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Wave-Field Shaping in Cavities by Tunable Metasurfaces

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Abstract

Cavities, because they trap waves for long times due to their reflecting walls, are used in a vast number of scientific domains. Indeed, in these closed media and due to interferences, the free space continuum of solutions becomes a discrete set of stationary eigenmodes. These enhanced stationary fields are commonly used in fundamental physics to increase wave-matter interactions. The eigenmodes and associated eigenfrequencies of a cavity are imposed by its geometrical properties through the boundary conditions. In this paper, we show that one can control the wave fields created by point sources inside cavities by tailoring only the boundaries of the cavities. This is achieved through the use of a tunable reflecting metasurface, which is part of the frontiers of the cavity, and can switch its boundary conditions from Dirichlet to Neumann. Based on the use of arrays of subwavelength resonators, a mathematical modeling of the physical mechanism underlying the concept of tunable metasurfaces is provided.

Mathematics Subject Classification (MSC2000). 35R30, 35C20.

Keywords. Subwavelength resonance, Helmholtz resonator, hybridization, metasurfaces, cavity.

Contents

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1 Introduction

Controlling waves in cavities, which are used in numerous domains of applied and fundamental physics, has become a major topic of interest [11, 12, 18, 21]. The wave fields established in cavities are fixed by their geometry. They are usually modified by using mechanical parts. Nevertheless, tailoring the cavity boundaries permits one to design at will the wave fields they support. Here, we show that it is achievable simply by using tunable metasurfaces that locally modify the boundaries, switching them from Dirichlet to Neumann conditions. The concept of metasurfaces is a powerful tool to shape waves by governing precisely the phase response of each constituting element through its subwavelength resonance properties [6, 9, 13, 24]. Subwavelength resonators have been also used as the building block of super-resolution imaging [3, 5, 10].

A metasurface is a thin sheet with patterned subwavelength structures, which nevertheless has a macroscopic effect on wave propagation. Based on the concept of hybridized resonators, a tunable metasurface can be designed. Hence, it can be transformed into a tunable component that allows shaping waves dynamically in unprecedented ways [14, 20, 21]. The mechanism is based on the very general concept of hybridized coupled resonant elements whose resonant frequencies can be tuned by adjusting the coupling strength. The idea from [20] is to design a metasurface that can be either resonant or not resonant at a given operating frequency. In the first case, the collective resonant behaviour of the subwavelength resonators provides a change of the boundary condition while in the second case, the metasurface is transparent to the incident wave. To that aim, one can take as unit cell of the metasurface a system made out of two individual subwavelength resonators: one static resonator (referred to as the main resonator), whose frequency is fixed to the operating frequency and one tunable parasitic resonator (referred to as the parasitic resonator) whose frequency can be wisely adjusted by a given tunable mechanism. In the first case, the resonance frequency of the parasitic resonator is different enough from the resonance frequency of the main resonator, so that the two resonant elements do not couple. At the operating frequency, the metasurface is then resonant. In the second case however, one sets the resonance frequency of the parasitic resonator to match that of the static resonator. In that case, the subwavelength resonators hybridize to create a dimer whose eigenfrequencies are respectively under and above the initial resonance frequency.

In this paper, we mathematically and numerically model the physical mechanism underlying the concept of tunable metasurfaces. We consider Helmholtz resonators. We show that an array of Helmholtz resonators behaves as an equivalent surface with

Neumann boundary condition at a resonant frequency which corresponds to a wavelength much greater than the size of the Helmholtz resonators. Analytical formulas for the hybridized resonances of coupled Helmholtz resonators are also derived. Numerical simulations confirm their accuracy. We also propose an efficient approach to characterize the Green's function of a cavity with mixed (Dirichlet and Neumann) boundary conditions. The use of tunable metasurfaces allows us to find a criterion ensuring that modifying parts of a cavity's boundaries turn it into a completely different one. We provide a new and simple procedure for maximizing the Green's function between two points at a chosen frequency in terms of the boundary conditions. Our algorithm is a one shot optimization algorithm and can then be used in real-time to focalize the wave on a given spot by maximizing the transmission between an emitter and a receiver through specific eigenmodes of the cavity or on the contrary, to minimize the field on a receiver.

The paper is organized as follows. Section 2 is devoted to give preliminary results on the so-called Neumann functions, which play a key role in proving the results in Sections 3 and 4. We first introduce the quasi-periodic fundamental solutions to the Helmholtz equation and recall in Lemmas 2.2–2.4 some key results from [6]. Then, we consider the Neumann functions, which depends crucially on certain remainder functions, for which we provide exact formulas in Lemmas 2.9–2.12.

In Section 3, we look at one periodically repeated Helmholtz resonator above a ground plate. After treating them with an incident wave, we obtain a scattered wave, whose resonant values are discussed and its behavior at the far-field are examined. We show in Theorem 3.2 that the structure behaves as an equivalent surface with Neumann boundary condition at the resonant frequencies characterized in Theorem 3.1. The proof uses a combined technique of [10] and [6].

Section 4 has the same objective as the previous section, but this time we have two periodically repeated Helmholtz resonator above a ground plate. As shown in Theorem 4.1, the strong coupling between the periodically repeated pair of resonators leads us to hybridized resonances. It is shown in Theorem 4.2 that at only these hybridized frequencies the structure behaves as an equivalent surface with Neumann boundary condition.

Section 5 is devoted to the derivation of an asymptotic formula of the Green's function of a cavity with mixed boundary conditions in terms of the size of the part of the cavity boundary where the boundary condition is switched from Dirichlet to Neumann. A closed form of the derivative of the Green's function with respect to changes in the boundary condition is given in Theorem 5.4.

In Section 6, we consider the problem where we have a source in a bounded domain and we want to determine whether we activate a small part of the boundary to be reflecting or not in such a way the signal at a given receiving point is significantly enhanced. Based on Theorem 5.4, we propose a simple strategy aiming to maximize the norm of the Green's function by nucleating Neumann boundary conditions. The basic idea follows the concept of topological derivative. The switching of parts of the boundary from Dirichlet to Neumann where the topological derivative of the norm of

the Green's function is positive allows for an increasing of the transmission between the point source and the receiver. Finally, we present some numerical experiments to show the applicability of the proposed methodology.

2 Preliminaries

In this section, we introduce the quasi-periodic fundamental solutions to the Helmholtz equation. The explicit formula derived in [6] will be helpful for us. Then we consider the Neumann functions and their remainders.

2.1 Quasi-Periodic Fundamental Solution to the Helmholtz Equation with Dirichlet Boundary Conditions

Let $\mathbf{k} := (k_1, k_2)^{\mathrm{T}} \in \mathbb{R}^2$ and $k := |\mathbf{k}| := (k_1^2 + k_2^2)^{1/2} \in [0, \infty)$, where T denotes the transpose. Then for $k \in (0, \infty)$ we define $\Gamma^k : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C}$ as the fundamental solution to $\left(\triangle + k^2\right) \Gamma^k(x) = \delta_{\mathbf{0}}(x)$ satisfying the Sommerfeld radiation condition. For $k = 0$ we define $\Gamma^0 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{C}$ as the fundamental solution to $\triangle \Gamma^0(x) = \delta_{\mathbf{0}}(x)$. For $k \in [0, \infty)$, we define $\Gamma^k(z, x) := \Gamma^k(z - x)$. For $x \in \mathbb{R}^2 \setminus \{0\}$ we have that

$$\Gamma^0(x) = \frac{1}{2\pi} \ln |x|,$$
$$\Gamma^k(x) = -\frac{i}{4} H_0^{(1)}(k|x|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind. For x near zero we have

$$-\frac{i}{4} H_0^{(1)}(k|x|) = \frac{1}{2\pi} \ln |x| + \eta_k + \sum_{j=1}^{+\infty} (b_j \ln(k|x|) + c_j)(k|x|)^{2j}, \tag{2.1}$$

where

$$b_j = \frac{(-1)^j}{2\pi} \frac{1}{2^{2j}(j!)^2}, \quad c_j = -b_j \left(\gamma - \ln 2 - \frac{i\pi}{2} - \sum_{l=1}^{j} \frac{1}{l} \right), \quad \eta_k = \frac{1}{2\pi} (\ln k + \gamma - \ln 2) - \frac{i}{4},$$

where γ is the Euler-Mascheroni constant.

The quasi-periodic fundamental solutions $\Gamma_\sharp^{\mathbf{k}} : \mathbb{R}^2 \setminus \{(np, 0)^{\mathrm{T}} | n \in \mathbb{Z}\} \to \mathbb{C}$ and $\Gamma_*^{\mathbf{k}} : \mathbb{R}^2 \setminus \{(np, 0)^{\mathrm{T}} | n \in \mathbb{Z}\} \to \mathbb{C}$ are defined by

$$\Gamma_\sharp^{\mathbf{k}}(x) = \sum_{n \in \mathbb{Z}} \Gamma^k \left(x + \begin{pmatrix} np \\ 0 \end{pmatrix} \right) e^{-ik_1 np},$$
$$\Gamma_*^{\mathbf{k}}(x) = \sum_{n \in \mathbb{Z}} \Gamma^k \left(x + \begin{pmatrix} np \\ 0 \end{pmatrix} \right) e^{ik_1 np}.$$

$\Gamma_\sharp^{\mathbf{k}}$ and $\Gamma_*^{\mathbf{k}}$ are solutions to

$$\left(\triangle + k^2\right)\Gamma_\sharp^{\mathbf{k}}(x) = \sum_{n\in\mathbb{Z}} e^{-ik_1np}\delta_{(np,0)^{\mathrm{T}}}(x),$$
$$\left(\triangle + k^2\right)\Gamma_*^{\mathbf{k}}(x) = \sum_{n\in\mathbb{Z}} e^{+ik_1np}\delta_{(np,0)^{\mathrm{T}}}(x).$$

Using Poisson summation formula, this leads us to the following formulas [6, Lemma 3.2]:

Lemma 2.1 *For the case* $\mathbf{k} = \mathbf{0}$ *we have*

$$\Gamma_\sharp^{\mathbf{0}}(x) \;=\; \Gamma_*^{\mathbf{0}}(x) \;=\; \frac{|x_2|}{2p} \;-\; \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \frac{1}{2p|l|} e^{-|l|\,|x_2|} e^{i(l\,x_1)}. \tag{2.2}$$

If $\mathbf{k}$ *satisfies* $k^2 < \inf\{|l-k_1|^2 \,|\, l := 2\pi n/p,\ n\in\mathbb{Z}\setminus\{0\}\}$, *we have*

$$\Gamma_\sharp^{\mathbf{k}}(x) = \frac{e^{-ik_1x_1}e^{ik_2\,|x_2|}}{2ik_2p} - \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \frac{e^{-ik_1x_1}}{2p\sqrt{|l-k_1|^2-k^2}} e^{-\sqrt{|l-k_1|^2-k^2}|x_2|} e^{i(lx_1)},$$
$$\Gamma_*^{\mathbf{k}}(x) = \frac{e^{ik_1x_1}e^{ik_2\,|x_2|}}{2ik_2p} - \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \frac{e^{ik_1x_1}}{2p\sqrt{|l+k_1|^2-k^2}} e^{-\sqrt{|l+k_1|^2-k^2}|x_2|} e^{i(lx_1)}.$$

Using Lemma 2.1 we can expand $\Gamma_\sharp^{\delta\mathbf{k}}$ and $\Gamma_*^{\delta\mathbf{k}}$ with respect to δ near 0 and obtain

$$\Gamma_\sharp^{\delta\mathbf{k}}(x) = \frac{1}{2i\delta k_2 p} + \Gamma_{0,\sharp}^{\mathbf{k}}(x) + \sum_{n=1}^{\infty} \delta^n \Gamma_{n,\sharp}^{\mathbf{k}}\,, \tag{2.3}$$
$$\Gamma_*^{\delta\mathbf{k}}(x) = \frac{1}{2i\delta k_2 p} + \Gamma_{0,*}^{\mathbf{k}}(x) + \sum_{n=1}^{\infty} \delta^n \Gamma_{n,*}^{\mathbf{k}}\,, \tag{2.4}$$

where the kernels $\Gamma_{0,\sharp}^{\mathbf{k}}$ and $\Gamma_{0,*}^{\mathbf{k}}$ for $n \geq 1$ can be computed explicitly and shown to be smoother than $\Gamma_{0,\sharp}^{\mathbf{k}}$ and $\Gamma_{0,*}^{\mathbf{k}}$, see [8, Chapters 7.3, 7.4]. For the zeroth kernel we have with Equation (2.2), for $\mathbf{k} \neq \mathbf{0}$, that

$$\Gamma_{0,\sharp}^{\mathbf{k}}(x) = \Gamma_\sharp^{\mathbf{0}}(x) - \frac{k_1x_1}{2k_2p}, \quad \Gamma_{0,*}^{\mathbf{k}}(x) = \Gamma_*^{\mathbf{0}}(x) + \frac{k_1x_1}{2k_2p}. \tag{2.5}$$

We define $\Gamma_\sharp^{\mathbf{k}}(z,x) := \Gamma_\sharp^{\mathbf{k}}(z-x)$ and $\Gamma_*^{\mathbf{k}}(z,x) := \Gamma_*^{\mathbf{k}}(z-x)$. Consider that $\Gamma_\sharp^{\mathbf{k}}$ and $\Gamma_*^{\mathbf{k}}$ are not symmetric for $\mathbf{k} \neq \mathbf{0}$ in general, that is $\Gamma_\sharp^{\mathbf{k}}(z,x) \neq \Gamma_\sharp^{\mathbf{k}}(x,z)$.

We define the parity operator $\mathtt{P} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\mathtt{P}(x_1,x_2) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}. \tag{2.6}$$

We introduce the quasi-periodic fundamental solutions to the Helmholtz equation with Dirichlet boundary condition $\Gamma^{\mathbf{k}}_{+}$, $\Gamma^{\mathbf{k}}_{\times} : \{(z,x) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, \neg(\exists n \in \mathbb{Z} : |z_1 - x_1| = np \wedge z_2 = x_2)\} \to \mathbb{C}$ by

$$\Gamma^{\mathbf{k}}_{+}(z,x) := \Gamma^{\mathbf{k}}_{\sharp}(z,x) - \Gamma^{\mathbf{k}}_{\sharp}(z,\mathtt{P}(x)), \qquad \Gamma^{\mathbf{k}}_{\times}(z,x) := \Gamma^{\mathbf{k}}_{*}(z,x) - \Gamma^{\mathbf{k}}_{*}(z,\mathtt{P}(x)). \tag{2.7}$$

From Lemma 2.1, we see that for $z \in \partial\mathbb{R}^2_+$ or $x \in \partial\mathbb{R}^2_+$, we have that $\Gamma^{\mathbf{k}}_{+}(z,x) = 0$ and $\Gamma^{\mathbf{k}}_{\times}(z,x) = 0$. Moreover, $\Gamma^{\mathbf{k}}_{+}$ and $\Gamma^{\mathbf{k}}_{\times}$ are not symmetric, in general, and $\Gamma^{\mathbf{k}}_{+}$ and $\Gamma^{\mathbf{k}}_{\times}$ are not translation invariant in general, that is $\Gamma^{\mathbf{k}}_{+}(z,x) \neq \Gamma^{\mathbf{k}}_{+}(z-x,0)$. The following results hold from [9].

Lemma 2.2 *$\Gamma^{\delta\mathbf{k}}_{+}$ and $\Gamma^{\delta\mathbf{k}}_{\times}$ admit the expansions of the form*

$$\Gamma^{\delta\mathbf{k}}_{+}(z,x) = \Gamma^{\mathbf{k}}_{+,0}(z,x) + \sum_{n=1}^{\infty} \delta^n \Gamma^{\mathbf{k}}_{+,n}(z,x), \quad \Gamma^{\delta\mathbf{k}}_{\times}(z,x) = \Gamma^{\mathbf{k}}_{\times,n}(z,x) + \sum_{n=1}^{\infty} \delta^n \Gamma^{\mathbf{k}}_{\times,n}(z,x),$$

where

$$\begin{aligned}
\Gamma^{\mathbf{0}}_{+}(z,x) &= \Gamma^{\mathbf{k}}_{+,0}(z,x) = \Gamma^{\mathbf{k}}_{\times,0}(z,x) \\
&= \frac{1}{4\pi}\Bigg[\log\left(\sinh\left(\frac{\pi}{p}(z_2 - x_2)\right)^2 + \sin\left(\frac{\pi}{p}(z_1 - x_1)\right)^2\right) \\
&\qquad - \log\left(\sinh\left(\frac{\pi}{p}(z_2 + x_2)\right)^2 + \sin\left(\frac{\pi}{p}(z_1 - x_1)\right)^2\right)\Bigg].
\end{aligned} \tag{2.8}$$

Lemma 2.3 *Let $z, x \in \mathbb{R}^2$, such that $x_2 > z_2 > 0$, and let k be small enough, then*

$$\Gamma^{\mathbf{k}}_{+}(z,x) = \Gamma^{\mathbf{k}}_{+,p}(z,x) + \Gamma^{\mathbf{k}}_{+,e}(z,x)\,,$$

where

$$\begin{aligned}
\Gamma^{\mathbf{k}}_{+,p}(z,x) &:= -\frac{\sin(k_2 z_2)}{k_2 p} e^{i(k_2 x_2 - k_1(z_1 - x_1))}, \\
\Gamma^{\mathbf{k}}_{+,e}(z,x) &:= -\sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ l := 2\pi n/p}} \left(\frac{e^{i(l-k_1)(z_1-x_1)}}{p\sqrt{|l-k_1|^2 - k^2}} \sinh\left(\sqrt{|l-k_1|^2 - k^2}\, z_2\right)\right) e^{-\sqrt{|l-k_1|^2-k^2}\, x_2}\,.
\end{aligned}$$

We also have that

$$\begin{aligned}
\nabla_x \Gamma^{\mathbf{k}}_{+,p}(z,x) &= i\,\mathbf{k}\,\Gamma^{\mathbf{k}}_{+,p}(z,x)\,, \\
\nabla_x \Gamma^{\mathbf{k}}_{+,e}(z,x) &= -\sum_{\substack{n \in \mathbb{Z}\setminus\{0\} \\ l := 2\pi n/p}} \Bigg(\begin{pmatrix} -i(l-k_1) \\ -\sqrt{|l-k_1|^2 - k^2} \end{pmatrix} \frac{e^{i(l-k_1)(z_1-x_1)}}{p\sqrt{|l-k_1|^2 - k^2}} \\
&\qquad \cdot \sinh\left(\sqrt{|l-k_1|^2 - k^2}\, z_2\right) e^{-\sqrt{|l-k_1|^2-k^2}\, x_2}\Bigg)\,.
\end{aligned}$$

Lemma 2.4 *Let $z, x \in \mathbb{R}^2$, such that $z_2 > x_2 > 0$, and let k be small enough, then*

$$\Gamma^{\mathbf{k}}_{+}(z,x) = \Gamma^{\mathbf{k}}_{+,p}(z,x) + \Gamma^{\mathbf{k}}_{+,e}(z,x)\,,$$

where

$$\Gamma^{\mathbf{k}}_{+,p}(z,x) := -\frac{\sin(k_2 x_2)}{k_2 p} e^{i(k_2 z_2 - k_1(z_1-x_1))},$$

$$\Gamma^{\mathbf{k}}_{+,e}(z,x) := -\sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \left(\frac{e^{i(l-k_1)(z_1-x_1)}}{p\sqrt{|l-k_1|^2-k^2}} \sinh\left(\sqrt{|l-k_1|^2-k^2}\,x_2\right)\right) e^{-\sqrt{|l-k_1|^2-k^2}\,z_2}\,.$$

We also have that

$$\nabla_x \Gamma^{\mathbf{k}}_{+,p}(z,x) = \begin{pmatrix} -\frac{ik_1}{k_2 p}\sin(k_2 x_2) \\ -\frac{1}{p}\cos(k_2 x_2)\end{pmatrix} e^{i(k_2 z_2 - k_1(z_1-x_1))},$$

$$\nabla_x \Gamma^{\mathbf{k}}_{+,e}(z,x) = -\sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \left(\begin{pmatrix} -i(l-k_1)\sinh\left(\sqrt{|l-k_1|^2-k^2}\,x_2\right) \\ \sqrt{|l-k_1|^2-k^2}\cosh\left(\sqrt{|l-k_1|^2-k^2}\,x_2\right)\end{pmatrix}\right.$$
$$\left. \cdot \frac{e^{i(l-k_1)(z_1-x_1)}}{p\sqrt{|l-k_1|^2-k^2}}\, e^{-\sqrt{|l-k_1|^2-k^2}\,z_2}\right)\,.$$

2.2 Layer Potentials

Let $E \subset \mathbb{R}^2$ be an open, bounded, simply connected Lipschitz domain. Then we define the double-layer potential $\mathcal{S}^k_{\partial E} : \mathcal{H}^{-1/2}(\partial E) \to \mathcal{H}^{1/2}(\mathbb{R}^2)$ and the Neumann-Poincaré operator $\mathcal{K}^k_{\partial E} : \mathcal{H}^{1/2}(\partial E) \to \mathcal{H}^{1/2}(\partial E)$ as

$$\mathcal{S}^k_{\partial E}[\varphi](x) := \int_{\partial E} \Gamma^k(x,y)\,\varphi(y)\mathrm{d}\sigma_y\,,$$
$$\mathcal{K}^k_{\partial E}[\varphi](x) := \text{p.v.}\int_{\partial E} \partial_{\nu_y}\Gamma^k(x,y)\,\varphi(y)\mathrm{d}\sigma_y\,,$$

where 'p.v.' denotes the principal value. $\mathcal{K}^k_{\partial E}$ is a bounded function and if E has a $\mathcal{C}^2$-boundary, $\mathcal{K}^k_{\partial E}$ is a compact operator. Moreover, we have the following result ([8, Propositions 2.5 and 2.6]):

Lemma 2.5 *The operator $(-\frac{1}{2})\,\mathcal{I} + \mathcal{K}^k_{\partial E}$ is invertible if and only if k is not a Neumann eigenvalue of the operator $-\triangle$. Then, $((-\frac{1}{2})\,\mathcal{I} + \mathcal{K}^k_{\partial E})^{-1}$ has a continuation to an operator-valued meromorphic function in $\mathbb{C}$. Also, all Neumann eigenvalues of the operator $-\triangle$ are characteristic values of the operator $(-\frac{1}{2})\,\mathcal{I} + \mathcal{K}^k_{\partial E}$.*

We define now the double-layer potential and the Neumann-Poincaré operator on a periodic structure. Let $q_1, q_2 \in \mathbb{R}^2$ and let $E \Subset \{y \in \mathbb{R}^2 \,|\, |y_1| < q_1/2 \text{ and } 0 < y_2 < q_2\}$

be a open, bounded, simply connected domain. We define $\Omega := \bigcup_{n\in\mathbb{Z}} E + (nq_1, 0)^{\mathrm{T}}$ and with that let $\mathcal{S}^{\mathbf{k}}_{\partial\Omega,\times} : \mathcal{H}^{-1/2}(\partial E) \to \mathcal{H}^{1/2}(\mathbb{R}^2)$ and $\mathcal{K}^{\mathbf{k}}_{\partial\Omega,\times} : \mathcal{H}^{1/2}(\partial E) \to \mathcal{H}^{1/2}(\partial E)$ be defined as

$$\mathcal{S}^{\mathbf{k}}_{\partial\Omega,\times}[\varphi](x) := \int_{\partial\Omega} \Gamma^{\mathbf{k}}_{\times}(x,y)\,\varphi(y)\mathrm{d}\sigma_y\,,$$
$$\mathcal{K}^{\mathbf{k}}_{\partial\Omega,\times}[\varphi](x) := \text{p.v.}\int_{\partial\Omega} \partial_{\nu_y}\Gamma^{\mathbf{k}}_{\times}(x,y)\,\varphi(y)\mathrm{d}\sigma_y\,.$$

We know that if E has a $\mathcal{C}^2$-boundary, then $\mathcal{K}^{\mathbf{k}}_{\partial\Omega,\times}$ is compact. We also have the following standard result (see [8, Lemma 7.4]):

Lemma 2.6 *The operator $\frac{1}{2}\mathcal{I} + \mathcal{K}^{\mathbf{k}}_{\partial\Omega,\times} : \mathcal{H}^{1/2}(\partial E) \to \mathcal{H}^{1/2}(\partial E)$ is invertible.*

2.3 Neumann Functions and their Remainders

Let $k_{E,\min,\triangle}$ be the first non-zero eigenvalue of the operator $-\triangle$ with Neumann conditions on the boundary ∂E of an open bounded domain $E \subset \mathbb{R}^2$. We now can define the following functions:

Definition 2.7 *Let $E \subset \mathbb{R}^2$ be a open, bounded, simply connected $\mathcal{C}^2$-domain. Let $0 \neq |\mathbf{k}| < k_{E,\min,\triangle}/2$. Then we define $\mathrm{N}^k_E : E \times \overline{E} \setminus \{(z,x) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid z = x\} \to \mathbb{C}$ and $\mathrm{N}^k_{\partial E} : \partial E \times \overline{E} \setminus \{(z,x) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, z = x\} \to \mathbb{C}$ as*

$$\mathrm{N}^k_E(z,x) := \frac{1}{2\pi}\log|z-x| + \frac{1/|E|}{k^2} + \mathrm{R}^k_E(z,x),$$
$$\mathrm{N}^k_{\partial E}(z,x) := \frac{1}{\pi}\log|z-x| + \frac{1/|E|}{k^2} + \mathrm{R}^k_{\partial E}(z,x),$$

where $|E|$ denotes the area of E and N^k_E and $\mathrm{N}^k_{\partial E}$ are solutions to

$$\begin{cases} (\triangle_x + k^2)\mathrm{N}^k_E(z,x) = \delta_{\mathbf{0}}(z-x) & \text{for} \quad x \in E, \\ \partial_{\nu_x}\mathrm{N}^k_E(z,x) = 0 & \text{for} \quad x \in \partial E, \end{cases}$$

where $z \in E$ is fixed and where ν_x denotes the outside normal on ∂E with respect to x, and

$$\begin{cases} (\triangle_x + k^2)\mathrm{N}^k_{\partial E}(z,x) = \delta_{\mathbf{0}}(z-x) & \text{for} \quad x \in E, \\ \partial_{\nu_x}\mathrm{N}^k_{\partial E}(z,x) = 0 & \text{for} \quad x \in \partial E, \end{cases}$$

where $z \in \partial E$ is fixed.

With this definition the function N^k_E is a solution to the Helmholtz equation with $\delta_{\mathbf{0}}(z-x)$ on the right-hand side and has a vanishing normal derivative on the boundary. The same is valid for the function $\mathrm{N}^k_{\partial E}$, although the source point z is on the boundary. The remainders $\mathrm{R}^k_E(z,\cdot)$ and $\mathrm{R}^k_{\partial E}(z,\cdot)$ are smooth enough. We refer to [8, Chapter 2.3.5.].

Definition 2.8 *Let $q_1, q_2 \in \mathbb{R}^2$ and let $E \Subset \{y \in \mathbb{R}^2 \mid |y_1| < q_1/2 \text{ and } 0 < y_2 < q_2\}$ be a open, bounded, simply connected $\mathcal{C}^2$-domain. Let $0 \neq |\mathbf{k}| < k_{E,\min,\triangle}/2$. We define $\Omega := \bigcup_{n\in\mathbb{Z}} E + (nq_1, 0)^{\mathrm{T}}$. Then we define*

$$\mathrm{N}^{\mathbf{k}}_{\Omega,+} : \{(z,x) \in (\mathbb{R}^2_+ \setminus \overline{\Omega}) \times (\mathbb{R}^2_+ \setminus \Omega) \,|\, \neg(\exists n \in \mathbb{Z} : z_1 - x_1 = nq_1 \wedge z_2 = x_2)\} \to \mathbb{C},$$
$$\mathrm{N}^{\mathbf{k}}_{\partial\Omega,+} : \{(z,x) \in \partial\Omega \times (\mathbb{R}^2_+ \setminus \Omega) \,|\, \neg(\exists n \in \mathbb{Z} : z_1 - x_1 = nq_1 \wedge z_2 = x_2)\} \to \mathbb{C},$$

as

$$\mathrm{N}^{\mathbf{k}}_{\Omega,+}(z,x) := \Gamma^{\mathbf{k}}_{+}(z,x) + \mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,x),$$
$$\mathrm{N}^{\mathbf{k}}_{\partial\Omega,+}(z,x) := 2\Gamma^{\mathbf{k}}_{+}(z,x) + \mathrm{R}^{\mathbf{k}}_{\partial\Omega,+}(z,x),$$

with $\mathrm{R}^{\mathbf{k}}_{\Omega,+}$, $\mathrm{R}^{\mathbf{k}}_{\partial\Omega,+}$ being solutions to

$$\begin{cases} (\triangle_x + k^2)\mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,x) = 0 & \text{for} \quad x \in \mathbb{R}^2_+ \setminus \overline{\Omega}, \\ \partial_{\nu_x}\mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,x) = -\partial_{\nu_x}\Gamma^{\mathbf{k}}_{+}(z,x) & \text{for} \quad x \in \partial\Omega, \\ \mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,x) = 0 & \text{for} \quad x \in \partial\mathbb{R}^2_+, \end{cases}$$

where $z \in \mathbb{R}^2_+ \setminus \overline{\Omega}$ is fixed, where ν_x denotes the outside normal on $\partial\Omega$ with respect to x, and

$$\begin{cases} (\triangle_x + k^2)\mathrm{R}^{\mathbf{k}}_{\partial\Omega,+}(z,x) = 0 & \text{for} \quad x \in \mathbb{R}^2_+ \setminus \overline{\Omega}, \\ \partial_{\nu_x}\mathrm{R}^{\mathbf{k}}_{\partial\Omega,+}(z,x) = -2\partial_{\nu_x}\Gamma^{\mathbf{k}}_{+}(z,x) & \text{for} \quad x \in \partial\Omega, \\ \mathrm{R}^{\mathbf{k}}_{\partial\Omega,+}(z,x) = 0 & \text{for} \quad x \in \partial\mathbb{R}^2_+. \end{cases}$$

Here, $z \in \partial\Omega$ is fixed, and $\mathrm{R}^{\mathbf{k}}_{\Omega,+}$, $\mathrm{R}^{\mathbf{k}}_{\partial\Omega,+}$ satisfy the outgoing radiation condition (see [16]), thus in particular

$$|\partial_{x_2}\mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,x) - ik_2\mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,x)| \to 0 \quad \text{for} \quad x_2 \to \infty.$$

Analogously, we define $\mathrm{N}^{\mathbf{k}}_{\Omega,\times}$, $\mathrm{N}^{\mathbf{k}}_{\partial\Omega,\times}$, $\mathrm{R}^{\mathbf{k}}_{\Omega,\times}$ and $\mathrm{R}^{\mathbf{k}}_{\partial\Omega,\times}$.

For the remainder functions we have the following formulas:

Lemma 2.9 *Let $z, x \in \mathbb{R}^2$, such that $z_2 > q_2$ and $x_2 \in \partial E$ then*

$$\mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,x) = -\int_{\partial E} \mathrm{N}^{\mathbf{k}}_{\partial\Omega,\times}(x,y)\partial_{\nu_y}\Gamma^{\mathbf{k}}_{+}(z,y)\mathrm{d}\sigma_y.$$

Lemma 2.10 *Let $z, x \in \partial E$ and $k \neq 0$, then*

$$\left(\frac{1}{2}\mathrm{I} + \mathcal{K}^{\mathbf{k}}_{\partial\Omega,\times}\right)\left[\mathrm{R}^{\mathbf{k}}_{\partial\Omega,+}(z,\cdot)\right](x) = \mathcal{S}^{\mathbf{k}}_{\partial\Omega,\times}\left[-2\partial_{\nu}\Gamma^{\mathbf{k}}_{+}(z,\cdot)\right](x).$$

Lemma 2.11 *Let $z, x \in \partial E$ and $0 \neq k < k_{E,\min,\triangle}/2$ then*

$$\left(\frac{1}{2}\mathrm{I} - \mathcal{K}^{k}_{\partial E}\right)\left[\mathrm{R}^{k}_{\partial E}(z,\cdot)\right](x) = -\int_E \left(2k^2\Gamma^0(z,y) + \frac{1}{|E|}\right)\Gamma^k(x,y)\mathrm{d}y + \mathcal{S}^{k}_{\partial E}\left[2\partial_{\nu_x}\Gamma^0(z,\cdot)\right](x).$$

Lemma 2.12 *For $z \in \partial E$ we have that*

$$\int_E \left(\frac{1}{|E|}\right)\frac{1}{2\pi}\log(k)\mathrm{d}y = 2\int_{\partial E} \partial_{\nu_y}\Gamma^0(z,y)\,\frac{1}{2\pi}\log(k)\mathrm{d}\sigma_y\,. \tag{2.9}$$

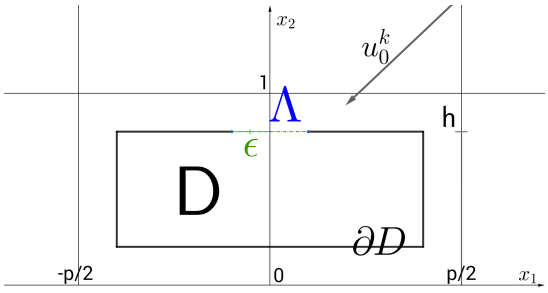
3 One Periodically Arranged Helmholtz Resonator

In this section, we look at a bounded, connected, domain D, which has height h. Additionally, D has a gap Λ at its boundary, which allows the incident wave U_0^k to pass through. The incident wave rebounds inside D and leaves at the gap Λ, which then leads to the scattered wave U_s^k. We repeat the geometry with periodicity p along the x_1-axis and scale it by a factor δ.

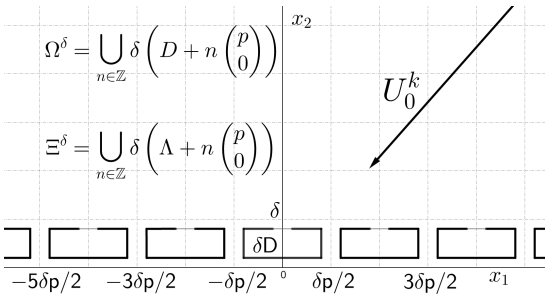
We look for an accurate approximation of the resonance as well as the scattered wave in the far-field. We will see that, this approximation satisfies the Helmholtz equation with a Robin boundary condition at the x_1-axis which approximates a Dirichlet boundary or a Neumann boundary depending on the magnitude of the incoming wave vector $\mathbf{k}$ and δ.

