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WELL-POSEDNESS OF HELMHOLTZ AND LAPLACE PROBLEMS IN UNBOUNDED DOMAINS WITH MULTIPLE SCREENS[*]

CARLOS JEREZ-HANCKES[†] AND JOSÉ PINTO[‡]

Abstract. We present proofs for the existence and uniqueness of solutions of Helmholtz and Laplace problems in unbounded domains with either Dirichlet or Neumann boundary conditions on finite collections of open arcs (in 2D) or screens (3D). This extends existent results for a single arc/screen and provides a constructive solution strategy based on boundary integral operators that is shown to apply to more general second order coercive equations.

Key words. Laplace equation, Helmholtz equation, boundary integral equations, screen problems, crack problems

AMS subject classifications. 65R20, 31A10, 45F05

1. Introduction. We study Laplace and time-harmonic wave equations in unbounded domains whose boundaries are a finite collection of disjoint open finite-length arcs or bounded screens when in two- or three-dimensional space, respectively. Since such problems arise in multiple applications, an extensive survey is impractical and we simply mention a few. In structural and mechanical engineering, fractures or cracks are represented as either screens (3D) or slits (2D) inside the volume domain [41, 42, 6, 28, 1, 5, 38]. Similar models are used in antenna design and acoustic engineering, wherein wave scattered by such structures is analyzed [24, 25]. Equivalently, thin apertures on unbounded screens can be reformulated into open arcs [44] or screens [27]. Other applications can be found in medicine such as imaging [2] or muscular strain detection due to sport injuries [43].

From a theoretical point of view, the main challenge is brought by solving partial differential equations on unbounded non-Lipschitz domains. One usually reformulates the infinite volume problem onto the domain boundaries (arcs or screens) via Boundary Integral Equations (BIEs). This leads to the appearance of Boundary Integral Operators (BIOs) with singular kernels whose analysis can be performed under different functional spaces depending on the regularity of solutions. More precisely, one could opt for either a classical approach based on Hölder continuous spaces, denoted $C^{m,\lambda}(\Gamma)$ with $m \in \mathbb{N}_0$ and $\lambda \in (0,1)$, or the relatively more recent L^2-based Sobolev or energy space setting, $H^s(\Gamma)$ for $s \in \mathbb{R}$ and their duals [34]. Heavily relying on complex analysis, the former has delivered many important results for two-dimensional problems over single and multiple open arcs such as solutions for Riemann-Hilbert problems and the Plemelj-Privalov theorem; we refer to the excellent monographs [32, 7] for detailed explanations. Notwithstanding, the Hölder framework is highly restrictive: it requires surface densities to be defined point-wise and thus losing connection to their physical interpretation, while for higher-dimensions results cannot be easily extended. On the other hand, there are special cases where the Hölder or related framework of continuous functions is able to find solutions while weak ones may

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not exist [26]. Hence, we will pursue our analysis in the Sobolev space framework.

For a single arc and screen, we highlight the work by Stephan *et al.* [38, 39] as the present contribution can be seen as extensions of their ideas to multiple screens. Therein, the authors consider arcs/screens than can be seen as parts of closed curves/surfaces enclosing Lipschitz domains as they allow the use of Costabel's continuity and coercivity results [11] for Lipschitz domains. Similar assumptions have been used for mixed problems [34] and for scattering by multiple connected screens in 3D [10]. Still, and to our knowledge, the first rigorous study of screens can be traced to Feld [13], though similar screen and aperture problems were already analyzed before [27]. From there, many authors have analyzed the problem introducing: extensions of the boundary integral formulations, characterizations of the operators spectra, and numerical discretization schemes (*cf.* Shestopalov *et al.* [35, 36]).

More recently, problems dealing with open arcs or screens have continued to receive attention for different aims. For instance, as the arising first kind BIEs yield ill-conditioned low-order Galerkin matrices, several preconditioning ideas have been put forward. For instance, strategies based on opposite order operator [31] and recent extensions of Calderón identities in two [20, 21] and three dimensions [19, 18, 33] have dramatically improved iteration counts for iterative solvers. In parallel, Bruno and collaborators [29, 9, 8] have contributed with similar ideas for Nyström-type strategies. Another line of research concerns the choice of approximation bases. Specifically, for Dirichlet problems with sufficiently smooth arcs/screens Γ, the solutions of the BIE –Neumann jumps– display singularities at the boundaries $\partial\Gamma$ of the form $\mathcal{O}(\mathrm{dist}(\mathbf{x},\Gamma)^{-1/2})$ [12, 14, 23]. Hence, standard low-order methods tend to fail at capturing these singularities if not properly meshed or if not embedded with bases portraying such behavior. Consequently, several methods have been proposed to either augment the standard discretization basis [40] or to include higher-order polynomials, portraying improved convergence rates [22, 16].

Yet, and to our knowledge, the question of existence and uniqueness of multiple arcs/screens has not been clearly stated, and to this we dedicate the present manuscript. In fact, the well-posedness result will be key for a series of works that dealing with the numerical resolution of both forward and inverse problem versions. Our paper is organized as follows. Section 2 puts forward formal definitions and properties needed throughout. In particular, uniqueness of the volume problem is given in Section 3. We then formulate the problem as a BIE (*cf.* Section 4) and provide conditions under which it can be uniquely solved, and how the volume solution can be recovered. Finally, Section 5 addresses the extensions to general second-order differential problems in two and three dimensions as well as the Neumann problem.

2. Matematical Tools.

2.1. General Notation. Extended integers are denoted by $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ and non-negative ones by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We employ the standard $\mathcal{O}(\cdot)$, and $o(\cdot)$ notation for asymptotics. Vectors are indicated by boldface symbols with Euclidean norm written as $\|\cdot\|_2$ while other norms are indicated by subscripts.

Let $G \subseteq \mathbb{R}^d$, for $d = 1, 2, 3$, be an open domain. The spaces $\mathcal{C}^k(G)$, for $k \in \mathbb{N}_0$, denote the set of continuous functions over G, along with their k derivatives. Compactly supported $\mathcal{C}^k(G)$ functions are designated by $\mathcal{C}^k_0(G)$. Let $\mathcal{D}(G) \equiv \mathcal{C}^\infty_0(G)$ be the space of infinitely differentiable functions with compact support on a non-empty measurable set G. Its dual, the so-called distributional space, is denoted by $\mathcal{D}^*(G)$.

The class of p-integrable functions over G is written $L^p(G)$. Duality pairings are denoted by $\langle\cdot,\cdot\rangle$ with subscripts indicating the domain involved, if not clear from the context. Similarly, inner products are written as $(\cdot,\cdot)$, only requiring integration domains as subscript. Finally, quantities defined over volume domains will be written in capital case whereas those on boundaries in small one, e.g., $U \in G$ while $u \in \partial G$.

2.2. Geometry. As explained, we carry out our analysis in two-dimensional space, leaving definitions for $\mathbb{R}^3$ extensions in Section 5. We denote by the canonical segment by $\widehat{\Gamma} := (-1,1)$. First, we formalize the notion of open arcs as follows:

Definition 2.1 (Jordan arc). *We say that $\Lambda \subset \mathbb{R}^2$ is a regular Jordan arc of class $\mathcal{C}^m$, for $m \in \mathbb{N}^*$, if there exists a bijective parametrization denoted by $\mathbf{r} = (r_1, r_2)$, such that its components are $\mathcal{C}^m(\widehat{\Gamma})$-functions; $\mathbf{r} : \overline{\widehat{\Gamma}} \to \overline{\Lambda}$ and $\|\mathbf{r}'(t)\|_2 > 0$, $\forall\ t \in \overline{\widehat{\Gamma}}$.*

Directly from this definition follows that $\overline{\Lambda}$ has to be a closed topological set in $\mathbb{R}^2$ while being open geometrically, i.e. not having self-intersection points, or equivalently $\partial\Lambda \neq \emptyset$. We also define the normal vector of an arc as $\mathbf{n} = (-r_2', r_1')/\|\mathbf{r}'\|_2$. For the class of problems of interest, we will assume the following condition on Jordan arcs.

Assumption 2.2. For any Λ regular Jordan arc of class $\mathcal{C}^m$, there exists an extension of Λ to $\tilde{\Lambda}$, with a $\mathcal{C}^m$-parametrization $\tilde{\mathbf{r}} : [0, 2\pi] \to \tilde{\Lambda}$, that is bijective in $[0, 2\pi)$ and satisfies $\tilde{\mathbf{r}}(0) = \tilde{\mathbf{r}}(2\pi)$ and $\|\tilde{\mathbf{r}}'(t)\|_2 > 0$, $\forall\ t \in [0, 2\pi]$. We also assume that the extension has the same orientation than Λ, and the normal vector of Λ is a restriction of the exterior normal of $\tilde{\Lambda}$.

