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IMPROVED EFFICIENCY OF A MULTI-INDEX FEM FOR COMPUTATIONAL UNCERTAINTY QUANTIFICATION*

JOSEF DICK†, MICHAEL FEISCHL‡, AND CHRISTOPH SCHWAB§

Abstract. We propose a multi-index algorithm for the Monte Carlo discretization of a linear, elliptic PDE with affine-parametric input. We prove an error vs. work analysis which allows a multi-level finite-element approximation in the physical domain, and apply the multi-index analysis with isotropic, unstructured mesh refinement in the physical domain for the solution of the forward problem, for the approximation of the random field, and for the Monte-Carlo quadrature error. Our approach allows Lipschitz domains and mesh hierarchies more general than tensor grids. The improvement in complexity is obtained from combining spacial discretization, dimension truncation and MC sampling in a multi-index fashion. Our analysis improves cost estimates compared to multi-level algorithms for similar problems and mathematically underpins the outstanding practical performance of multi-index algorithms for partial differential equations with random coefficients.

Key words. Multi-index, Monte Carlo, Finite Element Method, Uncertainty Quantification

AMS subject classifications. subject classification

1. Introduction.

The term *multi-index Monte Carlo method* (MIMC for short) was first coined in the work [14] as an extension of the multi-level Monte Carlo method (MLMC for short) developed in [12]. The MIMC idea abstracts sparse grids and sparse tensor products to approximate multivariate functions from sparse tensor products of univariate hierarchic approximations in each variable, see the surveys [5, 24] and the references there.

Since the appearance of [12], the multi-level idea has been applied in many areas including high-dimensional integration, stochastic differential equations, and several types of PDEs with random coefficients. We refer to [4, 11, 13, 23]. Most of these works addressed MLMC algorithms, while *multi-level quasi-Monte Carlo* (MLQMC for short) algorithms for PDEs with random field input data were addressed only more recently in [16, 10, 8, 9]. In the framework of PDEs with random coefficients, the idea of the multi-level approach is to introduce sequences of bisection refined grids and to compute finite element (FE) approximations of a given partial differential equation (PDE) with random coefficients on each discretization level. By varying the MC sample size on each level of the FE discretization and by judicious combincation of the individual approximations, it is possible to reduce the total cost (up to logarithmic factors) from $\mathrm{cost}(sampling) \times \mathrm{cost}(FEM)$ to $\mathrm{cost}(sampling) + \mathrm{cost}(FEM)$, where the individual cost terms are measured on the finest level.

For example, in linear, elliptic PDEs in divergence form in a bounded domain D, MLMC FEM were introduced in [6, 4]. It was shown there that MLMC FEM

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with continuous, piecewise affine ("P_1-FEM") finite elements in D can provide a numerically computed estimate of the mean field (or "ensemble average") of the random solution u (and, as explained in [4], also of its 2- and k-point correlations) which satisfies, in $H^1(D)$, essentially optimal (up to logarithmic terms) convergence rate bounds $O(h)$ in work which equals, in space dimension $d = 2$, essentially $O(h^{-2})$. These asymptotic orders equal the error vs. work relation for the solution of *one instance of the coresponding deterministic problem*. In [4], the random input was assumed to consist only of a single term in a KL epansion of the random diffusion coefficient. A similar result, again in space dimension $d = 2$, for *functionals* $G(\cdot) \in H^{-1}(D)$ *of the solution* was obtained in [17]. There, again P_1-FEM in D were employed, but in order to achieve the higher FE convergence rate $O(h^2)$ for $G(\cdot) \in L^2(D)$, *multi-level Quasi-Monte Carlo* integration over the ensemble was necessary.

This idea was further extended in [14] to include more than one parameter which is quantized into levels. One possible example for this approach, presented in [14], is to introduce anisotropic discretizations in the physical domain (as, e.g., sparse grid FE discretizations) for which two (three) parameters control the element size in the coordinate direction. This 'sparse grid' approach has been combined with a heuristic, adaptive algorithm and a Quasi-Monte Carlo algorithm in [22]. More examples of variations of this approach can be found in [15, 21]. In these approaches, the construction of sparse grid hierarchies in the physical domain to access the multi-index efficiency could impose obstructions on the shape of the physical domains which are amenable to this kind of discretization.

In the present work, we follow a different (but, as we will show, very natural) approach: we include the approximation of the random coefficients into the multi-index discretization and convergence analysis. As we show, this is effective due to the following consideration: apart from toy problems, it is often not possible to obtain exact samples of the random coefficients. This is usually due to the fact that the random coefficient is given in terms of some series expansion for which only finitely many terms can be computed. This particular approximation can constitute a major bottleneck in computations. It is therefore of practical importance to improve efficiency of algorithms.

Although the presently proposed approach is, in principle, more general, we develop it here for affine-parametric random coefficients in a standard, linear Poisson model problem

$$
-\mathrm{div}(A\nabla u) = f \quad \text{in } D\,, \qquad u = 0 \;\text{ on }\; \partial D \tag{1.1}
$$

for some Lipschitz domain $D \subseteq \mathbb{R}^d$. We parametrize the uncertain diffusion coefficient, assumed to belong to $W^{1,\infty}(D)$, by a dimensionally truncated Karhunen-Loeve expansion ("KL expansion" for short), i.e., for given $x \in D$ and $\omega \in \Omega$ (the probability space, see Section 2.1)

$$
A(x,\omega) = \phi_0(x) + \sum_{j=1}^{\infty} \phi_j(x)\psi_j(\omega_j) \approx A^\nu(x,\omega) := \phi_0(x) + \sum_{j=1}^{s_\nu} \phi_j(x)\psi_j(\omega_j),
$$

where $\{s_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{N}$ is an increasing sequence of "dimension truncation" parameters.

Given a quantity of interest in terms of a linear functional $G(\cdot)$, the idea is to approximate the expectation of the exact solution u of (1.1), i.e., $\mathbb{E}(G(u))$ (where the expectation is taken over Ω). This is done by computing several instances of the "double difference" $D_\ell^\nu = (u_\ell^\nu - u_{\ell-1}^\nu) - (u_\ell^{\nu-1} - u_{\ell-1}^{\nu-1})$, where u_ℓ^ν denotes the FEM approximation of u on a mesh of size h_ℓ and with respect to the approximation A^ν of the exact (i.e. withouth truncation) random coefficient. As for any multi-level approach, this requires a mesh hierarchy $h_0, h_1, \ldots, h_\ell$ as introduced in Section 2.2. This leads to

$$\mathbb{E}(G(u)) \approx \sum_{0\leq \ell+\nu\leq N} Q_{m_{N-\ell-\nu}}(G(D_\ell^\nu)),$$

where $Q_{m_{N-\ell-\nu}}$ denotes a MC sampling rule with given sample size $m_{N-\ell-\nu} \in \mathbb{N}$ such that $m_0 < m_1 < \cdots < m_N$. The main result of this work is to prove that the above approximation is (up to logarithmic factors) optimal in the sense that it is as good as the approximation given by the naive approach $Q_{m_N}(G(u_N^N))$, where all components are computed on the finest level, while reducing the computational cost.

The error/cost estimates from Section 6 show that the distribution of work among the individual levels is optimal up to logarithmic factors. This can be seen from the fact that the multi-index algorithm achieves the same (up to logarithmic factors) cost versus error ratio than the worst ratio of each of the involved algorithms (FEM, Monte Carlo, approximation of the random coefficient). Since a combined algorithm of this form cannot be more efficient than each of its components, this shows optimality.

2. Model problem.

We chose a simple Poisson model problem to give a concise presentation of the ideas and proof techniques. The authors are confident that very similar techniques can be used to include more general model problems. Moreover, we focus on the standard case of H^2-regularity of the Poisson problem. Intermediate cases with less regularity can be included with the same arguments, but are left out for the sake of clarity.

2.1. Abstract setting.

Consider a bounded "physical domain" $D \subseteq \mathbb{R}^d$ with Lipschitz boundary in dimension $d \in \{2, 3\}$. We model uncertain input data on a probability space $(\Omega, \Sigma, \mathbb{P})$. The mathematical expectation ("ensemble average") w.r. to the probability measure $\mathbb{P}$ is denoted by $\mathbb{E}$.

Define the parametrized bilinear form

$$a(A; w, v) := \int_D A(x)\nabla w(x) \cdot \nabla v(x)\, dx$$

for a scalar diffusion coefficient $A\colon D \to [0, \infty)$. To model uncertain input data, we consider random diffusion coefficients which satisfy $A(\cdot, \omega) \in L^\infty(D)$ for almost all $\omega \in \Omega$. Precisely, A is assumed a strongly measurable map from (Ω, Σ) to the Banach space $L^\infty(D)$, endowed with the Borel sigma algebra. For $A \in L^\infty(D)$, the bilinear form $a(A; ., .)$ is continuous on $H_0^1(D) \times H_0^1(D)$, the usual Sobolev space given by

$$H_0^1(D) := \big\{v \in L^2(D) \,:\, \nabla v \in L^2(D)^d,\ v|_{\partial D} = 0\big\}.$$

We assume at hand a sequence of approximate diffusion coefficients $(A^\nu)_{\nu\in\mathbb{N}}$ of $A = A^\infty$ which satisfy $A^\nu(\cdot, \omega) \in W^{1,\infty}(D)$ for almost all $\omega \in \Omega$ as well as

$$\lim_{\nu\to\infty} \|A - A^\nu\|_{L^\infty(\Omega; W^{1,\infty}(D))} = 0. \tag{2.1}$$

Furthermore, we assume the existence of deterministic bounds $A_{\min}$ and $A_{\max}$ such that for every $\nu \in \mathbb{N} \cup \{\infty\}$

$$0 < A_{\min} \leq \inf_{x\in D} A^\nu(x, \omega) \leq \sup_{x\in D} A^\nu(x, \omega) \leq A_{\max} < \infty. \tag{2.2}$$

To ease notation, we write $a^\nu_\omega(\cdot,\cdot) := a(A^\nu(\omega),\cdot,\cdot)$. Finally, suppose the right-hand side $f \in H^{-1}(D)$. We embed $L^2(D)$ in $H^{-1}(D)$ via the compact embedding $v \mapsto \langle v\,,\,\cdot\rangle_D$ for all $v \in L^2(D)$.

The assumptions imply ellipticity and continuity of the bilinear form, i.e., for almost all $\omega \in \Omega$

$$\inf_{\nu\in\mathbb{N}\cup\infty}\inf_{w\in H^1_0(D)}\frac{a^\nu_\omega(w,w)}{\|w\|^2_{H^1(D)}} \geq A_{\min} \tag{2.3}$$

as well as

$$\sup_{\nu\in\mathbb{N}\cup\infty}\sup_{w,v\in H^1_0(D)}\frac{a^\nu_\omega(w,v)}{\|w\|_{H^1(D)}\|v\|_{H^1(D)}} \leq A_{\max}. \tag{2.4}$$

The Lax-Milgram lemma implies with (2.3) and (2.4) unique solvability and continuity of the solution operator. This implies in particular the existence of a unique random solution u (i.e. a strongly measurable map $u : \Omega \to H^1_0(D)$) which is defined pathwise by: given $\omega \in \Omega$, find $u(\omega) \in H^1_0(D)$ such that

$$a(A(\omega); u(\omega), v) = \langle f\,,\,v\rangle_D \quad \text{for all } v \in H^1_0(D),\ \ \mathbb{P}\text{ a.e. } \omega \in \Omega.$$

The Lipschitz continuity of the data-to-solution operator $S_A : A \to u$ (for fixed source term f) on the data $A \in L^\infty(D)$ such that (2.2) holds implies the strong measurability of $u : \Omega \to H^1_0(D)$. We are interested in the expectation of a certain quantity of interest $G(\cdot)$ which is a deterministic, bounded linear functional $G(\cdot)\colon H^1_0(D) \to \mathbb{R}$, i.e.

$$\mathbb{E}(G(u)) \in \mathbb{R}.$$

We assume that G has an-L^2 representer, i.e., that there exists $g \in L^2(D)$ such that

$$G(v) = \int_D gv\,dx \quad \text{for all } v \in H^1_0(D).$$

2.2. Finite element discretization. We assume at our disposal a sequence of nested triangulations $\{\mathcal{T}_\ell\}_{\ell\in\mathbb{N}}$ with corresponding spaces $(\mathcal{X}_\ell)_{\ell\in\mathbb{N}}$ (such that $\mathcal{X}_\ell \subseteq \mathcal{X}_k \subset H^1_0(D)$ for all $\ell \leq k$). We assume the following approximation property of the spaces $\mathcal{X}_\ell$: There exists a constant $C_{\rm approx} > 0$ and a monotone sequence $\{h_\ell\}_{\ell\in\mathbb{N}}$ with $h_\ell > 0$ and with $\lim_\ell h_\ell = 0$ such that all $u \in H^2(D)$ satisfy

$$\inf_{v\in\mathcal{X}_\ell}\|u - v\|_{H^1(D)} \leq C_{\rm approx} h_\ell \|u\|_{H^2(D)}. \tag{2.5}$$

For convenience, we assume $h_{\ell+1} \geq C_{\rm unif} h_\ell$ for all $\ell \in \mathbb{N}$ and for some constant $C_{\rm unif} > 0$. A popular example would be based on the nested sequence $\{\mathcal{T}_\ell\}_{\ell\geq 0}$ of regular, uniform triangulations of D with corresponding decreasing sequence $\{h_\ell\}_{\ell\geq 0}$ of mesh-widths $h_\ell = \max\{\mathrm{diam}(T) : T \in \mathcal{T}_\ell\}$. The sequence $\{\mathcal{X}_\ell\}_{\ell\geq 0}$ of subspaces can then be chosen as spaces of continuous, piecewise-linear functions on $\mathcal{T}_\ell$.