3.1 Mathematical Description of the Physical Problem

3.1.1 Geometry



(a) *Microscopic, non-periodic view of our Helmholtz resonator. The Helmholtz resonator is contained in the unit cell* $(-p/2, p/2) \times (0,1)$*. It has a gap* Λ *of length* 2ε*, which is parallel to the* x_1 *axis and centered in* $(0,h)^{\mathrm{T}}$*, where* $h \in (0,1)$*.* u_0^k *denotes the incident wave. D has not to be rectangular in shape.*



(b) *Macroscopic view of our periodically arranged Helmholtz resonators, with periodicity* δp*. All Helmholtz resonators have the form of the Helmholtz resonator depicted in* (a)*, but are scaled with the factor* δ*.* U_0^k *denotes the incident wave.* Ω^δ *is the collection of all Helmholtz resonators and* Ξ^δ *is the collection of all gaps.*

Figure 1: *The physical setup. In* (a) *we have the microscopic, non-periodic view. In* (b) *we have the macroscopic, periodic view.*

Before we consider the periodic and macroscopic problem, we first define the geometry of our Helmholtz resonator in the unit cell. Let $D \Subset (-p/2, p/2) \times (0,1)$ be a open, bounded, simply connected and connected domain, where $p \in \mathbb{R}$ and p is close enough to 1. For sake of simplicity we assume that D is a C^2-domain. We define $\Lambda \subset \partial D$ to be the gap of D, where Λ is a line segment parallel to the x_1-axis. Λ is centered at $(0,h)^{\mathrm{T}}$, where $h \in (0,1)$ is the height of Λ, and Λ has length 2ε, where $\varepsilon \in (0,1)$ and it is small enough. To facilitate future computations we assume that $h/p \geq 1/2$.

Let us define the macroscopic view. We define the collection of periodically arranged Helmholtz resonators Ω^δ, with period δp, and the collection of gaps of those Helmholtz resonators Ξ^δ, where a single gap has length $2\delta\varepsilon$, as

$$\Omega^\delta := \bigcup_{n\in\mathbb{Z}} \delta\left(D + n\begin{pmatrix} p \\ 0 \end{pmatrix}\right),$$
$$\Xi^\delta := \bigcup_{n\in\mathbb{Z}} \delta\left(\Lambda + n\begin{pmatrix} p \\ 0 \end{pmatrix}\right).$$

3.1.2 Incident Wave

Let $\mathbf{k} := (k_1, k_2)^{\mathrm{T}} \in \mathbb{R}^2$ be the wave vector. We will fix the direction of the wave vector, that is, k_1/k and k_2/k, where $k := |\mathbf{k}| := (k_1^2 + k_2^2)^{1/2} \in [0, \infty)$, but let the magnitude k vary. With that, we define the function $U_0^k : \mathbb{R}^2 \to \mathbb{C}$ as

$$U_0^k(x) := a_0 e^{-ik_1x_1} e^{-ik_2x_2},$$

where $a_0 \in \mathbb{R}$ denotes the amplitude. U_0^k will be our incident wave.

From (2.6), it follows that

$$U_0^k \circ \mathtt{P}(x) = a_0 e^{-ik_1x_1} e^{ik_2x_2},$$

and

$$(U_0^k - U_0^k{\circ}\mathtt{P})(x) = -2ia_0 e^{-ik_1x_1} \sin(k_2x_2).$$

We will also need the following equation

$$\nabla(U_0^k - U_0^k{\circ}\mathtt{P})(x) = \begin{pmatrix} -2a_0k_1 e^{-ik_1x_1} \sin(k_2x_2) \\ -2ia_0k_2 e^{-ik_1x_1} \cos(k_2x_2) \end{pmatrix}.$$

Consider also that U_0^k and $U_0^k{\circ}\mathtt{P}$ are quasi-periodic with quasi-momentum $-k_1p$, that is,

$$U_0^k\left(x + \begin{pmatrix} p \\ 0 \end{pmatrix}\right) = e^{-ik_1p} U_0^k(x),$$
$$U_0^k{\circ}\mathtt{P}\left(x + \begin{pmatrix} p \\ 0 \end{pmatrix}\right) = e^{-ik_1p} U_0^k{\circ}\mathtt{P}(x).$$

3.1.3 The Resulting Wave

With the geometry and the incident wave, we model the electromagnetic scattering problem and the resulting wave $U^k : \mathbb{R}^2_+ \setminus \partial\Omega^\delta \to \mathbb{C}$ by the following system of equa-

tions:

$$\begin{cases} (\triangle + k^2)\, U^k = 0 & \text{in} \quad \mathbb{R}^2_+ \setminus \partial\Omega^\delta, \\ U^k|_+ - U^k|_- = 0 & \text{on} \quad \Xi^\delta, \\ \partial_\nu U^k|_+ - \partial_\nu U^k|_- = 0 & \text{on} \quad \Xi^\delta, \\ \partial_\nu U^k|_+ = 0 & \text{on} \quad \partial\Omega^\delta \setminus \Xi^\delta, \\ \partial_\nu U^k|_- = 0 & \text{on} \quad \partial\Omega^\delta \setminus \Xi^\delta, \\ U^k = 0 & \text{on} \quad \partial\mathbb{R}^2_+, \end{cases} \tag{3.1}$$

where $\cdot|_+$ denotes the limit from outside of Ω^δ and $\cdot|_-$ denotes the limit from inside of Ω^δ, and ∂_ν denotes the normal derivative on $\partial\Omega^\delta$. Similar to diffraction problems for gratings, the above system of equations is complemented by a certain outgoing radiation condition imposed on the scattered field $U^k_s := U^k - (U^k_0 - U^k_0 \circ \mathtt{P})$ and quasi-periodicity on U^k. More precisely, we are interested in the quasi-periodic solutions, that is,

$$U^k\left(x + \begin{pmatrix} p \\ 0 \end{pmatrix}\right) = e^{-ik_1 p} U^k(x) \quad \text{for} \quad x \in \mathbb{R}^2_+,$$

and solutions satisfying the outgoing radiation condition, thus we have

$$\left|\partial_{x_2} U^k_s - ik_2 U^k_s\right| \to 0 \quad \text{for} \quad x_2 \to \infty.$$

Then the outgoing radiation condition can be imposed by assuming that all the modes in the Rayleigh-Bloch expansion are either decaying exponentially or propagating along the x_2-direction. Since in our case we assume that the period of the resonator structure δp is much smaller than k, the outgoing radiation condition takes the following specific form:

$$\begin{aligned} (U^k - U^k_0)(x) &\sim a e^{-ik_1 x_1} e^{ik_2 x_2} \quad \text{as} \quad x_2 \to \infty, \\ U^k_s(x) &\sim (a+1) e^{-ik_1 x_1} e^{ik_2 x_2} \quad \text{as} \quad x_2 \to \infty, \end{aligned}$$

for some constant amplitude $a \in \mathbb{R}$.

As a remark, in the general case where U^k_0 is a superposition of plane-waves, we can decompose U^k_0 using Bloch-Flocquet theory [25, 8]. We obtain a family of problems to solve, each one with its own outgoing radiation condition. The final solution is then the superposition of all these solutions.

Consider also that in absence of Helmholtz resonators the solution to (3.1) is given by $U^k_0 - U^k_0 \circ \mathtt{P}$.

3.1.4 The Resulting Wave in the Microscopic View

Given the resulting wave $U^k(x)$, the function $u^{\delta k}(x) : \mathbb{R}^2_+ \setminus \partial\Omega^1 \to \mathbb{C}$, $u^{\delta k}(x) := U^k(\delta x)$ represents the resulting wave, but where the Helmholtz resonators are scaled-back

and thus are of height h and not δh. $u^{\delta k}$ satisfies

$$\begin{cases} \left(\triangle + (\delta k)^2\right) u^{\delta k} = 0 & \text{in} \quad \mathbb{R}^2_+ \setminus \partial\Omega^1, \\ u^{\delta k}|_+ - u^{\delta k}|_- = 0 & \text{on} \quad \Xi^1, \\ \partial_\nu u^{\delta k}|_+ - \partial_\nu u^{\delta k}|_- = 0 & \text{on} \quad \Xi^1, \\ \partial_\nu u^{\delta k}|_+ = 0 & \text{on} \quad \partial\Omega^1 \setminus \Xi^1, \\ \partial_\nu u^{\delta k}|_- = 0 & \text{on} \quad \partial\Omega^1 \setminus \Xi^1, \\ u^{\delta k} = 0 & \text{on} \quad \partial\mathbb{R}^2_+, \end{cases} \tag{3.2}$$

where $\cdot|_+$ denotes the limit from outside of Ω^1 and $\cdot|_-$ denotes the limit from inside of Ω^1, and ∂_ν denotes the normal derivative on $\partial\Omega^1$.

We can adopt the quasi-periodicity from the macroscopic view and obtain

$$u^{\delta k}\left(x + \begin{pmatrix} p \\ 0 \end{pmatrix}\right) = e^{-i\delta k_1 p} u^{\delta k}(x) \quad \text{for} \quad x \in \mathbb{R}^2_+.$$

Defining $u_s^{\delta k} := u^{\delta k} - (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})$ we also get that

$$\left|\partial_{x_2} u_s^{\delta k} - i\delta k_2 u_s^{\delta k}\right| \to 0 \quad \text{for} \quad x_2 \to \infty.$$

We see that $u^{\delta k}$ solves the same partial differential equation like U^k in the rescaled geometry, but with the scaled wave vector δk. We will see that we can express $u^{\delta k}$ as an expansion in terms of δ and we will give an analytic expression for the first order term.

3.2 Main Results

We assume that $\delta k \in \mathbb{K} := \{\hat{k} \in \mathbb{R} \,|\, 0 \neq |\hat{k}| < k_{D,\min,\triangle}/2 \text{ and } |\hat{k}|^2 < \inf\{|l - \hat{k}\, e_1|^2 \,|\, l := 2\pi n/p,\ n \in \mathbb{Z} \setminus \{0\}\}\}$, where $k_{D,\min,\triangle}$ is defined as the first non-zero eigenvalue of the operator $-\triangle$ with Neumann conditions on the boundary ∂D . If we would extend the domain $\mathbb{K}$ to $\mathbb{K}_{\mathbb{C}} := \{k_* \in \mathbb{C} \,|\, \sqrt{k_* \bar{k}_*} < k_{D,\min,\triangle}/2\}$, we would obtain following resonance values for our physical problem:

Theorem 3.1 *We have exactly two resonance values in $\mathbb{K}_{\mathbb{C}}$ for U^k. These are*

$$\begin{aligned} k_+^{\delta,\varepsilon} &= k_0^{\delta,\varepsilon} + \frac{\alpha_0 |D|}{2\delta} c_\varepsilon^{3/2} + \frac{\alpha_1 |D|}{2\delta} c_\varepsilon^2 + \frac{1}{\delta} \mathcal{O}\left(c_\varepsilon^{5/2}\right), \\ k_-^{\delta,\varepsilon} &= -k_0^{\delta,\varepsilon} - \frac{\alpha_0 |D|}{2\delta} c_\varepsilon^{3/2} + \frac{\alpha_1 |D|}{2\delta} c_\varepsilon^2 + \frac{1}{\delta} \mathcal{O}\left(c_\varepsilon^{5/2}\right), \\ \delta k_0^{\delta,\varepsilon} &:= c_\varepsilon^{1/2} := \sqrt{\frac{-\pi}{2|D| \log(\varepsilon/2)}}, \end{aligned}$$

where α_0 and α_1 are given by

$$\alpha_0 := \mathrm{R}^0_{\partial D}\left(\begin{pmatrix}0\\h\end{pmatrix},\begin{pmatrix}0\\h\end{pmatrix}\right)+\mathrm{R}^0_{\partial\Omega^1,+}\left(\begin{pmatrix}0\\h\end{pmatrix},\begin{pmatrix}0\\h\end{pmatrix}\right)+\frac{1}{\pi}\log\left(\frac{\pi}{p}\right)-\frac{1}{\pi}\log\left(\sinh\left(\frac{\pi}{p}2\,h\right)\right), \tag{3.3}$$

$$\alpha_1 := \partial_{\delta k}\mathrm{R}^0_{\partial D}\left(\begin{pmatrix}0\\h\end{pmatrix},\begin{pmatrix}0\\h\end{pmatrix}\right)+\partial_{\delta k}\mathrm{R}^0_{\partial\Omega^1,+}\left(\begin{pmatrix}0\\h\end{pmatrix},\begin{pmatrix}0\\h\end{pmatrix}\right). \tag{3.4}$$

We have the following approximation for the resulting wave U^k:

Theorem 3.2 *Let $V_r := \{z \in \mathbb{R}^2_+ \,|\, z_2 > r\}$. There exist constants $C_{(3.5)}, \widetilde{C}_{(3.5)} > 0$ such that*

$$\begin{aligned}&\left\|U^k_s-(U^k_{\mathcal{S}^\star_p}+U^k_{\mathcal{T}^\star_p}+U^k_{\mathrm{RHS},p})\right\|_{\mathrm{L}^\infty(V_r)}+\left\|\nabla\left[U^k_s-(U^k_{\mathcal{S}^\star_p}+U^k_{\mathcal{T}^\star_p}+U^k_{\mathrm{RHS},p})\right]\right\|_{\mathrm{L}^\infty(V_r)} \\ &\quad\le C_{(3.5)}\left(\frac{\delta}{|\log(\varepsilon)|^3}\left(\frac{1}{\delta k-\delta k^{\delta,\varepsilon}_+}-\frac{1}{\delta k-\delta k^{\delta,\varepsilon}_-}\right)+\frac{\delta}{|\log(\varepsilon)|^2}+\delta\varepsilon+\delta\,e^{-\widetilde{C}_{(3.5)}r}+\delta^2\right),\end{aligned} \tag{3.5}$$

for δ, ε small enough and r large enough, where

$$\begin{aligned}U^k_{\mathcal{S}^\star_p}(z) &= -e^{i(k_2z_2-k_1z_1)}\frac{h}{p}\int_\Lambda \mu^{\delta k}_\star(y)\mathrm{d}\sigma_y\,,\\ U^k_{\mathcal{T}^\star_p}(z) &= e^{i(k_2z_2-k_1z_1)}\frac{1}{p}\int_\Lambda\int_{\partial D}\mu^{\delta k}_\star(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,,\\ U^k_{\mathrm{RHS},p}(z) &= e^{i\delta(k_2z_2-k_1z_1)}\left[\int_{\partial D}\partial_\nu(u^{\delta k}_0-u^{\delta k}_0\circ\mathtt{P})(y)\frac{\sin(\delta k_2y_2)\,e^{i\delta k_1y_1}}{\delta k_2p}\mathrm{d}\sigma_y\right.\\ &\quad\left.-\int_{\partial D}\int_{\partial D}\partial_\nu(u^{\delta k}_0-u^{\delta k}_0\circ\mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}\frac{ik_1}{pk_2}\sin(\delta k_2w_2)\\ \frac{1}{p}\cos(\delta k_2w_2)\end{pmatrix}e^{i\delta k_1w_1}\mathrm{d}\sigma_w\mathrm{d}\sigma_y\right].\end{aligned}$$

and where, for $y \in \Lambda$, we have

$$\mu^{\delta k}_\star(y)=\frac{1}{\sqrt{\varepsilon^2-y_1^2}}\left(\frac{-\pi}{4|D|(\log(\varepsilon/2))^2}\frac{f^{\delta k}(0)}{\delta k^{\delta,\varepsilon}_+-\delta k^{\delta,\varepsilon}_-}\left(\frac{1}{\delta k-\delta k^{\delta,\varepsilon}_+}-\frac{1}{\delta k-\delta k^{\delta,\varepsilon}_-}\right)+\frac{f^{\delta k}(0)}{2\log(\varepsilon/2)}\right),$$

with the constant $f^{\delta k}(0) \in \mathbb{R}$ being given by

$$f^{\delta k}(0)=2ia_0\sin(\delta k_2h)-2a_0\delta\int_{\partial D}\nu_y\cdot\begin{pmatrix}k_1e^{-i\delta k_1y_1}\sin(\delta k_2y_2)\\ ik_2e^{-i\delta k_1y_1}\cos(\delta k_2y_2)\end{pmatrix}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}\left(\begin{pmatrix}0\\h\end{pmatrix},y\right)\mathrm{d}\sigma_y\,.$$

We see from Theorem 3.2 that the function

$$U^k_{\mathrm{app}}(z) := (U^k_0-U^k_0\circ\mathtt{P})(z)+U^k_{\mathcal{S}^\star_p}(z)+U^k_{\mathcal{T}^\star_p}(z)+U^k_{\mathrm{RHS},p}(z)\,,$$

gives an accurate approximation of U^k in the far-field. Moreover, it satisfies the Helmholtz equation in $\mathbb{R}^2_+$ with the boundary condition

$$U^k_{\mathrm{app}}(z)+\delta c_{\mathrm{IBC}}\partial_{z_2}U^k_{\mathrm{app}}(z)=0\,,\quad \text{for } z\in\partial\mathbb{R}^2_+\,,$$

and $U^k_{\text{app}} - U^k_0$ satisfies the outgoing radiation condition. The boundary condition is called 'Impedance Boundary Condition' and for $c_{\text{IBC}} = 0$ it yields a Dirichlet boundary condition while for $c_{\text{IBC}} \gg 1$ it approximates a Neumann boundary condition for U^k_{app}. Using Theorem 3.2, we can express c_{IBC} as follows.

Theorem 3.3 *The constant in the impedance boundary condition is given by*

$$c_{\text{IBC}} = \frac{1}{2ia_0\delta k_2}\, C^{\delta k}_{(3.6)} + \mathcal{O}(\delta)\,,$$

where $C^{\delta k}_{(3.6)} \in \mathbb{R}$ *is defined as*

$$\begin{aligned} C^{\delta k}_{(3.6)} =& -\frac{h}{p}\int_\Lambda \mu^{\delta k}_\star(y)\mathrm{d}\sigma_y + \frac{1}{p}\int_\Lambda\int_{\partial D} \mu^{\delta k}_\star(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}\sigma_w\mathrm{d}\sigma_y \\ &+ \int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ \mathtt{P})(y)\frac{\sin(\delta k_2 y_2)\, e^{i\delta k_1 y_1}}{\delta k_2 p}\mathrm{d}\sigma_y \qquad (3.6)\\ &- \int_{\partial D}\int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}\frac{ik_1}{pk_2}\sin(\delta k_2 w_2)\\ \frac{1}{p}\cos(\delta k_2 w_2)\end{pmatrix} e^{i\delta k_1 w_1}\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,. \end{aligned}$$

3.3 Proof of the Main Results

We want to proof Theorems 3.1 – 3.3. First, we express the resulting wave outside the Helmholtz resonators and the resulting wave inside the Helmholtz resonators through operators acting on the resulting wave, but restricted on the gap. This leads us to a condition with the linear operator $\mathcal{A}^{\delta k,\varepsilon}$, whose solution is the resulting wave on the gap up to a term of order δ^2. We solve this linear system based on the procedure given in [10]. We will see that it is solvable for a complex wave vector near 0 except in three points, two of which are the resonances of our system. With this we obtain the resulting wave on the gap. Then we recover the resulting wave outside the resonators up to a term of order δ^2. We will see, that we can split the resulting wave into a propagating wave and an evanescent one. The propagating one leads us to the impedance boundary condition constant c_{IBC}.

3.3.1 Collapsing the Wave-Informations on the Gap

Let us consider the resulting wave $u^{\delta k}$ in the microscopic view described in Subsection 3.1.4. We will keep the microscopic view until Subsection 3.3.6. We define the main strip $Y := \{y \in \mathbb{R}^2_+ \,|\, |y_1| < p/2\}$. D is the Helmholtz resonator on that strip and Λ the gap on ∂D. Furthermore, we fix $k_1/k =: e_1$ and $k_2/k =: e_2$ and assume that $\delta k \in \mathbb{K} := \{\hat{k} \in \mathbb{R} \,|\, 0 \neq |\hat{k}| < k_{D,\min,\triangle}/2 \text{ and } |\hat{k}|^2 < \inf\{|l - \hat{k}\, e_1|^2 \,|\, l := 2\pi n/p,\ n \in \mathbb{Z}\setminus\{0\}\}\}$. Consider that $u^{\delta k}$ is continuous on the gap Λ, thus $u^{\delta k}(z)$ is well-defined for $z \in \Lambda$.

Proposition 3.4 *Let* N^k_D *be as in Definition 2.7. Let* $z \in D$*, then we have*

$$u^{\delta k}(z) = -\int_\Lambda \partial_\nu u^{\delta k}(y)\mathrm{N}^{\delta k}_D(z,y)\mathrm{d}\sigma_y\,.$$

Let $z \in \Lambda$. Then we have

$$u^{\delta k}(z) = -\int_{\Lambda} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial D}(z,y) \mathrm{d}\sigma_y \,. \tag{3.7}$$

Proposition 3.5 *Let $\mathrm{N}^k_{\Omega,+}$, $\mathrm{N}^k_{\Omega,\times}$ be as in Definition 2.8. Let $z \in Y \setminus \overline{D}$. Then it follows that*

$$u^{\delta k}_s(z) = \int_{\Lambda} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\Omega^1,+}(z,y) \mathrm{d}\sigma_y - \int_{\partial D} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\Omega^1,+}(z,y) \mathrm{d}\sigma_y \,.$$

Let $z \in \Lambda$. Then we have

$$u^{\delta k}_s(z) = \int_{\Lambda} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y) \mathrm{d}\sigma_y - \int_{\partial D} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y) \mathrm{d}\sigma_y \,. \tag{3.8}$$

Using that $u^{\delta k}$ is continuous on the gap we can deduce from the following proposition a necessary condition for $\partial_\nu u^{\delta k}|_{\Lambda}$. Assume we can obtain a solution $\partial_\nu u^{\delta k}$ from that condition, then from Propositions 3.4 and 3.5 we can recover the resulting wave on Y.

Proposition 3.6 (Gap Formula) *Let $z \in \Lambda$ then*

$$\int_{\Lambda} \partial_\nu u^{\delta k}(y) \left(\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y) + \mathrm{N}^{\delta k}_{\partial D}(z,y) \right) \mathrm{d}\sigma_y = \int_{\partial D} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y) \mathrm{d}\sigma_y - (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(z). \tag{3.9}$$

Consider that the right-hand side in (3.9) does not depend on $u^{\delta k}$ and it is computable.

Proof (Proposition 3.4) Let us look at (3.7) first. Let $z \in \partial D$ then using Green's formula with $\left(\triangle + k^2\right) u^{\delta k} = 0$ we have

$$\begin{aligned} u^{\delta k}(z) &= \int_D u^{\delta k}(y) \left(\triangle_y + k^2\right) \mathrm{N}^{\delta k}_{\partial D}(z,y) \mathrm{d}y \\ &= \int_{\partial D} u^{\delta k}(y) \partial_{\nu_y} \mathrm{N}^{\delta k}_{\partial D}(z,y) \mathrm{d}\sigma_y - \int_{\partial D} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial D}(z,y) \mathrm{d}\sigma_y \,. \end{aligned}$$

Using that $\partial_{\nu_y} \mathrm{N}^{\delta k}_{\partial D}(z,y) = 0$ on ∂D and $\partial_\nu u^{\delta k}(y) = 0$ on $\partial D \setminus \Lambda$ we obtain the desired equation. We get the other equation analogously. ■

Proof (Proposition 3.5) Let us first look at (3.8). Let $r > 0$ and $U_r := \{y \in \mathbb{R}^2 \setminus \overline{D} \mid |y_1| < p/2 \wedge 0 < y_2 < r\}$, $\partial U_{r,0} := \{y \in \partial U_r \,|\, y_2 = 0\}$, $\partial U_{r,-} := \{y \in \partial U_r \,|\, y_1 = -p/2\}$, $\partial U_{r,+} := \{y \in \partial U_r \,|\, y_1 = +p/2\}$ and $\partial U_{r,r} := \{y \in \partial U_r \,|\, y_2 = r\}$. Then

$$u^{\delta k}_s(z) = \lim_{r\to\infty} \int_{U_r} u^{\delta k}_s(y) (\triangle_y + k^2) \mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y) dy.$$

Using Green's formula, we have

$$u_s^{\delta k}(z) = \lim_{r\to\infty} \Bigg(\int_{U_r} (\triangle + k^2) u_s^{\delta k}(y)\, \mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}y \tag{3.10}$$

$$- \int_{\partial D} u_s^{\delta k}(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y + \int_{\partial D} \partial_\nu u_s^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y \tag{3.11}$$

$$+ \int_{\partial U_{r,0}} u_s^{\delta k}(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,0}} \partial_\nu u_s^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y \tag{3.12}$$

$$+ \int_{\partial U_{r,-}} u_s^{\delta k}(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,-}} \partial_\nu u_s^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y \tag{3.13}$$

$$+ \int_{\partial U_{r,+}} u_s^{\delta k}(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,+}} \partial_\nu u_s^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y \tag{3.14}$$

$$+ \int_{\partial U_{r,r}} u_s^{\delta k}(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,r}} \partial_\nu u_s^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y \Bigg). \tag{3.15}$$

The right-hand side in (3.10) vanishes because $u_s^{\delta k}$ satisfies the homogeneous Helmholtz equation. The left term in (3.11) vanishes because $\mathrm{N}^{\delta k}_{\partial\Omega^1,+}$ has a vanishing normal derivative on $\partial\Omega^1$. Both terms in (3.12) vanish because of the Dirichlet boundary. The terms in (3.13) and in (3.14) cancel each other out because of the quasi-periodicity with quasi-momentum $-k_1 p$ for $u_s^{\delta k}$ and the quasi-periodicity with quasi-momentum $k_1 p$ for $\mathrm{N}^{\delta k}_{\partial\Omega^1,+}$, together with the explicit expression for the normal on $\partial U_{r,-}$, which is $(-1,0)^{\mathrm{T}}$, and the explicit expression for the normal on $\partial U_{r,+}$, which is $(1,0)^{\mathrm{T}}$. Thus we are left with

$$u_s^{\delta k}(z,x) = \int_{\partial D} \partial_\nu u_s^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y$$
$$+ \lim_{r\to\infty} \left(\int_{\partial U_{r,r}} u_s^{\delta k}(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,r}} \partial_\nu u_s^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y \right). \tag{3.16}$$

Using that $u_s^{\delta k}$ and $\mathrm{N}^{\delta k}_{\partial\Omega^1,+}$ satisfy the outgoing radiation condition, we can write $\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y) = \frac{1}{ik_2}\partial_{y_2}\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y) + o(1)$ and $\partial_{y_2} u_s^{\delta k}(y) = ik_2 u_s^{\delta k}(y) + 0(1)$ for $y_2 \to \infty$. With that we can eliminate the integrals within the limes.

Finally, using that $\partial_\nu u^{\delta k}|_{\partial D\setminus\Lambda} = 0$ and the definition of $u_s^{\delta k}$, we proved (3.8).

We get the other equation analogously. ■

Proof (Proposition 3.6) Using that $u^{\delta k}$ is continuous at Λ we have that $u^{\delta k}|_+(z) - u^{\delta k}|_-(z) = u^{\delta k}(z) - u^{\delta k}(z) = 0$, for $z \in \Lambda$. From (3.8) and (3.7), we obtain (3.9). ■

3.3.2 Expanding the Gap Formula in Terms of δ

We define $f^{\delta k} : \Lambda \to \mathbb{C}$ as the right-hand side of the Gap Formula 3.6, that is,

$$f^{\delta k}(z) := \int_{\partial D} \partial_\nu (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,+}(z,y)\mathrm{d}\sigma_y - (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(z). \tag{3.17}$$

We identify $f^{\delta k}(\tau)$ with $f^{\delta k}((\tau, h)^{\mathrm{T}})$, for $\tau \in (-\varepsilon, \varepsilon)$.

Let us define the following operator spaces and their respective norms:

Definition 3.7 *Let μ' represent the distributional derivative of μ. We define*

$$\begin{aligned}
\mathcal{X}^\varepsilon &:= \left\{ \mu \in L^2((-\varepsilon,\varepsilon)) \middle| \int_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2 - t^2} |\mu(t)|^2 \mathrm{d}t < \infty \right\}, \\
\|\mu\|_{\mathcal{X}^\varepsilon} &:= \left(\int_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2 - t^2} |\mu(t)|^2 \mathrm{d}t \right)^{1/2}, \\
\mathcal{Y}^\varepsilon &:= \left\{ \mu \in \mathcal{C}^0([-\varepsilon,\varepsilon]) | \mu' \in \mathcal{X}^\varepsilon \right\}, \\
\|\mu\|_{\mathcal{Y}^\varepsilon} &:= \left(\|\mu\|_{\mathcal{X}^\varepsilon}^2 + \|\mu'\|_{\mathcal{X}^\varepsilon}^2 \right)^{1/2}.
\end{aligned}$$

Consider that for $\mu \in \mathcal{X}^\varepsilon$

$$\int_{-\varepsilon}^{\varepsilon} \mu(t) \mathrm{d}t = \int_{-\varepsilon}^{\varepsilon} \mu(t) \frac{\sqrt[4]{\varepsilon^2 - t^2}}{\sqrt[4]{\varepsilon^2 - t^2}} \mathrm{d}t \leq \sqrt{\pi} \|\mu\|_{\mathcal{X}^\varepsilon}, \tag{3.18}$$

because of the L^2-Cauchy-Schwarz inequality.

Definition 3.8 *Let $\mu \in \mathcal{X}^\varepsilon$ and $\alpha > 0$. We say $\mu = \mathcal{O}_{\mathcal{X}^\varepsilon}(\alpha)$ for $\alpha \to 0$ if $\frac{\|\mu\|_{\mathcal{X}^\varepsilon}}{\alpha}$ is bounded as $\alpha \to 0$.*

With those spaces we can define the following operators:

Definition 3.9 *The following operators are defined as functions from $\mathcal{X}^\varepsilon$ to $\mathcal{Y}^\varepsilon$. Let $n \in \mathbb{N} \setminus \{0\}$, then*

$$\begin{aligned}
\mathcal{L}^\varepsilon[\mu](\tau) &:= \int_{-\varepsilon}^{\varepsilon} \mu(t) \log(|\tau - t|) \mathrm{d}t, \\
\mathcal{K}^\varepsilon[\mu](\tau) &:= \frac{1}{|D|} \int_{-\varepsilon}^{\varepsilon} \mu(t) \mathrm{d}t, \\
\mathcal{R}^{\delta k,\varepsilon}[\mu](\tau) &:= \int_{-\varepsilon}^{\varepsilon} \mu(t) \Bigg[\mathrm{R}_{\partial D}^{\delta k} \left(\begin{pmatrix} \tau \\ h \end{pmatrix}, \begin{pmatrix} t \\ h \end{pmatrix} \right) + \mathrm{R}_{\partial\Omega^1,+}^{\delta k} \left(\begin{pmatrix} \tau \\ h \end{pmatrix}, \begin{pmatrix} t \\ h \end{pmatrix} \right) \\
&\qquad + \frac{1}{\pi} \log\left(\frac{\pi}{p} \right) + \frac{1}{\pi} \log\left(\operatorname{sinc}\left| \frac{\pi}{p} (\tau - t) \right| \right) \\
&\qquad - \frac{1}{2\pi} \log\left(\sinh\left(\frac{\pi}{p} 2h \right)^2 + \sin\left(\frac{\pi}{p} (\tau - t) \right)^2 \right) \Bigg] \mathrm{d}t, \\
\widehat{\Gamma}_{+,n}^{k,\varepsilon}[\mu](\tau) &:= \int_{-\varepsilon}^{\varepsilon} \mu(t) \, \Gamma_{+,n}^k \left(\begin{pmatrix} \tau \\ h \end{pmatrix}, \begin{pmatrix} t \\ h \end{pmatrix} \right) \mathrm{d}t, \\
\mathcal{A}^{\delta k,\varepsilon}[\mu](\tau) &:= \frac{2}{\pi} \mathcal{L}^\varepsilon[\mu](\tau) + \frac{\mathcal{K}^\varepsilon[\mu](\tau)}{\delta^2 k^2} + \mathcal{R}^{\delta k,\varepsilon}[\mu](\tau),
\end{aligned}$$

where $\Gamma_{+,n}^k$ is given in Lemma 2.2.

Later, we will show that

$$\partial_\nu u^{\delta k}|_\Lambda = (\mathcal{A}^{\delta k,\varepsilon})^{-1}[f^{\delta k}] + \mathcal{O}(\delta^2).$$

Proposition 3.10 *Let $2\varepsilon < p$, let $\tau \in (-\varepsilon, \varepsilon)$, then*

$$\mathcal{A}^{\delta k,\varepsilon}[\partial_\nu u^{\delta k}|_\Lambda](\tau) + \sum_{n=1}^{\infty} \delta^n \, 2\,\widehat{\Gamma}^{k,\varepsilon}_{+,n}[\partial_\nu u^{\delta k}|_\Lambda](\tau) = f^{\delta k}((\tau,h)^{\mathrm{T}}).$$

Proof Let $z := (\tau,h)^{\mathrm{T}} \in \Lambda$. From Proposition 3.6, it follows that

$$\int_\Lambda \partial_\nu u^{\delta k}(y) \left(2\Gamma^{\delta k}_+(z,y) + \mathrm{R}^{\delta k}_{\partial\Omega^1,+}(z,y) + \frac{1}{\pi}\log|z-y| + \frac{2/|D|}{\delta^2 k^2} + \mathrm{R}^{\delta k}_{\partial D}(z,y)\right) \mathrm{d}\sigma_y = f^{\delta k}(z).$$

Using Lemma 2.2, we can rearrange the last equation and obtain

$$\begin{aligned}\int_\Lambda \partial_\nu u^{\delta k}(y) \Bigg(&2\Gamma^0_+(z,y) + \frac{1}{\pi}\log|z-y| + \frac{1/|D|}{\delta^2 k^2} \\ &+ \mathrm{R}^{\delta \mathbf{k}}_{\partial\Omega^1,+}(z,y) + \mathrm{R}^{\delta k}_{\partial D}(z,y) + \sum_{n=1}^{\infty} \delta^n \, 2\Gamma^k_{+,n}(z,y)\Bigg) \mathrm{d}\sigma_y = f^{\delta k}(z). \quad (3.19)\end{aligned}$$

Using that Λ is a line segment parallel to the x_1-axis, we have that $\mathrm{d}\sigma_y = \mathrm{d}t$, by writing $y = (t,h)^{\mathrm{T}}$. Using (2.8) for $\Gamma^0_+(z,y)$ and using that the expansion in δ (see Lemma 2.2) is uniform, we can interchange the infinite sum and the integration. Let $\mu(t) := \partial_\nu u^{\delta k}(y)$, we have that

$$\begin{aligned}f^{\delta k}(z) = \int_{-\varepsilon}^{\varepsilon} \mu(t) \Bigg[&\frac{1}{\pi}\log|\tau - t| + \frac{1}{2\pi}\log\left(\sin\left(\frac{\pi}{p}(\tau-t)\right)^2\right) + \mathrm{R}^{\delta k}_{\partial\Omega^1,+}(z,y) + \mathrm{R}^{\delta k}_{\partial D}(z,y) \\ &- \frac{1}{2\pi}\log\left(\sinh\left(\frac{\pi}{p}2h\right)^2 + \sin\left(\frac{\pi}{p}(\tau-t)\right)^2\right)\Bigg] \mathrm{d}t + \frac{\mathcal{K}^\varepsilon[\mu](\tau)}{\delta^2 k^2} + \sum_{n=1}^{\infty} \delta^n \, 2\widehat{\Gamma}^{k,\varepsilon}_{+,n}[\mu](\tau). \quad (3.20)\end{aligned}$$

Now consider that for $2\varepsilon < p$, we have

$$\begin{aligned}\frac{1}{2\pi}\log\left(\sin\left(\frac{\pi}{p}(\tau-t)\right)^2\right) &= \frac{1}{\pi}\log\left(\sin\left|\frac{\pi}{p}(\tau-t)\right|\right) \\ &= \frac{1}{\pi}\log\left(\frac{\pi}{p}\right) + \frac{1}{\pi}\log|\tau-t| + \frac{1}{\pi}\log\left(\operatorname{sinc}\left|\frac{\pi}{p}(\tau-t)\right|\right). \quad (3.21)\end{aligned}$$

Inserting the last equation into (3.20), we obtain that

$$\frac{2}{\pi}\mathcal{L}^\varepsilon[\mu](\tau) + \mathcal{R}^{\delta k,\varepsilon}[\mu](\tau) + \frac{\mathcal{K}^\varepsilon[\mu](\tau)}{\delta^2 k^2} + \sum_{n=1}^{\infty} \delta^n \, \widehat{\Gamma}^{k,\varepsilon}_{+,n}[\mu](\tau) = f^{\delta k}(z).$$

With that we have proven Proposition 3.10. ■

Let us show that $\mathcal{L}^\varepsilon$ is bijective and let us consider its inverse.