We will also define $\Lambda^c := \tilde{\Lambda} \setminus \overline{\Lambda}$.

Lemma 2.3. *Let Λ be a regular Jordan arc of class $\mathcal{C}^m$. Then, there exists a bounded domain D_Λ with boundary $\tilde{\Lambda}$, that is also of class $\mathcal{C}^m$.*

Proof. As existence follows from the Jordan curve theorem [15], we only need to prove the smoothness of the domain. For a given $\mathbf{x} \in \tilde{\Lambda}$, as $\|\tilde{\mathbf{r}}'\|_2 > 0$ we can find an $\epsilon > 0$ and a function $g \in \mathcal{C}^m(\mathbb{R})$ such that $D_\Lambda \cap B_\epsilon(\mathbf{x})$ is the hypograph of g by the implicit function theorem. Finally, we can find a finite cover of $\tilde{\Lambda}$ with open sets of the form $B_\epsilon(\mathbf{x})$, because $\tilde{\Lambda}$ is compact. ◻

Now consider a finite number $M \in \mathbb{N}$ of arcs $\{\Gamma_i\}_{i=1}^M$, each one of class $\mathcal{C}^m$, $m \geq 1$, such that their closures $\{\tilde{\Gamma}_i\}_{i=1}^M$, defined as in Assumption 2.2, are disjoint and set

$$\Gamma := \bigcup_{i=1}^{M} \Gamma_i \quad \text{and} \quad \Omega := \mathbb{R}^2 \setminus \overline{\Gamma}.$$

Clearly, Ω is an open unbounded non-Lipschitz domain. Furthermore, by Lemma 2.3, for each Γ_i there is at least one domain Ω_i whose boundary contains Γ_i. For $i \in \{1, \dots, M\}$, let us also define the complement domains $\Omega_i^c := \mathbb{R}^2 \setminus \overline{\Omega}_i$ If the arcs are disjoint, the domains Ω_i can also be disjointly selected. In fact, this will be our next working assumption.

Assumption 2.4. The M domains Ω_i originated by arcs Γ_i are mutually disjoint for all $i = 1, \dots, M$, i.e. $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ for $i \neq j$.

2.3. Sobolev spaces. Let $G \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, be an open domain. For $s \in \mathbb{R}$, we denote by $H^s(G)$ the standard Sobolev spaces and by $H^s_{loc}(G)$ their local integrable counterparts [34, Section 2.3]. We also define

$$H^s_\Delta(G) := \{U \in H^s(G) : \Delta U \in L^2(G)\}, \tag{1}$$

$$H^s_{\Delta,loc}(G) := \{U \in H^s_{loc}(G) : \Delta U \in L^2_{loc}(G)\}. \tag{2}$$

As in [21, Section 2.3], for any Lipschitz open arc Λ that can be extended to a closed curve $\tilde{\Lambda}$, we define tilde spaces $\widetilde{H}^s(\Lambda)$ as

$$\widetilde{H}^s(\Lambda) := \{u \in \mathcal{D}^*(\Lambda) : \tilde{u} \in H^s(\tilde{\Lambda})\}, \quad s > 0, \tag{3}$$

where $\tilde{u}$ denotes the extension by zero of u to $\widetilde{\Lambda}$. For $s > 0$ we can identify

$$\widetilde{H}^{-s}(\Lambda) = (H^s(\Lambda))^*, \quad \text{and} \quad H^{-s}(\Lambda) = (\widetilde{H}^s(\Lambda))^*. \tag{4}$$

We will also need the family zero-mean Sobolev spaces:

$$\widetilde{H}^s_{\langle 0 \rangle}(\Lambda) = \{u \in \widetilde{H}^s(\Lambda) : \langle u, 1 \rangle = 0\}. \tag{5}$$

Particular attention will be paid to the case $s = \frac{1}{2}$. The Sobolev-Slobodeckii definition is

$$H^{\frac{1}{2}}(\Lambda) = \left\{ u \in L^2(\Lambda) : \int_\Lambda \int_\Lambda \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|_2^2} \mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} < \infty \right\}. \tag{6}$$

This space can also be characterized in the following way [4, Section 5.1]:

$$H^{\frac{1}{2}}(\Lambda) = \left\{ u \in L^2(\Lambda) : \exists\ v \in H^{\frac{1}{2}}(\tilde{\Lambda}),\ v|_\Lambda = u \right\}, \tag{7}$$

with norm

$$\|u\|_{H^{\frac{1}{2}}(\Lambda)} = \inf_{\{v \in H^{\frac{1}{2}}(\tilde{\Lambda})\ :\ v|_\Lambda = u\}} \|v\|_{H^{\frac{1}{2}}(\tilde{\Lambda})}. \tag{8}$$

2.3.1. Norm Equivalences. For our forthcoming analysis, we require the following technical lemmas concerning $H^{\pm\frac{1}{2}}$- and $\widetilde{H}^{\pm\frac{1}{2}}$-spaces whose proofs are provided in Section A.

LEMMA 2.5. *If $u \in \widetilde{H}^{-\frac{1}{2}}(\Lambda)$ then $\tilde{u} \in H^{-\frac{1}{2}}(\tilde{\Lambda})$.*

LEMMA 2.6. *For $u \in \widetilde{H}^{-\frac{1}{2}}(\Lambda)$, it holds $\|\tilde{u}\|_{H^{-\frac{1}{2}}(\tilde{\Lambda})} = \|u\|_{\widetilde{H}^{-\frac{1}{2}}(\Lambda)}$.*

Though the following result is not used for the current analysis, it will turn to be helpful to the error estimation of numerical methods and we include it for completeness of exposition.

LEMMA 2.7. *Let $\zeta \in H^{\frac{1}{2}}(\Gamma_i)$ and recall that $\widehat{\Gamma} := (-1, 1) \times \{0\}$ with $\mathbf{r}_i : \widehat{\Gamma} \to \Gamma_i$. Then, the following bounds hold*

$$C_1 \|\zeta \circ \mathbf{r}_i\|_{H^{\frac{1}{2}}(\widehat{\Gamma})} \leq \|\zeta\|_{H^{\frac{1}{2}}(\Gamma_i)} \leq C_2 \|\zeta \circ \mathbf{r}_i\|_{H^{\frac{1}{2}}(\widehat{\Gamma})}, \tag{9}$$

where $C_1 = \min\left\{ \left\| \|\mathbf{r}_i' \circ \mathbf{r}_i^{-1}\|_2^{-1} \right\|_{L^\infty(\Gamma_i)}^{-\frac{1}{2}}, 1 \right\}$ *and* $C_2 = \max\left\{ \|\mathbf{r}_i'\|_{L^\infty(\widehat{\Gamma})}^{\frac{1}{2}}, 1 \right\}$.

We can conclude from Lemma 2.7 that the spaces $H^{\frac{1}{2}}(\Gamma_i)$ and $H^{\frac{1}{2}}(\widehat{\Gamma})$ are isomorphic as well as their duals $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ and $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$.

2.3.2. Cartesian Product Spaces. For the finite union of disjoint open arcs Γ, as defined in Section 2.2, we define piecewise spaces as

$$\mathbb{H}^s(\Gamma) := \{u \in \mathcal{D}^*(\Gamma) : u|_{\Gamma_i} \in H^s(\Gamma_i),\ i = 1, \dots, M\}. \tag{10}$$

From this definition, the identification:

$$\mathbb{H}^s(\Gamma) \cong H^s(\Gamma_1) \times H^s(\Gamma_2) \times \cdots \times H^s(\Gamma_M)$$

follows. The norm and dual products are naturally extended by the previous identification. The spaces $\widetilde{\mathbb{H}}^s(\Gamma)$ and $\widetilde{\mathbb{H}}^s_{\langle 0\rangle}(\Gamma)$ are defined in similar fashion, imposing their components to be in $\{\widetilde{H}^s(\Gamma_j)\}_{j=1}^M$ and $\{\widetilde{H}^s_{\langle 0\rangle}(\Gamma_j)\}_{j=1}^M$, respectively. Also, spaces $\mathbb{H}^s(\widehat{\Gamma})$ are to be understand as the Cartesian product $\prod_{i=1}^M H^s(\widehat{\Gamma})$.

