Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

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Research Report No. 2018-49
December 2018

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
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Funding: Fondecyt Regular 1171491 and grant Conicyt 21171479 Doctorado Nacional 2017

HIGH-ORDER GALERKIN METHOD FOR HELMHOLTZ AND LAPLACE PROBLEMS ON MULTIPLE OPEN ARCS*

CARLOS JEREZ-HANCKES† AND JOSÉ PINTO‡

Abstract. We present a spectral Galerkin numerical scheme for solving Helmholtz and Laplace problems with Dirichlet boundary conditions on a finite collection of open arcs in two-dimensional space. A boundary integral method is employed, giving rise to a first kind Fredholm equation whose variational form is discretized using weighted Chebyshev polynomials. Well-posedness of the discrete problems is established as well as algebraic or even exponential convergence rates depending on the regularities of both arcs and excitations. Moreover, our numerical experiments show the robustness of the method with respect to number of arcs and large wavenumber range.

Key words. Boundary integral equations, spectral methods, wave scattering problems, screens problems, non-Lipschitz domains

AMS subject classifications. 65R20, 65N22, 65N35, 65N38

1. Introduction. We present a spectral Galerkin method for solving weakly singular Boundary Integral Equations (BIEs) arising from Laplace or Helmholtz Dirichlet problems on unbounded domains and whose boundaries are a finite collections of disjoints finite open arcs in $\mathbb{R}^2$. Such problems are of particular interest in multiple contexts: in structural and mechanical engineering, wherein fractures or cracks are represented as slits [31, 32, 4, 23]; in the detection of micro-fractures [1, 3] and even for the imaging of muscular strains due to sport injuries [34]. For these applications, one is interested in developing a numerical scheme that can robustly deal with large numbers of arcs –from tens to thousands– for a broad range of wavenumbers –from zero to several hundreds.

Following similar arguments to those presented for the single arc case [30], we proved in [15] that the continuous volume problem for multiple arcs is uniquely solvable when enforcing wavenumber-dependent conditions at infinity. In fact, the volume solution is shown to be constructed as the superposition of single layer potentials applied to surface densities over each arc and which are in turn obtained by solving a system of BIEs. Numerical approximations of these boundary unknowns are traditionally obtained via either the Boundary Element Method (BEM) [27] or Nyström-type strategies [5]. Though we opt for the former, for the type of applications considered, several issues hinder standard schemes' performance. On one hand, solutions at the continuos level are well known to exhibit square-root singularities at the arcs' endpoints [7, 11, 22]. Consequently, convergence of low-order uniform-mesh discretizations is suboptimal with improvements relying on either graded [35] or adaptive mesh refinement [8], or on augmenting the approximation space [30]. Also, the Galerkin matrices derived from first kind Fredholm formulations are intrinsically ill-conditioned, thus heavily requiring preconditioning [13, 25]. Moreover, the minimal number of unknowns to ensure asymptotic convergence increases with the wavenumber [26] while matrix entries grow quadratically with the number of arcs in order to account for cross-interactions. Hence, for our problems of interest, one can expect

* This work was partially funded by Fondecyt Regular 1171491 and grant Conicyt 21171479 Doctorado Nacional 2017.

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extremely large numbers of degrees of freedom when using mesh-dependent methods and alternative ones must be sought after.

In [19] a spectral Galerkin-Bubnov discretization for a single arc was shown to greatly reduce the number of degrees of freedom in comparison to the case of locally defined low-order basis. Specifically, the approximation basis employed is given by weighted first kind Chebyshev polynomials, where the weight mimics the singular behavior at the endpoints. For this basis, we prove approximation properties, using as main tool the asymptotic decay of the Fourier-Chebyshev expansions coefficients of the solutions. With these tools, one can derive convergence rates for order p polynomial approximations that only depend on smoothness of the excitations and of the slits, with constants that may depend on the wavenumber. In particular, one obtains super-algebraic convergence when both arcs and sources can be represented by analytic functions. Interestingly, the convergence properties in [19] were established by different tools to the ones presented here. Furthermore, the current work presents a thorough analysis of the Helmholtz case and general arcs as well as a highly efficient numerical implementation. Indeed, we follow the scheme introduced in [14] wherein all integral kernel singularities are subtracted. This gives rise to smooth functions and singular functions whose integrals are respectively computed via the Fast Fourier Transform (FFT) [20] or analytically using a Chebyshev polynomial expansion of the fundamental solution [9]. The resulting scheme delivers convergence rates with respect to the polynomial order of the approximation depending on the regularity of the geometry, begin super-algebraic when arcs are described by analytic functions.

Recently, Slevinsky and Olver [28] use a similar construction based on Chebyshev polynomials for more general problems but limited to line segments with focus in the spectral properties of Nyström-discretized operators. Hewett *et al.* [12] propose a different numerical method for which they also obtain super-convergence. Their discretization basis captures explicitly the oscillatory behavior on a segment while employing a low polynomial order adaptive basis for the slow but singular part. Though this splitting leads to impressive results especially for high-frequency, its use is restricted to linear segments and not for general arcs. In this sense, our discretization scheme can be used in a more general context.

The paper is organized as follows. Section 2 puts forward formal definitions and properties needed throughout. In Section 3, we review some of the results presented in [15], in order to obtain well-posedness of the BIE system. Section 4 gives details on the Galerkin discretization method; in particular, we establish the convergence rates for the discrete problem assuming regularity conditions on the data. Numerical results and their corresponding discussion are found in Section 6. Finally, further improvements on computational efficiency and compression are sketched in Section 7, with appendices provided for technical lemmas.

2. Mathematical Tools.

2.1. General Notation. The set of extended integer values is denoted by $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$. We employ the standard $\mathcal{O}(\cdot)$ and $o(\cdot)$ notation for asymptotics. Also, for two sequences $\{a_n\}_{n\geq 0}$, $\{b_n\}_{n\geq 0}$, we say that both have the same asymptotic behavior if $\lim_{n\to\infty} |a_n b_n^{-1}|$ exists and is different from zero. If so, we write $a_n \sim b_n$. We also use the notation $a_n \lesssim b_n$ if there exists a positive constant C and an integer $N > 0$ such that $a_n \leq C b_n$ for all $n > N$.

Vectors are indicated by boldface symbols with Euclidean norm written as $\|\cdot\|_2$; other norms are signaled by subscripts. Quantities defined over volume domains will be written in capital case whereas those on boundaries in normal one, e.g., $U : G \to \mathbb{C}$

while $u : \partial G \to \mathbb{C}$.

Let $G \subseteq \mathbb{R}^d$, $d = 1, 2$, be an open domain. For $k \in \mathbb{N}_0$, $\mathcal{C}^k(G)$ denotes the set of k times continuously differentiable functions over G. Compactly supported $\mathcal{C}^k(G)$ functions are designated by $\mathcal{C}^k_0(G)$. Denote by $\mathcal{D}(G) \equiv \mathcal{C}^\infty_0(G)$ the space of infinitely differentiable functions with compact support on a non-empty measurable set G. Duals are indicated by asterisks, e.g., the space of distributions is $\mathcal{D}^*(G)$. The class of p-integrable functions over G is written $L^p(G)$. Duality pairings and inner products are written as $\langle \cdot, \cdot \rangle$ and $(\cdot, \cdot)$, respectively, with subscripts declaring the domain involved, if not clear from the context.

Let us define the canonical domain $\widehat{\Gamma} := (-1, 1) \times \{0\}$. We say that a function $g : \widehat{\Gamma} \to \mathbb{C}$ is analytic, if there exists a Bernstein ellipse of parameter $\rho > 1$, such that g is analytic in the complex ellipse containing $\widehat{\Gamma}$ (*cf.* [33, Chapter 8]). Throughout, the class of all the analytic functions will be denoted by $\mathcal{C}^\infty(\widehat{\Gamma})$. We remark that this space is different from the space of infinity differentiable functions equally denoted.

2.2. Arcs. We briefly recall the geometrical notions introduced in [15]. We say that $\Lambda \subset \mathbb{R}^2$ is a regular Jordan arc of class $\mathcal{C}^m$, for $m \in \mathbb{N}^*$, if there exists a bijective parametrization denoted by $\mathbf{r} : \widehat{\Gamma} \to \Lambda$, such that its components are $\mathcal{C}^m(\widehat{\Gamma})$-functions and $\|\mathbf{r}'(t)\|_2 > 0$, for all $t \in \overline{\widehat{\Gamma}}$.

Assumption 2.1. For any Λ regular Jordan arc of class $\mathcal{C}^m$, there exists an extension to $\tilde{\Lambda}$ with a $\mathcal{C}^m$-parametrization $\tilde{\mathbf{r}} : [0, 2\pi] \to \tilde{\Lambda}$, that is bijective in $[0, 2\pi)$ and satisfies $\tilde{\mathbf{r}}(0) = \tilde{\mathbf{r}}(2\pi)$ and $\|\tilde{\mathbf{r}}'(t)\|_2 > 0$, for all $t \in [0, 2\pi]$.

We consider a finite number $M \in \mathbb{N}$ of $\mathcal{C}^m$-arcs, for $m \geq 1$, written $\{\Gamma_i\}_{i=1}^M$, such that under Assumption 2.1 their closures are mutually disjoint. We define

$$\Gamma := \bigcup_{i=1}^M \Gamma_i \quad \text{and} \quad \Omega := \mathbb{R}^2 \setminus \overline{\Gamma}.$$

Assumption 2.2. There are M disjoint domains Ω_i whose boundaries are given by $\partial\Omega_i = \widetilde{\Gamma}_i$, for $i = 1, \ldots, M$.

For $m \in \mathbb{N}^*$, we say that the family of arcs Γ is at least of class $\mathcal{C}^m$, if each arc Γ_i is of class $\mathcal{C}^m$, and write $\Gamma \in \mathcal{C}^m$. Denote by $\mathbf{r}_i$ the mapping between $\widehat{\Gamma}$ to an arc Γ_i, $i \in \{1, \ldots, M\}$. For a vector function $\mathbf{g} = (g_1, \ldots, g_M)$ such that $g_i : \overline{\Gamma}_i \to \mathbb{C}$, for $i \in \{1, \ldots, M\}$, we state that $\mathbf{g}$ is of class $\mathcal{C}^m(\Gamma)$, if $g_i \circ \mathbf{r}_i \in \mathcal{C}^m(\widehat{\Gamma})$, for $i \in \{1 \ldots M\}$, and denote $\mathbf{g} \in \mathcal{C}^m(\Gamma)$.

2.3. Sobolev Spaces and Trace operator. Let $G \subseteq \mathbb{R}^d$, $d = 1, 2$, be an open domain. For $s \in \mathbb{R}$, we denote by $H^s(G)$ the standard Sobolev spaces in $L^2(G)$ and by $H^s_{loc}(G)$ their locally integrable counterparts [27, Section 2.3]. We also use the following Hilbert space:

$$W(G) := \left\{ U \in \mathcal{D}^*(G) : \frac{U(\mathbf{x})}{\sqrt{1 + \|\mathbf{x}\|_2^2} \log(2 + \|\mathbf{x}\|_2^2)} \in L^2(G), \nabla U \in L^2(G) \right\}, \tag{1}$$

which is a subspace of $H^s_{loc}(G)$ [15, Lemma 2.8]. Under Assumption 2.1 for a Jordan curve Λ, we also define

$$\widetilde{H}^s(\Lambda) := \{u \in \mathcal{D}^*(\Lambda) : \widetilde{u} \in H^s(\widetilde{\Lambda})\}, \quad s > 0, \tag{2}$$

wherein $\widetilde{u}$ denotes the extension by zero of u to $\widetilde{\Lambda}$. For $s > 0$, we can identify

$$\widetilde{H}^{-s}(\Lambda) = (H^s(\Lambda))^* \quad \text{and} \quad H^{-s}(\Lambda) = (\widetilde{H}^s(\Lambda))^*. \tag{3}$$

We will also need the family of mean-zero Sobolev spaces:

$$\widetilde{H}^s_{\langle 0\rangle}(\Lambda) = \{u \in \widetilde{H}^s(\Lambda) : \langle u, 1\rangle = 0\}. \tag{4}$$

The following result was proved in [15, Lemma 2.7] and will be used to establish convergence rates and error computations in our numerical experiments (*cf.* Section 6).

LEMMA 2.3. *Let $\zeta \in H^{\frac{1}{2}}(\Gamma_i)$, $\psi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$, and $\mathbf{r}_i : \widehat{\Gamma} \to \Gamma_i$. Then, we have the norm equivalences:*

$$c\,\|\zeta\|_{H^{\frac{1}{2}}(\Gamma_i)} \leq \|\zeta \circ \mathbf{r}_i\|_{H^{\frac{1}{2}}(\widehat{\Gamma})} \leq C\,\|\zeta\|_{H^{\frac{1}{2}}(\Gamma_i)},$$
$$c\,\|\psi\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq \|\psi \circ \mathbf{r}_i\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} \leq C\,\|\psi\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)},$$

with generic positive constants c and C.

