].\end{aligned} \tag{2.58}$$

This establishes Item (iv). The proof of Lemma 2.3 is thus completed. □

Lemma 2.4. *Assume Setting 2.1 and let $p_n\colon H \to [0, 1]$, $n \in \{1, 2, \ldots, N\}$, be the functions which satisfy for all $n \in \{1, 2, \ldots, N\}$, $v \in H$ that*

$$p_n(v) = \mathbb{P}\Big(\big\{\mathcal{V}(\Phi(v, Z_n)) \geq c[\mathcal{V}(v)]^{\alpha}\big\} \cap \big\{\Phi(v, Z_n) \in \mathbb{H}_n\big\}\Big). \tag{2.59}$$

Then

$$\begin{aligned}
&\mathbb{P}\Big(\cap_{n=0}^{N} \Big[\Big\{\mathcal{V}(Y_n) \geq c^{(\sum_{k=0}^{n-1}\alpha^k)} [\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\} \cap \Big\{Y_n \in \mathbb{H}_n\Big\}\Big] \,\Big|\, \sigma(Y_0)\Big) \\
&\quad \cdot \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big] \\
&\geq \Bigg[\prod_{n=1}^{N} \inf\Big(\big\{ p_n(v) \colon (v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}) \big\} \cup \{1\} \Big) \Bigg] \\
&\quad \cdot \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}\cap\{Y_0\in\mathbb{H}_0\}} \Big]\!\Big].
\end{aligned} \tag{2.60}$$

Proof of Lemma 2.4 . Throughout this proof let $A_n \subseteq \Omega$, $n \in \{0, 1, \ldots, N\}$, be the sets which satisfy for all $n \in \{1, 2, \ldots, N\}$ that $A_0 = \big\{Y_0 \in \mathbb{H}_0\big\}$ and

$$A_n = \big\{\mathcal{V}(Y_n) \geq c[\mathcal{V}(Y_{n-1})]^{\alpha}\big\} \cap \big\{Y_n \in \mathbb{H}_n\big\}. \tag{2.61}$$

Note that for all $k \in \{1, 2, \ldots, N\}$, $n \in \{0, 1, \ldots, k-1\}$, $\omega \in (\cap_{l=1}^{k}\{\mathcal{V}(Y_l) \geq c[\mathcal{V}(Y_{l-1})]^{\alpha}\})$ it holds that

$$\mathcal{V}(Y_{n+1}(\omega)) \geq c[\mathcal{V}(Y_n(\omega))]^{\alpha}. \tag{2.62}$$

This and Lemma 2.2 (with $c = c$, $\alpha = \alpha$, $N = k$, $e_n = \mathcal{V}(Y_n(\omega))$ for $k \in \{1, 2, \ldots, N\}$, $n \in \{0, 1, \ldots, k\}$, $\omega \in (\cap_{l=1}^{k}\{\mathcal{V}(Y_l) \geq c[\mathcal{V}(Y_{l-1})]^{\alpha}\})$ in the notation of Lemma 2.2) ensure for all $k \in \{1, 2, \ldots, N\}$, $n \in \{0, 1, \ldots, k\}$, $\omega \in (\cap_{l=1}^{k}\{\mathcal{V}(Y_l) \geq c[\mathcal{V}(Y_{l-1})]^{\alpha}\})$ that

$$\mathcal{V}(Y_n(\omega)) \geq c^{(\sum_{l=0}^{n-1}\alpha^l)} [\mathcal{V}(Y_0(\omega))]^{(\alpha^n)}. \tag{2.63}$$

Hence, we obtain for all $k \in \{1, 2, \ldots, N\}$ that

$$\big(\cap_{n=1}^{k} \big\{\mathcal{V}(Y_n) \geq c[\mathcal{V}(Y_{n-1})]^{\alpha}\big\}\big) \subseteq \Big(\cap_{n=0}^{k} \Big\{\mathcal{V}(Y_n) \geq c^{(\sum_{l=0}^{n-1}\alpha^l)} [\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\}\Big). \tag{2.64}$$

Moreover, observe that (2.61) establishes for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}
(\cap_{n=0}^{k} A_n) &= A_0 \cap (\cap_{n=1}^{k} A_n) \\
&= \big\{Y_0 \in \mathbb{H}_0\big\} \cap \Big(\cap_{n=1}^{k} \big(\big\{\mathcal{V}(Y_n) \geq c[\mathcal{V}(Y_{n-1})]^{\alpha}\big\} \cap \big\{Y_n \in \mathbb{H}_n\big\}\big)\Big) \\
&= \big(\cap_{n=1}^{k} \big\{\mathcal{V}(Y_n) \geq c[\mathcal{V}(Y_{n-1})]^{\alpha}\big\}\big) \cap \big(\cap_{n=0}^{k} \big\{Y_n \in \mathbb{H}_n\big\}\big).
\end{aligned} \tag{2.65}$$

This and (2.64) imply for all $k \in \{1, 2, \ldots, N\}$ that

$$\begin{aligned}
(\cap_{n=0}^{k} A_n) &\subseteq \Big(\cap_{n=0}^{k} \Big\{\mathcal{V}(Y_n) \geq c^{(\sum_{l=0}^{n-1}\alpha^l)} [\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\} \cap \big(\cap_{n=0}^{k} \big\{Y_n \in \mathbb{H}_n\big\}\big)\Big) \\
&= \cap_{n=0}^{k} \Big(\Big\{\mathcal{V}(Y_n) \geq c^{(\sum_{l=0}^{n-1}\alpha^l)} [\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\} \cap \Big\{Y_n \in \mathbb{H}_n\Big\}\Big).
\end{aligned} \tag{2.66}$$

Hence, we obtain that

$$\begin{aligned}
&\mathbb{P}\Big(\cap_{n=0}^{N} \Big[\Big\{\mathcal{V}(Y_n) \geq c^{(\sum_{k=0}^{n-1}\alpha^k)} [\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\} \cap \Big\{Y_n \in \mathbb{H}_n\Big\}\Big] \,\Big|\, \sigma(Y_0)\Big) \\
&\quad \cdot \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big] \\
&\geq \mathbb{P}\Big((\cap_{n=0}^{N} A_n) \,\big|\, \sigma(Y_0)\Big) \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big].
\end{aligned} \tag{2.67}$$

Next note that Item (iv) in Lemma 2.3 and induction establish that

$$\begin{aligned}
&\mathbb{P}\Big((\cap_{n=0}^N A_n) \,\big|\, \sigma(Y_0)\Big) \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big] \\
&\geq \inf\Big(\big\{ p_N(v) \colon (v \in \mathbb{H}_{N-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(N-1)})}) \big\} \cup \{1\} \Big) \\
&\quad \cdot \mathbb{P}\Big((\cap_{n=0}^{N-1} A_n) \,\big|\, \sigma(Y_0)\Big) \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big] \\
&\geq \left[\prod_{n=1}^N \inf\Big(\big\{ p_n(v) \colon (v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}) \big\} \cup \{1\} \Big) \right] \\
&\quad \cdot \mathbb{P}\Big(A_0 \,\big|\sigma(Y_0) \Big) \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big].
\end{aligned} \tag{2.68}$$

This ensures that

$$\begin{aligned}
&\mathbb{P}\Big((\cap_{n=0}^N A_n) \,\big|\, \sigma(Y_0)\Big) \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big] \\
&\geq \left[\prod_{n=1}^N \inf\Big(\big\{ p_n(v) \colon (v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}) \big\} \cup \{1\} \Big) \right] \\
&\quad \cdot \mathbb{P}\Big(\big\{ Y_0 \in \mathbb{H}_0 \big\} \,\big|\, \sigma(Y_0) \Big) \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big] \\
&= \left[\prod_{n=1}^N \inf\Big(\big\{ p_n(v) \colon (v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}) \big\} \cup \{1\} \Big) \right] \\
&\quad \cdot \mathbb{E}\Big[\mathbb{1}^{\Omega}_{\{Y_0 \in \mathbb{H}_0\}} \,\big|\, \sigma(Y_0) \Big] \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}} \Big]\!\Big] \\
&= \left[\prod_{n=1}^N \inf\Big(\big\{ p_n(v) \colon (v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}) \big\} \cup \{1\} \Big) \right] \\
&\quad \cdot \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\} \cap \{Y_0 \in \mathbb{H}_0\}} \Big]\!\Big].
\end{aligned} \tag{2.69}$$

Combining this with (2.67) establishes (2.60). The proof of Lemma 2.4 is thus completed. □

2.3 Reverse a priori bounds

Lemma 2.5. *Assume Setting 2.1. Then*

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{V}(Y_N)\big] \geq c^{(\sum_{k=0}^{N-1} \alpha^k)} \\
&\cdot \mathbb{E}\Big[[\mathcal{V}(Y_0)]^{(\alpha^N)} \mathbb{P}\Big(\cap_{n=0}^N \Big[\Big\{ \mathcal{V}(Y_n) \geq c^{(\sum_{k=0}^{n-1} \alpha^k)} \, [\mathcal{V}(Y_0)]^{(\alpha^n)} \Big\} \cap \big\{ Y_n \in \mathbb{H}_n \big\} \Big] \,\Big|\, \sigma(Y_0) \Big) \Big].
\end{aligned} \tag{2.70}$$

Proof of Lemma 2.5. First, note that the tower property for conditional expectations implies that

$$\begin{aligned}
\mathbb{E}\big[\mathcal{V}(Y_N)\big] &\geq \mathbb{E}\Big[\mathcal{V}(Y_N) \, \mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_N) \geq c^{(\sum_{k=0}^{N-1} \alpha^k)} \, [\mathcal{V}(Y_0)]^{(\alpha^N)}\}} \Big] \\
&\geq \mathbb{E}\Big[c^{(\sum_{k=0}^{N-1} \alpha^k)} \, [\mathcal{V}(Y_0)]^{(\alpha^N)} \, \mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_N) \geq c^{(\sum_{k=0}^{N-1} \alpha^k)} \, [\mathcal{V}(Y_0)]^{(\alpha^N)}\}} \Big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[c^{(\sum_{k=0}^{N-1} \alpha^k)} \, [\mathcal{V}(Y_0)]^{(\alpha^N)} \, \mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_N) \geq c^{(\sum_{k=0}^{N-1} \alpha^k)} \, [\mathcal{V}(Y_0)]^{(\alpha^N)}\}} \,\Big|\, \sigma(Y_0) \Big] \Big] \\
&= c^{(\sum_{k=0}^{N-1} \alpha^k)} \, \mathbb{E}\Big[[\mathcal{V}(Y_0)]^{(\alpha^N)} \, \mathbb{E}\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_N) \geq c^{(\sum_{k=0}^{N-1} \alpha^k)} \, [\mathcal{V}(Y_0)]^{(\alpha^N)}\}} \,\Big|\, \sigma(Y_0) \Big] \Big].
\end{aligned} \tag{2.71}$$

Hence, we obtain that

$$\begin{aligned}
&\mathbb{E}\big[\mathcal{V}(Y_N)\big] \\
&\geq c^{(\sum_{k=0}^{N-1}\alpha^k)}\,\mathbb{E}\Big[[\mathcal{V}(Y_0)]^{(\alpha^N)}\,\mathbb{P}\Big(\Big\{\mathcal{V}(Y_N)\geq c^{(\sum_{k=0}^{N-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^N)}\Big\}\,\Big|\,\sigma(Y_0)\Big)\Big] \\
&\geq c^{(\sum_{k=0}^{N-1}\alpha^k)} \\
&\cdot\mathbb{E}\Big[[\mathcal{V}(Y_0)]^{(\alpha^N)}\,\mathbb{P}\Big(\cap_{n=0}^{N}\Big[\Big\{\mathcal{V}(Y_n)\geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\}\cap\big\{Y_n\in\mathbb{H}_n\big\}\Big]\,\Big|\,\sigma(Y_0)\Big)\Big].
\end{aligned} \tag{2.72}$$

The proof of Lemma 2.5 is thus completed. □

Proposition 2.6. *Assume Setting 2.1. Then*

$$\begin{aligned}
\mathbb{E}\big[\mathcal{V}(Y_N)\big]\geq\theta^{(\alpha^N)}\,&\mathbb{P}\big(\big\{Y_0\in\mathbb{H}_0\big\}\cap\big\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\big\}\big) \\
&\cdot\left[\prod_{n=1}^{N}\inf\Big(\Big\{\mathbb{P}\Big(\big\{\mathcal{V}(\Phi(v,Z_1))\geq c[\mathcal{V}(v)]^{\alpha}\big\}\cap\big\{\Phi(v,Z_1)\in\mathbb{H}_n\big\}\Big)\right. \\
&\qquad\left.:\big(v\in\mathbb{H}_{n-1}\colon\mathcal{V}(v)\geq\theta^{(\alpha^{(n-1)})}\big)\Big\}\cup\{1\}\Big)\right].
\end{aligned} \tag{2.73}$$

Proof of Proposition 2.6 . First, note that Lemma 2.5 ensures that

$$\begin{aligned}
\mathbb{E}\big[\mathcal{V}(Y_N)\big]&\geq c^{(\sum_{k=0}^{N-1}\alpha^k)}\,\mathbb{E}\Big[[\mathcal{V}(Y_0)]^{(\alpha^N)} \\
&\quad\cdot\mathbb{P}\Big(\cap_{n=0}^{N}\Big[\Big\{\mathcal{V}(Y_n)\geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\}\cap\big\{Y_n\in\mathbb{H}_n\big\}\Big]\,\Big|\,\sigma(Y_0)\Big)\Big] \\
&\geq c^{(\frac{\alpha^N-1}{\alpha-1})}\,\mathbb{E}\Big[[\mathcal{V}(Y_0)]^{(\alpha^N)}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}} \\
&\quad\cdot\mathbb{P}\Big(\cap_{n=0}^{N}\Big[\Big\{\mathcal{V}(Y_n)\geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\}\cap\big\{Y_n\in\mathbb{H}_n\big\}\Big]\,\Big|\,\sigma(Y_0)\Big)\Big].
\end{aligned} \tag{2.74}$$

The assumption that $c\in(0,1]$, the fact that $\alpha>1$, and the fact that $\alpha^N>1$ therefore imply that

$$\begin{aligned}
\mathbb{E}\big[\mathcal{V}(Y_N)\big]&\geq c^{(\alpha^N/(\alpha-1))}\,\mathbb{E}\Big[\theta^{(\alpha^N)}c^{(\alpha^N/(1-\alpha))}\,\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}} \\
&\quad\cdot\mathbb{P}\Big(\cap_{n=0}^{N}\Big[\Big\{\mathcal{V}(Y_n)\geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\}\cap\big\{Y_n\in\mathbb{H}_n\big\}\Big]\,\Big|\,\sigma(Y_0)\Big)\Big].
\end{aligned} \tag{2.75}$$

This assures that

$$\begin{aligned}
\mathbb{E}\big[\mathcal{V}(Y_N)\big]&\geq\theta^{(\alpha^N)}\,c^{(\alpha^N/(1-\alpha))}\,c^{(\alpha^N/(\alpha-1))}\,\mathbb{E}\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}} \\
&\quad\cdot\mathbb{P}\Big(\cap_{n=0}^{N}\Big[\Big\{\mathcal{V}(Y_n)\geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\}\cap\big\{Y_n\in\mathbb{H}_n\big\}\Big]\,\Big|\,\sigma(Y_0)\Big)\Big] \\
&=\theta^{(\alpha^N)}\,\mathbb{E}\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}} \\
&\quad\cdot\mathbb{P}\Big(\cap_{n=0}^{N}\Big[\Big\{\mathcal{V}(Y_n)\geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\}\cap\big\{Y_n\in\mathbb{H}_n\big\}\Big]\,\Big|\,\sigma(Y_0)\Big)\Big].
\end{aligned} \tag{2.76}$$

Moreover, note that Lemma 2.4 ensures that

$$\begin{aligned}
&\mathbb{P}\Big(\cap_{n=0}^{N} \Big[\Big\{\mathcal{V}(Y_n) \geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\} \cap \big\{Y_n \in \mathbb{H}_n\big\}\Big] \,\Big|\, \sigma(Y_0)\Big) \\
&\quad\cdot \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\Big]\!\Big] \geq \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}\cap\{Y_0\in\mathbb{H}_0\}}\Big]\!\Big] \\
&\quad\cdot \Bigg[\prod_{n=1}^{N} \inf\Big(\Big\{\mathbb{P}\Big(\big\{\mathcal{V}(\Phi(v,Z_n)) \geq c[\mathcal{V}(v)]^{\alpha}\big\} \cap \big\{\Phi(v,Z_n) \in \mathbb{H}_n\big\}\Big) \\
&\qquad\qquad : \big(v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}\big)\Big\} \cup \{1\}\Big)\Bigg].
\end{aligned} \tag{2.77}$$

Furthermore, observe that the fact that $Z_1, Z_2, \ldots, Z_N$ are identically distributed random variables implies that

$$\begin{aligned}
&\mathbb{P}\Big(\cap_{n=0}^{N} \Big[\Big\{\mathcal{V}(Y_n) \geq c^{(\sum_{k=0}^{n-1}\alpha^k)}\,[\mathcal{V}(Y_0)]^{(\alpha^n)}\Big\} \cap \big\{Y_n \in \mathbb{H}_n\big\}\Big] \,\Big|\, \sigma(Y_0)\Big) \\
&\quad\cdot \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}}\Big]\!\Big] \geq \Big[\!\Big[\mathbb{1}^{\Omega}_{\{\mathcal{V}(Y_0)\geq c^{1/(1-\alpha)}\theta\}\cap\{Y_0\in\mathbb{H}_0\}}\Big]\!\Big] \\
&\quad\cdot \Bigg[\prod_{n=1}^{N} \inf\Big(\Big\{\mathbb{P}\Big(\big\{\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^{\alpha}\big\} \cap \big\{\Phi(v,Z_1) \in \mathbb{H}_n\big\}\Big) \\
&\qquad\qquad : \big(v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}\big)\Big\} \cup \{1\}\Big)\Bigg].
\end{aligned} \tag{2.78}$$

Combining this with (2.76) proves that

$$\begin{aligned}
\mathbb{E}\big[\mathcal{V}(Y_N)\big] &\geq \theta^{(\alpha^N)}\,\mathbb{P}\Big(\Big\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\Big\} \cap \big\{Y_0 \in \mathbb{H}_0\big\}\Big) \\
&\quad\cdot \Bigg[\prod_{n=1}^{N} \inf\Big(\Big\{\mathbb{P}\Big(\big\{\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^{\alpha}\big\} \cap \big\{\Phi(v,Z_1) \in \mathbb{H}_n\big\}\Big) \\
&\qquad\qquad : \big(v \in \mathbb{H}_{n-1} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}\big)\Big\} \cup \{1\}\Big)\Bigg].
\end{aligned} \tag{2.79}$$

The proof of Proposition 2.6 is thus completed. □

Corollary 2.7. *Let $(H, \mathcal{H})$ and $(U, \mathcal{U})$ be measurable spaces, let $\Phi\colon H \times U \to H$ be an $(\mathcal{H} \otimes \mathcal{U})/\mathcal{H}$-measurable function, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathbb{H} \in \mathcal{H}$, $N \in \mathbb{N}$, $c \in (0, 1]$, $\alpha, \theta \in (1, \infty)$, let $Z_1, Z_2, \ldots, Z_N \colon \Omega \to U$ be i.i.d. random variables, let $Y_0, Y_1, \ldots, Y_N \colon \Omega \to H$ be random variables which satisfy for all $n \in \{1, 2, \ldots, N\}$ that $Y_n = \Phi(Y_{n-1}, Z_n)$, assume that $\sigma(Y_0)$ and $\sigma(Z_1, Z_2, \ldots, Z_N)$ are independent on $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{V}\colon H \to [0, \infty)$ be an $\mathcal{H}/\mathcal{B}([0, \infty))$-measurable function. Then*

$$\begin{aligned}
\mathbb{E}\big[\mathcal{V}(Y_N)\big] &\geq \theta^{(\alpha^N)}\,\mathbb{P}\big(\big\{Y_0 \in \mathbb{H}\big\} \cap \big\{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\big\}\big) \\
&\quad\cdot \Bigg[\prod_{n=1}^{N} \inf\Big(\Big\{\mathbb{P}\Big(\big\{\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^{\alpha}\big\} \cap \big\{\Phi(v,Z_1) \in \mathbb{H}\big\}\Big) \\
&\qquad\qquad : \big(v \in \mathbb{H} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}\big)\Big\} \cup \{1\}\Big)\Bigg].
\end{aligned} \tag{2.80}$$

Proof of Corollary 2.7. First, note that Proposition 2.6 (with $(H, \mathcal{H}) = (H, \mathcal{H})$, $(U, \mathcal{U}) = (U, \mathcal{U})$, $\Phi = \Phi$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $N = N$, $c = c$, $\alpha = \alpha$, $\theta = \theta$, $\mathbb{H}_n = \mathbb{H}$

for $n \in \{0,1,\dots,N\}$, $Z_n = Z_n$ for $n \in \{1,2,\dots,N\}$, $Y_n = Y_n$ for $n \in \{0,1,\dots,N\}$, $\mathcal{V} = \mathcal{V}$ in the notation of Proposition 2.6) ensures that

$$\begin{aligned} \mathbb{E}\big[\mathcal{V}(Y_N)\big] &\geq \theta^{(\alpha^N)}\, \mathbb{P}\big(\{Y_0 \in \mathbb{H}\} \cap \{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}\big) \\ &\quad \cdot \left[\prod_{n=1}^{N} \inf\!\left(\left\{\mathbb{P}\Big(\{\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^\alpha\} \cap \{\Phi(v,Z_1) \in \mathbb{H}\}\Big)\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. : \big(v \in \mathbb{H} \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}\big)\right\} \cup \{1\}\right)\right]. \end{aligned} \tag{2.81}$$

The proof of Corollary 2.7 is thus completed. □

Corollary 2.8. *Let $(H,\mathcal{H})$ and $(U,\mathcal{U})$ be measurable spaces, let $\Phi\colon H \times U \to H$ be an $(\mathcal{H} \otimes \mathcal{U})/\mathcal{H}$-measurable function, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $N \in \mathbb{N}$, $c \in (0,1]$, $\alpha,\theta \in (1,\infty)$, let $Z_1, Z_2, \dots, Z_N\colon \Omega \to U$ be i.i.d. random variables, let $Y_0, Y_1, \dots, Y_N\colon \Omega \to H$ be random variables which satisfy for all $n \in \{1,2,\dots,N\}$ that $Y_n = \Phi(Y_{n-1}, Z_n)$, assume that $\sigma(Y_0)$ and $\sigma(Z_1,Z_2,\dots,Z_N)$ are independent on $(\Omega,\mathcal{F},\mathbb{P})$, and let $\mathcal{V}\colon H \to [0,\infty)$ be an $\mathcal{H}/\mathcal{B}([0,\infty))$-measurable function. Then*

$$\begin{aligned} \mathbb{E}\big[\mathcal{V}(Y_N)\big] &\geq \theta^{(\alpha^N)}\, \mathbb{P}\big(\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\big) \\ &\cdot \left[\prod_{n=1}^{N} \inf\!\left(\left\{\mathbb{P}\Big(\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^\alpha\Big) \colon \big(v \in H \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}\big)\right\} \cup \{1\}\right)\right]. \end{aligned} \tag{2.82}$$

Proof of Corollary 2.8. First, note that Corollary 2.7 (with $(H,\mathcal{H}) = (H,\mathcal{H})$, $(U,\mathcal{U}) = (U,\mathcal{U})$, $\Phi = \Phi$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $\mathbb{H} = H$, $N = N$, $c = c$, $\alpha = \alpha$, $\theta = \theta$, $Z_n = Z_n$ for $n \in \{1,2,\dots,N\}$, $Y_n = Y_n$ for $n \in \{0,1,\dots,N\}$, $\mathcal{V} = \mathcal{V}$ in the notation of Corollary 2.7) ensures that

$$\begin{aligned} \mathbb{E}\big[\mathcal{V}(Y_N)\big] &\geq \theta^{(\alpha^N)}\, \mathbb{P}\big(\{Y_0 \in H\} \cap \{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}\big) \\ &\quad \cdot \left[\prod_{n=1}^{N} \inf\!\left(\left\{\mathbb{P}\Big(\{\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^\alpha\} \cap \{\Phi(v,Z_1) \in H\}\Big)\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. : \big(v \in H \colon \mathcal{V}(v) \geq \theta^{(\alpha^{(n-1)})}\big)\right\} \cup \{1\}\right)\right]. \end{aligned} \tag{2.83}$$

Moreover, observe that the fact that $Y_0(\Omega) \subseteq H$ implies that

$$\{Y_0 \in H\} \cap \{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\} = \{\mathcal{V}(Y_0) \geq c^{1/(1-\alpha)}\theta\}. \tag{2.84}$$

In addition, note that the fact that $\Phi(H \times U) \subseteq H$ assures that for all $v \in H$ it holds that

$$\{\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^\alpha\} \cap \{\Phi(v,Z_1) \in H\} = \{\mathcal{V}(\Phi(v,Z_1)) \geq c[\mathcal{V}(v)]^\alpha\}. \tag{2.85}$$

Combining this with (2.84) and (2.83) establishes (2.82). The proof of Corollary 2.8 is thus completed. □

3 Divergence results for Euler-type approximation schemes for SPDEs with superlinearly growing nonlinearities

Throughout this section the following setting is frequently used.