For unbounded domains it is customary to use local Sobolev spaces. Notwithstanding, as these are not Hilbert spaces, we rather use weighted spaces as in [21, Section 2.5][1]

$$W(G) := \left\{ U \in \mathcal{D}^*(G) : \frac{U(\mathbf{x})}{\sqrt{1 + \|\mathbf{x}\|_2^2}\log(2 + \|\mathbf{x}\|_2^2)} \in L^2(G), \nabla U \in L^2(G) \right\}, \tag{11}$$

where G is assumed to be an unbounded open domain in $\mathbb{R}^2$.

Lemma 2.8. *The inclusion $W(G) \subset H^1_{loc}(G)$ holds.*

Proof. For any compact set $K \subset \mathbb{R}^d$ there is a finite $R \in \mathbb{R}_+$, with $R > \log 2$ such that

$$\log 2 < \left(1 + \|\mathbf{x}\|_2^2\right)^{\frac{1}{2}} \log(2 + \|\mathbf{x}\|_2^2) < R,$$

for every $\mathbf{x} \in \mathbb{R}^2$ in the given compact, thus bounding the H^1-norm in K. ∎

2.4. Dirichlet and Neumann traces. Given the family of arcs $\{\Gamma_i\}_{i=1}^M$ defined as in Section 2.2, we will need a Dirichlet trace operator γ_i over each Γ_i. Again, we recall that the three-dimensional version will be given in Section 5. In general, there are two possible versions of this operator. For a given arc Γ_i, consider the induced bounded domain Ω_i with boundary $\widetilde{\Gamma}_i = \partial\Omega_i$ as defined in Section 2.2. Then, for $U \in \mathcal{C}^\infty(\mathbb{R}^2)$ we can set

$$\widetilde{\gamma}_i^{\pm} = U(\mathbf{x}) := \lim_{\epsilon \uparrow 0} U(\mathbf{x} \pm \epsilon \mathbf{n}_i), \quad \forall\, \mathbf{x} \in \widetilde{\Gamma}_i, \tag{12}$$

where $\mathbf{n}_i$ denotes the outward unitary normal vector to the closed curve $\tilde{\Gamma}_i$. We will denote by $\gamma_i^{\pm}$ the restriction to Γ_i of the operator $\widetilde{\gamma}^{\pm}$, i.e. $\gamma_i^{\pm} U := \widetilde{\gamma}_i^{\pm} U|_{\Gamma_i}$. These definitions can be extended from continuous functions to more general Sobolev spaces by density. In particular, we have

Lemma 2.9. *For $i = 1, \dots, M$, the operators $\gamma_i^{\pm} : W(\Omega) \to H^{\frac{1}{2}}(\Gamma_i)$ are bounded.*

Proof. As $W(\Omega) \subset H^1_{loc}(\Omega)$ (*cf.* Lemma 2.8), by Theorem 2.21 in [37] we have that $\widetilde{\gamma}_i^{\pm} : W(\Omega) \to H^{\frac{1}{2}}(\widetilde{\Gamma}_i)$. Using the definition of Sobolev spaces on a subset as in [17], which implies the continuity of the restriction operator, we derive the stated mapping property. ∎

[1]For the sake of clarity, we have dropped superindices $W^{1,-1}$ used in the original work. There will no need to introduce a three-dimensional version of these spaces as explained in Section 5.

In the following, we will denote by $\gamma_{i^c}^{+}$ (resp. $\gamma_{i^c}^{-}$) the restriction to the arc Γ_i^c of the interior (resp. exterior) Dirichlet trace with respect to the domain Ω_i. Moreover, we define Neumann traces for smooth functions as

$$\widetilde{\gamma}_{\mathrm{N},i}^{\pm} U := \lim_{\epsilon \uparrow 0} \mathbf{n}_i \cdot \nabla U(\mathbf{x} \pm \epsilon \mathbf{n}_i), \quad \forall\ \mathbf{x} \in \widetilde{\Gamma}_i. \tag{13}$$

For any function $U \in H^1_{\Delta,loc}(\Omega_i^c) \cup H^1_{\Delta}(\Omega_i)$, its Neumann traces can be extended to linear mappings with range in $H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$ (*cf.* [34, Section 2.7], [30, Lemma 4.3]). We will also require the following result:

Lemma 2.10. *Let $F \in L^2_{loc}(\Omega)$, and $U \in W(\Omega)$ such that $\Delta U = F$ in Ω. Then, the normal traces $\widetilde{\gamma}_{N,i}^{\pm} U$ belong to $H^{-\frac{1}{2}}(\partial\Omega_i)$.*

Proof. By Lemma 2.8, it is direct that $U \in H^1_{\Delta,loc}(\Omega)$ Hence, as $\Omega_i \subset \Omega$ we get that $U \in H^1_{\Delta,loc}(\Omega_i^c) \cup H^1_{\Delta}(\Omega_i)$ and the result follows from the Neumann trace definition (13). □

For a function U defined as in the previous lemma, we will denote by $\gamma_{\mathrm{N},i}^{\pm} U$ the restriction of $\widetilde{\gamma}_{\mathrm{N},i}^{\pm} U$ to Γ_i and by $\gamma_{\mathrm{N},i^c}^{\pm} U$ the restriction to Γ_i^c. With these definitions, we can finally define the Neumann trace jumps on parts of the boundary:

$$[\gamma_{\mathrm{N}} U]_i := \gamma_{\mathrm{N},i}^{+} U - \gamma_{\mathrm{N},i}^{-} U \quad \text{on } \Gamma_i, \tag{14}$$

$$[\gamma_{\mathrm{N}} U]_{i^c} := \gamma_{\mathrm{N},i^c}^{+} U - \gamma_{\mathrm{N},i^c}^{-} U \quad \text{on } \Gamma_i^c. \tag{15}$$

3. Model Problem. In what follows, we present the specific volume boundary value problem we aim to solve as well as some of its characteristics.

Problem 3.1. Let $\mathbf{g} = (g_1, \ldots, g_M) \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$ and consider a time-harmonic excitation $\omega > 0$, leading to a bounded wavenumber k real and non-negative. We seek $U \in H^1_{loc}(\Omega)$ such that

$$-\Delta U - k^2 U = 0 \qquad \text{in } \Omega, \tag{16}$$

$$\gamma_i^{\pm} U = g_i \quad \text{for } i = 1, \ldots, M, \tag{17}$$

$$\text{Condition at infinity}(k) \tag{18}$$

The behavior at infinity (18) depends on k in the following way. If $k > 0$, we employ the classical Sommerfeld condition:

$$\frac{\partial U}{\partial r} - ikU = o\left(R^{-\frac{1}{2}}\right) \quad \text{for } R \to \infty, \tag{19}$$

where $R = \|\mathbf{x}\|_2$. If $k = 0$, we seek for solutions $U \in W(\Omega)$ and which we will discuss in detail later.

Remark 3.2. In Problem 3.1, it is also possible to have wavenumbers with non-zero imaginary part. As this would render the above problem elliptic, many results are easily derived via the Lax-Milgram theorem and we forgo the associated analysis.

We now aim to prove uniqueness of weak solutions for Problem 3.1; existence will be obtained by means of boundary integral potentials in the next section. For a single arc, the result was first established in Sobolev spaces by Stephan and Wendland [40]. Before we proceed, we recall preliminary results from [21, Section 2.4], which are also based on the ideas of [40]. Let us first define the subspace of $W(\Omega)$:

$$W_0(\Omega) = \left\{ U \in W(\Omega) : \gamma_i^{\pm} U = 0, \quad \text{for } i = 1, \ldots, M \right\}. \tag{20}$$

LEMMA 3.3 (Lemma 2.2, [21]). *The semi-norm $|U|_{W(\Omega)} := \|\nabla U\|_{L^2(\Omega)}$ bounds the $W(\Omega)$-norm for functions in $W_0(\Omega)$, i.e. there exists a constant $c > 0$ such that*

$$\|U\|_{W(\Omega)} \leq c\,|U|_{W(\Omega)}\,, \quad \forall\, U \in W_0(\Omega). \tag{21}$$

LEMMA 3.4 (Proposition 2.6, [21]). *Let U belong to $W(\Omega)$ and $F \in L^2_{loc}(\Omega)$, such that $-\Delta U = F$ in Ω. For $R > 0$, denote the ball of radius R centered at the origin by $B_R := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < R\}$. Then,*

$$\lim_{R\to\infty} \langle \gamma_{N,R} U, \gamma_R V\rangle_{\partial B_R} = 0, \quad \forall\, V \in W(\Omega), \tag{22}$$

where γ_R and $\gamma_{N,R}$ denote interior Dirichlet and Neumann traces on ∂B_R, respectively, the latter being equivalent to the radial derivative on the boundary.