For the finite union of disjoint open arcs Γ, we define piecewise spaces as

$$\mathbb{H}^s(\Gamma) := \{u \in \mathcal{D}^*(\Gamma) : u|_{\Gamma_i} \in H^s(\Gamma_i),\ i = 1, \ldots, M\}. \tag{5}$$

From this definition, the identification:

$$\mathbb{H}^s(\Gamma) \equiv H^s(\Gamma_1) \times H^s(\Gamma_2) \times \cdots \times H^s(\Gamma_M)$$

follows. Norms and dual products are naturally extended by the previous identification, similarly for spaces $\widetilde{\mathbb{H}}^s(\Gamma)$ and $\widetilde{\mathbb{H}}^s_{\langle 0\rangle}(\Gamma)$, while $\mathbb{H}^s(\widehat{\Gamma})$ is understood as the Cartesian product $\prod_{i=1}^M H^s(\widehat{\Gamma})$.

Consider Γ_i and the induced bounded domain Ω_i with boundary $\widetilde{\Gamma}_i = \partial\Omega_i$. For $u \in \mathcal{C}^\infty(\mathbb{R}^2)$, we can set the Dirichlet trace:

$$\widetilde{\gamma}_i^\pm u(\mathbf{x}) := \lim_{\epsilon \uparrow 0} u(\mathbf{x} \pm \epsilon \mathbf{n}_i), \quad \forall\ \mathbf{x} \in \widetilde{\Gamma}_i, \tag{6}$$

where $\mathbf{n}_i$ denotes the outward unitary normal vector to the closed curve $\widetilde{\Gamma}_i$ with direction of $(r'_{i,2}, -r'_{i,1})$. We will denote by $\gamma_i^\pm$ the restriction to Γ_i of the operator $\widetilde{\gamma}_i^\pm$, i.e. $\gamma_i^\pm u := \widetilde{\gamma}_i^\pm u|_{\Gamma_i}$, and if $\gamma_i^+ u = \gamma_i^- u$, we denote $\gamma_i u := \gamma_i^\pm u$. These definitions can be extended to more general Sobolev spaces by density.

LEMMA 2.4 (Lemma 2.9, [15]). *For $i = 1, \ldots, M$, the operators $\gamma_i^\pm : W(\Omega) \to H^{\frac{1}{2}}(\Gamma_i)$ are bounded.*

3. Boundary Integral Problem Formulation. We are interested in solving the following family of volume boundary value problems in Ω. As explained, we will first reduce these via suitable integral representations with unknowns densities over the boundaries Γ.

Problem 3.1 (Volume Problem). Let $\mathbf{g} = (g_1, \ldots, g_M) \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ and consider a bounded real wavenumber $\kappa \geq 0$. We seek $U \in H^1_{loc}(\Omega)$ such that

$$-\Delta U - \kappa^2 U = 0 \qquad \text{in } \Omega, \tag{7}$$
$$\gamma_i^\pm U = g_i \qquad \text{for } i = 1, \ldots, M, \tag{8}$$
$$\text{Condition at infinity}(\kappa). \tag{9}$$

The behavior at infinity (9) depends on κ in the following way: if $\kappa > 0$, we employ the classical Sommerfeld condition:

$$\frac{\partial U}{\partial r} - i\kappa U = o\left(R^{-\frac{1}{2}}\right) \quad \text{for } R \to \infty, \tag{10}$$

where $R = \|\mathbf{x}\|_2$. If $\kappa = 0$, we seek solutions $U \in W(\Omega)$, this last condition was discussed in detail in [15, Remarks 3.9, 4.2 and 4.9]. Moreover, uniqueness of solutions for Problem 3.1 was given in [15, Proposition 3.8 and 3.10].

For $\kappa \geq 0$, we can express the volume solution U as

$$U(\mathbf{x}) = \sum_{i=1}^{M} (\mathsf{SL}_i[\kappa]\lambda_i)(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Omega, \tag{11}$$

where

$$(\mathsf{SL}_i[\kappa]\lambda_i)(\mathbf{x}) := \int_{\Gamma_i} G_\kappa(\mathbf{x},\mathbf{y})\lambda_i(\mathbf{y})\mathrm{d}\Gamma_i(\mathbf{y}),$$

denotes the single layer potential generated at a curve Γ_i with fundamental solution:

$$G_\kappa(\mathbf{x},\mathbf{y}) = \begin{cases} \dfrac{-1}{2\pi}\log\|\mathbf{x}-\mathbf{y}\|_2 & k = 0, \\ \dfrac{i}{4}H_0^1(\kappa\|\mathbf{x}-\mathbf{y}\|_2) & k > 0. \end{cases} \tag{12}$$

Here $H_0^1(\cdot)$ denotes the zeroth-order first kind Hankel function [2, Chapter 9]. It is direct from this representation that U solves (7)–(8) in Ω. Also, by [15, Proposition 4.1] for $\kappa = 0$ and by [24, Theorem 9.6] for $\kappa > 0$, U displays the desired behavior at infinity when each λ_i is in the right functional space.

In order to find the boundary unknowns λ_i, we take Dirichlet traces of the single layers potentials and impose (8). This induces the definition of weakly singular Boundary Integral Operators (BIOs) as

$$\mathcal{L}_{ij}[\kappa] := \frac{1}{2}\left(\gamma_i^+\mathsf{SL}_j[\kappa] + \gamma_i^+\mathsf{SL}_j[\kappa]\right) = \gamma_i\mathsf{SL}_j[\kappa],$$

the last equation resulting from the continuity properties of the SL_i across Γ_i for each $i = 1, \ldots, M$ (*cf.* [27, 6]).

Problem 3.2. Let $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$. For $\kappa > 0$, we seek $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M) \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ such that

$$\boldsymbol{\mathcal{L}}[\kappa]\boldsymbol{\lambda} = \mathbf{g}, \tag{13}$$

or equivalently,

$$\langle \boldsymbol{\mathcal{L}}[\kappa]\boldsymbol{\lambda}, \boldsymbol{\phi}\rangle_\Gamma = \langle \mathbf{g}, \boldsymbol{\phi}\rangle_\Gamma, \quad \forall\, \boldsymbol{\phi} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma), \tag{14}$$

where

$$\boldsymbol{\mathcal{L}}[\kappa] := \begin{bmatrix} \mathcal{L}_{11}[\kappa] & \mathcal{L}_{12}[\kappa] & \ldots & \mathcal{L}_{1M}[\kappa] \\ \mathcal{L}_{21}[\kappa] & \mathcal{L}_{22}[\kappa] & \ldots & \mathcal{L}_{2M}[\kappa] \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{M1}[\kappa] & \mathcal{L}_{M2}[\kappa] & \ldots & \mathcal{L}_{MM}[\kappa] \end{bmatrix} : \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{\frac{1}{2}}(\Gamma). \tag{15}$$

In the case $\kappa = 0$, we look for $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$.

Remark 3.3. Problem 3.2 can be recast in the reference space $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\widehat{\Gamma})$: Find $\widehat{\boldsymbol{\lambda}} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\widehat{\Gamma})$ such that

$$\widehat{\boldsymbol{\mathcal{L}}}[\kappa]\widehat{\boldsymbol{\lambda}} = \widehat{\mathbf{g}}, \tag{16}$$

wherein $\widehat{g}_j := g_j \circ \mathbf{r}_j$, $\widehat{\mathcal{L}}_{ij}$ is the BIO integrating over the reference arc $\widehat{\Gamma}$ with integral kernel $G_\kappa(\mathbf{r}_i(t), \mathbf{r}_j(s))$, and unknowns $\widehat{\lambda}_j := (\lambda_j \circ \mathbf{r}_j)/\|\mathbf{r}'_j\|_2$.

Remark 3.4. Later on we will use the operator $\mathcal{L}_{ii}[\kappa]$ for the choice $\Gamma_i = \widehat{\Gamma}$, which we denote by $\check{\mathcal{L}}[\kappa]$. Observe that the difference with respect to $\widehat{\mathcal{L}}_{ii}[\kappa]$ relies on the absence of parametrizations $\mathbf{r}_i$ involved in the kernel. In the case of a single Jordan arc with parametrization $\mathbf{r}$, we will write $\widehat{\mathcal{L}}[\kappa] \equiv \widehat{\mathcal{L}}_{ii}[\kappa]$. In this case, and for $\kappa = 0$, one can deduce that the kernel function of the integral operator $\check{\mathcal{L}}[0] - \widehat{\mathcal{L}}[0]$ is given by

$$E_{\mathbf{r}}(t,s) := -\frac{1}{2\pi} \log\left(\frac{\|\mathbf{r}(t) - \mathbf{r}(s)\|_2}{|t-s|}\right)$$

for which we have the following result.

LEMMA 3.5. *Let $m \in \mathbb{N}^*$ with $m \geq 1$ and Γ be a single $\mathcal{C}^m$-arc. Then, for $(s,t) \in [-1,1]^2$, the function $E_{\mathbf{r}}(t,s)$ is a $\mathcal{C}^m(\widehat{\Gamma})$-function in each of its components.*

Proof. By Taylor expansion in t, we obtain

$$\Theta_{\mathbf{r}}(t,s) := \frac{\mathbf{r}(t) - \mathbf{r}(s)}{t-s} = \sum_{j=1}^{m-1} \frac{(t-s)^{j-1}\mathbf{r}^{(j)}(s)}{j!} + \frac{1}{t-s}\int_s^t \frac{(t-\xi)^m \mathbf{r}^{(m)}(\xi)}{m!} \mathrm{d}\xi.$$

This function admits m continuous derivatives in the t variable. As mentioned at the beginning of Section 2.2, Jordan arc parametrizations are inyective, and thus, the function can only be zero if $t = s$. However, as t approaches s, the above function behaves as $\mathbf{r}'(s)$, which is not zero. Hence, $\Theta_{\mathbf{r}}(t,s)$ does vanish and so $E_{\mathbf{r}}(t,s)$ is the composition of $\mathcal{C}^m$-functions, despite there being an absolute value. ∎

THEOREM 3.6 (Theorem 4.13 in [15]). *For $\kappa > 0$, Problem 3.2 has a unique solution $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, whereas for $\kappa = 0$ a unique solution exists in the subspace $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma)$. Also, we have the continuity estimate*

$$\|\boldsymbol{\lambda}\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma,\kappa)\|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma)}. \tag{17}$$

4. Numerical Analysis. We now describe a spectral Galerkin numerical scheme for solving Problem 3.2 and establish specific convergence rates. For this, let us recall an abstract definition of discrete or finite-dimensional spaces as in [29, 27].

DEFINITION 4.1. *Let H be a Hilbert space. For $\kappa \geq 0$, let $\{\mathsf{H}_N\}_{N\in\mathbb{N}_0}$ be a sequence of finite dimensional spaces, such that $\dim \mathsf{H}_N = (N+1)$. We say that the sequence of subspaces is a* conforming discretization *of H if $\mathsf{H}_N \subset \mathsf{H}$ and $\mathsf{H}_N \subset \mathsf{H}_{N+1}$ for all $N \in \mathbb{N}$. Furthermore, if it also holds*

$$\overline{\bigcup_{N\in\mathbb{N}_0} \mathsf{H}_N}^{\|\cdot\|_{\mathsf{H}}} = \mathsf{H},$$

we say that the sequence is a dense conforming discretization*.*

Let us define the discrete version of Problem 3.2:

Problem 4.2 (Discrete Boundary Integral Problem). For $\kappa > 0$, let $\{\mathbb{H}_N\}_{N\in\mathbb{N}_0}$ be a dense conforming discretization of $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$. For $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$, we seek $\boldsymbol{\lambda}_N \in \mathbb{H}_N$ such that

$$\langle \boldsymbol{\mathcal{L}}[\kappa]\boldsymbol{\lambda}_N, \boldsymbol{\vartheta}_N\rangle_\Gamma = \langle \mathbf{g}, \boldsymbol{\vartheta}_N\rangle_\Gamma\,, \quad \forall\ \boldsymbol{\vartheta}_N \in \mathbb{H}_N. \tag{18}$$

For $\kappa = 0$, we change $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ for $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$ and accordingly the discrete spaces.