Proposition 3.11 *Let $0 < \varepsilon < 2$. The operator $\mathcal{L}^\varepsilon : \mathcal{X}^\varepsilon \to \mathcal{Y}^\varepsilon$ is linear, bounded and invertible and has the inverse*

$$(\mathcal{L}^\varepsilon)^{-1}[\eta](t) = -\frac{1}{\pi^2\sqrt{\varepsilon^2 - t^2}} \fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2}\,\eta'(\tau)}{t-\tau} \mathrm{d}\tau + \frac{C_{\mathcal{L}}[\eta]}{\pi \log(\varepsilon/2)\sqrt{\varepsilon^2 - t^2}}$$

where

$$C_{\mathcal{L}}[\eta] \coloneqq \eta(\tau) - \mathcal{L}^\varepsilon\left[-\frac{1}{\pi^2\sqrt{\varepsilon^2 - t^2}} \fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - s^2}\,\eta'(s)}{t-s} \mathrm{d}s \right] \tag{3.22}$$

is a constant depending on η and it is linear in η.

Proof The proof for invertiblity is given in [26, Chapter 11.5], the exact formula is derived in [8, Chapter 5.2.3]. ■

Lemma 3.12 *We have that*

$$\mathcal{L}^\varepsilon\left[t \mapsto \frac{1}{\sqrt{\varepsilon^2 - t^2}} \right](\tau) = \pi \log(\varepsilon/2), \tag{3.23}$$

$$\mathcal{L}^\varepsilon[1](\tau) = -2\varepsilon + (\varepsilon+\tau)\log(\varepsilon+\tau) + (\varepsilon-\tau)\log(\varepsilon-\tau)\,, \tag{3.24}$$

$$\mathcal{L}^\varepsilon\left[\frac{t}{\sqrt{\varepsilon^2 - t^2}} \right](\tau) = -\pi\tau\,, \tag{3.25}$$

$$(\mathcal{L}^\varepsilon)^{-1}[1](t) = \frac{1}{\pi \log(\varepsilon/2)\sqrt{\varepsilon^2 - t^2}}\,. \tag{3.26}$$

Proof Equations (3.23)–(3.25) follow from straightforward calculations and (3.26) follows using Proposition 3.11. ■

From Lemma 3.12, we also readily compute the following lemma:

Lemma 3.13 *We have that*

$$\left\| (\mathcal{L}^\varepsilon)^{-1}[1] \right\|_{\mathcal{X}^\varepsilon} = \frac{1}{\sqrt{\pi}\,|\log(\varepsilon/2)|}\,,$$
$$\left\| (\mathcal{L}^\varepsilon)^{-1}[\tau] \right\|_{\mathcal{X}^\varepsilon} = \frac{\varepsilon}{\sqrt{2\pi}}\,.$$

Since $\mathrm{R}_{\partial D}^{\delta k}$ and $\mathrm{R}_{\partial\Omega^1,+}^{\delta \mathbf{k}}$ are continuous, $\mathcal{R}^{\delta k,\varepsilon}$ is a compact operator. Thus we have that $\frac{2}{\pi}\mathcal{L}^\varepsilon + \mathcal{R}^{\delta k,\varepsilon}$ is a Fredholm operator of index zero. Hence, for the operator $\mathcal{A}^{\delta k,\varepsilon}$, extending the domain $\mathbb{K}$ to the complex numbers in a disk-shaped form, we will see that $\mathcal{A}^{\delta k,\varepsilon}$ is invertible except for a finite amount of values of δk. Some of those values are the resonances of our physical system. To that end, we will need the following result.

Lemma 3.14 *Let $\mathcal{R}$ be the integral operator defined from $\mathcal{X}^\varepsilon$ into $\mathcal{Y}^\varepsilon$ by*

$$\mathcal{R}[\mu](\tau) \;=\; \int_{-\varepsilon}^{\varepsilon} R(\tau,t)\,\mu(t)\mathrm{d}t\,,$$

where $R(\tau,t)$ is of class $\mathcal{C}^{1,\eta}$ in τ and t, for $0<\eta<1$. There exists a positive constant $C_{(3.27)}$, independent of ε, such that

$$\left\|(\mathcal{L}^\varepsilon)^{-1}\mathcal{R}\right\|_{\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)} \leq \frac{C_{(3.27)}}{|\log(\varepsilon)|}, \tag{3.27}$$

where $\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)$ denotes the space of bounded linear operators from $\mathcal{X}^\varepsilon$ to $\mathcal{X}^\varepsilon$.

Proof The proof is given in [8, Lemma 5.4] ■

3.3.3 Characteristic Values of $\mathcal{A}^{\delta k,\varepsilon}$ and the two Resonance Values

Let us first look at the characteristic values of

$$\mathcal{Q}^{\delta k,\varepsilon}[\mu] := \frac{2}{\pi}\mathcal{L}^\varepsilon[\mu] + \frac{\mathcal{K}^\varepsilon[\mu]}{\delta^2 k^2},$$

where $\mu\in\mathcal{X}^\varepsilon$ and $\delta k\in\mathbb{K}_{\mathbb{C}} := \{k_*\in\mathbb{C}\,|\,\sqrt{k_*\bar{k}_*}<k_{D,\min,\triangle}/2\}$. Let $(\cdot\,,\cdot)_\varepsilon$ be defined as the $\mathrm{L}^2((-\varepsilon,\varepsilon))$ inner product.

Lemma 3.15 *$\mathcal{Q}^{\delta k,\varepsilon}$ has exactly the two characteristic values $\pm k_0^{\delta,\varepsilon}$ where both have the characteristic function $\mu_0^{\delta,\varepsilon}$, with*

$$\begin{aligned} k_0^{\delta,\varepsilon} =& \frac{1}{\delta}\left(-\frac{\pi}{2|D|}\left((\mathcal{L}^\varepsilon)^{-1}[1]\,,1\right)_\varepsilon\right)^{1/2} = \frac{1}{\delta}\left(-\frac{\pi}{2|D|\log(\varepsilon/2)}\right)^{1/2},\\ \mu_0^\varepsilon =& -\frac{\pi}{2|D|}\frac{(\mathcal{L}^\varepsilon)^{-1}[1]}{(\delta k_0^{\delta,\varepsilon})^2}\,, \end{aligned}$$

after imposing $(\mu,1)_\varepsilon=1$.

Consider that $k_0^{\delta,\varepsilon}$ is real and positive.

Proof We are looking for $\mu\in\mathcal{X}^\varepsilon$ such that

$$\mathcal{Q}^{\delta k,\varepsilon}[\mu]=0\,.$$

Since $(\mathcal{L}^\varepsilon)^{-1}[0]=0$, the last equation is equivalent to

$$\frac{2}{\pi}\mu + \frac{1}{|D|}\frac{(\mu\,,1)_\varepsilon\,(\mathcal{L}^\varepsilon)^{-1}[1]}{(\delta k)^2}=0\,. \tag{3.28}$$

Applying $(\cdot\,,1)_\varepsilon$ on both sides yields

$$(\mu\,,1)_\varepsilon\left(\frac{2}{\pi}+\frac{1}{|D|}\frac{((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon}{(\delta k)^2}\right)=0\,. \tag{3.29}$$

If $(\mu\,,1)_\varepsilon=0$, then $\mathcal{L}^\varepsilon[\mu]=0$, because of the condition $\mathcal{Q}^{\delta k,\varepsilon}[\mu]=0$, and then $\mu=0$, since $\mathcal{L}^\varepsilon$ is invertible and linear. But 0 cannot be a characteristic function, by definition. Hence $(\mu\,,1)\neq 0$. Thus the second factor in (3.29) has to be zero. This leads us to

$$\delta^2k^2=-\frac{\pi}{2|D|}\Big((\mathcal{L}^\varepsilon)^{-1}[1]\,,1\Big)_\varepsilon\,.$$

Using Lemma 3.12, we can calculate that $((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon=\frac{1}{\log(\varepsilon/2)}$, and obtain the characteristic values.

As for the characteristic functions, we rewrite (3.28) as

$$\frac{\mu}{(\mu,1)}=-\frac{\pi}{2|D|}\frac{(\mathcal{L}^\varepsilon)^{-1}[1]}{(\delta k)^2}\,.$$

Imposing the normalization on μ we have proven our statement. ■

To facilitate future expressions we define

$$c_\varepsilon\coloneqq-\frac{\pi}{2|D|\log(\varepsilon/2)}=-\frac{\pi}{2|D|}((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon\,.$$

Next we will look at the characteristic values of $\mathcal{A}^{\delta k,\varepsilon}$. Denote $\widetilde{\mathcal{L}}^{\delta k,\varepsilon}\coloneqq\mathcal{L}^\varepsilon+\frac{\pi}{2}\mathcal{R}^{\delta k,\varepsilon}$ and $\mathcal{S}^{\delta k,\varepsilon}\coloneqq\frac{\pi}{2}(\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}$, where we fixed the angles of the incoming wave vector, but let the magnitude be complex. Using that $\mathcal{R}^{\delta k,\varepsilon}$ is in $\mathcal{C}^{1,\eta}$, for $\eta\in(0,1)$, and $\mathcal{L}^\varepsilon$ invertible and using Lemma 3.14, we can apply the Neumann series, whenever ε is small enough, and thus we have

$$(\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}=\big(\mathcal{I}+\mathcal{S}^{\delta k,\varepsilon}\big)^{-1}(\mathcal{L}^\varepsilon)^{-1}=\sum_{l=0}^{\infty}(-\mathcal{S}^{\delta k,\varepsilon})^l(\mathcal{L}^\varepsilon)^{-1}, \tag{3.30}$$

where $\mathcal{I}$ denotes the identity function in $\mathcal{X}^\varepsilon$. We then define

$$A^{\delta k,\varepsilon}\coloneqq-\frac{\pi}{2|D|}\left((\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[1]\,,1\right)_\varepsilon\,. \tag{3.31}$$

Lemma 3.16 *The characteristic values of $\mathcal{A}^{\delta k,\varepsilon}$ are zeros of the function $\delta^2k^2-A^{\delta k,\varepsilon}$ and we have the asymptotic formula*

$$\delta^2k^2=(\delta k_0^{\delta,\varepsilon})^2+\delta^2\mathcal{O}(c_\varepsilon^2),\qquad\qquad \text{for } \varepsilon\to 0\,.$$

Proof We prove that the zeros of $\delta^2k^2-A^{\delta k,\varepsilon}$ are the characteristic values in the same way as in the proof of Lemma 3.15, but substitute $\mathcal{L}^\varepsilon$ with $\widetilde{\mathcal{L}}^{\delta k,\varepsilon}$. In order to derive the asymptotic formulas, we use (3.30) and Lemma 3.14. ■

Proposition 3.17 *There exist two characteristic values, counting multiplicity, for the operator $\mathcal{A}^{\delta k,\varepsilon}$ in $\mathbb{K}_{\mathbb{C}}$. Moreover, they have the asymptotic expansions*

$$\delta k = \pm \delta k_0^{\delta,\varepsilon} + \delta \mathcal{O}(c_\varepsilon), \qquad \text{for } \varepsilon \to 0\,.$$

Proof Recall that the operator-valued analytic function $\mathcal{Q}^{\delta k,\varepsilon}$ is finitely meromorphic and of Fredholm type. Moreover, it has two characteristic values $\pm k_0^{\delta k,\varepsilon}$, and has a pole at 0 with order two in $\mathbb{K}_{\mathbb{C}}$. Thus, the multiplicity of $\mathcal{Q}^{\delta k,\varepsilon}$ is zero in $\mathbb{K}_{\mathbb{C}}$. Note that for $\delta k \in \mathbb{K}_{\mathbb{C}} \setminus \{0, \pm \delta k_0^{\delta,\varepsilon}\}$, the operator $\mathcal{Q}^{\delta k,\varepsilon}$ is invertible, because it is of Fredholm type and because it is injective due to Lemma 3.15. With that,

$$(\mathcal{Q}^{\delta k,\varepsilon})^{-1}\mathcal{R}^{\delta k,\varepsilon} = \frac{2}{\pi}\left(\frac{2}{\pi}\mathcal{I} - \frac{(\mathcal{L}^\varepsilon)^{-1}\mathcal{K}^\varepsilon}{\delta^2 k^2}\right)^{-1}\mathcal{S}^{\delta k,\varepsilon}\,.$$

Thus, $\left\|(\mathcal{Q}^{\delta k,\varepsilon})^{-1}\mathcal{R}^{\delta k,\varepsilon}\right\|_{\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)} = \mathcal{O}(c_\varepsilon)$ uniformly for $\delta k \in \partial\mathbb{K}_{\mathbb{C}}$. By the generalized Rouché's theorem [8, Theorem 1.15], we can conclude that for ε sufficiently small, the operator $\mathcal{A}^{\delta k,\varepsilon}$ has the same multiplicity as the operator $\mathcal{Q}^{\delta k,\varepsilon}$ in $\mathbb{K}_{\mathbb{C}}$, which is zero. Since $\mathcal{A}^{\delta k,\varepsilon}$ has a pole of order two, we derive that $\mathcal{A}^{\delta k,\varepsilon}$ has either one characteristic value of order two or two characteristic values of order one. This completes the proof of the proposition. ■

Now, let us give an asymptotic expression for those resonances. We recall that α_0 and α_1, defined in (3.3) and (3.4), are used in the decomposition of $\mathcal{R}^{\delta k,\varepsilon}$, that is, $\mathcal{R}^{\delta k,\varepsilon} = \mathcal{R}_1^{\delta k,\varepsilon} + \mathcal{R}_2^{\delta k,\varepsilon} + \mathcal{R}_3^{\delta k,\varepsilon}$, with

$$\mathcal{R}_1^{\delta k,\varepsilon}[\mu] := \alpha_0(\,\mu, 1)_\varepsilon\,, \tag{3.32}$$

$$\mathcal{R}_2^{\delta k,\varepsilon}[\mu] := \delta k\, \alpha_1\,(\mu, 1)_\varepsilon\,, \tag{3.33}$$

$$\mathcal{R}_3^{\delta k,\varepsilon}[\mu](\tau) := \int_{-\varepsilon}^{\varepsilon} \mu(t)\left(\tau\,\widetilde{\partial}_\tau R^{\delta k}(\tau,t) + t\,\widetilde{\partial}_t R^{\delta k}(\tau,t) + \delta^2 k^2\,\widetilde{\partial_{\delta k}^2} R^{\delta k}(\tau,t)\right)\mathrm{d}t\,, \tag{3.34}$$

with $R^{\delta k}(\tau,t)$ denotes the kernel of $\mathcal{R}^{\delta k,\varepsilon}$, see Definition 3.9, and $\widetilde{\partial}$ denotes the derivative part of the remainder in Taylor's theorem in the Peano form.

Lemma 3.18 *For all $\delta k \in \mathbb{K}_{\mathbb{C}}$, $\mathcal{R}_3^{\delta k,\varepsilon}$ satisfies the estimate*

$$\left\|(\mathcal{L}^\varepsilon)^{-1}\mathcal{R}_3^{\delta k,\varepsilon}\right\|_{\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)} = \mathcal{O}(\varepsilon\cdot c_\varepsilon + c_\varepsilon\cdot\delta^2 k^2)\,. \tag{3.35}$$

Proof Using that $(\mathcal{L}^\varepsilon)^{-1}$ is linear we divide the proof according to the three terms in (3.34). The proof for the term with $\delta^2 k^2$ in it, can be seen immediately using Lemma 3.14.

For the $\int_{-\varepsilon}^{\varepsilon} \mu(t) t\, \partial_t R^{\delta k}(\tau, t) \mathrm{d}t =: \eta(\tau)$ term we have

$$\begin{aligned}\left\| (\mathcal{L}^\varepsilon)^{-1} \eta \right\|_{\mathcal{X}^\varepsilon} &= \left\| -\frac{1}{\pi^2 \sqrt{\varepsilon^2 - t^2}} \, \mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2}\, \eta'(\tau)}{t - \tau} \mathrm{d}\tau + \frac{C_{\mathcal{L}}[\eta]}{\pi \log(\varepsilon/2) \sqrt{\varepsilon^2 - t^2}} \right\|_{\mathcal{X}^\varepsilon} \\ &\leq \left\| \mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2}\, \eta'(\tau)}{\pi^2 \sqrt{\varepsilon^2 - t^2}(t - \tau)} \mathrm{d}\tau \right\|_{\mathcal{X}^\varepsilon} + \frac{|C_{\mathcal{L}}[\eta]|}{\log(\varepsilon/2)} \,. \end{aligned} \tag{3.36}$$

Let us consider the left term first. We split $\eta'(\tau) = \eta'(t) + (\eta'(\tau) - \eta'(t))$ and thus can split the principal value integral into an principal value integral with $\eta'(t)$ and into a normal integral with $\eta'(\tau) - \eta'(t)$. Using $\mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - s^2}}{t - s} \mathrm{d}s = \pi t$ and that

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2}}{\sqrt{|t - \tau|}} \mathrm{d}\tau \leq C_{(3.37)} \varepsilon \sqrt{\varepsilon},$$

where $C_{(3.37)}$ is independent of ε and t, see also [8, Proof of Lemma 5.4], we can estimate the left term in Inequality (3.37) to be smaller or equal to

$$\pi |t| \, \left\| \eta'(\tau) \right\|_{\mathcal{C}^0([-\varepsilon,\varepsilon])} + C_{(3.37)} \varepsilon \sqrt{\varepsilon} \left| \eta'(\tau) \right|_{\mathcal{C}^{0,1/2}([-\varepsilon,\varepsilon])} \,. \tag{3.37}$$

As for the right term in Inequality (3.36), consider that

$$|C_{\mathcal{L}}[\eta]| \leq |\eta(0)| + \mathcal{L}^\varepsilon \left[\frac{1/\pi^2}{\sqrt{\varepsilon^2 - t^2}} \, \mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tilde{\tau}^2}\, \eta'(\tilde{\tau})}{t - \tilde{\tau}} \mathrm{d}\tilde{\tau} \right] (0) \,.$$

Similarly to the argumentation above and using $|\eta(0)| \leq \varepsilon$, we can infer that

$$\begin{aligned} |C_{\mathcal{L}}[\eta]| &\leq \varepsilon + \int_{-\varepsilon}^{\varepsilon} \frac{|\log(|t|)|/\pi^2}{\sqrt{\varepsilon^2 - t^2}} \left(\pi |t| \, \left\| \eta'(\tau) \right\|_{\mathcal{C}^0} + C_{(3.37)} \varepsilon \sqrt{\varepsilon} \left| \eta'(\tau) \right|_{\mathcal{C}^{0,1/2}} \right) \mathrm{d}t \\ &= \varepsilon + \frac{2\varepsilon}{\pi} (-\log(2\varepsilon) + 1) \left\| \eta'(\tau) \right\|_{\mathcal{C}^0} + \frac{|\log(2\varepsilon)|}{\pi} C_{(3.37)} \varepsilon \sqrt{\varepsilon} \left| \eta'(\tau) \right|_{\mathcal{C}^{0,1/2}} \,, \end{aligned}$$

thus we have shown with the term in (3.36) the desired estimation for $\int_{-\varepsilon}^{\varepsilon} \mu(t) t\, \partial_t R^{\delta k}(\tau, t) \mathrm{d}t$.

For the term $\int_{-\varepsilon}^{\varepsilon} \mu(t) \tau\, \partial_t R^{\delta k}(\tau, t) \mathrm{d}t$, we can use the same argumentation, where this time $|\eta(0)| = 0$. This concludes the proof. ■

Let $l \geq 1$ be an integer, we define

$$S_{(l)}^{\delta k, \varepsilon} := -\frac{\pi}{2|D|} \left((-\mathcal{S}^{\delta k, \varepsilon})^l (\mathcal{L}^\varepsilon)^{-1}[1] \,, 1 \right)_\varepsilon \,.$$

Because of (3.31), we can write

$$A^{\delta k,\varepsilon} = c_\varepsilon + \sum_{l=1}^{\infty} S_{(l)}^{\delta k,\varepsilon}\,.$$

We want to give a second order analytic expression for $S_{(1)}^{\delta k,\varepsilon} = \frac{\pi}{2|D|}\left(\frac{\pi}{2}(\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}(\mathcal{L}^\varepsilon)^{-1}[1]\,,1\right)_\varepsilon$. To this end, we define

$$S_{(1,1)}^{\delta k,\varepsilon} = \frac{\pi}{2|D|}\left(\frac{\pi}{2}(\mathcal{L}^\varepsilon)^{-1}\mathcal{R}_1^{\delta k,\varepsilon}(\mathcal{L}^\varepsilon)^{-1}[1]\,,1\right)_\varepsilon\,, \tag{3.38}$$

$$S_{(1,2)}^{\delta k,\varepsilon} = \frac{\pi}{2|D|}\left(\frac{\pi}{2}(\mathcal{L}^\varepsilon)^{-1}\mathcal{R}_2^{\delta k,\varepsilon}(\mathcal{L}^\varepsilon)^{-1}[1]\,,1\right)_\varepsilon\,, \tag{3.39}$$

$$S_{(1,3)}^{\delta k,\varepsilon} = \frac{\pi}{2|D|}\left(\frac{\pi}{2}(\mathcal{L}^\varepsilon)^{-1}\mathcal{R}_3^{\delta k,\varepsilon}(\mathcal{L}^\varepsilon)^{-1}[1]\,,1\right)_\varepsilon\,. \tag{3.40}$$

We obtain that second order analytic expression with the following lemma:

Lemma 3.19 *For all $l \in \mathbb{N}$, we have that*

$$\begin{aligned}
|S_{(l)}^{\delta k,\varepsilon}| =& \mathcal{O}(c_\varepsilon^{l+1})\,,\\
S_{(1,1)}^{\delta k,\varepsilon} =& \alpha_0 c_\varepsilon^2|D|\,,\\
S_{(1,2)}^{\delta k,\varepsilon} =& \alpha_1\,\delta k\, c_\varepsilon^2|D|\,,\\
S_{(1,3)}^{\delta k,\varepsilon} =& \mathcal{O}(\varepsilon\, c_\varepsilon^2 + c_\varepsilon^2\,\delta^2 k^2)\,.
\end{aligned}$$

Proof The proof follows from straightforward calculation using Lemma 3.14 and the expressions in (3.35), (3.32)–(3.34) and (3.38)–(3.40). ■

Now we can deduce that

$$A^{\delta k,\varepsilon} = c_\varepsilon + \alpha_0\, c_\varepsilon^2|D| + \alpha_1\,\delta k\, c_\varepsilon^2|D| + \mathcal{O}(\varepsilon\, c_\varepsilon^2 + c_\varepsilon^2\,\delta^2 k^2)\,.$$

Now we are able to solve for the zeros in $\delta^2 k^2 - A^{\delta k,\varepsilon}$, and thus get the characteristic values of $\mathcal{A}^{\delta k,\varepsilon}$. To this end, consider that $\sqrt{1+x} = 1 + \frac{1}{2}x - \mathcal{O}(x^2)$ for $|x| < 1$, thus

$$A_\star^{\delta k,\varepsilon} := \sqrt{A^{\delta k,\varepsilon}} = \sqrt{c_\varepsilon}\left(1 + c_\varepsilon\frac{\alpha_0}{2}|D| + \delta k\, c_\varepsilon\frac{\alpha_1}{2}|D| + \mathcal{O}(\varepsilon\, c_\varepsilon + c_\varepsilon\,\delta^2 k^2)\right)\,, \tag{3.41}$$

which leads us to the factorization,

$$\delta^2 k^2 - A^{\delta k,\varepsilon} = (\delta k - A_\star^{\delta k,\varepsilon})\cdot(\delta k + A_\star^{\delta k,\varepsilon}).$$

Thus, the roots for $\delta^2 k^2 - A^{\delta k,\varepsilon}$ are those for $\delta k - A_\star^{\delta k,\varepsilon}$ and $k + A_\star^{\delta k,\varepsilon}$.

Proposition 3.20 *There are exactly two characteristic values for the operator $\mathcal{A}^{\delta k,\varepsilon}$ in $\mathbb{K}_C$. Those are approximated by*

$$\delta k_+^{\delta,\varepsilon} = \delta k_0^{\delta,\varepsilon} + \frac{\alpha_0 |D|}{2} c_\varepsilon^{3/2} + \frac{\alpha_1 |D|}{2} c_\varepsilon^2 + \mathcal{O}(c_\varepsilon^{5/2}),$$
$$\delta k_-^{\delta,\varepsilon} = -\delta k_0^{\delta,\varepsilon} - \frac{\alpha_0 |D|}{2} c_\varepsilon^{3/2} + \frac{\alpha_1 |D|}{2} c_\varepsilon^2 + \mathcal{O}(c_\varepsilon^{5/2}),$$
$$\delta k_0^{\delta,\varepsilon} := c_\varepsilon^{1/2} := \sqrt{\frac{-\pi}{2|D| \log(\varepsilon/2)}}.$$

Proof We define

$$A_{\star,(0)}^{\delta k,\varepsilon} := \sqrt{c_\varepsilon} \left(1 + c_\varepsilon \frac{\alpha_0}{2} |D| + \delta k \, c_\varepsilon \frac{\alpha_1}{2} |D|\right).$$

For ε small enough, the zeros of $\delta k - A_{\star,(0)}^{\delta k,\varepsilon}$ are exactly the zeros of $\delta k - A_\star^{\delta k,\varepsilon}$, up to a term in $\mathcal{O}(c_\varepsilon^{5/2})$. This follows readily from Rouché's Theorem.
As for the zeros of $\delta k - A_{\star,(0)}^{\delta k,\varepsilon}$, we obtain them through the approach

$$\delta k = \gamma_1 c_\varepsilon^{1/2} + \gamma_2 c_\varepsilon^1 + \gamma_3 c_\varepsilon^{3/2} + \gamma_4 c_\varepsilon^2.$$

Inserting the approach into $A_{\star,(0)}^{\delta k,\varepsilon}$ and solving $\delta k - A_{\star,(0)}^{\delta k,\varepsilon} = 0$, we obtain

$$\gamma_1 = 1, \quad \gamma_2 = 0, \quad \gamma_3 = \frac{\alpha_0 |D|}{2}, \quad \gamma_4 = \frac{\alpha_1 |D|}{2}.$$

An analogous argumentation leads to the zeros of $\delta k + A_{\star,(0)}^{\delta k,\varepsilon}$. ■

With Proposition 3.20 we immediately obtain Theorem 3.1.

3.3.4 Inversion of $\mathcal{A}^{\delta k,\varepsilon}$ - Solving the First Order Linear Equation

We know now that $\mathcal{A}^{\delta k,\varepsilon}$ is invertible for $\delta k \in \mathbb{K}_C$, except at the characteristic values $k = k_+^{\delta,\varepsilon}$ and $k = k_-^{\delta,\varepsilon}$ and at the pole, $k = 0$. Let us examine, how we can express $(\mathcal{A}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]$, where $f^{\delta k}$ is given by (3.17).

First consider that we already know that the equation $\mathcal{A}^{\delta k,\varepsilon}[\mu] = f^{\delta k}$ is equivalent to

$$\frac{2}{\pi}\mu + \frac{(\mu, 1)_\varepsilon (\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[1]}{|D| \, \delta^2 k^2} = (\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]. \tag{3.42}$$

Applying $(\cdot, 1)_\varepsilon$ on both sides, we obtain

$$(\mu, 1)_\varepsilon \left(1 - \frac{A^{\delta k,\varepsilon}}{\delta^2 k^2}\right) = \frac{\pi}{2} ((\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}], 1)_\varepsilon.$$

And thus

$$(\mu, 1)_\varepsilon = \frac{\delta^2 k^2}{\delta^2 k^2 - A^{\delta k,\varepsilon}} \frac{\pi}{2} ((\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}], 1)_\varepsilon. \tag{3.43}$$

Lemma 3.21 *For $\delta k \in \mathbb{K}_{\mathbb{C}}$ and $\varepsilon \to 0$ we have that*

$$\frac{1}{\delta^2 k^2 - A^{\delta k,\varepsilon}} = \frac{1}{(\delta k - \delta k_+^{\delta,\varepsilon})(\delta k - \delta k_-^{\delta,\varepsilon})} \left(1 + \mathcal{O}(c_\varepsilon)\right) .$$

Proof Recall that $\delta^2 k^2 - A^{\delta k,\varepsilon} = (\delta k - A_\star^{\delta k,\varepsilon})(\delta k + A_\star^{\delta k,\varepsilon})$. Let us first investigate the function $\delta k - A_\star^{\delta k,\varepsilon}$. We already established that it has the unique root $\delta k_+^{\delta,\varepsilon}$ in the set $\mathbb{K}_{\mathbb{C}}$, for ε small enough. Thus $\delta k - A_\star^{\delta k,\varepsilon}$ can be written as

$$\delta k - A_\star^{\delta k,\varepsilon} = (\delta k - \delta k_+^{\delta,\varepsilon})(1 + g^\varepsilon(\delta k)) , \tag{3.44}$$

where g^ε is an analytic function in $\mathbb{K}_{\mathbb{C}}$, for ε small enough. Considering the definition of $A_\star^{\delta k,\varepsilon}$, (3.41), and of $\delta k_+^{\delta,\varepsilon}$, Proposition 3.20, we can conclude that g^ε is smooth with respect to $\sqrt{c_\varepsilon}$. By the Taylor expansion, we can write g^ε in the form

$$g^\varepsilon(\delta k) = g_0(\delta k) + g_1(\delta k)\sqrt{c_\varepsilon} + g_2(\delta k, \sqrt{c_\varepsilon})c_\varepsilon ,$$

where g_0 and g_1 are analytic in δk and the function g_2 is analytic in the first variable and is smooth in the second one. By comparing coefficients of different orders of $\sqrt{c_\varepsilon}$ on both sides of (3.44), we can deduce that $g_0(\delta k) = g_1(\delta k) = 0$. Hence, we can conclude that $g^\varepsilon(\delta k) = g_2(\delta k, \sqrt{c_\varepsilon})c_\varepsilon = \mathcal{O}(c_\varepsilon)$. Similarly, we can prove that

$$\delta k + A_\star^{\delta k,\varepsilon} = (\delta k - \delta k_-^{\delta,\varepsilon})(1 + \mathcal{O}(c_\varepsilon)) .$$

Thus, we can conclude that

$$\begin{aligned}
\frac{1}{\delta^2 k^2 - A^{\delta k,\varepsilon}} =& \frac{1}{\delta k - A_\star^{\delta k,\varepsilon}} \frac{1}{\delta k + A_\star^{\delta k,\varepsilon}} \\
=& \frac{1}{(\delta k - \delta k_+^{\delta,\varepsilon})(\delta k - \delta k_-^{\delta,\varepsilon})} \frac{1}{(1 + \mathcal{O}(c_\varepsilon))(1 + \mathcal{O}(c_\varepsilon))} \\
=& \frac{1}{(\delta k - \delta k_+^{\delta,\varepsilon})(\delta k - \delta k_-^{\delta,\varepsilon})} (1 + \mathcal{O}(c_\varepsilon)) ,
\end{aligned}$$

which completes the proof of the lemma. ■

Lemma 3.22 *We have*

$$\begin{aligned}
(\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}](t) =& f^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1] + \mathcal{O}(\delta)\mathcal{O}_{\mathcal{X}^\varepsilon}(c_\varepsilon^2) + \mathcal{O}(\delta)\mathcal{O}_{\mathcal{X}^\varepsilon}(\varepsilon) , \\
(\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[1](t) =& (\mathcal{L}^\varepsilon)^{-1}[1] + \mathcal{O}_{\mathcal{X}^\varepsilon}(c_\varepsilon^2) ,
\end{aligned}$$

for $\varepsilon \to 0$ in the $\mathcal{X}^\varepsilon$ norm and $\delta \to 0$.