3.1. Laplace case. We now focus on the case $k = 0$ with homogeneous Dirichlet conditions on all interfaces Γ_i, for all $i \in \{1, \dots, M\}$ and no source term.

Problem 3.5. Seek $U \in W_0(\Omega)$ such that

$$-\Delta U(\mathbf{x}) = 0, \quad \forall\, \mathbf{x} \in \Omega. \tag{23}$$

LEMMA 3.6 (Theorem 1.7.1, [14]). *Let $V \in W_0(\Omega)$. Then, for any $\Gamma_i^c = \widetilde{\Gamma}_i \backslash \overline{\Gamma}_i$ as defined in Section 2.3, it holds*

$$\gamma^+_{i^c} V = \gamma^-_{i^c} V.$$

for $i \in \{1 \dots, M\}$.

Hence, we can denote indistinctly by γ_{i^c} the trace defined over Γ_i^c on $W_0(\Omega)$.

LEMMA 3.7 (Section 2.6.1, [21]). *Let a function $U \in W_0(\Omega)$ solve $-\Delta U = 0$ in Ω. Then, the normal jump on Γ_i^c is null, i.e. $[\gamma_N U]_{i^c} = 0$.*

Finally, we can prove the uniqueness of the Laplace Problem 3.5.

PROPOSITION 3.8. *Problem 3.5 has at most one solution.*

Proof. Let $\Omega_* := \bigcup_{j=1}^M \Omega_j$, where the collection is disjoint by Assumption 2.4, and choose $R > 0$ such that $\Omega_* \subset B_R$. Set $\Omega_0(R) := B_R \cap \overline{\Omega}_*^c$, so that for any $V \in W_0(\Omega)$, it holds that

$$\langle -\Delta U, V\rangle_{B_R\cap\Omega} = \langle -\Delta U, V\rangle_{B_R} = \langle -\Delta U, V\rangle_{\Omega_0(R)} + \sum_{j=1}^M \langle -\Delta U, V\rangle_{\Omega_j}\,, \tag{24}$$

where the first equality follows directly from the null condition of v in Γ. Performing integration by parts on both terms on the right-hand side above yields

$$\begin{aligned} \langle -\Delta U, V\rangle_{\Omega_0(R)} &= (\nabla U, \nabla V)_{\Omega_0(R)} - \langle \gamma_{\mathrm{N},R} U, \gamma_R V\rangle_{\partial B_R} \\ &\quad + \sum_{j=1}^M \left\langle \gamma^-_{\mathrm{N},j} U, \gamma^-_j V\right\rangle_{\Gamma_j} + \left\langle \gamma^-_{\mathrm{N},j^c} U, \gamma^-_{j^c} V\right\rangle_{\Gamma^c_j} \end{aligned} \tag{25}$$

and

$$\langle -\Delta U, V\rangle_{\Omega_j} = (\nabla U, \nabla V)_{\Omega_j} - \left\langle \gamma^+_{\mathrm{N},j} U, \gamma^+_j V\right\rangle_{\Gamma_j} - \left\langle \gamma^+_{\mathrm{N},j^c} U, \gamma^+_{j^c} V\right\rangle_{\Gamma^c_j}. \tag{26}$$

Since $\gamma_{j^c}^{\pm} V = \gamma_{j^c} V$ and $\gamma_j^{\pm} V = 0$, adding terms gives

$$\langle -\Delta U, V \rangle_{B_R \cap \Omega} = (\nabla U, \nabla V)_{B_R} - \langle \gamma_{\mathrm{N},R} U, \gamma_R V \rangle_{\partial B_R} - \sum_{j=1}^{M} \langle [\gamma_{\mathrm{N}} U]_{j^c}, \gamma_{j^c} V \rangle_{\Gamma_j^c}, \tag{27}$$

where $(\nabla U, \nabla V)_{B_R} = (\nabla U, \nabla V)_{\cup_{j \geq 0} \Omega_j}$ as gradients belong to $L^2(\Omega)$. By Lemma 3.7, the terms inside the sum vanish. Thus, we arrive at

$$\langle -\Delta U, V \rangle_{B_R \cap \Omega} = (\nabla U, \nabla V)_{B_R} - \langle \gamma_{\mathrm{N},R} U, \gamma_R V \rangle_{\partial B_R}. \tag{28}$$

Letting $R \to \infty$, by Lemma 3.4 it holds

$$\langle -\Delta U, V \rangle_{\Omega} = (\nabla U, \nabla V)_{\Omega} = 0. \tag{29}$$

Finally, we conclude by the norm equivalence in Lemma 3.3. ∎

Remark 3.9. Uniqueness for the Laplace problem ($k = 0$) can be proved with the simpler radiation condition:

$$|U(\mathbf{x})| = \mathcal{O}(1) \quad \text{for} \quad \|\mathbf{x}\|_2 \to \infty. \tag{30}$$

Yet, and as we will see in the next section, this condition will not be the most convenient for working with the underlying BIEs.

3.2. Helmholtz case. Uniqueness for the Helmholtz problem can be established by using the variational formulation on B_R, derived as before, and by carrying out the same procedure as in closed domains so as to take into account the radiation condition.

PROPOSITION 3.10. *For $k > 0$, if $\mathbf{g} = 0$ and if the Sommerfeld radiation condition (19) is enforced, Problem 3.1 can only have the trivial solution.*

Proof. Following the steps of the proof of Proposition 3.8, we have the following variational formulation:

$$\int_{B_R} \left(\|\nabla U(\mathbf{x})\|_2^2 - k^2 |U(\mathbf{x})|^2 \right) \mathrm{d}\mathbf{x} - \langle \gamma_{\mathrm{N},R} U, \gamma_R U \rangle_{\partial B_R} = 0. \tag{31}$$

From here the proof follows verbatim as in [30, Chapter 9]. ∎

Hence, by combining Propositions 3.8 and 3.10, we have proven uniqueness for Problem 3.1 yet not existence. To this end we now turn our attention.

4. Boundary Integral Equation. Due to the unbounded domain in Problem 3.1, the boundary integral framework turns out to be a good alternative to study and solve the problem.

4.1. Preliminary results. Let $G_k(\mathbf{x}, \mathbf{y})$ denote the free space fundamental solution associated to the partial differential equation (16). We seek solutions of Problem 3.1 of the form:

$$U(\mathbf{x}) = \sum_{i=1}^{M} (\mathsf{SL}_i[k] \lambda_i)(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Omega, \tag{32}$$

where

$$(\mathsf{SL}_i[k] \lambda)(\mathbf{x}) := \int_{\Gamma_i} G_k(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) \mathrm{d}\Gamma_i(\mathbf{y}),$$

denotes the single layer potential generated at a curve Γ_i, with a fundamental solution

$$G_k(\mathbf{x},\mathbf{y}) = \begin{cases} -\dfrac{1}{2\pi}\log\|\mathbf{x}-\mathbf{y}\|_2 & \text{if } k=0, \\ \dfrac{i}{4}H_0^1(k\|\mathbf{x}-\mathbf{y}\|_2) & \text{if } k>0, \end{cases} \tag{33}$$

and wherein $H_0^1(\cdot)$ denotes the first kind Hankel function of zeroth-order [3, Chapter 9].