As in Remark 3.3, Problem 4.2 can also be reformulated in $\widehat{\Gamma}$. Here, we consider a sequence $\{\widehat{\mathbb{H}}_N\}_{N\in\mathbb{N}_0}$ of dense and conforming discretizations of $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\widehat{\Gamma})$, for $\kappa > 0$, or $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$, for $\kappa = 0$, and seek for a solution $\widehat{\boldsymbol{\lambda}}_N \in \widehat{\mathbb{H}}_N$ such that

$$\left\langle \widehat{\boldsymbol{\mathcal{L}}}[\kappa]\widehat{\boldsymbol{\lambda}}_N, \boldsymbol{\vartheta}_N \right\rangle_{\widehat{\Gamma}} = \langle \widehat{\mathbf{g}}, \boldsymbol{\vartheta}_N\rangle_{\widehat{\Gamma}}\,, \quad \forall\ \boldsymbol{\vartheta}_N \in \widehat{\mathbb{H}}_N, \tag{19}$$

where $\widehat{\boldsymbol{\mathcal{L}}}[\kappa]$, and $\widehat{\mathbf{g}}$ are defined as in Remark 3.3.

THEOREM 4.3. *For every bounded $\kappa > 0$ and given a dense conforming discretization $\{\mathbb{H}_N\}_{N\in\mathbb{N}}$ of $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, there exists a $N_0 \in \mathbb{N}$ such that, for every $N > N_0$, there exists a unique solution of the discrete Problem 4.2. Furthermore, if $\boldsymbol{\lambda}$ denotes the unique solution of the continuous Problem 3.2, then the best approximation error:*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma,\kappa) \inf_{\boldsymbol{\vartheta}_N\in\mathbb{H}_N} \|\boldsymbol{\lambda} - \boldsymbol{\vartheta}_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)}$$

holds for $N > N_0$, with a positive constant C that depends on the family of arcs $\{\Gamma_i\}_{i=1}^M$ and wavenumber κ. For $\kappa = 0$, the same result holds with $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}$ instead of $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, and similarly for the discrete spaces.

Proof. The proof follows standard arguments for coercive operators together with the hypothesis concerning the approximation properties of $\mathbb{H}_N$ (*cf.* [29, Theorem 8.11]). ∎

4.1. Approximation Spaces. We now construct a dense conforming discretization for the spaces required in Problem 4.2. Certainly, one could use traditional low-order bases built on the arc meshes and for which approximation properties are well known. However, this would imply a large number of degrees of freedom to solve problems with many arcs and/or large values of κ. Thus, we opt for high-order global polynomial bases such as weighted Chebyshev polynomials per arc. This finite-dimensional bases were applied successfully for a single slit [19] and shown to largely reduce the degrees of freedom for high frequency without losing stability [20, 21]. To properly define these approximation basis, we start by introducing some technical tools.

4.1.1. Fourier-Chebyshev expansions. Every function in $\mathcal{C}^1([-1,1])$ can be expanded as a Chebyshev series (*cf.* [33, Theorem 3.1]),

$$f(s) = \sum_{n=0}^{\infty} f_n T_n(s), \quad \forall\ s \in [-1,1] \quad \text{with} \quad f_n := c_n \left\langle f, w^{-1}T_n\right\rangle_{\widehat{\Gamma}}\,, \tag{20}$$

with $c_0 = \pi$ and $c_n = \pi/2$ for $n > 0$. For a given $N \in \mathbb{N}$, the Fourier-Chebyshev coefficients $\{f_n\}_{n\in\mathbb{N}_0}$ can be approximated using the FFT as follows:

(i) Construct a vector $\mathbf{f}^N \in \mathbb{C}^{N+1}$ with entries $f(s_n^N)$, for $n = 0,\dots,N$, and where the $s_n^N = \cos(n\pi/N)$ correspond to the Chebyshev points of order N.

(ii) Apply the FFT to a periodic extension of the vector $\mathbf{f}^N$.
(iii) Extract the first N terms of the resulting vector and update the entries of $\mathbf{f}^N$. Thus, after scaling the first and last terms by a factor of $\frac{1}{2}$, we obtain

$$f_n \approx f_n^N, \quad n = 0, \ldots, N,$$

where $\approx$ denotes numerical approximation.

As in the discrete Fourier transform –which is, in fact, an approximation of the Fourier transform–, the above procedure is an approximation of the coefficients by interpolation (*cf.* [20, 21] or [33, Chapter 1]).

Remark 4.4. By Theorem [33, Theorem 3.1], only Lipschitz continuity is required for a function to be expressed as a Chebyshev series. Moreover, the series converges absolutely and uniformly.

An expansion similar to the one above holds for the fundamental solution $G_0(\mathbf{x}, \mathbf{y})$ when $\kappa = 0$ over $\widehat{\Gamma}$. Specifically, for collinear vectors, i.e. $\mathbf{x} = (t, 0)$ and $\mathbf{y} = (s, 0)$, $(s, t) \in [-1, 1]^2$, it holds [18, Theorem 4.4]

$$G_0(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log |t - s| = \frac{1}{2\pi} \log 2 + \sum_{n \geq 1} \frac{1}{\pi n} T_n(t) T_n(s), \quad \forall\ s \neq t. \tag{21}$$

This series expansion converges point-wise for $t \neq s$ as the fundamental solution is then smooth. We will heavily rely on this representation for our numerical implementation (*cf.* Section 5).

4.1.2. Single Arc Approximation. For an open arc, the density λ_i corresponding to the solution of the Laplace problem for a single slit Γ_i, can be represented as [19, Section 2.7]:

$$\lambda_i \sim \rho_i^{-\frac{1}{2}} (C_1 + C_2 \rho_i + \psi), \tag{22}$$

where $\rho_i(\mathbf{x}) := \operatorname{dist}(\mathbf{x}, \partial\Gamma_i)$ with $\mathbf{x} \in \Gamma_i$, ψ is a smooth function, and C_1 and C_2 are positive constants. Hence, instead of using only polynomial approximations, one should resort to functions that behave like $\rho_i^{-\frac{1}{2}}$.

Let $\mathbb{T}_N(\widehat{\Gamma})$ be the space spanned by first kind Chebyshev polynomials [33], denoted $\{T_n\}_{n=0}^N$, of degree lower or equal to N over on $\widehat{\Gamma}$, orthogonal under the weight w^{-1} with $w(t) := \sqrt{1 - t^2}$. With these, we can construct elements $p_n^i = T_n \circ \mathbf{r}_i^{-1}$ over each arc Γ_i, and thus spanning the space $\mathbb{T}_N(\Gamma_i)$. For practical reasons, we define the normalized space:

$$\overline{\mathbb{T}}_N(\Gamma_i) := \left\{ \bar{p}^i \in C(\Gamma_i) : \ \bar{p}_n^i := \frac{p_n^i}{\left\| \mathbf{r}_i' \circ \mathbf{r}_i^{-1} \right\|_2}, \quad p_n^i \in \mathbb{T}_N(\Gamma_i) \right\}. \tag{23}$$

We account for edge singularities by multiplying the basis $\{\bar{p}_n^i\}_{n=0}^N$ by a suitable weight:

$$\mathbb{Q}_N(\Gamma_i) := \left\{ q_n^i := w_i^{-1} \bar{p}_n^i : \ \bar{p}_n^i \in \overline{\mathbb{T}}_N(\Gamma_i) \right\}, \tag{24}$$

wherein $w_i := w \circ \mathbf{r}_i^{-1}$. The corresponding basis for $\mathbb{Q}_N(\Gamma_i)$ will be denoted $\{q_n^i\}_{n=0}^N$. By Chebyshev orthogonality, we can easily define the mean-zero subspace:
$\mathbb{Q}_{N,\langle 0 \rangle}(\Gamma_i) := \mathbb{Q}_N(\Gamma_i) \setminus \mathbb{Q}_0(\Gamma_i)$, spanned by $\{q_n^i\}_{n=1}^N$.

4.1.3. Multiple Arcs Approximation. With the above definitions, we give concrete proxies to the spaces $\mathbb{H}_N$ in Problem 4.2. Specifically, let us define

$$\mathbb{H}_N[\kappa] := \begin{cases} \prod_{i=1}^{M} \mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i) & \text{for } \kappa = 0, \\ \prod_{i=1}^{M} \mathbb{Q}_{N}(\Gamma_i) & \text{for } \kappa > 0. \end{cases} \tag{25}$$

Remark 4.5. Though polynomial degrees in each arc could be selected differently, the analysis remains the same as norms for $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ are defined piecewise per arc. In fact, we only need to check that for every $i = 1,\dots,M$, $\{\mathbb{Q}_N(\Gamma_i)\}_{N\in\mathbb{N}}$ (resp. $\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)$) is a dense discretization of $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma_i)$ (resp. $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$). In practice, the polynomial degrees could be set depending on arc length for example.

With the previously defined discrete spaces, one can rewrite Problem 4.2 to obtain the following discrete Galerkin linear system:

Problem 4.6 (Linear System). For $\kappa \geq 0$, let $N \in \mathbb{N}_0$ and $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ be the same as in Problem 4.2. Then, we seek coefficients $\mathfrak{u} = (\mathfrak{u}_1,\dots,\mathfrak{u}_M) \in \mathbb{C}^{M(N+1)}$, such that

$$\mathsf{L}[\kappa]\mathfrak{u} = \mathfrak{g}, \tag{26}$$

wherein we have defined the Galerkin matrix $\mathsf{L}[\kappa] \in \mathbb{C}^{M(N+1)\times M(N+1)}$ with matrix blocks $\mathsf{L}_{ij} \in \mathbb{C}^{(N+1)\times(N+1)}$ whose entries are

$$(\mathsf{L}_{ij}[\kappa])_{lm} = \left\langle \mathcal{L}_{ij} q_m^j, q_l^i \right\rangle_{\Gamma_i}, \quad \forall\, i,j = 1,\dots,M, \text{ and } \forall\, l,m = 0,\dots,N, \tag{27}$$

and functions $q_m^j \in \mathbb{Q}_N(\Gamma_j)$ (resp. $\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_j)$), $q_l^i \in \mathbb{Q}_N(\Gamma_i)$ (resp. $\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)$). The right-hand $\mathfrak{g} = (\mathfrak{g}_1,\dots,\mathfrak{g}_M) \in \mathbb{C}^{M(N+1)}$ has components $(\mathfrak{g}_i)_l = \left\langle g_i, q_l^i \right\rangle_{\Gamma_i}$.

For a given $N \in \mathbb{N}_0$, the solutions of Problems 4.2 and 4.6, $\boldsymbol{\lambda}_N$ and $\mathfrak{u}$, respectively, satisfy the following relation:

$$(\lambda_N)_i = \sum_{l=0}^{N} (\mathfrak{u}_i)_l q_l^i \ \text{ in } \Gamma_i, \quad \text{for all } i \in \{1,\dots,M\}.$$

Proposition 4.7. *For $\kappa > 0$, the family $\{\mathbb{H}_N[\kappa]\}_{N\in\mathbb{N}}$ is a dense conforming discretization of $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$. For $\kappa = 0$, the result holds in $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$.*

Proof. This is a direct consequence of Lemmas A.1 and A.2 (*cf.* Appendix). ∎

Remark 4.8. The discrete problem can also be reformulated in $\widehat{\Gamma}$ as Problems 3.2, and 4.2. The problem becomes: find $\widehat{\mathfrak{u}} = (\widehat{\mathfrak{u}}_1,\dots,\widehat{\mathfrak{u}}_M) \in \mathbb{C}^{M(N+1)}$, such that

$$\widehat{\mathsf{L}}[\kappa]\widehat{\mathfrak{u}} = \widehat{\mathfrak{g}}, \tag{28}$$

where

$$(\widehat{\mathsf{L}}_{ij}[\kappa])_{lm} = \left\langle \widehat{\mathcal{L}}_{ij} w^{-1} T_m, w^{-1} T_l \right\rangle_{\widehat{\Gamma}}, \quad \forall\, i,j = 1,\dots,M, \text{ and } \forall\, l,m = 0,\dots,N, \tag{29}$$

with $w(t) = \sqrt{1-t^2}$, and right-hand term $\widehat{\mathfrak{g}} = (\hat{\mathfrak{g}}_1,\dots,\hat{\mathfrak{g}}_M) \in \mathbb{C}^{M(N+1)}$ with components $(\widehat{\mathfrak{g}}_i)_l = \langle \widehat{g}_i, w^{-1} T_l \rangle_{\widehat{\Gamma}}$. We have the corresponding series expansion:

$$(\widehat{\lambda}_N)_i = \sum_{l=0}^{N} (\widehat{\mathfrak{u}}_i)_l w^{-1} T_l \ \text{ in } \widehat{\Gamma}, \quad \text{for all } i \in \{1,\dots,M\}$$

Notice also that by a simple change of variables $\mathfrak{u}, \mathfrak{g}, \mathsf{L}$, defined as in Problem 4.6, are equal to $\widehat{\mathfrak{u}}, \widehat{\mathfrak{g}}, \widehat{\mathsf{L}}$. Thus, henceforth we drop the hat notation for this variables.