Proof We write $f^{\delta k}(\tau) = f^{\delta k}(0) + \tau\, g^{\delta k}(\tau)$ with a smooth function $g^{\delta k} \in \mathcal{C}^{1,1/2}([-\varepsilon, \varepsilon])$. We readily see that $f^{\delta k} = \mathcal{O}(\delta)$ and $g^{\delta k} = \mathcal{O}(\delta)$. Recall that

$$(\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1} = \sum_{l=0}^{\infty} (-\mathcal{S}^{\delta k,\varepsilon})^l (\mathcal{L}^\varepsilon)^{-1} .$$

Using that $\left\|\mathcal{S}^{\delta k,\varepsilon}\right\|_{\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)} = \mathcal{O}(c_\varepsilon)$ and according to Lemma 3.13, $\left\|(\mathcal{L}^\varepsilon)^{-1}[1]\right\|_{\mathcal{X}^\varepsilon} = \mathcal{O}(c_\varepsilon)$, we have for $n \in \mathbb{N}$ that

$$\begin{aligned}\left\|(\mathcal{S}^{\delta k,\varepsilon})^n(\mathcal{L}^\varepsilon)^{-1}[1]\right\|_{\mathcal{X}^\varepsilon} &= \mathcal{O}(c_\varepsilon^{n+1})\,,\\ \left\|(\mathcal{S}^{\delta k,\varepsilon})^n(\mathcal{L}^\varepsilon)^{-1}[\tau\, g^{\delta k}(\tau)]\right\|_{\mathcal{X}^\varepsilon} &= \mathcal{O}(c_\varepsilon^n)\left\|(\mathcal{L}^\varepsilon)^{-1}[\tau\, g^{\delta k}(\tau)]\right\|_{\mathcal{X}^\varepsilon}\,.\end{aligned}$$

Consider that $\varepsilon = \mathcal{O}(c_\varepsilon^n)$, thus it is enough to show

$$\left\|(\mathcal{L}^\varepsilon)^{-1}[\tau\, g^{\delta k}(\tau)]\right\|_{\mathcal{X}^\varepsilon} = \mathcal{O}(\delta)\mathcal{O}(\varepsilon)\,.$$

So, let us show that. Using Proposition 3.11 we have

$$\left\|(\mathcal{L}^\varepsilon)^{-1}[\tau\, g^{\delta k}(\tau)]\right\|^2_{\mathcal{X}^\varepsilon} = \int_{-\varepsilon}^{\varepsilon}\sqrt{\varepsilon^2-t^2}\left|(\mathcal{L}^\varepsilon)^{-1}[\tau g^{\delta k}(\tau)]\right|^2\mathrm{d}t \tag{3.45}$$

$$= \int_{-\varepsilon}^{\varepsilon}\sqrt{\varepsilon^2-t^2}\left|\frac{-1}{\pi^2\sqrt{\varepsilon^2-t^2}}\,\mathrm{p\!\!\int\!\!\!\!v.}_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau(\tau g^{\delta k}(\tau))}{t-\tau}\mathrm{d}\tau + \frac{C_{\mathcal{L}}[\tau g^{\delta k}(\tau)]}{\pi\log(\varepsilon/2)\sqrt{\varepsilon^2-t^2}}\right|^2\mathrm{d}t\,. \tag{3.46}$$

Using $(a+b)^2 \le 2(a^2+b^2)$ and pulling the $\sqrt{\varepsilon^2-t^2}$ term inside, we obtain that the term in (3.46) is smaller or equal to

$$\int_{-\varepsilon}^{\varepsilon}\left(\frac{1/\pi^2}{\sqrt{\varepsilon^2-t^2}}\left|\mathrm{p\!\!\int\!\!\!\!v.}_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau(\tau g^{\delta k}(\tau))}{t-\tau}\mathrm{d}\tau\right|^2 + \frac{C_{\mathcal{L}}[\tau g^{\delta k}(\tau)]^2/\pi^2}{(\log(\varepsilon/2))^2\sqrt{\varepsilon^2-t^2}}\right)\mathrm{d}t\,. \tag{3.47}$$

Using $\int_{-\varepsilon}^{\varepsilon} 1/\sqrt{\varepsilon^2-s^2}\mathrm{d}s = \pi$ and $C_{\mathcal{L}}[\tau g^{\delta k}(\tau)]$ is a constant, the term is equal to

$$\int_{-\varepsilon}^{\varepsilon}\left(\frac{1/\pi^2}{\sqrt{\varepsilon^2-t^2}}\left|\mathrm{p\!\!\int\!\!\!\!v.}_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau(\tau g^{\delta k}(\tau))}{t-\tau}\mathrm{d}\tau\right|^2\right)\mathrm{d}t + \frac{C_{\mathcal{L}}[\tau g^{\delta k}(\tau)]^2}{(\log(\varepsilon/2))^2\pi}\,. \tag{3.48}$$

Let us estimate both terms in the sum.

Observe that

$$\begin{aligned}
&\left| \fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau(\tau g^{\delta k}(\tau))}{t-\tau}\mathrm{d}\tau \right| \\
&= \left| \fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_t(t g^{\delta k}(t))}{t-\tau}\mathrm{d}\tau + \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,(\partial_\tau(\tau g^{\delta k}(\tau)) - \partial_t(t g^{\delta k}(t)))}{t-\tau}\mathrm{d}\tau \right| \\
&\leq \pi|t|\,|\partial_t(t g^{\delta k}(t))| + \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}}{\sqrt{|t-\tau|}} \frac{|\partial_\tau(\tau g^{\delta k}(\tau)) - \partial_t(t g^{\delta k}(t))|}{\sqrt{|t-\tau|}}\mathrm{d}\tau \\
&\leq \pi|t|\,\left\|\partial_\tau(\tau g^{\delta k}(\tau))\right\|_{\mathcal{C}^0([-\varepsilon,\varepsilon])} + C_{(3.49)}\varepsilon\sqrt{\varepsilon}\left|\partial_\tau(\tau g^{\delta k}(\tau))\right|_{\mathcal{C}^{0,1/2}([-\varepsilon,\varepsilon])}\,,
\end{aligned} \tag{3.49}$$

where we used that $\fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-s^2}}{t-s}\mathrm{d}s = \pi t$, and that $\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}}{\sqrt{|t-\tau|}}\mathrm{d}\tau \leq C_{(3.49)}\varepsilon\sqrt{\varepsilon}$, with $C_{(3.49)}$ being independent of ε and t, see [8, Proof of Lemma5.4].

With that, we can estimate the left term in (3.48) to be smaller or equal to

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} \frac{1/\pi^2}{\sqrt{\varepsilon^2-t^2}} 2\left(\pi^2|t|^2\,\left\|\partial_\tau(\tau g^{\delta k}(\tau))\right\|^2_{\mathcal{C}^0([-\varepsilon,\varepsilon])} + C^2_{(3.49)}\varepsilon^3\left|\partial_\tau(\tau g^{\delta k}(\tau))\right|^2_{\mathcal{C}^{0,1/2}([-\varepsilon,\varepsilon])}\right)\mathrm{d}t \\
&\qquad \leq \varepsilon^2\pi\left\|\partial_\tau(\tau g^{\delta k}(\tau))\right\|^2_{\mathcal{C}^0([-\varepsilon,\varepsilon])} + \frac{2C^2_{(3.49)}}{\pi}\varepsilon^3\left|\partial_\tau(\tau g^{\delta k}(\tau))\right|^2_{\mathcal{C}^{0,1/2}([-\varepsilon,\varepsilon])}\,,
\end{aligned} \tag{3.50}$$

where we used that $\int_{-\varepsilon}^{\varepsilon} s^2/\sqrt{\varepsilon^2-s^2}\mathrm{d}s = \varepsilon^2\pi/2$.

Let us consider the right term in (3.48). To compute $C_{\mathcal{L}}[\tau g^{\delta k}(\tau)]$ we can pick any $\tau \in (-\varepsilon,\varepsilon)$. We pick $\tau = 0$. Thus,

$$C_{\mathcal{L}}[\tau g^{\delta k}(\tau)] = 0 + \mathcal{L}^\varepsilon\left[\frac{1/\pi^2}{\sqrt{\varepsilon^2-t^2}} \fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tilde{\tau}^2}\,\partial_{\tilde{\tau}}(\tilde{\tau} g^{\delta k}(\tilde{\tau}))}{t-\tilde{\tau}}\mathrm{d}\tilde{\tau}\right](0)\,. \tag{3.51}$$

With the observation in Inequality (3.49), we can infer that

$$\begin{aligned}
|C_{\mathcal{L}}[\tau g^{\delta k}(\tau)]| &\leq \int_{-\varepsilon}^{\varepsilon} \frac{|\log(|t|)|/\pi^2}{\sqrt{\varepsilon^2-t^2}}\left(\pi|t|\,\left\|\partial_\tau(\tau g^{\delta k}(\tau))\right\|_{\mathcal{C}^0} + C_{(3.49)}\varepsilon\sqrt{\varepsilon}\left|\partial_\tau(\tau g^{\delta k}(\tau))\right|_{\mathcal{C}^{0,1/2}}\right)\mathrm{d}t \\
&= \frac{2\varepsilon}{\pi}(-\log(2\varepsilon)+1)\left\|\partial_\tau(\tau g^{\delta k}(\tau))\right\|_{\mathcal{C}^0} + \frac{|\log(2\varepsilon)|}{\pi}C_{(3.49)}\varepsilon\sqrt{\varepsilon}\left|\partial_\tau(\tau g^{\delta k}(\tau))\right|_{\mathcal{C}^{0,1/2}}\,,
\end{aligned} \tag{3.52}$$

where we used that

$$\int_{-\varepsilon}^{\varepsilon} |\log(|t|)\,t|/\sqrt{\varepsilon^2-t^2} = 2\varepsilon(-\log(2\varepsilon)+1)$$

and where we assumed that ε is small enough, so that $|\log(|t|)| = -\log(|t|)$. Thus, we have

$$\frac{C_{\mathcal{L}}[\tau g^{\delta k}(\tau)]^2}{\log(\varepsilon/2)^2\pi} \leq \frac{8\varepsilon^2}{\pi^3}\frac{(1-\log(2\varepsilon))^2}{\log(\varepsilon/2)^2}\left\|\partial_\tau(\tau g^{\delta k}(\tau))\right\|_{\mathcal{C}^0}^2 + \frac{2C_{(3.49)}^2\varepsilon^3}{\pi^3}\frac{\log(2\varepsilon)^2}{\log(\varepsilon/2)^2}\left|\partial_\tau(\tau g^{\delta k}(\tau))\right|_{\mathcal{C}^{0,1/2}}^2. \tag{3.53}$$

Combining Inequality (3.53) and Inequality (3.50) and using that $\left\|\partial_\tau(\tau g^{\delta k}(\tau))\right\|_{\mathcal{C}^0} = \mathcal{O}(\delta)$ and $\left|\partial_\tau(\tau g^{\delta k}(\tau))\right|_{\mathcal{C}^{0,1/2}} = \mathcal{O}(\delta)$, we have shown that

$$\left\|(\mathcal{L}^\varepsilon)^{-1}[\tau\, g^{\delta k}(\tau)]\right\|_{\mathcal{X}^\varepsilon} = \mathcal{O}(\delta)\mathcal{O}(\varepsilon)$$

and proved Lemma 3.22. ■

Proposition 3.23 *Let $\delta k \in \mathbb{K}_{\mathbb{C}} \setminus \{0, \delta k_+^{\delta,\varepsilon}, \delta k_-^{\delta,\varepsilon}\}$, there exists a unique solution to the equation $\mathcal{A}^{\delta k,\varepsilon}[\mu] = f^{\delta k}$. Moreover, the solution can be written as $\mu = \mu_\star + \mu_\sim$, where*

$$\begin{aligned}
\mu_\star =& \frac{\pi\, c_\varepsilon}{2}\frac{f^{\delta k}(0)\,(\mathcal{L}^\varepsilon)^{-1}[1]}{\delta k_+^{\delta,\varepsilon} - \delta k_-^{\delta,\varepsilon}}\left(\frac{1}{\delta k - \delta k_+^{\delta,\varepsilon}} - \frac{1}{\delta k - \delta k_-^{\delta,\varepsilon}}\right) + \frac{\pi}{2}f^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1]\,,\\
\mu_\sim =& \mathcal{O}(\delta)\left(\mathcal{O}_{\mathcal{X}^\varepsilon}(c_\varepsilon^3)\left(\frac{1}{\delta k - \delta k_+^{\delta,\varepsilon}} - \frac{1}{\delta k - \delta k_-^{\delta,\varepsilon}}\right) + \mathcal{O}_{\mathcal{X}^\varepsilon}(c_\varepsilon^2) + \mathcal{O}_{\mathcal{X}^\varepsilon}(\varepsilon)\right).
\end{aligned}$$

Proof From $\mathcal{A}^{\delta k,\varepsilon}[\mu] = f^{\delta k}$ we get that

$$\frac{2}{\pi}\mu + \frac{(\mu\,,1)_\varepsilon(\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[1]}{|D|\,\delta^2k^2} = (\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]\,. \tag{3.54}$$

After rearranging, see (3.42), we derive

$$(\mu\,,1)_\varepsilon = \frac{\delta^2k^2}{\delta^2k^2 - A^{\delta k,\varepsilon}}\frac{\pi}{2}((\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]\,,1)_\varepsilon\,.$$

Then, we obtain by applying Lemma 3.21 that

$$(\mu\,,1)_\varepsilon = \frac{\pi}{2}((\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]\,,1)_\varepsilon\,\frac{\delta^2k^2}{(\delta k - \delta k_+^{\delta,\varepsilon})(\delta k - \delta k_-^{\delta,\varepsilon})}\,(1+\mathcal{O}(c_\varepsilon))\,.$$

Using that $\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$, we have

$$(\mu\,,1)_\varepsilon = \frac{\pi}{2}((\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]\,,1)_\varepsilon\,\frac{\delta^2k^2}{\delta k_+^{\delta,\varepsilon} - \delta k_-^{\delta,\varepsilon}}\left(\frac{1}{\delta k - \delta k_+^{\delta,\varepsilon}} - \frac{1}{\delta k - \delta k_-^{\delta,\varepsilon}}\right)(1+\mathcal{O}(c_\varepsilon))\,. \tag{3.55}$$

To get the the solution we use (3.54), but insert (3.55) into it to arrive at

$$\begin{aligned}\frac{2}{\pi}\mu =& -\frac{\pi}{2}\frac{((\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]\,,1)_\varepsilon\,(\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[1]}{|D|(\delta k_+^{\delta,\varepsilon}-\delta k_-^{\delta,\varepsilon})}\left(\frac{1}{\delta k-\delta k_+^{\delta,\varepsilon}}-\frac{1}{\delta k-\delta k_-^{\delta,\varepsilon}}\right)(1+\mathcal{O}(c_\varepsilon))\\ &+(\widetilde{\mathcal{L}}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]\,.\end{aligned}$$

Then from Lemma 3.22 it follows that

$$\begin{aligned}\frac{2}{\pi}\mu =& -\frac{\pi}{2}\frac{((f^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon+\mathcal{O}(\delta c_\varepsilon^2)+\mathcal{O}(\delta\varepsilon))\cdot((\mathcal{L}^\varepsilon)^{-1}[1]+\mathcal{O}_{\mathcal{X}^\varepsilon}(c_\varepsilon^2))}{|D|(\delta k_+^{\delta,\varepsilon}-\delta k_-^{\delta,\varepsilon})}\\ &\cdot\left(\frac{1}{\delta k-\delta k_+^{\delta,\varepsilon}}-\frac{1}{\delta k-\delta k_-^{\delta,\varepsilon}}\right)(1+\mathcal{O}(c_\varepsilon))+f^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1]+\mathcal{O}_{\mathcal{X}^\varepsilon}(\delta c_\varepsilon^2)+\mathcal{O}_{\mathcal{X}^\varepsilon}(\delta\varepsilon)\\ =& -\frac{\pi}{2|D|}\frac{f^{\delta k}(0)((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon\,(\mathcal{L}^\varepsilon)^{-1}[1]+\mathcal{O}_{\mathcal{X}^\varepsilon}(\delta c_\varepsilon^3)+\mathcal{O}_{\mathcal{X}^\varepsilon}(\delta\varepsilon c_\varepsilon^3)}{\delta k_+^{\delta,\varepsilon}-\delta k_-^{\delta,\varepsilon}}\\ &\cdot\left(\frac{1}{\delta k-\delta k_+^{\delta,\varepsilon}}-\frac{1}{\delta k-\delta k_-^{\delta,\varepsilon}}\right)(1+\mathcal{O}(c_\varepsilon))+f^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1]+\mathcal{O}_{\mathcal{X}^\varepsilon}(\delta c_\varepsilon^2)+\mathcal{O}_{\mathcal{X}^\varepsilon}(\delta\varepsilon)\\ =&\frac{-\pi}{2|D|}\frac{f^{\delta k}(0)((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon\,(\mathcal{L}^\varepsilon)^{-1}[1]}{\delta k_+^{\delta,\varepsilon}-\delta k_-^{\delta,\varepsilon}}\left(\frac{1}{\delta k-\delta k_+^{\delta,\varepsilon}}-\frac{1}{\delta k-\delta k_-^{\delta,\varepsilon}}\right)+f^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1]\\ &+\mathcal{O}(\delta)\mathcal{O}_{\mathcal{X}^\varepsilon}(c_\varepsilon^3)\left(\frac{1}{\delta k-\delta k_+^{\delta,\varepsilon}}-\frac{1}{\delta k-\delta k_-^{\delta,\varepsilon}}\right)+\mathcal{O}(\delta)\mathcal{O}_{\mathcal{X}^\varepsilon}(c_\varepsilon^2)+\mathcal{O}(\delta)\mathcal{O}_{\mathcal{X}^\varepsilon}(\varepsilon)\,.\end{aligned}$$

With that, and using $((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon=-\frac{2|D|}{\pi}c_\varepsilon$, the proof for Proposition 3.23 readily follows. ■

3.3.5 Asymptotic Expansion of our Solution to the Physical Problem

In Proposition 3.10 we established

$$\mathcal{A}^{\delta k,\varepsilon}[\partial_\nu u^{\delta k}\,|_\Lambda](\tau)+\sum_{n=1}^{\infty}\delta^n\,2\,\widehat{\Gamma}_{+,n}^{k,\varepsilon}[\partial_\nu u^{\delta k}\,|_\Lambda](\tau)=f^{\delta k}(\tau)\,,\tag{3.56}$$

and we know that for $\delta k\in\mathbb{K}_{\mathbb{C}}\setminus\{0,\delta k_+^{\delta,\varepsilon},\delta k_-^{\delta,\varepsilon}\}$ that $\mathcal{A}^{\delta k,\varepsilon}$ is invertible, see Proposition 3.23. Then for δ small enough we can use the Neumann series and obtain

$$\begin{aligned}(\mathcal{A}^{\delta k,\varepsilon}+\sum_{n=1}^{\infty}2\,\delta^n\,\widehat{\Gamma}_{+,n}^{k,\varepsilon})^{-1}&=\left(\mathcal{I}+(\mathcal{A}^{\delta k,\varepsilon})^{-1}\sum_{n=1}^{\infty}2\,\delta^n\,\widehat{\Gamma}_{+,n}^{k,\varepsilon}\right)^{-1}(\mathcal{A}^{\delta k,\varepsilon})^{-1}\\ &=\sum_{m=0}^{\infty}\left(-(\mathcal{A}^{\delta k,\varepsilon})^{-1}\sum_{n=1}^{\infty}2\,\delta^n\,\widehat{\Gamma}_{+,n}^{k,\varepsilon}\right)^m(\mathcal{A}^{\delta k,\varepsilon})^{-1}\\ &=\sum_{m=0}^{\infty}\left(\sum_{n=1}^{\infty}-2\,\delta^n\,(\mathcal{A}^{\delta k,\varepsilon})^{-1}\widehat{\Gamma}_{+,n}^{k,\varepsilon}\right)^m(\mathcal{A}^{\delta k,\varepsilon})^{-1}\\ &=(\mathcal{A}^{\delta k,\varepsilon})^{-1}-2\delta(\mathcal{A}^{\delta k,\varepsilon})^{-1}\widehat{\Gamma}_{+,1}^{k,\varepsilon}+\mathcal{O}(\delta^2)\mathcal{O}_{\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)}(1)\,.\end{aligned}$$

Thus we have from solving (3.56)

$$\partial_\nu u^{\delta k}\,|_\Lambda = (\mathcal{A}^{\delta k,\varepsilon})^{-1}[f^{\delta k}] + \mathcal{O}(\delta^2)\,,$$

where we use that $f^{\delta k} = \mathcal{O}(\delta)$, that $\widehat{\Gamma}^{k,\varepsilon}_{+,1}$ is linear, and the formula for $(\mathcal{A}^{\delta k,\varepsilon})^{-1}$. With Proposition 3.23 we split $(\mathcal{A}^{\delta k,\varepsilon})^{-1}[f^{\delta k}]$ into $\mu^{\delta k}_\star$ and $\mu^{\delta k}_\sim$ and thus we have for $t \in (-\varepsilon\,,\varepsilon)$

$$\partial_\nu u^{\delta k}\left(\!\!\binom{t}{h}\!\!\right) = \mu^{\delta k}_\star(t) + \mu^{\delta k}_\sim(t) + \mathcal{O}(\delta^2)\,,$$

and we see from Proposition 3.23 also that $\partial_\nu u^{\delta k}\,|_\Lambda = \mathcal{O}(\delta)$.

Now we want to calculate the first order expansion term in δ for the solution in the far-field. Let $z \in \mathbb{R}^2$, where $z_2 \gg 1$. The following asymptotic expansion holds.

Lemma 3.24 *We have for $z \in \mathbb{R}^2_+$, $z_2 > 1$,*

$$\begin{aligned} u^{\delta k}_s(z) = u^{\delta k}_{\mathrm{RHS},p}(z) + u^{\delta k}_{\mathrm{RHS},e}(z) + \mathcal{S}^{\delta k}_{+,p}[\partial_\nu u^{\delta k}\,|_\Lambda](z) + \mathcal{S}^{\delta k}_{+,e}[\partial_\nu u^{\delta k}\,|_\Lambda](z) \\ + \mathcal{T}^{\delta k}_{+,p}[\partial_\nu u^{\delta k}\,|_\Lambda](z) + \mathcal{T}^{\delta k}_{+,e}[\partial_\nu u^{\delta k}\,|_\Lambda](z) + \mathcal{O}(\delta^2)\,, \quad (3.57) \end{aligned}$$

where for $\mu \in \mathcal{X}^\varepsilon$

$$\begin{aligned} \mathcal{S}^{\delta k}_{+,p}[\mu](z) &:= \int_\Lambda \mu(y)\Gamma^{\delta k}_{+,p}(z,y)\mathrm{d}\sigma_y\,, \\ \mathcal{S}^{\delta k}_{+,e}[\mu](z) &:= \int_\Lambda \mu(y)\Gamma^{\delta k}_{+,e}(z,y)\mathrm{d}\sigma_y\,, \\ \mathcal{T}^{\delta k}_{+,p}[\mu](z) &:= -\int_\Lambda\int_{\partial D} \mu(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\partial_{\nu_w}\Gamma^{\delta k}_{+,p}(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,, \\ \mathcal{T}^{\delta k}_{+,e}[\mu](z) &:= -\int_\Lambda\int_{\partial D} \mu(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\partial_{\nu_w}\Gamma^{\delta k}_{+,e}(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,, \end{aligned}$$

and

$$\begin{aligned} u^{\delta k}_{\mathrm{RHS},p}(z) &= e^{i\delta(k_2z_2-k_1z_1)}\left[\int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\frac{\sin(\delta k_2 y_2)\,e^{i\delta k_1 y_1}}{\delta k_2 p}\mathrm{d}\sigma_y\right. \\ &\quad \left. -\int_{\partial D}\int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}\frac{ik_1}{pk_2}\sin(\delta k_2 w_2)\\ \frac{1}{p}\cos(\delta k_2 w_2)\end{pmatrix}e^{i\delta k_1 w_1}\mathrm{d}\sigma_w\mathrm{d}\sigma_y\right]\,, \\ u^{\delta k}_{\mathrm{RHS},e}(z) &= \left[-\int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\Gamma^{\delta k}_{+,e}(z,y)\mathrm{d}\sigma_y\right. \\ &\quad \left. +\int_{\partial D}\int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\,\partial_{\nu_w}\Gamma^{\delta k}_{+,e}(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\right]\,. \end{aligned}$$

Proof We define $u_f^{\delta k}(z) := u^{\delta k}(z) + f^{\delta k}(z) =: u_s^{\delta k}(z) - u_{\mathrm{RHS}}^{\delta k}(z)$. Then we have from Proposition 3.5 that

$$\begin{aligned} u_f^{\delta k}(z) &= \int_\Lambda \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\Omega^1,+}(z,y)\mathrm{d}\sigma_y \\ &= \int_\Lambda \partial_\nu u^{\delta k}(y) \Gamma^{\delta k}_+(z,y)\mathrm{d}\sigma_y + \int_\Lambda \partial_\nu u^{\delta k}(y) \mathrm{R}^{\delta k}_{\Omega^1,+}(z,y)\mathrm{d}\sigma_y \\ &=: \mathcal{S}^{\delta k}_+[\partial_\nu u^{\delta k}\,|_\Lambda] + \mathcal{T}^{\delta k}_+[\partial_\nu u^{\delta k}\,|_\Lambda]\,. \end{aligned}$$

Using the splitting $\Gamma^{\mathbf{k}}_+ = \Gamma^{\mathbf{k}}_{+,p} + \Gamma^{\mathbf{k}}_{+,e}$, see Proposition 2.4, we already obtain $\mathcal{S}^{\delta k}_{+,p}$ and $\mathcal{S}^{\delta k}_{+,e}$. To study the terms $u^{\delta k}_{\mathrm{RHS}}$ and $\mathcal{T}^{\delta k}_+$, consider that

$$\mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,y) = -\int_{\partial E} \mathrm{N}^{\mathbf{k}}_{\partial\Omega,\times}(y,w)\partial_{\nu_y}\Gamma^{\mathbf{k}}_+(z,w)\mathrm{d}\sigma_w\,,$$

from Proposition 2.9. Thus

$$\int_\Lambda \partial_\nu u^{\delta k}(y)\mathrm{R}^{\delta k}_{\Omega^1,+}(z,y)\mathrm{d}\sigma_y = -\int_\Lambda\int_{\partial D} \partial_\nu u^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\partial_{\nu_w}\Gamma^{\delta k}_+(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,,$$

and

$$\begin{aligned} &\int_{\partial D} \partial_\nu (u_0^{\delta k} - u_0^{\delta k}\circ \mathtt{P})(y)\mathrm{R}^{\delta k}_{\Omega^1,+}(z,y)\mathrm{d}\sigma_y \\ &\qquad = -\int_\Lambda\int_{\partial D} \partial_\nu (u_0^{\delta k} - u_0^{\delta k}\circ \mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\partial_{\nu_w}\Gamma^{\delta k}_+(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,. \quad (3.58) \end{aligned}$$

Using again the splitting $\nabla\Gamma^{\mathbf{k}}_+ = \nabla\Gamma^{\mathbf{k}}_{+,p} + \nabla\Gamma^{\mathbf{k}}_{+,e}$ and the explicit formula for $\Gamma^{\mathbf{k}}_{+,p}$ and $\nabla\Gamma^{\mathbf{k}}_{+,p}$ we obtain the formulas in Lemma 3.24. ■

Let us approximate $\mathcal{S}^{\delta k}_+$ and $\mathcal{T}^{\delta k}_+$. Let $z_2 > 1$. We define

$$\begin{aligned} \mathcal{S}^{\delta k}_{+,p,0}[\mu](z) &:= -e^{i(\delta k_2 z_2 - \delta k_1 z_1)}\frac{1}{p}\int_\Lambda y_2\mu(y)\mathrm{d}\sigma_y\,, \\ \mathcal{S}^{\delta k}_{+,e,0}[\mu](z) &:= e^{-i\delta k_1 z_1}\left(\int_\Lambda \mu(y)\Gamma^0_+(z,y)\mathrm{d}\sigma_y + \frac{1}{p}\int_\Lambda y_2\mu(y)\mathrm{d}\sigma_y\right) \\ &= -e^{-i\delta k_1 z_1}\int_\Lambda \mu(y)\sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \frac{1}{p|l|}e^{il(z_1-y_1)}e^{-|l|z_2}\sinh(|l|y_2)\mathrm{d}\sigma_y\,, \quad (3.59) \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}^{\delta k}_{+,p,0}[\mu](z) &:= e^{i(\delta k_2 z_2 - \delta k_1 z_1)}\frac{1}{p}\int_\Lambda\int_{\partial D}\mu(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,, \\ \mathcal{T}^{\delta k}_{+,e,0}[\mu](z) &:= -e^{-i\delta k_1 z_1}\left(\int_\Lambda\int_{\partial D}\mu(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\partial_{\nu_w}\Gamma^0_+(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\right. \\ &\qquad \left. + e^{-i\delta k_2 z_2}\mathcal{T}^{\delta k}_{+,p,0}[\mu](z)\right). \end{aligned}$$

Lemma 3.25 *There exist $C_{(3.60)} > 0$, $C_{(3.61)} > 0$, $C_{(3.62)} > 0$ and $C_{(3.63)} > 0$ such that, for all $z \in \mathbb{R}^2_+$ with $z_2 > 1$ and all $\mu \in \mathcal{X}^\varepsilon$, it holds that*

$$\left|\left(\mathcal{S}^{\delta k}_{+,p} - \mathcal{S}^{\delta k}_{+,p,0}\right)[\mu](z)\right| + \frac{1}{\delta}\left|\nabla\left(\mathcal{S}^{\delta k}_{+,p} - \mathcal{S}^{\delta k}_{+,p,0}\right)[\mu](z)\right| \leq C_{(3.60)} \|\mu\|_{\mathcal{X}^\varepsilon} \delta^2 , \tag{3.60}$$

$$\left|\left(\mathcal{S}^{\delta k}_{+,e} - \mathcal{S}^{\delta k}_{+,e,0}\right)[\mu](z)\right| + \frac{1}{\delta}\left|\nabla\left(\mathcal{S}^{\delta k}_{+,e} - \mathcal{S}^{\delta k}_{+,e,0}\right)[\mu](z)\right| \leq C_{(3.61)} \|\mu\|_{\mathcal{X}^\varepsilon} \delta , \tag{3.61}$$

and

$$\left|\left(\mathcal{T}^{\delta k}_{+,p} - \mathcal{T}^{\delta k}_{+,p,0}\right)[\mu](z)\right| + \frac{1}{\delta}\left|\nabla\left(\mathcal{T}^{\delta k}_{+,p} - \mathcal{T}^{\delta k}_{+,p,0}\right)[\mu](z)\right| \leq C_{(3.62)} \|\mu\|_{\mathcal{X}^\varepsilon} \delta , \tag{3.62}$$

$$\left|\left(\mathcal{T}^{\delta k}_{+,e} - \mathcal{T}^{\delta k}_{+,e,0}\right)[\mu](z)\right| + \frac{1}{\delta}\left|\nabla\left(\mathcal{T}^{\delta k}_{+,e} - \mathcal{T}^{\delta k}_{+,e,0}\right)[\mu](z)\right| \leq C_{(3.63)} \|\mu\|_{\mathcal{X}^\varepsilon} \delta . \tag{3.63}$$

Proof Let us consider $\mathcal{S}^{\delta k}_+$ first. Using the following splitting for $\Gamma^{\delta k}_+(z,y)$, which is given in Proposition 2.4, we have

$$\Gamma^{\delta k}_+(z,y) = \Gamma^{\delta k}_{+,p}(z,x) + \Gamma^{\delta k}_{+,e}(z,x) ,$$

where

$$\Gamma^{\delta k}_{+,p}(z,x) = -\frac{\sin(\delta k_2 x_2)}{\delta k_2 p} e^{i\delta(k_2 z_2 - k_1 z_1)} e^{i\delta k_1 x_1} , \tag{3.64}$$

$$\begin{aligned}\Gamma^{\delta k}_{+,e}(z,x) =& e^{-i\delta k_1 z_1} \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \left(\frac{-e^{il(z_1-x_1)+i\delta k_1 x_1}}{p\sqrt{|l-\delta k_1|^2 - \delta^2 k^2}} \right. \\ & \left. \cdot \sinh\left(\sqrt{|l-\delta k_1|^2 - \delta^2 k^2}\, x_2\right)\right) e^{-\sqrt{|l-\delta k_1|^2-\delta^2 k^2}\, z_2} . \end{aligned} \tag{3.65}$$

$\mathcal{S}^{\delta k}_{+,p,0}$ is the zeroth order term of the Taylor expansion with respect to δ of $\Gamma^{\delta k}_{+,p}(z,x)$, but without the $e^{i\delta(k_2 z_2 - k_1 z_1)}$ term, and $\mathcal{S}^{\delta k}_{+,e,0}$ is the zeroth order term of the Taylor expansion with respect to δ of $\Gamma^{\delta k}_{+,e}(z,x)$, but without the $e^{-i\delta k_1 z_1}$ term, which is located before the evanescent sum in (3.65). To see that, write $\mathcal{S}^{\delta k}_{+,e,0}$ using

$$\Gamma^0_+(z,x) = -\frac{x_2}{p} - \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \frac{1}{p|l|} e^{il(z_1-x_1)} e^{-|l|z_2} \sinh(|l|x_2) , \tag{3.66}$$

which is given through (2.2) for $z_2 > 1$, and rewrite $\int_\Lambda \partial_\nu u^{\delta k}(y)\Gamma^0_+(z,y)\mathrm{d}\sigma_y$ with it. Then we see that the sum in (3.66) is exactly the zeroth order term of the Taylor expansion. With that, we obtain the formula in (3.59).