Given the sequence $\{\mathcal{X}_\ell\}_{\ell\geq 0}$ of subspaces, the Galerkin approximation $u^\nu_\ell(\omega) \in \mathcal{X}_\ell$ is the solution of

$$a^\nu_\omega(u^\nu_\ell(\omega), v) = \langle f\,,\,v\rangle_D \quad \text{for all } v \in \mathcal{X}_\ell \text{ and almost all } \omega \in \Omega.$$

Unique solvability follows from the Lax-Milgram lemma and (2.3)–(2.4). Consider the solution operators $\mathbb{S}_\ell^\nu(\omega)\colon H^{-1}(D) \to \mathcal{X}_\ell$ defined by $\mathbb{S}_\ell^\nu(\omega) f := u_\ell^\nu(\omega)$. Moreover, let $(\mathbb{S}_\ell^\nu(\omega))^{-1}\colon \mathcal{X}_\ell \to H^{-1}(D)$ be defined by

$$((\mathbb{S}_\ell^\nu(\omega))^{-1} u)(v) := a_\omega^\nu(u, v) \quad \text{for all } u \in \mathcal{X}_\ell,\ v \in H_0^1(D).$$

For brevity, we will omit the random parameter and just write $\mathbb{S}_\ell^\nu := \mathbb{S}_\ell^\nu(\omega)$. Moreover, we write $\mathbb{S}_\infty^\nu f := u^\nu$, where $u^\nu(\omega) \in H_0^1(D)$ is the unique solution of

$$a_\omega^\nu(u^\nu(\omega), v) = \langle f\ ,\ v\rangle_D \quad \text{for all } v \in H_0^1(D).$$

Thus, u^ν denotes the exact solution corresponding to A^ν and $((\mathbb{S}_\infty^\nu(\omega))^{-1}\cdot)(v) := a_\omega^\nu(\cdot, v) \in H^{-1}(D)$.

For simplicity of presentation, we restrict to domains $D \subseteq \mathbb{R}^d$ which admit uniform (w.r.t. to all MC samples) H^2-regularity of the exact solution as long as $f \in L^2(D)$: there exists a constant $C_{\mathrm{reg}} > 0$ such that for all $\omega \in \Omega$

$$\sup_{\nu\in\mathbb{N}} \|\mathbb{S}_\infty^\nu f\|_{H^2(D)} \leq \frac{C_{\mathrm{reg}}}{{A_{\min}}^2}(1 + \|A^\nu(\omega)\|_{W^{1,\infty}(D)})\|f\|_{L^2(D)}. \tag{2.6}$$

We remark that when the solution of the Poisson equation is H^2-regular, (2.6) follows as an immediate consequence. Possible examples of domains D which satisfy this property include domains with C^2-boundary ∂D or convex domains.

Lemma 2.1. *The discrete solution operators $\mathbb{S}_\ell^\nu\colon H^{-1}(D) \to \mathcal{X}_\ell$ as defined above satisfy for almost all $\omega \in \Omega$ that*

$$\|\mathbb{S}_\ell^\nu\|_{H^{-1}(D)\to H^1(D)} \leq {A_{\min}}^{-1}$$

as well as

$$\|(\mathbb{S}_\ell^\nu)^{-1}\|_{\mathcal{X}_\ell\to H^{-1}(D)} \leq A_{\max}.$$

Proof. The result follows immediately from (2.3)–(2.4). ∎

3. Product structure of the approximation error. The main purpose of this section is to prove the product error estimate of Theorem 3.9 below at the end of this section. This error estimate factors the total error into error contributions of the approximation of the random coefficient $A \approx A^\nu$ and finite element approximation error $h_\ell \to 0$. We will restate several well-known results from finite-element analysis, as we will make use of the exact dependence on the constants.

In view of the multi-index decomposition in Section 6, we consider the "difference of differences"

$$D_\ell^\nu := (u_\ell^\nu - u_{\ell-1}^\nu) - (u_\ell^{\nu-1} - u_{\ell-1}^{\nu-1})\colon \Omega \to \mathcal{X}_\ell.$$

The goal is to get an error estimate of product form, as this allows us to obtain nearly optimal complexity estimates. The key observation is that the definition of D_ℓ^ν and $\mathbb{S}_\ell^\nu$ implies that

$$\begin{aligned} D_\ell^\nu &= (\mathbb{S}_\ell^\nu - \mathbb{S}_{\ell-1}^\nu) f - (\mathbb{S}_\ell^{\nu-1} - \mathbb{S}_{\ell-1}^{\nu-1}) f \\ &= (\mathbb{S}_\ell^\nu - \mathbb{S}_{\ell-1}^\nu)(\mathbb{S}_\ell^\nu)^{-1}(\mathbb{S}_\ell^\nu - \mathbb{S}_\ell^{\nu-1}) f + \text{remainder}, \end{aligned}$$

where the remainder term can be controlled in Lemma 3.5, below. The product form of the first term already suggest the product error estimate which is the goal of this section.

In the following, we use the operator norm for bilinear forms $b(\cdot,\cdot)\colon \mathcal{X}\times\mathcal{X}\to\mathbb{R}$ for a Hilbert space $\mathcal{X}$, i.e.,

$$\|b\| := \sup_{x,y\in\mathcal{X}\setminus\{0\}} \frac{|b(x,y)|}{\|x\|_{\mathcal{X}}\|y\|_{\mathcal{X}}}.$$

LEMMA 3.1. *Given $A, B\colon \Omega\to L^\infty(D)$, there holds the estimate*

$$\|a(A(\omega),\cdot,\cdot) - a(B(\omega),\cdot,\cdot)\| \leq \|A(\omega)-B(\omega)\|_{L^\infty(D)} \quad \text{for all } \omega\in\Omega.$$

as well as

$$\|\mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f\|_{H^1(D)} \leq {A_{\min}}^{-2}\|A^\nu(\omega)-A^\mu(\omega)\|_{L^\infty(D)}\|f\|_{L^2(D)}$$

for all $\ell,\nu,\mu\in\mathbb{N}$.

Proof. The first estimate follows since we have for almost all $\omega\in\Omega$ that

$$\begin{aligned}|a(A(\omega),u,v) - a(B(\omega),u,v)| &\leq \int_D |A(x,\omega)-B(x,\omega)||\nabla u||\nabla v|\,dx\\ &\leq \|A(\omega)-B(\omega)\|_{L^\infty(D)}\|u\|_{H^1(D)}\|v\|_{H^1(D)}.\end{aligned}$$

For the second statement, we combine the above with (2.3), and Lemma 2.1, to obtain

$$\begin{aligned}A_{\min}\|\mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f\|_{H^1(D)}^2 &\leq a_\omega^\nu(\mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f, \mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f)\\ &= \langle f\,,\,\mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f\rangle_D - a_\omega^\nu(\mathbb{S}_\ell^\mu f, \mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f)\\ &= (a_\omega^\mu - a_\omega^\nu)(\mathbb{S}_\ell^\mu f, \mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f)\\ &\leq {A_{\min}}^{-1}\|A^\nu - A^\mu\|_{L^\infty(D)}\|f\|_{L^2(D)}\|\mathbb{S}_\ell^\nu f - \mathbb{S}_\ell^\mu f\|_{H^1(D)}\end{aligned}$$

for all $\omega\in\Omega$. This concludes the proof. ∎

LEMMA 3.2 (Galerkin orthogonality). *There holds Galerkin orthogonality for all $k,\ell\in\mathbb{N}\cup\{\infty\}$, $\nu\in\mathbb{N}$ and all $f\in H^{-1}(D)$ in the form*

$$a_\omega^\nu(\mathbb{S}_k^\nu f, v) = a_\omega^\nu(\mathbb{S}_\ell^\nu f, v) \quad \text{for all } v\in\mathcal{X}_{\min\{\ell,k\}} \text{ and all } \omega\in\Omega.$$

Particularly, this implies $\mathbb{S}_\ell^\nu(\mathbb{S}_k^\nu)^{-1} = \mathrm{id}_{\mathcal{X}_k}$ for all $\ell\geq k$ and $k<\infty$.

Proof. By definition, we have

$$a_\omega^\nu(\mathbb{S}_k^\nu f, v) = \langle f\,,\,v\rangle_D = a_\omega^\nu(\mathbb{S}_\ell^\nu f, v).$$

To see the second statement, note that for $v\in\mathcal{X}_k$ and $w\in\mathcal{X}_\ell$, there holds by definition of the inverse

$$a_\omega^\nu(\mathbb{S}_\ell^\nu(\mathbb{S}_k^\nu)^{-1}v, w) = ((\mathbb{S}_k^\nu)^{-1}v)(w) = a_\omega^\nu(v,w).$$

This and the positive definiteness of the bilinear form $a_\omega^\nu(\cdot,\cdot)$ conclude the proof. ∎

For the next lemma, we define the energy norm

$$\|u\|_{\omega,\nu} := (a^\nu_\omega(u,u))^{1/2}.$$

Note that (2.3)–(2.4) ensure ${A_{\min}}^{1/2}\|\cdot\|_{H^1(D)} \le \|\cdot\|_{\omega,\nu} \le {A_{\max}}^{1/2}\|\cdot\|_{H^1(D)}$ for almost all $\omega\in\Omega$ and for all $\nu\in\mathbb{N}$.

There holds the following variant of Céa's lemma:

Lemma 3.3 (Céa's lemma). *For $v\colon\Omega\to\mathcal{X}_\ell$, $\omega\in\Omega$, and $k\le\ell$, we have*

$$\begin{aligned}\|(\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega)\|_{H^1(D)} &\le {A_{\min}}^{-1/2}\inf_{w\in\mathcal{X}_k}\|v(\omega)-w\|_{\omega,\mu}\\ &\le {A_{\min}}^{-1/2}{A_{\max}}^{1/2}\inf_{w\in\mathcal{X}_k}\|v(\omega)-w\|_{H^1(D)}.\end{aligned}$$

Proof. For almost all $\omega\in\Omega$, Galerkin orthogonality guarantees

$$\begin{aligned}&a^\mu_\omega\big((\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v, (\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v\big)\\ &\qquad = a^\mu_\omega\big((\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v, \mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1}v - w\big)\end{aligned}$$

for all $w\in\mathcal{X}_k$. Since a^ν_ω is a scalar product with respective norm $\|\cdot\|_{\omega,\nu}$, we have

$$\begin{aligned}&a^\mu_\omega\big((\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v, \mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1}v - w\big)\\ &\qquad \le \|(\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v\|_{\omega,\mu}\|\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1}v - w\|_{\omega,\mu}.\end{aligned}$$

Ellipticity (2.3), norm equivalence $A_{\min}^{1/2}\|\cdot\|_{H^1(D)} \le \|\cdot\|_{\omega,\nu} \le {A_{\max}}^{1/2}\|\cdot\|_{H^1(D)}$, and the fact that ω was arbitrary conclude the proof. ∎

The following lemma bounds the difference of the Galerkin projections $\mathbb{S}^\nu_k(\mathbb{S}^\nu_\ell)^{-1}$ for different parameters ν.

Lemma 3.4. *There holds for $\ell,k,\nu,\mu\in\mathbb{N}$, all $v\colon\Omega\to\mathcal{X}_\ell$, and all $\omega\in\Omega$*

$$\begin{aligned}&\|(\mathbb{S}^\nu_k(\mathbb{S}^\nu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega)\|_{H^1(D)}\\ &\qquad \le C_{\rm proj}(\omega)\|(A^\nu - A^\mu)(\omega)\|_{L^\infty(D)}\inf_{w\in\mathcal{X}_k}\|v(\omega)-w\|_{H^1(D)},\end{aligned}$$

where $C_{\rm proj}(\omega) := {A_{\min}}^{-2}A_{\max}$.