4.2. Convergence Results. The above density property (Proposition 4.7) in combination with Theorem 4.3 allows to conclude that when N goes to infinity convergence occurs in the general context. However, this does not provide any insight on convergence rates. Under regularity assumptions for both $\mathbf{g}$ and Γ, such estimates can be found by analyzing simpler versions of the general Problem 3.2 –namely Problems 4.9 and 4.14 for single arcs– to then combine these intermediate results –Lemmas 4.13 and 4.20– for the general case of single and multiple arcs, Theorems 4.21 and 4.24, respectively. Still, explicit convergence rates with respect to κ are not analyzed and we leave this as future work.

By Proposition 4.7, every function $\widehat{\lambda}$ in $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$, can be expressed as a convergent series:

$$\widehat{\lambda}(s) = w^{-1} \sum_{n\geq 0} \lambda_n T_n(s), \quad s \in (-1,1), \tag{30}$$

Furthermore, we have an explicit expression for the norm when such representation is used

$$\left\|\widehat{\lambda}\right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} = \sum_{n\geq 0} |\lambda_n|^2 d_n, \tag{31}$$

where $d_0 = 1$, and $d_n = n^{-1}$ for $n > 0$ [17, proof of Proposition 3.5].

4.2.1. Chebyshev Coefficients Behavior: Laplace Case. We recall operators $\check{\mathcal{L}}[0]$ and $\widehat{\mathcal{L}}[0]$ defined over $\widehat{\Gamma}$ (*cf.* Remark 3.4). In this subsection, we will consider the pullback problem:

Problem 4.9. For $m \geq 1$ and given $\widehat{g} \in C^m(\widehat{\Gamma})$ and a C^m-parametrization $\mathbf{r}$, we seek $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\widehat{\Gamma})$ such that

$$\widehat{\mathcal{L}}[0]\widehat{\lambda} = \widehat{g} \quad \text{on } \widehat{\Gamma}, \tag{32}$$

which is equivalent to Problem 3.2 with $\kappa = 0$ and $M = 1$.

We aim to characterize the mapping properties of these weakly singular BIOs (defined as in Section 3) acting on weighted Chebyshev polynomials.

Lemma 4.10. *For n and l in $\mathbb{N}$, it holds*

$$\left\langle \check{\mathcal{L}}[0]\frac{T_n}{w}, \frac{T_l}{w} \right\rangle = \frac{\pi}{4n}\delta_{nl}.$$

Proof. Direct consequence of the kernel expansion (21) and the orthogonality property of Chebyshev polynomials. ∎

One can interpret this result as follows: given an element in $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$, its image by $\check{\mathcal{L}}[0]$ is a function whose Chebyshev coefficients decay as $\mathcal{O}(n^{-1})$. The rest of this section extends this idea to more general arcs.

Lemma 4.11. *For $m \in \mathbb{N}^*$, let $h : [-1,1]^2 \to \mathbb{C}$ be such that $h(t,\cdot)$ and $h(\cdot,s)$ are $\mathcal{C}^m(\widehat{\Gamma})$-functions as functions of s and t, respectively. Thus, we can write h as*

$$h(t,s) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{nk} T_n(t) T_k(s),$$

with coefficients decaying as follows:

(i) if $m < \infty$, $b_{nk} = \mathcal{O}(\min\{n^{-m}, k^{-m}\})$.
(ii) if $m = \infty$, there exists a $\rho > 1$ such that $b_{nk} = \mathcal{O}(\rho^{\min\{-n,-k\}})$.

Proof. Given that $m \geq 1$, for any $s \in [-1,1]$, we can write the univariate Fourier-Chebyshev expansion in t:

$$h(t,s) = \sum_{n=0}^{\infty} a_n(s) T_n(t), \quad \forall\, t \in [-1,1]. \tag{33}$$

In fact, the regularity of $h(t,\cdot)$ implies that the functions $a_n(s)$ belong to $\mathcal{C}^m(\widehat{\Gamma})$, and consequently, one can write down expansions:

$$a_n(s) = \sum_{k=0}^{\infty} b_{nk} T_k(s), \quad \forall\, s \in [-1,1], \quad \forall\, n \in \mathbb{N}_0. \tag{34}$$

If $m < \infty$, by [33, Theorem 7.1], we have that $b_{nk} \lesssim k^{-m}$, where the constant depends on the m-th derivative of $a_n(s)$, which is bounded by the m-th derivative of h in s.

For $m = \infty$, we have by [33, Theorem 8.1] that $b_{nk} \lesssim \hat{\rho}_n^{-k}$, with $\hat{\rho}_n > 1$. However, the coefficients $a_n(s)$ are given by

$$a_n(s) = c_n \int_{-1}^{1} h(t,s) w^{-1}(t) T_n(t) dt,$$

where $c_0 = \pi^{-1}$, and $c_n = 2\pi^{-1}$, for $n \in \mathbb{N}$. Hence, since $h(t,\cdot)$ is analytic in the Bernstein ellipse of parameter $\rho_1 > 1$, denoted $\mathcal{E}_{\rho_1}$, we have that, for every $z \in \mathcal{E}_{\rho_1}$, we can write

$$a_n(z) = \sum_{p \geq 0} z^p \int_{-1}^{1} A_p(t) w^{-1}(t) T_n(t) dt,$$

where $A_p(t)$ are the coefficients of the power series of $h(t,\cdot)$. From this last expression, we have that a_n is analytic in $\mathcal{E}_{\rho_1}$ for every n, and thus, we can take $\hat{\rho}_n = \rho_1$ for every $n \in \mathbb{N}_0$.

The final result is obtained by repeating the above arguments –for $m < \infty$ and $m = \infty$– inverting the roles of n and k. ∎

LEMMA 4.12. *Let $m \in \mathbb{N}^*$ with $m \geq 2$ and $h : [-1,1]^2 \to \mathbb{C}$ be a $\mathcal{C}^m(\widehat{\Gamma})$-function per argument. Consider the integral operator taking as kernel the bivariate function h:*

$$(\mathcal{H} f)(s) := \int_{\widehat{\Gamma}} h(t,s) f(t) dt,$$

Now, let $\widehat{\lambda} \in \widetilde{H}^{-1/2}(\widehat{\Gamma})$ have the expansion $\widehat{\lambda} = w^{-1} \sum_{n \geq 0} a_n T_n$, $a_n \in \mathbb{C}$. Then, the following asymptotic behaviors for Fourier-Chebyshev coefficients of $H\widehat{\lambda}$, denoted $\{v_l\}_{l \in \mathbb{N}_0}$, hold
(i) if $m < \infty$ and $1 - \frac{1}{m} - \epsilon > 0$ for an $\epsilon > 0$, then $v_l = \mathcal{O}(l^{-m+1+m\epsilon})$;
(ii) if $m = \infty$, there exists a $\rho > 1$, such that, $v_l = \mathcal{O}(\rho^{-l})$.

Proof. By Lemma 4.11, we expand $h(t,s)$ as the series $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_{nk} T_n(t) T_k(s)$. Hence, by the Chebyshev polynomials' orthogonality property, we can write

$$v_l = \frac{\pi^2}{4} \sum_{n=0}^{\infty} b_{nl} a_n \quad l > 0.$$

Thus, by definition of constants d_n (31) and the series expression for $\widetilde{H}^{-1/2}(\widehat{\Gamma})$-norm, we obtain the following bound:

$$|v_l|^2 \lesssim \|\widehat{\lambda}\|^2_{\widetilde{H}^{-1/2}(\widehat{\Gamma})} \sum_{n=0}^{\infty} |b_{nl}|^2 d_n^{-1}, \quad \forall\ \mathbb{N}_0. \tag{35}$$

The convergence result for $m = \infty$ is immediate from Lemma 4.11. For $m < \infty$, using again Lemma 4.11, it holds

$$|b_{nl}|^2 \lesssim l^{-2m\mu} n^{-2m(1-\mu)}, \quad \forall\ \mu \in (0, 1).$$

With the above bound and the estimate $d_n \sim n^{-1}$ (35), we arrive to

$$|v_l|^2 \lesssim \|\widehat{\lambda}\|^2_{\widetilde{H}^{-1/2}(\widehat{\Gamma})} l^{-m\mu} \sum_{n=1}^{\infty} n^{-2m(1-\mu)+1}. \tag{36}$$

Since $m > 2$ by choosing $\mu = 1 - \frac{1}{m} - \epsilon$, the series in the right-hand side converges and we get the stated result. ∎

With this, we can now estimate bounds for the Chebyshev coefficients of solutions of the BIE associated to the Laplace problem for any sufficiently smooth single arc.

Lemma 4.13. *For $m \in \mathbb{N}^*$ with $m > 2$ and $\widehat{g} \in \mathcal{C}^m(\widehat{\Gamma})$, let $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\widehat{\Gamma})$ be the unique solution of Problem 4.9. If we expand $\widehat{\lambda}$ as*

$$\widehat{\lambda} = w^{-1} \sum_{n=1}^{\infty} a_n T_n,$$

we obtain the following coefficient asymptotic behaviors:
(i) If $m < \infty$, $a_n = \mathcal{O}(n^{-m+1})$;
(ii) If $m = \infty$, there is a $\rho > 1$ such that $a_n = \mathcal{O}(n\rho^{-n})$.

Proof. Since $\widehat{g} \in C^1(\widehat{\Gamma})$, we can expand it as a Fourier-Chebyshev series with coefficients $\widehat{g}_l$ leading to

$$(\widehat{\mathcal{L}}[0]\widehat{\lambda})_l = \widehat{g}_l, \quad \forall\ l \in \mathbb{N}.$$

Left-hand side coefficients can be computed by adding and subtracting the term $\check{\mathcal{L}}[0]\widehat{\lambda}$. Then, by combining Lemmas 4.10, 4.11, 4.12 and 3.5, we obtain the following expression:

$$\frac{\pi^2}{4} \frac{a_l}{l} + v_l = \widehat{g}_l, \quad \forall\ l \in \mathbb{N},$$

where the coefficient v_l corresponds to that in the expansion of $(\widehat{\mathcal{L}}[0] - \check{\mathcal{L}}[0])\widehat{\lambda}$. By the regularity conditions, it holds $\widehat{g}_l = \mathcal{O}(l^{-m})$, and therefore,

$$\frac{\pi^2}{4} a_l l^{-1} + v_l = \mathcal{O}(l^{-m}).$$

Hence, there are two alternatives: either (i) $a_l = \mathcal{O}(l^{-m+1})$ and $v_l = O(l^{-m})$, or (ii) both have the same decay order. As the first implies the result directly, we assume the second alternative in what follows.

Let $3 < m < \infty$. By Lemma 4.12 (i), we have that $v_l = \mathcal{O}(l^{-m+1+m\epsilon})$, and under our current assumption, this implies that

$$a_l = \mathcal{O}(l^{-m+2+m\epsilon}).$$

Since $m > 3$, we can choose ϵ such that $\sum_{n=1}^{\infty} a_n$ is finite and a new estimate for v_l holds

$$v_l = \sum_{n=1}^{\infty} b_{nl} a_n \lesssim l^{-m}.$$

Here, b_{nl} are the coefficients detailed in Lemma 4.11 for the function $E_{\mathbf{r}}$ defined in Remark 3.4 This last equality implies the result directly. For $m = \infty$, the result is retrieved in a similar manner.

The case $m = 3$ is slightly more complicated as one can not directly ensure that the coefficients a_l are summable. However, by Lemma 4.12, for a small $\delta > 0$, then $v_l = \mathcal{O}(l^{-2+\delta})$, which implies that $a_l = \mathcal{O}(l^{-1+\delta})$. By re-estimating bounds on v_l, we now obtain that $v_l = \mathcal{O}(l^{-3+2\delta})$. Hence, $a_l = \mathcal{O}(l^{-2+2\delta})$ which are summable from where one can argue as before. ◻

4.2.2. Chebyshev Coefficients Behavior: Helmholtz Case. We will we now consider the following single arc problem:

Problem 4.14. For $m \geq 1$ and given $\widehat{g} \in C^m(\widehat{\Gamma})$ and a C^m-parametrization $\mathbf{r}$, we seek $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ such that

$$\widehat{\mathcal{L}}[\kappa]\widehat{\lambda} = \widehat{g} \quad \text{on } \widehat{\Gamma}, \tag{37}$$

which is equivalent to Problem 3.2 with $\kappa > 0$ and $M = 1$.

One could see the Helmholtz case as a perturbation of the previous one, but this perturbation is not smooth as the operator difference $\widehat{\mathcal{L}}[\kappa] - \check{\mathcal{L}}[0]$ (*cf.* Remark 3.4) only has a $\mathcal{C}^1$-kernel, even for smooth arcs. Thus, we can not replicate the previous arguments and need to examine in depth $\widehat{\mathcal{L}}[\kappa] - \check{\mathcal{L}}[0]$ in terms of Chebyshev coefficients.