Setting 3.1. *Let $\lambda\colon \mathcal{B}((0,1)) \to [0,\infty]$ be the Lebesgue-Borel measure on $(0,1)$, let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H) = (L^2(\lambda;\mathbb{R}), \|\cdot\|_{L^2(\lambda;\mathbb{R})}, \langle\cdot,\cdot\rangle_{L^2(\lambda;\mathbb{R})})$, let $e_n \in H$, $n \in \mathbb{Z}$, satisfy for all $n \in \mathbb{N}$ that $e_0(\cdot) = 1$, $e_n(\cdot) = \sqrt{2}\cos(2n\pi(\cdot))$, and $e_{-n}(\cdot) = \sqrt{2}\sin(2n\pi(\cdot))$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies that*

$$D(A) = \left\{ v \in H \colon \sum_{n\in\mathbb{Z}} n^4 \left|\langle e_n, v\rangle_H\right|^2 < \infty \right\} \tag{3.1}$$

and

$$\forall\, v \in D(A)\colon \quad Av = \sum_{n\in\mathbb{Z}} -4\pi^2 n^2 \langle e_n, v\rangle_H \, e_n, \tag{3.2}$$

let $\eta \in (0,\infty)$, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, and let $P_N \in L(H_{-1}, H_1)$, $N \in \mathbb{N}$, be the linear operators which satisfy for all $N \in \mathbb{N}$, $v \in H$ that $P_N(v) = \sum_{n=-N}^{N} \langle e_n, v\rangle_H \, e_n$.

3.1 A continuous embedding based on the Sobolev embedding theorem

The next elementary and well-known result, Lemma 3.2 below, presents a special case of the Sobolev embedding theorem. Lemma 3.2 is used in our proof of Proposition 3.6 in Section 3.3 below. For completeness we also include in this article the short proof of Lemma 3.2.

Lemma 3.2. *Assume Setting 3.1 and let $p \in [2,\infty)$, $\chi \in [1/4 - 1/(2p), \infty)$. Then it holds that $H_\chi \subseteq L^p(\lambda;\mathbb{R})$ and*

$$0 < \sup_{v\in H_\chi\setminus\{0\}} \frac{\|v\|_{L^p(\lambda;\mathbb{R})}}{\|v\|_{H_\chi}} < \infty. \tag{3.3}$$

Proof of Lemma 3.2. Throughout this proof let $C \in [0,\infty]$ be the extended real number which satisfies that

$$C = \sup_{v\in(L^p(\lambda;\mathbb{R})\cap H_\chi)\setminus\{0\}} \frac{\|v\|_{L^p(\lambda;\mathbb{R})}}{\|v\|_{H_\chi}}. \tag{3.4}$$

Note that

$$C \geq \frac{\|e_0\|_{L^p(\lambda;\mathbb{R})}}{\|e_0\|_{H_\chi}} = \frac{1}{|\eta|^\chi} > 0. \tag{3.5}$$

Next observe that the hypothesis that $\chi \geq 1/4 - 1/(2p)$ ensures that

$$2\chi \geq \max\{1/2 - 1/p, 0\}. \tag{3.6}$$

Combining this with the Sobolev embedding theorem implies that $H_\chi \subseteq L^p(\lambda;\mathbb{R})$ and

$$C = \sup_{v\in H_\chi\setminus\{0\}} \frac{\|v\|_{L^p(\lambda;\mathbb{R})}}{\|v\|_{H_\chi}} < \infty. \tag{3.7}$$

This and (3.5) establish (3.3). The proof of Lemma 3.2 is thus completed. □

3.2 Lower bounds for the probabilities of certain rare events

In the next result, Lemma 3.3 below, we establish an elementary property for certain normally distributed random variables. We use Lemma 3.3 to establish in Lemma 3.4 below lower bounds for the probabilities of certain rare events. We refer to the statement and the proof of Lemma 3.4 below for more details.

Lemma 3.3. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $c\in\mathbb{R}$, $T,\varepsilon\in(0,\infty)$, $N\in\mathbb{N}$, and let $Y\colon\Omega\to\mathbb{R}$ be a normalkly distributed random variable with mean 0 and variance T/N. Then*

$$\mathbb{P}\big(|c-Y|\le\varepsilon\big) \ge \tfrac{\varepsilon}{\sqrt{2\pi T}} \exp\Big(-\tfrac{N(c^2+\varepsilon^2)}{T}\Big). \tag{3.8}$$

Proof of Lemma 3.3. First, note that

$$\begin{aligned}
&\mathbb{P}\big(|c-Y|\le\varepsilon\big) = \mathbb{P}\big(|Y-c|\le\varepsilon\big)\\
&= \mathbb{P}\big(Y-c\in[-\varepsilon,\varepsilon]\big) = \mathbb{P}\big(Y\in[c-\varepsilon,c+\varepsilon]\big)\\
&= \int_{c-\varepsilon}^{c+\varepsilon} \tfrac{\sqrt{N}}{\sqrt{2\pi T}}\, e^{-\frac{Ny^2}{2T}}\,dy.
\end{aligned} \tag{3.9}$$

This and the fact that $\sup_{x\in[c-\varepsilon,c+\varepsilon]} x^2 = \max\{(c-\varepsilon)^2,(c+\varepsilon)^2\}$ ensure that

$$\mathbb{P}\big(|c-Y|\le\varepsilon\big) \ge \tfrac{2\varepsilon\sqrt{N}}{\sqrt{2\pi T}}\, e^{-\frac{N\max\{(c-\varepsilon)^2,(c+\varepsilon)^2\}}{2T}}. \tag{3.10}$$

Moreover, observe that the fact that $\forall\, a,b\in[0,\infty)\colon\ \max\{a,b\}\le a+b$ assures that

$$\begin{aligned}
&\max\big\{(c-\varepsilon)^2,(c+\varepsilon)^2\big\} \le (c-\varepsilon)^2+(c+\varepsilon)^2\\
&= c^2+\varepsilon^2-2c\varepsilon+c^2+\varepsilon^2+2c\varepsilon = 2(c^2+\varepsilon^2).
\end{aligned} \tag{3.11}$$

This implies that

$$\frac{N\max\{(c-\varepsilon)^2,(c+\varepsilon)^2\}}{2T} \le \frac{N(c^2+\varepsilon^2)}{T}. \tag{3.12}$$

Combining this with (3.10) proves that

$$\mathbb{P}\big(|c-Y|\le\varepsilon\big) \ge \tfrac{2\varepsilon\sqrt{N}}{\sqrt{2\pi T}}\, e^{-\frac{N(c^2+\varepsilon^2)}{T}} \ge \tfrac{\varepsilon}{\sqrt{2\pi T}} \exp\Big(-\tfrac{N(c^2+\varepsilon^2)}{T}\Big). \tag{3.13}$$

The proof of Lemma 3.3 is thus completed. □

Lemma 3.4. *Assume Setting 3.1, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $T, x, \gamma \in (0, \infty)$*,* $\nu \in (1/4, 1]$*,* $N \in \mathbb{N}$*,* $v \in H$ *satisfy that*

$$\gamma = \textstyle\sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu}, \tag{3.14}$$

and let $W\colon [0, T] \times \Omega \to H_{-\nu}$ *be an* Id_H*-cylindrical Wiener process. Then*

$$\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \leq x\big) \geq \Big[\tfrac{x}{\sqrt{2\pi\gamma T}}\Big]^{(2N+1)} \exp\Big(-\tfrac{3N^2}{T}\Big[\|v\|_H^2 + \tfrac{x^2}{\gamma}\Big]\Big). \tag{3.15}$$

Proof of Lemma 3.4. Throughout this proof let $\beta^n\colon \Omega \to \mathbb{R}$, $n \in \{-N, -N+1, \ldots, N-1, N\}$, be the random variables which satisfy for all $n \in \{-N, -N+1, \ldots, N-1, N\}$ that

$$\beta^n = \big\langle e_n, P_N(W_{T/N})\big\rangle_H \tag{3.16}$$

and let $v_n \in \mathbb{R}$, $n \in \mathbb{Z}$, be the real numbers which satisfy for all $n \in \mathbb{Z}$ that

$$v_n = \langle e_n, v\rangle_H\,. \tag{3.17}$$

Note that Parseval's identity assures that

$$\begin{aligned}
&\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \leq x\big) \\
&= \mathbb{P}\Big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H^2 \leq x^2\Big) \\
&= \mathbb{P}\Big(\textstyle\sum_{n=-\infty}^{\infty} \big|\big\langle(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big), e_n\big\rangle_H\big|^2 \leq x^2\Big).
\end{aligned} \tag{3.18}$$

This implies that

$$\begin{aligned}
&\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \leq x\big) \\
&= \mathbb{P}\Big(\textstyle\sum_{n=-\infty}^{\infty} \big|\big\langle P_N(v) - P_N(W_{T/N}), (\eta - A)^{-\nu} e_n\big\rangle_H\big|^2 \leq x^2\Big) \\
&= \mathbb{P}\Big(\textstyle\sum_{n=-\infty}^{\infty} \big|\big\langle P_N(v) - P_N(W_{T/N}), (\eta + 4\pi^2 n^2)^{-\nu} e_n\big\rangle_H\big|^2 \leq x^2\Big).
\end{aligned} \tag{3.19}$$

Hence, we obtain that

$$\begin{aligned}
&\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \leq x\big) \\
&= \mathbb{P}\Big(\textstyle\sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu} \big|\big\langle P_N(v) - P_N(W_{T/N}), e_n\big\rangle_H\big|^2 \leq x^2\Big) \\
&= \mathbb{P}\Big(\textstyle\sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu} \big|\langle e_n, v\rangle_H - \big\langle P_N(W_{T/N}), e_n\big\rangle_H\big|^2 \leq x^2\Big) \\
&= \mathbb{P}\Big(\textstyle\sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu} |v_n - \beta^n|^2 \leq x^2\Big).
\end{aligned} \tag{3.20}$$

This proves that

$$\begin{aligned}
&\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \leq x\big) \\
&\geq \mathbb{P}\left(\Big(\textstyle\sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu}\Big) \sup_{n \in \{-N, \ldots, N\}} |v_n - \beta^n|^2 \leq x^2\right) \\
&= \mathbb{P}\left(\sup_{n \in \{-N, \ldots, N\}} |v_n - \beta^n|^2 \leq \tfrac{x^2}{\gamma}\right).
\end{aligned} \tag{3.21}$$

The fact that $v_n - \beta^n$, $n \in \{-N, -N+1, \dots, N-1, N\}$, are independent random variables (cf., e.g., Proposition 2.5.2 in [113]) therefore implies that

$$\begin{aligned} &\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \le x\big) \\ &\ge \mathbb{P}\Big(\cap_{n=-N}^{N} \Big\{|v_n - \beta^n|^2 \le \tfrac{x^2}{\gamma}\Big\}\Big) \\ &= \textstyle\prod_{n=-N}^{N} \mathbb{P}\Big(|v_n - \beta^n|^2 \le \tfrac{x^2}{\gamma}\Big). \end{aligned} \tag{3.22}$$

Next note that Lemma 3.3 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $c = v_n$, $T = T$, $\varepsilon = \frac{x}{\sqrt{\gamma}}$, $N = N$, $Y = \beta^n$ for $n \in \{-N, \dots, N\}$ in the notation of Lemma 3.3) ensures for all $n \in \{-N, \dots, N\}$ that

$$\begin{aligned} &\mathbb{P}\Big(|v_n - \beta^n|^2 \le \tfrac{x^2}{\gamma}\Big) = \mathbb{P}\Big(|v_n - \beta^n| \le \tfrac{x}{\sqrt{\gamma}}\Big) \\ &\ge \tfrac{x}{\sqrt{2\pi\gamma T}} \exp\Big(-\tfrac{N}{T}\Big[|v_n|^2 + \tfrac{x^2}{\gamma}\Big]\Big). \end{aligned} \tag{3.23}$$

Combining this with (3.22) establishes that

$$\begin{aligned} &\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \le x\big) \\ &\ge \textstyle\prod_{n=-N}^{N} \Big[\tfrac{x}{\sqrt{2\pi\gamma T}} \exp\Big(-\tfrac{N}{T}\Big[|v_n|^2 + \tfrac{x^2}{\gamma}\Big]\Big)\Big] \\ &= \Big[\tfrac{x}{\sqrt{2\pi\gamma T}}\Big]^{(2N+1)} \exp\Big(-\textstyle\sum_{n=-N}^{N} \frac{N\big[|v_n|^2 + \frac{x^2}{\gamma}\big]}{T}\Big). \end{aligned} \tag{3.24}$$

Hence, we obtain that

$$\begin{aligned} &\mathbb{P}\big(\big\|(\eta - A)^{-\nu}\big(P_N(v) - P_N(W_{T/N})\big)\big\|_H \le x\big) \\ &\ge \Big[\tfrac{x}{\sqrt{2\pi\gamma T}}\Big]^{(2N+1)} \exp\Big(-\tfrac{N}{T}\Big[\|v\|_H^2 + \tfrac{(2N+1)x^2}{\gamma}\Big]\Big) \\ &\ge \Big[\tfrac{x}{\sqrt{2\pi\gamma T}}\Big]^{(2N+1)} \exp\Big(-\tfrac{N}{T}\Big[3N\|v\|_H^2 + \tfrac{3Nx^2}{\gamma}\Big]\Big) \\ &\ge \Big[\tfrac{x}{\sqrt{2\pi\gamma T}}\Big]^{(2N+1)} \exp\Big(-\tfrac{3N^2}{T}\Big[\|v\|_H^2 + \tfrac{x^2}{\gamma}\Big]\Big). \end{aligned} \tag{3.25}$$

The proof of Lemma 3.4 is thus completed. □

Corollary 3.5. *Assume Setting 3.1, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $T, x, \gamma, y \in (0, \infty)$, $p \in [2, \infty)$, $\delta \in [1, \infty)$, $\nu \in (1/4, 3/4)$, $s \in (1/4, 1 - \nu]$, $N \in \mathbb{N}$, $v \in L^p(\lambda; \mathbb{R})$, $S \in L(H, H_{\nu+s})$ satisfy that*

$$\gamma = \textstyle\sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu}, \tag{3.26}$$

$$\|(\eta - A)^{(\nu+s)} S\|_{L(H)} \le \delta \left[\tfrac{N}{T}\right]^{(\nu+s)}, \tag{3.27}$$

$$\forall\, u \in H \colon (\eta - A)^{-\nu} S u = S(\eta - A)^{-\nu} u, \tag{3.28}$$

$$y = \tfrac{x}{\delta} \left[\tfrac{T}{N}\right]^{(\nu+s)} \big\|(\eta - A)^{-s}\big\|_{L(H, L^p(\lambda;\mathbb{R}))}^{-1}, \tag{3.29}$$

and let $W \colon [0, T] \times \Omega \to H_{-\nu}$ be an Id_H-cylindrical Wiener process. Then

$$\begin{aligned} &\mathbb{P}\Big(\big\|S\big(P_N(v) - P_N(W_{T/N})\big)\big\|_{L^p(\lambda;\mathbb{R})} \le x\Big) \\ &\ge \Big[\tfrac{y}{\sqrt{2\pi\gamma T}}\Big]^{(2N+1)} \exp\Big(-\tfrac{3N^2}{T}\Big[\|v\|_H^2 + \tfrac{y^2}{\gamma}\Big]\Big). \end{aligned} \tag{3.30}$$

Proof of Corollary 3.5. First, note that (3.28) ensures that

$$\begin{aligned}&\big\|S(P_N(v)-P_N(W_{T/N}))\big\|_{L^p(\lambda;\mathbb{R})}\\&=\big\|(\eta-A)^{\nu}(\eta-A)^{-\nu}S(P_N(v)-P_N(W_{T/N}))\big\|_{L^p(\lambda;\mathbb{R})}\\&=\big\|(\eta-A)^{\nu}S(\eta-A)^{-\nu}(P_N(v)-P_N(W_{T/N}))\big\|_{L^p(\lambda;\mathbb{R})}.\end{aligned}\tag{3.31}$$

This implies that

$$\begin{aligned}&\big\|S(P_N(v)-P_N(W_{T/N}))\big\|_{L^p(\lambda;\mathbb{R})}\\&\leq\big\|(\eta-A)^{s}(\eta-A)^{-s}(\eta-A)^{\nu}S\big\|_{L(H,L^p(\lambda;\mathbb{R}))}\big\|(\eta-A)^{-\nu}(P_N(v)-P_N(W_{T/N}))\big\|_H\\&\leq\big\|(\eta-A)^{(\nu+s)}S\big\|_{L(H)}\big\|(\eta-A)^{-s}\big\|_{L(H,L^p(\lambda;\mathbb{R}))}\big\|(\eta-A)^{-\nu}(P_N(v)-P_N(W_{T/N}))\big\|_H.\end{aligned}\tag{3.32}$$

Combining this with (3.27) proves that

$$\begin{aligned}&\big\|S(P_N(v)-P_N(W_{T/N}))\big\|_{L^p(\lambda;\mathbb{R})}\\&\leq\delta\left[\tfrac{N}{T}\right]^{(\nu+s)}\big\|(\eta-A)^{-s}\big\|_{L(H,L^p(\lambda;\mathbb{R}))}\big\|(\eta-A)^{-\nu}(P_N(v)-P_N(W_{T/N}))\big\|_H.\end{aligned}\tag{3.33}$$

Hence, we obtain that

$$\begin{aligned}&\mathbb{P}\Big(\big\|S\big(P_N(v)-P_N(W_{T/N})\big)\big\|_{L^p(\lambda;\mathbb{R})}\leq x\Big)\\&\geq\mathbb{P}\Bigg(\big\|(\eta-A)^{-\nu}(P_N(v)-P_N(W_{T/N}))\big\|_H\leq\tfrac{x}{\delta}\left[\tfrac{T}{N}\right]^{(\nu+s)}\big\|(\eta-A)^{-s}\big\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigg).\end{aligned}\tag{3.34}$$

Combining this with Lemma 3.4 (with $(\Omega,\mathcal{F},\mathbb{P})=(\Omega,\mathcal{F},\mathbb{P})$, $T=T$, $x=y$, $\nu=\nu$, $N=N$, $v=v$, $W=W$ in the notation of Lemma 3.4) establishes (3.30). The proof of Corollary 3.5 is thus completed. □

3.3 Divergence results for general Euler-type approximation schemes for SPDEs with superlinearly growing nonlinearities

Proposition 3.6. *Assume Setting 3.1, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $T\in(0,\infty)$, $q\in\{2,3,\dots\}$, $a_0,a_1,\dots,a_{q-1}\in\mathbb{R}$, $a_q\in\mathbb{R}\backslash\{0\}$, $\chi\in(1/4,1]$, $\nu\in(1/4,3/4)$, $\xi\in H_\chi$, let $W\colon[0,T]\times\Omega\to H_{-\nu}$ be an Id_H-cylindrical Wiener process, let $S_N\in L(H_{-\nu})$, $N\in\mathbb{N}$, be linear operators which satisfy for all $N\in\mathbb{N}$, $r\in[-\nu,\infty)$, $v,u\in H$ that*

$$S_N(H_r)\subseteq H_{r+1},\qquad S_Ne_0=e_0,\qquad\langle S_Nu,v\rangle_H=\langle u,S_Nv\rangle_H,\tag{3.35}$$

$$\sup\nolimits_{M\in\mathbb{N}}\sup\nolimits_{s\in[0,1]}\sup\nolimits_{w\in H,\|w\|_H\leq1}\big(M^{-s}\|S_Mw\|_{H_s}\big)<\infty,\tag{3.36}$$

$$(\eta-A)^{-\nu}S_Nv=S_N(\eta-A)^{-\nu}v,\qquad\text{and}\qquad P_NS_Nv=S_NP_Nv,\tag{3.37}$$

and let $Y^N\colon\{0,1,\dots,N\}\times\Omega\to H$, $N\in\mathbb{N}$, be the stochastic processes which satisfy for all $N\in\mathbb{N}$, $n\in\{0,1,\dots,N-1\}$ that $Y^N_0=P_N(\xi)$ and

$$Y^N_{n+1}=P_NS_N\Big(Y^N_n+\tfrac{T}{N}\Big(\textstyle\sum_{k=0}^q a_k\big[Y^N_n\big]^k\Big)+\big(W_{\frac{(n+1)T}{N}}-W_{\frac{nT}{N}}\big)\Big).\tag{3.38}$$

Then it holds for all $r\in(0,\infty)$ that $\liminf_{N\to\infty}\mathbb{E}\big[\|Y^N_N\|^r_H\big]=\infty$.