Proof. For $k\ge\ell$, we have $\mathbb{S}^\nu_k(\mathbb{S}^\nu_\ell)^{-1} = \mathrm{id}_{\mathcal{X}_\ell} = \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1}$ and thus the assertion holds trivially. Assume $k<\ell$. Define $v_k := (\mathbb{S}^\nu_k(\mathbb{S}^\nu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v\colon\Omega\to\mathcal{X}_\ell$. Ellipticity (2.3) of $a^\nu_\omega(\cdot,\cdot)$ together with Galerkin orthogonality shows for $\omega\in\Omega$

$$A_{\min}\|v_k(\omega)\|^2_{H^1(D)} \le a^\nu_\omega(v_k(\omega),v_k(\omega)) = a^\nu_\omega((\mathbb{S}^\nu_\ell(\mathbb{S}^\nu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega), v_k(\omega)).$$

Since $\mathbb{S}^\nu_\ell(\mathbb{S}^\nu_\ell)^{-1} = \mathrm{id}_{\mathcal{X}_\ell} = \mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1}$, we have

$$\begin{aligned}A_{\min}\|v_k(\omega)\|^2_{H^1(D)} &\le a^\nu_\omega((\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega), v_k(\omega))\\ &= a^\mu_\omega((\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega), v_k(\omega))\\ &\quad + (a^\nu_\omega - a^\mu_\omega)((\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega), v_k(\omega)).\end{aligned}$$

The first term on the right-hand side above is zero due to Galerkin orthogonality. Therefore, we obtain

(3.1)
$$\|v_k(\omega)\|^2_{H^1(D)} \lesssim A_{\min}^{-1}\|a^\nu_\omega - a^\mu_\omega\|\,\|(\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega)\|_{H^1(D)}\|v_k(\omega)\|_{H^1(D)}.$$

As shown in Lemma 3.1, there holds $\|a^\nu_\omega - a^\mu_\omega\| \le \|(A^\nu - A^\mu)(\omega)\|_{L^\infty(D)}$. Moreover, we have by Céa's lemma (Lemma 3.3)

$$\|(\mathbb{S}^\mu_\ell(\mathbb{S}^\mu_\ell)^{-1} - \mathbb{S}^\mu_k(\mathbb{S}^\mu_\ell)^{-1})v(\omega)\|_{H^1(D)} \le {A_{\min}}^{-1}{A_{\max}}^{1/2} \inf_{w\in\mathcal{X}_k} \|v(\omega) - w\|_{H^1(D)}.$$

This together with (3.1) concludes the proof. ∎

For the statement of the next result, we recall the definition of the double difference

$$D^\nu_\ell := (u^\nu_\ell - u^\nu_{\ell-1}) - (u^{\nu-1}_\ell - u^{\nu-1}_{\ell-1})\colon \Omega \to \mathcal{X}_\ell.$$

Lemma 3.5. *There holds for all $\omega \in \Omega$*

$$\begin{aligned}\|D^\nu_\ell(\omega)\|_{H^1(D)} &\le \|(\mathbb{S}^\nu_\ell - \mathbb{S}^\nu_{\ell-1})(\mathbb{S}^\nu_\ell)^{-1}(\mathbb{S}^\nu_\ell - \mathbb{S}^{\nu-1}_\ell)f\|_{H^1(D)} \\ &\quad + C_{\mathrm{proj}}(\omega)\|(A^\nu - A^{\nu-1})(\omega)\|_{L^\infty(D)} \inf_{w\in\mathcal{X}_k} \|u^{\nu-1}_\ell(\omega) - v\|_{H^1(D)}.\end{aligned} \tag{3.2}$$

Proof. Straightforward expansion of the equation and $\mathbb{S}^\nu_\ell(\mathbb{S}^\nu_k)^{-1} = \mathrm{id}_{\mathcal{X}_k}$, $k \le \ell$ from Lemma 3.2 show

$$\begin{aligned}D^\nu_\ell &= ((\mathbb{S}^\nu_\ell - \mathbb{S}^\nu_{\ell-1}) - (\mathbb{S}^{\nu-1}_\ell - \mathbb{S}^{\nu-1}_{\ell-1}))f \\ &= (\mathbb{S}^\nu_\ell - \mathbb{S}^\nu_{\ell-1})(\mathbb{S}^\nu_\ell)^{-1}(\mathbb{S}^\nu_\ell - \mathbb{S}^{\nu-1}_\ell)f - (\mathbb{S}^\nu_{\ell-1}(\mathbb{S}^\nu_\ell)^{-1}\mathbb{S}^{\nu-1}_\ell - \mathbb{S}^{\nu-1}_{\ell-1})f.\end{aligned}$$

The last term on the right-hand side satisfies

$$\begin{aligned}&\|(\mathbb{S}^\nu_{\ell-1}(\mathbb{S}^\nu_\ell)^{-1}\mathbb{S}^{\nu-1}_\ell - \mathbb{S}^{\nu-1}_{\ell-1})f\|_{H^1(D)} \\ &\qquad \le \|(\mathbb{S}^{\nu-1}_{\ell-1}(\mathbb{S}^{\nu-1}_\ell)^{-1}\mathbb{S}^{\nu-1}_\ell - \mathbb{S}^{\nu-1}_{\ell-1})f\|_{H^1(D)} \\ &\qquad\quad + \|(\mathbb{S}^\nu_{\ell-1}(\mathbb{S}^\nu_\ell)^{-1} - \mathbb{S}^{\nu-1}_{\ell-1}(\mathbb{S}^{\nu-1}_\ell)^{-1})\mathbb{S}^{\nu-1}_\ell f\|_{H^1(D)}.\end{aligned} \tag{3.3}$$

The first term on the right-hand side satisfies for all $v \in \mathcal{X}_{\ell-1}$

$$a^\nu_\omega((\mathbb{S}^{\nu-1}_{\ell-1}(\mathbb{S}^{\nu-1}_\ell)^{-1}\mathbb{S}^{\nu-1}_\ell - \mathbb{S}^{\nu-1}_{\ell-1})f, v) = a^\nu_\omega((\mathbb{S}^{\nu-1}_\ell(\mathbb{S}^{\nu-1}_\ell)^{-1}\mathbb{S}^{\nu-1}_\ell - \mathbb{S}^{\nu-1}_\ell)f, v) = 0$$

and thus $\|(\mathbb{S}^{\nu-1}_{\ell-1}(\mathbb{S}^{\nu-1}_\ell)^{-1}\mathbb{S}^{\nu-1}_\ell - \mathbb{S}^{\nu-1}_{\ell-1})f\|_{H^1(D)} = 0$. For the second term on the right-hand side of (3.3), Lemma 3.4 with $\mu = \nu - 1$ and $k = \ell - 1$ proves

$$\begin{aligned}&\|(\mathbb{S}^\nu_{\ell-1}(\mathbb{S}^\nu_\ell)^{-1} - \mathbb{S}^{\nu-1}_{\ell-1}(\mathbb{S}^{\nu-1}_\ell)^{-1})\mathbb{S}^{\nu-1}_\ell f\|_{H^1_0(D)} \\ &\qquad \lesssim \|A^\nu(\omega) - A^{\nu-1}(\omega)\|_{L^\infty(D)} \inf_{v\in\mathcal{X}_{\ell-1}} \|u^{\nu-1}_\ell(\omega) - v\|_{H^1(D)}.\end{aligned}$$

Altogether, this concludes the proof. ∎

The following result is well-known and we reprove it in our setting for the convenience of the reader.

Lemma 3.6 (Aubin-Nitsche duality). *There holds for all $v \in H^1_0(D)$ that*

$$\|v - \mathbb{S}^\nu_\ell(\mathbb{S}^\nu_\infty)^{-1}v\|_{L^2(D)} \le C_{\mathrm{approx}}\frac{C_{\mathrm{reg}}}{{A_{\min}}^2}(1 + \|A^\nu(\omega)\|_{W^{1,\infty}(D)})h_\ell\|v\|_{H^1(D)}.$$

Proof. Let $\iota\colon L^2(D) \to H^{-1}(D)$ be the usual embedding via the $L^2(D)$-scalar product. Define $V := v - \mathbb{S}_\ell^\nu(\mathbb{S}_\infty^\nu)^{-1}v$. We have with Galerkin orthogonality and by symmetry of a_ω^ν for all $w \in \mathcal{X}_\ell$

$$\begin{aligned}\|v - \mathbb{S}_\ell^\nu(\mathbb{S}_\infty^\nu)^{-1}v\|_{L^2(D)}^2 &= a_\omega^\nu(\mathbb{S}_\infty^\nu \circ \iota(V), V) = a_\omega^\nu(\mathbb{S}_\infty^\nu \circ \iota(V) - w, V)\\ &\leq \|\mathbb{S}_\infty^\nu \circ \iota(V) - w\|_{H^1(D)}\|V\|_{H^1(D)}.\end{aligned}$$

Since $w \in \mathcal{X}_\ell$ was arbitrary, we get with (2.5) and (2.6)

$$\begin{aligned}&\|v - \mathbb{S}_\ell^\nu(\mathbb{S}_\infty^\nu)^{-1}v\|_{L^2(D)}^2\\ &\quad\leq C_{\mathrm{approx}} h_\ell \|\mathbb{S}_\infty^\nu \circ \iota(V)\|_{H^2(D)}\|V\|_{H^1(D)}\\ &\quad\leq C_{\mathrm{approx}} \frac{C_{\mathrm{reg}}}{{A_{\min}}^2}(1 + \|A^\nu(\omega)\|_{W^{1,\infty}(D)}) h_\ell \|v - \mathbb{S}_\ell^\nu(\mathbb{S}_\infty^\nu)^{-1}v\|_{L^2(D)}\|V\|_{H^1(D)}.\end{aligned}$$

With Lemma 2.1, we show $\|V\|_{H^1(D)} \leq (1 + {A_{\min}}^{-1}A_{\min})\|v\|_{H^1(D)}$ and thus conclude the proof. ∎

The following result bounds the first term on the right-hand side of the estimate in Lemma 3.5 by an error estimate in product form.

LEMMA 3.7. *There holds for all* $\omega \in \Omega$

$$\begin{aligned}&\|(\mathbb{S}_\ell^\nu - \mathbb{S}_{\ell-1}^\nu)(\mathbb{S}_\ell^\nu)^{-1}(\mathbb{S}_\ell^\nu - \mathbb{S}_\ell^{\nu-1})f\|_{H^1(D)}\\ &\qquad\leq \widetilde{C}_{\mathrm{prod}}(\omega) h_\ell \|(A^\nu - A^{\nu-1})(\omega)\|_{W^{1,\infty}(D)}\|f\|_{L^2(D)},\end{aligned}$$

where $\widetilde{C}_{\mathrm{prod}}(\omega) \simeq C_{\mathrm{unif}} {A_{\min}}^{-5} {A_{\max}}^{1/2}(1 + \max_{i\in\{0,1\}} \|A^{\nu-i}(\omega)\|_{W^{1,\infty}(D)})^2 > 0.$

Proof. First, Céa's lemma (Lemma 3.3) shows for $v\colon \Omega \to \mathcal{X}_\ell$

$$\|(\mathbb{S}_\ell^\nu - \mathbb{S}_{\ell-1}^\nu(\omega))(\mathbb{S}_\ell^\nu)^{-1}v\|_{H^1(D)} \leq {A_{\min}}^{-1} \inf_{w\in\mathcal{X}_{\ell-1}} \|v(\omega) - w\|_{\omega,\nu}.$$

Let $v := (\mathbb{S}_\ell^\nu - \mathbb{S}_\ell^{\nu-1})f$ and choose $w := \mathbb{S}_{\ell-1}^\nu(\mathbb{S}_\infty^\nu)^{-1}v$. Then, there holds with Galerkin orthogonality $a_\omega^\nu(w, v - w) = a_\omega^\nu(v - \mathbb{S}_{\ell-1}^\nu(\mathbb{S}_\infty^\nu)^{-1}v, w) = 0$ and hence

$$\begin{aligned}\|v - w\|_{\omega,\nu}^2 &= a_\omega^\nu(v, v - w) = a_\omega^\nu(u^\nu - \mathbb{S}_\ell^{\nu-1}f, v - w)\\ &= a_\omega^{\nu-1}(u^\nu - \mathbb{S}_\ell^{\nu-1}f, v - w) + (a_\omega^\nu - a_\omega^{\nu-1})(u^\nu - \mathbb{S}_\ell^{\nu-1}f, v - w)\\ &= a_\omega^{\nu-1}(u^\nu, v - w) - \langle f\,,\, v - w\rangle_D + (a_\omega^\nu - a_\omega^{\nu-1})(u^\nu - \mathbb{S}_\ell^{\nu-1}f, v - w),\end{aligned}$$

where we inserted and subtracted $a_\omega^{\nu-1}(\cdot,\cdot)$. This leads to

$$\begin{aligned}\|v - w\|_{\omega,\nu}^2 &= a_\omega^{\nu-1}(u^\nu, v - w) - a_\omega^\nu(u^\nu, v - w) + (a_\omega^\nu - a_\omega^{\nu-1})(u^\nu - \mathbb{S}_\ell^{\nu-1}f, v - w)\\ &= -(a_\omega^\nu - a_\omega^{\nu-1})(\mathbb{S}_\ell^{\nu-1}f, v - w)\\ &= -(a_\omega^\nu - a_\omega^{\nu-1})(u^{\nu-1}, v - w) - (a_\omega^\nu - a_\omega^{\nu-1})(u_\ell^{\nu-1} - u^{\nu-1}, v - w),\end{aligned}$$

where we used $\mathbb{S}_\ell^{\nu-1}f = u_\ell^{\nu-1}$ and we added and subtracted the corresponding exact solution $u^{\nu-1}$. Using the definition of the bilinear forms as well as integration by parts, the above reads

$$\begin{aligned}\|v - w\|_{\omega,\nu}^2 = \int_D &\big(\nabla(A^\nu - A^{\nu-1})\cdot\nabla u^{\nu-1} + (A^\nu - A^{\nu-1})\Delta u^{\nu-1}\big)(v - w)\,dx\\ &- (a_\omega^\nu - a_\omega^{\nu-1})(u_\ell^{\nu-1} - u^{\nu-1}, v - w)\\ \leq\ &\|A^\nu - A^{\nu-1}\|_{W^{1,\infty}(D)}\|u^{\nu-1}\|_{H^2(D)}\|v - w\|_{L^2(D)}\\ &+ \|a_\omega^\nu - a_\omega^{\nu-1}\|\|u_\ell^{\nu-1} - u^{\nu-1}\|_{H^1(D)}\|v - w\|_{H^1(D)}.\end{aligned}$$