Inserting these exact formulas into the expressions in Lemma 3.25 and using

$$\int_{-\varepsilon}^{\varepsilon} \mu(t)\varphi(t)\mathrm{d}t = \int_{-\varepsilon}^{\varepsilon} \varphi(t)\frac{\sqrt[4]{\varepsilon^2 - t^2}}{\sqrt[4]{\varepsilon^2 - t^2}}\mu(t)\mathrm{d}t \leq \|\varphi\|_{\mathcal{C}^0} \|\mu\|_{\mathcal{X}^\varepsilon} \sqrt{\pi} ,$$

where $\varphi \in \mathcal{C}^0([-\varepsilon,\varepsilon])$ and where we used the L^2-Cauchy-Schwarz inequality, yields the desired estimations for $\mathcal{S}_+^{\delta k}$. For $\mathcal{T}_+^{\delta k}$ it works analogously. ■

We define

$$\begin{aligned}
u^{\delta k}_{\mathcal{S}^\star_p}(z) &:= \mathcal{S}^{\delta k}_{+,p,0}[\mu^{\delta k}_\star](z)\,,\\
u^{\delta k}_{\mathcal{S}^\star_e}(z) &:= \mathcal{S}^{\delta k}_{+,e,0}[\mu^{\delta k}_\star](z)\,,\\
u^{\delta k}_{\mathcal{T}^\star_p}(z) &:= \mathcal{T}^{\delta k}_{+,p,0}[\mu^{\delta k}_\star](z)\,,\\
u^{\delta k}_{\mathcal{T}^\star_e}(z) &:= \mathcal{T}^{\delta k}_{+,e,0}[\mu^{\delta k}_\star](z)\,.
\end{aligned}$$

We then have the following proposition:

Proposition 3.26 *Let $V_r := \{z \in \mathbb{R}^2_+ \,|\, z_2 > r\}$. There exists a constant $C_{(3.67)} > 0$ such that*

$$\begin{aligned}
&\left\| u^{\delta k}_{\mathrm{s}} - (u^{\delta k}_{\mathcal{S}^\star_p} + u^{\delta k}_{\mathcal{S}^\star_e} + u^{\delta k}_{\mathcal{T}^\star_p} + u^{\delta k}_{\mathcal{T}^\star_e} + u^{\delta k}_{\mathrm{RHS},p} + u^{\delta k}_{\mathrm{RHS},e}) \right\|_{\mathrm{L}^\infty(V_r)}\\
&\quad + \frac{1}{\delta} \left\| \nabla \left[u^{\delta k}_{\mathrm{s}} - (u^{\delta k}_{\mathcal{S}^\star_p} + u^{\delta k}_{\mathcal{S}^\star_e} + u^{\delta k}_{\mathcal{T}^\star_p} + u^{\delta k}_{\mathcal{T}^\star_e} + u^{\delta k}_{\mathrm{RHS},p} + u^{\delta k}_{\mathrm{RHS},e}) \right] \right\|_{\mathrm{L}^\infty(V_r)} \qquad (3.67)\\
&\quad \leq C_{(3.67)} \left(\delta c_\varepsilon^3 \left(\frac{1}{\delta k - \delta k_+^{\delta,\varepsilon}} - \frac{1}{\delta k - \delta k_-^{\delta,\varepsilon}} \right) + \delta c_\varepsilon^2 + \delta \varepsilon^1 + \delta^2 \right),
\end{aligned}$$

for δ, ε small enough and r large enough.

Proof According to (3.57), we have

$$u^{\delta k}_{\mathrm{s}} - u^{\delta k}_{\mathrm{RHS},p} - u^{\delta k}_{\mathrm{RHS},e} = \left(\mathcal{S}^{\delta k}_{+,e} + \mathcal{S}^{\delta k}_{+,p} + \mathcal{T}^{\delta k}_{+,e} + \mathcal{T}^{\delta k}_{+,p} \right) [u^{\delta k}|_\Lambda] + \mathcal{O}(\delta^2)\,.$$

With Lemma 3.25 and the fact that $u^{\delta k}\,|_\Lambda = \mu^{\delta k}_\star + \mu^{\delta k}_\sim + \mathcal{O}(\delta^2)$, $\left\|\mu^{\delta k}_\star\right\|_{\mathcal{X}^\varepsilon} = \mathcal{O}(\delta)$ and $\left\|\mu^{\delta k}_\sim\right\|_{\mathcal{X}^\varepsilon} = \mathcal{O}(\delta)$, we readily proved Proposition 3.26. ■

Proof (Theorem 3.2) We see that $u^{\delta k}_{\mathcal{S}^\star_e}$, $u^{\delta k}_{\mathcal{T}^\star_e}$ and $u^{\delta k}_{\mathrm{RHS},e}$ are exponentially decaying in z_2 and are of order $\mathcal{O}(\delta)$, thus their $\mathrm{L}^\infty(V_r)$-norm is of order $\delta e^{-C\,r}$ for some constant $C > 0$. Then with Proposition 3.26 and the change into the macroscopic view with $U^k(x) = u^{\delta k}(x/\delta)$, we obtain Theorem 3.2. ■

3.3.6 Evaluating the Impedance Boundary Condition

We switch back to the macroscopic variable $U^k(x) = u^{\delta k}(x/\delta)$. We approximate our solution in the far-field with the function

$$\begin{aligned}
U^k_{\mathrm{app}}(z) :=& (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathsf{P})(z/\delta) + u^{\delta k}_{\mathcal{S}^\star_p}(z/\delta) + u^{\delta k}_{\mathcal{T}^\star_p}(z/\delta) + u^{\delta k}_{\mathrm{RHS},p}(z/\delta)\\
=& -2ia_0 e^{-ik_1 z_1} \sin(k_2 z_2) + e^{i(k_2 z_2 - k_1 z_1)}\, C^{\delta k}_{(3.68)}\,,
\end{aligned}$$

where

$$
\begin{aligned}
C^{\delta k}_{(3.68)} := &- \frac{h}{p}\int_\Lambda \mu^{\delta k}_\star(y)\mathrm{d}\sigma_y + \frac{1}{p}\int_\Lambda\int_{\partial D}\mu^{\delta k}_\star(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}\sigma_w\mathrm{d}\sigma_y \\
&+ \int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ \mathtt{P})(y)\frac{\sin(\delta k_2 y_2)\, e^{i\delta k_1 y_1}}{\delta k_2 p}\mathrm{d}\sigma_y \qquad (3.68)\\
&- \int_{\partial D}\int_{\partial D}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\Omega^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}\frac{i k_1}{p k_2}\sin(\delta k_2 w_2)\\ \frac{1}{p}\cos(\delta k_2 w_2)\end{pmatrix} e^{i\delta k_1 w_1}\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,.
\end{aligned}
$$

Consider that $C^{\delta k}_{(3.68)} = \mathcal{O}(\delta)$ since $\mu^{\delta k}_\star = \mathcal{O}(\delta)$ and $(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P}) = \mathcal{O}(\delta)$ and the other factors are of size $\mathcal{O}(1)$.

We want that U^k_{app} satisfies the impedance boundary condition, that is $U^k_{\mathrm{app}}(z) + \delta c_{\mathrm{IBC}}\partial_{z_2}U^k_{\mathrm{app}}(z) = 0$ at $\partial\mathbb{R}^2_+$. From (3.3.6), we obtain the condition

$$
e^{-ik_1 z_1}\, C^{\delta k}_{(3.68)} + \delta c_{\mathrm{IBC}}(-2ia_0 k_2 e^{-ik_1 z_1} + i\, k_2 e^{-ik_1 z_1}\, C^{\delta k}_{(3.68)}) = 0\,, \quad \text{for all } z_2 \in \mathbb{R}\,.
$$

After rearranging the terms, we obtain

$$
c_{\mathrm{IBC}} = \frac{-C^{\delta k}_{(3.68)}}{i\,\delta k_2(\, C^{\delta k}_{(3.68)} - 2a_0)}\,.
$$

Using that $\frac{1}{1+\mathcal{O}(\delta)} = 1 - \mathcal{O}(\delta)$, we have

$$
c_{\mathrm{IBC}} = \frac{C^{\delta k}_{(3.68)}}{2ia_0\,\delta k_2} + \mathcal{O}(\delta)\,.
$$

This proves Theorem 3.3.

3.4 Numerical Illustrations

In this subsection we compute the impedance boundary condition constant c_{IBC} with numerical means using Theorem 3.3. We use two geometries, both rectangles, but with different sizes and for each geometry we have different ranges for the wave vector k and the gap length ε. This is because the resonance value $k^{\delta,\varepsilon}_{\mathrm{res}} := k^{\delta,\varepsilon}_+$ is proportional to the square root of the geometry area, and the same holds for the width of the resonance peak of c_{IBC}. However, we do not have to consider different values of δ, since according to Theorem 3.3 all computations are done with the input δk, thus a different δ would only scale the first coordinate axis, but would not change the shape of the curve itself. We fix $\delta = 0.01$.

We implement $\Gamma^{\mathbf{k}}_\sharp$ using Edwald's Method see [17] or [8, Chapter 7.3.2]. We implement the remainders using Lemmas 2.10, 2.11 and 2.12.

The first geometry has the following set-up. It is a rectangle with length 0.9 and height 0.9. The period is $p = 1$ and $h = 1$. The amplitude of the incident wave

is $a_0 = 1$. The number of points, with which we approximate the boundary of the rectangle, is 200. For the wave vector $k = k_2$, we take 341 equidistant points on the interval $[30, 200]$. For ε we pick 5 values, those are $\{0.1, 0.05, 0.03, 0.01, 0.001\}$. The result can be seen in Figure 2.

The second geometry has the following set-up. It is a rectangle with length 0.2 and height 0.3. The period is $p = 1$ and $h = 0.5$. The amplitude of the incident wave is $a_0 = 1$. The number of points, with which we approximate the boundary of the rectangle, is 200. For the wave vector $k = k_2$, we take 301 equidistant points on the interval $[100, 400]$. For ε we pick 5 values, those are $\{0.1, 0.05, 0.03, 0.01, 0.001\}$. Consider that the case $\varepsilon = 0.1$ means that the whole upper boundary of our rectangle is the gap. The result can be seen in Figure 3.

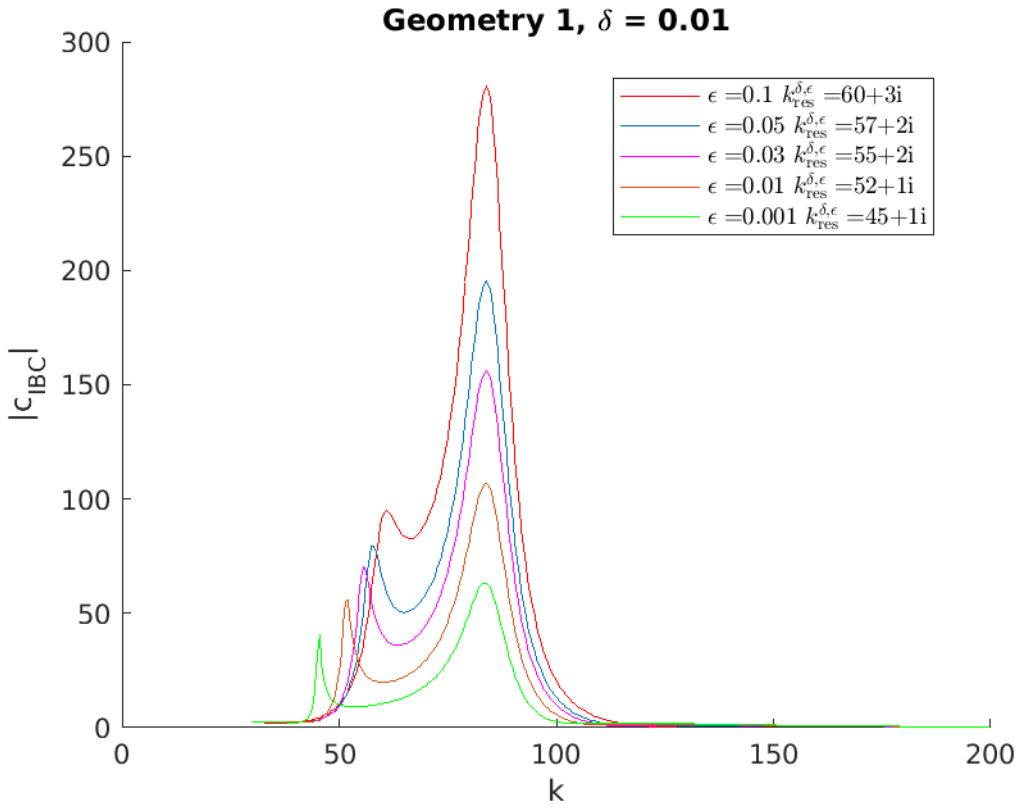


Figure 2: *The plot of the absolute value of the variable c_{IBC} depending on the wave vector k for the first geometry. For every value of ε, the rounded value of the resonance value $k_{\mathrm{res}}^{\delta\varepsilon}$ is displayed.*

Consider that in Figure 2, for $\varepsilon \leq 0.01$, the resonance splits, so we get two peaks, due to the geometry, which is large enough that the neighbouring resonators have an effect on the main one. In the second geometry however, it seems the resonators have a large enough distance from each other with respect to their width, such that no the

neighbouring resonators do not affect the main one. See Figure 3.

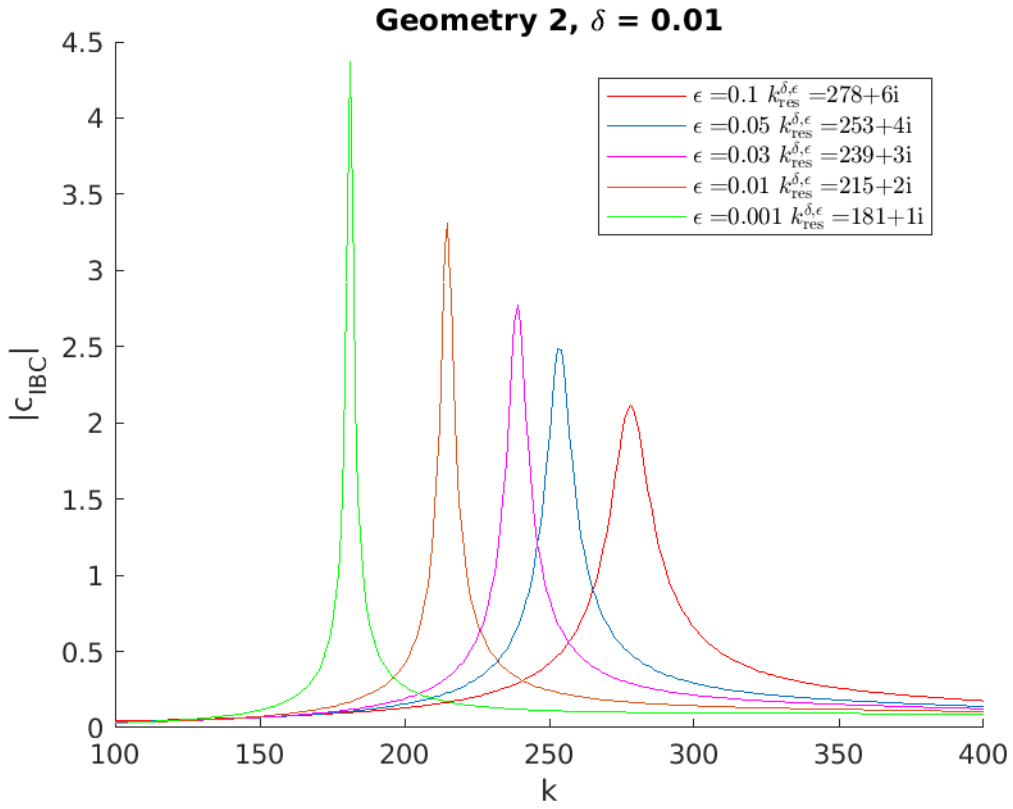


Figure 3: *The plot of the absolute value of the variable* c_{IBC} *depending on the wave vector* k *for the second geometry. For every value of* ε*, the rounded value of the resonance value* $k_{\mathrm{res}}^{\delta\varepsilon}$ *is displayed.*

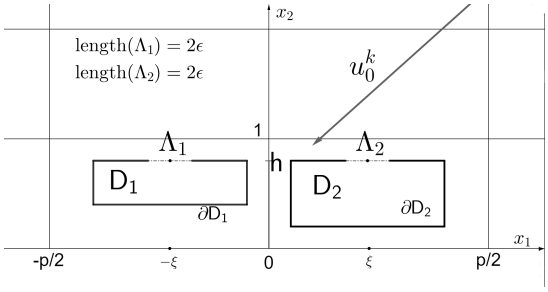
4 Two Periodically Arranged Helmholtz Resonators

In this section, we look at two domains D_1 and D_2 both bounded and simply connected domains, which have the same height h. We repeat $D_1 \cup D_2$ periodically along the x_1-axis, with period p and scale the whole geometry by a factor of δ. Additionally, D_1 and D_2 have each a gap called Λ_1 and Λ_2 on their boundary, which allows the incident wave U_0^k to pass through. The incident wave rebounds inside D_1 and D_2 and leaves at the gaps, which then leads to the scattered wave U_s^k.

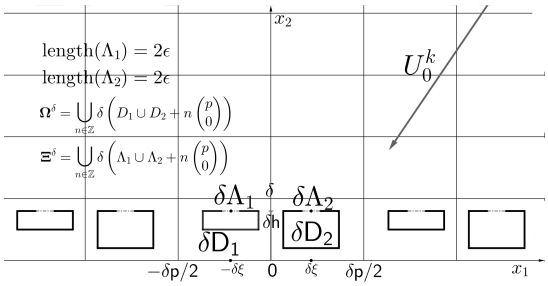
We will have a good approximation of the scattered wave in the far-field. Moreover, this approximation satisfies the Helmholtz equation with a Robin boundary condition at the x_1-axis which approximates a Dirichlet or a Neumann boundary condition depending on the magnitude of the incoming wave vector $\mathbf{k}$ and δ.

4.1 Mathematical Description of the Physical Problem

4.1.1 Geometry



(a) *Microscopic, non-periodic view of our two Helmholtz resonators. The Helmholtz resonators are contained in the unit cell* $(-p/2, p/2) \times (0, 1)$. *They have the gaps* Λ_1 *and* Λ_2*, both of length* 2ε*, which are parallel to the* x_1 *axis and centered at* $(-\xi, h)^{\mathrm{T}}$ *respectively* $(\xi, h)^{\mathrm{T}}$*, where* $h \in (0, 1)$. u_0^k *denotes the incident wave.* D_1 *and* D_2 *have not to be rectangular in shape.*



(b) *Macroscopic view of our periodically arranged Helmholtz resonators, with periodicity* δp. *All Helmholtz resonators have the form of the Helmholtz resonators depicted in* (a)*, but are scaled with the factor* δ. U_0^k *denotes the incident wave.* $\mathbf{\Omega}^\delta$ *is the collection of all Helmholtz resonators and* $\mathbf{\Xi}^\delta$ *is the collection of all gaps.*

Figure 4: *The physical setup. In* (a) *we have the microscopic, non-periodic view. In* (b) *we have the macroscopic, periodic view.*

Before we consider the periodic and macroscopic problem, we first define the geometry of our Helmholtz resonators in the unit cell. Let $D_1, D_2 \Subset (-p/2, p/2) \times (0, 1)$ be two open, bounded, and simply connected domains, such that $\overline{D_1}$ and $\overline{D_2}$ do not intersect and where $p \in \mathbb{R}$ is close enough to 1. We assume that $D_1 \cup D_2$ is a $\mathcal{C}^2$-domain. We define $\Lambda_1 \subset \partial D$ and $\Lambda_2 \subset \partial D$ to be the gap of D_1 and D_2 respectively, where Λ_1 and Λ_2 are both line segment parallel to the x_1-axis. Λ_1 is centered at $(-\xi, h)^{\mathrm{T}}$ and Λ_2 is centered at $(\xi, h)^{\mathrm{T}}$, where $h \in (0, 1)$ and $\xi \in (0, p)$, and Λ_1 and Λ_2 have both length 2ε, where $\varepsilon \in (0, 1)$ small enough. To facilitate future computations we assume that $h/p \geq 1/2$.

Now we define the macroscopic view, that is, we shrink our domain by the factor $\delta \in (0, \infty)$. We define the collection of periodically arranged Helmholtz resonators $\mathbf{\Omega}^\delta$, with period δp, and the collection of gaps of those Helmholtz resonators $\mathbf{\Xi}^\delta$, where a single gap has length $2\delta\varepsilon$, as

$$\mathbf{\Omega}^\delta := \bigcup_{n\in\mathbb{Z}} \delta\left(D_1 \cup D_2 + n\begin{pmatrix}p\\0\end{pmatrix}\right),$$
$$\mathbf{\Xi}^\delta := \bigcup_{n\in\mathbb{Z}} \delta\left(\Lambda_1 \cup \Lambda_2 + n\begin{pmatrix}p\\0\end{pmatrix}\right).$$

4.1.2 Incident Wave

Let $\mathbf{k} := (k_1, k_2)^{\mathrm{T}} \in \mathbb{R}^2$ be the wave vector. We will fix the direction of the wave vector, that is k_1/k and k_2/k, where $k := |\mathbf{k}| := (k_1^2 + k_2^2)^{1/2} \in [0, \infty)$, but let the magnitude k vary. With that, we define the function $U_0^k : \mathbb{R}^2 \to \mathbb{C}$ as

$$U_0^k(x) := a_0 e^{-ik_1 x_1} e^{-ik_2 x_2},$$

where $a_0 \in \mathbb{R}$ denotes the amplitude. U_0^k will be our incident wave. We define the parity operator $\mathtt{P} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$\mathtt{P}(x_1, x_2) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}.$$

With this we have the reflected incident wave

$$U_0^k \circ \mathtt{P}(x) = a_0 e^{-ik_1 x_1} e^{ik_2 x_2}.$$

Moreover,

$$(U_0^k - U_0^k \circ \mathtt{P})(x) = -2ia_0 e^{-ik_1 x_1} \sin(k_2 x_2).$$

We will also need the following equation

$$\nabla(U_0^k - U_0^k \circ \mathtt{P})(x) = \begin{pmatrix} -2a_0 k_1 e^{-ik_1 x_1} \sin(k_2 x_2) \\ -2ia_0 k_2 e^{-ik_1 x_1} \cos(k_2 x_2) \end{pmatrix}.$$

Consider also that U_0^k and $U_0^k \circ \mathtt{P}$ are quasi-periodic with quasi-momentum $-k_1 p$, that is

$$U_0^k \left(x + \begin{pmatrix} p \\ 0 \end{pmatrix} \right) = e^{-ik_1 p} U_0^k(x),$$
$$U_0^k \circ \mathtt{P} \left(x + \begin{pmatrix} p \\ 0 \end{pmatrix} \right) = e^{-ik_1 p} U_0^k \circ \mathtt{P}(x).$$

4.1.3 The Resulting Wave

With the geometry and the incident wave, we model the electromagnetic scattering problem and the resulting wave $U^k : \mathbb{R}^2_+ \setminus \partial\mathbf{\Omega}^\delta \to \mathbb{C}$ by the following system of equations:

$$\begin{cases} \left(\triangle + k^2\right) U^k = 0 & \text{in} \quad \mathbb{R}^2_+ \setminus \partial\mathbf{\Omega}^\delta, \\ U^k|_+ - U^k|_- = 0 & \text{on} \quad \Xi^\delta, \\ \partial_\nu U^k|_+ - \partial_\nu U^k|_- = 0 & \text{on} \quad \Xi^\delta, \\ \partial_\nu U^k|_+ = 0 & \text{on} \quad \partial\mathbf{\Omega}^\delta \setminus \Xi^\delta, \\ \partial_\nu U^k|_- = 0 & \text{on} \quad \partial\mathbf{\Omega}^\delta \setminus \Xi^\delta, \\ U^k = 0 & \text{on} \quad \partial\mathbb{R}^2_+, \end{cases} \tag{4.1}$$

where $\cdot|_+$ denotes the limit from outside of $\mathbf{\Omega}^\delta$ and $\cdot|_-$ denotes the limit from inside of $\mathbf{\Omega}^\delta$, and ∂_ν denotes the normal derivative on $\partial\mathbf{\Omega}^\delta$. Similar to diffraction problems for gratings, the above system of equations is complemented by a certain outgoing radiation condition imposed on the scattered field $U_s^k := U^k - (U_0^k - U_0^k \circ \mathtt{P})$ and quasi-periodicity on U^k. More precisely, we are interested in the quasi-periodic solutions, that is

$$U^k\left(x + \begin{pmatrix} p \\ 0 \end{pmatrix}\right) = e^{-ik_1 p} U^k(x) \quad \text{for} \quad x \in \mathbb{R}^2_+,$$

and solutions satisfying the outgoing radiation condition, thus

$$\left|\partial_{x_2} U_s^k - ik_2 U_s^k\right| \to 0 \quad \text{for} \quad x_2 \to \infty.$$

Then the outgoing radiation condition can be imposed by assuming that all the modes in the Rayleigh-Bloch expansion are either decaying exponentially or propagating along the x_2-direction. Since in our case we assume that the period of the resonator structure δp is much smaller than k, the outgoing radiation condition takes the following specific form:

$$\begin{aligned} (U^k - U_0^k)(x) &\sim a e^{-ik_1 x_1} e^{ik_2 x_2} \quad \text{as} \quad x_2 \to \infty, \\ U_s^k(x) &\sim (a+1) e^{-ik_1 x_1} e^{ik_2 x_2} \quad \text{as} \quad x_2 \to \infty, \end{aligned}$$

for some constant amplitude $a \in \mathbb{R}$.

Consider also that in absence of Helmholtz resonators the solution to (4.1) is given by $U_0^k - U_0^k \circ \mathtt{P}$.

4.1.4 The Resulting Wave in the Microscopic View

Given the resulting wave $U^k(x)$, the function $u^{\delta k}(x) : \mathbb{R}^2_+ \setminus \partial\mathbf{\Omega}^1 \to \mathbb{C}$, $u^{\delta k}(x) := U^k(\delta x)$ represents the resulting wave, but where the Helmholtz resonators are scaled-back and thus are of height h and not δh. $u^{\delta k}$ satisfies

$$\begin{cases} \left(\triangle + (\delta k)^2\right) u^{\delta k} = 0 & \text{in} \quad \mathbb{R}^2_+ \setminus \partial\mathbf{\Omega}^1, \\ u^{\delta k}|_+ - u^{\delta k}|_- = 0 & \text{on} \quad \mathbf{\Xi}^1, \\ \partial_\nu u^{\delta k}|_+ - \partial_\nu u^{\delta k}|_- = 0 & \text{on} \quad \mathbf{\Xi}^1, \\ \partial_\nu u^{\delta k}|_+ = 0 & \text{on} \quad \partial\mathbf{\Omega}^1 \setminus \mathbf{\Xi}^1, \\ \partial_\nu u^{\delta k}|_- = 0 & \text{on} \quad \partial\mathbf{\Omega}^1 \setminus \mathbf{\Xi}^1, \\ u^{\delta k} = 0 & \text{on} \quad \partial\mathbb{R}^2_+, \end{cases} \tag{4.2}$$

where $\cdot|_+$ denotes the limit from outside of $\mathbf{\Omega}^1$ and $\cdot|_-$ denotes the limit from inside of $\mathbf{\Omega}^1$, and ∂_ν denotes the normal derivative on $\partial\mathbf{\Omega}^1$.

We can adopt the quasi-periodicity from the macroscopic view and obtain

$$u^{\delta k}\left(x+\begin{pmatrix}p\\0\end{pmatrix}\right)=e^{-i\delta k_1 p}u^{\delta k}(x)\quad\text{for}\quad x\in\mathbb{R}^2_+.$$

Defining $u_s^{\delta k}:=u^{\delta k}-(u_0^{\delta k}-u_0^{\delta k}\circ\mathrm{P})$ we also get that

$$\left|\partial_{x_2}u_s^{\delta k}-i\delta k_2 u_s^{\delta k}\right|\to 0\quad\text{for}\quad x_2\to\infty.$$

We see that $u^{\delta k}$ solves the same partial differential equation like U^k in the re-scaled geometry, but with the scaled wave vector δk. We will see that we can express $u^{\delta k}$ as an expansion in terms of δ and we will give an analytic expression for the first order term.

4.2 Main Results

We assume that $\delta k\in\mathbb{K}:=\{\hat{k}\in\mathbb{R}\mid 0\neq|\hat{k}|<k_{D_1\cup D_2,\min,\triangle}/2 \text{ and } |\hat{k}|^2<\inf\{|l-\hat{k}\,e_1|^2\mid l:=2\pi n/p,\ n\in\mathbb{Z}\setminus\{0\}\}\}$, where $k_{D_1\cup D_2,\min,\triangle}$ is defined as the first non-zero eigenvalue of the operator $-\triangle$ with Neumann conditions on the boundary $\partial D_1\cup\partial D_2$. If we would extend the domain $\mathbb{K}$ to $\mathbb{K}_{\mathbb{C}}:=\{k_*\in\mathbb{C}\mid\sqrt{k_*\bar{k}_*}<k_{D_1\cup D_2,\min,\triangle}/2\}$, we would obtain the following resonance values for our physical problem:

Theorem 4.1 *There exists exactly four resonance values in $\mathbb{K}_{\mathbb{C}}$ for U^k. These are for $j\in\{1,2\}$*

$$\begin{aligned}\delta k_{j,+}^{\delta,\varepsilon}=&\delta k_{\star,j}^{\delta,\varepsilon}\left(1+\frac{(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\mathrm{e}_j^{\mathrm{T}}(\mathbf{Y}^\varepsilon_{\star,\star})^{-1}\boldsymbol{A}_{\star,(1)}(\mathrm{Y}^\varepsilon_{\star,\star})_j\right)+\mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}),\\ \delta k_{j,-}^{\delta,\varepsilon}=&\delta k_{\star,j}^{\delta,\varepsilon}\left(-1+\frac{(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\mathrm{e}_j^{\mathrm{T}}(\mathbf{Y}^\varepsilon_{\star,\star})^{-1}\boldsymbol{A}_{\star,(1)}(\mathrm{Y}^\varepsilon_{\star,\star})_j\right)+\mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}),\end{aligned}$$

where

$$\begin{aligned}\delta k_{\star,1}^{\delta,\varepsilon}=&\sqrt{\frac{\pi}{2}\mathfrak{c}_\varepsilon}\left(\frac{1}{2}\left(k^\varepsilon_{\star,\mathrm{tr}1}+k^\varepsilon_{\star,\mathrm{tr}2}\right)+\left[\frac{1}{4}\left(k^\varepsilon_{\star,\mathrm{tr}1}-k^\varepsilon_{\star,\mathrm{tr}2}\right)^2-k^\varepsilon_{\star,\det}\right]^{1/2}\right),\\ \delta k_{\star,2}^{\delta,\varepsilon}=&\sqrt{\frac{\pi}{2}\mathfrak{c}_\varepsilon}\left(\frac{1}{2}\left(k^\varepsilon_{\star,\mathrm{tr}1}+k^\varepsilon_{\star,\mathrm{tr}2}\right)-\left[\frac{1}{4}\left(k^\varepsilon_{\star,\mathrm{tr}1}-k^\varepsilon_{\star,\mathrm{tr}2}\right)^2-k^\varepsilon_{\star,\det}\right]^{1/2}\right),\end{aligned}$$

$$\begin{aligned}k^\varepsilon_{\star,\mathrm{tr}1}=&\frac{1}{\sqrt{|D_1|}}\left(1+\frac{(\boldsymbol{\alpha}_{(0)})_{1,1}\pi\mathfrak{c}_\varepsilon}{4}\right),\quad k^\varepsilon_{\star,\mathrm{tr}2}=\frac{1}{\sqrt{|D_2|}}\left(1+\frac{(\boldsymbol{\alpha}_{(0)})_{2,2}\pi\mathfrak{c}_\varepsilon}{4}\right),\\ k^\varepsilon_{\star,\det}=&\frac{\pi^2\mathfrak{c}_\varepsilon^2}{4}\frac{(\mathbf{T}_{(0)})_{1,2}(\mathbf{T}_{(0)})_{2,1}}{(\sqrt{|D_1|}+\sqrt{|D_2|})},\quad \mathfrak{c}_\varepsilon=\frac{-1}{\log(\varepsilon/2)}.\end{aligned}$$

Here, $\mathbf{Y}^{\varepsilon}_{\star,\star} = [(\mathrm{Y}^{\varepsilon}_{\star,\star})_1, (\mathrm{Y}^{\varepsilon}_{\star,\star})_2]$*, with* $(\mathrm{Y}^{\varepsilon}_{\star,\star})_j$ *being the normalized eigenvector to the eigenvalue* $\delta k^{\delta,\varepsilon}_{\star,j}$ *of* $\boldsymbol{A}^{\varepsilon}_{\star,\star}$*, where* $\boldsymbol{A}^{\varepsilon}_{\star,\star}$ *is given in (4.43) and* $\boldsymbol{A}_{\star,D}$*,* $\boldsymbol{A}_{\star,(0)}$ *and* $\boldsymbol{A}_{\star,(1)}$ *are given in Lemma 4.15, where* $(\mathbf{T}_{(\mathbf{1})})_{2,1}$*,* $(\mathbf{T}_{(\mathbf{1})})_{1,2}$*,* $(\mathbf{T}_{(\mathbf{0})})_{2,1}$*,* $(\mathbf{T}_{(\mathbf{0})})_{1,2}$*,* $(\boldsymbol{\alpha}_{(\mathbf{1})})_{2,2}$*,* $(\boldsymbol{\alpha}_{(\mathbf{1})})_{1,1}$*,* $(\boldsymbol{\alpha}_{(\mathbf{0})})_{2,2}$ *and* $(\boldsymbol{\alpha}_{(\mathbf{0})})_{1,1}$ *are given in (4.33)-(4.40).*

We have the following approximation for the resulting wave U^k:

Theorem 4.2 *Let* $V_r := \{z \in \mathbb{R}^2_+ \,|\, z_2 > r\}$*. There exist constants* $C_{(4.3)}, \widetilde{C}_{(4.3)} > 0$ *such that*

$$\begin{aligned}\left\|U^k_s - (U^k_{\mathcal{S}^\star_p} + U^k_{\mathcal{T}^\star_p} + U^k_{\mathrm{RHS},p})\right\|_{\mathrm{L}^\infty(V_r)} + \left\|\nabla\left[U^k_s - (U^k_{\mathcal{S}^\star_p} + U^k_{\mathcal{T}^\star_p} + U^k_{\mathrm{RHS},p})\right]\right\|_{\mathrm{L}^\infty(V_r)} \\ \leq C_{(4.3)}\left(\frac{\delta}{|\log(\varepsilon)|^3}\left(\|(\mathbf{M}^{\delta k}_+\mathbf{M}^{\delta k}_-)^{-1}\|\right) + \frac{\delta}{|\log(\varepsilon)|^2} + \delta\varepsilon + \delta\, e^{-\widetilde{C}_{(4.3)}r} + \delta^2\right),\end{aligned} \tag{4.3}$$

for ε *small enough, where*

$$U^k_{\mathcal{S}^\star_p}(z) = -e^{i(k_2z_2-k_1z_1)}\frac{h}{p}\sum_{j\in\{1,2\}}\int_{\Lambda_j}(\boldsymbol{\mu}^{\delta k}_\star)_j(y)\mathrm{d}\sigma_y,$$

$$U^k_{\mathcal{T}^\star_p}(z) = e^{i(k_2z_2-k_1z_1)}\frac{1}{p}\sum_{j,\hat{j}\in\{1,2\}}\int_{\Lambda_j}\int_{\partial D_{\hat{j}}}(\boldsymbol{\mu}^{\delta k}_\star)_j(y)\mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}0\\1\end{pmatrix}\mathrm{d}\sigma_w\mathrm{d}\sigma_y,$$

$$\begin{aligned}U^k_{\mathrm{RHS},p}(z) = e^{i\delta(k_2z_2-k_1z_1)}\Bigg[&\sum_{j\in\{1,2\}}\int_{\partial D_j}\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\frac{\sin(\delta k_2y_2)\,e^{i\delta k_1y_1}}{\delta k_2 p}\mathrm{d}\sigma_y \\ -\sum_{j,\hat{j}\in\{1,2\}}\int_{\partial D_j}\int_{\partial D_{\hat{j}}}&\partial_\nu(u^{\delta k}_0 - u^{\delta k}_0\circ\mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,\times}(y,w)\nu_w\cdot\begin{pmatrix}\frac{ik_1}{pk_2}\sin(\delta k_2w_2)\\ \frac{1}{p}\cos(\delta k_2w_2)\end{pmatrix}e^{i\delta k_1w_1}\mathrm{d}\sigma_w\mathrm{d}\sigma_y\Bigg].\end{aligned}$$

Here, for $y = ((t_1-\xi,h)^{\mathrm{T}},(t_2+\xi,h)^{\mathrm{T}}) \in \Lambda_1\times\Lambda_2$*, we have*

$$\boldsymbol{\mu}_\star(y) = \begin{bmatrix}\frac{-1}{\sqrt{\varepsilon^2-t_1^2}}\left(\mathfrak{c}_\varepsilon^2\frac{\pi}{4}\left(\boldsymbol{A}_D(\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_+\mathbf{M}^{\delta k}_-)^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\left[f^{\delta k}_{D_1},f^{\delta k}_{D_2}\right]^{\mathrm{T}}\right)_1 + \frac{1}{2}\mathfrak{c}_\varepsilon f^{\delta k}_{D_1}\right)\\ \frac{-1}{\sqrt{\varepsilon^2-t_2^2}}\left(\mathfrak{c}_\varepsilon^2\frac{\pi}{4}\left(\boldsymbol{A}_D(\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_+\mathbf{M}^{\delta k}_-)^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\left[f^{\delta k}_{D_1},f^{\delta k}_{D_2}\right]^{\mathrm{T}}\right)_2 + \frac{1}{2}\mathfrak{c}_\varepsilon f^{\delta k}_{D_2}\right)\end{bmatrix},$$

where $f^{\delta k}_{D_1}, f^{\delta k}_{D_2} \in \mathbb{C}$ *are given by*

$$\begin{aligned}f^{\delta k}_{D_1} =& 2ia_0\sin(\delta k_2h)e^{-i\delta k_1\xi} - 2a_0\delta\int_{\partial D_1\cup\partial D_2}\nu_y\cdot\begin{pmatrix}k_1e^{-i\delta k_1y_1}\sin(\delta k_2y_2)\\ ik_2e^{-i\delta k_1y_1}\cos(\delta k_2y_2)\end{pmatrix}\mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}-\xi\\h\end{pmatrix},y\right)\mathrm{d}\sigma_y,\\ f^{\delta k}_{D_2} =& 2ia_0\sin(\delta k_2h)e^{i\delta k_1\xi} - 2a_0\delta\int_{\partial D_1\cup\partial D_2}\nu_y\cdot\begin{pmatrix}k_1e^{-i\delta k_1y_1}\sin(\delta k_2y_2)\\ ik_2e^{-i\delta k_1y_1}\cos(\delta k_2y_2)\end{pmatrix}\mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}\xi\\h\end{pmatrix},y\right)\mathrm{d}\sigma_y.\end{aligned}$$

Here, $\Upsilon^{\varepsilon}_{\star,\star}$ *is given as in Theorem 4.1 and*

$$\mathbf{M}^{\delta k}_{+} = \begin{bmatrix} \delta k - \delta k^{\delta,\varepsilon}_{1,+} & 0 \\ 0 & \delta k - \delta k^{\delta,\varepsilon}_{2,+} \end{bmatrix}, \quad \mathbf{M}^{\delta k}_{-} = \begin{bmatrix} \delta k - \delta k^{\delta,\varepsilon}_{1,-} & 0 \\ 0 & \delta k - \delta k^{\delta,\varepsilon}_{2,-} \end{bmatrix},$$
$$\boldsymbol{A}_D = \begin{bmatrix} 1/|D_1| & 0 \\ 0 & 1/|D_2| \end{bmatrix}.$$

We see from Theorem 4.2 that the function

$$U^k_{\text{app}}(z) := (U^k_0 - U^k_0 \circ \mathtt{P})(z) + U^k_{\mathcal{S}^\star_p}(z) + U^k_{\mathcal{T}^\star_p}(z) + U^k_{\text{RHS},p}(z)\,,$$

gives an accurate approximation of U^k in the far-field. Moreover, it satisfies the Helmholtz equation in $\mathbb{R}^2_+$ with the boundary condition

$$U^k_{\text{app}}(z) + \delta c_{\text{IBC}} \partial_{z_2} U^k_{\text{app}}(z) = 0\,, \quad \text{for } z \in \partial\mathbb{R}^2_+\,,$$

and $U^k_{\text{app}} - U^k_0$ satisfies the outgoing radiation condition.