Proof of Proposition 3.6. Throughout this proof let $p \in [2q, \infty)$, $s \in (1/4, 1-\nu]$, let $\zeta_r \in [1, \infty)$, $r \in [0,1]$, be real numbers which satisfy for all $r \in [0,1]$ that

$$\sup_{N\in\mathbb{N}} \left(N^{-r} \left\|(\eta - A)^r S_N\right\|_{L(H)}\right) \leq \zeta_r T^{-r}, \tag{3.39}$$

let $C \in (0, \infty)$ be the real number which satisfies that

$$C = \sup_{v\in(L^p(\lambda;\mathbb{R})\cap H_\chi)\setminus\{0\}} \frac{\|v\|_{L^p(\lambda;\mathbb{R})}}{\|v\|_{H_\chi}} \tag{3.40}$$

(cf. Lemma 3.2), for every $N \in \mathbb{N}$, $r \in (0, \infty)$ let $\kappa, \vartheta, \rho_{N,r}, \theta_{N,r} \in (1, \infty)$, $c_{N,r} \in (0, 1]$, $\gamma_N, y_N, z_{N,r}, g_{N,r} \in (0, \infty)$ be the real numbers which satisfy that

$$\kappa = (q+2)\,|\max\{C,1\}|^q \max\{T,1\} \max_{k\in\{0,1,\ldots,q\}} \{1, |a_k|\} \max\{1, \|\xi\|_{H_\chi}^q\}, \tag{3.41}$$

$$\vartheta = 2^{(q-1)} \max\{C,1\} \max\{T,1\} \max_{k\in\{0,1,\ldots,q\}} \{8, |a_k|\}, \tag{3.42}$$

$$\rho_{N,r} = \max\left\{8\vartheta^2 \max\{C,1\} \max\{T,1\} \tfrac{\zeta_\chi N^\chi}{|c_{N,r}|^{1/r} \min\{T,1\}}, \tfrac{1}{2^{1/q}-1}\right\}, \tag{3.43}$$

$$\theta_{N,r} = \max\left\{\left[\tfrac{4T\vartheta+8N}{T|a_q|}\right]^r, 2^r\right\}, \tag{3.44}$$

$$c_{N,r} = \min\left\{\left[\tfrac{T|a_q|}{4N}\right]^r, 1\right\}, \quad \gamma_N = \textstyle\sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu}, \tag{3.45}$$

$$z_{N,r} = \tfrac{y_N}{|\rho_{N,r}|^{(N+1)}}, \qquad g_{N,r} = \tfrac{y_N}{2|\rho_{N,r}|^N}, \tag{3.46}$$

$$\text{and} \qquad y_N = \tfrac{T^{(\nu+s)}}{\zeta_{\nu+s} N^{(\nu+s)} \|(\eta-A)^{-s}\|_{L(H, L^p(\lambda;\mathbb{R}))}} \tag{3.47}$$

(cf. Lemma 3.2), let $P_0, \mathcal{R} \colon H \to H$ be the linear operators which satisfy for all $v \in H$ that

$$P_0(v) = \langle e_0, v\rangle_H \, e_0 \qquad \text{and} \qquad \mathcal{R}[v] = v - P_0(v), \tag{3.48}$$

let $\Phi_N \colon H \times H_{-\nu} \to H$, $N \in \mathbb{N}$, be the functions which satisfy for all $N \in \mathbb{N}$, $(v, u) \in H \times H_{-\nu}$ that

$$\Phi_N(v,u) = P_N S_N\big(v + \tfrac{T}{N}\big(\textstyle\sum_{k=0}^{q} a_k [v]^k\big) + u\big) \mathbb{1}_{L^{2q}(\lambda;\mathbb{R})\times H_{-\nu}}^{H\times H_{-\nu}}(v,u), \tag{3.49}$$

let $\mathcal{V}_r \colon H \to [0, \infty)$, $r \in (0, \infty)$, be the functions which satisfy for all $r \in (0, \infty)$, $v \in H$ that $\mathcal{V}_r(v) = \|P_0(v)\|_H^r$, let $Z_n^N \colon \Omega \to H_{-\nu}$, $n \in \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$, be the random variables which satisfy for all $N \in \mathbb{N}$, $n \in \{1, 2, \ldots, N\}$ that

$$Z_n^N = W_{\frac{nT}{N}} - W_{\frac{(n-1)T}{N}}, \tag{3.50}$$

let $(v_n^u)_{n\in\mathbb{Z}} \subseteq H$, $u \in H_{-\nu}$, satisfy for all $u \in H_{-\nu}$ that

$$\limsup_{n\to\infty} \|u - v_n^u\|_{H_{-\nu}} = 0, \tag{3.51}$$

and let $\mathbb{H}^N_{n,r} \subseteq H_\chi$, $n \in \{0,1,\ldots,N\}$, $N \in \mathbb{N}$, $r \in (0,\infty)$, be the sets which satisfy for all $r \in (0,\infty)$, $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N\}$ that

$$\mathbb{H}^N_{n,r} = \big\{ v \in H_\chi \colon \|\mathcal{R}[v]\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2}|\rho_{N,r}|^{(n-N)}\|P_0(v)\|_H \big\}. \tag{3.52}$$

Note that Lemma 3.2 (with $p = p$, $\chi = \chi$ in the notation of Lemma 3.2) ensures that the function $(H_\chi \ni v \mapsto v \in L^p(\lambda;\mathbb{R}))$ is continuous. Combining this with the fact that the functions $(H_\chi \ni v \mapsto \mathcal{R}[v] \in H_\chi)$ and $(H_\chi \ni v \mapsto P_0(v) \in H)$ are continuous assures that

$$(H_\chi \ni v \mapsto \mathcal{R}[v] \in L^p(\lambda;\mathbb{R})) \in \mathcal{M}(\mathcal{B}(H_\chi), \mathcal{B}(L^p(\lambda;\mathbb{R}))) \tag{3.53}$$

and

$$(H_\chi \ni v \mapsto P_0(v) \in H) \in \mathcal{M}(\mathcal{B}(H_\chi), \mathcal{B}(H)). \tag{3.54}$$

This implies for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N\}$, $r \in (0,\infty)$ that

$$\mathbb{H}^N_{n,r} \in \mathcal{B}(H_\chi). \tag{3.55}$$

Furthermore, observe that the fact that $H_\chi \subseteq H$ continuously and Lemma 2.2 in [3] (with $V_0 = H$ and $V_1 = H_\chi$ in the notation of Lemma 2.2 in [3]) establish that

$$\mathcal{B}(H_\chi) \subseteq \mathcal{B}(H). \tag{3.56}$$

Combining this with (3.55) proves for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N\}$, $r \in (0,\infty)$ that

$$\mathbb{H}^N_{n,r} \in \mathcal{B}(H). \tag{3.57}$$

In addition, note that Lemma 5.3 in [6] (with $V = L^{2q}(\lambda;\mathbb{R}) \times H_{-\nu}$, $W = H \times H_{-\nu}$, $(S,\mathcal{S}) = (H,\mathcal{B}(H))$, $s = 0$, $\psi(v,u) = P_N S_N \big(v + \frac{T}{N}\sum_{k=0}^q a_k[v]^k + u\big)$ for $(v,u) \in L^{2q}(\lambda;\mathbb{R}) \times H_{-\nu}$, $N \in \mathbb{N}$ in the notation of Lemma 5.3 in [6]) ensures for all $N \in \mathbb{N}$ that

$$\Phi_N \in \mathcal{M}(\mathcal{B}(H \times H_{-\nu}), \mathcal{B}(H)). \tag{3.58}$$

Combining this with the fact that $\mathcal{B}(H \times H_{-\nu}) = \mathcal{B}(H) \otimes \mathcal{B}(H_{-\nu})$ proves for all $N \in \mathbb{N}$ that

$$\Phi_N \in \mathcal{M}(\mathcal{B}(H) \otimes \mathcal{B}(H_{-\nu}), \mathcal{B}(H)). \tag{3.59}$$

Moreover, note that it holds for all $r \in (0,\infty)$ that

$$\mathcal{V}_r \in \mathcal{M}(\mathcal{B}(H), \mathcal{B}([0,\infty))). \tag{3.60}$$

Next observe that it holds for all $N \in \mathbb{N}$ that $\sigma(Y_1^N)$ and $\sigma(Z_2^N, Z_3^N, \ldots, Z_N^N)$ are independent on $(\Omega,\mathcal{F},\mathbb{P})$ and $Z_2^N, Z_3^N, \ldots, Z_N^N$ are i.i.d. random variables. This, (3.59), (3.60), and Proposition 2.6 (with $(H,\mathcal{H}) = (H,\mathcal{B}(H))$, $(U,\mathcal{U}) = (H_{-\nu},\mathcal{B}(H_{-\nu}))$, $\Phi = \Phi_M$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $N = M-1$, $c = c_{M,r}$, $\alpha = q$, $\theta = \theta_{M,r}$, $\mathbb{H}_0 = \mathbb{H}^M_{0,r}$, $\mathbb{H}_1 = \mathbb{H}^M_{1,r}$, $\ldots$, $\mathbb{H}_N = \mathbb{H}^M_{M-1,r}$, $(Z_1,Z_2,\ldots,Z_N) = (Z_2^M, Z_3^M, \ldots, Z_M^M)$, $Y_0 = Y_1^M$,

$Y_1 = Y_2^M, \ldots, Y_N = Y_M^M$, $\mathcal{V} = \mathcal{V}_r$ for $r \in (0,\infty)$, $M \in \{2,3,\ldots\}$ in the notation of Proposition 2.6) ensure for all $r \in (0,\infty)$, $M \in \{2,3,\ldots\}$ that

$$\begin{aligned}
&\mathbb{E}\Big[\big|\big\langle e_0, Y_M^M\big\rangle_H\big|^r\Big] \\
&\geq |\theta_{M,r}|^{(q^{(M-1)})}\,\mathbb{P}\Big(\Big\{\big|\big\langle e_0, Y_1^M\big\rangle_H\big|^r \geq |c_{M,r}|^{1/(1-q)}\theta_{M,r}\Big\} \cap \big\{Y_1^M \in \mathbb{H}_{0,r}^M\big\}\Big) \\
&\cdot \Bigg[\prod_{n=1}^{M-1} \inf\Bigg(\Bigg\{\mathbb{P}\Big(\Big\{\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big|^r \geq c_{M,r}\,|\langle e_0, v\rangle_H|^{rq}\Big\} \\
&\qquad\qquad \cap \Big\{P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}_{n,r}^M\Big\}\Big) \\
&\qquad\qquad\quad : \Big(v \in \mathbb{H}_{n-1,r}^M \colon |\langle e_0, v\rangle_H|^r \geq |\theta_{M,r}|^{(q^{(n-1)})}\Big)\Bigg\} \cup \{1\}\Bigg)\Bigg].
\end{aligned} \tag{3.61}$$

This implies for all $r \in (0,\infty)$, $M \in \{2,3,\ldots\}$ that

$$\begin{aligned}
&\mathbb{E}\Big[\big|\big\langle e_0, Y_M^M\big\rangle_H\big|^r\Big] \\
&\geq |\theta_{M,r}|^{(q^{(M-1)})}\,\mathbb{P}\Big(\Big\{\big|\big\langle e_0, Y_1^M\big\rangle_H\big|^r \geq |c_{M,r}|^{1/(1-q)}\theta_{M,r}\Big\} \cap \big\{Y_1^M \in \mathbb{H}_{0,r}^M\big\}\Big) \\
&\cdot \Bigg[\prod_{n=1}^{M-1} \inf\Bigg(\Bigg\{\mathbb{P}\Big(\Big\{\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^{q}\Big\} \\
&\qquad\qquad \cap \Big\{P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}_{n,r}^M\Big\}\Big) \\
&\qquad\qquad\quad : \Big(v \in \mathbb{H}_{n-1,r}^M \colon |\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}\Big)\Bigg\} \cup \{1\}\Bigg)\Bigg].
\end{aligned} \tag{3.62}$$

Next observe that Lemma 3.2 (with $p = pq$, $\chi = \chi$ in the notation of Lemma 3.2) ensures that for all $v \in H_\chi$ it holds that $[v]^q \in L^p(\lambda;\mathbb{R})$. This proves that for all $M \in \{2,3,\ldots\}$, $v \in H_\chi$ it holds that

$$v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k[v]^k \in L^p(\lambda;\mathbb{R}). \tag{3.63}$$

Furthermore, note that for all $M \in \{2,3,\ldots\}$, $v \in H_\chi$ it holds that

$$\begin{aligned}
&\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H \\
&= \big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k[v]^k\big) + P_M S_M Z_2^M\big\rangle_H \\
&= \big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k[v]^k\big)\big\rangle_H + \big\langle e_0, P_M S_M Z_2^M\big\rangle_H \\
&= \big\langle P_M(e_0), S_M\big(v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k[v]^k\big)\big\rangle_H + \big\langle e_0, P_M S_M Z_2^M\big\rangle_H .
\end{aligned} \tag{3.64}$$

This, (3.35), and (3.63) imply for all $M \in \{2,3,\ldots\}$, $v \in H_\chi$ that

$$\begin{aligned}
&\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H \\
&= \big\langle e_0, S_M\big(v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k[v]^k\big)\big\rangle_H + \big\langle e_0, P_M S_M Z_2^M\big\rangle_H \\
&= \big\langle S_M(e_0), v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k[v]^k\big\rangle_H + \big\langle e_0, P_M S_M Z_2^M\big\rangle_H \\
&= \big\langle e_0, v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k[v]^k\big\rangle_H + \big\langle e_0, P_M S_M Z_2^M\big\rangle_H .
\end{aligned} \tag{3.65}$$

Next observe that (3.37) and the fact that $\forall\, u \in H_{-\nu}$, $N \in \mathbb{N}$, $n \in \mathbb{Z}$: $P_N(u - v_n^u) \in H_\nu$ ensure for all $u \in H_{-\nu}$, $N \in \mathbb{N}$, $n \in \mathbb{Z}$ that

$$\begin{aligned} S_N P_N u - S_N P_N v_n^u &= S_N P_N (u - v_n^u) \\ &= S_N (\eta - A)^{-\nu} (\eta - A)^\nu P_N (u - v_n^u) \\ &= (\eta - A)^{-\nu} S_N (\eta - A)^\nu P_N (u - v_n^u). \end{aligned} \tag{3.66}$$

The fact that $\forall\, N \in \mathbb{N}$: $(\eta - A)^{-\nu} S_N \in L(H_{-\nu}, H)$ therefore assures for all $u \in H_{-\nu}$, $N \in \mathbb{N}$, $n \in \mathbb{Z}$ that

$$\begin{aligned} \|S_N P_N u - S_N P_N v_n^u\|_H &\leq \left\|(\eta - A)^{-\nu} S_N\right\|_{L(H_{-\nu},H)} \|(\eta - A)^\nu P_N (u - v_n^u)\|_{H_{-\nu}} \\ &= \left\|(\eta - A)^{-\nu} S_N\right\|_{L(H_{-\nu},H)} \|P_N (u - v_n^u)\|_H \,. \end{aligned} \tag{3.67}$$

This, the fact that $\forall\, N \in \mathbb{N}$: $P_N \in L(H_{-1}, H_1)$, and the fact that $L(H_{-1}, H_1) \subseteq L(H_{-1}, H)$ prove for all $u \in H_{-\nu}$, $N \in \mathbb{N}$, $n \in \mathbb{Z}$ that

$$\|S_N P_N u - S_N P_N v_n^u\|_H \leq \left\|(\eta - A)^{-\nu} S_N\right\|_{L(H_{-\nu},H)} \|P_N\|_{L(H_{-1},H)} \|u - v_n^u\|_{H_{-1}} \,. \tag{3.68}$$

Combining this with (3.51) and the fact that $H_{-\nu} \subseteq H_{-1}$ continuously establishes for all $u \in H_{-\nu}$, $N \in \mathbb{N}$ that

$$\limsup_{n \to \infty} \|S_N P_N u - S_N P_N v_n^u\|_H = 0. \tag{3.69}$$

In addition, observe that it holds for all $u \in H_{-\nu}$, $N \in \mathbb{N}$, $n \in \mathbb{Z}$ that

$$\begin{aligned} &\|P_N S_N u - P_N S_N v_n^u\|_H = \|P_N S_N (u - v_n^u)\|_H \\ &\leq \|P_N\|_{L(H_{-1},H)} \|S_N (u - v_n^u)\|_{H_{-1}} \\ &\leq \left[\sup_{w \in H_{-\nu} \setminus \{0\}} \tfrac{\|w\|_{H_{-1}}}{\|w\|_{H_{-\nu}}}\right] \|P_N\|_{L(H_{-1},H)} \|S_N (u - v_n^u)\|_{H_{-\nu}} \\ &\leq \left[\sup_{w \in H_{-\nu} \setminus \{0\}} \tfrac{\|w\|_{H_{-1}}}{\|w\|_{H_{-\nu}}}\right] \|P_N\|_{L(H_{-1},H)} \|S_N\|_{L(H_{-\nu})} \|u - v_n^u\|_{H_{-\nu}} \,. \end{aligned} \tag{3.70}$$

Combining this with (3.51) and the fact that $H_{-\nu} \subseteq H_{-1}$ continuously proves for all $u \in H_{-\nu}$, $N \in \mathbb{N}$ that

$$\limsup_{n \to \infty} \|P_N S_N u - P_N S_N v_n^u\|_H = 0. \tag{3.71}$$

Moreover, note that (3.37) assures for all $u \in H_{-\nu}$, $N \in \mathbb{N}$, $n \in \mathbb{N}$ that

$$S_N P_N v_u^N = P_N S_N v_u^N. \tag{3.72}$$

Combining this, (3.69), and (3.71) establishes for all $u \in H_{-\nu}$, $N \in \mathbb{N}$ that

$$S_N P_N u = P_N S_N u. \tag{3.73}$$

This and (3.65) ensure for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned} &\left\langle e_0, P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^q a_k [v]^k\big) + Z_2^M\big)\right\rangle_H \\ &= \left\langle e_0, v + \tfrac{T}{M} \textstyle\sum_{k=0}^q a_k [v]^k\right\rangle_H + \left\langle e_0, S_M P_M Z_2^M\right\rangle_H \,. \end{aligned} \tag{3.74}$$

Combining this with the reverse triangle inequality proves for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned} &\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k [v]^k\big) + Z_2^M\big)\big\rangle_H\big| \\ &\geq \big|\big\langle e_0, v + \tfrac{T}{M}\textstyle\sum_{k=0}^{q} a_k [v]^k\big\rangle_H\big| - \big|\big\langle e_0, S_M P_M Z_2^M\big\rangle_H\big| \\ &\geq \big|\big\langle e_0, \tfrac{T}{M} a_q [v]^q\big\rangle_H\big| - \big|\big\langle e_0, v + \tfrac{T}{M}\textstyle\sum_{k=0}^{q-1} a_k [v]^k\big\rangle_H\big| - \big|\big\langle e_0, S_M P_M Z_2^M\big\rangle_H\big| \\ &= \tfrac{T|a_q|}{M}\big|\big\langle e_0, [v]^q\big\rangle_H\big| - \big|\langle e_0, v\rangle_H + \tfrac{T}{M}\textstyle\sum_{k=0}^{q-1} a_k \big\langle e_0, [v]^k\big\rangle_H\big| - \big|\big\langle e_0, S_M P_M Z_2^M\big\rangle_H\big|. \end{aligned} \tag{3.75}$$

This and the triangle inequality imply for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned} &\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k [v]^k\big) + Z_2^M\big)\big\rangle_H\big| \\ &\geq \tfrac{T|a_q|}{M}\big|\big\langle e_0, [v]^q\big\rangle_H\big| - \big|\tfrac{T}{M}\textstyle\sum_{k=0}^{q-1} a_k \big\langle e_0, [v]^k\big\rangle_H\big| - |\langle e_0, v\rangle_H| - \big|\big\langle e_0, S_M P_M Z_2^M\big\rangle_H\big| \\ &\geq \tfrac{T|a_q|}{M}\big|\big\langle e_0, [v]^q\big\rangle_H\big| - \tfrac{T}{M}\textstyle\sum_{k=0}^{q-1} |a_k| \big|\big\langle e_0, [v]^k\big\rangle_H\big| - |\langle e_0, v\rangle_H| - \big|\big\langle e_0, S_M P_M Z_2^M\big\rangle_H\big|. \end{aligned} \tag{3.76}$$

Moreover, observe that it holds for all $v \in H_\chi$ that

$$[v]^q = (P_0(v) + \mathcal{R}[v])^q = \textstyle\sum_{m=0}^{q} \binom{q}{m} (P_0(v))^{(q-m)} (\mathcal{R}[v])^m. \tag{3.77}$$

This proves for all $v \in H_\chi$ that

$$\begin{aligned} &\big|\big\langle e_0, [v]^q\big\rangle_H\big| = \big|\big\langle e_0, \textstyle\sum_{m=0}^{q} \binom{q}{m} (P_0(v))^{(q-m)} (\mathcal{R}[v])^m\big\rangle_H\big| \\ &= \big|\big\langle e_0, (P_0(v))^q + \textstyle\sum_{m=1}^{q} \binom{q}{m} (P_0(v))^{(q-m)} (\mathcal{R}[v])^m\big\rangle_H\big| \\ &\geq |\langle e_0, (P_0(v))^q\rangle_H| - \big|\big\langle e_0, \textstyle\sum_{m=1}^{q} \binom{q}{m} (P_0(v))^{(q-m)} (\mathcal{R}[v])^m\big\rangle_H\big| \\ &\geq |\langle e_0, (P_0(v))^q\rangle_H| - \textstyle\sum_{m=1}^{q} \binom{q}{m} \big|\big\langle e_0, (P_0(v))^{(q-m)} (\mathcal{R}[v])^m\big\rangle_H\big|. \end{aligned} \tag{3.78}$$