In Ω, G_k is a smooth function such that $(-\Delta_{\mathbf{x}}-k^2)G_k(\mathbf{x},\mathbf{y})=0$ for $\mathbf{y}\in\Gamma,\mathbf{x}\in\Omega$. By the dominated convergence theorem, we can conclude that $(-\Delta-k^2)U=0$. Taking Dirichlet traces in (32) and imposing boundary conditions (17), we are lead to the system of BIEs:

$$\mathcal{L}_{ii}[k]\lambda_i + \sum_{j\neq i}\mathcal{L}_{ij}[k]\lambda_j = g_i, \quad \forall\, i\in\{1,\dots,M\}, \tag{34}$$

where have introduced the BIOs:

$$\mathcal{L}_{ij}[k] := \gamma_i \mathsf{SL}_j[k] = \gamma_i^{\pm}\mathsf{SL}_j[k],$$

with indices i and j in $\{1,\dots,M\}$. These are well defined as the following result shows. We also recall the classical weakly singular BIO $\mathcal{V}_i[k]$ over the closed curve $\widetilde{\Gamma}_i$:

$$\mathcal{V}_i[k] := \frac{1}{2}\left(\tilde{\gamma}_i^{+}+\tilde{\gamma}_i^{-}\right)\mathsf{SL}_{\widetilde{\Gamma}_i}[k]. \tag{35}$$

Let us denote $R_{\Gamma_i}: H^{\frac{1}{2}}(\widetilde{\Gamma}_i)\to H^{\frac{1}{2}}(\Gamma_i)$ the restriction operator, which is continuos by space definitions. Then, we have that

$$R_{\Gamma_i}\mathcal{V}_i[k]\tilde{u} = \mathcal{L}_{ii}[k]u. \tag{36}$$

Proposition 4.1. *For each arc Γ_i, with $i\in\{1,\dots,M\}$, and $k\geq 0$, the single layer potential $\mathsf{SL}_i[k]: \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)\to H^1_{loc}(\mathbb{R}^2)$ is a linear bounded map. For the Laplace case, it also holds $\mathsf{SL}_i[0]: \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)\to W(\mathbb{R}^2\setminus\overline{\Gamma}_i)$.*

Proof. Let $k\geq 0$ and choose $u\in\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ for any curve Γ_i. By Assumption 2.2, there exists a closed curve $\widetilde{\Gamma}_i$ including Γ_i and an associated extension-by-zero denoted by $\tilde{u}\in H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$. Furthermore,

$$\mathsf{SL}_i[k]u = \mathsf{SL}_{\widetilde{\Gamma}_i}[k]\tilde{u} \quad \text{in } \mathbb{R}^2\setminus\widetilde{\Gamma}_i.$$

From classical mapping properties of layer potentials [11, Theorem 1] we know that $\mathsf{SL}_{\widetilde{\Gamma}_i}[k]\in H^1_{loc}(\mathbb{R}^2)$. Thus, it follows that traces on both sides are equal, and so the BIOs $\mathcal{L}_{ij}[k]:=\gamma_i\mathsf{SL}_j[k]$ are well defined.

For the case $k=0$, we need to show that $\mathsf{SL}_i[0]$ maps into $W(\mathbb{R}^2\setminus\overline{\Gamma}_i)$, i.e.

$$\frac{(\mathsf{SL}_i[0]u)(\mathbf{x})}{\sqrt{1+\|\mathbf{x}\|_2^2\log(2+\|\mathbf{x}\|_2^2)}}\in L^2(\mathbb{R}^2\setminus\overline{\Gamma}_i), \quad \text{and} \quad \nabla\left(\mathsf{SL}_i[0]u\right)\in L^2(\mathbb{R}^2\setminus\overline{\Gamma}_i).$$

From [30, Corollary 8.11], we know that the asymptotic behavior of the single layer potential for large arguments is

$$(\mathsf{SL}_i[0]u)(\mathbf{x}) = -\frac{1}{2\pi}\langle u,1\rangle\log\|\mathbf{x}\|_2 + \mathcal{O}(\|\mathbf{x}\|_2^{-1}), \quad \text{for } \|\mathbf{x}\|_2\to\infty. \tag{37}$$

Hence, if $u \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$ then

$$(\mathsf{SL}_i[0]u)(\mathbf{x}) = \mathcal{O}(\|\mathbf{x}\|_2^{-1}), \quad \text{for } \|\mathbf{x}\|_2 \to \infty. \tag{38}$$

By using polar coordinates, it can be shown that it is enough to have $\mathsf{SL}_i[0]u \in W(\mathbb{R}^2 \setminus \overline{\Gamma}_i)$. ∎

Remark 4.2. In (37), we introduced the behavior at infinity of the single layer potential for the case $k = 0$. As stated in Remark 3.9, there are two possibilities to ensure uniqueness of solutions for the volume Problem 3.1 in this setting:

(i) Ask for the solution to be in $U \in W(\Omega)$;
(ii) Incorporate the condition $U(\mathbf{x}) = \mathcal{O}(1)$ for $\|\mathbf{x}\|_2 \to \infty$.

When representing U as a single layer potential acting on a surface density λ in a suitable trace space, we have a different condition, namely, the one stated in (37). Hence, in order for U to fulfill this condition at infinity, one must require that $\langle \lambda, 1\rangle = 0$. This condition is indeed stronger as it implies that we are only able to represent solutions that decay at infinity. Fortunately, this condition ensures invertibility of the weakly singular integral operator $\mathcal{L}_{ii}[0]$, for $i = 1, \dots, M$ (*cf.* Lemma 4.4). Still, the Dirichlet Laplace problem could be solved with only a boundedness condition by means of a different integral equation to the one here presented. This alternative boundary equation and its corresponding analysis are provided in [40], and its generalization to multiple arcs can be done with the same arguments shown in this section.

LEMMA 4.3. *For $k \geq 0$, the operator $\mathcal{L}_{ii}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$ is linear and bounded for $i \in \{1, \dots, M\}$.*

Proof. Let u and v belong to $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ and denote by $\tilde{u}$ and $\tilde{v}$ in $H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$ their suitable extensions by zero over the closed curve $\widetilde{\Gamma}_i$. Then, by (35), it holds

$$\langle \mathcal{L}_{ii}[k]u, v\rangle_{\Gamma_i} = \langle \mathcal{V}_i[k]\tilde{u}, \tilde{v}\rangle_{\widetilde{\Gamma}_i}.$$

By [11, Theorem 1], we have

$$\langle \mathcal{L}_{ii}[k]u, v\rangle_{\Gamma_i} \leq C(k)\,\|\tilde{u}\|_{H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)}\,\|\tilde{v}\|_{H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)},$$

for a positive constant C depending on k. By using Lemma 2.6, we find

$$\langle \mathcal{L}_{ii}[k]u, v\rangle_{\Gamma_i} \leq C(k)\,\|u\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}\,\|v\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}$$

as stated. ∎

LEMMA 4.4. *For $k = 0$, let $u \in \widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$. Then, there exist constants $c_{e,i} > 0$ such that*

$$\langle \mathcal{L}_{ii}[0]u, v\rangle_{\Gamma_i} \geq c_{e,i}\,\|u\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}, \quad i = 1, \dots, M. \tag{39}$$

The same inequality holds if $u \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ and $\operatorname{diam}(\Omega_i) < 1$, i.e. if the analytic capacity of Γ_i is less than one.

Proof. We follow the proof of Lemma 4.3 and conclude by Theorems 6.22 and 6.23 in [37]. ∎

Remark 4.5. In Remark 4.2 we stated that the condition $\lambda_i \in H^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$, is necessary to obtain a decaying solution $U_i := \mathsf{SL}_i[0]\lambda_i$. Unfortunately, this does not hold if only the condition $\operatorname{diam}(\Omega_i) < 1$ is enforced. The most simple case is to consider just one arc and use as right-hand side $g_i = \mathcal{L}_{ii}[0]1$. Thus, the volume solution is given by $U_i = \mathsf{SL}_i[0]1$, which no longer solves the Laplace problem. In other words, we not only need a condition that ensures the invertibility of the BIOs, but also that ensures that the volume solution has the correct behavior at infinity. The situation is more delicate when we consider more than one domain as we will see.