Using Theorem 4.2 we can express c_{IBC} as follows.

Theorem 4.3 *The constant in the impedance boundary condition is given as*

$$c_{\text{IBC}} = \frac{1}{2ia_0\delta k_2}\, C^{\delta k}_{(4.4)} + \mathcal{O}(\delta)\,,$$

where $C^{\delta k}_{(4.4)} \in \mathbb{R}$ *is defined as*

$$\begin{aligned} C^{\delta k}_{(4.4)} =& -\frac{h}{p} \sum_{j\in\{1,2\}} \int_{\Lambda_j} (\boldsymbol{\mu}^{\delta k}_\star)_j(y)\mathrm{d}\sigma_y + \sum_{j\in\{1,2\}} \int_{\partial D_j} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \frac{\sin(\delta k_2 y_2)\, e^{i\delta k_1 y_1}}{\delta k_2 p} \mathrm{d}\sigma_y \\ &+ \frac{1}{p} \sum_{j,\hat{j}\in\{1,2\}} \int_{\Lambda_j} \int_{\partial D_{\hat{j}}} (\boldsymbol{\mu}^{\delta k}_\star)_j(y) \mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,\times}(y,w)\nu_w \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}\sigma_w \mathrm{d}\sigma_y \qquad (4.4) \\ &- \sum_{j,\hat{j}\in\{1,2\}} \int_{\partial D_j} \int_{\partial D_{\hat{j}}} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,\times}(y,w)\nu_w \cdot \begin{pmatrix} \frac{ik_1}{pk_2} \sin(\delta k_2 w_2) \\ \frac{1}{p}\cos(\delta k_2 w_2) \end{pmatrix} e^{i\delta k_1 w_1} \mathrm{d}\sigma_w \mathrm{d}\sigma_y\,. \end{aligned}$$

4.3 Proof of the Main Results

We want to proof Theorem 3.1 - 3.3. First, we express the resulting wave outside the Helmholtz resonators and the resulting wave inside of the Helmholtz resonators through operators acting on the resulting wave, but restricted on the gap. This leads us to a condition with the linear operator $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$, whose solution is the resulting wave on the gap up to a term of order δ^2. We solve this linear system based on the procedure given in [10]. We will see that it is solvable for a complex wave vector near 0 except in five points, two of which are the resonances of our system. Then we recover the resulting wave outside the resonators up to a term of order δ^2. We will see, that we can split the resulting wave into a propagating wave and a evanescent one. The propagating one leads us to the impedance boundary condition constant c_{IBC}.

4.3.1 Collapsing the Wave-Informations on the Two Gaps

Let us consider the resulting wave $u^{\delta k}$ in the microscopic view, recall Subsection 4.1.4. We will keep the microscopic view until Subsection 4.3.6. We only look on the main strip $Y := \{y \in \mathbb{R}^2_+ \,|\, |y_1| < p/2\}$. D is the Helmholtz resonator on that strip and Λ the gap on ∂D. Furthermore, we fix $k_1/k =: e_1$ and $k_2/k =: e_2$ and assume that $\delta k \in \mathbb{K} := \{\hat{k} \in \mathbb{R} \,|\, 0 \neq |\hat{k}| < k_{D_1 \cup D_2, \min, \triangle}/2 \text{ and } |\hat{k}|^2 < \inf\{|l - \hat{k}\, e_1|^2 \,|\, l := 2\pi n/p,\ n \in \mathbb{Z} \setminus \{0\}\}\}$. Consider that $u^{\delta k}$ is continuous on the gap $\Lambda_1 \cup \Lambda_2$, thus $u^{\delta k}(z)$ is well-defined for $z \in \Lambda_1 \cup \Lambda_2$.

Proposition 4.4 *Let $j \in \{1,2\}$ and let $\mathrm{N}^k_{D_j}$ be as in Definition 2.7. Let $z \in D_j$. Then,*

$$u^{\delta k}(z) = -\int_{\Lambda_j} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{D_j}(z,y) \mathrm{d}\sigma_y \,.$$

Let $z \in \Lambda_j$. Then,

$$u^{\delta k}(z) = -\int_{\Lambda_j} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial D_j}(z,y) \mathrm{d}\sigma_y \,. \tag{4.5}$$

Proposition 4.5 *Let $\mathrm{N}^k_{\Omega,+}$, $\mathrm{N}^k_{\Omega,\times}$ be as in Definition 2.8. Let $z \in Y \setminus \overline{D}$. Then,*

$$u^{\delta k}_s(z) = \int_{\Lambda_1 \cup \Lambda_2} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y - \int_{\partial D_1 \cup \partial D_2} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y \,.$$

Let $z \in \Lambda$. Then,

$$u^{\delta k}_s(z) = \int_{\Lambda_1 \cup \Lambda_2} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y - \int_{\partial D_1 \cup \partial D_2} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y \,. \tag{4.6}$$

Using that $u^{\delta k}$ is continuous on the gap we can deduce from the following proposition a necessary condition for $\partial_\nu u^{\delta k}|_\Lambda$. Assume we can obtain a solution $\partial_\nu u^{\delta k}$ from that condition, then from Propositions 4.4 and 4.5 we can recover the resulting wave on Y.

Proposition 4.6 (Gap-Formula) *Let $j, \hat{j} \in \{1,2\}$, $j \neq \hat{j}$ and let $z \in \Lambda_j$ then*

$$\begin{aligned} &\int_{\Lambda_j} \partial_\nu u^{\delta k}(y) \left(\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) + \mathrm{N}^{\delta k}_{\partial D_j}(z,y) \right) \mathrm{d}\sigma_y + \int_{\Lambda_{\hat{j}}} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y \\ &\qquad = \int_{\partial D_1 \cup \partial D_2} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y - (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(z) \,. \end{aligned} \tag{4.7}$$

Consider that the right-hand-side in (4.7) does not depend on $u^{\delta k}$ and it is computable.

Proof (Proposition 4.4) Let us look at (4.5) first. Let $z \in \partial D_j$ then using Green's formula with $\left(\triangle + k^2\right) u^{\delta k} = 0$ we obtain that

$$\begin{aligned} u^{\delta k}(z) &= \int_{D_j} u^{\delta k}(y) \left(\triangle_y + k^2\right) \mathrm{N}^{\delta k}_{\partial D_j}(z,y)\mathrm{d}y \\ &= \int_{\partial D_j} u^{\delta k}(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial D_j}(z,y)\mathrm{d}\sigma_y - \int_{\partial D_j} \partial_\nu u^{\delta k}(y)\mathrm{N}^{\delta k}_{\partial D_j}(z,y)\mathrm{d}\sigma_y\,. \end{aligned}$$

Using that $\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial D_j}(z,y) = 0$ on ∂D_j and $\partial_\nu u^{\delta k}(y) = 0$ on $\partial D_j \setminus \Lambda_j$ we obtain the desired equation.
We get the other equation analogously. ■

Proof (Proposition 4.5) Let us look at (4.6) first.
Let $r > 0$ and $U_r := \{y \in \mathbb{R}^2 \setminus \overline{D_1 \cup D_2} \mid |y_1| < p/2 \wedge 0 < y_2 < r\}$, $\partial U_{r,0} := \{y \in \partial U_r \mid y_2 = 0\}$, $\partial U_{r,-} := \{y \in \partial U_r \mid y_1 = -p/2\}$, $\partial U_{r,+} := \{y \in \partial U_r \mid y_1 = +p/2\}$ and $\partial U_{r,r} := \{y \in \partial U_r \mid y_2 = r\}$. Then

$$u^{\delta k}_s(z) = \lim_{r\to\infty} \int_{U_r} u^{\delta k}_s(y)(\triangle_y + k^2)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)dy,$$

because of the Dirac measure. Using Green's formula, we have

$$u^{\delta k}_s(z) = \lim_{r\to\infty} \Bigg(\int_{U_r} (\triangle + k^2) u^{\delta k}_s(y)\, \mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}y \tag{4.8}$$

$$- \int_{\partial D_1\cup\partial D_2} u^{\delta k}_s(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y + \int_{\partial D_1\cup\partial D_2} \partial_\nu u^{\delta k}_s(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y \tag{4.9}$$

$$+ \int_{\partial U_{r,0}} u^{\delta k}_s(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,0}} \partial_\nu u^{\delta k}_s(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y \tag{4.10}$$

$$+ \int_{\partial U_{r,-}} u^{\delta k}_s(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,-}} \partial_\nu u^{\delta k}_s(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y \tag{4.11}$$

$$+ \int_{\partial U_{r,+}} u^{\delta k}_s(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,+}} \partial_\nu u^{\delta k}_s(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y \tag{4.12}$$

$$+ \int_{\partial U_{r,r}} u^{\delta k}_s(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,r}} \partial_\nu u^{\delta k}_s(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y \Bigg). \tag{4.13}$$

The right-hand-side in (4.8) vanishes because $u^{\delta k}_s$ satisfies the homogeneous Helmholtz equation. The left term in (4.9) vanishes because $\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}$ has a vanishing normal derivative on $\partial\mathbf{\Omega}^1$. Both terms in (4.10) vanish because of the Dirichlet boundary. The terms in (4.11) and in (4.12) cancel each other out because of the quasi-periodicity with quasi-momentum $-k_1 p$ for $u^{\delta k}_s$ and the quasi-periodicity with quasi-momentum $k_1 p$ for $\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}$, together with the explicit expression for the normal on $\partial U_{r,-}$, which is $(-1,0)^{\mathrm{T}}$, and the explicit expression for the normal on $\partial U_{r,+}$, which is $(1,0)^{\mathrm{T}}$. Thus we are left with

$$\begin{aligned} u^{\delta k}_s(z,x) &= \int_{\partial D_1\cup\partial D_2} \partial_\nu u^{\delta k}_s(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y \\ &\quad + \lim_{r\to\infty} \left(\int_{\partial U_{r,r}} u^{\delta k}_s(y)\partial_{\nu_y}\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y - \int_{\partial U_{r,r}} \partial_\nu u^{\delta k}_s(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y \right). \end{aligned} \tag{4.14}$$

Using that $u_s^{\delta k}$ and $\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}$ satisfy the outgoing radiation condition, we can write $\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) = \frac{1}{ik_2}\partial_{y_2}\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) + o(1)$ and $\partial_{y_2}u_s^{\delta k}(y) = ik_2u_s^{\delta k}(y) + o(1)$ for $y_2 \to \infty$. With that we can eliminate the integrals within the limes.

Finally, using that $\partial_\nu u^{\delta k} \mid_{\partial D_1 \cup \partial D_2 \setminus \Lambda_1 \cup \Lambda_2} = 0$ and the definition of $u_s^{\delta k}$, we proved (4.6).

We get the other equation analogously. ■

Proof (Proposition 3.6) Using that $u^{\delta k}$ is continuous at Λ_j we have that $u^{\delta k}|_+(z) - u^{\delta k}|_-(z) = u^{\delta k}(z) - u^{\delta k}(z) = 0$, for $z \in \Lambda_j$. Inserting (4.6) and (4.5) we obtain (4.7). ■

4.3.2 Expanding the Gap-Formula in Terms of Delta

We define $f^{\delta k}_{\partial\mathbf{\Omega}^1} : \Lambda_1 \cup \Lambda_2 \to \mathbb{C}$ as the right-hand-side of the Gap-Formula 3.6, that is

$$f^{\delta k}_{\partial\mathbf{\Omega}^1}(z) \coloneqq \int_{\partial D_1 \cup \partial D_2} \partial_\nu(u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y)\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y)\mathrm{d}\sigma_y - (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(z). \tag{4.15}$$

We define $f^{\delta k}_{D_1}(\tau)$ as $f^{\delta k}_{\partial\mathbf{\Omega}^1}((\tau - \xi, h)^{\mathrm{T}})$ and $f^{\delta k}_{D_2}(\tau)$ as $f^{\delta k}_{\partial\mathbf{\Omega}^1}((\tau + \xi, h)^{\mathrm{T}})$, for $\tau \in (-\varepsilon, \varepsilon)$.

Let us define the following operator-spaces and their respective norms:

Definition 4.7 *Recall Definition 3.7, we define*

$$\begin{aligned}
\boldsymbol{\mathcal{X}}^\varepsilon &\coloneqq \mathcal{X}^\varepsilon \times \mathcal{X}^\varepsilon, \\
\|\mu\|_{\boldsymbol{\mathcal{X}}^\varepsilon} &\coloneqq \|\mu_1\|_{\mathcal{X}^\varepsilon} + \|\mu_2\|_{\mathcal{X}^\varepsilon}, \quad \text{for } \mu = (\mu_1, \mu_2) \in \boldsymbol{\mathcal{X}}^\varepsilon, \\
\boldsymbol{\mathcal{Y}}^\varepsilon &\coloneqq \mathcal{Y}^\varepsilon \times \mathcal{Y}^\varepsilon, \\
\|\mu\|_{\boldsymbol{\mathcal{Y}}^\varepsilon} &\coloneqq \|\mu_1\|_{\mathcal{Y}^\varepsilon} + \|\mu_2\|_{\mathcal{Y}^\varepsilon}, \quad \text{for } \mu = (\mu_1, \mu_2) \in \boldsymbol{\mathcal{Y}}^\varepsilon.
\end{aligned}$$

Definition 4.8 *Let $\mu \in \boldsymbol{\mathcal{X}}^\varepsilon$ and $\alpha > 0$. We say $\mu = \mathcal{O}_{\boldsymbol{\mathcal{X}}^\varepsilon}(\alpha)$ for $\alpha \to 0$ if $\frac{\|\mu\|_{\boldsymbol{\mathcal{X}}^\varepsilon}}{\alpha}$ is bounded as $\alpha \to 0$.*

With those spaces we can define the following operators:

Definition 4.9 *The following operators are defined as functions from $\boldsymbol{\mathcal{X}}^\varepsilon$ to $\boldsymbol{\mathcal{Y}}^\varepsilon$ or from $\mathcal{X}^\varepsilon$ to $\mathcal{Y}^\varepsilon$. Let $n \in \mathbb{N} \setminus \{0\}$, and $\mu = (\mu_1, \mu_2) \in \boldsymbol{\mathcal{X}}^\varepsilon$ then*

$$\boldsymbol{\mathcal{L}}^\varepsilon[\mu](\tau) \coloneqq \begin{bmatrix} \mathcal{L}^\varepsilon & 0 \\ 0 & \mathcal{L}^\varepsilon \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}(\tau) = \begin{bmatrix} \mathcal{L}^\varepsilon[\mu_1](\tau) \\ \mathcal{L}^\varepsilon[\mu_2](\tau) \end{bmatrix},$$

$$\boldsymbol{\mathcal{K}}^\varepsilon[\mu](\tau) \coloneqq \begin{bmatrix} \mathcal{K}_1^\varepsilon[\mu_1](\tau) \\ \mathcal{K}_2^\varepsilon[\mu_2](\tau) \end{bmatrix} \coloneqq \begin{bmatrix} \int_{-\varepsilon}^{\varepsilon} \frac{\mu_1(t)}{|D_1|}\mathrm{d}t \\ \int_{-\varepsilon}^{\varepsilon} \frac{\mu_2(t)}{|D_2|}\mathrm{d}t \end{bmatrix},$$

$$
\begin{aligned}
\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}[\mu](\tau) &:= \begin{bmatrix} \mathcal{R}_{1,1}^{\delta k,\varepsilon} & \mathcal{R}_{1,2}^{\delta k,\varepsilon} \\ \mathcal{R}_{2,1}^{\delta k,\varepsilon} & \mathcal{R}_{2,2}^{\delta k,\varepsilon} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}(\tau)\,, \\
\mathcal{R}_{0,0}^{\delta k,\varepsilon}[\mu_1](\tau) &:= \int_{-\varepsilon}^{\varepsilon} \mu_1(t) \left[\frac{1}{\pi} \log\left(\frac{\pi}{p}\right) + \frac{1}{\pi} \log\left(\operatorname{sinc}\left| \frac{\pi}{p}(\tau - t) \right| \right) \right. \\
&\qquad \left. - \frac{1}{2\pi} \log\left(\sinh\left(\frac{\pi}{p} 2h\right)^2 + \sin\left(\frac{\pi}{p}(\tau - t)\right)^2 \right) \right] \mathrm{d}t\,, \\
\mathcal{R}_{1,1}^{\delta k,\varepsilon}[\mu_1](\tau) &:= \mathcal{R}_{0,0}^{\delta k,\varepsilon}[\mu_1](\tau) \\
&\qquad + \int_{-\varepsilon}^{\varepsilon} \mu_1(t) \left[\mathrm{R}_{\partial D_1}^{\delta k}\left(\begin{pmatrix} \tau - \xi \\ h \end{pmatrix}, \begin{pmatrix} t - \xi \\ h \end{pmatrix} \right) + \mathrm{R}_{\partial \Omega^1,+}^{\delta k}\left(\begin{pmatrix} \tau - \xi \\ h \end{pmatrix}, \begin{pmatrix} t - \xi \\ h \end{pmatrix} \right) \right] \mathrm{d}t\,, \\
\mathcal{R}_{1,2}^{\delta k,\varepsilon}[\mu_2](\tau) &:= \int_{-\varepsilon}^{\varepsilon} \mu_2(t) \mathrm{N}_{\partial \boldsymbol{\Omega}^1,+}^{\delta k}\left(\begin{pmatrix} \tau - \xi \\ h \end{pmatrix}, \begin{pmatrix} t + \xi \\ h \end{pmatrix} \right) \mathrm{d}t\,, \\
\mathcal{R}_{2,1}^{\delta k,\varepsilon}[\mu_1](\tau) &:= \int_{-\varepsilon}^{\varepsilon} \mu_1(t) \mathrm{N}_{\partial \boldsymbol{\Omega}^1,+}^{\delta k}\left(\begin{pmatrix} \tau + \xi \\ h \end{pmatrix}, \begin{pmatrix} t - \xi \\ h \end{pmatrix} \right) \mathrm{d}t\,, \\
\mathcal{R}_{2,2}^{\delta k,\varepsilon}[\mu_2](\tau) &:= \mathcal{R}_{0,0}^{\delta k,\varepsilon}[\mu_2](\tau) \\
&\qquad + \int_{-\varepsilon}^{\varepsilon} \mu_2(t) \left[\mathrm{R}_{\partial D_2}^{\delta k}\left(\begin{pmatrix} \tau + \xi \\ h \end{pmatrix}, \begin{pmatrix} t + \xi \\ h \end{pmatrix} \right) + \mathrm{R}_{\partial \Omega^1,+}^{\delta k}\left(\begin{pmatrix} \tau + \xi \\ h \end{pmatrix}, \begin{pmatrix} t + \xi \\ h \end{pmatrix} \right) \right] \mathrm{d}t\,, \\
\widehat{\boldsymbol{\Gamma}}_{+,n}^{k,\varepsilon}[\mu](\tau) &:= \begin{bmatrix} \int_{-\varepsilon}^{\varepsilon} \mu_1(t)\, \Gamma_{+,n}^k \left(\begin{pmatrix} \tau - \xi \\ h \end{pmatrix}, \begin{pmatrix} t - \xi \\ h \end{pmatrix} \right) \mathrm{d}t \\ \int_{-\varepsilon}^{\varepsilon} \mu_2(t)\, \Gamma_{+,n}^k \left(\begin{pmatrix} \tau + \xi \\ h \end{pmatrix}, \begin{pmatrix} t + \xi \\ h \end{pmatrix} \right) \mathrm{d}t \end{bmatrix}, \\
\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}[\mu](\tau) &:= \frac{2}{\pi} \boldsymbol{\mathcal{L}}^{\varepsilon}[\mu](\tau) + \frac{\boldsymbol{\mathcal{K}}^{\varepsilon}[\mu](\tau)}{\delta^2 k^2} + \boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}[\mu](\tau)\,,
\end{aligned}
$$

where $\Gamma_{+,n}^k$ is given in Lemma 2.2.

For $\tau \in (\varepsilon, \varepsilon)^2$, we also define

$$
\boldsymbol{f}^{\delta k}(\tau) := \begin{bmatrix} f_{D_1}^{\delta k}(\tau_1) \\ f_{D_2}^{\delta k}(\tau_2) \end{bmatrix}. \tag{4.16}
$$

Later, we will show that

$$
\begin{bmatrix} \partial_\nu u^{\delta k}\big|_{\Lambda_1} \\ \partial_\nu u^{\delta k}\big|_{\Lambda_2} \end{bmatrix} = (\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}] + \mathcal{O}(\delta^2).
$$

Proposition 4.10 *Let $2\varepsilon < p$, let $\tau \in (-\varepsilon, \varepsilon)^2$, then*

$$\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon} \begin{bmatrix} \partial_\nu u^{\delta k} \big|_{\Lambda_1} \\ \partial_\nu u^{\delta k} \big|_{\Lambda_2} \end{bmatrix} (\tau) + \sum_{n=1}^{\infty} \delta^n \, 2\,\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,n} \begin{bmatrix} \partial_\nu u^{\delta k} \big|_{\Lambda_1} \\ \partial_\nu u^{\delta k} \big|_{\Lambda_2} \end{bmatrix} (\tau) = \boldsymbol{f}^{\delta k}(\tau).$$

Proof Let $z := (\tau \pm \xi, h)^{\mathrm{T}} \in \Lambda_j$, where $j, \hat{j} \in \{1,2\}$, $j \neq \hat{j}$. From Proposition 4.6 we have

$$\int_{\Lambda_j} \partial_\nu u^{\delta k}(y) \left(\mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) + \mathrm{N}^{\delta k}_{\partial D_j}(z,y) \right) \mathrm{d}\sigma_y + \int_{\Lambda_{\hat{j}}} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y = f^{\delta k}_{\partial\mathbf{\Omega}^1}(z)\,.$$

Using Lemma 2.2 we can rearrange the last equation and obtain

$$\begin{aligned} f^{\delta k}_{\partial\mathbf{\Omega}^1}(z) = \int_{\Lambda_j} \partial_\nu u^{\delta k}(y) \Bigg(& 2\Gamma^0_+(z,y) + \frac{1}{\pi} \log|z-y| + \frac{1/|D_j|}{\delta^2 k^2} + \mathrm{R}^{\delta\mathbf{k}}_{\partial\mathbf{\Omega}^1,+}(z,y) + \mathrm{R}^{\delta k}_{\partial D_j}(z,y) \\ & + \sum_{n=1}^{\infty} \delta^n \, 2\Gamma^k_{+,n}(z,y) \Bigg) \mathrm{d}\sigma_y + \int_{\Lambda_{\hat{j}}} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y. \quad (4.17) \end{aligned}$$

Using that Λ is a line segment parallel to the x_1-axis and writing $y = y_1 = (t - \xi, h)^{\mathrm{T}}$, on the gap Λ_1, and writing $y = y_2 = (t + \xi, h)^{\mathrm{T}}$, on the gap Λ_2, we have that $\mathrm{d}\sigma_y = \mathrm{d}t$. Using (2.8) for $\Gamma^0_+(z,y)$ and using that the expansion in δ (Lemma 2.2) is uniform, we can interchange the infinite sum and the integration. Let $\mu_j(t) := \partial_\nu u^{\delta k}(y)\,|_{\Lambda_j}$, we have that

$$\begin{aligned} f^{\delta k}_{\partial\mathbf{\Omega}^1}(z) = & \int_{-\varepsilon}^{\varepsilon} \mu_j(t) \Bigg[\frac{1}{\pi} \log|\tau - t| + \frac{1}{2\pi} \log\left(\sin\left(\frac{\pi}{p}(\tau - t) \right)^2 \right) \\ & - \frac{1}{2\pi} \log\left(\sinh\left(\frac{\pi}{p} 2h \right)^2 + \sin\left(\frac{\pi}{p}(\tau - t) \right)^2 \right) + \mathrm{R}^{\delta k}_{\partial\mathbf{\Omega}^1,+}(z, y_j) + \mathrm{R}^{\delta k}_{\partial D_j}(z, y_j) \Bigg] \mathrm{d}t \\ & + \mathcal{R}^{\delta k,\varepsilon}_{j,\hat{j}}[\mu_{\hat{j}}](\tau) + \frac{\mathcal{K}^\varepsilon[\mu_j](\tau)}{\delta^2 k^2} + \sum_{n=1}^{\infty} \delta^n \, 2\widehat{\Gamma}^{k,\varepsilon}_{+,n}[\mu_j](\tau). \quad (4.18) \end{aligned}$$

Now consider that for $2\varepsilon < p$, we have

$$\begin{aligned} \frac{1}{2\pi} \log\left(\sin\left(\frac{\pi}{p}(\tau - t) \right)^2 \right) &= \frac{1}{\pi} \log\left(\sin\left| \frac{\pi}{p}(\tau - t) \right| \right) \\ &= \frac{1}{\pi} \log\left(\frac{\pi}{p} \right) + \frac{1}{\pi} \log|\tau - t| + \frac{1}{\pi} \log\left(\operatorname{sinc}\left| \frac{\pi}{p}(\tau - t) \right| \right). \quad (4.19) \end{aligned}$$

Inserting last equation into (4.18) we obtain that

$$\frac{2}{\pi} \mathcal{L}^\varepsilon[\mu_j](\tau) + \mathcal{R}^{\delta k,\varepsilon}_{j,j}[\mu_j](\tau) + \mathcal{R}^{\delta k,\varepsilon}_{j,\hat{j}}[\mu_{\hat{j}}](\tau) + \frac{\mathcal{K}^\varepsilon_j[\mu_j](\tau)}{\delta^2 k^2} + \sum_{n=1}^{\infty} \delta^n \, \widehat{\Gamma}^{k,\varepsilon}_{+,n}[\mu_j](\tau) = f^{\delta k}_{\partial\mathbf{\Omega}^1}(z).$$

With Definition 4.9 we have proven Proposition 4.10. ■

From Lemma 3.11 we readily get that for $0 < \varepsilon < 2$, the operator $\boldsymbol{\mathcal{L}}^\varepsilon : \boldsymbol{\mathcal{X}}^\varepsilon \to \boldsymbol{\mathcal{Y}}^\varepsilon$ is linear, bounded and invertible and has the inverse

$$(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}[\boldsymbol{\mu}](t) = \begin{bmatrix} (\mathcal{L}^\varepsilon)^{-1}[\mu_1](t_1) \\ (\mathcal{L}^\varepsilon)^{-1}[\mu_2](t_2) \end{bmatrix} . \tag{4.20}$$

Since $\mathrm{R}^{\delta k}_{\partial D_j}$ and $\mathrm{R}^{\delta \mathbf{k}}_{\partial \boldsymbol{\Omega}^1,+}$ are continuous, $\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}$ is a compact operator. Thus we have that $\frac{2}{\pi}\boldsymbol{\mathcal{L}}^\varepsilon + \boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}$ is a Fredholm operator of index zero. Thus for the operator $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$, extending the domain $\mathbb{K}$ to the complex numbers in a disk-shaped form, we will see that $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ is invertible except for a finite amount of values of δk. Some of those values are the resonances of our physical system. To that end, we will need the following result.

4.3.3 Characteristic Values of $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ and the four Resonance Values

Let us first look at the characteristic values of

$$\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon}[\boldsymbol{\mu}] := \frac{2}{\pi}\boldsymbol{\mathcal{L}}^\varepsilon[\boldsymbol{\mu}] + \frac{\boldsymbol{\mathcal{K}}^\varepsilon[\boldsymbol{\mu}]}{\delta^2 k^2},$$

where $\boldsymbol{\mu} \in \boldsymbol{\mathcal{X}}^\varepsilon$ and $\delta k \in \mathbb{K}_{\mathbb{C}} := \{k_* \in \mathbb{C} \,|\, \sqrt{k_*\bar{k}_*} < k_{D_1 \cup D_2,\min,\triangle}/2\}$. For $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \boldsymbol{\mathcal{X}}^\varepsilon$ we define $(\boldsymbol{\mu}\,, \boldsymbol{\lambda})_{\varepsilon\oplus\varepsilon} := (\mu_1\,, \lambda_1)_\varepsilon + (\mu_2\,, \lambda_2)_\varepsilon$, where $(\cdot\,,\cdot)_\varepsilon$ is the $\mathrm{L}^2((-\varepsilon,\varepsilon))$ inner-product. We also define $\mathbf{1} := [1,1]^{\mathrm{T}} \in \boldsymbol{\mathcal{X}}^\varepsilon$, $\mathbf{e}_1 := [1,0]^{\mathrm{T}} \in \boldsymbol{\mathcal{X}}^\varepsilon$ and $\mathbf{e}_2 := [0,1]^{\mathrm{T}} \in \boldsymbol{\mathcal{X}}^\varepsilon$.

Lemma 4.11 *$\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon}$ has exactly the four characteristic values $\pm k^{\delta,\varepsilon}_{j,0}$ for $j \in \{1,2\}$ with the characteristic functions $\mu^{\delta,\varepsilon}_{j,0}$, where*

$$\begin{aligned} k^{\delta,\varepsilon}_{j,0} =& \frac{1}{\delta}\left(-\frac{\pi}{2|D_j|}\Big((\mathcal{L}^\varepsilon)^{-1}[1]\,, 1\Big)_\varepsilon\right)^{1/2} = \frac{1}{\delta}\left(-\frac{\pi}{2|D_j|\log(\varepsilon/2)}\right)^{1/2}, \\ \mu^{\varepsilon}_{j,0} =& -\frac{\pi}{2|D_j|}\frac{(\mathcal{L}^\varepsilon)^{-1}[1]}{(\delta k^{\delta,\varepsilon}_{j,0})^2}\,, \end{aligned}$$

after imposing $(\mu_1, 1)_\varepsilon = 1$, $(\mu_2, 1)_\varepsilon = 1$.

Consider that $k^{\delta,\varepsilon}_{j,0}$ is real and positive.