Finally, Lemma 3.6 shows

$$\|v-w\|_{L^2(D)} \lesssim h_{\ell-1}{A_{\min}}^{-2}(1+\|A^{\nu-1}(\omega)\|_{W^{1,\infty}(D)})\|v\|_{H^1(D)}$$
$$\lesssim h_{\ell-1}{A_{\min}}^{-3}(1+\|A^{\nu-1}(\omega)\|_{W^{1,\infty}(D)})\|f\|_{L^2(D)},$$

where the last estimate uses Lemma 2.1. Assumption (2.5), together with the Céa lemma (Lemma 3.3), implies

$$\|u_\ell^{\nu-1}-u^{\nu-1}\|_{H^1(D)} \lesssim {A_{\min}}^{-1}{A_{\max}}^{1/2}h_\ell\|u^{\nu-1}\|_{H^2(D)}.$$

Together with (2.6), we obtain

$$\|u^{\nu-1}\|_{H^2(D)} \lesssim {A_{\min}}^{-2}(1+\|A^{\nu-1}(\omega)\|_{W^{1,\infty}(D)}\|f\|_{L^2(D)} \qquad \square$$

and thus conclude the proof.

Finally, we have collected all the ingredients to obtain the combined discretization error estimate in product form.

PROPOSITION 3.8. *There holds for all* $\omega\in\Omega$

$$\|D_\ell^\nu(\omega)\|_{H^1(D)} \le C_{\rm prod}(\omega)h_\ell\|(A^\nu-A^{\nu-1})(\omega)\|_{W^{1,\infty}(D)}\|f\|_{L^2(D)},$$

where $C_{\rm prod}(\omega)\simeq \widetilde{C}_{\rm prod}(\omega)(1+A_{\max})>0$.

Proof. The first term on the right-hand side of (3.2) is bounded by Lemma 3.7. For the second term, we use (2.5) together with (2.6) to obtain a similar bound. Finally, we exploit that $h_\ell\ge C_{\rm unif}h_{\ell-1}$ and conclude the proof. $\square$

Since we are interested in the error of the goal functional $G(\cdot)$, we may exploit a standard Aubin-Nitsche duality argument to double the rate of convergence.

THEOREM 3.9. *There holds for all* $\omega\in\Omega$

$$|G(D_\ell^\nu(\omega))| \le \overline{C}_{\rm prod}(\omega)h_\ell^2\min\big\{1,\|(A^\nu-A^{\nu-1})(\omega)\|_{W^{1,\infty}(D)}\big\}\|f\|_{L^2(D)}\|g\|_{L^2(D)}$$

with $\overline{C}_{\rm prod}(\omega)>0$ *depending on* $C_{\rm prod}(\omega)$ *from Proposition 3.8 via* $\overline{C}_{\rm prod}(\omega)\simeq {A_{\min}}^{-5}A_{\max}\|A^\nu(\omega)\|_{W^{1,\infty}(D)}\|A^{\nu-1}(\omega)\|_{W^{1,\infty}(D)}C_{\rm prod}(\omega)$.

Proof. Let $g^\nu\in H_0^1(\Omega)$ such that $G(\cdot)=a_\omega^\nu(\cdot,g^\nu)$ (note that such a function always exists due to the ellipticity (2.3) of $a_\omega^{\nu-1}$). There holds for $v,w\in\mathcal{X}_{\ell-1}$

$$G(D_\ell^\nu) = a_\omega^\nu(u_\ell^\nu-u_{\ell-1}^\nu,g^\nu)-a_\omega^{\nu-1}(u_\ell^{\nu-1}-u_{\ell-1}^{\nu-1},g^{\nu-1})$$
$$= a_\omega^\nu(u_\ell^\nu-u_{\ell-1}^\nu,g^\nu-v)-a_\omega^{\nu-1}(u_\ell^{\nu-1}-u_{\ell-1}^{\nu-1},g^{\nu-1}-v),$$

where we used Galerkin orthogonality (Lemma 3.2) to insert $v\in\mathcal{X}_{\ell-1}$. Adding and subtracting of $a_\omega^\nu(\cdot,\cdot)$ leads to

$$G(D_\ell^\nu) = a_\omega^\nu(u_\ell^\nu-u_{\ell-1}^\nu,g^\nu-v)-a_\omega^\nu(u_\ell^{\nu-1}-u_{\ell-1}^{\nu-1},g^{\nu-1}-v)$$
$$+(a_\omega^\nu-a_\omega^{\nu-1})(u_\ell^{\nu-1}-u_{\ell-1}^{\nu-1},g^{\nu-1}-v)$$
$$= a_\omega^\nu(u_\ell^\nu-u_{\ell-1}^\nu,g^{\nu-1}-v)-a_\omega^\nu(u_\ell^{\nu-1}-u_{\ell-1}^{\nu-1},g^{\nu-1}-v)$$
$$+(a_\omega^\nu-a_\omega^{\nu-1})(u_\ell^{\nu-1}-u_{\ell-1}^{\nu-1},g^{\nu-1}-v)+a_\omega^\nu(u_\ell^\nu-u_{\ell-1}^\nu,g^\nu-g^{\nu-1}-w),$$

where we added and subtracted $a^\nu_\omega(u^\nu_\ell - u^\nu_{\ell-1}, g^{\nu-1})$ and inserted $w \in \mathcal{X}_{\ell-1}$ using Galerkin orthogonality (Lemma 3.2). Recalling the definition of D^ν_ℓ, we arrive at

$$\begin{aligned} G(D^\nu_\ell) &= a^\nu_\omega(D^\nu_\ell, g^{\nu-1} - v) + (a^\nu_\omega - a^{\nu-1}_\omega)(u^{\nu-1}_\ell - u^{\nu-1}_{\ell-1}, g^{\nu-1} - v) \\ &\quad + a^\nu_\omega(u^\nu_\ell - u^\nu_{\ell-1}, g^\nu - g^{\nu-1} - w). \end{aligned}$$

Lemma 3.1 and the Céa lemma (Lemma 3.3) together with (2.5) and (2.6) allows us to estimate

$$\begin{aligned} |G(D^\nu_\ell)| &\lesssim A_{\max}\|D^\nu_\ell\|_{H^1(D)}\|g^{\nu-1} - v\|_{H^1(D)} \\ &\quad + \|A^\nu - A^{\nu-1}\|_{L^\infty(D)}\|u^{\nu-1}_\ell - u^{\nu-1}_{\ell-1}\|_{H^1(D)}\|g^{\nu-1} - v\|_{H^1(D)} \\ &\quad + \|u^\nu_\ell - u^\nu_{\ell-1}\|_{H^1(D)}\|g^\nu - g^{\nu-1} - w\|_{H^1(D)} \\ &\lesssim A_{\max}\|D^\nu_\ell\|_{H^1(D)}\|g^{\nu-1} - v\|_{H^1(D)} \\ &\quad + {A_{\min}}^{-3}{A_{\max}}^{1/2}(1 + \|A^{\nu-1}(\omega)\|_{W^{1,\infty}(D)})\|f\|_{L^2(D)}h_\ell \\ &\quad \Big(\|A^\nu - A^{\nu-1}\|_{L^\infty(D)}\|g^{\nu-1} - v\|_{H^1(D)} + \|g^\nu - g^{\nu-1} - w\|_{H^1(D)}\Big). \end{aligned} \tag{3.4}$$

Since $G(\cdot) = \int_D g(x)(\cdot)\,dx$ for some $g \in L^2(D)$, we obtain from (2.6) that $g^\nu, g^{\nu-1} \in H^2(D)$. Therefore, and since $v \in \mathcal{X}_{\ell-1}$ was arbitrary, (2.5) and (2.6) show

$$\inf_{v\in\mathcal{X}_{\ell-1}} \|g^{\nu-1} - v\|_{H^1(D)} \lesssim {A_{\min}}^{-2}(1 + \|A^{\nu-1}(\omega)\|_{W^{1,\infty}(D)})h_\ell\|g\|_{L^2(D)}.$$

Moreover, there holds for all $v \in H^1_0(D)$

$$\begin{aligned} a^\nu_\omega(g^\nu - g^{\nu-1}, v) &= \langle g\,,\, v\rangle_D - a^\nu_\omega(g^{\nu-1}, v) = (a^{\nu-1} - a^\nu)(g^{\nu-1}, v) \\ &= \int_D \big(\nabla(A^\nu - A^{\nu-1})\cdot\nabla g^{\nu-1} + (A^\nu - A^{\nu-1})\Delta g^{\nu-1}\big)v\,dx. \end{aligned}$$

It is easy to see that the right-hand side is of the form $\langle r\,,\, v\rangle_D$ for some $r \in L^2(D)$ with

$$\|r\|_{L^2(D)} \le 2\|A^\nu - A^{\nu-1}\|_{W^{1,\infty}(D)}\|g^{\nu-1}\|_{H^2(D)} \lesssim \|A^\nu - A^{\nu-1}\|_{W^{1,\infty}(D)}\|g\|_{L^2(D)}.$$

Therefore, (2.6) shows

$$\|g^\nu - g^{\nu-1}\|_{H^2(D)} \lesssim {A_{\min}}^{-2}(1 + \|A^\nu(\omega)\|_{W^{1,\infty}(D)})\|A^\nu - A^{\nu-1}\|_{W^{1,\infty}(D)}\|g\|_{L^2(D)}.$$

Since $w \in \mathcal{X}_{\ell-1}$ in (3.4) was arbitrary, the same argument and (2.5) show

$$\begin{aligned} &\inf_{w\in\mathcal{X}_{\ell-1}} \|g^\nu - g^{\nu-1} - w\|_{H^1(D)} \\ &\qquad \lesssim h_\ell {A_{\min}}^{-2}(1 + \|A^\nu(\omega)\|_{W^{1,\infty}(D)})\|A^\nu - A^{\nu-1}\|_{W^{1,\infty}(D)}\|g\|_{L^2(D)}. \end{aligned}$$

Altogether, we conclude the proof by use of Proposition 3.8, the above estimates, and insertion in (3.4). The minimum in the statement follows from standard arguments which we will sketch briefly. There holds

$$G(u^\nu_\ell - u^\nu_{\ell-1}) = a^\nu_\omega(u^\nu_\ell - u^\nu_{\ell-1}, g^\nu) = a^\nu_\omega(u^\nu_\ell - u^\nu_{\ell-1}, g^\nu - v)$$

for all $v \in \mathcal{X}_{\ell-1}$. As above, choosing $v = \mathbb{S}^\nu_\ell(\mathbb{S}^\nu_\infty)^{-1}g^\nu$ and Lemma 3.3 together with (2.5) leads to

$$\begin{aligned} |G(u^\nu_\ell - u^\nu_{\ell-1})| &\lesssim \|u^\nu_\ell - u^\nu_{\ell-1}\|_{H^1(D)}h_{\ell-1}\|g\|_{L^2(D)} \\ &\lesssim h^2_{\ell-1}\|f\|_{L^2(D)}\|g\|_{L^2(D)}. \end{aligned}$$

This concludes the proof. ∎

4. Approximation of the random coefficient. This section gives two examples of how to choose the random coefficient $A(x,\omega)$ as well as the approximations $A^\nu(x,\omega)$ in terms of the KL-expansion.