Next note that it holds for all $v \in H_\chi$ that

$$\begin{aligned} &\big|\big\langle e_0, (P_0(v))^q\big\rangle_H\big| = \big|\big\langle e_0, \big(\langle e_0, v\rangle_H\, e_0\big)^q\big\rangle_H\big| \\ &= \big|\big\langle e_0, \big(\langle e_0, v\rangle_H\big)^q (e_0)^q\big\rangle_H\big| = \big|\big\langle e_0, \big(\langle e_0, v\rangle_H\big)^q e_0\big\rangle_H\big| \\ &= \big|\big(\langle e_0, v\rangle_H\big)^q \langle e_0, e_0\rangle_H\big| = \big|\langle e_0, v\rangle_H\big|^q. \end{aligned} \tag{3.79}$$

Furthermore, observe that it holds for all $v \in H_\chi$, $k \in \{0, 1, \dots, q\}$, $m \in \{0, 1, \dots, k\}$ that

$$\begin{aligned} &\big|\big\langle e_0, (P_0(v))^{(k-m)} (\mathcal{R}[v])^m\big\rangle_H\big| \leq \big\langle e_0, \big|(P_0(v))^{(k-m)} (\mathcal{R}[v])^m\big|\big\rangle_H \\ &= \big\langle e_0, \big|(\langle e_0, v\rangle_H\, e_0)^{(k-m)} (\mathcal{R}[v])^m\big|\big\rangle_H = \big\langle e_0, \big|(\langle e_0, v\rangle_H)^{(k-m)} (\mathcal{R}[v])^m\big|\big\rangle_H \\ &= |\langle e_0, v\rangle_H|^{(k-m)} \langle e_0, |(\mathcal{R}[v])^m|\rangle_H = |\langle e_0, v\rangle_H|^{(k-m)} \langle e_0, |\mathcal{R}[v]|^m\rangle_H \\ &= |\langle e_0, v\rangle_H|^{(k-m)} \|\mathcal{R}[v]\|_{L^m(\lambda;\mathbb{R})}^m \leq |\langle e_0, v\rangle_H|^{(k-m)} \|\mathcal{R}[v]\|_{L^{2q}(\lambda;\mathbb{R})}^m. \end{aligned} \tag{3.80}$$

This, (3.52), and the fact that $2q \leq p$ prove for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in \mathbb{H}_{n-1,r}^M$ that

$$\begin{aligned} &\textstyle\sum_{m=1}^{q} \binom{q}{m} \big|\big\langle e_0, (P_0(v))^{(q-m)} (\mathcal{R}[v])^m\big\rangle_H\big| \\ &\leq \textstyle\sum_{m=1}^{q} \binom{q}{m} |\langle e_0, v\rangle_H|^{(q-m)} \|\mathcal{R}[v]\|_{L^p(\lambda;\mathbb{R})}^m \\ &\leq \textstyle\sum_{m=1}^{q} \binom{q}{m} \frac{1}{2^m} |\rho_{M,r}|^{m(n-1-M)} |\langle e_0, v\rangle_H|^{(q-m)} |\langle e_0, v\rangle_H|^m \\ &= \textstyle\sum_{m=1}^{q} \binom{q}{m} \frac{1}{2^m} |\rho_{M,r}|^{m(n-1-M)} |\langle e_0, v\rangle_H|^q. \end{aligned} \tag{3.81}$$

Hence, we obtain for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ that

$$\begin{aligned}
&\textstyle\sum_{m=1}^{q} \binom{q}{m} \left|\left\langle e_0, (P_0(v))^{(q-m)}(\mathcal{R}[v])^m \right\rangle_H\right| \\
&\le |\langle e_0, v\rangle_H|^q \textstyle\sum_{m=1}^{q} \binom{q}{m} \left(\frac{1}{2}\right)^m \left(|\rho_{M,r}|^{(n-1-M)}\right)^m \\
&= |\langle e_0, v\rangle_H|^q \textstyle\sum_{m=1}^{q} \binom{q}{m} \left(\frac{1}{2}|\rho_{M,r}|^{(n-1-M)}\right)^m \cdot 1^{(q-m)} \\
&= |\langle e_0, v\rangle_H|^q \left[\textstyle\sum_{m=0}^{q} \binom{q}{m} \left(\frac{1}{2}|\rho_{M,r}|^{(n-1-M)}\right)^m \cdot 1^{(q-m)} - 1\right] \\
&= |\langle e_0, v\rangle_H|^q \left[\left(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\right)^q - 1\right].
\end{aligned} \tag{3.82}$$

Combining this with (3.78) and (3.79) establishes for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ that

$$\begin{aligned}
\left|\left\langle e_0, [v]^q \right\rangle_H\right| &\ge |\langle e_0, v\rangle_H|^q - \textstyle\sum_{m=1}^{q} \binom{q}{m} \left|\left\langle e_0, (P_0(v))^{(q-m)}(\mathcal{R}[v])^m \right\rangle_H\right| \\
&\ge |\langle e_0, v\rangle_H|^q - |\langle e_0, v\rangle_H|^q \left[\left(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\right)^q - 1\right] \\
&= \left[2 - \left(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\right)^q\right] |\langle e_0, v\rangle_H|^q.
\end{aligned} \tag{3.83}$$

Next observe that for all $v \in H_\chi$, $k \in \{0,1,\dots,q-1\}$ it holds that

$$\begin{aligned}
\left|\left\langle e_0, [v]^k \right\rangle_H\right| &= \left|\left\langle e_0, (P_0(v) + \mathcal{R}[v])^k \right\rangle_H\right| \\
&= \left|\left\langle e_0, \textstyle\sum_{m=0}^{k} \binom{k}{m} (P_0(v))^{(k-m)}(\mathcal{R}[v])^m \right\rangle_H\right| \\
&= \left|\textstyle\sum_{m=0}^{k} \left\langle e_0, \binom{k}{m} (P_0(v))^{(k-m)}(\mathcal{R}[v])^m \right\rangle_H\right| \\
&\le \textstyle\sum_{m=0}^{k} \left|\left\langle e_0, \binom{k}{m} (P_0(v))^{(k-m)}(\mathcal{R}[v])^m \right\rangle_H\right|.
\end{aligned} \tag{3.84}$$

This, (3.52), and (3.80) prove for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$, $k \in \{0,1,\dots,q-1\}$ that

$$\begin{aligned}
\left|\left\langle e_0, [v]^k \right\rangle_H\right| &\le \textstyle\sum_{m=0}^{k} \binom{k}{m} |\langle e_0, v\rangle_H|^{(k-m)} \|\mathcal{R}[v]\|^m_{L^{2q}(\lambda;\mathbb{R})} \\
&\le \textstyle\sum_{m=0}^{k} \binom{k}{m} |\langle e_0, v\rangle_H|^{(k-m)} \|\mathcal{R}[v]\|^m_{L^{p}(\lambda;\mathbb{R})} \\
&\le \textstyle\sum_{m=0}^{k} \binom{k}{m} |\langle e_0, v\rangle_H|^{(k-m)} |\langle e_0, v\rangle_H|^m \\
&= \textstyle\sum_{m=0}^{k} \binom{k}{m} |\langle e_0, v\rangle_H|^k = |\langle e_0, v\rangle_H|^k \textstyle\sum_{m=0}^{k} \binom{k}{m} \\
&= |\langle e_0, v\rangle_H|^k \textstyle\sum_{m=0}^{k} \binom{k}{m} 1^{(k-m)} \cdot 1^m \\
&= |\langle e_0, v\rangle_H|^k (1+1)^k = |\langle e_0, v\rangle_H|^k 2^k.
\end{aligned} \tag{3.85}$$

This implies for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$, $k \in \{0,1,\dots,q-1\}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\left|\left\langle e_0, [v]^k \right\rangle_H\right| \le 2^{(q-1)} |\langle e_0, v\rangle_H|^{(q-1)}. \tag{3.86}$$

This and (3.42) ensure for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$,

$v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned}
&\tfrac{T}{M}\textstyle\sum_{k=0}^{q-1} |a_k| \big|\big\langle e_0, [v]^k\big\rangle_H\big| \leq \tfrac{T}{M}\textstyle\sum_{k=0}^{q-1} 2^{(q-1)}|a_k|\, |\langle e_0, v\rangle_H|^{(q-1)} \\
&\leq \tfrac{2^{(q-1)}T}{M}\textstyle\sum_{k=0}^{q-1} \max\{|a_0|, |a_1|, \ldots, |a_{q-1}|\}\, |\langle e_0, v\rangle_H|^{(q-1)} \\
&\leq \tfrac{2^{(q-1)}T}{M} \max\{|a_0|, |a_1|, \ldots, |a_{q-1}|\}\, |\langle e_0, v\rangle_H|^{(q-1)} \\
&\leq \tfrac{T\vartheta}{M}\, |\langle e_0, v\rangle_H|^{(q-1)}.
\end{aligned} \tag{3.87}$$

Combining this with (3.76) establishes for all $r \in (0,\infty)$, $M \in \{2, 3, \ldots\}$, $n \in \{1, 2, \ldots, M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned}
&\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \\
&\geq \tfrac{T|a_q|}{M}\big|\big\langle e_0, [v]^q\big\rangle_H\big| - \tfrac{T\vartheta}{M}\, |\langle e_0, v\rangle_H|^{(q-1)} - |\langle e_0, v\rangle_H| - \big|\big\langle e_0, S_M P_M Z_2^M\big\rangle_H\big|.
\end{aligned} \tag{3.88}$$

This and (3.83) prove for all $r \in (0,\infty)$, $M \in \{2, 3, \ldots\}$, $n \in \{1, 2, \ldots, M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned}
&\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \\
&\geq \tfrac{T|a_q|}{M}\Big[2 - \big(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\big)^q\Big]\, |\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta}{M}\, |\langle e_0, v\rangle_H|^{(q-1)} \\
&\quad - |\langle e_0, v\rangle_H| - \big|\big\langle e_0, S_M P_M Z_2^M\big\rangle_H\big|.
\end{aligned} \tag{3.89}$$

Hence, we obtain for all $r \in (0,\infty)$, $M \in \{2, 3, \ldots\}$, $n \in \{1, 2, \ldots, M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned}
&\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \\
&\geq \tfrac{T|a_q|}{M}\Big[2 - \big(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\big)^q\Big]\, |\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta}{M}\, |\langle e_0, v\rangle_H|^{(q-1)} \\
&\quad - |\langle e_0, v\rangle_H| - \|S_M P_M Z_2^M\|_H.
\end{aligned} \tag{3.90}$$

This establishes for all $r \in (0,\infty)$, $M \in \{2, 3, \ldots\}$, $n \in \{1, 2, \ldots, M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned}
&\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \\
&\geq \tfrac{T|a_q|}{M}\Big[2 - \big(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\big)^q\Big]\, |\langle e_0, v\rangle_H|^q \\
&\quad - \tfrac{T\vartheta}{M}\, |\langle e_0, v\rangle_H|^{(q-1)} - |\langle e_0, v\rangle_H|^{(q-1)} - \|S_M P_M Z_2^M\|_H.
\end{aligned} \tag{3.91}$$

Next note that the fact that $\forall\, r \in (0,\infty), N \in \mathbb{N}\colon \rho_{N,r} \geq \frac{1}{2^{1/q}-1}$ ensures for all $r \in (0,\infty)$, $M \in \{2, 3, \ldots\}$, $n \in \{1, 2, \ldots, M\}$ that

$$\begin{aligned}
&2 - \big(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\big)^q \geq 2 - \big(\tfrac{1}{2}|\rho_{M,r}|^{-1} + 1\big)^q \geq 2 - \big(\tfrac{1}{2}\,\big(2^{1/q} - 1\big) + 1\big)^q \\
&= 2 - \big(\tfrac{1}{2}\cdot 2^{1/q} + \tfrac{1}{2}\big)^q = 2 - \big(2^{(1/q)-1} + \tfrac{1}{2}\big)^q = 2 - \big(2^{(1-q)/q} + \tfrac{1}{2}\big)^q.
\end{aligned} \tag{3.92}$$

Moreover, observe that the fact that $\forall\, x, y \in \mathbb{R}\colon |x + y|^q \leq 2^{(q-1)}(|x|^q + |y|^q)$ assures that

$$\big(2^{(1-q)/q} + \tfrac{1}{2}\big)^q \leq 2^{(q-1)}(2^{(1-q)} + 2^{-q}) = 1 + \tfrac{1}{2} = \tfrac{3}{2}. \tag{3.93}$$

This and (3.92) establish for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$ that

$$2 - \left(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + 1\right)^q \geq 2 - \tfrac{3}{2} = \tfrac{1}{2}. \tag{3.94}$$

Combining this with (3.91) proves for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned} &\left|\left\langle e_0, P_M S_M\left(v + \tfrac{T}{M}\left(\textstyle\sum_{k=0}^q a_k[v]^k\right) + Z_2^M\right)\right\rangle_H\right| \\ &\geq \tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} - |\langle e_0, v\rangle_H|^{(q-1)} - \|S_M P_M Z_2^M\|_H \\ &= \tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} - \|S_M P_M Z_2^M\|_H. \end{aligned} \tag{3.95}$$

Hence, we obtain for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned} &\mathbb{P}\Big(\left|\left\langle e_0, P_M S_M\left(v + \tfrac{T}{M}\left(\textstyle\sum_{k=0}^q a_k[v]^k\right) + Z_2^M\right)\right\rangle_H\right| \geq |c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^q\Big) \\ &\geq \mathbb{P}\Big(\tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} - \|S_M P_M Z_2^M\|_H \geq |c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^q\Big) \\ &\geq \mathbb{P}\Big(\Big\{\tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} \\ &\qquad - \|S_M P_M Z_2^M\|_H \geq |c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^q\Big\} \cap \big\{\|S_M P_M Z_2^M\|_H \leq 1\big\}\Big). \end{aligned} \tag{3.96}$$

This implies for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned} &\mathbb{P}\Big(\left|\left\langle e_0, P_M S_M\left(v + \tfrac{T}{M}\left(\textstyle\sum_{k=0}^q a_k[v]^k\right) + Z_2^M\right)\right\rangle_H\right| \geq |c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^q\Big) \\ &\geq \mathbb{P}\Big(\tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} - 1 \geq |c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^q\Big) \\ &\quad \cdot \mathbb{P}\Big(\|S_M P_M Z_2^M\|_H \leq 1\Big). \end{aligned} \tag{3.97}$$

Moreover, note that it holds for all $M \in \{2,3,\dots\}$, $v \in H$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned} &\tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} - 1 \\ &\geq \tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} - |\langle e_0, v\rangle_H|^{(q-1)} \\ &= \tfrac{T|a_q|}{2M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+2M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} \\ &= \tfrac{T|a_q|}{4M}\,|\langle e_0, v\rangle_H|^q + \tfrac{T|a_q|}{4M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+2M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)}. \end{aligned} \tag{3.98}$$

Next observe that it holds for all $M \in \{2,3,\dots\}$, $v \in H$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned} &\tfrac{T|a_q|}{4M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+2M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} \geq 0 \\ &\Leftrightarrow \tfrac{T|a_q|}{4M}\,|\langle e_0, v\rangle_H| \geq \tfrac{T\vartheta+2M}{M} \\ &\Leftrightarrow |\langle e_0, v\rangle_H| \geq \tfrac{4T\vartheta+8M}{T|a_q|}. \end{aligned} \tag{3.99}$$

The fact that $\forall\, r \in (0,\infty), M \in \{2,3,\dots\}, n \in \{1,2,\dots,M\}, v \in H$ with $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}$: $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{1/r}$ and (3.44) therefore assure for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in H$ with $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}$ that

$$\tfrac{T|a_q|}{4M}\,|\langle e_0, v\rangle_H|^q - \tfrac{T\vartheta+2M}{M}\,|\langle e_0, v\rangle_H|^{(q-1)} \geq 0. \tag{3.100}$$

Combining this with (3.98) proves for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in H$ with $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}$ that

$$\begin{aligned} &\tfrac{T|a_q|}{2M} \left|\langle e_0, v\rangle_H\right|^q - \tfrac{T\vartheta+M}{M} \left|\langle e_0, v\rangle_H\right|^{(q-1)} - 1 \geq \tfrac{T|a_q|}{4M} \left|\langle e_0, v\rangle_H\right|^q \\ &\geq \min\Bigl\{ \tfrac{T|a_q|}{4M}, 1\Bigr\} \left|\langle e_0, v\rangle_H\right|^q = \min\Bigl\{ \Bigl[\tfrac{T|a_q|}{4M}\Bigr]^r, 1\Bigr\}^{1/r} \left|\langle e_0, v\rangle_H\right|^q \\ &= |c_{M,r}|^{1/r} \left|\langle e_0, v\rangle_H\right|^q . \end{aligned} \tag{3.101}$$

This establishes for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in H$ with $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}$ that

$$\tfrac{T|a_q|}{2M} \left|\langle e_0, v\rangle_H\right|^q - \tfrac{T\vartheta+M}{M} \left|\langle e_0, v\rangle_H\right|^{(q-1)} - 1 \geq |c_{M,r}|^{1/r} \left|\langle e_0, v\rangle_H\right|^q . \tag{3.102}$$

Next observe that for all $M \in \{2,3,\dots\}$ it holds that

$$\mathbb{P}\bigl(\|S_M P_M Z_2^M\|_H \leq 1\bigr) \geq \mathbb{P}\bigl(\|S_M P_M Z_2^M\|_{L^p(\lambda;\mathbb{R})} \leq 1\bigr). \tag{3.103}$$

Corollary 3.5 (with $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $T = T$, $x = 1$, $\gamma = \gamma_M$, $y = y_M$, $p = p$, $\delta = \zeta_{\nu+s}$, $\nu = \nu$, $s = s$, $N = M$, $v = 0$, $S = (H \ni w \mapsto S_M w \in H_{\nu+s})$, $W = W$ for $M \in \{2,3,\dots\}$ in the notation of Corollary 3.5) therefore establishes for all $M \in \{2,3,\dots\}$ that

$$\begin{aligned} &\mathbb{P}\bigl(\|S_M P_M Z_2^M\|_H \leq 1\bigr) \geq \mathbb{P}\Bigl(\bigl\|S_M P_M (W_{2T/M} - W_{T/M})\bigr\|_{L^p(\lambda;\mathbb{R})} \leq 1\Bigr) \\ &= \mathbb{P}\Bigl(\bigl\|S_M P_M (W_{T/M})\bigr\|_{L^p(\lambda;\mathbb{R})} \leq 1\Bigr) \geq \Bigl[\tfrac{y_M}{\sqrt{2\pi\gamma_M T}}\Bigr]^{(2M+1)} \exp\Bigl(-\tfrac{3M^2|y_M|^2}{\gamma_M T}\Bigr). \end{aligned} \tag{3.104}$$

Combining this with (3.97) and (3.102) assures for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}$ that

$$\begin{aligned} &\mathbb{P}\Bigl(\bigl|\bigl\langle e_0, P_M S_M \bigl(v + \tfrac{T}{M}\bigl(\textstyle\sum_{k=0}^q a_k [v]^k\bigr) + Z_2^M\bigr)\bigr\rangle_H\bigr| \geq |c_{M,r}|^{1/r} \left|\langle e_0, v\rangle_H\right|^q \Bigr) \\ &\geq \Bigl[\tfrac{y_M}{\sqrt{2\pi\gamma_M T}}\Bigr]^{(2M+1)} \exp\Bigl(-\tfrac{3M^2|y_M|^2}{\gamma_M T}\Bigr). \end{aligned} \tag{3.105}$$

Next note that the triangle inequality and (3.37) establish for all $M \in \{2,3,\dots\}$, $v \in H_\chi$ that

$$\begin{aligned} &\Bigl\|\mathcal{R}\Bigl[P_M S_M \bigl(v + \tfrac{T}{M}\bigl(\textstyle\sum_{k=0}^q a_k [v]^k\bigr) + Z_2^M\bigr)\Bigr]\Bigr\|_{L^p(\lambda;\mathbb{R})} \\ &= \Bigl\|\mathcal{R}\Bigl[P_M S_M \bigl(v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k [v]^k\bigr) + P_M S_M Z_2^M\Bigr]\Bigr\|_{L^p(\lambda;\mathbb{R})} \\ &\leq \Bigl\|\mathcal{R}\Bigl[P_M S_M \bigl(v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k [v]^k\bigr)\Bigr]\Bigr\|_{L^p(\lambda;\mathbb{R})} + \Bigl\|\mathcal{R}\Bigl[P_M S_M Z_2^M\Bigr]\Bigr\|_{L^p(\lambda;\mathbb{R})} \\ &= \Bigl\|P_M \mathcal{R}\bigl[S_M \bigl(v + \tfrac{T}{M}\textstyle\sum_{k=0}^q a_k [v]^k\bigr)\bigr]\Bigr\|_{L^p(\lambda;\mathbb{R})} + \Bigl\|\mathcal{R}\Bigl[S_M P_M Z_2^M\Bigr]\Bigr\|_{L^p(\lambda;\mathbb{R})}. \end{aligned} \tag{3.106}$$

Moreover, observe that the triangle inequality proves for all $v \in L^p(\lambda;\mathbb{R})$ that

$$\begin{aligned} \|\mathcal{R}[v]\|_{L^p(\lambda;\mathbb{R})} &= \|v - \langle e_0, v\rangle_H\, e_0\|_{L^p(\lambda;\mathbb{R})} \leq \|v\|_{L^p(\lambda;\mathbb{R})} + \|\langle e_0, v\rangle_H\, e_0\|_{L^p(\lambda;\mathbb{R})} \\ &= \|v\|_{L^p(\lambda;\mathbb{R})} + |\langle e_0, v\rangle_H| \leq \|v\|_{L^p(\lambda;\mathbb{R})} + \|v\|_H \leq 2\|v\|_{L^p(\lambda;\mathbb{R})}. \end{aligned} \tag{3.107}$$