Proof We are looking for $\boldsymbol{\mu} = [\mu_1\,, \mu_2]^{\mathrm{T}} \in \boldsymbol{\mathcal{X}}^\varepsilon$ such that

$$\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon}[\boldsymbol{\mu}] = \mathbf{0}\,.$$

Since $(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}[\mathbf{0}] = \mathbf{0}$, the last equation is equivalent to

$$\frac{2}{\pi}\boldsymbol{\mu} + \frac{(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}\boldsymbol{\mathcal{K}}^\varepsilon[\boldsymbol{\mu}]}{(\delta k)^2} = \mathbf{0}\,. \tag{4.21}$$

Applying once $(\cdot\,,\mathbf{e}_1)_{\varepsilon\oplus\varepsilon}$ and once $(\cdot\,,\mathbf{e}_2)_{\varepsilon\oplus\varepsilon}$ on both sides, we obtain

$$(\mu_j\,,1)_\varepsilon\left(\frac{2}{\pi}+\frac{1}{|D_j|}\frac{((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon}{(\delta k)^2}\right)=0\,,\quad \text{for } j\in\{1,2\}. \tag{4.22}$$

If $(\mu_j\,,1)_\varepsilon=0$, then $\mathcal{L}^\varepsilon[\mu_j]=0$, because of the condition $\mathcal{Q}^{\delta k,\varepsilon}[\boldsymbol{\mu}]=0$, and then $\mu_j=0$, since $\mathcal{L}^\varepsilon$ is invertible and linear. But 0 cannot be a characteristic function, by definition. Hence $(\mu_j\,,1)\neq 0$. Thus the second factor in (4.22) has to be zero. This leads us to

$$\delta^2k^2=-\frac{\pi}{2|D_j|}\Big((\mathcal{L}^\varepsilon)^{-1}[1]\,,1\Big)_\varepsilon\,,\quad \text{for } j\in\{1,2\}.$$

Using Lemma 3.12, we can calculate that $((\mathcal{L}^\varepsilon)^{-1}[1]\,,1)_\varepsilon=\frac{1}{\log(\varepsilon/2)}$, and obtain the characteristic values.

As for the characteristic functions, we rewrite (4.21) as

$$\frac{\mu_j}{(\mu_j,1)_\varepsilon}=-\frac{\pi}{2|D_j|}\frac{(\mathcal{L}^\varepsilon)^{-1}[1]}{(\delta k)^2}\,,\quad \text{for } j\in\{1,2\}.$$

Imposing the normalization on μ we have proven our statement. ■

Let us look at the characteristic values of $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$. Denote $\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon}:=\boldsymbol{\mathcal{L}}^\varepsilon+\frac{\pi}{2}\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}$ and $\boldsymbol{\mathcal{S}}^{\delta k,\varepsilon}:=\frac{\pi}{2}(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}$, where we fixed the angles of the incoming wave vector, but let the magnitude be complex. Using that $\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}$ is in $\mathcal{C}^{1,\eta}$, for $\eta\in(0,1)$, and $\boldsymbol{\mathcal{L}}^\varepsilon$ invertible and using Lemma 3.14, we can apply the Neumann series, whenever ε is small enough, and thus we have

$$(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}=\left(\boldsymbol{\mathcal{I}}+\boldsymbol{\mathcal{S}}^{\delta k,\varepsilon}\right)^{-1}(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}=\sum_{l=0}^{\infty}(-\boldsymbol{\mathcal{S}}^{\delta k,\varepsilon})^l(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}, \tag{4.23}$$

where $\boldsymbol{\mathcal{I}}$ denotes the identity function in $\boldsymbol{\mathcal{X}}^\varepsilon$.

We then define the $\mathbb{R}^{2\times2}$-matrix $A^{\delta k,\varepsilon}$ as

$$(A^{\delta k,\varepsilon})_{j,\hat{j}}:=-\frac{1}{|D_{\hat{j}}|}\left((\boldsymbol{\mathcal{S}}^{\delta k,\varepsilon})^{-1}[\mathbf{e}_j]\,,\mathbf{e}_{\hat{j}}\right)_{\varepsilon\oplus\varepsilon}=-\frac{\pi}{2|D_{\hat{j}}|}((\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}_{j,\hat{j}}[1]\,,1)_\varepsilon\,, \tag{4.24}$$

for $j,\hat{j}\in\{1,2\}$.

Lemma 4.12 *Any characteristic value of $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ is a characteristic value of the $\mathbb{R}^{2\times2}$-matrix $\delta^2k^2\mathbb{I}_2-A^{\delta k,\varepsilon}$.*

Proof Suppose $(\delta k)^2$ is a characteristic value of $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$. Substituting $\boldsymbol{\mathcal{L}}^\varepsilon$ with $\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon}$ in Lemma 4.11, we readily see that

$$\frac{2}{\pi}\boldsymbol{\mu}+\frac{(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}\boldsymbol{\mathcal{K}}^\varepsilon[\boldsymbol{\mu}]}{(\delta k)^2}=\mathbf{0}\,. \tag{4.25}$$

Applying $(\cdot\,,\mathbf{e}_1)_{\varepsilon\oplus\varepsilon}$ on both sides, we obtain

$$\left(\frac{2}{\pi}(\mu_1\,,1)_\varepsilon+\frac{(\mu_1\,,1)_\varepsilon}{|D_1|}\frac{((\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}_{1,1}[1]\,,1)_\varepsilon}{(\delta k)^2}+\frac{(\mu_2\,,1)_\varepsilon}{|D_2|}\frac{((\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}_{1,2}[1]\,,1)_\varepsilon}{(\delta k)^2}\right)=0\,. \tag{4.26}$$

Thus,

$$(\delta k)^2(\mu_1\,,1)_\varepsilon=-\frac{\pi\,(\mu_1\,,1)_\varepsilon}{2|D_1|}((\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}_{1,1}[1]\,,1)_\varepsilon-\frac{\pi\,(\mu_2\,,1)_\varepsilon}{2|D_2|}((\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}_{1,2}[1]\,,1)_\varepsilon\,.$$

Analogously,

$$(\delta k)^2(\mu_2\,,1)_\varepsilon=-\frac{\pi\,(\mu_1\,,1)_\varepsilon}{2|D_1|}((\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}_{2,1}[1]\,,1)_\varepsilon-\frac{\pi\,(\mu_2\,,1)_\varepsilon}{2|D_2|}((\mathcal{L}^\varepsilon)^{-1}\mathcal{R}^{\delta k,\varepsilon}_{2,2}[1]\,,1)_\varepsilon\,.$$

Thus,

$$(\delta k)^2\begin{bmatrix}(\mu_1\,,1)_\varepsilon\\(\mu_2\,,1)_\varepsilon\end{bmatrix}=\boldsymbol{A}^{\delta k,\varepsilon}\begin{bmatrix}(\mu_1\,,1)_\varepsilon\\(\mu_2\,,1)_\varepsilon\end{bmatrix}\,.$$

Hence, if $(\delta k)^2$ is a characteristic value of $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ then it also is a characteristic value of $(\delta k)^2-\boldsymbol{A}^{\delta k,\varepsilon}$. ■

We define

$$\mathfrak{c}_\varepsilon:=\frac{-1}{\log(\varepsilon/2)}\,.$$

Proposition 4.13 *There exist four characteristic values, counting multiplicity, for the operator $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ function in $\mathbb{K}_\mathbb{C}$. Moreover, they have the asymptotic*

$$\delta k=\pm\delta k_0^{\delta,\varepsilon}+\delta\mathcal{O}(\mathfrak{c}_\varepsilon),\qquad\qquad \text{for } \varepsilon\to 0\,.$$

Proof Recall that the operator-valued analytic function $\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon}$ is finitely meromorphic and of Fredholm type. Moreover, it has four characteristic values $\pm k_{j,0}^{\delta k,\varepsilon}$, and has a pole at 0 with order two in $\mathbb{K}_\mathbb{C}$. Thus, the multiplicity of $\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon}$ is 2 in $\mathbb{K}_\mathbb{C}$. Note that for $\delta k\in\mathbb{K}_\mathbb{C}\setminus\{0,\pm\delta k_{1,0}^{\delta,\varepsilon},\pm\delta k_{2,0}^{\delta\varepsilon}\}$, the operator $\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon}$ is invertible, because it is of Fredholm type and because it is injective due to Lemma 4.11. With that,

$$(\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon})^{-1}\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}=\frac{2}{\pi}\left(\frac{2}{\pi}\boldsymbol{\mathcal{I}}-\frac{(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}\boldsymbol{\mathcal{K}}^\varepsilon}{\delta^2k^2}\right)^{-1}\boldsymbol{\mathcal{S}}^{\delta k,\varepsilon}\,.$$

Thus, $\left\|(\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon})^{-1}\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}\right\|_{\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)}=\mathcal{O}(\mathfrak{c}_\varepsilon)$ uniformly for $\delta k\in\partial\mathbb{K}_\mathbb{C}$. By the generalized Rouché's theorem [8, Theorem 1.15], we can conclude that for ε sufficiently small, the operator $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ has the same multiplicity as the operator $\boldsymbol{\mathcal{Q}}^{\delta k,\varepsilon}$ in $\mathbb{K}_\mathbb{C}$, which is 2. Since $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ has a pole of order two, we derive that $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ has four characteristic values counting multiplicity. This completes the proof of the proposition. ■

Let us give an asymptotic expression for those characteristic values.
Let $l \geq 1$ be an integer, we define the $\mathbb{R}^{2\times 2}$-matrix $\boldsymbol{S}_{(l)}^{\delta k,\varepsilon}$ as

$$(\boldsymbol{S}_{(l)}^{\delta k,\varepsilon})_{j,\hat{j}} := -\frac{\pi}{2|D_{\hat{j}}|}\left((-\boldsymbol{\mathcal{S}}^{\delta k,\varepsilon})^l(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[\mathbf{e}_j]\,,\mathbf{e}_{\hat{j}}\right)_{\varepsilon\oplus\varepsilon}\,,$$

for $j,\hat{j}\in\{1,2\}$. Because of (4.23), we can write

$$\boldsymbol{A}^{\delta k,\varepsilon} = \frac{\pi}{2}\mathfrak{c}_\varepsilon\begin{bmatrix}\frac{1}{|D_1|} & 0\\ 0 & \frac{1}{|D_2|}\end{bmatrix} + \sum_{l=1}^{\infty}\boldsymbol{S}_{(l)}^{\delta k,\varepsilon}\,.$$

We want to give a second order analytic expression for $(\boldsymbol{S}_{(1)}^{\delta k,\varepsilon})_{j,\hat{j}} = \frac{\pi}{2|D_{\hat{j}}|}\left(\frac{\pi}{2}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[\mathbf{e}_j]\,,\mathbf{e}_{\hat{j}}\right)_{\varepsilon\oplus\varepsilon}$. To this end, we define

$$(\boldsymbol{S}_{(1,1)}^{\delta k,\varepsilon})_{j,\hat{j}} = \frac{\pi}{2|D_{\hat{j}}|}\left(\frac{\pi}{2}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}\boldsymbol{\mathcal{R}}_1^{\delta k,\varepsilon}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[\mathbf{e}_j]\,,\mathbf{e}_{\hat{j}}\right)_{\varepsilon\oplus\varepsilon}\,, \tag{4.27}$$

$$(\boldsymbol{S}_{(1,2)}^{\delta k,\varepsilon})_{j,\hat{j}} = \frac{\pi}{2|D_{\hat{j}}|}\left(\frac{\pi}{2}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}\boldsymbol{\mathcal{R}}_2^{\delta k,\varepsilon}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[\mathbf{e}_j]\,,\mathbf{e}_{\hat{j}}\right)_{\varepsilon\oplus\varepsilon}\,, \tag{4.28}$$

$$(\boldsymbol{S}_{(1,3)}^{\delta k,\varepsilon})_{j,\hat{j}} = \frac{\pi}{2|D_{\hat{j}}|}\left(\frac{\pi}{2}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}\boldsymbol{\mathcal{R}}_3^{\delta k,\varepsilon}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[\mathbf{e}_j]\,,\mathbf{e}_{\hat{j}}\right)_{\varepsilon\oplus\varepsilon}\,, \tag{4.29}$$

where

$$\boldsymbol{\mathcal{R}}_1^{\delta k,\varepsilon}[\boldsymbol{\mu}](\tau) := \begin{bmatrix}(\boldsymbol{\alpha}_{(\mathbf{0})})_{1,1}(\mu_1\,,1)_\varepsilon & (\mathbf{T}_{(\mathbf{0})})_{1,2}(\mu_2\,,1)_\varepsilon\\ (\mathbf{T}_{(\mathbf{0})})_{2,1}(\mu_1\,,1)_\varepsilon & (\boldsymbol{\alpha}_{(\mathbf{0})})_{2,2}(\mu_2\,,1)_\varepsilon\end{bmatrix}\,, \tag{4.30}$$

$$\boldsymbol{\mathcal{R}}_2^{\delta k,\varepsilon}[\boldsymbol{\mu}](\tau) := \delta k\begin{bmatrix}(\boldsymbol{\alpha}_{(\mathbf{1})})_{1,1}(\mu_1\,,1)_\varepsilon & (\mathbf{T}_{(\mathbf{1})})_{1,2}(\mu_2\,,1)_\varepsilon\\ (\mathbf{T}_{(\mathbf{1})})_{2,1}(\mu_1\,,1)_\varepsilon & (\boldsymbol{\alpha}_{(\mathbf{1})})_{2,2}(\mu_2\,,1)_\varepsilon\end{bmatrix}\,, \tag{4.31}$$

$$(\boldsymbol{\mathcal{R}}_3^{\delta k,\varepsilon})_{j,\hat{j}}[\mu_{\hat{j}}](\tau) := \int_{-\varepsilon}^{\varepsilon}\mu_{\hat{j}}(t)\left(t\widetilde{\partial}_t R_{j,\hat{j}}^{\delta k}(\tau,t) + \tau\widetilde{\partial}_\tau R_{j,\hat{j}}^{\delta k}(\tau,t) + \delta^2k^2\widetilde{\partial}_{\delta k}^2 R_{j,\hat{j}}^{\delta k}(\tau,t)\right)\mathrm{d}t\,. \tag{4.32}$$

Here, $R_{j,\hat{j}}^{\delta k}(\tau,t)$ denotes the kernel of $(\boldsymbol{\mathcal{R}}^{\delta k,\varepsilon})_{j,\hat{j}}$, see Definition 4.9, and $\widetilde{\partial}$ denotes the derivative part of the remainder in Taylor's theorem in the Peano form and where

$$\begin{aligned}(\boldsymbol{\alpha}_{(\mathbf{0})})_{1,1} :=& \mathrm{R}^0_{\partial D_1}\left(\begin{pmatrix}-\xi\\ h\end{pmatrix},\begin{pmatrix}-\xi\\ h\end{pmatrix}\right) + \mathrm{R}^0_{\partial\boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}-\xi\\ h\end{pmatrix},\begin{pmatrix}-\xi\\ h\end{pmatrix}\right)\\ &+\frac{1}{\pi}\log\left(\frac{\pi}{p}\right) - \frac{1}{2\pi}\log\left(\sinh\left(\frac{\pi}{p}2\,h\right)^2\right)\,,\end{aligned} \tag{4.33}$$

$$\begin{aligned}(\boldsymbol{\alpha}_{(\mathbf{0})})_{2,2} :=& \mathrm{R}^0_{\partial D_2}\left(\begin{pmatrix}\xi\\ h\end{pmatrix},\begin{pmatrix}\xi\\ h\end{pmatrix}\right) + \mathrm{R}^0_{\partial\boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}\xi\\ h\end{pmatrix},\begin{pmatrix}\xi\\ h\end{pmatrix}\right)\\ &+\frac{1}{\pi}\log\left(\frac{\pi}{p}\right) - \frac{1}{2\pi}\log\left(\sinh\left(\frac{\pi}{p}2\,h\right)^2\right)\,,\end{aligned} \tag{4.34}$$

$$(\boldsymbol{\alpha}_{(\mathbf{1})})_{1,1} := \partial_{\delta k} \mathrm{R}^0_{\partial D_1}\left(\begin{pmatrix}-\xi\\ h\end{pmatrix}, \begin{pmatrix}-\xi\\ h\end{pmatrix}\right) + \partial_{\delta k} \mathrm{R}^0_{\partial \boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}-\xi\\ h\end{pmatrix}, \begin{pmatrix}-\xi\\ h\end{pmatrix}\right), \tag{4.35}$$

$$(\boldsymbol{\alpha}_{(\mathbf{1})})_{2,2} := \partial_{\delta k} \mathrm{R}^0_{\partial D_2}\left(\begin{pmatrix}\xi\\ h\end{pmatrix}, \begin{pmatrix}\xi\\ h\end{pmatrix}\right) + \partial_{\delta k} \mathrm{R}^0_{\partial \boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}\xi\\ h\end{pmatrix}, \begin{pmatrix}\xi\\ h\end{pmatrix}\right), \tag{4.36}$$

$$(\mathbf{T}_{(\mathbf{0})})_{1,2} := \mathrm{N}^0_{\partial \boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}-\xi\\ h\end{pmatrix}, \begin{pmatrix}+\xi\\ h\end{pmatrix}\right), \tag{4.37}$$

$$(\mathbf{T}_{(\mathbf{0})})_{2,1} := \mathrm{N}^0_{\partial \boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}+\xi\\ h\end{pmatrix}, \begin{pmatrix}-\xi\\ h\end{pmatrix}\right), \tag{4.38}$$

$$(\mathbf{T}_{(\mathbf{1})})_{1,2} := \partial_{\delta k} \mathrm{N}^0_{\partial \boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}-\xi\\ h\end{pmatrix}, \begin{pmatrix}+\xi\\ h\end{pmatrix}\right), \tag{4.39}$$

$$(\mathbf{T}_{(\mathbf{1})})_{2,1} := \partial_{\delta k} \mathrm{N}^0_{\partial \boldsymbol{\Omega}^1,+}\left(\begin{pmatrix}+\xi\\ h\end{pmatrix}, \begin{pmatrix}-\xi\\ h\end{pmatrix}\right). \tag{4.40}$$

We define

$$\begin{aligned}
\boldsymbol{\alpha}_{(\mathbf{0})} &:= \begin{bmatrix} (\boldsymbol{\alpha}_{(\mathbf{0})})_{1,1}/|D_1| & 0 \\ 0 & (\boldsymbol{\alpha}_{(\mathbf{0})})_{2,2}/|D_2| \end{bmatrix}, \\
\mathbf{T}_{(\mathbf{0})} &:= \begin{bmatrix} 0 & (\mathbf{T}_{(\mathbf{0})})_{1,2}/|D_2| \\ (\mathbf{T}_{(\mathbf{0})})_{2,1}/|D_1| & 0 \end{bmatrix}, \\
\boldsymbol{\alpha}_{(\mathbf{1})} &:= \begin{bmatrix} (\boldsymbol{\alpha}_{(\mathbf{1})})_{1,1}/|D_1| & 0 \\ 0 & (\boldsymbol{\alpha}_{(\mathbf{1})})_{2,2}/|D_2| \end{bmatrix}, \\
\mathbf{T}_{(\mathbf{1})} &:= \begin{bmatrix} 0 & (\mathbf{T}_{(\mathbf{1})})_{1,2}/|D_2| \\ (\mathbf{T}_{(\mathbf{1})})_{2,1}/|D_1| & 0 \end{bmatrix}.
\end{aligned}$$

We obtain the second order analytic expression with the following lemma:

Lemma 4.14 *For all $l \in \mathbb{N}$, we have that*

$$\boldsymbol{S}^{\delta k,\varepsilon}_{(l)} = \mathcal{O}(\mathfrak{c}_\varepsilon^{l+1}),$$

and for $\boldsymbol{S}^{\delta k,\varepsilon}_{(1,1)}$, $\boldsymbol{S}^{\delta k,\varepsilon}_{(1,2)}$ and $\boldsymbol{S}^{\delta k,\varepsilon}_{(1,3)}$, we have

$$\begin{aligned}
\boldsymbol{S}^{\delta k,\varepsilon}_{(1,1)} &= \frac{\pi^2}{2}\mathfrak{c}_\varepsilon^2(\boldsymbol{\alpha}_{(\mathbf{0})} + \mathbf{T}_{(\mathbf{0})}), \\
\boldsymbol{S}^{\delta k,\varepsilon}_{(1,2)} &= \delta k\, \frac{\pi^2}{2}\mathfrak{c}_\varepsilon^2(\boldsymbol{\alpha}_{(\mathbf{1})} + \mathbf{T}_{(\mathbf{1})}), \\
\boldsymbol{S}^{\delta k,\varepsilon}_{(1,3)} &= \mathcal{O}(\varepsilon\, \mathfrak{c}_\varepsilon^2 + \mathfrak{c}_\varepsilon^2\, \delta^2 k^2).
\end{aligned}$$

Proof The proof follows from straightforward calculation using Lemma 3.14 and the expressions in (4.30)-(4.32) and (4.27)-(4.29). For $\boldsymbol{S}^{\delta k,\varepsilon}_{(1,3)}$, we can use the argument in (3.35). ■

Now we can deduce that

$$A^{\delta k,\varepsilon} = \frac{\pi}{2}\mathfrak{c}_\varepsilon \begin{bmatrix} \frac{1}{|D_1|} & 0 \\ 0 & \frac{1}{|D_2|} \end{bmatrix} + \frac{\pi^2}{4}\mathfrak{c}_\varepsilon^2(\boldsymbol{\alpha}_{(0)} + \mathbf{T}_{(0)}) + \delta k \frac{\pi^2}{4}\mathfrak{c}_\varepsilon^2(\boldsymbol{\alpha}_{(1)} + \mathbf{T}_{(1)}) + \mathcal{O}(\varepsilon\,\mathfrak{c}_\varepsilon^2 + \mathfrak{c}_\varepsilon^2\,\delta^2 k^2)\,. \tag{4.41}$$

Lemma 4.15 *There exists a $\mathbb{R}^{2\times 2}$-matrix $A_\star^{\delta k,\varepsilon}$ such that $A^{\delta k,\varepsilon} = (A_\star^{\delta k,\varepsilon})^2$ and*

$$A_\star^{\delta k,\varepsilon} = \sqrt{\frac{\pi}{2}\mathfrak{c}_\varepsilon}\left(A_{\star,D} + \frac{\pi\mathfrak{c}_\varepsilon}{2}A_{\star,(0)} + \frac{\delta k\pi\mathfrak{c}_\varepsilon}{2}A_{\star,(1)}\right) + \mathcal{O}(\mathfrak{c}_\varepsilon^{3/2}\,\delta^2 k^2) + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2})\,,$$

where

$$A_{\star,D} = \begin{bmatrix} \frac{1}{\sqrt{|D_1|}} & 0 \\ 0 & \frac{1}{\sqrt{|D_2|}} \end{bmatrix},$$

$$A_{\star,(0)} = \begin{bmatrix} \frac{(\boldsymbol{\alpha}_{(0)})_{1,1}}{2\sqrt{|D_1|}} & \frac{(\mathbf{T}_{(0)})_{1,2}\sqrt{|D_1|/|D_2|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} \\ \frac{(\mathbf{T}_{(0)})_{2,1}\sqrt{|D_2|/|D_1|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} & \frac{(\boldsymbol{\alpha}_{(0)})_{2,2}}{2\sqrt{|D_2|}} \end{bmatrix},$$

$$A_{\star,(1)} = \begin{bmatrix} \frac{(\boldsymbol{\alpha}_{(1)})_{1,1}}{2\sqrt{|D_1|}} & \frac{(\mathbf{T}_{(1)})_{1,2}\sqrt{|D_1|/|D_2|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} \\ \frac{(\mathbf{T}_{(1)})_{2,1}\sqrt{|D_2|/|D_1|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} & \frac{(\boldsymbol{\alpha}_{(1)})_{2,2}}{2\sqrt{|D_2|}} \end{bmatrix}.$$

Proof We use the following approach:

$$A_\star^{\delta k,\varepsilon} = \mathfrak{c}_\varepsilon^{1/2}A_{\star,1}^{\delta k} + \mathfrak{c}_\varepsilon A_{\star,2}^{\delta k} + \mathfrak{c}_\varepsilon^{3/2}A_{\star,3}^{\delta k} + \mathfrak{c}_\varepsilon^2 A_{\star,4}^{\delta k} + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}).$$

Thus,

$$\begin{aligned} A^{\delta k,\varepsilon} = (A_\star^{\delta k,\varepsilon})^2 &= (\mathfrak{c}_\varepsilon^{1/2}A_{\star,1}^{\delta k} + \mathfrak{c}_\varepsilon A_{\star,2}^{\delta k} + \mathfrak{c}_\varepsilon^{3/2}A_{\star,3}^{\delta k} + \mathfrak{c}_\varepsilon^2 A_{\star,4}^{\delta k} + \mathcal{O}(\mathfrak{c}_\varepsilon^2))^2 \\ &= (A_{\star,1}^{\delta k})^2\mathfrak{c}_\varepsilon + (A_{\star,1}^{\delta k}A_{\star,2}^{\delta k} + A_{\star,2}^{\delta k}A_{\star,1}^{\delta k})\mathfrak{c}_\varepsilon^{3/2} + ((A_{\star,2}^{\delta k})^2 + A_{\star,1}^{\delta k}A_{\star,3}^{\delta k} + A_{\star,3}^{\delta k}A_{\star,1}^{\delta k})\mathfrak{c}_\varepsilon^2 \\ &+ (A_{\star,4}^{\delta k}A_{\star,1}^{\delta k} + A_{\star,1}^{\delta k}A_{\star,4}^{\delta k} + A_{\star,2}^{\delta k}A_{\star,3}^{\delta k} + A_{\star,3}^{\delta k}A_{\star,2}^{\delta k}) + \mathcal{O}(\mathfrak{c}_\varepsilon^3). \end{aligned}$$

Comparing this equation to (4.41), it follows that

$$\begin{aligned} (A_{\star,1}^{\delta k})^2 &= \frac{\pi}{2}\begin{bmatrix} \frac{1}{|D_1|} & 0 \\ 0 & \frac{1}{|D_2|} \end{bmatrix}, \\ (A_{\star,1}^{\delta k}A_{\star,2}^{\delta k} + A_{\star,2}^{\delta k}A_{\star,1}^{\delta k}) &= 0\,, \\ ((A_{\star,2}^{\delta k})^2 + A_{\star,1}^{\delta k}A_{\star,3}^{\delta k} + A_{\star,3}^{\delta k}A_{\star,1}^{\delta k}) &= \frac{\pi^2}{4}\left((\boldsymbol{\alpha}_{(0)} + \mathbf{T}_{(0)}) + \delta k(\boldsymbol{\alpha}_{(1)} + \mathbf{T}_{(1)})\right) + \mathcal{O}(\delta^2 k^2)\,, \\ (A_{\star,4}^{\delta k}A_{\star,1}^{\delta k} + A_{\star,1}^{\delta k}A_{\star,4}^{\delta k} + A_{\star,2}^{\delta k}A_{\star,3}^{\delta k} + A_{\star,3}^{\delta k}A_{\star,2}^{\delta k}) &= 0\,, \end{aligned}$$

which implies

$$
\begin{aligned}
\boldsymbol{A}^{\delta k}_{\star,1} =& \sqrt{\frac{\pi}{2}} \begin{bmatrix} \frac{1}{\sqrt{|D_1|}} & 0 \\ 0 & \frac{1}{\sqrt{|D_2|}} \end{bmatrix}, \\
\boldsymbol{A}^{\delta k}_{\star,2} =& 0, \\
\boldsymbol{A}^{\delta k}_{\star,3} =& \frac{\pi^{\frac{3}{2}}}{2\sqrt{2}} \begin{bmatrix} \frac{(\boldsymbol{\alpha}_{(\mathbf{0})})_{1,1}+\delta k(\boldsymbol{\alpha}_{(\mathbf{1})})_{1,1}}{2\sqrt{|D_1|}} & \frac{((\mathbf{T}_{(\mathbf{0})})_{1,2}+\delta k(\mathbf{T}_{(\mathbf{1})})_{1,2})\sqrt{|D_1|/|D_2|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} \\ \frac{((\mathbf{T}_{(\mathbf{0})})_{2,1}+\delta k(\mathbf{T}_{(\mathbf{1})})_{2,1})\sqrt{|D_2|/|D_1|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} & \frac{(\boldsymbol{\alpha}_{(\mathbf{0})})_{2,2}+\delta k(\boldsymbol{\alpha}_{(\mathbf{1})})_{2,2}}{2\sqrt{|D_2|}} \end{bmatrix} + \mathcal{O}(\delta^2 k^2), \\
\boldsymbol{A}^{\delta k}_{\star,4} =& 0.
\end{aligned}
$$

This leads us to

$$
\begin{aligned}
\boldsymbol{A}^{\delta k,\varepsilon}_{\star} =& \sqrt{\frac{\pi}{2}\mathfrak{c}_\varepsilon} \left(\begin{bmatrix} \frac{1}{\sqrt{|D_1|}} & 0 \\ 0 & \frac{1}{\sqrt{|D_2|}} \end{bmatrix} + \frac{\pi\mathfrak{c}_\varepsilon}{2} \begin{bmatrix} \frac{(\boldsymbol{\alpha}_{(\mathbf{0})})_{1,1}}{2\sqrt{|D_1|}} & \frac{(\mathbf{T}_{(\mathbf{0})})_{1,2}\sqrt{|D_1|/|D_2|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} \\ \frac{(\mathbf{T}_{(\mathbf{0})})_{2,1}\sqrt{|D_2|/|D_1|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} & \frac{(\boldsymbol{\alpha}_{(\mathbf{0})})_{2,2}}{2\sqrt{|D_2|}} \end{bmatrix} \right. \\
& \left. + \frac{\delta k\,\pi\mathfrak{c}_\varepsilon}{2} \begin{bmatrix} \frac{(\boldsymbol{\alpha}_{(\mathbf{1})})_{1,1}}{2\sqrt{|D_1|}} & \frac{(\mathbf{T}_{(\mathbf{1})})_{1,2}\sqrt{|D_1|/|D_2|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} \\ \frac{(\mathbf{T}_{(\mathbf{1})})_{2,1}\sqrt{|D_2|/|D_1|}}{\sqrt{|D_1|}+\sqrt{|D_2|}} & \frac{(\boldsymbol{\alpha}_{(\mathbf{1})})_{2,2}}{2\sqrt{|D_2|}} \end{bmatrix} \right) + \mathcal{O}(\mathfrak{c}_\varepsilon^{3/2}\,\delta^2 k^2) + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}).
\end{aligned}
\tag{4.42}
$$

■

With $\boldsymbol{A}^{\delta k,\varepsilon}_{\star}$ we can write $\delta^2 k^2 \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon} = (\delta k \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}_{\star})(\delta k \mathbb{I}_2 + \boldsymbol{A}^{\delta k,\varepsilon}_{\star})$, thus it is enough to find the characteristic values of $(\delta k \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}_{\star})$ and $(\delta k \mathbb{I}_2 + \boldsymbol{A}^{\delta k,\varepsilon}_{\star})$ to get the characteristic values of $\delta^2 k^2 \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}$.

We define

$$
\boldsymbol{A}^{\varepsilon}_{\star,\star} := \sqrt{\frac{\pi}{2}\mathfrak{c}_\varepsilon} \left(\boldsymbol{A}_{\star,D} + \frac{\pi\mathfrak{c}_\varepsilon}{2} \boldsymbol{A}_{\star,(0)} \right). \tag{4.43}
$$

Consider that $\boldsymbol{A}^{\delta k,\varepsilon}_{\star} = \boldsymbol{A}^{\varepsilon}_{\star,\star} + \frac{\delta k(\pi\mathfrak{c}_\varepsilon)^{3/2}}{2}\boldsymbol{A}_{\star,(1)} + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2} + \mathfrak{c}_\varepsilon^{3/2}\,\delta^2 k^2)$

Now for ε small enough, $\boldsymbol{A}^{\varepsilon}_{\star,\star}$ is diagonalizable with $\boldsymbol{A}^{\varepsilon}_{\star,\star} = \mathbf{Y}^{\varepsilon}_{\star,\star}\mathbf{M}^{\varepsilon}_{\star,\star}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}$ where $\mathbf{Y}^{\varepsilon}_{\star,\star} = [(\mathrm{Y}^{\varepsilon}_{\star,\star})_1, (\mathrm{Y}^{\varepsilon}_{\star,\star})_2]$, with $(\mathrm{Y}^{\varepsilon}_{\star,\star})_1$ being the normalized eigenvector to the eigenvalue $\delta k^{\delta,\varepsilon}_{\star,1}$ of $\boldsymbol{A}^{\varepsilon}_{\star,\star}$ and $(\mathrm{Y}^{\varepsilon}_{\star,\star})_2$ being the normalized eigenvector to the eigenvalue $\delta k^{\delta,\varepsilon}_{\star,2}$ of $\boldsymbol{A}^{\varepsilon}_{\star,\star}$, and for ε small enough, but not zero, $\delta k^{\delta,\varepsilon}_{\star,1}$ and $\delta k^{\delta,\varepsilon}_{\star,2}$ are distinct, since $\boldsymbol{A}_{\star,(0)}$ is not zero and thus $\boldsymbol{A}^{\varepsilon}_{\star,\star}$ is not similar to $\mathbb{I}_2$. Then we sort $\delta k^{\delta,\varepsilon}_{\star,1}$ and $\delta k^{\delta,\varepsilon}_{\star,2}$ such that the real part of $\delta k^{\delta,\varepsilon}_{\star,1}$ is greater or equal to $\delta k^{\delta,\varepsilon}_{\star,2}$ and

$$
\mathbf{M}^{\varepsilon}_{\star,\star} = \begin{bmatrix} \delta k^{\delta,\varepsilon}_{\star,1} & 0 \\ 0 & \delta k^{\delta,\varepsilon}_{\star,2} \end{bmatrix}.
$$

We can explicitly compute those values:

Lemma 4.16 *We have that*

$$\delta k^{\delta,\varepsilon}_{\star,1} = \sqrt{\pi \mathfrak{c}_\varepsilon}\left(\frac{1}{2}\left(k^{\varepsilon}_{\star,\mathrm{tr}1} + k^{\varepsilon}_{\star,\mathrm{tr}2}\right) + \left[\frac{1}{4}\left(k^{\varepsilon}_{\star,\mathrm{tr}1} - k^{\varepsilon}_{\star,\mathrm{tr}2}\right)^2 - k^{\varepsilon}_{\star,\mathrm{det}}\right]^{1/2}\right),$$
$$\delta k^{\delta,\varepsilon}_{\star,2} = \sqrt{\pi \mathfrak{c}_\varepsilon}\left(\frac{1}{2}\left(k^{\varepsilon}_{\star,\mathrm{tr}1} + k^{\varepsilon}_{\star,\mathrm{tr}2}\right) - \left[\frac{1}{4}\left(k^{\varepsilon}_{\star,\mathrm{tr}1} - k^{\varepsilon}_{\star,\mathrm{tr}2}\right)^2 - k^{\varepsilon}_{\star,\mathrm{det}}\right]^{1/2}\right),$$

where

$$k^{\varepsilon}_{\star,\mathrm{tr}1} = \frac{1}{\sqrt{|D_1|}}\left(1 + \frac{(\boldsymbol{\alpha}_{(\mathbf{0})})_{1,1}\pi \mathfrak{c}_\varepsilon}{4}\right),$$
$$k^{\varepsilon}_{\star,\mathrm{tr}2} = \frac{1}{\sqrt{|D_2|}}\left(1 + \frac{(\boldsymbol{\alpha}_{(\mathbf{0})})_{2,2}\pi \mathfrak{c}_\varepsilon}{4}\right),$$
$$k^{\varepsilon}_{\star,\mathrm{det}} = \frac{\pi^2 \mathfrak{c}_\varepsilon^2}{4}\frac{(\mathbf{T}_{(\mathbf{0})})_{1,2}(\mathbf{T}_{(\mathbf{0})})_{2,1}}{(\sqrt{|D_1|} + \sqrt{|D_2|})}.$$

With this lemma we especially see that $\delta k^{\delta,\varepsilon}_{\star,1} = \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})$ and $\delta k^{\delta,\varepsilon}_{\star,2} = \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})$.