4.1. KL expansion. In this section, we assume $\Omega = [0,1]^{\mathbb{N}}$, and define $\omega = (\omega_i)_{i\in\mathbb{N}}$. We assume that A^ν is of the form

$$A^\nu(x,\omega) := \phi_0(x) + \sum_{j=1}^{s_\nu} \psi_j(\omega_j)\phi_j(x) \tag{4.1}$$

for functions $\phi_j \in W^{1,\infty}(D)$ and $\psi_j \in L^\infty([0,1],[-C_\psi, C_\psi])$ for some fixed $C_\psi > 0$. While the literature often deals with the uniform case $\psi_j(\omega) := \omega - 1/2$ (see next subsection), we allow this slightly more general case. We assume that the series converges absolutely in $W^{1,\infty}(D)$ for all $\omega \in \Omega$ and hence define

$$A(x,\omega) := A^\infty(x,\omega) := \phi_0(x) + \sum_{j=1}^{\infty} \psi_j(\omega_j)\phi_j(x).$$

Moreover, we assume that (2.2) holds.

THEOREM 4.1. *Under the assumptions of the current section, there holds*

$$\|G(D_\ell^\nu)\|_{L^\infty(\Omega)} \le C_{\mathrm{KL}} h_\ell^2 \sum_{i=s_{\nu-1}+1}^{s_\nu} \|\phi_i\|_{W^{1,\infty}(D)} \|f\|_{L^2(D)} \|g\|_{L^2(D)}. \tag{4.2}$$

The constant $C_{\mathrm{KL}} > 0$ depends on C_ψ but is independent of ℓ, ν, and ω.

Proof. The estimate follows immediately by definition of A^ν and Theorem 3.9. ∎

4.2. KL expansion with uniform random variables. In many cases, it is possible to reduce (4.1) to the simplified form

$$A^\nu(x,\omega) := \phi_0(x) + \sum_{j=1}^{s_\nu} \omega_j\phi_j(x), \tag{4.3}$$

where now $\Omega = [-1/2, 1/2]^{\mathbb{N}}$ and $\operatorname{ess\,inf}_{x\in D} \phi_0(x) > 0$. This means setting $\psi_j(\omega) := \omega - 1/2$ in (4.1).

Remark 4.2. Note that theoretically, the case from Section 4.1 can always be reduced to the present case. However, in many cases, this requires the user to precompute all functions ϕ_j which is computationally impractical.

It turns out that in this case, an improved version of Theorem 3.9 (see Theorem 4.7 at the end of this section) can be derived by arguments already used for quasi-Monte Carlo estimates (see, e.g., the works [8, 9] and the references therein). Given a subset $\Omega' \subseteq \prod_{j\in\mathbb{N}} \mathbb{C}$, we define for all $j \in \mathbb{N}$

$$\Omega'_j := \big\{\omega_j \in \mathbb{C} : \exists \omega_i \in \mathbb{C},\, i \in \mathbb{N}\setminus\{j\} \text{ such that } \omega = (\omega_1,\omega_2,\ldots) \in \Omega'\big\}.$$

LEMMA 4.3. *Assume that $\Omega' \supseteq \Omega$ is such that all results of Section 3 hold true with Ω' instead of Ω. This is particularly the case if the random coefficient remains uniformly bounded away from zero and infinity also in Ω'. Then the map $F\colon \Omega'_j \to \mathbb{C}$, $\omega_j \mapsto G(\mathbb{S}_\ell^\nu(\omega)f)$ is holomorphic for all $j \in \mathbb{N}$.*

Proof. Along the lines of the argument in [7], we verify complex differentiability of the parametric solutions. Fix $j \in \mathbb{N}$. Given $z \in \mathbb{C}$, define $\omega + z \in \mathbb{C}^{\mathbb{N}}$ by $(\omega+z)_i = \omega_i$ for all $i \neq j$ and $(\omega + z)_j = \omega_j + z$. Let z be sufficiently small such that there exists $\varepsilon \geq 2|z|$ with $B_\varepsilon(\omega) \subseteq \Omega'$. By definition, we have for $v \in \mathcal{X}_\ell$

$$\begin{aligned} 0 &= a^\nu_{\omega+z}(\mathbb{S}^\nu_\ell(\omega+z)f, v) - a^\nu_\omega(\mathbb{S}^\nu_\ell(\omega)f, v) \\ &= \int_D (A^\nu(x,\omega+z) - A^\nu(x,\omega))\nabla\mathbb{S}^\nu_\ell(\omega+z)f \cdot \nabla v \,dx + a^\nu_\omega(\mathbb{S}^\nu_\ell(\omega+z)f - \mathbb{S}^\nu_\ell(\omega)f, v). \end{aligned}$$

Let $g^\nu \in \mathcal{X}_\ell$ denote the representer of $G(\cdot)|_{\mathcal{X}_\ell}$ with respect to a^ν_ω. This and the above allows us to compute

$$\begin{aligned} \frac{G(\mathbb{S}^\nu_\ell(\omega+z)f) - G(\mathbb{S}^\nu_\ell(\omega)f)}{z} &= \frac{a^\nu_\omega(\mathbb{S}^\nu_\ell(\omega+z)f - \mathbb{S}^\nu_\ell(\omega)f, g^\nu)}{z} \\ &= -\int_D \frac{A^\nu(x,\omega+z) - A^\nu(x,\omega)}{z}\nabla\mathbb{S}^\nu_\ell(\omega+z)f\cdot\nabla g^\nu\,dx. \end{aligned} \tag{4.4}$$

Since A^ν is holomorphic, Cauchy's integral formula shows for $B_\varepsilon(\omega_j) \subset \Omega'_j$ that

$$\begin{aligned} &\Big|\frac{A^\nu(x,\omega+z) - A^\nu(x,\omega)}{z} - \partial_{\omega_j}A^\nu(x,\omega)\Big| \\ &\quad= \frac{1}{2\pi}\Big|\int_{\partial B_\varepsilon(\omega_j)} \frac{1}{z}\Big(\frac{A^\nu(x,y)}{(y-(\omega_j+z))} - \frac{A^\nu(x,y)}{(y-\omega_j)}\Big) - \frac{A^\nu(x,y)}{(y-\omega_j)^2}\,dy\Big| \\ &\quad= \frac{1}{2\pi}\Big|\int_{\partial B_\varepsilon(\omega_j)} \frac{A^\nu(x,y)}{(y-\omega_j-z)(y-\omega_j)} - \frac{A^\nu(x,y)}{(y-\omega_j)^2}\,dy\Big| \\ &\quad= \frac{1}{2\pi}\Big|\int_{\partial B_\varepsilon(\omega_j)} \frac{A^\nu(x,y)z}{(y-\omega_j-z)(y-\omega_j)^2}\,dy\Big| \\ &\quad\lesssim \varepsilon^{-2}\|A^\nu\|_{L^\infty(\Omega\times D)}|z|\,. \end{aligned}$$

This uniform convergence in $|z|$ together with Lemma 3.1 shows that passing to the limit $z \to 0$ in $\mathbb{C}$ in (4.4) leads to

$$\partial_{\omega_j}G(\mathbb{S}^\nu_\omega f) = -\int_D \partial_{\omega_j}A^\nu(x,\omega)\nabla\mathbb{S}^\nu_\ell(\omega)f\cdot\nabla g^\nu\,dx \in \mathbb{C}.$$

This shows that F is complex differentiable and thus holomorphic. ◻

LEMMA 4.4. *Let $(\varrho_j)_{j\in\mathbb{N}}$ be a positive sequence such that*

$$\Omega \subset \Omega' := \prod_{j\in\mathbb{N}} B_{1+\varrho_j}(0)$$

and that all the results of Section 3 hold true with Ω' instead of Ω. Given $\ell,\nu \in \mathbb{N}$, the map $F^\nu_\ell\colon \Omega \to \mathbb{R}$, $\omega \mapsto G(D^\nu_\ell(\omega))$ satisfies

$$\begin{aligned} &\frac{\|\partial^\alpha_\omega F^\nu_\ell\|_{L^\infty(\Omega)}}{\|f\|_{L^2(D)}\|g\|_{L^2(D)}} \\ &\quad\leq \begin{cases} 0 & \sum_{i=s_\nu+1}^\infty \alpha_i > 0, \\ C_{\mathrm{der}}\frac{\alpha! h_\ell^2}{\prod_{i=1}^\infty \varrho_i^{\alpha_i}}\min\{1, \sup_{\omega\in\Omega'}\|A^\nu - A^{\nu-1}\|_{W^{1,\infty}(D)}\} & \textit{else,} \end{cases} \end{aligned}$$

for all multi-indices $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $|\alpha| < \infty$. The constant $C_{\mathrm{der}} > 0$ depends only on C_{prod} from Theorem 3.9.

Proof. For brevity of presentation, we fix ℓ and ν and write $F := F_\ell^\nu$. Lemma 4.3 shows that F can be extended to a function $F\colon \Omega' \to \mathbb{C}$, which is holomorphic in each coordinate ω_j. Moreover, Lemma 3.1 proves that F is uniformly continuous in Ω. Therefore, we obtain immediately by induction that F satisfies the multidimensional analog of Cauchy's integral formula for all $\omega \in \Omega'$

$$F(\omega) = (2\pi \mathrm{i})^{-n} \int_{\partial B_{\varepsilon_1}(\omega_{d_1})} \cdots \int_{\partial B_{\varepsilon_n}(\omega_{d_n})} \frac{F(z)}{(z_1-\omega_{d_1})\ldots(z_n-\omega_{d_n})}\, dz_1 \ldots dz_n,$$

where $(d_1,\ldots,d_n) \in \mathbb{N}^n$ contains exactly n distinct dimensions and the parameters $\varepsilon_i > 0$, $i = 1,\ldots,n$ are chosen so small that the integration domains of the contour integrals above are contained in Ω'. This shows immediately that for any multi-index $\alpha \in {\mathbb{N}_0}^{\mathbb{N}}$ with $|\alpha| < \infty$, $\partial_\omega^\alpha F$ is holomorphic in each variable. Iterated application of Cauchy's integral formula shows for all $\omega \in \Omega$ that

$$\partial_\omega^\alpha F(\omega) = \Big(\prod_{\substack{i=1\\ \alpha_i \neq 0}}^\infty \frac{\alpha_i!}{2\pi \mathrm{i}} \Big) \int_{\prod_{\substack{i=1\\ \alpha_i\neq 0}}^\infty \partial B_{\varrho_i}(\omega_i)} \frac{F(z)}{\prod_{\substack{i=1\\ \alpha_i\neq 0}}^\infty (z_i-\omega_i)^{\alpha+1}}\, dz\ .$$

This shows immediately

$$|\partial_\omega^\alpha F(\omega)| \leq \Big(\prod_{\substack{i=1\\ \alpha_i \neq 0}}^\infty \frac{\alpha_i!}{2\pi} 2\pi \varrho_i^{-\alpha_i} \Big) \|F\|_{L^\infty(\Omega')} \leq \alpha! \Big(\prod_{i=1}^\infty \varrho_i^{-\alpha_i} \Big) \|F\|_{L^\infty(\Omega')}.$$

This and Theorem 3.9 with $A^\nu(\omega) = \phi_0 + \sum_{i=1}^\nu \omega_i \phi_i$ conclude the proof. ∎

LEMMA 4.5. *Define for sufficiently small $\delta > 0$*

$$\beta_i := \frac{\|\phi_i\|_{W^{1,\infty}(D)}}{(\operatorname{ess\,inf}_{x\in D} \phi_0(x) - 2\delta)}.$$

Given $\ell, \nu \in \mathbb{N}$, the map $F\colon \Omega \to \mathbb{R}$, $\omega \mapsto G(D_\ell^\nu(\omega))$ satisfies

$$\|\partial_\omega^\alpha F\|_{L^\infty(\Omega)} \leq \widetilde{C}_{\mathrm{der}} \begin{cases} 0 & \sum_{i=s_\nu+1}^\infty \alpha_i > 0, \\ \Big(\prod_{i=1}^{s_\nu} \beta_i^{\alpha_i} \Big) h_\ell^2 \|f\|_{L^2(D)} \|g\|_{L^2(D)} & \textit{else}, \end{cases}$$

for all multi-indices $\alpha \in \mathbb{N}_0^{\mathbb{N}}$ with $|\alpha| \leq 2$. The constant $\widetilde{C}_{\mathrm{der}} > 0$ depends only on C_{der}, δ, and $(\phi_j)_{j\in\mathbb{N}}$.