Next note that (3.35) ensures that for all $M \in \mathbb{N}$, $v \in H$ it holds that

$$\begin{aligned}\mathcal{R}[S_M v] &= (\mathrm{Id}_H - P_0) S_M v = S_M v - P_0 S_M v \\ &= S_M v - \langle e_0, S_M v \rangle_H \, e_0 = S_M v - \langle S_M e_0, v \rangle_H \, e_0 \\ &= S_M v - \langle e_0, v \rangle_H \, e_0 = S_M v - \langle e_0, v \rangle_H \, S_M e_0 \\ &= S_M v - S_M(\langle e_0, v \rangle_H \, e_0) = S_M(v - \langle e_0, v \rangle_H \, e_0) \\ &= S_M \mathcal{R}[v].\end{aligned} \tag{3.108}$$

Combining this with (3.63), (3.106), and (3.107) proves for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned}&\Big\| \mathcal{R}\Big[P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^{q} a_k [v]^k \big) + Z_2^M \big) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \\ &\leq \Big\| P_M S_M \mathcal{R}\big[v + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [v]^k \big] \Big\|_{L^p(\lambda;\mathbb{R})} + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})}.\end{aligned} \tag{3.109}$$

This establishes for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned}&\Big\| \mathcal{R}\Big[P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^{q} a_k [v]^k \big) + Z_2^M \big) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \Big\| P_M S_M \mathcal{R}\big[v + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [v]^k \big] \Big\|_{H_\chi} + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})} \\ &= C \Big\| (\eta - A)^\chi P_M S_M \mathcal{R}\big[v + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [v]^k \big] \Big\|_{H} + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})}.\end{aligned} \tag{3.110}$$

Combining this with (3.37) implies for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned}&\Big\| \mathcal{R}\Big[P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^{q} a_k [v]^k \big) + Z_2^M \big) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \Big\| (\eta - A)^\chi S_M P_M \mathcal{R}\big[v + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [v]^k \big] \Big\|_{H} + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \Big\| (\eta - A)^\chi S_M \Big\|_{L(H)} \big\| P_M \mathcal{R}\big[v + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [v]^k \big] \big\|_{H} + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})}.\end{aligned} \tag{3.111}$$

This and the fact that $\forall\, N \in \mathbb{N}, w \in H \colon \|P_N(w)\|_H \leq \|w\|_H$ prove for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned}&\Big\| \mathcal{R}\Big[P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^{q} a_k [v]^k \big) + Z_2^M \big) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \Big\| (\eta - A)^\chi S_M \Big\|_{L(H)} \big\| \mathcal{R}\big[v + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [v]^k \big] \big\|_{H} + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \zeta_\chi \big[\tfrac{M}{T} \big]^\chi \big\| \mathcal{R}\big[v + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [v]^k \big] \big\|_{H} + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})}.\end{aligned} \tag{3.112}$$

The triangle inequality and the linearity of $\mathcal{R}$ hence ensure for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned}&\Big\| \mathcal{R}\Big[P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^{q} a_k [v]^k \big) + Z_2^M \big) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \zeta_\chi \big[\tfrac{M}{T} \big]^\chi \big(\| \mathcal{R}[v] \|_H + \tfrac{T}{M} \big\| \mathcal{R}\big[\textstyle\sum_{k=0}^{q} a_k [v]^k \big] \big\|_H \big) + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})} \\ &= C \zeta_\chi \big[\tfrac{M}{T} \big]^\chi \big(\| \mathcal{R}[v] \|_H + \tfrac{T}{M} \big\| \textstyle\sum_{k=0}^{q} a_k \mathcal{R}\big[[v]^k \big] \big\|_H \big) + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})}.\end{aligned} \tag{3.113}$$

The triangle inequality therefore implies for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned} &\Big\| \mathcal{R}\Big[P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^{q} a_k [v]^k \big) + Z_2^M \big) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \zeta_\chi \big[\tfrac{M}{T} \big]^\chi \big(\| \mathcal{R}[v] \|_H + \tfrac{T}{M} \max\{|a_0|, |a_1|, \dots, |a_q|\} \textstyle\sum_{k=0}^{q} \big\| \mathcal{R}\big[[v]^k\big] \big\|_H \big) \\ &\quad + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})}. \end{aligned} \tag{3.114}$$

This and (3.42) ensure for all $M \in \{2, 3, \dots\}$, $v \in H_\chi$ that

$$\begin{aligned} &\Big\| \mathcal{R}\Big[P_M S_M \big(v + \tfrac{T}{M} \big(\textstyle\sum_{k=0}^{q} a_k [v]^k \big) + Z_2^M \big) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \\ &\leq C \zeta_\chi \big[\tfrac{M}{T} \big]^\chi \Big(\| \mathcal{R}[v] \|_H + \tfrac{T}{M} \vartheta \textstyle\sum_{k=0}^{q} \big\| \mathcal{R}\big[[v]^k\big] \big\|_H \Big) + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})} \\ &\leq \max\{C, 1\} \zeta_\chi \tfrac{|M|^\chi}{\min\{T,1\}} \Big(\| \mathcal{R}[v] \|_H + \tfrac{T\vartheta}{M} \textstyle\sum_{k=0}^{q} \big\| \mathcal{R}\big[[v]^k\big] \big\|_H \Big) + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})} \\ &\leq \max\{C, 1\} \zeta_\chi \tfrac{|M|^\chi}{\min\{T,1\}} \big(\| \mathcal{R}[v] \|_H + \tfrac{T\vartheta}{M} \textstyle\sum_{k=0}^{q} \big\| \mathcal{R}\big[[v]^k\big] \big\|_H + 2 \| S_M P_M Z_2^M \|_{L^p(\lambda;\mathbb{R})} \big). \end{aligned} \tag{3.115}$$

Furthermore, note that (3.107) and the fact that $\|\mathcal{R}\|_{L(H)} \leq 2$ ensure for all $v \in H_\chi$ that

$$\begin{aligned} \textstyle\sum_{k=0}^{q} \big\| \mathcal{R}\big[[v]^k\big] \big\|_H &\leq \textstyle\sum_{k=0}^{q} \|\mathcal{R}\|_{L(H)} \big\| [v]^k \big\|_H \leq 2 \textstyle\sum_{k=0}^{q} \big\| [v]^k \big\|_H \\ &= 2 \textstyle\sum_{k=0}^{q} \big\| \textstyle\sum_{m=0}^{k} \binom{k}{m} (P_0(v))^{(k-m)} (\mathcal{R}[v])^m \big\|_H \\ &\leq 2 \textstyle\sum_{k=0}^{q} \textstyle\sum_{m=0}^{k} \binom{k}{m} \big\| (P_0(v))^{(k-m)} (\mathcal{R}[v])^m \big\|_H \\ &= 2 \textstyle\sum_{k=0}^{q} \textstyle\sum_{m=0}^{k} \binom{k}{m} |\langle e_0, v \rangle_H|^{(k-m)} \big\| (\mathcal{R}[v])^m \big\|_H. \end{aligned} \tag{3.116}$$

This assures for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in \mathbb{H}_{n-1,r}^M$ that

$$\begin{aligned} &\textstyle\sum_{k=0}^{q} \big\| \mathcal{R}\big([v]^k\big) \big\|_H \leq 2 \textstyle\sum_{k=0}^{q} \textstyle\sum_{m=0}^{k} \binom{k}{m} |\langle e_0, v \rangle_H|^{(k-m)} \big\| \mathcal{R}[v] \big\|_{L^{2m}(\lambda,\mathbb{R})}^m \\ &\leq 2 \textstyle\sum_{k=0}^{q} \textstyle\sum_{m=0}^{k} \binom{k}{m} |\langle e_0, v \rangle_H|^{(k-m)} \big\| \mathcal{R}[v] \big\|_{L^p(\lambda,\mathbb{R})}^m \\ &\leq 2 \textstyle\sum_{k=0}^{q} \textstyle\sum_{m=0}^{k} \binom{k}{m} \tfrac{1}{2^m} |\rho_{M,r}|^{m(n-1-M)} |\langle e_0, v \rangle_H|^{(k-m)} |\langle e_0, v \rangle_H|^m \\ &\leq 2 \textstyle\sum_{k=0}^{q} \textstyle\sum_{m=0}^{k} \binom{k}{m} |\rho_{M,r}|^{(n-1-M)} |\langle e_0, v \rangle_H|^k \\ &= 2 |\rho_{M,r}|^{(n-1-M)} \Big(\textstyle\sum_{k=0}^{q} |\langle e_0, v \rangle_H|^k \Big[\textstyle\sum_{m=0}^{k} \binom{k}{m} \Big] \Big) \\ &= 2 |\rho_{M,r}|^{(n-1-M)} \Big(\textstyle\sum_{k=0}^{q} 2^k |\langle e_0, v \rangle_H|^k \Big). \end{aligned} \tag{3.117}$$

Therefore, we obtain for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in \mathbb{H}_{n-1,r}^M$ with $|\langle e_0, v \rangle_H| > 1$ that

$$\begin{aligned} &\textstyle\sum_{k=0}^{q} \big\| \mathcal{R}\big([v]^k\big) \big\|_H \leq 2 |\rho_{M,r}|^{(n-1-M)} |\langle e_0, v \rangle_H|^q \big(\textstyle\sum_{k=0}^{q} 2^k \big) \\ &= 2 |\rho_{M,r}|^{(n-1-M)} |\langle e_0, v \rangle_H|^q \, (2^{q+1} - 1) \leq 2^{q+2} |\rho_{M,r}|^{(n-1-M)} |\langle e_0, v \rangle_H|^q \\ &\leq \vartheta |\rho_{M,r}|^{(n-1-M)} |\langle e_0, v \rangle_H|^q. \end{aligned} \tag{3.118}$$

Combining this with (3.115) proves for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in \mathbb{H}_{n-1,r}^M$, $\omega \in \Omega$ with $|\langle e_0, v \rangle_H| > 1$ and $\| S_M P_M Z_2^M(\omega) \|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)}$

that

$$\begin{aligned}
&\Big\|\mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M(\omega)\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})} \qquad (3.119)\\
&\leq \max\{C,1\}\tfrac{\zeta_\chi |M|^\chi}{\min\{T,1\}}\Big(\|\mathcal{R}[v]\|_H + \tfrac{T\vartheta^2}{M}|\rho_{M,r}|^{(n-1-M)}\,|\langle e_0, v\rangle_H|^q + 2|\rho_{M,r}|^{(n-1-M)}\Big)\\
&\leq \max\{C,1\}\tfrac{\zeta_\chi |M|^\chi}{\min\{T,1\}}\Big(\|\mathcal{R}[v]\|_{L^p(\lambda;\mathbb{R})} + \tfrac{T\vartheta^2}{M}|\rho_{M,r}|^{(n-1-M)}\,|\langle e_0, v\rangle_H|^q + 2|\rho_{M,r}|^{(n-1-M)}\Big)\\
&\leq \max\{C,1\}\tfrac{\zeta_\chi |M|^\chi}{\min\{T,1\}}\Big(\tfrac{1}{2}|\rho_{M,r}|^{(n-1-M)} + \tfrac{T\vartheta^2}{M}|\rho_{M,r}|^{(n-1-M)}\,|\langle e_0, v\rangle_H|^q + 2|\rho_{M,r}|^{(n-1-M)}\Big).
\end{aligned}$$

Hence, we obtain for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}_{n-1,r}^M$, $\omega \in \Omega$ with $|\langle e_0, v\rangle_H| > 1$ and $\|S_M P_M Z_2^M(\omega)\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)}$ that

$$\begin{aligned}
&\Big\|\mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M(\omega)\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})}\\
&\leq \max\{C,1\}\tfrac{\zeta_\chi |M|^\chi}{\min\{T,1\}}|\rho_{M,r}|^{(n-1-M)}\Big(\tfrac{T\vartheta^2}{M} + 3\Big)\,|\langle e_0, v\rangle_H|^q\,.
\end{aligned} \tag{3.120}$$

Next note that it holds for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$\begin{aligned}
&\max\{C,1\}\tfrac{\zeta_\chi |M|^\chi}{|c_{M,r}|^{1/r}\min\{T,1\}}\Big(\tfrac{T\vartheta^2}{M} + 3\Big)\\
&\leq \max\{C,1\}\tfrac{\zeta_\chi |M|^\chi}{|c_{M,r}|^{1/r}\min\{T,1\}}\big(\max\{T,1\}\vartheta^2 + 3\max\{T,1\}\vartheta^2\big)\\
&= \vartheta^2\max\{C,1\}\max\{T,1\}\tfrac{\zeta_\chi |M|^\chi}{|c_{M,r}|^{1/r}\min\{T,1\}} \leq \tfrac{1}{2}\rho_{M,r}.
\end{aligned} \tag{3.121}$$

Combining this with (3.120) therefore establishes for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}_{n-1,r}^M$, $\omega \in \Omega$ with $|\langle e_0, v\rangle_H| > 1$ and $\|S_M P_M Z_2^M(\omega)\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)}$ that

$$\begin{aligned}
&\Big\|\mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M(\omega)\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})}\\
&\leq \tfrac{1}{2}|\rho_{M,r}|^{(n-M)}|c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^q\,.
\end{aligned} \tag{3.122}$$

This implies for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in H_\chi$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{\Big\|\mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})}\\
&\quad\leq \tfrac{1}{2}|\rho_{M,r}|^{(n-M)}|c_{M,r}|^{1/r}\,|\langle e_0, v\rangle_H|^q\Big\} \cap \big\{\|S_M P_M Z_2^M\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)}\big\}\Big)\\
&= \mathbb{P}\big(\|S_M P_M Z_2^M\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)}\big).
\end{aligned} \tag{3.123}$$

Furthermore, note that (3.63) ensures for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}_{n-1,r}^M$ that

$$\begin{aligned}
&\big\{\omega \in \Omega\colon P_M S_M\,\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M(\omega)\big) \in \mathbb{H}_{n,r}^M\big\}\\
&= \Big\{\omega \in \Omega\colon \Big\|\mathcal{R}\Big[P_M S_M\,\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M(\omega)\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})}\\
&\leq \tfrac{1}{2}|\rho_{M,r}|^{(n-M)}\big\|P_0\big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^{q} a_k[v]^k\big) + Z_2^M(\omega)\big)\big]\big\|_H\Big\}.
\end{aligned} \tag{3.124}$$

This implies for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} |\left\langle e_0, v\right\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\
&= \mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} |\left\langle e_0, v\right\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ \Big\| \mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})} \\
&\quad\quad \leq \tfrac{1}{2} |\rho_{M,r}|^{(n-M)} \big\| P_0 \left[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\right] \big\|_H \Big\}\Big).
\end{aligned} \tag{3.125}$$

Hence, we obtain for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} |\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\
&= \mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} |\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ \Big\| \mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})} \\
&\quad\quad \leq \tfrac{1}{2} |\rho_{M,r}|^{(n-M)} \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \Big\}\Big).
\end{aligned} \tag{3.126}$$

This assures for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} |\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\
&\geq \mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} |\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ \Big\| \mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})} \\
&\quad\quad \leq \tfrac{1}{2} |\rho_{M,r}|^{(n-M)} |c_{M,r}|^{1/r} |\left\langle e_0, v\right\rangle_H|^q \Big\}\Big).
\end{aligned} \tag{3.127}$$

Next observe that (3.73), (3.35) prove for all $N \in \mathbb{N}$, $v \in H_{-\nu}$ that

$$\langle e_0, P_N S_N v\rangle_H = \langle e_0, S_N P_N v\rangle_H = \langle S_N e_0, P_N v\rangle_H = \langle e_0, P_N v\rangle_H . \tag{3.128}$$

This implies that it holds for all $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in H_{-\nu}$ that

$$\begin{aligned}
\big\langle e_0, P_M S_M\big(v + Z_n^M\big)\big\rangle_H &= \langle e_0, P_M S_M v\rangle_H + \big\langle e_0, P_M S_M Z_n^M\big\rangle_H \\
&= \langle e_0, P_M S_M v\rangle_H + \big\langle e_0, P_M Z_n^M\big\rangle_H .
\end{aligned} \tag{3.129}$$

Therefore, we obtain for all $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in H_{-\nu}$, $x \in \mathbb{R}$ that

$$\big\{\omega \in \Omega \colon \big|\big\langle e_0, P_M S_M\big(v + Z_n^M(\omega)\big)\big\rangle_H\big| \geq x\big\} \in \sigma\big(\big\langle e_0, P_M Z_n^M\big\rangle_H\big) . \tag{3.130}$$

Moreover, observe that it holds for all $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in H_{-\nu}$ that

$$\mathcal{R}\big[P_M S_M \big(v + Z_n^M\big)\big] = \mathcal{R}[P_M S_M v] + \mathcal{R}\big[P_M S_M Z_n^M\big]. \tag{3.131}$$

In addition, note that (3.73), (3.108), and the fact that $\forall\, u \in H_{-\nu}$, $M \in \{2, 3, \dots\}$: $P_M u \in H$ ensure for all $M \in \{2, 3, \dots\}$, $v \in H_{-\nu}$ that

$$\mathcal{R}[P_M S_M v] = \mathcal{R}[S_M P_M v] = S_M \mathcal{R}[P_M v]\,. \tag{3.132}$$

Hence, we obtain for all $M \in \{2, 3, \dots\}$, $v \in H_{-\nu}$ that

$$\mathcal{R}[P_M S_M v] = S_M \Big(\textstyle\sum_{k \in \{-M, \dots, M\} \setminus \{0\}} \langle e_k, P_M v \rangle_H \, e_k \Big). \tag{3.133}$$

This and (3.131) establish for all $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in H_{-\nu}$ that

$$\begin{aligned} &\mathcal{R}\big[P_M S_M \big(v + Z_n^M\big)\big] \\ &= \mathcal{R}[P_M S_M v] + S_M \Big(\textstyle\sum_{k \in \{-M, \dots, M\} \setminus \{0\}} \big\langle e_k, P_M Z_n^M \big\rangle_H \, e_k \Big). \end{aligned} \tag{3.134}$$

Therefore, we obtain that for all $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in H_{-\nu}$, $x \in \mathbb{R}$ it holds that

$$\begin{aligned} &\Big\{\omega \in \Omega \colon \big\|\mathcal{R}\big[P_M S_M \big(v + Z_n^M(\omega)\big)\big]\big\|_{L^p(\lambda;\mathbb{R})} \leq x\Big\} \\ &= \Big\{\omega \in \Omega \colon \Big\|\mathcal{R}[P_M S_M v] + S_M \Big(\textstyle\sum_{k \in \{-M, \dots, M\} \setminus \{0\}} \big\langle e_k, P_M Z_n^M(\omega) \big\rangle_H \, e_k \Big)\Big\|_{L^p(\lambda;\mathbb{R})} \leq x\Big\} \\ &\quad \in \sigma\big(\big\{\big\langle e_m, P_M Z_n^M \big\rangle_H : m \in \{-M, \dots, M\} \setminus \{0\}\big\}\big)\,. \end{aligned} \tag{3.135}$$

Moreover, observe that for all $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$ it holds that $\sigma\big(\big\langle e_m, P_M Z_n^M \big\rangle_H\big)$, $m \in \mathbb{Z}$, are independent sigma algebras (cf., e.g., Proposition 2.5.2 in [113]). Combining this with (3.130) and (3.135) ensures for all $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in H_{-\nu}$, $x_1, x_2 \in \mathbb{R}$ that

$$\big\{\omega \in \Omega \colon \,\big|\big\langle e_0, P_M S_M \big(v + Z_n^M(\omega)\big)\big\rangle_H\big| \geq x_1\big\} \tag{3.136}$$

and

$$\Big\{\omega \in \Omega \colon \big\|\mathcal{R}\big[P_M S_M \big(v + Z_n^M(\omega)\big)\big]\big\|_{L^p(\lambda;\mathbb{R})} \leq x_2\Big\} \tag{3.137}$$

are independent events. Hence, we obtain for all $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in H_{-\nu}$, $x_1, x_2 \in \mathbb{R}$ that

$$\begin{aligned} &\mathbb{P}\Big(\big\{\big|\big\langle e_0, P_M S_M \big(v + Z_n^M\big)\big\rangle_H\big| \geq x_1\big\} \cap \Big\{\big\|\mathcal{R}\big[P_M S_M \big(v + Z_n^M\big)\big]\big\|_{L^p(\lambda;\mathbb{R})} \leq x_2\Big\} \Big) \\ &= \mathbb{P}\Big(\,\big|\big\langle e_0, P_M S_M \big(v + Z_n^M\big)\big\rangle_H\big| \geq x_1\Big)\mathbb{P}\Big(\big\|\mathcal{R}\big[P_M S_M \big(v + Z_n^M\big)\big]\big\|_{L^p(\lambda;\mathbb{R})} \leq x_2\Big). \end{aligned} \tag{3.138}$$

Combining this with (3.127) establishes for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\
&\geq \mathbb{P}\Big(\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big) \\
&\quad \cdot \mathbb{P}\Big(\Big\| \mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})} \\
&\qquad \leq \tfrac{1}{2} |\rho_{M,r}|^{(n-M)} |c_{M,r}|^{1/r} |\, \langle e_0, v\rangle_H \,|^q \Big).
\end{aligned} \tag{3.139}$$

This implies for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\
&\geq \mathbb{P}\Big(\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big) \\
&\quad \cdot \mathbb{P}\Big(\Big\{ \Big\| \mathcal{R}\Big[P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\Big]\Big\|_{L^p(\lambda;\mathbb{R})} \\
&\quad \leq \tfrac{1}{2} |\rho_{M,r}|^{(n-M)} |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big\} \cap \big\{ \|S_M P_M Z_2^M\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)} \big\} \Big).
\end{aligned} \tag{3.140}$$

Combining this with (3.123) proves for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| > 1$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\
&\geq \mathbb{P}\Big(\big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big) \\
&\quad \cdot \mathbb{P}\big(\|S_M P_M Z_2^M\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)} \big).
\end{aligned} \tag{3.141}$$

This and (3.105) assure for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$, $n \in \{1,2,\dots,M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \,|\langle e_0, v\rangle_H|^q \Big\} \\
&\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k[v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\
&\geq \mathbb{P}\big(\|S_M P_M Z_2^M\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(n-1-M)} \big) \Big[\tfrac{y_M}{\sqrt{2\pi\gamma_M T}} \Big]^{(2M+1)} \exp\!\Big(-\tfrac{3M^2 |y_M|^2}{\gamma_M T} \Big) \\
&\geq \mathbb{P}\big(\|S_M P_M Z_2^M\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{(-1-M)} \big) \Big[\tfrac{y_M}{\sqrt{2\pi\gamma_M T}} \Big]^{(2M+1)} \exp\!\Big(-\tfrac{3M^2 |y_M|^2}{\gamma_M T} \Big).
\end{aligned} \tag{3.142}$$