LEMMA 4.6. *For $i \in \{1,\dots,M\}$ and $k \geq 0$, there exist constants $c_{e,i} > 0$ and compact boundary operators $\mathcal{K}_{ii}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$, such that*

$$\langle (\mathcal{L}_{ii}[k] + \mathcal{K}_{ii}[k])u, u\rangle_{\Gamma_i} \geq c_{e,i} \|u\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}, \qquad \forall u \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i). \tag{40}$$

Proof. For any Γ_i, take a $u \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ and its extension by zero to $\widetilde{\Gamma}_i$, $\tilde{u} \in H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$. By [11, Theorem 2], there exists a positive constant $c_{e,i}$ and a compact operator $\mathcal{K}_i : H^{\frac{1}{2}}(\widetilde{\Gamma}_i) \to H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$, such that

$$\langle (\mathcal{V}_i[k] + \mathcal{K}_i[k])\tilde{u}, \tilde{u}\rangle_{\widetilde{\Gamma}_i} \geq c_{e,i} \|\tilde{u}\|^2_{H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)}. \tag{41}$$

Define the operator $\mathcal{K}_{ii} := R_{\Gamma_i}\mathcal{K}_i$. Thus, by the compactness of $\mathcal{K}_i$, we have that $\mathcal{K}_{ii} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$ is also compact, and consequently,

$$\langle (\mathcal{L}_{ii}[k] + \mathcal{K}_{ii}[k])u, u\rangle_{\Gamma_i} = \langle (\mathcal{V}_i[k] + \mathcal{K}_i[k])\tilde{u}, \tilde{u}\rangle_{\widetilde{\Gamma}_i} \geq c_{e,i}\|u\|^2_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)}, \tag{42}$$

concluding the proof. ∎

PROPOSITION 4.7. *Assume that k is not an eigenvalue of the Laplace operator, with Dirichlet conditions, for at least one domain enclosed by $\widetilde{\Gamma}_i$, for all $i = 1,\dots,M$. Then, the self-interaction operators $\mathcal{L}_{ii}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_i) \to H^{\frac{1}{2}}(\Gamma_i)$ are coercive and injective for $k > 0$, and elliptic for $k = 0$ in $\widetilde{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma_i)$, for $i \in \{1,\dots,M\}$.*

Proof. Coercivity and ellipticity follow directly from Lemmas 4.4 and 4.6. Injectivity comes from the injectivity on closed curves. Specifically, for any $u \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$, we can build an extension by zero $\tilde{u} \in H^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$, such that

$$(\mathcal{L}_{ii}[k]u)(\mathbf{x}) = \int_{\Gamma_i} G_k(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\Gamma_i(\mathbf{y}) = \int_{\widetilde{\Gamma}_i} G_k(\mathbf{x},\mathbf{y})\tilde{u}(\mathbf{y})\mathrm{d}\widetilde{\Gamma}_i(\mathbf{y}) = 0.$$

Since k is not an eigenvalue, by Theorem 3.9.1 in [34], it holds $\tilde{u} = 0$, and we conclude that $u = 0$. ∎

Remark 4.8. Assumption 2.2 allows us to build an extension $\widetilde{\Gamma}_i$ for every arc Γ_i but it is possible to have several extensions and thus different possibilities for enclosed domains Ω_i. Therefore, if for at least one of these domains k is not eigenvalue of the Laplace operator with Dirichlet condition, we have injectivity for $\mathcal{L}_{ii}[k]$, for every $i = 1,\dots,M$. Furthermore, in [40, Theorem 1.7], it is shown for a single arc that the integral equation and the volume problem are equivalent for all $k > 0$.

PROPOSITION 4.9. *For $k \geq 0$, the cross-interaction operators $\mathcal{L}_{ij}[k] : \widetilde{H}^{-\frac{1}{2}}(\Gamma_j) \to H^{\frac{1}{2}}(\Gamma_i)$ defined over disjoint interfaces are compact for all $i, j \in \{1,\dots,M\}$ with $i \neq j$.*

Proof. Since $\Gamma_i \cap \Gamma_j = \emptyset$, for any $i \neq j$, $\mathcal{L}_{ij}[k]$ has a smooth kernel function, as it is the composition of an analytic function G_k –at separate points– and curve parametrizations are at least $\mathcal{C}^1$. Consequently, it holds $\mathcal{L}_{ij} : \widetilde{H}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma_j) \to H^s(\Gamma_i)$, for all $s > \frac{1}{2}$. Then, by the compact embedding theorem [34, Section 2.5] we have the stated result. ∎

4.2. Multiple Arcs Boundary Integral Equation. We can now state the BIE associated to Problem 3.1. Based on our the Cartesian identification for $\mathbb{H}^{\pm\frac{1}{2}}(\Gamma)$ and its tilde counterparts, we introduce the following BIO:

$$\boldsymbol{\mathcal{L}}[k] := \begin{bmatrix} \mathcal{L}_{11}[k] & \mathcal{L}_{12}[k] & \dots & \mathcal{L}_{1M}[k] \\ \mathcal{L}_{21}[k] & \mathcal{L}_{22}[k] & \dots & \mathcal{L}_{2M}[k] \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{L}_{M1}[k] & \mathcal{L}_{M2}[k] & \dots & \mathcal{L}_{MM}[k] \end{bmatrix} : \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{\frac{1}{2}}(\Gamma). \tag{43}$$

Problem 4.10 (Boundary Integral Problem). For $k > 0$, let $\mathbf{g} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)$. We seek $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ such that

$$\boldsymbol{\mathcal{L}}[k]\boldsymbol{\lambda} = \mathbf{g}. \tag{44}$$

In the case $k = 0$, we look for $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma)$.

Remark 4.11. Problem 4.10 can be recast in the reference space $\mathbb{H}^{-\frac{1}{2}}(\widehat{\Gamma})$, so as to find $\hat{\boldsymbol{\lambda}} \in \mathbb{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ such that

$$\hat{\boldsymbol{\mathcal{L}}}[k]\hat{\boldsymbol{\lambda}} = \hat{\mathbf{g}}, \tag{45}$$

wherein the elements of $\hat{\mathbf{g}}$, $\hat{g}_j := g_j \circ \mathbf{r}_j$, $\hat{\mathcal{L}}_{ij}$ is the integral operator integrating over the reference arc $\widehat{\Gamma}$ with integral kernel $G_k(\mathbf{r}_i(t), \mathbf{r}_j(s))$, and unknown coordinates $\hat{\lambda}_j := (\lambda_j \circ \mathbf{r}_j)/\|\mathbf{r}'_j\|_2$.

Proposition 4.12. *The weakly singular operator $\boldsymbol{\mathcal{L}}[k] : \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma) \to \mathbb{H}^{\frac{1}{2}}(\Gamma)$ is bounded, coercive and injective for all $k > 0$. For $k = 0$, this result holds on the subspace $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0 \rangle}(\Gamma)$.*

Proof. As the case $M = 1$ was covered in Lemma 4.3 and Proposition 4.7 already, we focus on the general case $M > 1$. Boundeness is a direct consequence of the norm definition for $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, Lemma 4.3 and Proposition 4.9. Coercivity follows from Proposition 4.7 and 4.9 by taking duality product with functions in $\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$ and writing the operator matrix as a sum of diagonal coercive and off-diagonal compact terms. Hence, we only need to show injectivity for $k > 0$. The proof for the case $k = 0$ follows the steps verbatim but on mean-zero spaces.

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M) \in \mathrm{Ker}(\boldsymbol{\mathcal{L}}[k]) \subset \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$. Then, we have

$$\sum_{j=1}^{M} \mathcal{L}_{ij}[k]\lambda_j = 0, \quad \forall\, i = 1, \dots, M. \tag{46}$$

For $j \in \{1, \dots, M\}$, define potentials $U_j := \mathsf{SL}_j[k]\lambda_j$, solutions of individual homogenous Helmholtz problems over $\mathbb{R}^2 \setminus \overline{\Gamma}_j$ as well as the superposition $U_\sigma := \sum_{j=1}^{M} U_j$

defined over Ω. Then,

$$\gamma_i U_\sigma = \gamma_i \sum_{j=1}^{M} U_j = \sum_{i=1}^{M} \mathcal{L}_{ij}[k]\lambda_j = 0, \quad \forall\ i = 1, \dots, M. \tag{47}$$

However, U_σ is also the solution of Problem 3.1, with zero Dirichlet boundary condition. Hence, by uniqueness results of Propositions 3.10 (case $k > 0$) or 3.8 (case $k = 0$), we conclude that

$$U_\sigma = \sum_{j=1}^{M} \mathsf{SL}_j[k]\lambda_j = 0, \tag{48}$$

and consequently, for all $i = 1, \dots, M$, it holds

$$U_i = \mathsf{SL}_i[k]\lambda_i = -\sum_{j\neq i} \mathsf{SL}_j[k]\lambda_j. \tag{49}$$

Let us now consider the closed curve $\widetilde{\Gamma}_i$, and denote by $\tilde{\lambda}_i \in \widetilde{H}^{-\frac{1}{2}}(\widetilde{\Gamma}_i)$ the extension by zero of λ_i. It holds,

$$U_i = \mathsf{SL}_i[k]\lambda_i = \mathsf{SL}_{\widetilde{\Gamma}_i}[k]\tilde{\lambda}_i \tag{50}$$

where the last potential is defined on the closed curve $\widetilde{\Gamma}_i$. If we take normal jumps, by [34, Theorem 3.3.1], we obtain

$$[\gamma_{\mathrm{N}} U_i]_{\widetilde{\Gamma}_i} = [\mathsf{SL}_{\widetilde{\Gamma}_i}[k]\tilde{\lambda}_i]_{\widetilde{\Gamma}_i} = -\tilde{\lambda}_i. \tag{51}$$

Using (49) in the expression above yields

$$[\gamma_{\mathrm{N}} U_i]_{\widetilde{\Gamma}_i} = -\left[\sum_{j\neq i} \mathsf{SL}_j[k]\lambda_j\right]_{\widetilde{\Gamma}_i} = 0 \tag{52}$$

where the last equality comes from the smoothness of the integral kernel since $\widetilde{\Gamma}_i \cap \widetilde{\Gamma}_j = \emptyset$, for $j \neq i$. Thus, we can conclude that $\lambda_j = 0$ and the same follows for the other components. ∎

Using the classical Fredholm alternative [34, Theorem 2.1.36], one can easily prove the next result.