Using [2, Formula 9.1.13], the kernel of $\widehat{\mathcal{L}}[\kappa]$, given in (12), can be also be written as

$$\widehat{G}_k(t,s) = \frac{i}{4} H_0^1\left(k \left\|\mathbf{r}(t) - \mathbf{r}(s)\right\|_2\right) = \sum_{p=0}^{\infty} z_p R_p(t,s) |t-s|^{2p} \log|t-s| + \psi_R(t,s), \tag{38}$$

wherein $\mathbf{r} : \widehat{\Gamma} \to \Gamma_i$ is a suitable parametrization and

$$z_p := \frac{1}{2\pi}(-1)^p \left(\frac{k}{2}\right)^{2p} (p!)^{-2}, \tag{39}$$

$$R_p(t,s) := \left(\frac{\left\|\mathbf{r}(t) - \mathbf{r}(s)\right\|_2}{|t-s|}\right)^{2p}, \tag{40}$$

and ψ_R is C^m-regular. Notice that the term $|t-s|^{2p} \log|t-s|$ is a $\mathcal{C}^{2p-1}(\widehat{\Gamma})$-function in each component.

We begin by analyzing the Helmholtz case for $\widehat{\Gamma}$ following similar techniques to those in [9]. To simplify notation, we define kernels $\widehat{G}_k^p(t,s) := z_p R_p(t,s)|t-s|^{2p} \log|t-s|$ and their corresponding BIOs:

$$\widehat{\mathcal{L}}^p[\kappa] f := \int_{\widehat{\Gamma}} \widehat{G}_k^p(t,s) f(t) dt.$$

Extensive use will be given to the following lemma:

Lemma 4.15. *For $p \in \mathbb{N}_0$, we have*

$$|t-s|^{2p}\log|t-s| = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} b_{nl}^{p} T_n(t)T_l(s)$$

where

$$b_{nl}^{p} \sim \begin{cases} l^{-(2p+1)} & n = l, l\pm 2, \ldots, l \pm 2p \\ 0 & \text{any other case.} \end{cases}$$

Proof. We proceed by induction. As the case $p = 0$ was proven in Lemma 4.10, we start by setting $p = 1$. By Lemma A.4, it holds

$$|t-s|^{2}\log|t-s| = \sum_{j\in\{-1,0,1\}}\sum_{n=0}^{\infty} \beta_n^{(j)} T_n(t)T_{|n+2j|}(s). \tag{41}$$

Bounds for coefficients $\beta_n^{(j)}$ are found by using Lemma A.4. Since in this case $a_n := b_n^0 \sim \frac{2}{n}$ (*cf.* Lemma 4.10), we obtain the stated result.

Assuming now that the result holds for p, we prove it for $p+1$. Indeed,

$$\begin{aligned} |t-s|^{2}(|t-s|^{2p}\log|t-s|) &= |t-s|^{2}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} b_{nl}^{p} T_n(t)T_l(s) \\ &= |t-s|^{2}\sum_{j\in\{-1,0,1\}^n}\sum_{n=0}^{\infty} \beta_n^{(j)} T_n(t)T_{|n+2\sum j|}(s) \end{aligned} \tag{42}$$

and we proceed as in the proof of Lemma A.4 to obtain the expansion. The asymptotic behavior is obtained by a direct computation using expressions of Lemma A.4. ∎

Lemma 4.16. *Let $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ with expansion*

$$\widehat{\lambda} = w^{-1}\sum_{n=0}^{\infty} a_n T_n.$$

Then, the Fourier-Chebyshev coefficients of $\widehat{\mathcal{L}}^p[\kappa]\widehat{\lambda}$, denoted $\{v_l^p\}_{l\in\mathbb{N}_0}$, are given by

$$v_l^p = z_p \sum_{n=0}^{\infty} b_{nl}^{p} a_n,$$

where the coefficients b_{nl}^p are given by Lemma 4.15, z_p are defined in (39). Moreover, it holds

$$v_l^p = \mathcal{O}(l^{-2p-\frac{1}{2}}).$$

Proof. The representation is a direct consequence of the Fourier-Chebyshev expansion of $\widehat{\lambda}$ and the kernel function given by Lemma 4.15. The asymptotic behavior is deduced as follows

$$|v_l^p| \sim \left|\sum_{n=0}^{\infty} b_{nl}^{p} a_n\right| \leq \|\widehat{\lambda}\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} \left|\sum_{n=0}^{\infty} (b_{nl}^{p})^2 d_n^{-1}\right|^{\frac{1}{2}}, \lesssim l^{-2p-\frac{1}{2}},$$

with d_n coming from (31) and where the last inequality is obtained using Lemma 4.15. ∎

With the above results, we can estimate the asymptotic order of the Chebyshev coefficients of $\check{\mathcal{L}}[\kappa] - \check{\mathcal{L}}[0]$, where $\check{\mathcal{L}}[\kappa]$ is the weakly singular Helmholtz operator for the special case $\Gamma \equiv \widehat{\Gamma}$. This bound turns out to be crucial in proving the convergence of the proposed method.

LEMMA 4.17. *For $m \in \mathbb{N}^*$ with $m \geq 2$ and $g \in \mathcal{C}^m(\widehat{\Gamma})$. Let $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ be the only solution of*

$$\check{\mathcal{L}}[\kappa]\widehat{\lambda} = g,$$

which is a special case of Problem, 3.2 with $\Gamma = \{\widehat{\Gamma}\}$ and $\kappa > 0$. Then, if a_n are the series coefficients of $\widehat{\lambda}$, the following asymptotic behaviors hold

(i) if $m < \infty$, $a_n = \mathcal{O}(n^{-m+1})$;

(ii) if $m = \infty$, there exists a $\rho > 1$ such that $a_n = \mathcal{O}(n\rho^{-n})$.

Proof. Consider first $m < \infty$. By the regularity of g, we have

$$(\check{\mathcal{L}}[\kappa]\lambda)_l = g_l = \mathcal{O}(l^{-m}), \quad l \in \mathbb{N}_0. \tag{43}$$

On the other hand, using the integral kernel expansion and Lemma 4.10, for any $Q \in \mathbb{N}$, with $Q > 1$, we derive

$$(\check{\mathcal{L}}[\kappa]\lambda)_l = \frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q-1} v_l^j + v_l^{R(Q)}, \tag{44}$$

where coefficients v_l^j are given by Lemma 4.16 and $v_l^{R(Q)}$ is the remainder of order $\mathcal{O}(l^{-2Q-\frac{1}{2}})$. Thus, if we choose Q as the upper integer part of $\frac{m}{2}$, we have that

$$\frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q-1} v_l^j = \mathcal{O}(l^{-m}). \tag{45}$$

The proof follows by induction in m. For $m = 2$, it holds

$$\frac{\pi^2}{4}\frac{a_l}{l} = \mathcal{O}(l^{-2}), \tag{46}$$

which directly implies $a_l = \mathcal{O}(l^{-1})$. For the induction hypothesis we denote $Q(n)$ the corresponding value of Q given a natural number n. Then, the induction hypothesis reads as: if

$$\frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q(n)-1} v_l^j = \mathcal{O}(l^{-n}), \tag{47}$$

then $a_l = \mathcal{O}(l^{-n+1})$. Now, we prove for $m = n+1$ and for which we have two options: $Q(n+1) = Q(n)$ or $Q(n+1) = Q(n)+1$. If the latter is true, there is a new term of order $-n$. Thus, without loss of generality we can assume that

$$\frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q(n)-1} v_l^j = \mathcal{O}(l^{-n}). \tag{48}$$

By the induction hypothesis, $a_l = O(l^{-n+1})$. Then, by definition of coefficients v_l^j and Lemma 4.15, one has

$$\begin{aligned} v_l^1 &= \mathcal{O}(l^{n-2}) \\ v_l^2 &= \mathcal{O}(l^{n-4}) \\ \vdots &= \vdots \\ v_l^{Q(n)-1} &= \mathcal{O}(l^{n-2(Q(n)-1)}) \end{aligned}$$

and from (47) we obtain the desire order for a_l.

The case $m = \infty$ employs the same argument. As $a_l l^{-1}$ and $\sum_{j=1}^{\infty} v_l^j$ cannot have the same decay order, the only option is for both terms to decay geometrically. ∎

To end this section, we consider the Helmholtz case for arcs different from the canonical segment. Our main ingredients here are the bounds for Chebyshev coefficients of the product of two functions. For one-dimensional $\mathcal{C}^1$-functions, this can be done easily: let $f(t) = \sum_{k\in\mathbb{N}_0} f_k T_k(t)$ and $g(t) = \sum_{l\in\mathbb{N}_0} g_l T_l(t)$. One can write

$$f(t)g(t) = \sum_{n\in\mathbb{N}_0} e_n c_n T_n(t), \qquad \text{where } e_n = \int_{-1}^{1} f(t)g(t)\frac{T_n(t)}{w(t)}dt,$$

and $c_0 = \pi^{-1}$, $c_n = 2\pi^{-1}$, for $n > 0$. By replacing the series expansion for f above, we derive

$$e_n = \sum_{k\in\mathbb{N}_0} f_k \int_{-1}^{1} g(t)T_k(t)\frac{T_n(t)}{w(t)}dt,$$

Using now Lemma A.3 and Chebyshev orthogonality, it holds

$$e_n = \sum_{k\in\mathbb{N}_0} f_k \int_{-1}^{1} g(t)\frac{T_{k+n}(t)+T_{|k-n|}(t)}{2w(t)}dt = \sum_{k\in\mathbb{N}_0} \frac{f_k}{2}\left(\frac{g_{k+n}}{c_{k+n}} + \frac{g_{|k-n|}}{c_{|k-n|}}\right).$$

Consequently, we can estimate the decay of e_n by the properties of f_n and g_n. In two dimensions we have a similar result.

Lemma 4.18. *Let $m \in \mathbb{N}^*$ with $m > 2$, $p \in \mathbb{N}$, and recall the definition of $R_p(t,s)$ given in (40). Then, the series*

$$R_p(t,s)|t-s|^{2p}\log|t-s| = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C_{ij}^p T_i(t)T_j(t), \quad \forall\ (t,s) \in [-1,1]^2, \tag{49}$$

holds, with coefficients

$$C_{ij}^p = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{b_{nl}^p}{4}\left(r_{n+i,l+j} + r_{n+i,|l-j|} + r_{|n-i|,l+j} + r_{|n-i|,|l-j|}\right) \tag{50}$$

with coefficients b_{nl}^p being those of Lemma 4.15 and $r_{i,j}$ the Chebyshev coefficients of $R_p(t,s)$. Moreover, the following asymptotic behaviors hold

(i) If $m < \infty$, $C_{ij}^p = \mathcal{O}(\min\{i^{-\min(m,2p+1)}, j^{-\min(m,2p+1)}\})$;

(ii) If $m = \infty$, then $C_{ij}^p = \mathcal{O}(\min\{i^{-(2p+1)}, j^{-(2p+1)}\})$.

Proof. We proceed as in the one-dimensional case and assume, for simplicity, that the Chebyshev polynomials are normalized, thus omitting constants c_n.

First, let us consider $m < \infty$. The coefficients C^p_{ij} are given by

$$\begin{aligned} C^p_{ij} &= \int_{-1}^{1}\int_{-1}^{1} R_p(t,s)|t-s|^{2p}\log|t-s|\frac{T_i(t)}{w(t)}\frac{T_j(s)}{w(s)}dtds \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} b^p_{nl}\int_{-1}^{1}\int_{-1}^{1} R_p(t,s)\frac{1}{4}\frac{T_{n+i}(t)+T_{|n-i|}(t)}{w(t)}\frac{T_{l+j}(s)+T_{|l-j|}(s)}{w(s)}dtds \qquad (51) \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\frac{b^p_{nl}}{4}(r_{n+i,l+j}+r_{n+i,|l-j|}+r_{|n-i|,l+j}+r_{|n-i|,|l-j|}). \end{aligned}$$

Now, we have to find the decay order for the different terms. Define the index set $I_p(l) := \{l, l\pm 2, l\pm 4, \dots, l\pm 2p\}$. By Lemma 4.15, we have the estimate:

$$C^p_{ij} \sim \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1}(r_{n+i,l+j}+r_{n+i,|l-j|}+r_{|n-i|,l+j}+r_{|n-i|,|l-j|}). \qquad (52)$$

By Lemma 4.11, it holds

$$r_{\nu,\mu} = \mathcal{O}\left(\min\{\nu^{-m},\mu^{-m}\}\right), \quad \text{for } \nu,\mu\in\mathbb{N}, \qquad (53)$$

and we can estimate each term in C^p_{ij} as follows, we provide details for the first two.