Corollary 3.5 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $x = |\rho_{M,r}|^{-1-M}$, $\gamma = \gamma_M$, $y = z_{M,r}$, $p = p$, $\delta = \zeta_{\nu+s}$, $\nu = \nu$, $s = s$, $N = M$, $v = 0$, $S = (H \ni w \mapsto S_M w \in H_{\nu+s})$,

$W = W$ for $M \in \{2, 3, \dots\}$, $r \in (0, \infty)$ in the notation of Corollary 3.5) therefore implies for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$, $v \in \mathbb{H}^M_{n-1,r}$ with $|\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}$ that

$$\begin{aligned} &\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k [v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \left|\langle e_0, v\rangle_H\right|^q \Big\} \\ &\quad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k [v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\ &\geq \Big[\tfrac{z_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big]^{(2M+1)} \exp\Big(-\tfrac{3M^2|z_{M,r}|^2}{\gamma_M T}\Big) \Big[\tfrac{y_M}{\sqrt{2\pi\gamma_M T}}\Big]^{(2M+1)} \exp\Big(-\tfrac{3M^2|y_M|^2}{\gamma_M T}\Big) \\ &= \Big[\tfrac{z_{M,r}\, y_M}{2\pi\gamma_M T}\Big]^{(2M+1)} \exp\Big(-\tfrac{3M^2}{\gamma_M T}\big(|z_{M,r}|^2 + |y_M|^2\big)\Big). \end{aligned} \tag{3.143}$$

Moreover, observe that for all $r \in (0, \infty)$, $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N\}$, $\alpha \in \mathbb{R}$ it holds that $\alpha e_0 \in \mathbb{H}^N_{n,r}$. This ensures for all $r \in (0, \infty)$, $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N\}$ that $\mathbb{H}^N_{n,r} \neq \emptyset$ and

$$\sup\big(\big\{|\langle e_0, v\rangle_H| \colon v \in \mathbb{H}^N_{n,r}\big\}\big) = \infty. \tag{3.144}$$

Hence, we obtain that for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$, $n \in \{1, 2, \dots, M\}$ it holds that

$$\big\{v \in \mathbb{H}^M_{n-1,r} \colon\ |\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}\big\} \neq \emptyset. \tag{3.145}$$

This and (3.143) assure for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned} &\Bigg[\prod_{n=1}^{M-1} \inf\Bigg(\Bigg\{\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k [v]^k\big) + Z_2^M\big)\big\rangle_H\big| \geq |c_{M,r}|^{1/r} \left|\langle e_0, v\rangle_H\right|^q \Big\} \\ &\qquad\qquad \cap \Big\{ P_M S_M\big(v + \tfrac{T}{M}\big(\textstyle\sum_{k=0}^q a_k [v]^k\big) + Z_2^M\big) \in \mathbb{H}^M_{n,r} \Big\}\Big) \\ &\qquad\qquad\qquad : \Big(v \in \mathbb{H}^M_{n-1,r} \colon |\langle e_0, v\rangle_H| \geq |\theta_{M,r}|^{(q^{(n-1)})/r}\Big)\Bigg\} \cup \{1\}\Bigg)\Bigg] \\ &\geq \Big[\tfrac{z_{M,r}\, y_M}{2\pi\gamma_M T}\Big]^{M(2M+1)} \exp\Big(-\tfrac{3M^3}{\gamma_M T}\big(|z_{M,r}|^2 + |y_M|^2\big)\Big). \end{aligned} \tag{3.146}$$

Combining this with (3.62) proves for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned} &\mathbb{E}\Big[\big|\big\langle e_0, Y_M^M\big\rangle_H\big|^r\Big] \geq |\theta_{M,r}|^{(q^{(M-1)})} \,\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, Y_1^M\big\rangle_H\big|^r \geq |c_{M,r}|^{1/(1-q)}\theta_{M,r}\Big\} \\ &\quad \cap \big\{Y_1^M \in \mathbb{H}^M_{0,r}\big\}\Big) \Big[\tfrac{z_{M,r} y_M}{2\pi\gamma_M T}\Big]^{M(2M+1)} \exp\Big(-\tfrac{3M^3}{\gamma_M T}\big(|z_{M,r}|^2 + |y_M|^2\big)\Big). \end{aligned} \tag{3.147}$$

Next note that it holds for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned} &\mathbb{P}\Big(\Big\{ \big|\big\langle e_0, Y_1^M\big\rangle_H\big|^r \geq |c_{M,r}|^{1/(1-q)}\theta_{M,r}\Big\} \cap \big\{Y_1^M \in \mathbb{H}^M_{0,r}\big\}\Big) \\ &= \mathbb{P}\Big(\Big\{ \big|\big\langle e_0, Y_1^M\big\rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]} \left|\theta_{M,r}\right|^{1/r}\Big\} \cap \big\{Y_1^M \in \mathbb{H}^M_{0,r}\big\}\Big) \\ &= \mathbb{P}\Big(\Big\{ \big|\big\langle e_0, Y_1^M\big\rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]} \left|\theta_{M,r}\right|^{1/r}\Big\} \cap \big\{Y_1^M \in H_\chi\big\} \\ &\quad \cap \Big\{\big\|Y_1^M - \big\langle e_0, Y_1^M\big\rangle_H e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2}|\rho_{M,r}|^{-M} \big|\big\langle e_0, Y_1^M\big\rangle_H\big|\Big\}\Big). \end{aligned} \tag{3.148}$$

In addition, note that (3.63) ensures for all $M \in \{2, 3, \dots\}$ that

$$P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k \in H. \tag{3.149}$$

The fact that $\forall\, N \in \mathbb{N}, \omega \in \Omega\colon W_{\frac{T}{N}}(\omega) \in H_{-\nu}$ and (3.35) therefore establish for all $M \in \{2, 3, \dots\}$, $\omega \in \Omega$ that

$$S_M\Big(P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k\Big) \in H_1 \qquad \text{and} \qquad S_M W_{\frac{T}{M}}(\omega) \in H_{-\nu+1}. \tag{3.150}$$

This, the fact that $\forall\, j \in \{1, -\nu+1\}\colon H_j \subseteq H_{-1}$, and the fact that $\forall\, N \in \mathbb{N}\colon P_N(H_{-1}) \subseteq H_1$ prove for all $M \in \{2, 3, \dots\}$, $\omega \in \Omega$ that

$$P_M S_M\Big(P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k\Big) \in H_1 \qquad \text{and} \qquad P_M S_M W_{\frac{T}{M}}(\omega) \in H_1. \tag{3.151}$$

This implies for all $M \in \{2, 3, \dots\}$, $\omega \in \Omega$ that

$$Y_1^M(\omega) = P_M S_M\Big(P_M(\xi) + \tfrac{T}{M}\Big(\textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k\Big) + W_{\frac{T}{M}}(\omega)\Big) \in H_1. \tag{3.152}$$

The fact that $\chi \leq 1$ therefore ensures for all $M \in \{2, 3, \dots\}$, $\omega \in \Omega$ that

$$Y_1^M(\omega) \in H_\chi. \tag{3.153}$$

Combining this with (3.148) therefore proves for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\langle e_0, Y_1^M \rangle_H\big|^r \geq |c_{M,r}|^{1/(1-q)} \theta_{M,r}\Big\} \cap \big\{Y_1^M \in \mathbb{H}_{0,r}^M\big\}\Big) \\
&= \mathbb{P}\Big(\Big\{ \big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]} \, |\theta_{M,r}|^{1/r}\Big\} \\
&\quad \cap \Big\{\big\|Y_1^M - \langle e_0, Y_1^M \rangle_H \, e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2}|\rho_{M,r}|^{-M} \, \big|\langle e_0, Y_1^M \rangle_H\big|\Big\}\Big).
\end{aligned} \tag{3.154}$$

Moreover, note that the fact that $\forall\, r \in (0, \infty), M \in \{2, 3, \dots\}\colon c_{M,r} \in (0, 1]$ and $r(1-q) < 0$ ensures for all $r \in (0, \infty), M \in \{2, 3, \dots\}$ that

$$|c_{M,r}|^{1/[r(1-q)]} \geq 1. \tag{3.155}$$

Furthermore, observe that (3.44) establishes for all $r \in (0, \infty), M \in \{2, 3, \dots\}$ that

$$|\theta_{M,r}|^{1/r} \geq 2. \tag{3.156}$$

Combining this with (3.155) proves for all $r \in (0, \infty), M \in \{2, 3, \dots\}$ that

$$\tfrac{1}{2}|c_{M,r}|^{1/[r(1-q)]} \, |\theta_{M,r}|^{1/r} \geq 1. \tag{3.157}$$

This and (3.154) imply for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{ \big|\langle e_0, Y_1^M \rangle_H\big|^r \geq |c_{M,r}|^{1/(1-q)} \theta_{M,r}\Big\} \cap \big\{Y_1^M \in \mathbb{H}_{0,r}^M\big\}\Big) \\
&\geq \mathbb{P}\Big(\Big\{ \big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]} \, |\theta_{M,r}|^{1/r}\Big\} \\
&\quad \cap \Big\{\big\|Y_1^M - \langle e_0, Y_1^M \rangle_H \, e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2}|\rho_{M,r}|^{-M} |c_{M,r}|^{1/[r(1-q)]} \, |\theta_{M,r}|^{1/r}\Big\}\Big) \\
&\geq \mathbb{P}\Big(\Big\{\big| \langle e_0, Y_1^M \rangle_H \big| \geq |c_{M,r}|^{1/[r(1-q)]} \, |\theta_{M,r}|^{1/r}\Big\} \\
&\quad \cap \Big\{\big\|Y_1^M - \langle e_0, Y_1^M \rangle_H \, e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{-M}\Big\}\Big).
\end{aligned} \tag{3.158}$$

Next note that it holds for all $M \in \{2, 3, \dots\}$ that

$$Y_1^M = P_M S_M \Big(P_M(\xi) + \tfrac{T}{M} \Big(\textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k \Big) + Z_1^M \Big) . \tag{3.159}$$

Combining this with (3.138) and (3.158) ensures that for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ it holds that

$$\begin{aligned} &\mathbb{P}\Big(\Big\{ \big|\langle e_0, Y_1^M \rangle_H\big|^r \geq |c_{M,r}|^{1/(1-q)} \theta_{M,r} \Big\} \cap \big\{ Y_1^M \in \mathbb{H}_{0,r}^M \big\} \Big) \\ &\geq \mathbb{P}\big(\big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]} \, |\theta_{M,r}|^{1/r} \big) \\ &\quad \cdot \mathbb{P}\left(\big\|Y_1^M - \langle e_0, Y_1^M \rangle_H \, e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{-M} \right) . \end{aligned} \tag{3.160}$$

Furthermore, observe that it holds for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned} &\mathbb{P}\Big(\big\|Y_1^M - \langle e_0, Y_1^M \rangle_H \, e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{-M} \Big) \\ &\geq \mathbb{P}\Big(\big\|Y_1^M\big\|_{L^p(\lambda;\mathbb{R})} + \big|\langle e_0, Y_1^M \rangle_H\big| \leq |\rho_{M,r}|^{-M} \Big) \\ &\geq \mathbb{P}\Big(\big\|Y_1^M\big\|_{L^p(\lambda;\mathbb{R})} + \big\|Y_1^M\big\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{-M} \Big) \\ &= \mathbb{P}\Big(\big\|Y_1^M\big\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2} |\rho_{M,r}|^{-M} \Big). \end{aligned} \tag{3.161}$$

This, (3.63), and (3.73) assure for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned} &\mathbb{P}\Big(\big\|Y_1^M - \langle e_0, Y_1^M \rangle_H \, e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{-M} \Big) \\ &\geq \mathbb{P}\bigg(\Big\| P_M S_M \big(P_M(\xi) + \tfrac{T}{M} \Big(\textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k \Big) + W_{\frac{T}{M}} \big) \Big\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2} |\rho_{M,r}|^{-M} \bigg) \\ &= \mathbb{P}\bigg(\Big\| S_M \Big[P_M \Big(P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k \Big) + P_M(W_{T/M}) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2} |\rho_{M,r}|^{-M} \bigg) \\ &= \mathbb{P}\bigg(\Big\| S_M \Big[P_M \Big(P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k \Big) - P_M(W_{T/M}) \Big] \Big\|_{L^p(\lambda;\mathbb{R})} \leq \tfrac{1}{2} |\rho_{M,r}|^{-M} \bigg). \end{aligned} \tag{3.162}$$

Combining this with Corollary 3.5 (with $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $T = T$, $x = \frac{1}{2} |\rho_{M,r}|^{-M}$, $\gamma = \gamma_M$, $y = g_{M,r}$, $p = p$, $\delta = \zeta_{\nu+s}$, $\nu = \nu$, $s = s$, $N = M$, $v = P_M(\xi) + \frac{T}{M} \sum_{k=0}^{q} a_k [P_M(\xi)]^k$, $S = (H \ni w \mapsto S_M w \in H_{\nu+s})$, $W = W$ for $M \in \{2, 3, \dots\}$, $r \in (0, \infty)$ in the notation of Corollary 3.5) ensures for all $r \in (0, \infty)$, $M \in \{2, 3, \dots\}$ that

$$\begin{aligned} &\mathbb{P}\Big(\big\|Y_1^M - \langle e_0, Y_1^M \rangle_H \, e_0\big\|_{L^p(\lambda;\mathbb{R})} \leq |\rho_{M,r}|^{-M} \Big) \\ &\geq \left[\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}} \right]^{(2M+1)} \exp\Big(-\tfrac{3M^2}{T} \Big(\big\| P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k \big\|_H^2 + \tfrac{|g_{M,r}|^2}{\gamma_M} \Big) \Big) \\ &\geq \left[\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}} \right]^{(2M+1)} \exp\Big(-\tfrac{3M^2}{T} \Big[\Big(\tfrac{T}{M} \Big[\textstyle\sum_{k=0}^{q} |a_k| \big\| [P_M(\xi)]^k \big\|_H \Big] + \|P_M(\xi)\|_H \Big)^2 + \tfrac{|g_{M,r}|^2}{\gamma_M} \Big] \Big) \\ &\geq \left[\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}} \right]^{(2M+1)} \exp\Big(-\tfrac{3M^2}{T} \Big[\Big(T \Big[\textstyle\sum_{k=0}^{q} |a_k| \big\| [P_M(\xi)]^k \big\|_H \Big] + \|\xi\|_H \Big)^2 + \tfrac{|g_{M,r}|^2}{\gamma_M} \Big] \Big) . \end{aligned} \tag{3.163}$$

Moreover, note that it holds for all $M \in \{2,3,\dots\}$ that

$$
\begin{aligned}
&T\left(\textstyle\sum_{k=0}^{q} |a_k| \big\| [P_M(\xi)]^k \big\|_H\right) + \|\xi\|_H \le T\left(\textstyle\sum_{k=0}^{q} |a_k| \|P_M(\xi)\|_{L^{2k}(\lambda;\mathbb{R})}^k\right) + \|\xi\|_H \\
&\le T\left(\textstyle\sum_{k=0}^{q} |a_k| \|P_M(\xi)\|_{L^{2q}(\lambda;\mathbb{R})}^k\right) + \|\xi\|_H \\
&\le T\left(\textstyle\sum_{k=0}^{q} |a_k| \|P_M(\xi)\|_{L^{p}(\lambda;\mathbb{R})}^k\right) + \|\xi\|_{L^p(\lambda;\mathbb{R})} \\
&\le T\left(\textstyle\sum_{k=0}^{q}\Big[|a_k| \left(\sup_{N\in\mathbb{N}} \|P_N(\xi)\|_{L^{p}(\lambda;\mathbb{R})}\right)^k\Big]\right) + \|\xi\|_{L^p(\lambda;\mathbb{R})}.
\end{aligned}
\tag{3.164}
$$

Next note that the fact that $H_\chi \subseteq L^p(\lambda;\mathbb{R})$ and the fact that $\forall\, N \in \mathbb{N}\colon \|P_N\|_{L(H)} \le 1$ ensure that

$$
\begin{aligned}
&\sup_{M\in\mathbb{N}} \|P_M(\xi)\|_{L^p(\lambda;\mathbb{R})} \le C \sup_{M\in\mathbb{N}} \|P_M(\xi)\|_{H_\chi} \\
&= C \sup_{M\in\mathbb{N}} \big\|(\eta - A)^\chi P_M(\xi)\big\|_H = C \sup_{M\in\mathbb{N}} \big\|P_M(\eta - A)^\chi \xi\big\|_H \\
&\le C \sup_{M\in\mathbb{N}} \|P_M\|_{L(H)} \big\|(\eta - A)^\chi \xi\big\|_H \le C \sup_{M\in\mathbb{N}} \big\|(\eta - A)^\chi \xi\big\|_H \\
&= C\big\|(\eta - A)^\chi \xi\big\|_H = C\|\xi\|_{H_\chi}.
\end{aligned}
\tag{3.165}
$$

This and (3.164) prove for all $M \in \{2,3,\dots\}$ that

$$
\begin{aligned}
&T\left(\textstyle\sum_{k=0}^{q} |a_k| \big\| [P_M(\xi)]^k \big\|_H\right) + \|\xi\|_H \le T\left(\textstyle\sum_{k=0}^{q} |C|^k |a_k| \|\xi\|_{H_\chi}^k\right) + \|\xi\|_{L^p(\lambda;\mathbb{R})} \\
&\le T\left(\textstyle\sum_{k=0}^{q} |C|^k |a_k| \|\xi\|_{H_\chi}^k\right) + C\|\xi\|_{H_\chi} \\
&\le (q+2)|\max\{C,1\}|^q \max\{T,1\} \max\{1, |a_0|, \dots, |a_q|\} \max\{\|\xi\|_{H_\chi}^q, 1\} = \kappa.
\end{aligned}
\tag{3.166}
$$

Combining this with (3.163) assures for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$
\begin{aligned}
&\mathbb{P}\Big(\big\|Y_1^M - \langle e_0, Y_1^M\rangle_H\, e_0\big\|_{L^p(\lambda;\mathbb{R})} \le |\rho_{M,r}|^{-M}\Big) \\
&\ge \left[\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\right]^{(2M+1)} \exp\!\Big(-\tfrac{3M^2}{T}\Big(\kappa^2 + \tfrac{|g_{M,r}|^2}{\gamma_M}\Big)\Big).
\end{aligned}
\tag{3.167}
$$

In addition, note that for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ it holds that

$$
\begin{aligned}
&\mathbb{P}\Big(\,\big|\big\langle e_0, Y_1^M\big\rangle_H\big| \ge |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&= \mathbb{P}\Big(\,\Big|\Big\langle e_0, P_M S_M\Big(P_M(\xi) + \tfrac{T}{M}\Big(\textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k\Big) + W_{\frac{T}{M}}\Big)\Big\rangle_H\Big| \ge |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big).
\end{aligned}
\tag{3.168}
$$

This, (3.37), and (3.128) prove for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$
\begin{aligned}
&\mathbb{P}\Big(\,\big|\big\langle e_0, Y_1^M\big\rangle_H\big| \ge |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&= \mathbb{P}\Big(\,\Big|\Big\langle e_0, P_M\Big(P_M(\xi) + \tfrac{T}{M}\Big(\textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k\Big) + W_{\frac{T}{M}}\Big)\Big\rangle_H\Big| \ge |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&\ge \mathbb{P}\Big(\,\Big|\Big\langle e_0, P_M W_{\frac{T}{M}}\Big\rangle_H\Big| - \Big|\Big\langle e_0, P_M(\xi) + \tfrac{T}{M}\textstyle\sum_{k=0}^{q} a_k [P_M(\xi)]^k\Big\rangle_H\Big| \\
&\qquad \ge |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big).
\end{aligned}
\tag{3.169}
$$

Hence, we obtain for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&\geq \mathbb{P}\Big(\Big|\Big\langle e_0, P_M W_{\frac{T}{M}} \Big\rangle_H\Big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \Big|\Big\langle e_0, P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^q a_k [P_M(\xi)]^k \Big\rangle_H\Big|\Big) \\
&\geq \mathbb{P}\Big(\Big|\Big\langle e_0, P_M W_{\frac{T}{M}} \Big\rangle_H\Big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \big\|P_M(\xi) + \tfrac{T}{M} \textstyle\sum_{k=0}^q a_k [P_M(\xi)]^k \big\|_H\Big).
\end{aligned} \tag{3.170}$$

This ensures for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&\geq \mathbb{P}\Big(\Big|\Big\langle e_0, P_M W_{\frac{T}{M}} \Big\rangle_H\Big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \tfrac{T}{M}\Big(\textstyle\sum_{k=0}^q |a_k| \big\| [P_M(\xi)]^k \big\|_H\Big) + \|\xi\|_H\Big) \\
&\geq \mathbb{P}\Big(\Big|\Big\langle e_0, P_M W_{\frac{T}{M}} \Big\rangle_H\Big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + T\Big(\textstyle\sum_{k=0}^q |a_k| \big\| [P_M(\xi)]^k \big\|_H\Big) + \|\xi\|_H\Big).
\end{aligned} \tag{3.171}$$

Combining this with (3.166) proves for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&\geq \mathbb{P}\Big(\Big|\Big\langle e_0, P_M W_{\frac{T}{M}} \Big\rangle_H\Big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\Big) \\
&= \mathbb{P}\Big(M^{-1/2}\, |\langle e_0, P_M W_T \rangle_H| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\Big) \\
&= \mathbb{P}\Big(T^{-1/2}\, |\langle e_0, P_M W_T \rangle_H| \geq M^{1/2} T^{-1/2} \big(|c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\big)\Big).
\end{aligned} \tag{3.172}$$

Therefore, we obtain for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&\geq 2 \int_{M^{1/2} T^{-1/2}\big(|c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\big)}^{\infty} \tfrac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \\
&\geq \tfrac{2}{\sqrt{2\pi}} \int_{\left(\frac{T}{M}\right)^{-1/2}\big(|c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\big)}^{2\left(\frac{T}{M}\right)^{-1/2}\big(|c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\big)} e^{-\frac{y^2}{2}}\, dy.
\end{aligned} \tag{3.173}$$

This and the fact that $\forall\, a,b \in \mathbb{R}\colon |a+b|^2 \leq 2|a|^2 + 2|b|^2$ imply for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$ that

$$\begin{aligned}
&\mathbb{P}\Big(\big|\langle e_0, Y_1^M \rangle_H\big| \geq |c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}\Big) \\
&\geq \tfrac{2}{\sqrt{2\pi}} \big(\tfrac{T}{M}\big)^{-1/2} \big(|c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\big) \exp\Bigg(-\frac{4M\big(|c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r} + \kappa\big)^2}{2T}\Bigg) \\
&\geq \frac{|c_{M,r}|^{1/[r(1-q)]}\, |\theta_{M,r}|^{1/r}}{\sqrt{2\pi T}} \exp\Bigg(-\frac{4M\big(|c_{M,r}|^{2/[r(1-q)]}\, |\theta_{M,r}|^{2/r} + \kappa^2\big)}{T}\Bigg).
\end{aligned} \tag{3.174}$$