Proof. Given $\alpha \in \mathbb{N}^{\mathbb{N}_0}$ with $|\alpha| \leq 2$ an admissible sequence $(\varrho_j)_{j\in\mathbb{N}}$ in Lemma 4.4 is, given $\varepsilon > 0$,

$$\varrho_j := \begin{cases} (\inf_{x\in D} \phi_0(x) - 2\delta)\alpha_j/2\|\phi_j\|_{W^{1,\infty}(D)}^{-1} & \text{for all } j \in \mathbb{N} \text{ with } \alpha_j > 0, \\ \varepsilon & \text{for all } j \in \mathbb{N} \text{ with } \alpha_j = 0. \end{cases}$$

This sequence satisfies

$$\inf_{\omega_i \in B_{1+\varrho_i}(0) : i \in \mathbb{N}} \Re\big(\phi_0 + \sum_{i=1}^\nu \omega_i \phi_i\big) \geq \phi_0 - (\operatorname*{ess\,inf}_{x\in D} \phi_0(x) - 2\delta) - \varepsilon \sum_{i=1}^\infty \|\phi_j\|_{L^\infty(D)} \geq \delta$$

for sufficiently small $\varepsilon > 0$ (here $\Re$ denotes the real part). Moreover, the term $\|\phi_0 + \sum_{i=1}^\nu \omega_i \phi_i\|_{W^{1,\infty}(D)}$ remains uniformly bounded in $\Omega' := \prod_{i=1}^\infty B_{1+\varrho_i}(0)$. This ensures that Ω' satisfies all the assumptions required for Ω and thus all results of Section 3 remain valid for Ω' instead of Ω. In particular, the constant $C_{\mathrm{prod}}(\omega)$ from Theorem 3.9 is uniformly bounded in $\omega \in \Omega'$. The affine-parametric map $\omega \mapsto A^\nu(x,\omega)$ is holomorphic in each coordinate in Ω', with constant derivative

$$\partial_{\omega_j} A^\nu(x,\omega) = \begin{cases} \phi_j(x) & \text{for } j \le s_\nu, \\ 0 & \text{else.} \end{cases}$$

Moreover, since $|\alpha| \le 2$ there holds

$$\prod_{i=1}^\infty \varrho_i^{-\alpha_i} \le \prod_{i=1}^\infty \beta_i^{\alpha_i}.$$

This, together with Lemma 4.4 concludes the proof. ◻

LEMMA 4.6. *Let $g \in L^\infty(\Omega)$ be sufficiently smooth and let g depend only on the first $s \in \mathbb{N}$ dimensions, i.e., $\partial_{\omega_i} g = 0$ for all $i > s$. For $0 \le r \le s$ and $x = (x_1, x_2, \dots, x_s) \in \Omega^s$, define the function space*

$$\mathcal{P}_r^s(\Omega) := \operatorname{span}\Big\{ f \in L^\infty(\Omega) \,:\, f(x) = \sum_{i=r+1}^s \alpha(x_1,\dots,x_r) x_i,\ \alpha(x_1,\dots,x_r) \in \mathbb{R} \Big\}.$$

Assume that $\omega \in \Omega$ with $\omega_i = 0$ for all $i > r$ implies $g(\omega) = 0$. Then, there holds

$$\|g(\omega)\|_{L^\infty(\Omega)} \le \sum_{i=r+1}^s \|\partial_{\omega_i} g\|_{L^\infty(\Omega)}.$$

Moreover, there exists $g_0 \in \mathcal{P}_r^s(\Omega)$ such that

$$\|g(\omega) - g_0(\omega)\|_{L^\infty(\Omega)} \le \frac{1}{2} \sum_{i=r+1}^s \sum_{j=r+1}^i \|\partial_{\omega_i}\partial_{\omega_j} g\|_{L^\infty(\Omega)}.$$

Proof. Let $\omega \in \mathbb{R}^s$. There holds

$$\begin{aligned} g(\omega) &= \underbrace{g(\omega_1,\dots,\omega_r,0,\dots)}_{=0} + \sum_{i=r+1}^s \int_0^{\omega_i} \partial_{\omega_i} g(\omega_1,\dots,\omega_{i-1},t_i,0,\dots)\,dt_i \\ &= \sum_{i=r+1}^s \int_0^{\omega_i} \Big(\partial_{\omega_i} g(\omega_1,\dots,\omega_r,0,\dots) \\ &\qquad + \int_0^{t_i} \partial_{\omega_i}^2 g(\omega_1,\dots,\omega_{i-1},s_i,0,\dots)\,ds_i \\ &\qquad + \sum_{j=r+1}^{i-1} \int_0^{\omega_j} \partial_{\omega_j}\partial_{\omega_i} g(\omega_1,\dots,\omega_{j-1},s_j,0,\dots)\,ds_j \Big) dt_i. \end{aligned}$$

Since the first integrand on the right-hand side does not depend on ω_i, the above implies

$$
\begin{aligned}
g(\omega) = \sum_{i=r+1}^{s} \Big(& \omega_i \partial_{\omega_i} g(\omega_1, \ldots, \omega_r, 0, \ldots) \\
& + \int_0^{\omega_i} \Big(\int_0^{t_i} \partial_{\omega_i}^2 g(\omega_1, \ldots, \omega_{i-1}, s_i, 0, \ldots)\, ds_i \\
& \qquad + \sum_{j=r+1}^{i-1} \int_0^{\omega_j} \partial_{\omega_j} \partial_{\omega_i} g(\omega_1, \ldots, \omega_{j-1}, s_j, 0, \ldots)\, ds_j \Big) dt_i \Big).
\end{aligned}
$$

Since there holds $(\omega \mapsto \omega_i \partial_{\omega_i} g(\omega_1, \ldots, \omega_r, 0, \ldots)) \in \mathcal{P}_r^s(\Omega)$ for all $i \geq r+1$, we conclude the proof. ∎

THEOREM 4.7. *Under the assumptions of the current section, there holds*

$$
\|G(D_\ell^\nu)\|_{L^\infty(\Omega)} \leq C_{\mathrm{KL}} h_\ell^2 \sum_{i=s_{\nu-1}+1}^{s_\nu} \|\phi_i\|_{W^{1,\infty}(D)} \|f\|_{L^2(D)} \|g\|_{L^2(D)}. \tag{4.5}
$$

Moreover, there exists $g_0 \in \mathcal{P}_{s_{\nu-1}}^{s_\nu}(\Omega)$ such that

$$
\begin{aligned}
&\|G(D_\ell^\nu) - g_0\|_{L^\infty(\Omega)} \\
&\leq C_{\mathrm{KL}} h_\ell^2 \sum_{i=s_{\nu-1}+1}^{s_\nu} \sum_{j=s_{\nu-1}+1}^{s_\nu} \|\phi_i\|_{W^{1,\infty}(D)} \|\phi_j\|_{W^{1,\infty}(D)} \|f\|_{L^2(D)} \|g\|_{L^2(D)}.
\end{aligned} \tag{4.6}
$$

The constant $C_{\mathrm{KL}} > 0$ is independent of ℓ, ν, and ω.

Proof. The first estimate (4.5) follows from the definition of A^ν and Theorem 3.9. For (4.6), the map $g(\omega) := D_\ell^\nu(\omega)$ satisfies the requirements of Lemma 4.6 with $r = s_{\nu-1}$. Hence, the result follows immediately from Lemma 4.6 and Lemma 4.5. ∎

5. Monte Carlo integration. This section discusses the Monte Carlo quadrature rules. The uniform KL-expansion case (Section 4.2) allows us to increase the order of convergence by symmetrization of the Monte Carlo rule. This section defines the Monte Carlo integration for the case that the random coefficient is given by a KL-expansion as discussed in Sections 4.1–4.2.

We make the standard assumption that the functions ϕ_i from (4.3) satisfy

$$
\|\phi_j\|_{W^{1,\infty}(D)} \leq C_{\mathrm{KL}} j^{-r} \quad \text{for all } j \in \mathbb{N} \tag{5.1}
$$

for some $r > 1$.

LEMMA 5.1. *Define the Monte Carlo rule*

$$
Q_M(g) := \frac{1}{M} \sum_{i=1}^{M} g(X^i)
$$

for uniformly distributed i.i.d $X^i \in [-1/2, 1/2]^{s_\nu}$. Then, under the assumptions of Section 4.1 given $\ell, \nu \in \mathbb{N}$, the function $F\colon \Omega \to \mathbb{R}$, $\omega \mapsto G(D_\ell^\nu(\omega))$ satisfies

$$
\sqrt{\mathbb{E}_{\mathrm{MC}} |\mathbb{E}(F) - Q_M(F)|^2} \leq C_{\mathrm{MC}} s_{\nu-1}^{1-r} \frac{h_\ell^2}{\sqrt{M}} \|f\|_{L^2(D)} \|g\|_{L^2(D)}.
$$

Here, $\mathbb{E}_{\mathrm{MC}}(\cdot)$ denotes integration over the combined probability spaces of the X^i, $i = 1, \ldots, M$, whereas $\mathbb{E}(\cdot)$ denotes integration over Ω_ν.

Proof. The statement follows immediately from the standard Monte Carlo error estimate, Theorem 4.1, and the fact that $\sum_{j=s_{\nu-1}+1}^{s_\nu} j^{-r} \lesssim s_{\nu-1}^{1-r}$. ∎

By symmetrization of the Monte Carlo sequence, we are able to increase the order of convergence in the truncation parameter ν.

LEMMA 5.2. *Define the symmetric Monte Carlo rule*

$$Q_M(g) := \frac{1}{2M}\sum_{i=1}^{M}(g(X_1^i,\dots,X_{s_\nu}^i) + g(X_1^i,\dots,X_{s_{\nu-1}}^i,-X_{s_{\nu-1}+1}^i,\dots,-X_{s_\nu}^i)),$$

where the $X^i \in [-1/2,1/2]^{s_\nu}$ are i.i.d. and uniformly distributed. Under the assumptions of Section 4.2, there holds $Q_M(g_0)=0$ for all $g_0 \in \mathcal{P}_{s_{\nu-1}}^{s_\nu}(\Omega)$. Moreover, given $\ell,\nu\in\mathbb{N}$, the map $F\colon\Omega\to\mathbb{R}$, $\omega\mapsto G(D_\ell^\nu(\omega))$ satisfies

$$\sqrt{\mathbb{E}_{\rm MC}|\mathbb{E}(F)-Q_M(F)|^2} \le C_{\rm MC} s_{\nu-1}^{2(1-r)}\frac{h_\ell^2}{\sqrt{M}}\|f\|_{L^2(D)}\|g\|_{L^2(D)}.$$

Here, $\mathbb{E}_{\rm MC}(\cdot)$ denotes integration over the combined probability spaces of the X_i, $i=1,\dots,2^m$, whereas $\mathbb{E}(\cdot)$ denotes integration over Ω_ν.

Proof. First, we notice that for $g_0\in\mathcal{P}_{s_{\nu-1}}^1(\Omega)$, there holds

$$g_0(X_1^i,\dots,X_{s_\nu}^i) = -g_0(X_1^i,\dots,X_{s_{\nu-1}}^i,-X_{s_{\nu-1}+1}^i,\dots,-X_{s_\nu}^i).$$

Therefore, we have $Q_M(g_0)=0$ for all $g_0\in\mathcal{P}_{s_{\nu-1}}^1(\Omega)$. Thus, the statement follows from the standard Monte Carlo error estimate and Theorem 4.7, where we note with (5.1)

$$\begin{aligned}\sum_{i=s_{\nu-1}+1}^{s_\nu}\sum_{j=s_{\nu-1}+1}^{s_\nu}&\|\phi_i\|_{W^{1,\infty}(D)}\|\phi_j\|_{W^{1,\infty}(D)}\\ &\lesssim \sum_{i=s_{\nu-1}+1}^{\infty}\sum_{j=s_{\nu-1}+1}^{\infty} i^{-r}j^{-r} \lesssim (s_{\nu-1})^{2(-r+1)}.\end{aligned}$$ ∎

6. Multi-Index error control.

The multi-index decomposition allows us to exploit the product error estimates and, hence, to improve the complexity of the finite-element/Monte Carlo algorithm.

6.1. Complexity of MIMCFEM.

To quantify the complexity, i.e., the error vs. work, of the presently proposed MIMCFEM, we rewrite the exact solution as (Q_m denotes one of the MC sample averages Q_M from Section 5 with $M=2^m$ samples)

$$\begin{aligned}\mathbb{E}(G(u)) &= \sum_{j=0}^{\infty}(Q_{m_j}-Q_{m_j-1})(G(u))\\ &= \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}(Q_{m_j}-Q_{m_{j-1}})(G(u_\ell-u_{\ell-1}))\\ &= \sum_{j=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{\nu=0}^{\infty}(Q_{m_j}-Q_{m_{j-1}})(G(D_\ell^\nu)),\end{aligned}$$

where $m_j \in \mathbb{N}$ and $Q_{m_{-1}} := 0$. By truncation of the series, we achieve a sparse approximation, i.e., given $N \in \mathbb{N}$

$$\mathbb{E}(G(u)) \approx G_N := \sum_{0\le j+\ell+\nu\le N} (Q_{m_j} - Q_{m_j-1})G(D_\ell^\nu) = \sum_{0\le \ell+\nu\le N} Q_{m_{N-\ell-\nu}}(G(D_\ell^\nu)).$$

Recall the expectation of the Monte Carlo integration $\mathbb{E}_{\mathrm{MC}}(\cdot)$ and the expectation over Ω denoted by $\mathbb{E}(\cdot)$. We define two quantities to quantify the efficiency of the presently proposed method: the MC sampling error is defined by

$$E_N := \sqrt{\mathbb{E}_{\mathrm{MC}}|\mathbb{E}(G(u)) - G_N|^2}$$

whereas the cost model is defined by

$$C_N := (\text{The number of computational operations necessary to compute } G_N)$$

and obviously depends on the chosen method discussed below.