Define $K_1 := \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1}r_{n+i,l+j}$. Assume that $r_{n+i,l+j} = \mathcal{O}((l+j)^{-m})$, then

$$K_1 \lesssim 2p\sum_{l=1}^{\infty} l^{-2p-1}(l+j)^{-m} = \mathcal{O}(j^{-m}).$$

Alternatively, we can use that $r_{n+i,l+j} = \mathcal{O}((n+i)^{-m})$ so that

$$K_1 \lesssim \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1}(n+i)^{-m} = \mathcal{O}(i^{-m}).$$

Thus, we then conclude that

$$K_1 = \mathcal{O}\left(\min\{i^{-m},j^{-m}\}\right)$$

Now set $K_2 := \sum_{l=1}^{\infty}\sum_{n\in I_p(l)} l^{-2p-1}r_{n+i,|l-j|}$. Let $r_{n+i,|l-j|} = \mathcal{O}((|l-j|+1)^{-m})$, we obtain

$$K_2 \lesssim \sum_{l=1}^{\infty} l^{-2p-1}(|l-j|+1)^{-m},$$

where we added one to avoid infinity. Thus, we can split this last sum into two terms

$$K_2 \lesssim \sum_{l=1}^{j/2} l^{-2p-1}(j-l)^{-m} + \sum_{l>j/2} l^{-2p-1}(|l-j|+1)^{-m}.$$

The first one is bounded as

$$\sum_{l=0}^{j/2} l^{-2p-1}(j-l)^{-m} \lesssim j^{-m}\sum_{l=0}^{j/2} l^{-2p-1} \lesssim j^{-m},$$

whereas the second one

$$\sum_{l>j/2} l^{-2p-1}(|l-j|+1)^{-m} \lesssim j^{-2p-1}.$$

Hence, we have

$$K_2 = \mathcal{O}(j^{-m}) + \mathcal{O}(j^{-2p-1}) = \mathcal{O}\left(j^{-\min\{m,2p+1\}}\right).$$

If alternatively we use $r_{n+i,|l-j|} = \mathcal{O}((n+i)^{-m})$, then

$$K_2 \lesssim \sum_{l=0}^{\infty} l^{-2p-1}(n+i)^{-m} = \mathcal{O}(i^{-m}).$$

Combining both results yields

$$K_2 = \mathcal{O}\left(\min\{i^{-m}, j^{-\min\{m,2p+1\}}\right).$$

The remaining two terms in (52) are bounded in a similar manner so that

$$K_3 := \sum_{l=0}^{\infty} \sum_{n\in I_p(l)} l^{-2p-1} r_{|n-i|,l+j} = \mathcal{O}\left(\min\{j^{-m}, i^{-\min\{m,2p+1\}}\right)$$
$$K_4 := \sum_{l=0}^{\infty} \sum_{n\in I_p(l)} l^{-2p-1} r_{|n-i|,|l-j|} = \mathcal{O}\left(\min\{j^{-\min\{m,2p+1\}}, i^{-\min\{m,2p+1\}}\right)$$

Finally, considering all the bounds yields the stated result. The case $m = \infty$ follows the same arguments. ∎

Lemma 4.19. *For $m \in \mathbb{N}^*$ with $m \geq 2$, let Γ be a $\mathcal{C}^m$-arc and $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ have the representation:*

$$\widehat{\lambda} = w^{-1} \sum_{n=0}^{\infty} a_n T_n.$$

Then, the Fourier-Chebyshev coefficients of $\widehat{\mathcal{L}}^p[\kappa]\widehat{\lambda}$, denoted $\{v_l^p\}_{l\in\mathbb{N}_0}$, satisfy

$$v_l^p = z_p \sum_{n=0}^{\infty} C_{nl}^p a_n,$$

where the coefficients C_{nl}^p are given in Lemma 4.18, z_p are defined in (39), and the asymptotic behaviors hold

(i) If $m \leq 2p+1$ and for $\epsilon > 0$ such that $1 - \frac{1}{m} - \epsilon > 0$, $v_l^p = \mathcal{O}(l^{-m+1+m\epsilon})$
(ii) If $2p+1 < m$ and for $\epsilon > 0$ such that $1 - \frac{1}{2p+1} - \epsilon > 0$, then

$$v_l^p = \mathcal{O}(l^{-2p+(2p+1)\epsilon}).$$

Proof. The proof follows the steps of Lemma 4.12 but by using Lemma 4.18 instead of Lemma 4.15. ∎

Lemma 4.20. *For $m \in \mathbb{N}^*$ with $m > 2$, let $\widehat{\lambda} \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ be the unique solution of Problem 4.14. Then, if the solution is expanded as $\widehat{\lambda} = \sum_{n=0}^{\infty} a_n w^{-1} T_n$, the following asymptotic behaviors for coefficients a_n holds*

(i) If $m < \infty$, $a_n = \mathcal{O}(n^{-m+1})$*;*
(ii) If $m = \infty$ *there exists a* $\rho > 1$ *such that* $a_n = \mathcal{O}(n\rho^{-n})$*.*

Proof. We follow similar steps of those for Lemmas 4.17 and 4.13. For $m < \infty$, we have

$$(\widehat{\mathcal{L}}[\kappa]\widehat{\lambda})_l = \frac{\pi^2}{4}\frac{a_l}{l} + \sum_{j=1}^{Q} v_l^j + v_l^R = \mathcal{O}(l^{-m}), \tag{54}$$

where v_l^j are defined as in Lemma 4.19, and Q is fixed such that the remainder is given by a $\mathcal{C}^m(\widehat{\Gamma})$-function. Thus, for $\epsilon \in (0, 1 - \frac{1}{m})$, $v_l^R = \mathcal{O}(l^{-m+1+m\epsilon})$. Moreover, we can assume that, for $\delta \in (0, 1-\frac{1}{3})$, by Lemma 4.19, it holds $v_l^j = \mathcal{O}(l^{-2j+(2j+1)\delta})$, for all $j = 1, \ldots, Q$. The proof then follows that of Lemma 4.13.

The proof for the case $m = \infty$ is as the one presented in Lemma 4.17 with the corresponding modifications. ∎

4.2.3. Convergence rates for a single arc. From the decay proprieties of Chebyshev coefficients, we can obtain bounds for the approximation error. First, notice that by the norm equivalence (*cf.* Lemma 2.3), we can do all the estimates in $\widehat{\Gamma}$ and transform $\lambda \mapsto \widehat{\lambda}$. On the other hand, we have the quasi-optimality result (*cf.* Theorem 4.3): there exists $N_0 > 0$ and a constant $C(\Gamma, \kappa) > 0$, such that for all $N > N_0$:

$$\|\lambda - \lambda_N\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq C(\Gamma,\kappa) \inf_{q_N \in \mathbb{Q}_N(\widehat{\Gamma})} \left\|\widehat{\lambda} - q_N\right\|_{\widetilde{H}^{-1/2}(\widehat{\Gamma})}. \tag{55}$$

For $\widehat{\lambda}$ we have an expansion of the form $\widehat{\lambda} = \sum a_n w^{-1} T_n$. Hence, we can choose $q_N = \sum_{n \leq N} a_n w^{-1} T_n$, and use the norm representation to estimate the error as

$$\left\|\widehat{\lambda} - q_N\right\|^2_{\widetilde{H}^{-1/2}(\widehat{\Gamma})} = \sum_{n>N} \frac{|a_n|^2}{n}. \tag{56}$$

Finally, using the bounds from Lemmas 4.20, and 4.13 for the behavior of coefficients a_n, we can establish convergence rates.

THEOREM 4.21. *Let* $\kappa \geq 0$, $m \in \mathbb{N}^*$ *with* $m > 2$, Γ *be a* $\mathcal{C}^m$*-arc. For* $g \in \mathcal{C}^m(\Gamma)$*, let* λ *be the unique solution of Problem 3.2. Then, there exists* $N_0 \in \mathbb{N}$ *such that, for every* $N \in \mathbb{N}$ *with* $N > N_0$*, there is a unique* λ_N *solution of Problem 4.2 using the discrete spaces detailed in Section 4.1. Moreover,*

- *if* $m < \infty$*, then*
$$\|\lambda - \lambda_N\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq C(\Gamma,\kappa) N^{-m+1};$$
- *if* $m = \infty$*, there exist* $\rho > 1$ *such that*
$$\|\lambda - \lambda_N\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq C(\Gamma,\kappa) \rho^{-N+2}\sqrt{N}.$$

Proof. Following the above discussion, we have to estimate $\sum_{n>N} \frac{|a_n|^2}{n}$, where the a_n are characterized in Lemmas 4.20 and 4.13. Since these are decreasing, the results follows from the following elementary estimation:

$$\sum_{n>N} \frac{|a_n|^2}{n} \leq \int_N^\infty \frac{a(\xi)^2}{\xi} \mathrm{d}\xi,$$

where $a(\xi)$ is a monotonously continuous decreasing function such that $a(n) = |a_n|$. ∎

4.2.4. Multiple arcs approximation. Since the existence of more than one arc translates into perturbations of the Chebyshev coefficients with decay rates given by arc regularity, convergence rates for the case of multiple arcs are given by those of the single arc case. To see this, let us recall Problem 3.2 for the case of two $\mathcal{C}^m$-arcs pullbacked onto $\widehat{\Gamma}$: for $g_1,\, g_2 \in C^m(\widehat{\Gamma})$, find $\widehat{\lambda}_1, \widehat{\lambda}_2 \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ such that

$$\widehat{\mathcal{L}}_{11}[\kappa]\widehat{\lambda}_1 + \widehat{\mathcal{L}}_{12}[\kappa]\widehat{\lambda}_2 = \widehat{g}_1,$$
$$\widehat{\mathcal{L}}_{21}[\kappa]\widehat{\lambda}_1 + \widehat{\mathcal{L}}_{22}[\kappa]\widehat{\lambda}_2 = \widehat{g}_2.$$

By Assumption 2.2, the arcs cannot touch nor intersect. Hence, there is always $d > 0$ such that for all $(\mathbf{x}, \mathbf{y}) \in \Gamma_1 \times \Gamma_2$, $\|\mathbf{x} - \mathbf{y}\|_2 > d$. This leads to the next result.

LEMMA 4.22. *Let $m \in \mathbb{N}^*$ with $m > 2$ and consider two open $\mathcal{C}^m$-arcs fulfilling Assumption 2.2. Then, for $i = 1, 2$, there are $\lambda_i \in H^{-\frac{1}{2}}(\Gamma_i)$ such that*

$$\widehat{\lambda}_i = \sum a_n^i \frac{T_n}{w},$$

and, for $i \neq j$, it holds

$$(\widehat{\mathcal{L}}_{ij}[\kappa]\widehat{\lambda})_l = \sum_n b_{nl} a_n,$$

with asymptotic decay rates:

(i) if $m < \infty$, $b_{nl} = \mathcal{O}(\min\{n^{-m}, l^{-m}\})$;
(ii) if $m = \infty$, there is $\rho > 1$ such that $b_{nl} = \mathcal{O}\left(\rho^{\min\{-m,-l\}}\right)$.

Proof. As the distance between two disjoint arcs is strictly positive, the kernel $G_\kappa(\mathbf{r}_i(t), \mathbf{r}_j(s))$ is $\mathcal{C}^m$ and the proof follows verbatim from Lemma 4.11.

LEMMA 4.23. *Let $\kappa > 0$, $m \in \mathbb{N}^*$ with $m > 2$, and Γ be a family of $\mathcal{C}^m$ arcs. For $\mathbf{g} \in \mathcal{C}^m(\Gamma)$, let $\boldsymbol{\lambda}$ be the only solution of Problem 3.2. Then, for $\widehat{\lambda}_j := \frac{\lambda_j \circ \mathbf{r}_j}{\|\mathbf{r}_j'\|_2}$, with series expansion $\sum_{n \geq \mathbb{N}_0} a_n^j w^{-1} T_n$, it holds*

- *If $m < \infty$, $a_n^j = \mathcal{O}(n^{-m+1})$;*
- *If $m = \infty$, there is a $\rho > 1$, such that $a_n^j = \mathcal{O}(n\rho^{-n})$.*

Proof. The proof is similar to that of Lemma 4.20, now taking care of cross-interaction terms by Lemma 4.22 and using the same arguments from Lemma 4.13. ∎

THEOREM 4.24. *Let $\kappa \geq 0$, $m \in \mathbb{N}^*$ with $m > 2$, Γ a family of $\mathcal{C}^m$-arcs, $\mathbf{g} \in \mathcal{C}^m(\Gamma)$ and $\boldsymbol{\lambda}$ the only solution of Problem 3.2. Then, there exists $N_0 \in \mathbb{N}$ such that for every $N \in \mathbb{N} : N > N_0$ there is a unique $\boldsymbol{\lambda}_N$ solution of Problem 4.2 contructed with the discrete spaces detailed in Section 4.1. Moreover, the following convergence rates hold*

(i) If $m < \infty$, then

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) N^{-m+1};$$

(ii) If $m = \infty$, there exists $\rho > 1$ such that

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_N\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(\Gamma, \kappa) \rho^{-N+2}\sqrt{N}.$$

Proof. The proof follows that of Theorem 4.21, as the norm on $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ is equivalent to the Cartesian product of M times the space $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$, and the corresponding bounds for the coefficients are established in Lemma 4.23. ∎

Remark 4.25. In this section, we heavily used the fact that $f \in \mathcal{C}^m$ implies $f_l = O(l^{-m})$. However, this estimate is not optimal. For example, if $f^{(m+1)} \in L^1([-1,1])$ in distributional sense, then one can show that $f_l = \mathcal{O}(l^{-m-1})$ (*cf.* [33, Chapter 7]). This property is called bounded variation. Hence, if we require data not only to be $\mathcal{C}^m$-continuity but to also have bounded variation in their $(m+1)$-th derivatives, we gain one additional power in our convergence estimates.