Combining this with (3.160) and (3.167) ensures for all $r \in (0,\infty)$, $M \in \{2,3,\dots\}$

that

$$\mathbb{P}\Big(\Big\{\,\big|\big\langle e_0, Y_1^M\big\rangle_H\big|^r \geq |c_{M,r}|^{1/1-q}\theta_{M,r}\Big\}\cap\big\{Y_1^M\in\mathbb{H}_{0,r}^M\big\}\Big) \\ \geq \tfrac{|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}}{\sqrt{2\pi T}}\,\exp\Big(-\tfrac{4M\left(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r}+\kappa^2\right)}{T}\Big) \\ \cdot\Big[\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big]^{(2M+1)}\exp\Big(-\tfrac{3M^2}{T}\Big(\kappa^2+\tfrac{|g_{M,r}|^2}{\gamma_M}\Big)\Big). \tag{3.175}$$

This and (3.147) establish for all $r\in(0,\infty)$, $M\in\{2,3,\dots\}$ that

$$\mathbb{E}\Big[\,\big|\big\langle e_0, Y_M^M\big\rangle_H\big|^r\,\Big] \geq |\theta_{M,r}|^{(q^{(M-1)})}\,\tfrac{|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}}{\sqrt{2\pi T}}\,\exp\Big(-\tfrac{4M\left(|c_{M,r}|^{2/[r(1-q)]}|\theta_{M,r}|^{2/r}+\kappa^2\right)}{T}\Big) \\ \cdot\Big[\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big]^{(2M+1)}\exp\Big(-\tfrac{3M^2}{T}\Big(\kappa^2+\tfrac{|g_{M,r}|^2}{\gamma_M}\Big)\Big)\Big[\tfrac{z_{M,r}y_M}{2\pi\gamma_M T}\Big]^{M(2M+1)} \\ \cdot\exp\Big(-\tfrac{3M^3}{\gamma_M T}\big(|z_{M,r}|^2+|y_M|^2\big)\Big). \tag{3.176}$$

Hence, we obtain that for all $r\in(0,\infty)$, $M\in\{2,3,\dots\}$ it holds that

$$\mathbb{E}\Big[\,\big|\big\langle e_0, Y_M^M\big\rangle_H\big|^r\,\Big] \geq \exp\Big(q^{(M-1)}\ln(\theta_{M,r})+\ln\big(|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}\big) \\ -\tfrac{1}{2}\ln(2\pi T)-\tfrac{4M}{T}\big(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r}+\kappa^2\big)+(2M+1)\ln\Big(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big) \\ -\tfrac{3M^2}{T}\Big(\kappa^2+\tfrac{|g_{M,r}|^2}{\gamma_M}\Big)+(2M^2+M)\ln\Big(\tfrac{z_{M,r}y_M}{2\pi\gamma_M T}\Big)-\tfrac{3M^3}{\gamma_M T}\big(|z_{M,r}|^2+|y_M|^2\big)\Big). \tag{3.177}$$

The fact that $\forall\, N\in\mathbb{N}, r\in(0,\infty)\colon \theta_{N,r}\geq 2^r$ therefore assures for all $r\in(0,\infty)$, $M\in\{2,3,\dots\}$ that

$$\mathbb{E}\Big[\,\big|\big\langle e_0, Y_M^M\big\rangle_H\big|^r\,\Big] \geq \exp\Big(q^{(M-1)}r\ln(2)+\ln\big(|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}\big) \\ -\tfrac{1}{2}\ln(2\pi T)-\tfrac{4M}{T}\big(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r}+\kappa^2\big)+(2M+1)\ln\Big(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big) \\ -\tfrac{3M^2}{T}\Big(\kappa^2+\tfrac{|g_{M,r}|^2}{\gamma_M}\Big)+(2M^2+M)\ln\Big(\tfrac{z_{M,r}y_M}{2\pi\gamma_M T}\Big)-\tfrac{3M^3}{\gamma_M T}\big(|z_{M,r}|^2+|y_M|^2\big)\Big). \tag{3.178}$$

Combining this with the fact that $\ln(2)\geq\frac{1}{2}$ and the fact that $\frac{1}{2}\ln(2\pi T)\leq\pi T$ proves for all $r\in(0,\infty)$ that

$$\liminf_{M\to\infty}\mathbb{E}\Big[\,\big|\big\langle e_0, Y_M^M\big\rangle_H\big|^r\,\Big] \geq \liminf_{M\to\infty}\Big[\exp\Big(\tfrac{r}{2}q^{(M-1)} \\ +\ln\big(|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}\big)-\pi T-\tfrac{4M}{T}\big(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r}+\kappa^2\big) \\ +(2M+1)\ln\Big(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big)-\tfrac{3M^2}{T}\Big(\kappa^2+\tfrac{|g_{M,r}|^2}{\gamma_M}\Big)+(2M^2+M)\ln\Big(\tfrac{z_{M,r}y_M}{2\pi\gamma_M T}\Big) \\ -\tfrac{3M^3}{\gamma_M T}\big(|z_{M,r}|^2+|y_M|^2\big)\Big)\Big]. \tag{3.179}$$

This ensures for all $r \in (0,\infty)$ that

$$\begin{aligned}
\liminf_{M\to\infty} \mathbb{E}\Big[\big|\langle e_0, Y_M^M\rangle_H\big|^r \Big] \geq \liminf_{M\to\infty} \Bigg[\exp\Bigg(\tfrac{r}{10} q^{(M-1)} + \ln\big(|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}\big) \\
- \pi T - \tfrac{4M}{T}\big(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r} + \kappa^2\big) + \tfrac{r}{10} q^{(M-1)} \\
+ (2M+1)\ln\Big(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big) + \tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2}{T}\Big(\kappa^2 + \tfrac{|g_{M,r}|^2}{\gamma_M}\Big) \\
+ \tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln\Big(\tfrac{z_{M,r} y_M}{2\pi\gamma_M T}\Big) + \tfrac{r}{10} q^{(M-1)} - \tfrac{3M^3}{\gamma_M T}\big(|z_{M,r}|^2 + |y_M|^2\big) \Bigg)\Bigg].
\end{aligned} \tag{3.180}$$

Hence, we obtain for all $r \in (0,\infty)$ that

$$\begin{aligned}
\liminf_{M\to\infty} \mathbb{E}\Big[\big|\langle e_0, Y_M^M\rangle_H\big|^r \Big] \geq \exp\Bigg(\liminf_{M\to\infty} \Big[\tfrac{r}{10} q^{(M-1)} + \ln\big(|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}\big) \\
- \pi T - \tfrac{4M}{T}\big(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r} + \kappa^2\big) \Big] + \liminf_{M\to\infty} \Big[\tfrac{r}{10} q^{(M-1)} + (2M+1) \\
\cdot \ln\Big(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big)\Big] + \liminf_{M\to\infty} \Big[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2}{T}\Big(\kappa^2 + \tfrac{|g_{M,r}|^2}{\gamma_M}\Big) \Big] \\
+ \liminf_{M\to\infty} \Big[\tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln\Big(\tfrac{z_{M,r} y_M}{2\pi\gamma_M T}\Big) \Big] \\
+ \liminf_{M\to\infty} \Big[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^3}{\gamma_M}\,\big(|z_{M,r}|^2 + |y_M|^2\big) \Big] \Bigg).
\end{aligned} \tag{3.181}$$

Moreover, note that it holds for all $r \in (0,\infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty} \Big[\tfrac{r}{10} q^{(M-1)} + \ln\big(|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}\big) \\
&\quad - \pi T - \tfrac{4M}{T}\big(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r} + \kappa^2\big) \Big] \\
&= \liminf_{M\to\infty} \Bigg[\tfrac{r}{10} q^{(M-1)} + \ln\Bigg(\Big[\tfrac{T|a_q|}{4M}\Big]^{1/(1-q)}\ |\theta_{M,r}|^{1/r} \Bigg) \\
&\quad - \pi T - \tfrac{4M}{T}\Bigg(\Big[\tfrac{T|a_q|}{4M}\Big]^{2/(1-q)}\ |\theta_{M,r}|^{2/r} + \kappa^2 \Bigg) \Bigg] \\
&= \liminf_{M\to\infty} \Bigg[\tfrac{r}{10} q^{(M-1)} + \ln\Bigg(\Big[\tfrac{T|a_q|}{4M}\Big]^{1/(1-q)} \Big[\tfrac{4T\vartheta+8M}{T|a_q|}\Big] \Bigg) \\
&\quad - \pi T - \tfrac{4M}{T}\Bigg(\Big[\tfrac{T|a_q|}{4M}\Big]^{2/(1-q)} \Big[\tfrac{4T\vartheta+8M}{T|a_q|}\Big]^2 + \kappa^2 \Bigg) \Bigg].
\end{aligned} \tag{3.182}$$

In addition, observe that it holds for all $M \in \{2,3,\dots\}$ that

$$\begin{aligned}
&\ln\Bigg(\Big[\tfrac{T|a_q|}{4M}\Big]^{1/(1-q)} \Big[\tfrac{4T\vartheta+8M}{T|a_q|}\Big] \Bigg) \geq \ln\Bigg(\Big[\tfrac{T|a_q|}{4M}\Big]^{1/(1-q)} \Big[\tfrac{4T\vartheta+8}{T|a_q|}\Big] \Bigg) \\
&= \ln\Bigg(M^{1/(q-1)} \Big[\tfrac{T|a_q|}{4}\Big]^{1/(1-q)} \Big[\tfrac{4T\vartheta+8}{T|a_q|}\Big] \Bigg) \\
&= \ln\big(M^{1/(q-1)}\big) + \ln\Bigg(\Big[\tfrac{T|a_q|}{4}\Big]^{1/(1-q)} \Big[\tfrac{4T\vartheta+8}{T|a_q|}\Big] \Bigg) \\
&= \tfrac{1}{q-1}\ln(M) + \ln\Bigg(\Big[\tfrac{T|a_q|}{4}\Big]^{1/(1-q)} \Big[\tfrac{4T\vartheta+8}{T|a_q|}\Big] \Bigg).
\end{aligned} \tag{3.183}$$

The fact that $q > 1$ and the fact that $\forall\, M \in \{2, 3, \dots\}\colon \ln(M) > 0$ therefore imply for all $M \in \{2, 3, \dots\}$ that

$$\ln\!\left(\left[\tfrac{T|a_q|}{4M}\right]^{1/(1-q)}\left[\tfrac{4T\vartheta+8M}{T|a_q|}\right]\right) \geq \ln\!\left(\left[\tfrac{T|a_q|}{4}\right]^{1/(1-q)}\left[\tfrac{4T\vartheta+8}{T|a_q|}\right]\right). \tag{3.184}$$

Combining this with (3.182) proves for all $r \in (0, \infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty}\Big[\tfrac{r}{10}q^{(M-1)} + \ln\!\big(|c_{M,r}|^{1/[r(1-q)]}\,|\theta_{M,r}|^{1/r}\big) \\
&\quad - \pi T - \tfrac{4M}{T}\big(|c_{M,r}|^{2/[r(1-q)]}\,|\theta_{M,r}|^{2/r} + \kappa^2\big)\Big] \\
&\geq \liminf_{M\to\infty}\left[\tfrac{r}{10}q^{(M-1)} + \ln\!\left(\left[\tfrac{T|a_q|}{4}\right]^{1/(1-q)}\left[\tfrac{4T\vartheta+8}{T|a_q|}\right]\right)\right. \\
&\quad \left. - \pi T - \tfrac{4M}{T}\left(\left[\tfrac{T|a_q|}{4M}\right]^{2/(1-q)}\left[\tfrac{4T\vartheta+8M}{T|a_q|}\right]^2 + \kappa^2\right)\right] = \infty.
\end{aligned} \tag{3.185}$$

Furthermore, observe that it holds for all $r \in (0, \infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty}\left[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\!\left(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\right)\right] \\
&= \liminf_{M\to\infty}\left[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln(g_{M,r}) - (2M+1)\ln\!\big(\sqrt{2\pi\gamma_M T}\big)\right] \\
&= \liminf_{M\to\infty}\Big[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\!\Big(\tfrac{1}{2\zeta_{\nu+s}}\left[\tfrac{T}{M}\right]^{(\nu+s)}\big\|(\eta-A)^{-s}\big\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}|\rho_{M,r}|^{-M}\Big) \\
&\quad - (2M+1)\ln\!\big(\sqrt{2\pi\gamma_M T}\big)\Big] \\
&= \liminf_{M\to\infty}\left[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\!\left(\tfrac{1}{2\zeta_{\nu+s}}\left[\tfrac{T}{M}\right]^{(\nu+s)}\big\|(\eta-A)^{-s}\big\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\right.\right. \\
&\left.\left.\left(8\vartheta^2\max\{C,1\}\max\{T,1\}\tfrac{\zeta_\chi|M|^\chi}{|c_{M,r}|^{1/r}\min\{T,1\}}\right)^{-M}\right) - (2M+1)\ln\!\big(\sqrt{2\pi\gamma_M T}\big)\right].
\end{aligned} \tag{3.186}$$

Hence, we obtain for all $r \in (0, \infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty}\left[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\!\left(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\right)\right] \\
&= \liminf_{M\to\infty}\left[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\!\left(\tfrac{1}{2\zeta_{\nu+s}}\left[\tfrac{T}{M}\right]^{(\nu+s)}\big\|(\eta-A)^{-s}\big\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\right.\right. \\
&\left.\left.\left(\tfrac{T|a_q|\min\{T,1\}}{32M\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi|M|^\chi}\right)^{M}\right) - (2M+1)\ln\!\big(\sqrt{2\pi\gamma_M T}\big)\right].
\end{aligned} \tag{3.187}$$

Moreover, observe that it holds for all $M \in \mathbb{N}$ that

$$\begin{aligned}
&\ln\!\left(\tfrac{1}{2\zeta_{\nu+s}}\left[\tfrac{T}{M}\right]^{(\nu+s)}\big\|(\eta-A)^{-s}\big\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\right)^{M}\right) \\
&= \ln\!\left(\tfrac{1}{2\zeta_{\nu+s}}\left[\tfrac{T}{M}\right]^{(\nu+s)}\big\|(\eta-A)^{-s}\big\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\right) \\
&\qquad + \ln\!\left(\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\right)^{M}\right).
\end{aligned} \tag{3.188}$$

Next note that it holds for all $M \in \mathbb{N}$ that

$$\begin{aligned}
&\ln\Bigl(\Bigl(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\Bigr)^M\Bigr) = M\ln\Bigl(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\Bigr) \\
&= M\ln(T|a_q|\min\{T,1\}) - M\ln\bigl(32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}\bigr).
\end{aligned} \tag{3.189}$$

Combining this with (3.187) and (3.188) proves for all $r \in (0,\infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty}\Bigl[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\Bigl(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Bigr)\Bigr] \\
&= \liminf_{M\to\infty}\Bigl(\tfrac{r}{10}q^{(M-1)} + (2M+1)\Bigl[\ln\Bigl(\tfrac{1}{2\zeta_{\nu+s}}\bigl[\tfrac{T}{M}\bigr]^{(\nu+s)}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigr) \\
&\quad + M\ln(T|a_q|\min\{T,1\}) - M\ln\bigl(32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}\bigr) \\
&\quad - \ln\bigl(\sqrt{2\pi\gamma_M T}\bigr)\Bigr]\Bigr).
\end{aligned} \tag{3.190}$$

Furthermore, observe that the fact that $\forall\, x \in (0,\infty)\colon \ln(x) \le x$ assures that for all $M \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\ln\Bigl(\tfrac{1}{2\zeta_{\nu+s}}\bigl[\tfrac{T}{M}\bigr]^{(\nu+s)}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigr) \\
&= \ln\Bigl(\tfrac{T^{(\nu+s)}}{2\zeta_{\nu+s}}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}M^{(-\nu-s)}\Bigr) \\
&= \ln\Bigl(\tfrac{T^{(\nu+s)}}{2\zeta_{\nu+s}}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigr) + \ln\bigl(M^{(-\nu-s)}\bigr) \\
&= \ln\Bigl(\tfrac{T^{(\nu+s)}}{2\zeta_{\nu+s}}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigr) - (\nu+s)\ln(M) \\
&\ge \ln\Bigl(\tfrac{T^{(\nu+s)}}{2\zeta_{\nu+s}}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigr) - (\nu+s)M.
\end{aligned} \tag{3.191}$$

This and (3.190) imply for all $r \in (0,\infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty}\Bigl[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\Bigl(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Bigr)\Bigr] \\
&\ge \liminf_{M\to\infty}\Bigl(\tfrac{r}{10}q^{(M-1)} + (2M+1)\Bigl[\ln\Bigl(\tfrac{T^{(\nu+s)}}{2\zeta_{\nu+s}}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigr) - (\nu+s)M \\
&\quad + M\ln(T|a_q|\min\{T,1\}) - M\ln\bigl(32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}\bigr) \\
&\quad - \ln\bigl(\sqrt{2\pi\gamma_M T}\bigr)\Bigr]\Bigr).
\end{aligned} \tag{3.192}$$

The fact that $\forall\, x \in (0,\infty)\colon \ln(x) \le x$ therefore ensures for all $r \in (0,\infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty}\Bigl[\tfrac{r}{10}q^{(M-1)} + (2M+1)\ln\Bigl(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Bigr)\Bigr] \\
&\ge \liminf_{M\to\infty}\Bigl(\tfrac{r}{10}q^{(M-1)} + (2M+1)\Bigl[\ln\Bigl(\tfrac{T^{(\nu+s)}}{2\zeta_{\nu+s}}\bigl\|(\eta-A)^{-s}\bigr\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Bigr) - (\nu+s)M \\
&\quad + M\ln(T|a_q|\min\{T,1\}) - 32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(2+\chi)} \\
&\quad - \ln\bigl(\sqrt{2\pi\gamma_M T}\bigr)\Bigr]\Bigr).
\end{aligned} \tag{3.193}$$

Moreover, note that it holds for all $N \in \mathbb{N}$ that

$$\gamma_N = \sum_{n=-N}^{N} (\eta + 4\pi^2 n^2)^{-2\nu} = \sum_{n=-N}^{N} \frac{1}{(\eta + 4\pi^2 n^2)^{2\nu}} \le \sum_{n=-N}^{N} \frac{1}{\eta^{2\nu}} = \frac{2N+1}{\eta^{2\nu}} \le \frac{3N}{\eta^{2\nu}}. \tag{3.194}$$

This and the fact that $\forall\, x \in (0,\infty)\colon\ \ln(x) \le x$ imply for all $M \in \mathbb{N}$ that

$$\begin{aligned} &(2M+1)\ln\!\big(\sqrt{2\pi\gamma_M T}\big) \le 3M \ln\!\big(\sqrt{2\pi\gamma_M T}\big) = \tfrac{3M}{2}\ln(2\pi\gamma_M T) \\ &\le 3M\pi\gamma_M T \le 3M\pi T \tfrac{3M}{\eta^{2\nu}} = \tfrac{9M^2\pi T}{\eta^{2\nu}}. \end{aligned} \tag{3.195}$$

Combining this with (3.193) demonstrates for all $r \in (0,\infty)$ that

$$\begin{aligned} &\liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} + (2M+1)\ln\!\Big(\tfrac{g_{M,r}}{\sqrt{2\pi\gamma_M T}}\Big)\right] \\ &\ge \liminf_{M\to\infty} \Big[\tfrac{r}{10} q^{(M-1)} + (2M+1)\ln\!\Big(\tfrac{T^{(\nu+s)}}{2\zeta_{\nu+s}} \big\|(\eta - A)^{-s}\big\|^{-1}_{L(H,L^p(\lambda;\mathbb{R}))}\Big) - (\nu+s)M(2M+1) \\ &\quad + M(2M+1)\ln(T|a_q|\min\{T,1\}) - 32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(2+\chi)}(2M+1) \\ &\quad - \tfrac{9M^2\pi T}{\eta^{2\nu}}\Big] = \infty. \end{aligned} \tag{3.196}$$

Furthermore, observe that it holds for all $N \in \mathbb{N}$ that

$$\begin{aligned} \gamma_N &= \sum_{n=-N}^{N} \frac{1}{(\eta + 4\pi^2 n^2)^{2\nu}} \ge \sum_{n=-N}^{N} \frac{1}{(\eta N^2 + 4\pi^2 N^2)^{2\nu}} \\ &= \sum_{n=-N}^{N} \frac{1}{N^{4\nu}(\eta + 4\pi^2)^{2\nu}} = \frac{(2N+1)}{N^{4\nu}(\eta + 4\pi^2)^{2\nu}} \ge \frac{1}{N^{4\nu}(\eta + 4\pi^2)^{2\nu}}. \end{aligned} \tag{3.197}$$

This implies for all $r \in (0,\infty)$ that

$$\begin{aligned} &\liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2}{T}\Big(\kappa^2 + \tfrac{|g_{M,r}|^2}{\gamma_M}\Big)\right] = \liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2\kappa^2}{T} - \tfrac{3M^2 |g_{M,r}|^2}{T\gamma_M}\right] \\ &= \liminf_{M\to\infty} \Bigg[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2\kappa^2}{T} - \tfrac{3M^2}{4T\gamma_M|\zeta_{\nu+s}|^2}\left[\tfrac{T}{M}\right]^{2(\nu+s)} \big\|(\eta - A)^{-s}\big\|^{-2}_{L(H,L^p(\lambda;\mathbb{R}))} \\ &\quad \cdot \Big(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\Big)^{2M}\Bigg] \ge \liminf_{M\to\infty} \Bigg[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2\kappa^2}{T} \\ &\quad - \tfrac{3M^2(\eta+4\pi^2)^{2\nu} M^{4\nu}}{4T|\zeta_{\nu+s}|^2}\left[\tfrac{T}{M}\right]^{2(\nu+s)} \big\|(\eta - A)^{-s}\big\|^{-2}_{L(H,L^p(\lambda;\mathbb{R}))} \Big(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\Big)^{2M}\Bigg] \\ &= \liminf_{M\to\infty} \Bigg[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2\kappa^2}{T} - \tfrac{3(\eta+4\pi^2)^{2\nu} T^{(2(\nu+s)-1)}}{4|\zeta_{\nu+s}|^2} \\ &\quad \cdot \big\|(\eta - A)^{-s}\big\|^{-2}_{L(H,L^p(\lambda;\mathbb{R}))} M^{2(1+\nu-s)} \Big(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\Big)^{2M}\Bigg]. \end{aligned} \tag{3.198}$$