First, we establish the cost model. A standard FEM will ensure $h_\ell \simeq 2^{-\ell}$ which implies $\#\mathcal{T}_\ell \simeq 2^{d\ell}$. We assume a linear iterative solver such that solving the sparse FEM system costs $\mathcal{O}(2^{d\ell})$.

Under the assumptions of Section 4.1 and 4.2, we assume that we can compute the bilinear forms

$$a_j(v,w) := \int_D \phi_j(x)\nabla v(x)\nabla w(x)\,dx \quad \text{for all } v,w \in \mathcal{X}_\ell$$

exactly in $\mathcal{O}(\#\mathcal{T}_\ell)$. Depending on the truncation parameters s_ν, we have to compute s_ν bilinear forms $a_j(\cdot,\cdot)$ to obtain in the affine case

$$a_\omega^\nu(v,w) = \sum_{j=1}^{s_\nu} \omega_j a_j(v,w),$$

resulting in a cost of $\mathcal{O}(2^{d\ell}s_\nu)$. Altogether, this yields

$$C_N \simeq \sum_{0\le j+\ell+\nu\le N} 2^{m_j}2^{d\ell}s_\nu$$

Using Lemma 5.1 as well as linear operator notation for $\mathbb{E}(\cdot)$ and Q_{m_j}, we see that the multi-index error satisfies

$$\begin{aligned}
E_N &= \mathbb{E}_{\mathrm{MC}}\Big(\Big|\sum_{N<j+\ell+\nu} (Q_{m_j} - Q_{m_{j-1}})G(D_\ell^\nu)\Big|^2\Big)^{1/2} \\
&\le \sum_{0\le \ell+\nu} \mathbb{E}_{\mathrm{MC}}\big(|(\mathbb{E} - Q_{m_{\max\{0,N-\ell-\nu+1\}}})G(D_\ell^\nu)|^2\big)^{1/2} \\
&\lesssim \|f\|_{L^2(D)}\|g\|_{L^2(D)} \sum_{0\le \ell+\nu} 2^{-m_{\max\{0,N-\ell-\nu+1\}}/2}2^{-2\ell}s_{\nu-1}^{1-r}.
\end{aligned}$$

An obvious choice of the parameters s_ν and m_j is to balance the work spent on each of the two tasks such that the three error contributions (FEM-discretization error, truncation error, quadrature error) are of equal asymptotic order. We define

$$m_j := \lceil 4j \rceil \quad \text{and} \quad s_\nu := \lceil 2^{\frac{2\nu}{r-1}} \rceil.$$

With this, we have

$$E_N \lesssim \|f\|_{L^2(D)}\|g\|_{L^2(D)} \sum_{0\le \ell+\nu} 2^{-2\max\{0,N-\ell-\nu+1\}}2^{-2\ell}2^{-2\nu} \tag{6.1}$$
$$\lesssim \|f\|_{L^2(D)}\|g\|_{L^2(D)}(N+1)^2 2^{-2N}$$

as well as

$$C_N \simeq \sum_{0\le j+\ell+\nu\le N} 2^{4j}2^{d\ell}2^{\frac{2\nu}{r-1}} \lesssim 2^{\max\{4,d,\frac{2}{r-1}\}N}. \tag{6.2}$$

Using the symmetrized Monte Carlo rule from Lemma 5.2, we see that the multi-index error improves to

$$E_N \lesssim \|f\|_{L^2(D)}\|g\|_{L^2(D)} \sum_{0\le \ell+\nu} 2^{-m_{\max\{0,N-\ell-\nu+1\}}/2}2^{-2\ell}s_{\nu-1}^{2(1-r)}.$$

As above, we balance the contributions by

$$m_j := \lceil 4j\rceil \quad \text{and} \quad s_\nu := \lceil 2^{\frac{\nu}{r-1}}\rceil.$$

With this, we obtain the same error estimate as for the plain Monte Carlo rule (6.1), but with an improved cost estimate of

$$C_N^{\mathrm{symm}} \lesssim 2^{\max\{4,d,\frac{1}{r-1}\}N}. \tag{6.3}$$

6.2. Comparison to multi-level (quasi-) Monte Carlo FEM. The main difference to multi-level Monte Carlo is that the present method can capitalize on the approximation of the random coefficient, whereas the multi-level method has to treat this term in an a-priori fashion. However, the multi-level method can exploit symmetry properties of the exact operator to improve the rate of convergence in the approximation of the random coefficient, i.e., it achieves the same accuracy with a cost $\mathcal{O}(2^{\frac{1}{r-1}N})$ instead of $\mathcal{O}(2^{\frac{2}{r-1}N})$. This is worked out in the quasi-Monte Carlo case in [10] but transfers verbatim to the Monte Carlo case. Therefore, the multi-level (quasi-) Monte Carlo method with the same level structure as described in the previous section will achieve a cost versus error relation given by (see [18, Theorem 12] with $p = q = 1/r - \varepsilon$ for all $\varepsilon > 0$ and $\tau = 2$ in their notation)

$$E_N^{\mathrm{ML}} \lesssim (N+1)^\alpha 2^{-2N} \quad \text{with} \quad C_N^{\mathrm{ML}} \lesssim 2^{\max\{4\lambda,d\}N+\frac{1}{r-1}N},$$

where $\alpha > 0$ is a constant and $1/(2\lambda)$ for $\lambda \in (1/2, 1]$ is the convergence rate of the QMC quadrature (with the Monte Carlo rate *formally* corresponding here to the choice $1/(2\lambda) = 1/2$). Comparing the above estimates with the error vs. work estimates for the MIMCFEM from Section 6.1, we aim to identify parameter regimes in which the presently proposed MIMCFEM improves over alternative multi-level methods in terms of asymptotic error versus cost. We observe that standard multi-index Monte Carlo improves the multi-level Monte Carlo in case that

$$\max\{4,d,\frac{2}{r-1}\} < \max\{4,d\} + \frac{1}{r-1} \quad \text{equivalent to} \quad \max\{4,d\} > \frac{1}{r-1},$$

i.e., when the sampling and the FEM computations dominate the approximation of the random coefficient. We conclude that the symmetric multi-index Monte Carlo method from Lemma 5.2 *improves the multi-index Monte Carlo method for all parameter combinations.* For $\lambda \in (1/2, 1]$,

$$\max\{4, d, \frac{1}{r-1}\} < \max\{4\lambda, d\} + \frac{1}{r-1} \quad \text{equivalent to} \quad 4 - 4\lambda < \frac{1}{r-1}$$

the presently proposed, symmetric multi-index Monte Carlo FE method even improves in terms of error vs. work as compared to the first order multi-level quasi-Monte Carlo method based on e.g. a randomly shifted lattice rule as in [17]. This setting represents the case when the approximation of the random coefficient dominates the sampling and the FEM computations.

7. Extension of the MIFEM convergence to Reduced Regularity in D. Up to this point, the presentation and the error vs. work analysis assumed "full elliptic regularity" for data and solutions of the model problem in Section 2. Specifically, we assumed that the random diffusion coefficient A and the deterministic right hand side f in (1.1) belong to $W^{1,\infty}(D)$ and to $L^2(D)$, respectively. This, together with the convexity of the domain D and the homogeneous Dirichlet boundary conditions is well known to ensure $\mathbb{P}$-a.s. that $u \in L^2(\Omega; H^2(D))$. This, in turn, implies first order convergence of conforming P_1-FEM on regular, quasiuniform meshes, and second order (super)convergence for continuous linear functionals in $L^2(D)$. *These somewhat restrictive assumptions were made in order to present the MIFEM approach in the most explicit and transparent way.* The present MIFEM error analysis is, however, valid under more general assumptions, which we now indicate.

Still considering conforming P_1-FEM on regular meshes of triangles, *mixed boundary conditions* and nonconvex polygons D will allow verbatim the same line of argument and results, provided that the following modifications of the FE error analysis are taken into account: (i) *elliptic regularity*: as is well-known, the $L^2 - H^2$ regularity result which we used will, in general, cease to be valid for nonconvex D, or for mixed boundary value problems. A corresponding theory is available and uses weighted Sobolev spaces. We describe it to the extent necessary for extending our error analysis for conforming P_1-FEM. In polygonal domains $D \subset \mathbb{R}^2$, *weighted, hilbertian Kondrat'ev spaces of order $m \in \mathbb{N}_0$ with shift $a \in \mathbb{R}$* are defined by

$$\mathcal{K}^m_a(D) := \{v : D \to \mathbb{R} | r_D^{|\alpha|-a} \partial^\alpha v \in L^2(D), |\alpha| \leq m\} \tag{7.1}$$

In (7.1), $\alpha \in \mathbb{N}_0^2$ denotes a multi-index and ∂^α the usual mixed weak derivative of order $\alpha = (\alpha_1, \alpha_2)$. In these spaces, there holds the following regularity result [2, Thm. 1.1].

Proposition 7.1. *Assume that $D \subset \mathbb{R}^2$ is a bounded polygon with straight sides. In D consider the Dirichlet problem (1.1) with random coefficient $A \in L^\infty(\Omega; W^{1,\infty}(D))$ satisfying (2.2). Then the following holds:*

1. *There exists $\eta > 0$ such that for every $|a| < \eta$, and for every $f \in \mathcal{K}^0_{a-1}(D)$, the unique solution $u \in H^1_0(D)$ of (1.1) belongs to $\mathcal{K}^2_{a+1}(D)$.*
2. *For every fixed $f \in \mathcal{K}^0_{a-1}(D)$, the data-to-solution map $\mathbb{S} : W^{1,\infty}(D) \to \mathcal{K}^2_{a+1}(D) : A \mapsto u$ is analytic for every $|a| < \eta$.*
3. *There exists a sequence $\{\mathcal{T}^\ell\}_{\ell \geq 0}$ of regular, simplicial triangulations with refinements towards the corners of D such that there holds the approximation*

property

$$(7.2) \qquad \forall w \in \mathcal{K}^2_{a+1}(D): \quad \inf_{v \in S^1(D;\mathcal{T}^\ell)} \|w - v\|_{H^1(D)} \le C h_\ell \|w\|_{\mathcal{K}^2_{a+1}(D)} \,,$$

where $h_\ell := \max\{\mathrm{diam}(T) : T \in \mathcal{T}^\ell\}$ *and* $\mathcal{X}_\ell = \#(\mathcal{T}^\ell) \lesssim h_\ell^{-2}$.

We refer to [2, Thm. 1.1] for the proof of items 1. and 2., and to [1, 3, 20] for a proof of item 3.; we note in passing that [1, 3] cover so-called *graded* meshes, whereas item 3. for nested, bisection-tree meshes as generated e.g. by adaptive FEM is proved in [20].

With Proposition 7.1 at hand, the preceding MIFEM error analysis extends *verbatim* to the present, more general setting: the $H^2(D)$ regularity results for the forward problem as well as for the adjoint problem extend to $\mathcal{K}^2_{a+1}(D)$, under the assumption $f, g \in \mathcal{K}^0_{a-1}(D)$, and under identical assumptions on the random coefficient A. The use of the Cauchy integral theorem in the weighted function space setting is justified by item 2. combined with the (obvious) observation that affine-parametric functions such as (4.3) depend analytically on the parameters ω_j.

We also note that other discretizations, such as the symmetric IP DG FEM, admit corresponding error bounds on graded meshes including the superconvergence error bound in $L^2(D)$ [19]. A corresponding MIFEM algorithm and error analysis with exactly the same error vs. work bounds could also be obtained for SIPDG discretization of the forward problem.

We finally mention that Proposition 7.1 also extends verbatim to homogeneous, mixed boundary conditions, to symmetric matrix-valued random diffusion coefficients $A = (a_{ij})_{i,j=1,2} \in W^{1,\infty}(D; \mathbb{R}^{2\times 2})$ (the space $W^{1,\infty}(D)$ could even be slightly larger, admitting singular behaviour near corners of D) and to higher orders $m \ge 2$ of differentiation, allowing for Lagrangean FEM of polynomial degree $p = m \ge 2$ on locally refined meshes in D. A precise statement of these regularity results is available in [2, Thm. 4.4].

8. Numerical experiments.

We provide numerical tests in space dimension 2 to verify the theoretical results. In the first example, we choose uniform mesh refinement in a convex domain D and irregular forcing function f (which is to say in the present setting of first order FEM that $f \notin L^2(D)$). The second example will feature a non-convex domain with re-entrant corner and sequences $\{\mathcal{T}_\ell\}_\ell$ of locally refined, nested regular triangulations of D.