5. Matrix computations. We now explicitly describe numerically how to solve the Problem 4.6 using the discrete spaces defined in Section 4.1. By definition (27), the matrix entries are

$$(\mathsf{L}_{ij}[\kappa])_{ln} = \left\langle \mathcal{L}_{ij}[\kappa] q_n^j, q_l^i \right\rangle_{\Gamma_i}. \tag{57}$$

In Remark 4.8, we showed that this can computed as

$$(\mathsf{L}_{ij}[\kappa])_{ln} = \left\langle \widehat{\mathcal{L}}_{ij}[\kappa] w^{-1} T_n, w^{-1} T_l \right\rangle_{\widehat{\Gamma}}. \tag{58}$$

For its implementation, we will distinguish between the cases when i and j are equal or not.

5.1. Case $i \neq j$. In this case, the kernel function associated with this operator is smooth, and consequently, we can expand it as a Chebyshev series using the FFT. To this end, we consider a two-dimensional version of the procedure presented in Section 4.1.1 :

(i) Evaluate the function $F(t,s) := G_\kappa(\mathbf{r}_i(t), \mathbf{r}_j(s))$ in a grid of Chebyshev points (t_i^N, s_j^N), obtaining a matrix $\mathsf{F} \in \mathbb{C}^{(N+1)\times(N+1)}$.

(ii) For each row, we follow steps (i) and (ii) of the one-dimensional procedure detailed in Section 4.1.1. This leads to the following expansion:

$$F(t,s) = \sum_{n \geq 0} a_n(s) T_n(t),$$

where the coefficients of the matrix are approximations at the Chebyshev points, i.e. $\mathsf{F}_{jn} \approx a_n(x_j^N)$, $n = 0, \ldots, N$.

(iii) We repeat the last step but with the columns of the new matrix F, i.e. the same one-dimensional procedure for the functions $a_n(s)$, $n = 0, \ldots, N$. The matrix F is updated such that $\mathsf{F}_{ln} \approx a_{ln}$, where

$$F(t,s) = \sum_{l \geq 0} \sum_{n \geq 0} a_{ln} T_l(s) T_n(t).$$

Notice that this procedure requires $2(N+1)$ FFTs. Once the expansion is obtained, the integrals are computed directly using the orthogonality property of Chebyshev polynomials.

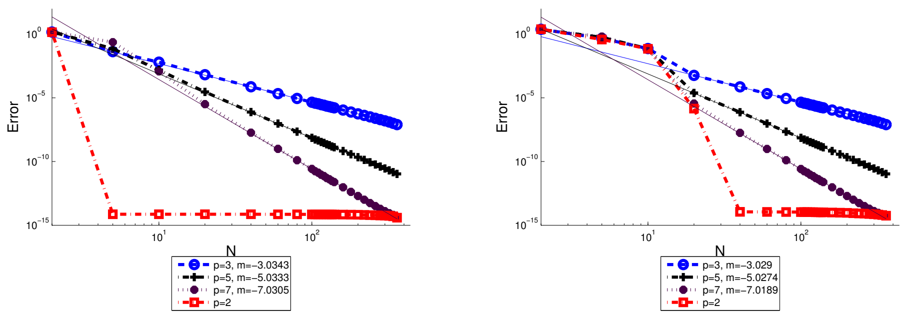
5.2. Case $i = j$. In this setting, we can extract the singularity by subtracting the purely logarithmic term:

$$R_k^i(t,s) := -\frac{1}{2\pi} \log|t-s| J_0(\kappa \left\| \mathbf{r}_i(t) - \mathbf{r}_i(s) \right\|_2), \tag{59}$$

and obtain two family of integrals:

$$I_{ln}^1 := \int_{-1}^{1} \int_{-1}^{1} (G_\kappa(\mathbf{r}_i(t), \mathbf{r}_i(s)) - R_k^i(t,s)) w^{-1} T_n(t) w^{-1} T_l(s) dt ds, \tag{60}$$

$$I_{ln}^2 := \int_{-1}^{1} \int_{-1}^{1} R_k^i(t,s) w^{-1} T_n(t) w^{-1} T_l(s) dt ds. \tag{61}$$



(a) Laplace (b) Helmholtz $\kappa = 10$

FIGURE 1. $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ *errors, for* $g(t) = |t|^p$. *Values* m *are slopes of* $\log_{10}(Error)$ *respect to* $\log_{10} N$. *Errors are computed with respect to an overkill solution with* $N = 440$.

Using the expansion [2, 9.1.13], we find that $G_\kappa(\mathbf{r}_i(t), \mathbf{r}_i(s)) - R^i_k(t, s)$ has the same regularity of $\mathbf{r}_i$, and thus we can compute I^1_{ln} as in the case $i \neq j$. For I^2_{ln}, we notice that $R^i_k(t, s)$ is a product of two functions: $-\frac{1}{2\pi} \log|t - s|$, with known Chebyshev expansion (21) and $J_0(\kappa \|\mathbf{r}_i(t) - \mathbf{r}_i(s)\|_2)$, which by [2, 9.1.12] has the same regularity of $\mathbf{r}_i$. Thus, its Chebyshev expansion can be computed using FFT. Finally, the Chebyshev expansion of $R^i_k(t, s)$ is computed using the technique shown in Lemma 4.18.

Remark 5.1. The evaluation of the Chebyshev expansion of $R^i_k(t, s)$ can be accelerated by extrapolation techniques like de-aliasing [14].

6. Numerical Results. In what follows, we show experimental results confirming the convergence rates proven in Theorem 4.24. Moreover, we show the quick computability of total fields, for different scenarios, by employing the FFT for the integral representation formula.

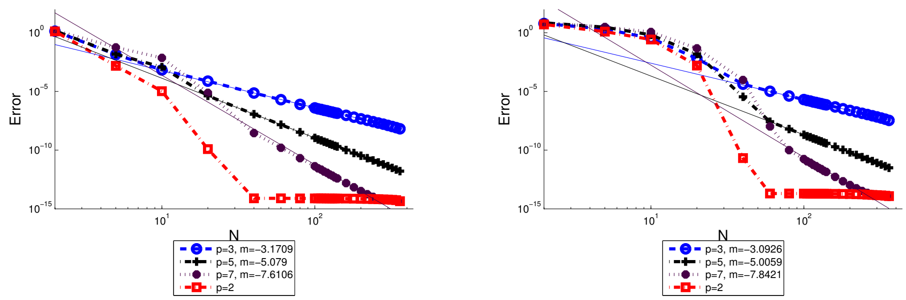
6.1. Convergence results. Let us first consider the case of a single arc $\widehat{\Gamma}$ and an excitation g with limited regularity. Figure 1 presents convergence results for different excitation functions. The first three are of the form $g(t) = |t|^p$, with $p = 3, 5, 7$. For these, g is at most in $\mathcal{C}^p(\widehat{\Gamma})$. Hence, by Theorem 4.24, we should observe the following error bounds:

$$Error := \|\lambda - \lambda_N\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} = \mathcal{O}(N^{-p+1}).$$

However, as discussed in Remark 4.25, the function g has bounded variation and so numerically it is equivalent to $g \in \mathcal{C}^{p+1}(\widehat{\Gamma})$ when interpreting Theorem 4.24. Thus, we have that the error as a function of N has a slope of p in logaritmic scale. The fourth case has as right-hand side $g(t) = t^2$, and, being an entire function, we observe the corresponding super-convergence.

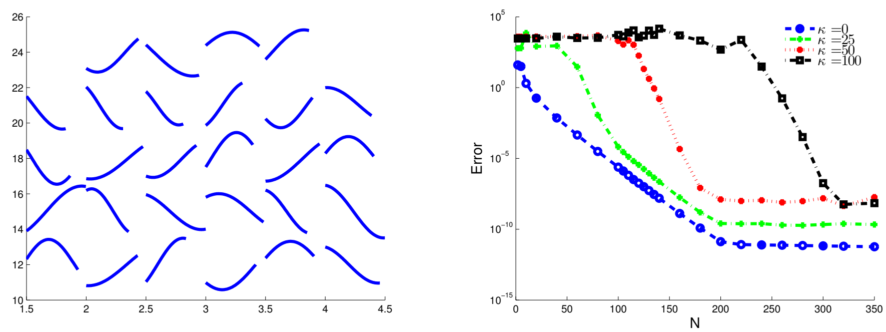
In Figure 2, we show results for geometries with limited regularity and smooth excitation. Just as in the case of excitation of limited regularity, we obtain the convergence rates stated in Theorem 4.24.

Lastly, we consider the case of multiple arcs and where the excitation function and the geometry are smooth (see Figure 3). We observe exponential convergence in the polynomial degree used per arc as predicted. We also observe that, as a function of κ,



(a) Laplace (b) Helmholtz $\kappa = 10$

FIGURE 2. $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ *errors, for* Γ *given by* $\mathbf{r}(t) = (t, |t|^p)$ *and* $g(t) = t^2$*. Values* m *are slopes of* $\log_{10}(Error)$ *respect to* $\log_{10} N$*. Errors are computed with respect to an overkill solution with* $N = 440$.



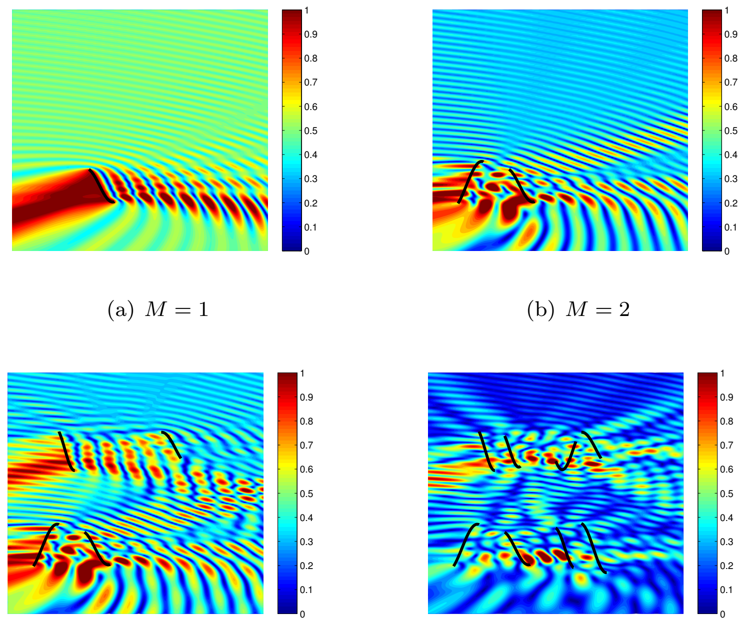
(a) Geometry (b) Convergence $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\widehat{\Gamma})$-norm

FIGURE 3. *In (a), a smooth geometry with* $M = 28$ *open arcs, each with a parametrization* $(at, c\sin(bt)) + cte$*, where* $a \in [0.45, 0.50]$*,* $b \in [1.0, 1.5]$*,* $c \in [1.0, 1.3]$*, and* $t \in [-1, 1]$*. In (b), convergence for the corresponding geometry and different wavenumbers using as right-hand side the trace of* $g(\mathbf{x}) = \exp -i\hat{k}\mathbf{x}\cdot\mathbf{y}$*, where* $\hat{k} = k$ *for* $k > 0$*,* $\hat{0} = 5$*,* $\mathbf{y} = (\cos\alpha, \sin\alpha)$*, and* $\alpha = \pi/4$*. The x-axis denotes the number of polynomials used per arc. Errors are computed with respect to an overkill solution with* $N = 500$ *per arc.*

the errors are increasingly bounded by below. Our experiments shows that this effect is caused by numerical pollution errors in the solution of the linear system, which is currently solved by a direct method. For the sake of brevity, we will not attempt to solve this anomaly, as it is a common issue when computing waves scattered by disjoint domains (*cf.* [10]).