Next observe that for all $x_1, x_2, x_3, \alpha \in (0,\infty)$ it holds that

$$\begin{aligned} \liminf_{M\to\infty} M^{x_1}\Big(\frac{\alpha}{M^{x_2}}\Big)^{x_3 M} &= \liminf_{M\to\infty}\Big(\frac{\alpha M^{x_1/(x_3 M)}}{M^{x_2}}\Big)^{x_3 M} \le \liminf_{M\to\infty}\Big(\frac{\alpha M^{x_2/2}}{M^{x_2}}\Big)^{x_3 M} \\ &= \liminf_{M\to\infty}\Big(\frac{\alpha}{M^{x_2/2}}\Big)^{x_3 M} \le \liminf_{M\to\infty}\big(\tfrac{1}{2}\big)^{x_3 M} = 0. \end{aligned} \tag{3.199}$$

Combining this with (3.198) establishes for all $r \in (0, \infty)$ that

$$\liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^2}{T} \left(\kappa^2 + \tfrac{|g_{M,r}|^2}{\gamma_M} \right) \right] = \infty. \tag{3.200}$$

Moreover, note that it holds for all $M \in \mathbb{N}$ that

$$\begin{aligned}
&\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}} \right)^{(1+M)} |y_M|^2 \\
&= \left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}} \right)^{(1+M)} \tfrac{1}{|\zeta_{\nu+s}|^2} \left[\tfrac{T}{M} \right]^{2(\nu+s)} \left\| (\eta - A)^{-s} \right\|_{L(H, L^p(\lambda;\mathbb{R}))}^{-2}.
\end{aligned} \tag{3.201}$$

This implies for all $M \in \mathbb{N}$ that

$$\begin{aligned}
&\ln\!\left(\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}} \right)^{(1+M)} |y_M|^2 \right) \\
&= \ln\!\left(\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}} \right)^{(1+M)} \right) \\
&\quad + \ln\!\left(\tfrac{1}{|\zeta_{\nu+s}|^2} \left[\tfrac{T}{M} \right]^{2(\nu+s)} \left\| (\eta - A)^{-s} \right\|_{L(H, L^p(\lambda;\mathbb{R}))}^{-2} \right) \Big] \\
&= (1+M) \ln\!\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}} \right) \\
&\quad + \ln\!\left(\tfrac{T^{2(\nu+s)}}{|\zeta_{\nu+s}|^2} \left\| (\eta - A)^{-s} \right\|_{L(H, L^p(\lambda;\mathbb{R}))}^{-2} \right) + \ln\!\left(M^{-2(\nu+s)} \right).
\end{aligned} \tag{3.202}$$

Hence, we obtain for all $M \in \mathbb{N}$ that

$$\begin{aligned}
&\ln\!\left(\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}} \right)^{(1+M)} |y_M|^2 \right) \\
&= (1+M) \ln\!\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi} \right) + (1+M)\ln\!\left(M^{-(1+\chi)} \right) \\
&\quad + \ln\!\left(\tfrac{T^{2(\nu+s)}}{|\zeta_{\nu+s}|^2} \left\| (\eta - A)^{-s} \right\|_{L(H, L^p(\lambda;\mathbb{R}))}^{-2} \right) - 2(\nu+s)\ln(M) \\
&= (1+M) \ln\!\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi} \right) - (1+\chi)(1+M)\ln(M) \\
&\quad + \ln\!\left(\tfrac{T^{2(\nu+s)}}{|\zeta_{\nu+s}|^2} \left\| (\eta - A)^{-s} \right\|_{L(H, L^p(\lambda;\mathbb{R}))}^{-2} \right) - 2(\nu+s)\ln(M).
\end{aligned} \tag{3.203}$$

This and the fact that $\forall\, x \in (0, \infty)\colon \ln(x) \leq x$ ensure for all $M \in \mathbb{N}$ that

$$\begin{aligned}
&\ln\!\left(\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}} \right)^{(1+M)} |y_M|^2 \right) \geq (1+M) \ln\!\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2 \max\{C,1\}\max\{T,1\}\zeta_\chi} \right) \\
&\quad - (1+\chi)(1+M)M + \ln\!\left(\tfrac{T^{2(\nu+s)}}{|\zeta_{\nu+s}|^2} \left\| (\eta - A)^{-s} \right\|_{L(H, L^p(\lambda;\mathbb{R}))}^{-2} \right) - 2(\nu+s)M.
\end{aligned} \tag{3.204}$$

In addition, note that (3.194) implies for all $r \in (0,\infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln\!\left(\tfrac{z_{M,r}y_M}{2\pi\gamma_M T}\right)\right] \\
&= \liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln(z_{M,r}y_M) - (2M^2+M)\ln(2\pi\gamma_M T)\right] \\
&\geq \liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln(z_{M,r}y_M) - 2(2M^2+M)\pi\gamma_M T\right] \\
&\geq \liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln(z_{M,r}y_M) - \tfrac{6M}{\eta^{2\nu}}(2M^2+M)\pi T\right] \\
&= \liminf_{M\to\infty} \Bigg[\tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln\!\left(\left(\tfrac{T|a_q|\min\{T,1\}}{32\vartheta^2\max\{C,1\}\max\{T,1\}\zeta_\chi M^{(1+\chi)}}\right)^{(1+M)} |y_M|^2\right) \\
&\quad - \tfrac{6M}{\eta^{2\nu}}(2M^2+M)\pi T\Bigg].
\end{aligned} \tag{3.205}$$

This and (3.204) assure for all $r \in (0,\infty)$ that

$$\liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} + (2M^2+M)\ln\!\left(\tfrac{z_{M,r}y_M}{2\pi\gamma_M T}\right)\right] = \infty. \tag{3.206}$$

Furthermore, observe that the fact that $\forall\, M \in \mathbb{N}, r \in (0,\infty)\colon |\rho_{M,r}|^{-2(1+M)} \leq 1$ ensures for all $M \in \mathbb{N}$, $r \in (0,\infty)$ that

$$\begin{aligned}
&|z_{M,r}|^2 + |y_M|^2 = |\rho_{M,r}|^{-2(1+M)}|y_M|^2 + |y_M|^2 \\
&= |y_M|^2\left(|\rho_{M,r}|^{-2(1+M)} + 1\right) \leq 2|y_M|^2 \\
&= \tfrac{2}{|\zeta_{\nu+s}|^2}\left[\tfrac{T}{M}\right]^{2(\nu+s)} \left\|(\eta - A)^{-s}\right\|_{L(H,L^p(\lambda;\mathbb{R}))}^{-2}.
\end{aligned} \tag{3.207}$$

This and (3.197) assure for all $r \in (0,\infty)$ that

$$\begin{aligned}
&\liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^3}{\gamma_M T}\left(|z_{M,r}|^2 + |y_M|^2\right)\right] \\
&\geq \liminf_{M\to\infty} \left[\tfrac{r}{10} q^{(M-1)} - \tfrac{3M^{(3+4\nu)}(\eta+4\pi^2)^{2\nu}}{T}\left[\tfrac{2}{|\zeta_{\nu+s}|^2}\left[\tfrac{T}{M}\right]^{2(\nu+s)} \left\|(\eta - A)^{-s}\right\|_{L(H,L^p(\lambda;\mathbb{R}))}^{-2}\right]\right] \\
&= \infty.
\end{aligned} \tag{3.208}$$

Combining this with (3.181), (3.185), (3.196), (3.200), and (3.206) proves for all $r \in (0,\infty)$ that

$$\liminf_{M\to\infty} \mathbb{E}\Big[\left|\langle e_0, Y_M^M\rangle_H\right|^r\Big] = \infty. \tag{3.209}$$

The fact that $\forall\, N \in \mathbb{N}\colon |\langle e_0, Y_N^N\rangle_H| \leq \|Y_N^N\|_H$ therefore establishes for all $r \in (0,\infty)$ that

$$\liminf_{N\to\infty} \mathbb{E}\big[\|Y_N^N\|_H^r\big] \geq \liminf_{N\to\infty} \mathbb{E}\big[\left|\langle e_0, Y_N^N\rangle_H\right|^r\big] = \infty. \tag{3.210}$$

The proof of Proposition 3.6 is thus completed. □

Theorem 3.7. *Let* $\lambda\colon \mathcal{B}((0,1)) \to [0,\infty]$ *be the Lebesgue-Borel measure on* $(0,1)$*, let* $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H) = (L^2(\lambda;\mathbb{R}), \|\cdot\|_{L^2(\lambda;\mathbb{R})}, \langle\cdot,\cdot\rangle_{L^2(\lambda;\mathbb{R})})$*, let* $e_n \in H$*,* $n \in \mathbb{Z}$*, satisfy* *for all $n \in \mathbb{N}$ that $e_0(\cdot) = 1$, $e_n(\cdot) = \sqrt{2}\cos(2n\pi(\cdot))$, and $e_{-n}(\cdot) = \sqrt{2}\sin(2n\pi(\cdot))$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies that*

$$D(A) = \left\{ v \in H \colon \sum_{n\in\mathbb{Z}} n^4 \left|\langle e_n, v\rangle_H\right|^2 < \infty \right\} \tag{3.211}$$

and

$$\forall\, v \in D(A)\colon \quad Av = \sum_{n\in\mathbb{Z}} -4\pi^2 n^2 \langle e_n, v\rangle_H \, e_n, \tag{3.212}$$

let $T, \eta \in (0,\infty)$, let $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $P_N \in L(H_{-1}, H_1)$, $N \in \mathbb{N}$, be the linear operators which satisfy for all $N \in \mathbb{N}$, $v \in H$ that $P_N(v) = \sum_{n=-N}^{N} \langle e_n, v\rangle_H \, e_n$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $q \in \{2,3,\dots\}$, $a_0, a_1, \dots, a_{q-1} \in \mathbb{R}$, $a_q \in \mathbb{R}\backslash\{0\}$, $\chi \in (1/4, \infty)$, $\nu \in (1/4, 3/4)$, $\xi \in H_\chi$, let $W\colon [0,T]\times\Omega \to H_{-\nu}$ be an Id_H-cylindrical Wiener process, let $S_N \in L(H_{-\nu})$, $N \in \mathbb{N}$, be linear operators which satisfy for all $N \in \mathbb{N}$, $r \in [-\nu,\infty)$, $v, u \in H$ that $S_N(H_r) \subseteq H_{r+1}$, $\sup_{M\in\mathbb{N}} \sup_{s\in[0,1]} \sup_{w\in H, \|w\|_H \le 1} M^{-s} \|S_M w\|_{H_s} < \infty$, $S_N e_0 = e_0$, $(\eta - A)^{-\nu} S_N v = S_N (\eta - A)^{-\nu} v$, $\langle S_N u, v\rangle_H = \langle u, S_N v\rangle_H$, and $P_N S_N v = S_N P_N v$, and let $Y^N\colon \{0,1,\dots,N\}\times\Omega \to H$, $N \in \mathbb{N}$, be stochastic processes which satisfy for all $N \in \mathbb{N}$, $n \in \{0,1,\dots,N-1\}$ that $Y_0^N = P_N(\xi)$ and

$$Y_{n+1}^N = P_N S_N \Big(Y_n^N + \tfrac{T}{N} \Big(\textstyle\sum_{k=0}^{q} a_k \big[Y_n^N\big]^k \Big) + \big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}} \big) \Big). \tag{3.213}$$

Then it holds for all $p \in (0,\infty)$ that $\liminf_{N\to\infty} \mathbb{E}\big[\|Y_N^N\|_H^p\big] = \infty$.

Proof of Theorem 3.7. Note that Proposition 3.6 (with $\lambda = \lambda$, $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H) = (H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$, $e_n = e_n$, $A = A$, $T = T$, $\eta = \eta$, $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r}) = (H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $P_N = P_N$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $q = q$, $a_0 = a_0, a_1 = a_1$, $\dots$, $a_{q-1} = a_{q-1}, a_q = a_q$, $\chi = \min\{\chi, 1\}$, $\nu = \nu$, $\xi = \xi$, $W = W$, $S_N = S_N$, $Y^N = Y^N$ for $n \in \mathbb{Z}$, $r \in \mathbb{R}$, $N \in \mathbb{N}$ in the notation of Proposition 3.6) establishes Theorem 3.7. The proof of Theorem 3.7 is thus completed. ☐

3.4 Divergence results for specific Euler-type approximation schemes for SPDEs with superlinearly growing nonlinearities

The next result, Corollary 3.8 below, follows from Theorem 3.7.

Corollary 3.8. *Let $\lambda\colon \mathcal{B}((0,1)) \to [0,\infty]$ be the Lebesgue-Borel measure on $(0,1)$, let $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H) = (L^2(\lambda;\mathbb{R}), \|\cdot\|_{L^2(\lambda;\mathbb{R})}, \langle\cdot,\cdot\rangle_{L^2(\lambda;\mathbb{R})})$, let $e_n \in H$, $n \in \mathbb{Z}$, satisfy for all $n \in \mathbb{N}$ that $e_0(\cdot) = 1$, $e_n(\cdot) = \sqrt{2}\cos(2n\pi(\cdot))$, and $e_{-n}(\cdot) = \sqrt{2}\sin(2n\pi(\cdot))$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies that*

$$D(A) = \left\{ v \in H \colon \sum_{n\in\mathbb{Z}} n^4 \left|\langle e_n, v\rangle_H\right|^2 < \infty \right\} \tag{3.214}$$

and

$$\forall\, v \in D(A)\colon \quad Av = \sum_{n\in\mathbb{Z}} -4\pi^2 n^2 \langle e_n, v\rangle_H \, e_n, \tag{3.215}$$

let $T, \eta \in (0,\infty)$, let $(H_r, \|\cdot\|_{H_r}, \langle \cdot,\cdot \rangle_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$, let $P_N \colon H \to H$, $N \in \mathbb{N}$, be the linear operators which satisfy for all $N \in \mathbb{N}$, $v \in H$ that $P_N(v) = \sum_{n=-N}^{N} \langle e_n, v\rangle_H \, e_n$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $q \in \{2, 3, \dots\}$, $a_0, a_1, \dots, a_{q-1} \in \mathbb{R}$, $a_q \in \mathbb{R}\backslash\{0\}$, $\chi \in (1/4, \infty)$, $\nu \in (1/4, 3/4)$, $\xi \in H_\chi$, let $W \colon [0,T] \times \Omega \to H_{-\nu}$ be an Id_H-cylindrical Wiener process, let $S_N \colon H_{-\nu} \to H$, $N \in \mathbb{N}$, be linear operators which satisfy for all $N \in \mathbb{N}$ that $S_N \in \{e^{T/NA}, (I - T/NA)^{-1}\}$, and let $Y^N \colon \{0, 1, \dots, N\} \times \Omega \to H$, $N \in \mathbb{N}$, be the stochastic processes which satisfy for all $N \in \mathbb{N}$, $n \in \{0, 1, \dots, N-1\}$ that $Y_0^N = P_N(\xi)$ and

$$Y_{n+1}^N = P_N S_N\Big(Y_n^N + \tfrac{T}{N}\Big(\textstyle\sum_{k=0}^{q} a_k \big[Y_n^N\big]^k\Big) + \big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big)\Big). \tag{3.216}$$

Then it holds for all $p \in (0,\infty)$ that $\liminf_{N\to\infty} \mathbb{E}\big[\|Y_N^N\|_H^p\big] = \infty$.

Proof of Corollary 3.8. Throughout this proof let $\tilde{P}_N \in L(H_{-1}, H_1)$, $N \in \mathbb{N}$, be the linear operators which satisfy for all $N \in \mathbb{N}$, $v \in H$ that $\tilde{P}_N(v) = P_N(v)$ and let $\tilde{S}_N \in L(H_{-\nu})$, $N \in \mathbb{N}$, be the linear operators which satisfy for all $N \in \mathbb{N}$, $v \in H_{-\nu}$ that $\tilde{S}_N v = S_N v$. Note that for all $N \in \mathbb{N}$, $r \in [-\nu, \infty)$, $v \in H_r$ it holds that

$$e^{T/NA} v \in H_{r+1} \qquad \text{and} \qquad (I - T/NA)^{-1} v \in H_{r+1}. \tag{3.217}$$

This proves that for all $N \in \mathbb{N}$, $r \in [-\nu, \infty)$ it holds that

$$\tilde{S}_N(H_r) = S_N(H_r) \subseteq H_{r+1}. \tag{3.218}$$

Next observe that the fact that $\forall\, r \in [0,1], t \in (0,\infty) \colon \|(t(\eta - A))^r \, e^{tA}\|_{L(H)} \leq e^{t\eta}$ (cf., e.g., Renardy & Rogers [141, Lemma 11.36]) ensures that for all $M \in \mathbb{N}$, $s \in [0,1]$ it holds that

$$\begin{aligned}
\sup_{w \in H, \|w\|_H \leq 1} \big(M^{-s} \|e^{T/MA} w\|_{H_s}\big) &= \sup_{w \in H, \|w\|_H \leq 1} \big(M^{-s} \|(\eta - A)^s \, e^{T/MA} w\|_H\big) \\
&= T^{-s} \sup_{w \in H, \|w\|_H \leq 1} \big(\big\|(T/M(\eta - A))^s \, e^{T/MA} w\big\|_H\big) \\
&\leq T^{-s} \big\|(T/M(\eta - A))^s \, e^{T/MA}\big\|_{L(H)} \\
&\leq T^{-s} \, e^{T\eta/M} \leq \max\{1, T^{-1}\} \, e^{T\eta}.
\end{aligned} \tag{3.219}$$

Moreover, note that for all $M \in \mathbb{N}$, $s \in [0,1]$ it holds that

$$\begin{aligned}
&\bigg[\sup_{w \in H, \|w\|_H \leq 1} \big(M^{-s} \|(I - T/MA)^{-1} w\|_{H_s}\big)\bigg]^2 \\
&= \sup_{w \in H, \|w\|_H \leq 1} \big(M^{-2s} \|(\eta - A)^s (I - T/MA)^{-1} w\|_H^2\big) \\
&= \sup_{w \in H, \|w\|_H \leq 1} \left(\sum_{n \in \mathbb{Z}} \frac{M^{-2s} (\eta + 4\pi^2 n^2)^{2s}}{(1 + 4\pi^2 n^2 T/M)^2} \langle w, e_n\rangle_H^2\right) \\
&\leq \sup_{w \in H, \|w\|_H \leq 1} \left(\left[\sup_{n \in \mathbb{Z}} \frac{M^{-2s} (\eta + 4\pi^2 n^2)^{2s}}{(1 + 4\pi^2 n^2 T/M)^2}\right] \left[\sum_{m \in \mathbb{Z}} \langle w, e_m\rangle_H^2\right]\right) \\
&\leq \sup_{n \in \mathbb{Z}} \left(\frac{M^{-2s} (\eta + 4\pi^2 n^2)^{2s}}{(1 + 4\pi^2 n^2 T/M)^2}\right) = \sup_{n \in \mathbb{Z}} \left(\frac{(\eta/M + 4\pi^2 n^2/M)^{2s}}{(1 + 4\pi^2 n^2 T/M)^2}\right).
\end{aligned} \tag{3.220}$$

This demonstrates that for all $M \in \mathbb{N}$, $s \in [0,1]$ it holds that

$$
\begin{aligned}
&\sup_{w\in H, \|w\|_H \leq 1} \big(M^{-s} \|(I - {}^{T}\!/\!_{M} A)^{-1} w\|_{H_s}\big) \leq \sup_{n\in\mathbb{Z}} \left(\frac{({}^{\eta}\!/\!_{M} + {}^{4\pi^2 n^2}\!/\!_{M})^s}{(1 + {}^{4\pi^2 n^2 T}\!/\!_{M})} \right) \\
&\leq \sup_{n\in\mathbb{Z}} \left(\frac{({}^{\eta}\!/\!_{\min\{T,1\}} + {}^{4\pi^2 n^2 T}\!/\!_{[\min\{T,1\}M]})^s}{(1 + {}^{4\pi^2 n^2 T}\!/\!_{M})} \right) \\
&\leq \left[\frac{\max\{\eta,1\}}{\min\{T,1\}}\right]^s \left[\sup_{n\in\mathbb{Z}} \left(\frac{(1 + {}^{4\pi^2 n^2 T}\!/\!_{M})^s}{(1 + {}^{4\pi^2 n^2 T}\!/\!_{M})} \right)\right] \leq \left[\frac{\max\{\eta,1\}}{\min\{T,1\}}\right] < \infty.
\end{aligned}
\tag{3.221}
$$

Combining this with (3.219) assures that

$$
\sup_{M\in\mathbb{N}} \sup_{s\in[0,1]} \sup_{w\in H, \|w\|_H \leq 1} \big(M^{-s} \|\tilde{S}_M w\|_{H_s}\big) < \infty. \tag{3.222}
$$

The fact that $\forall\, N \in \mathbb{N}, u, v \in H\colon \big(\tilde{S}_N e_0 = e_0$, $(\eta - A)^{-\nu} \tilde{S}_N v = \tilde{S}_N (\eta - A)^{-\nu} v$, $\langle \tilde{S}_N u, v\rangle_H = \langle u, \tilde{S}_N v\rangle_H$, and $\tilde{P}_N \tilde{S}_N v = \tilde{S}_N \tilde{P}_N v\big)$, (3.218), and Theorem 3.7 (with $\lambda = \lambda$, $(H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H) = (H, \|\cdot\|_H, \langle\cdot,\cdot\rangle_H)$, $e_n = e_n$, $A = A$, $T = T$, $\eta = \eta$, $(H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r}) = (H_r, \|\cdot\|_{H_r}, \langle\cdot,\cdot\rangle_{H_r})$, $P_N = \tilde{P}_N$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $q = q$, $a_0 = a_0, a_1 = a_1, \ldots, a_{q-1} = a_{q-1}, a_q = a_q$, $\chi = \chi$, $\nu = \nu$, $\xi = \xi$, $W = W$, $S_N = \tilde{S}_N$, $Y^N = Y^N$ for $n \in \mathbb{Z}$, $r \in \mathbb{R}$, $N \in \mathbb{N}$ in the notation of Theorem 3.7) hence establish Corollary 3.8. The proof of Corollary 3.8 is thus completed. □

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