THEOREM 4.13. *For $k > 0$, Problem 4.10 has a unique solution $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)$, whereas for $k = 0$ a unique solution exists in the subspace $\boldsymbol{\lambda} \in \widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$. Also, we have the estimate*

$$\|\boldsymbol{\lambda}\|_{\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)} \leq C(k)\|\mathbf{g}\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma)}. \tag{53}$$

Remark 4.14. For $k = 0$, the condition at infinity described in Problem 3.1, imposed on the single layer potential is fulfilled if the global condition:

$$\sum_{i=1}^{M} \langle 1, \lambda_i\rangle_{\Gamma_i} = 0, \tag{54}$$

is satisfied. This is less restrictive than searching for a solution in $\widetilde{\mathbb{H}}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$, which implies that each component satisfies $\langle 1, \lambda_i\rangle_{\Gamma_i} = 0$. However, and though coercivity is retrieved, an injectivity proof for this configuration is missing.

5. Further extensions. We generalize the above problem by first extending the notion of open arcs Γ_i to proper Lipschitz subsets of the boundary of a domain $\Omega_i \in \mathbb{R}^d$, for $d = 2, 3$, and whose normal vector is continuous. Define Ω as the exterior of a finite set of generalized open arcs Γ. As in [30, Chapter 4], consider any strongly elliptic second-order self-adjoint partial differential operator, denoted by $\mathcal{P}$, with smooth coefficients, acting on vector fields of $\mathbb{C}^m$. Thus, for a given Dirichlet or Neumann datum, $\mathbf{g} \in [\mathbb{H}^{\frac{1}{2}}(\Gamma)]^m$ or $\mathbf{h} \in [\mathbb{H}^{-\frac{1}{2}}(\Gamma)]^m$, respectively, we seek for $\mathbf{U} \in [H^1_{loc}(\Omega)]^m$ such that,

$$\mathcal{P}\mathbf{U} = 0 \quad \text{in } \Omega, \tag{55}$$

$$\gamma\mathbf{U} = \mathbf{g} \quad \text{or} \quad B_{\mathbf{n}}\mathbf{U} = \mathbf{h} \quad \text{on } \Gamma, \tag{56}$$

and where the conormal trace $B_{\mathbf{n}}$ defined as in [30, Chapter 4]. The following points are needed in order to establish the existence and uniqueness of an equivalent boundary integral formulation for Cauchy data.

(i) An adequate condition at infinity that ensures the uniqueness of the volume problem.
(ii) A fundamental solution $G(\mathbf{x}, \mathbf{y})$, such that $\mathcal{P}_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}-\mathbf{y}} \mathsf{I}$, where I is the identity operator in $\mathbb{R}^{m\times m}$.
(iii) Layer potentials:

$$(\mathsf{SL}_i\boldsymbol{\lambda})(\mathbf{x}) := \int_{\Gamma_i} G(\mathbf{x}, \mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\mathrm{d}\Gamma_i(\mathbf{y}) \qquad \text{(Dirichlet trace)},$$

$$(\mathsf{DL}_i\boldsymbol{\lambda})(\mathbf{x}) := \int_{\Gamma_i} B_{\mathbf{n}(\mathbf{y})} G(\mathbf{x}, \mathbf{y})\boldsymbol{\lambda}(\mathbf{y})\mathrm{d}\Gamma_i(\mathbf{y}) \quad \text{(conormal trace)},$$

that display the adequate behavior at infinity specified by the first point in the trace spaces. Specifically, $\boldsymbol{\lambda} \in \left[\widetilde{\mathbb{H}}^{-\frac{1}{2}}(\Gamma)\right]^m$ for the Dirichlet problem and $\boldsymbol{\lambda} \in \left[\widetilde{\mathbb{H}}^{\frac{1}{2}}(\Gamma)\right]^m$ for the conormal trace case.

With the above, the integral equation is constructed by the imposing the boundary condition to the following representations:

$$\mathbf{U} = \sum_{i=1}^{M} \mathsf{SL}_i\boldsymbol{\lambda}_i \quad \text{(Dirichlet trace)},$$

$$\mathbf{U} = \sum_{i=1}^{M} \mathsf{DL}_i\boldsymbol{\lambda}_i \quad \text{(conormal trace)}.$$

If the previously stated conditions are satisfied, then the construction of the arising BIEs as well as their wellposedness proofs is done as in Section 4 for the case $k > 0$. It worth notice that the Laplace case presented in this work is slightly different as the condition at infinity of the potential only holds in a subspace. We now focus in two specific examples.

5.1. Three-Dimensional case. Consider $\widehat{\Gamma} = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 < 1\}$, the notion of open arcs, is changed for open surfaces, also called screens. As in the two dimensional case, we restrict ourselves to screens that can be enclosed by a closed surfaces which is the boundary of a domain as in [39]. Also, we require our screens to be orientables so that under these assumptions, the only difference between the two and tree dimensional case occurs for Laplace condition at infinity. The new condition is

$$U(\mathbf{x}) = \mathcal{O}(\|\mathbf{x}\|_2^{-1}) \quad \text{for } \|\mathbf{x}\|_2 \to 0. \tag{57}$$

We need to show that using this condition in Problem 3.1 we have unique solution. The only difference with Proposition 3.8 is the derivation of a similar result to that Lemma 3.4, but this can be obtained by a direct computation. In fact, for smooth functions it holds

$$\langle \gamma_{\mathrm{N},R} U, \gamma_R V\rangle_{\partial B_R} = \int_0^{2\pi}\int_0^{\pi} R^{-1} d\varphi d\theta \to 0 \text{ as } R \to 0, \tag{58}$$

and since the new condition implies that $\nabla U \in L^2(\Omega)$, uniqueness follows.

Secondly, the integral kernel from in Section 4 has to be changed to

$$G_k(\mathbf{x}, \mathbf{y}) = \frac{e^{ik\|\mathbf{x}-\mathbf{y}\|_2}}{4\pi\|\mathbf{x}-\mathbf{y}\|_2}. \tag{59}$$

One can easily show that the single layer potential has the required condition at infinity ([30, Theorems 8.9, 9.6]). Thus, both cases (Laplace and Helmholtz) can be considered in $\mathbb{H}^{-\frac{1}{2}}(\Gamma)$ and the sub-space $\mathbb{H}^{-\frac{1}{2}}_{\langle 0\rangle}(\Gamma)$ is not needed. We mention again that more general cases were considered in the work of Claeys [10].

5.2. Neumannn Case. The two-dimensional Neumann problem is defined as follows.

Problem 5.1. Let $\mathbf{h} = (h_1, \dots, h_M) \in \mathbb{H}^{-\frac{1}{2}}(\Gamma)$ and consider a time-harmonic excitation $\omega > 0$, leading to a bounded wavenumber k real and non-negative. We seek $U \in H^1_{loc}(\Omega)$ such that

$$-\Delta U - k^2 U = 0 \qquad \text{in } \Omega, \tag{60}$$

$$\gamma^{\pm}_{\mathrm{N},i} U = h_i \qquad \text{for } i = 1, \dots, M, \tag{61}$$

$$\text{condition at infinity}(k), \tag{62}$$

and where the conditions at infinity are those for Problem 3.1.

5.2.1. Uniqueness Result. The uniqueness follows from the variational formulation formulation but with some minors modifications.

Proposition 5.2. *For $k = 0$, Problem 5.1, has a unique solution $U \in W(\Omega)\backslash\mathbb{C}$.*

Proof. We do the same procedure as in the proof of Proposition 3.8, but with $U, V \in W(\Omega)\backslash\mathbb{C}$, since the semi-norm is a norm in this subspace the results follows. □

For the case $k = 0$, the proof follows the same steps as for the Dirichlet condition version (*cf.* Proposition 3.10).