8.1. Irregular forcing and uniform mesh refinement.

For purposes of comparison, we use a similar example as in [8, Section 5.2]. We choose the convex domain $D = [0,1]^2$ and define the scalar random coefficient function A by

$$A(x,\omega) := 1/2 + \sum_{k_1,k_2=1}^{\infty} \frac{\omega_{k_1,k_2}}{(k_1^2 + k_2^2)^2} \sin(k_1 \pi x_1) \sin(k_2 \pi x_2)$$

$$:= 1/2 + \sum_{j=1}^{\infty} \frac{\omega_j}{\mu_j} \sin(k_{1,j} \pi x_1) \sin(k_{2,j} \pi x_2),$$

where $\mu_j := (k_{1,j}^2 + k_{2,j}^2)^2$ such that $\mu_i \le \mu_j$ for all $i \le j$ and ties are broken in an arbitrary fashion. This ensures that the ϕ_j satisfy (5.1) with $r = 2$. The variational form of the problem then reads

$$\text{Find } \; u \in H^1_0(D): \quad a(A(\cdot,\omega); u, v) = f(v) \quad \forall v \in H^1_0(D) \,.$$

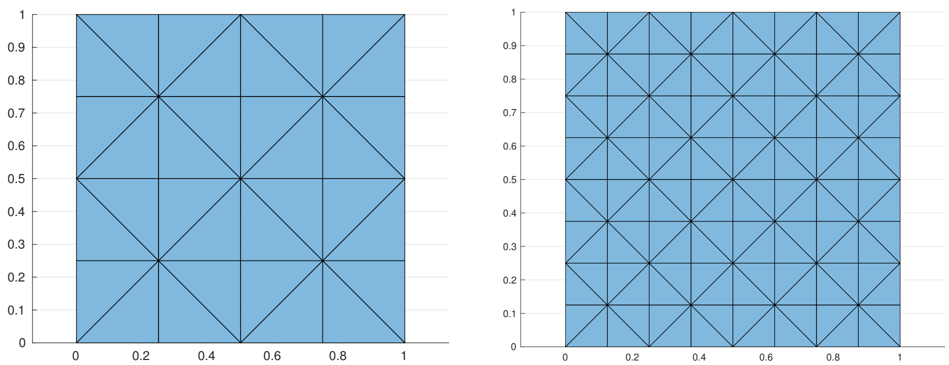

FIG. 1. *Two levels of mesh-refinement for the unit-square domain.*

where $f \in H^{-1/2-\varepsilon}(D)$ for all $\varepsilon > 0$ is defined by

$$f(v) := \int_\Gamma v(x_1, x_2)x_1 \, d\Gamma(x_1, x_2) = \sqrt{2}\int_0^1 t\, v(t, 1-t)\, dt$$

for $\Gamma = \big\{(0,1) + r(1,-1) \,:\, 0 \le r \le 1\big\}$ being a diagonal of D. Note that we choose the weight x_1 in the integral in the definition of the right-hand side to introduce some non-symmetric quantities and thus avoid super-convergence effects. We consider the quantity of interest $G(u) := \int_{D'} u\, dx$, where $D' = (1/2, 1)^2 \subset D$. Whereas the analysis of the present paper is focused on the full regularity case with right-hand side $f \in L^2(D)$, all arguments remain valid in case of reduced regularity of the right-hand side $f \in H^{-1/2-\varepsilon}(\boldsymbol{D})$ (for the case of reduced regularity due to re-entrant corners, see the second experiment).

The finite element discretization is based on first order, nodal continuous, piecewise affine finite elements $\mathcal{X}_\ell$ on a uniform partition of $[0,1]^2$ into $2^{2\ell+1}$ many congruent triangles (one example is shown in Figure 1). The meshwidth of this triangulation is $h_\ell = \mathcal{O}(2^{-\ell})$. Note that the cost model applies as we can compute the stiffness matrix exactly since the gradients of the shape functions are constants and the anti-derivatives of products of sine functions are known over triangles. The error expected by theory for the FEM on mesh-level ℓ is $\mathcal{O}(h_\ell) = \mathcal{O}(2^{-3/2\ell})$ (due to the reduced regularity of the right-hand side f). Thus we choose the $m_j := 3j$ as well as $s_\nu = \lceil 2^{\nu/(r-1)} \rceil$ for the original algorithm and $s_\nu = \lceil 2^{\nu/(2(r-1))} \rceil$ for the symmetrized version. Therefore we expect that the errors for both algorithms satisfy $E_N = \mathcal{O}(2^{-3/2N}) = \mathcal{O}(C_N^{-1/2})$, where C_N as defined in (6.2), (6.3) denotes the cost of the multi-index FEM on level N. This is confirmed in Figure 2. For the numerical experiments, we compare with a reference solution computed with a higher-order Quasi-Monte Carlo method proposed in [8]. The reference value is computed with a higher order QMC rule[1] To smooth out the effects of MC sampling, the plotted relative errors are averaged over 20 runs of the respective multi-index algorithm (we also plot empirical 90%-confidence intervals for each error point).

8.2. Local mesh refinement. The regularity of the exact solution can also be reduced by re-entrant corners with corresponding reduced rates of FE convergence for quasiuniform meshes. As is well-known (e.g. [3, 1]), in two space dimensions, this is

[1] The authors thank F. Henriquez, a PhD student at the Seminar for Applied Mathematics of ETH, for computing the reference value.

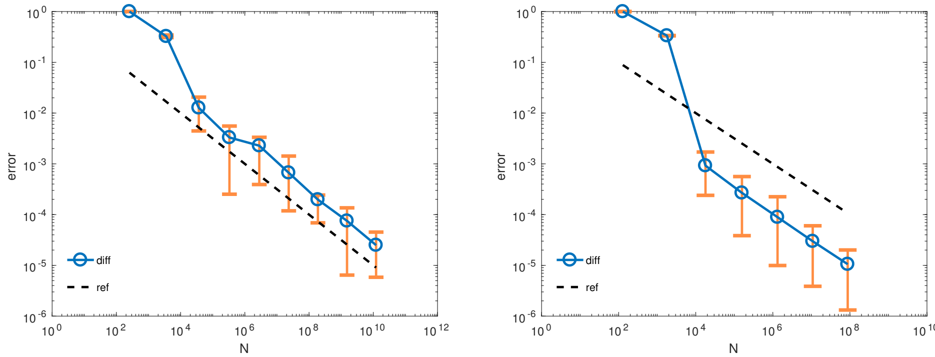

FIG. 2. *Averaged relative errors or the multi-index algorithms with respect to the reference solution G compared with the theoretical error bound $\mathcal{O}(C_N^{-1/2})$ (original algorithm (left) and symmetrized version (right)). Both plots shows the average error curve of 20 runs of the algorithms as well as the empirical 90%-confidence intervals of the computed error. The symmetrized version reaches the accuracy of the non-symmetric version already for $N=6$ instead of $N=9$.*

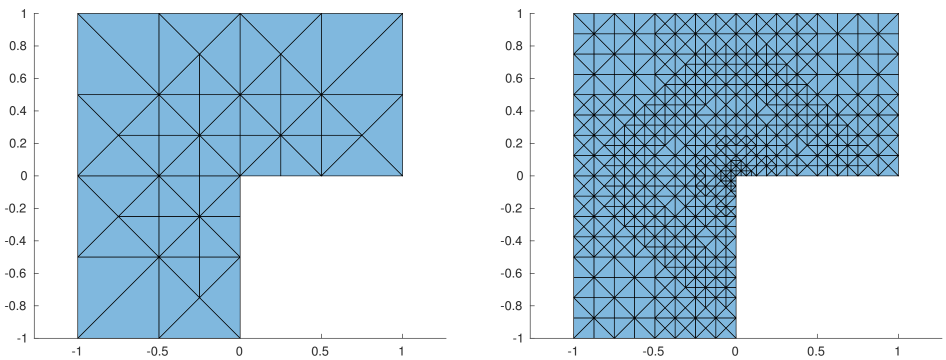

FIG. 3. *Two levels of graded mesh-refinement for the L-shaped domain.*

due to point-singularities in the solution. These can be compensated by a-priori local mesh-refinement in D. Using hierarchies of so-called graded or suitable bisection-tree meshes, and expressing regularity of solutions in terms of weighted $H^2(D)$ spaces, the present regularity and FE convergence analysis remains valid verbatim, with full convergence rates (see Section 7 for details).

This is demonstrated on the following example on the L-shaped domain $D := [-1,1]^2 \setminus (1,0) \times (-1,0)$ depicted in Figure 3 with the same coefficient and PDE as in the previous example. However, as a right-hand side, we use $f = 1$ and the quantity of interest is now defined by $G(u) := \int_{(0,1/2)^2} u \, dx$. The graded meshes $\mathcal{T}^\ell$ from Proposition 7.1 are generated by newest vertex bisection by iteratively refining all elements T which are coarser than the theoretically optimal grading of $\mathcal{O}(\mathrm{dist}(\{0\},T)^{1/3} h_\ell)$. This results in a sequence of meshes with $\#(\mathcal{T}^\ell) = \mathcal{O}(2^{2/3\ell})$. Figure 3 shows one instance of this sequence of meshes. Figure 4 confirms the correct distribution of element diameters within the mesh.

The performance of the multi-index Monte Carlo method is shown in Figure 5 for the symmetrized version. Since we aim for the full convergence rate $\mathcal{O}(2^{-2N})$ in this example, we choose the level parameters $m_j := 8/3j$ as well as $s_\nu = \lceil 2^{2\nu/(3r-3)} \rceil$. Due to the much higher number of Monte-Carlo samples required in this example, we

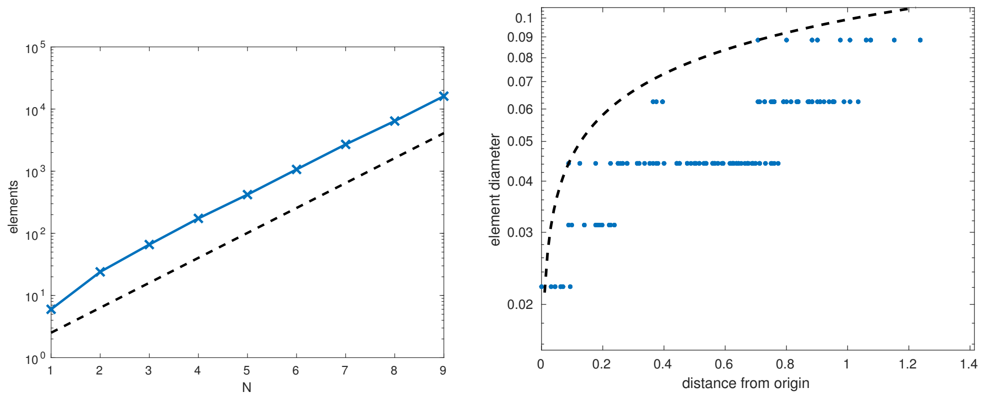

FIG. 4. *We see statistics of several graded meshes for levels $N = 1, \dots, 9$. The left-hand side plot shows that the number of elements behaves as $\mathcal{O}(2^{2/3N})$. The right-hand side plot shows for the mesh $\mathcal{T}^8$ that the distribution of element diameters with respect to their distance to the singularity behaves like $\mathcal{O}(\mathrm{dist}(\{0\}, T)^{1/3} h_N)$, where h_N is the maximal element diameter.*

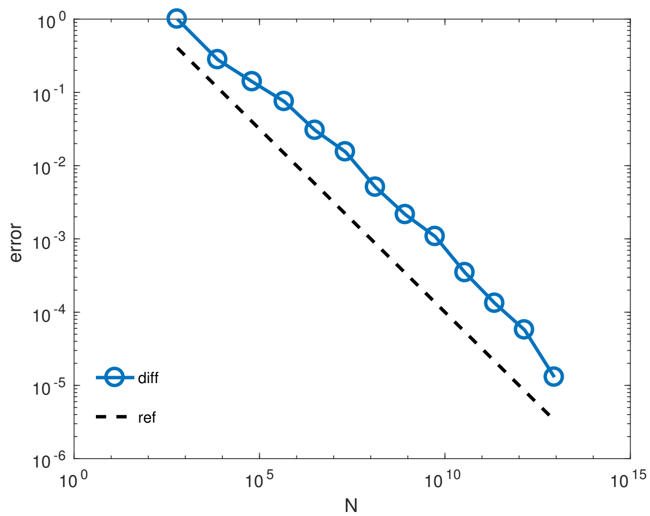

FIG. 5. *Averaged relative errors or the multi-index algorithms with respect to the reference solution G compared with the theoretical error bound $\mathcal{O}(C_N^{-1/2})$. The error curve is the average of four Monte-Carlo runs.*

only performed four Monte-Carlo runs and show the averaged error in Figure 5. We observe optimal convergence behaviour despite the presence of corner singularities in the exact solution. As a reference solution, we use the approximation on the next higher level $N = 14$.

9. Conclusion. The present work shows that the multi-index Monte Carlo algorithm with the indices being the discretization parameters of the finite element method, of the Monte Carlo method, and of the approximation of the random field is superior to its multi-level counterpart. The error estimates are rigorous and the product error estimate from Theorem 3.9 might be of independent interest. The method can be combined with existing multi-index techniques which focus on sparse grids in the physical domain D to further reduce the computational effort under the provision of appropriate extra regularity.

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