6.2. Field plots. Once the density $\boldsymbol{\lambda}$ in Problem 3.2 is approximated by the solution $\boldsymbol{\lambda}_N$ of the discrete Problem 4.2, the field solution U of Problem 3.1 can be also approximated by

$$U_N = \sum_{j=1}^{M} \mathsf{SL}_j[\kappa]\boldsymbol{\lambda}_N^j, \tag{62}$$



(a) $M = 1$ (b) $M = 2$

(c) $M = 4$ (d) $M = 8$

FIGURE 4. *Total volume solution intensity for different arc numbers. Polynomial order per arc was to set* $N = 150$ *for an incident plane wave,* U_{inc}*, with* $\kappa = 10$ *in the direction* $(1,1)$*. Color scales have been normalized.*

where the λ_N^j are the discrete solution components corresponding to arcs Γ_j. Evaluations of the potential SL_j acting over the discrete basis in points not in Γ_j can be obtained easily using the FFT as they are smooth functions –similar to the computations in Section 5. Once these are computed, the volume solution is reconstructed by doing a matrix-vector product with the coefficients vector of the solution in the discrete basis.

In Figures 4(a), 4(b), 4(c) and 4(d) show the normalized absolute value of the total volume fields for four different geometrical configurations. The total field is defined as $U_N + U_{inc}$, where U_{inc} is a plane wave with fixed wavenumber and whose trace constitutes the excitation $\mathbf{g}$. The number of curves M are 1,2,4 and 8, respectively, corresponding to subsets of the curves presented in Figure 3

7. Concluding remarks. The present work presents a high-order discretization method for the wave scattering by multiple disjoint arcs based on weighted polynomials bases with proven convergence rates similar to the classical interpolation theory of smooth functions. As an efficient solver for the forward problem, our method could be easily used for solving optimization or inverse problems, tasks which are currently under development. Still, for increasing frequencies and numbers of arcs, we remark that the solution of the resulting linear system can become a bottleneck, thus requiring further improvements.

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Appendix A. Technical Lemmas.

Lemma A.1. *The discretization (25) is conforming, i.e.* $\mathbb{Q}_N(\Gamma_i) \subset \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ *(resp.* $\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i) \subset \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$*).*

Proof. For any $\zeta^i \in \mathbb{Q}_N(\Gamma_i)$, by (24), the representation:

$$\zeta^i = \frac{\widehat{p} \circ \mathbf{r}_i^{-1}}{w_i \left\| \mathbf{r}_i' \circ \mathbf{r}_i^{-1} \right\|_2}, \tag{63}$$

holds, where $\widehat{p} \in \mathbb{P}_N(\widehat{\Gamma})$. By definition of dual norms, one can write

$$\left\|\zeta^i\right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} = \sup_{\vartheta \in H^{\frac{1}{2}}(\Gamma_i)} \frac{\left\langle \zeta^i, \vartheta \right\rangle_{H^{\frac{1}{2}}(\Gamma_i)}}{\left\|\vartheta\right\|_{H^{\frac{1}{2}}(\Gamma_i)}}.$$

At the same time, it holds

$$\begin{aligned} \left\langle \zeta^i, \vartheta \right\rangle_{\Gamma_i} = \int_{\widehat{\Gamma}} \frac{\widehat{p}(t)}{\sqrt{1-t^2}} (\vartheta \circ \mathbf{r}_i)(t) dt &\le \|\widehat{p}\|_{L^\infty(\widehat{\Gamma})} \int_{\widehat{\Gamma}} \frac{(\vartheta \circ \mathbf{r}_i)(t)}{w(t)} dt \\ &\le \|\widehat{p}\|_{L^\infty(\widehat{\Gamma})} \left\| w^{-1} \right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} \left\| \vartheta \circ \mathbf{r}_i \right\|_{H^{\frac{1}{2}}(\widehat{\Gamma})}, \end{aligned} \tag{64}$$

where $w(t) := \sqrt{1-t^2}$. Applying Lemma 2.3, we only need to check that the $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$-norm of w^{-1} is finite, which was already proved in [16, Lemma 6.1.19]. The inclusion for the mean-zero spaces is immediate from the Chebyshev polynomials' orthogonality property. ∎

Lemma A.2. *The family* $\left\{\mathbb{Q}_N(\Gamma_i)\right\}_{N\in\mathbb{N}_0}$ *is dense in* $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$*, while* $\left\{\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)\right\}_{N\in\mathbb{N}}$ *is dense in* $\widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$.

Proof. We only need to prove that there is a fix constant C such that, for a given $\epsilon > 0$ and $\phi \in \mathcal{D}(\Gamma_i)$, there exists $\zeta^i \in \mathbb{Q}_N(\Gamma_i)$ satisfying

$$\left\|\zeta^i - \phi\right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq C\epsilon.$$

By [16, Lemma 6.1.20], there exists a polynomial $\widehat{p} \in \mathbb{P}_N(\widehat{\Gamma})$ satisfying

$$\left\|w^{-1}\widehat{p} - \|\mathbf{r}_i'\|_2 \,(\phi \circ \mathbf{r}_i)\right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} < \epsilon.$$

Let $\zeta^i = \dfrac{\widehat{p} \circ \mathbf{r}_i}{w_i \left\|\mathbf{r}_i' \circ \mathbf{r}_i^{-1}\right\|_2}$. Again, we take the dual norm

$$\left\|\zeta^i - \phi\right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} = \sup_{\vartheta \in H^{\frac{1}{2}}(\Gamma_i)} \frac{\left\langle \zeta^i - \phi, \vartheta \right\rangle_{\Gamma_i}}{\|\vartheta\|_{H^{\frac{1}{2}}(\Gamma_i)}}.$$

However, we can write

$$\begin{aligned} \left\langle \zeta^i - \phi, \vartheta \right\rangle_{\Gamma_i} &= \int_{\Gamma_i} (\zeta^i - \phi)(\mathbf{x})\vartheta(\mathbf{x})\mathrm{d}\Gamma_i(\mathbf{x}) \\ &= \int_{\widehat{\Gamma}} \left(w^{-1}(t)\widehat{p}(t) - \|\mathbf{r}_i'\|_2 \,(t)(\phi \circ \mathbf{r}_i)(t)\right)(\vartheta \circ \mathbf{r}_i)(t)dt. \end{aligned} \tag{65}$$

By Lemma 2.3, there exists a constant C independent of ϵ such that

$$\left\langle \zeta^i - \phi, \vartheta \right\rangle_{\Gamma_i} \leq C \|\vartheta\|_{H^{\frac{1}{2}}(\Gamma_i)} \left\|w^{-1}\widehat{p} - \|\mathbf{r}_i'\|_2 \,(\phi \circ \mathbf{r}_i)\right\|_{\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})} \leq C\epsilon \|\vartheta\|_{H^{\frac{1}{2}}(\Gamma_i)},$$

and thus $\left\|\zeta^i - \phi\right\|_{\mathcal{H}^i} \leq C\epsilon$ as stated.

For the family $\left\{\mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)\right\}_{N\in\mathbb{N}}$, by the previous result, we observe that, given $\phi \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$ and $\epsilon > 0$. there exists $N \in \mathbb{N}$ and $\zeta^i \in \mathbb{Q}_N(\Gamma_i)$, such that

$$\left\|\zeta^i - \phi\right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq \epsilon. \tag{66}$$

Thus, by the definition of the norm in $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$, it holds

$$\left\langle \zeta^i, 1 \right\rangle_{\Gamma_i} = \left\langle \zeta^i - \phi, 1 \right\rangle_{\Gamma_i} \leq \left\|\zeta^i - \phi\right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)},$$

Hence, we can define $\zeta_0^i := \zeta^i - |\Gamma_i|^{-1}\left\langle \zeta^i, 1 \right\rangle_{\Gamma_i}$, where $|\Gamma_i|$ is the length of the arc Γ_i. Now, it is direct that $\zeta_0^i \in \mathbb{Q}_{N,\langle 0\rangle}(\Gamma_i)$ and

$$\left\|\zeta_0^i - \phi\right\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)} \leq 2\epsilon,$$

which gives the desired density. ∎

A.1. Some properties of Chebyshev polynomials. The next two identities follow directly from the explicit definition of Chebyshev polynomials as $T_n(t) = \cos(n \arccos(t))$.

LEMMA A.3. *For $n, k \in \mathbb{N}_0$, let T_n and T_k denote two Chebyshev polynomials of first kind. Then,*

TABLE 1
Coefficients used in Lemma A.3.

	$\beta_n^{(-1)}$	$\beta_n^{(0)}$
$n=0$	$\frac{1}{4}a_0$	$a_0 - \frac{1}{2}a_1$
$n=1$	$-a_0 + \frac{1}{4}a_1$	$-a_0 + \frac{5}{4}a_1 - \frac{1}{2}a_2$
$n=2$	$\frac{1}{2}a_0 - \frac{1}{2}a_1 + \frac{1}{4}a_2$	$-\frac{1}{2}a_1 + a_2 - \frac{1}{2}a_3$
$n\geq 3$	$\frac{1}{4}a_{n-2} - \frac{1}{2}a_{n-1} + \frac{1}{4}a_n$	$-\frac{1}{2}a_{n-1} + a_n - \frac{1}{2}a_{n+1}$

$$T_nT_k = \frac{1}{2}(T_{n+k} + T_{|n-k|}). \tag{67}$$

Moreover, for $(t,s)\in[-1,1]^2$*, it holds*

$$|t-s|^2 = 1 + \frac{1}{2}\left(T_2(t)+T_2(s)\right) - 2T_1(t)T_1(s). \tag{68}$$

LEMMA A.4. *Consider a function of the form:*

$$U(t,s) = \sum_{n=0}^{\infty} a_nT_n(t)T_{|n-k|}(s).$$

Then,

$$|t-s|^2U(t,s) = \sum_{j\in\{-1,0,1\}}\sum_{n=0}^{\infty}\beta_n^{(j)}T_n(t)T_{|n-k+2j|}(s),$$

wherein

$$\beta_n^{(1)} := \frac{1}{4}a_n - \frac{1}{2}a_{n+1} + \frac{1}{4}a_{n+2},$$

and coefficients $\beta_n^{(-1)}$ *and* $\beta_n^{(0)}$ *are given in Table 1 for* $n\in\mathbb{N}_0$*.*

Proof. Using Lemma A.3, we have that

$$\begin{aligned}|t-s|^2U(t,s) = \sum_{n=0}^{\infty} a_n\Big(T_n(t)T_{|n-k|}(s) &+ \frac{1}{4}T_{n+2}(t)T_{|n-k|}(s) + \frac{1}{4}T_{|n-2|}(t)T_{|n-k|}(s)\\ &+\frac{1}{4}T_n(t)T_{||n-k|+2|} + \frac{1}{4}T_n(t)T_{|n-k-2|}\\ &-\frac{1}{2}[T_{||n-k|+1|}(s)+T_{||n-k|-1|}(s)][T_{|n-1|}(t)+T_{n+1}]\Big)\end{aligned}$$

Observe that, for $i\in\{1,2\}$, the index sums

$$|n-k|+i = \begin{cases}|n-k+i| & n\geq k,\\ |n-k-i| & n<k,\end{cases} \qquad ||n-k|-i| = \begin{cases}|n-k-i| & n\geq k,\\ |n-k+i| & n<k.\end{cases} \tag{69}$$

Employing this in writing $|t-s|^2U(t,s)$ as a series expansion, we find expressions for different $u_n(s)$:

$$u_0 = \frac{a_0}{4}T_{|k+2|}(s) + \left(a_0 - \frac{a_1}{2}\right)T_{|k|}(s) + \left(\frac{a_0}{4} - \frac{a_1}{2} + \frac{a_2}{4}\right)T_{|k-2|}(s)$$

$$u_1 = \left(-a_0 + \frac{a_1}{4}\right) T_{|k+1|}(s) - \left(a_0 + \frac{5a_1}{4} + \frac{a_2}{2}\right) T_{|1-k|}(s) + \left(\frac{a_1}{4} - \frac{a_2}{2} + \frac{a_3}{4}\right) T_{|k-3|}(s)$$

$$u_2 = \left(\frac{a_0}{2} - \frac{a_1}{2} + \frac{a_2}{4}\right) T_{|k|}(s) - \left(\frac{a_1}{2} - a_2 + \frac{a_3}{2}\right) T_{|k-2|}(s) + \left(\frac{a_2}{4} - \frac{a_3}{2} + \frac{a_4}{4}\right) T_{|k-4|}(s)$$

and

$$u_n = \left(\frac{a_{n-2}}{4} - \frac{a_{n-1}}{2} + \frac{a_n}{4}\right) T_{|n-k-2|}(s) + \left(-\frac{a_{n-1}}{2} + a_n - \frac{a_{n+1}}{2}\right) T_{|n-k|}(s) + \left(\frac{a_n}{4} - \frac{a_{n+1}}{2} + \frac{a_{n+2}}{4}\right) T_{|n-k+2|}(s)$$

for $n \geq 3$, yielding the stated result. ∎