Proposition 5.3. *For $k > 0$, Problem 5.1 has a unique solution $U \in H^1_{loc}(\Omega)$.*

Remark 5.4. Notice that, in contrast to the classical Neumann problem, compatibility conditions are not required. This can be seen from the variational formulation (*cf.* proof of Proposition 3.8), as it involves the jump of the Neumann traces which is assumed to be zero.

5.2.2. Boundary Integral formulation. In order to obtain an integral equation for the Neumann problem we define the double layer potential as

$$(\mathsf{DL}_i[k]\lambda)(\mathbf{x}) := \int_{\Gamma_i} \partial_{\mathbf{n}(\mathbf{y})} G_k(\mathbf{x}, \mathbf{y}) \lambda(\mathbf{y}) \mathrm{d}\Gamma_i(\mathbf{y}). \tag{63}$$

We seek solutions of the form:

$$U(\mathbf{x}) = \sum_{i=1}^{M} (\mathsf{DL}_i[k]\lambda_i)(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{64}$$

We notice that for $\lambda \in \widetilde{H}^{\frac{1}{2}}(\Gamma_i)$ from the classical properties of the double layer potential for Lipchitz domains [11, Theorem 1], and using the zero extension $\tilde{\lambda}_i \in H^{\frac{1}{2}}(\widetilde{\Gamma}_i)$, we can establish that

$$\mathsf{DL}_i[k] : \widetilde{H}^{\frac{1}{2}}(\Gamma_i) \to H^1_{\Delta,loc}(\mathbb{R}^2 \setminus \overline{\Gamma}_i), \quad i = 1, \ldots, M, \tag{65}$$

also, for $k = 0$ and by using the explicit formula for the fundamental solution, we can conclude that $U \in W(\Omega)\backslash\mathbb{C}$. In the Helmholtz case from [30, Theorem 9.6] we have that u has the desire condition at infinity. Hence, the equivalent of the operator $\mathcal{L}_{ij}[k]$ is

$$\mathcal{J}_{ij}[k] := \gamma_{\mathrm{N},i}\mathsf{DL}_j[k], \tag{66}$$

all the properties of this operator can be shown in the similar way of the ones for $\mathcal{L}_{ij}[k]$, but replacing the space $\widetilde{H}^{-\frac{1}{2}}(\Gamma_i)$ for $\widetilde{H}^{\frac{1}{2}}(\Gamma_i)$.

Appendix A. Proofs of Lemmas in Section 2.3.1.

A.1. Proof of Lemma 2.5. By direct calculation, it holds

$$\|\tilde{u}\|_{H^{-\frac{1}{2}}(\tilde{\Lambda})} = \sup_{v \in H^{\frac{1}{2}}(\tilde{\Lambda})} \frac{\langle \tilde{u}, v\rangle}{\|v\|_{H^{\frac{1}{2}}(\tilde{\Lambda})}} = \sup_{v \in H^{\frac{1}{2}}(\tilde{\Lambda})} \frac{\langle u, v|_\Lambda\rangle}{\|v\|_{H^{\frac{1}{2}}(\tilde{\Lambda})}}.$$

It is easy to check that the semi-norm:

$$|v|_\Lambda|_{H^{\frac{1}{2}}(\Lambda)} = \int_\Lambda \int_\Lambda \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|_2^2} d\mathbf{y}d\mathbf{x} \le \int_{\tilde{\Lambda}} \int_{\tilde{\Lambda}} \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|_2^2} d\mathbf{y}d\mathbf{x} = |v|_{H^{\frac{1}{2}}(\tilde{\Lambda})}.$$

Using the same argument for the L^2-norm, we can conclude that $v|_\Lambda \in H^{\frac{1}{2}}(\Lambda)$, with

$$\|v|_\Lambda\|_{H^{\frac{1}{2}}(\Lambda)} \le \|v\|_{H^{\frac{1}{2}}(\tilde{\Lambda})}.$$

By duality between $\widetilde{H}^{-\frac{1}{2}}(\Lambda)$ and $H^{\frac{1}{2}}(\Lambda)$, we have

$$\|\tilde{u}\|_{H^{-\frac{1}{2}}(\tilde{\Lambda})} \le \frac{\|u\|_{\widetilde{H}^{-\frac{1}{2}}(\Lambda)} \|v|_\Lambda\|_{H^{\frac{1}{2}}(\Lambda)}}{\|v\|_{H^{\frac{1}{2}}(\tilde{\Lambda})}} \le \|u\|_{\widetilde{H}^{-\frac{1}{2}}(\Lambda)} < \infty,$$

as stated.

A.2. Proof of Lemma 2.6. Using the equivalent definition of the space $H^{\frac{1}{2}}(\Lambda)$ as restrictions of functions in $H^{\frac{1}{2}}(\tilde{\Lambda})$, we have

$$\|\tilde{u}\|_{H^{-\frac{1}{2}}(\tilde{\Lambda})} = \sup_{z \in H^{\frac{1}{2}}(\tilde{\Lambda})} \frac{\langle u, z|_\Lambda\rangle_\Lambda}{\|z\|_{H^{\frac{1}{2}}(\tilde{\Lambda})}} \ge \sup_{v \in H^{\frac{1}{2}}(\Lambda)} \frac{\langle u, v\rangle_\Lambda}{\|v\|_{H^{\frac{1}{2}}(\Lambda)}} = \|u\|_{\widetilde{H}^{-\frac{1}{2}}(\Lambda)}.$$

Combining this with the estimate from Lemma 2.5 gives the desired result.

A.3. Proof of Lemma 2.7. We only show the first inequality as for the second one similar steps are followed. By definition, it holds

$$\|\zeta\circ\mathbf{r}_i\|^2_{H^{\frac{1}{2}}(\hat{\Gamma})} = \int_{\hat{\Gamma}} |\zeta\circ\mathbf{r}_i(t)|^2 dt + \int_{\hat{\Gamma}}\int_{\hat{\Gamma}} \frac{|\zeta\circ\mathbf{r}_i(t)-\zeta\circ\mathbf{r}_i(s)|^2}{|t-s|^2} dtds. \tag{67}$$

For the first integral on the right-hand side, we deduce

$$\begin{aligned}\int_{\widehat{\Gamma}} |\zeta\circ\mathbf{r}_i(t)|^2 dt = \int_{\widehat{\Gamma}} |\zeta\circ\mathbf{r}_i|^2 \frac{\|\mathbf{r}_i'(t)\|_2}{\|\mathbf{r}_i'(t)\|_2} dt &= \int_{\Gamma_i} \frac{|\zeta|^2}{\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}\right\|_2} \mathrm{d}\Gamma_i \\ &\leq \left\|\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}\right\|_2^{-1}\right\|_{L^\infty(\Gamma_i)} \int_{\Gamma_i} |\zeta|^2 \mathrm{d}\Gamma_i.\end{aligned} \tag{68}$$

Similarly, by changing variables, the second term in (67) becomes

$$\int_{\Gamma_i}\int_{\Gamma_i} \frac{|\zeta(\mathbf{x})-\zeta(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|_2^2} \left(\frac{\|\mathbf{x}-\mathbf{y}\|_2^2}{\left\|\mathbf{r}_i^{-1}(\mathbf{x})-\mathbf{r}_i^{-1}(\mathbf{y})\right\|_2^2}\right) \frac{\mathrm{d}\Gamma_i(\mathbf{x})\mathrm{d}\Gamma_i(\mathbf{y})}{\left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}(\mathbf{x})\right\|_2 \left\|\mathbf{r}_i'\circ\mathbf{r}_i^{-1}(\mathbf{y})\right\|_2}. \tag{69}$$

By using the mean value theorem for $\mathbf{r}_i^{-1}$, we arrive at

$$\int_{\widehat{\Gamma}}\int_{\widehat{\Gamma}} \frac{|\zeta\circ\mathbf{r}_i(t)-\zeta\circ\mathbf{r}_i(s)|^2}{|t-s|^2} dtds \leq \int_{\Gamma_i}\int_{\Gamma_i} \frac{|\zeta(\mathbf{x})-\zeta(\mathbf{y})|^2}{\|\mathbf{x}-\mathbf{y}\|_2^2} \mathrm{d}\Gamma_i(\mathbf{x})\mathrm{d}\Gamma_i(\mathbf{y})$$

and thus, we get the stated result.

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