Proposition 4.17 *There exists exactly 2 characteristic values for each of the matrix-valued functions $\delta k\mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}_\star$ and $\delta k\mathbb{I}_2 + \boldsymbol{A}^{\delta k,\varepsilon}_\star$. For $j \in \{1,2\}$, these are*

$$\delta k^{\delta,\varepsilon}_{j,+} = \delta k^{\delta,\varepsilon}_{\star,j}\left(1 + \frac{(\pi \mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\mathrm{e}_j^{\mathrm{T}}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\boldsymbol{A}_{\star,(1)}(\mathrm{Y}^{\varepsilon}_{\star,\star})_j\right) + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}), \tag{4.44}$$

$$\delta k^{\delta,\varepsilon}_{j,-} = \delta k^{\delta,\varepsilon}_{\star,j}\left(-1 + \frac{(\pi \mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\mathrm{e}_j^{\mathrm{T}}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\boldsymbol{A}_{\star,(1)}(\mathrm{Y}^{\varepsilon}_{\star,\star})_j\right) + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}). \tag{4.45}$$

Proof Step 1: Non-perturbed characteristic values. Let us find the characteristic values and the corresponding vectors for the matrix-valued function $\delta k - \boldsymbol{A}^{\delta k,\varepsilon}_{\star,\star}$. By definition,

$$(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}(\delta k\mathbb{I}_2 - \boldsymbol{A}^{\varepsilon}_{\star,\star})\mathbf{Y}^{\varepsilon}_{\star,\star} = \delta k\mathbb{I}_2 - \mathbf{M}^{\varepsilon}_{\star,\star}.$$

Thus we see that the characteristic values of $(\delta k\mathbb{I}_2 - \boldsymbol{A}^{\varepsilon}_{\star,\star})$ are $k^{\delta,\varepsilon}_{\star,1}$ and $k^{\delta,\varepsilon}_{\star,2}$ with the characteristic vectors $(\mathrm{Y}^{\varepsilon}_{\star,\star})_1$ and $(\mathrm{Y}^{\varepsilon}_{\star,\star})_2$ and similarly we see that the characteristic values of $(\delta k\mathbb{I}_2 + \boldsymbol{A}^{\varepsilon}_{\star,\star})$ are $-k^{\delta,\varepsilon}_{\star,1}$ and $-k^{\delta,\varepsilon}_{\star,2}$ with the characteristic vectors $(\mathrm{Y}^{\varepsilon}_{\star,\star})_1$ and $(\mathrm{Y}^{\varepsilon}_{\star,\star})_2$.

Step 2: Existence of the perturbed characteristic values near the unperturbed ones. We now apply the generalized Rouché's theorem to obtain the existence of the characteristic values for $\delta k\mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}_\star$. Observe that $(\delta k\mathbb{I}_2 - \boldsymbol{A}^{\varepsilon}_{\star,\star})^{-1} = \mathbf{Y}^{\varepsilon}_{\star,\star}(\delta k\mathbb{I}_2 - \mathbf{M}^{\varepsilon}_{\star,\star})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}$, where $\left\|\mathbf{Y}^{\varepsilon}_{\star,\star}\right\| = \mathcal{O}(1)$, because of the normalization, thus $\left\|(\delta k^{\delta,\varepsilon}_{\star,j}\mathbb{I}_2 - \boldsymbol{A}^{\varepsilon}_{\star,\star})^{-1}\right\| = \mathcal{O}(\mathfrak{c}_\varepsilon^{-\frac{1}{2}})$. We define the domains $W_1 := \{k \in \mathbb{K}_{\mathbb{C}} \mid |k - k^{\delta,\varepsilon}_{\star,1}| < C_1\mathfrak{c}_\varepsilon^2\}$ and $W_2 := \{k \in \mathbb{K}_{\mathbb{C}} \mid$

$|k - k^{\delta,\varepsilon}_{\star,2}| < C_2 \mathfrak{c}_\varepsilon^2\}$ where $C_1, C_2 > 0$. Since $\boldsymbol{A}^{\delta k,\varepsilon}_\star - \boldsymbol{A}^{\varepsilon}_{\star,\star} = \delta k \mathcal{O}(\mathfrak{c}_\varepsilon^{3/2}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2} + \mathfrak{c}_\varepsilon^{3/2}\,\delta^2 k^2)$ and $k^{\delta,\varepsilon}_{\star,1}$ and $k^{\delta,\varepsilon}_{\star,2}$ are pairwise different, we can conclude that for ε sufficiently small, there exists C_1 such that the following inequality holds

$$\left\| (\delta k \mathbb{I}_2 - \boldsymbol{A}^{\varepsilon}_{\star,\star})^{-1} (\boldsymbol{A}^{\delta k,\varepsilon}_\star - \boldsymbol{A}^{\varepsilon}_{\star,\star}) \right\| < 1 \quad \text{for } k \in \partial W_1\,,$$

and the same holds for C_2. Then the generalized Rouché's theorem yields that there exists exactly one characteristic value $k^{\delta,\varepsilon}_{1,+}$ and one $k^{\delta,\varepsilon}_{2,+}$ in the domain $W_1 \cup W_2$ for $\delta k \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}_\star$, thus $k^{\delta,\varepsilon}_{1,+} = k^{\delta,\varepsilon}_{\star,1} + \mathcal{O}(\mathfrak{c}_\varepsilon^2)$ and $k^{\delta,\varepsilon}_{2,+} = k^{\delta,\varepsilon}_{\star,2} + \mathcal{O}(\mathfrak{c}_\varepsilon^2)$. For ε small enough, $W_1 \cup W_2 \subset \mathbb{K}_{\mathbb{C}}$ and with Proposition 4.13, $k^{\delta,\varepsilon}_{1,+}$ and one $k^{\delta,\varepsilon}_{2,+}$ are two characteristic values of the four of $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ in $\mathbb{K}_{\mathbb{C}}$. We can apply the same argument for $\delta k + \boldsymbol{A}^{\delta k,\varepsilon}_\star$.

Step 3: Expansion of the characteristic vectors. Let Y^ε_1 and Y^ε_2 be the characteristic vectors to $\delta k - \boldsymbol{A}^{\delta k,\varepsilon}_\star$ to the characteristic values $k^{\delta,\varepsilon}_{1,+}$ and $k^{\delta,\varepsilon}_{2,+}$. We show that

$$\mathrm{Y}^\varepsilon_j = (\mathrm{Y}^\varepsilon_{\star,\star})_j + \mathcal{O}(\sqrt{\mathfrak{c}_\varepsilon})$$

for $j \in \{1,2\}$. Indeed, note that

$$\delta k^{\delta,\varepsilon}_{j,+} \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}_\star \big|_{k = k^{\delta,\varepsilon}_{j,+}} = \delta k^{\delta,\varepsilon}_{j,+} \mathbb{I}_2 - \boldsymbol{A}^{\varepsilon}_{\star,\star} + \mathcal{O}(\mathfrak{c}_\varepsilon^2)\,,$$

since $\boldsymbol{A}^{\delta k,\varepsilon}_\star - \boldsymbol{A}^{\varepsilon}_{\star,\star} = \mathcal{O}(\delta k \mathfrak{c}_\varepsilon^{3/2}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2} + \mathfrak{c}_\varepsilon^{3/2}\,\delta^2 k^2)$ and $k^{\delta,\varepsilon}_{1,+} = k^{\delta,\varepsilon}_{\star,1} + \mathcal{O}(\mathfrak{c}_\varepsilon^2)$ and

$$k^{\delta,\varepsilon}_{2,+} = k^{\delta,\varepsilon}_{\star,2} + \mathcal{O}(\mathfrak{c}_\varepsilon^2).$$

Using that $\delta k - \boldsymbol{A}^{\varepsilon}_{\star,\star} = \mathbf{Y}^{\varepsilon}_{\star,\star}(\delta k \mathbb{I}_2 - \mathbf{M}^{\varepsilon}_{\star,\star})(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}$ we get

$$(\delta k^{\delta,\varepsilon}_{j,+} \mathbb{I}_2 - \mathbf{M}^{\varepsilon}_{\star,\star})(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1} \mathrm{Y}^\varepsilon_j = \mathcal{O}(\mathfrak{c}_\varepsilon^2)\,.$$

Using again that $k^{\delta,\varepsilon}_{1,+} = k^{\delta,\varepsilon}_{\star,1} + \mathcal{O}(\mathfrak{c}_\varepsilon^2)$ and $k^{\delta,\varepsilon}_{2,+} = k^{\delta,\varepsilon}_{\star,2} + \mathcal{O}(\mathfrak{c}_\varepsilon^2)$ and the definition of $\mathbf{M}^{\varepsilon}_{\star,\star}$ we see with $(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1} = ((\mathbf{Y}^{\varepsilon}_{\star,\star})^{\mathrm{T}}\,\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\,(\mathbf{Y}^{\varepsilon}_{\star,\star})^{\mathrm{T}}$ and $\left\|((\mathbf{Y}^{\varepsilon}_{\star,\star})^{\mathrm{T}}\,\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\right\| = \mathcal{O}(1)$ that

$$\begin{aligned}
&(\mathrm{Y}^\varepsilon_{\star,\star})^{\mathrm{T}}_1 \mathrm{Y}^\varepsilon_1 = \mathcal{O}(1)\,, && (\mathrm{Y}^\varepsilon_{\star,\star})^{\mathrm{T}}_1 \mathrm{Y}^\varepsilon_2 = \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})\,,\\
&(\mathrm{Y}^\varepsilon_{\star,\star})^{\mathrm{T}}_2 \mathrm{Y}^\varepsilon_1 = \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})\,, && (\mathrm{Y}^\varepsilon_{\star,\star})^{\mathrm{T}}_2 \mathrm{Y}^\varepsilon_2 = \mathcal{O}(1)\,.
\end{aligned}$$

It follows that Y^ε_j can be written as

$$\mathrm{Y}^\varepsilon_j = (\mathrm{Y}^\varepsilon_{\star,\star})_j + \mathfrak{c}_\varepsilon^{1/2} \mathrm{Y}^\varepsilon_{j,(1)} + \mathcal{O}(\mathfrak{c}_\varepsilon)\,. \tag{4.46}$$

Step 4: Expansions for the characteristic values

Using $k_{j,+}^{\delta,\varepsilon} = k_{\star,j}^{\delta,\varepsilon} + \mathcal{O}(\mathfrak{c}_\varepsilon^2)$, we obtain that

$$\begin{aligned}
\delta k_{j,+}^{\delta,\varepsilon} - \boldsymbol{A}_\star^{\delta k,\varepsilon}\,|_{k=k_{j,+}^{\delta,\varepsilon}} &= \delta k_{j,+}^{\delta,\varepsilon} - \boldsymbol{A}_{\star,\star}^{\varepsilon} + \boldsymbol{A}_{\star,\star}^{\varepsilon} - \boldsymbol{A}_\star^{\delta k,\varepsilon}\,|_{k=k_{j,+}^{\delta,\varepsilon}} \\
&= \mathbf{Y}_{\star,\star}^{\varepsilon}(\delta k_{j,+}^{\delta,\varepsilon}\mathbb{I}_2 - \mathbf{M}_{\star,\star}^{\varepsilon})(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1} - \delta k_{j,+}^{\delta,\varepsilon}\frac{(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\boldsymbol{A}_{\star,(1)} + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2} + \mathfrak{c}_\varepsilon^{3/2}\,\delta^2(k_{j,+}^{\delta,\varepsilon})^2) \\
&= \mathbf{Y}_{\star,\star}^{\varepsilon}(\delta k_{j,+}^{\delta,\varepsilon}\mathbb{I}_2 - \mathbf{M}_{\star,\star}^{\varepsilon})(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1} - \delta k_{\star,j}^{\delta,\varepsilon}\frac{(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\mathbf{Y}_{\star,\star}^{\varepsilon}(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\boldsymbol{A}_{\star,(1)}\mathbf{Y}_{\star,\star}^{\varepsilon}(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1} + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}) \\
&= \mathbf{Y}_{\star,\star}^{\varepsilon}\left(\delta k_{j,+}^{\delta,\varepsilon}\mathbb{I}_2 - \mathbf{M}_{\star,\star}^{\varepsilon} - \frac{\delta k_{\star,j}^{\delta,\varepsilon}(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\boldsymbol{A}_{\star,(1)}\mathbf{Y}_{\star,\star}^{\varepsilon}\right)(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1} + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2})\,.
\end{aligned}$$

Since $(\delta k_{j,+}^{\delta,\varepsilon} - \boldsymbol{A}_\star^{\delta k,\varepsilon}\,|_{k=k_{j,+}^{\delta,\varepsilon}})\mathbf{Y}_j^{\varepsilon} = 0$ and $\mathrm{Y}_j^{\varepsilon} = (\mathrm{Y}_{\star,\star}^{\varepsilon})_j + \mathfrak{c}_\varepsilon^{1/2}\mathrm{Y}_{j,(1)}^{\varepsilon} + \mathcal{O}(\mathfrak{c}_\varepsilon)$ we have

$$\begin{aligned}
\left(\delta k_{j,+}^{\delta,\varepsilon}\mathbb{I}_2 - \mathbf{M}_{\star,\star}^{\varepsilon} - \frac{\delta k_{\star,j}^{\delta,\varepsilon}(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\boldsymbol{A}_{\star,(1)}\mathbf{Y}_{\star,\star}^{\varepsilon}\right)(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\left((\mathrm{Y}_{\star,\star}^{\varepsilon})_j + \mathfrak{c}_\varepsilon^{1/2}\mathrm{Y}_{j,(1)}^{\varepsilon} + \mathcal{O}(\mathfrak{c}_\varepsilon)\right) \\
+ \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2}) = 0 \quad (4.47)
\end{aligned}$$

We can rewrite this as

$$\begin{aligned}
\left(\delta k_{j,+}^{\delta,\varepsilon}\mathbb{I}_2 - \mathbf{M}_{\star,\star}^{\varepsilon}\right)(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\left((\mathrm{Y}_{\star,\star}^{\varepsilon})_j + \mathfrak{c}_\varepsilon^{1/2}\mathrm{Y}_{j,(1)}^{\varepsilon} + \mathcal{O}(\mathfrak{c}_\varepsilon)\right) \\
= \frac{\delta k_{\star,j}^{\delta,\varepsilon}(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\boldsymbol{A}_{\star,(1)}(\mathrm{Y}_{\star,\star}^{\varepsilon})_j + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2})\,. \quad (4.48)
\end{aligned}$$

Consider that $(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}(\mathrm{Y}_{\star,\star}^{\varepsilon})_1 = \mathrm{e}_1 := [1,0]^{\mathrm{T}}$ and $(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}(\mathrm{Y}_{\star,\star}^{\varepsilon})_2 = \mathrm{e}_2 := [0,1]^{\mathrm{T}}$. This results in

$$\delta(k_{j,+}^{\delta,\varepsilon} - k_{\star,j}^{\delta,\varepsilon}) = \delta k_{\star,j}^{\delta,\varepsilon}\frac{(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\mathrm{e}_j\cdot(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\boldsymbol{A}_{\star,(1)}(\mathrm{Y}_{\star,\star}^{\varepsilon})_j + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2})\,.$$

Thus

$$\delta k_{j,+}^{\delta,\varepsilon} = \delta k_{\star,j}^{\delta,\varepsilon}\left(1 + \frac{(\pi\mathfrak{c}_\varepsilon)^{\frac{3}{2}}}{4}\mathrm{e}_j^{\mathrm{T}}(\mathbf{Y}_{\star,\star}^{\varepsilon})^{-1}\boldsymbol{A}_{\star,(1)}(\mathrm{Y}_{\star,\star}^{\varepsilon})_j\right) + \mathcal{O}(\mathfrak{c}_\varepsilon^{5/2})\,.$$

By a similar procedure, we can prove (4.45). ■

4.3.4 Inversion of $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$- Solving the First Order Linear Equation

We know now that $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ is invertible for $\delta k \in \mathbb{K}_{\mathbb{C}}$, except at the characteristic values $k = k_{j,+}^{\delta,\varepsilon}$ and $k = k_{j,-}^{\delta,\varepsilon}$ and at the pole, $k = 0$. Let us examine, how we can express $(\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}]$, where $\boldsymbol{f}^{\delta k}$ is given by (4.16), which uses (4.15).

First consider that we already know that the equation $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}[\boldsymbol{\mu}] = \boldsymbol{f}^{\delta k}$ is equivalent to

$$\frac{2}{\pi}\boldsymbol{\mu} + \frac{(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}\boldsymbol{\mathcal{K}}^{\varepsilon}[\boldsymbol{\mu}]}{(\delta k)^2} = (\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}]\,. \tag{4.49}$$

Similar to the proof of Lemma 4.12, we get that

$$(\delta^2 k^2 \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}) \begin{bmatrix} (\mu_1\,,1)_\varepsilon \\ (\mu_2\,,1)_\varepsilon \end{bmatrix} = \frac{\pi}{2}(\delta^2 k^2) \begin{bmatrix} ((\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}]\,,\mathbf{e}_1)_{\varepsilon\oplus\varepsilon} \\ ((\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}]\,,\mathbf{e}_2)_{\varepsilon\oplus\varepsilon} \end{bmatrix}. \tag{4.50}$$

We define

$$f^{\delta k}_{\mathcal{L}} := \begin{bmatrix} ((\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}]\,,\mathbf{e}_1)_{\varepsilon\oplus\varepsilon} \\ ((\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}]\,,\mathbf{e}_2)_{\varepsilon\oplus\varepsilon} \end{bmatrix}, \quad f^{\delta k}_{\mathbf{0}} := \begin{bmatrix} f^{\delta k}_{D_1}(0) \\ f^{\delta k}_{D_2}(0) \end{bmatrix}.$$

Lemma 4.18 *The inverse of $\delta^2 k^2 \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}$ has the following representation:*

$$(\delta^2 k^2 \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon})^{-1} = (\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1} + \mathbf{M}^{\delta k,\varepsilon}_{\text{rest}},$$

where

$$\mathbf{M}^{\delta k}_{+} = \begin{bmatrix} \delta k - \delta k^{\delta,\varepsilon}_{1,+} & 0 \\ 0 & \delta k - \delta k^{\delta,\varepsilon}_{2,+} \end{bmatrix}, \quad \mathbf{M}^{\delta k}_{-} = \begin{bmatrix} \delta k - \delta k^{\delta,\varepsilon}_{1,-} & 0 \\ 0 & \delta k - \delta k^{\delta,\varepsilon}_{2,-} \end{bmatrix},$$

$$\mathbf{M}^{\delta k,\varepsilon}_{\text{rest}} = \begin{bmatrix} \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{1,+}} + \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{1,-}} & \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{1,+}} + \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{2,-}} \\ \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{2,+}} + \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{1,-}} & \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{2,+}} + \frac{\mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})}{\delta k - \delta k^{\delta,\varepsilon}_{2,-}} \end{bmatrix}.$$

Proof As we know,

$$\delta^2 k^2 \mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon} = (\delta k - \boldsymbol{A}^{\delta k,\varepsilon}_{\star})(\delta k + \boldsymbol{A}^{\delta k,\varepsilon}_{\star}).$$

Since $(\delta k^{\delta,\varepsilon}_{j,+}\mathbb{I}_2 - \boldsymbol{A}^{\delta k,\varepsilon}_{\star}|_{k=k^{\delta,\varepsilon}_{j,+}})\mathrm{Y}^{\varepsilon}_{j} = 0$, thus $\delta k^{\delta,\varepsilon}_{j,+}\mathrm{Y}^{\varepsilon}_{j} = \boldsymbol{A}^{\delta k,\varepsilon}_{\star}|_{k=k^{\delta,\varepsilon}_{j,+}}\mathrm{Y}^{\varepsilon}_{j}$, hence

$$\boldsymbol{A}^{\delta k,\varepsilon}_{\star}\mathrm{Y}^{\varepsilon}_{j} = \delta k^{\delta,\varepsilon}_{j,+}\mathrm{Y}^{\varepsilon}_{j} + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}),$$

we have then with (4.46)

$$(\delta k - \boldsymbol{A}^{\delta k,\varepsilon}_{\star})\mathrm{Y}^{\varepsilon}_{j} = (\delta k - \delta k^{\delta,\varepsilon}_{j,+})(\mathrm{Y}^{\varepsilon}_{j} + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})) = (\delta k - \delta k^{\delta,\varepsilon}_{j,+})((\mathrm{Y}^{\varepsilon}_{\star,\star})_j + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))\,,$$

which implies

$$(\delta k - \boldsymbol{A}^{\delta k,\varepsilon}_{\star})\mathbf{Y}^{\varepsilon} = ((\mathbf{Y}^{\varepsilon}_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))\mathbf{M}^{\delta k}_{+},$$

that is,

$$(\delta k - A_\star^{\delta k,\varepsilon})^{-1} = ((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))(\mathbf{M}_+^{\delta k})^{-1}((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))^{-1},$$

and analogously we get

$$(\delta k + A_\star^{\delta k,\varepsilon})^{-1} = ((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))(\mathbf{M}_-^{\delta k})^{-1}((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))^{-1}.$$

This leads us to

$$\begin{aligned}(\delta^2 k^2 \mathbb{I}_2 - A^{\delta k,\varepsilon})^{-1} = ((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))(\mathbf{M}_+^{\delta k})^{-1}((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))^{-1} \\ ((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))(\mathbf{M}_-^{\delta k})^{-1}((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))^{-1}. \quad (4.51)\end{aligned}$$

Using

$$\begin{aligned}((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))^{-1}((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2})) &= \mathbb{I}_2 + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}), \\ ((\mathbf{Y}^\varepsilon_{\star,\star}) + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}))^{-1} &= (\mathbf{Y}^\varepsilon_{\star,\star})^{-1} + \mathcal{O}(\mathfrak{c}_\varepsilon^{1/2}),\end{aligned}$$

we proved Lemma 4.18. ■

Proposition 4.19 *Let $\delta k \in \mathbb{K}_{\mathbb{C}} \setminus \{0, \delta k_{1,+}^{\delta,\varepsilon}, \delta k_{2,+}^{\delta,\varepsilon}, \delta k_{1,-}^{\delta,\varepsilon}, \delta k_{2,-}^{\delta,\varepsilon}\}$, there exists a unique solution to the equation $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}[\boldsymbol{\mu}] = \boldsymbol{f}^{\delta k}$. Moreover, the solution can be written as $\boldsymbol{\mu} = \boldsymbol{\mu}_\star + \boldsymbol{\mu}_\sim$, where*

$$\begin{aligned}\boldsymbol{\mu}_\star &= \mathfrak{c}_\varepsilon \frac{\pi^2}{4}\Big[(\mathcal{L}^\varepsilon)^{-1}[1]\Big(A_D\left((\mathbf{Y}^\varepsilon_{\star,\star})(\mathbf{M}_+^{\delta k}\mathbf{M}_-^{\delta k})^{-1}(\mathbf{Y}^\varepsilon_{\star,\star})^{-1}\right) f_{\mathbf{0}}^{\delta k}\Big)_j\Big]_{j=1,2} + \frac{\pi}{2}(\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}[f_{\mathbf{0}}^{\delta k}], \\ \boldsymbol{\mu}_\sim &= \mathcal{O}(\delta)\Big((\mathcal{O}_{\boldsymbol{\mathcal{X}}^\varepsilon}(\mathfrak{c}_\varepsilon^{5/2}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^\varepsilon}(\varepsilon))\Big(\|(\mathbf{M}_+^{\delta k}\mathbf{M}_-^{\delta k})^{-1}\| + \|\mathbf{M}_{\text{rest}}^{\delta k,\varepsilon}\|\Big) + (\mathcal{O}_{\boldsymbol{\mathcal{X}}^\varepsilon}(\mathfrak{c}_\varepsilon^2) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^\varepsilon}(\varepsilon))\Big).\end{aligned}$$

where

$$A_D \coloneqq \begin{bmatrix} 1/|D_1| & 0 \\ 0 & 1/|D_2| \end{bmatrix}.$$

Proof From (4.50), we get that

$$(\delta^2 k^2 \mathbb{I}_2 - A^{\delta k,\varepsilon}) \begin{bmatrix} (\mu_1, 1)_\varepsilon \\ (\mu_2, 1)_\varepsilon \end{bmatrix} = \frac{\pi}{2}(\delta^2 k^2) f_{\mathcal{L}}^{\delta k}.$$

Thus, Lemma 4.18 gives that

$$\begin{bmatrix} (\mu_1, 1)_\varepsilon \\ (\mu_2, 1)_\varepsilon \end{bmatrix} = ((\mathbf{Y}^\varepsilon_{\star,\star})(\mathbf{M}_+^{\delta k}\mathbf{M}_-^{\delta k})^{-1}(\mathbf{Y}^\varepsilon_{\star,\star})^{-1} + \mathbf{M}_{\text{rest}}^{\delta k,\varepsilon})\frac{\pi}{2}(\delta^2 k^2) f_{\mathcal{L}}^{\delta k}.$$

Note that similarly to Lemma 3.22, we have

$$\begin{aligned}(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}] &= \begin{bmatrix} f_{D_1}^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1] + \mathcal{O}(\delta)(\mathcal{O}_{\mathcal{X}^\varepsilon}(\mathfrak{c}_\varepsilon^2) + \mathcal{O}_{\mathcal{X}^\varepsilon}(\varepsilon)) \\ f_{D_2}^{\delta k}(0)(\mathcal{L}^\varepsilon)^{-1}[1] + \mathcal{O}(\delta)(\mathcal{O}_{\mathcal{X}^\varepsilon}(\mathfrak{c}_\varepsilon^2) + \mathcal{O}_{\mathcal{X}^\varepsilon}(\varepsilon)) \end{bmatrix} \\ &= (\boldsymbol{\mathcal{L}}^\varepsilon)^{-1}[f_{\mathbf{0}}^{\delta k}] + \mathcal{O}(\delta)(\mathcal{O}_{\boldsymbol{\mathcal{X}}^\varepsilon}(\mathfrak{c}_\varepsilon^2) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^\varepsilon}(\varepsilon)).\end{aligned}$$

From (4.49), it follows that

$$\begin{aligned}
\boldsymbol{\mu} =& -\frac{\pi}{2}\frac{(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}\boldsymbol{\mathcal{K}}^{\varepsilon}[\mu]}{(\delta k)^2} + \frac{\pi}{2}(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}] \\
=& -\frac{\pi}{2(\delta k)^2}(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}\left[\boldsymbol{A}_D((\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1} + \mathbf{M}^{\delta k,\varepsilon}_{\text{rest}})\frac{\pi}{2}(\delta^2 k^2) f^{\delta k}_{\mathcal{L}}\right] \\
&+ \frac{\pi}{2}(\widetilde{\boldsymbol{\mathcal{L}}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}] \\
=& -\frac{\pi^2}{4}\Big[(\mathcal{L}^{\varepsilon})^{-1}[1]\Big(\boldsymbol{A}_D\left((\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1} + \mathbf{M}^{\delta k,\varepsilon}_{\text{rest}}\right) f^{\delta k}_{\mathcal{L}}\Big)_j\Big]_{j=1,2} \\
&\cdot\left(1 + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^1_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon)\right) + \frac{\pi}{2}\left((\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[f^{\delta k}_{\mathbf{0}}] + \mathcal{O}(\delta)(\mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^2_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon))\right).
\end{aligned}$$

Consider that

$$f^{\delta k}_{\mathcal{L}} = -\mathfrak{c}_{\varepsilon} f^{\delta k}_{\mathbf{0}} + \mathcal{O}(\delta)(\mathcal{O}(\mathfrak{c}^2_{\varepsilon}) + \mathcal{O}(\varepsilon)).$$

This leads us to

$$\begin{aligned}
\boldsymbol{\mu} =& -\frac{\pi^2}{4}\Big[(\mathcal{L}^{\varepsilon})^{-1}[1]\Big(\boldsymbol{A}_D\left((\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\right) f^{\delta k}_{\mathcal{L}}\Big)_j\Big]_{j=1,2} + \frac{\pi}{2}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[f^{\delta k}_{\mathbf{0}}] \\
&+ \left((\mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^1_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon)\right)\Big[(\mathcal{L}^{\varepsilon})^{-1}[1]\Big(\boldsymbol{A}_D\left((\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1} + \mathbf{M}^{\delta k,\varepsilon}_{\text{rest}}\right) f^{\delta k}_{\mathcal{L}}\Big)_j\Big]_{j=1,2} \\
&+ \mathcal{O}(\delta)(\mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^2_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon)) + \mathcal{O}_{\mathcal{X}^{\varepsilon}}(\mathfrak{c}_{\varepsilon})\Big[\boldsymbol{A}_D\mathbf{M}^{\delta k,\varepsilon}_{\text{rest}} f^{\delta k}_{\mathcal{L}}\Big]_{j=1,2} \\
=& \mathfrak{c}_{\varepsilon}\frac{\pi^2}{4}\Big[(\mathcal{L}^{\varepsilon})^{-1}[1]\Big(\boldsymbol{A}_D\left((\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\right) f^{\delta k}_{\mathbf{0}}\Big)_j\Big]_{j=1,2} + \frac{\pi}{2}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[f^{\delta k}_{\mathbf{0}}] \\
&+ \mathcal{O}(\delta)(\mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^3_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon\mathfrak{c}_{\varepsilon}))\left(\|(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}\| + \|\mathbf{M}^{\delta k,\varepsilon}_{\text{rest}}\|\right) \\
&+ \mathcal{O}(\delta)(\mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^2_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon)) + \mathcal{O}(\delta)\mathcal{O}_{\mathcal{X}^{\varepsilon}}(\mathfrak{c}^{5/2}_{\varepsilon})\|\mathbf{M}^{\delta k,\varepsilon}_{\text{rest}}\|, \\
=& \mathfrak{c}_{\varepsilon}\frac{\pi^2}{4}\Big[(\mathcal{L}^{\varepsilon})^{-1}[1]\Big(\boldsymbol{A}_D\left((\mathbf{Y}^{\varepsilon}_{\star,\star})(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}(\mathbf{Y}^{\varepsilon}_{\star,\star})^{-1}\right) f^{\delta k}_{\mathbf{0}}\Big)_j\Big]_{j=1,2} + \frac{\pi}{2}(\boldsymbol{\mathcal{L}}^{\varepsilon})^{-1}[f^{\delta k}_{\mathbf{0}}] \\
&+ \mathcal{O}(\delta)(\mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^{5/2}_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon))\left(\|(\mathbf{M}^{\delta k}_{+}\mathbf{M}^{\delta k}_{-})^{-1}\| + \|\mathbf{M}^{\delta k,\varepsilon}_{\text{rest}}\|\right) \\
&+ \mathcal{O}(\delta)(\mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\mathfrak{c}^2_{\varepsilon}) + \mathcal{O}_{\boldsymbol{\mathcal{X}}^{\varepsilon}}(\varepsilon)).
\end{aligned}$$

This proves Proposition 4.19. ■

4.3.5 Asymptotic Expansion of our Solution to the Physical Problem

In Proposition 4.10 we established

$$\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}\begin{bmatrix}\partial_\nu u^{\delta k}|_{\Lambda_1}\\ \partial_\nu u^{\delta k}|_{\Lambda_2}\end{bmatrix}(\tau) + \sum_{n=1}^{\infty}\delta^n\, 2\,\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,n}\begin{bmatrix}\partial_\nu u^{\delta k}|_{\Lambda_1}\\ \partial_\nu u^{\delta k}|_{\Lambda_2}\end{bmatrix}(\tau) = \boldsymbol{f}^{\delta k}(\tau). \tag{4.52}$$

On the other hand, we know that for $\delta k \in \mathbb{K}_{\mathbb{C}} \setminus \{0, \delta k^{\delta,\varepsilon}_{1,+}, \delta k^{\delta,\varepsilon}_{2,+}, \delta k^{\delta,\varepsilon}_{1,-}, \delta k^{\delta,\varepsilon}_{2,-}\}$ that $\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon}$ is invertible, Proposition 4.19. Then for δ small enough we can use the Neumann series

and obtain

$$
\begin{aligned}
(\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon} + \sum_{n=1}^{\infty} 2\,\delta^n\,\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,n})^{-1} &= \left(\boldsymbol{\mathcal{I}} + (\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1} \sum_{n=1}^{\infty} 2\,\delta^n\,\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,n}\right)^{-1} (\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1} \\
&= \sum_{m=0}^{\infty} \left(-(\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1} \sum_{n=1}^{\infty} 2\,\delta^n\,\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,n}\right)^{m} (\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1} \\
&= \sum_{m=0}^{\infty} \left(\sum_{n=1}^{\infty} -2\,\delta^n\,(\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1}\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,n}\right)^{m} (\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1} \\
&= (\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1} - 2\delta(\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1}\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,1} + \mathcal{O}(\delta^2)\mathcal{O}_{\mathcal{L}(\mathcal{X}^\varepsilon,\mathcal{X}^\varepsilon)}(1)\,.
\end{aligned}
$$

Thus we have from solving (4.52)

$$
\begin{bmatrix} \partial_\nu u^{\delta k}\,|_{\,\Lambda_1} \\ \partial_\nu u^{\delta k}\,|_{\,\Lambda_2} \end{bmatrix} = (\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}] + \mathcal{O}(\delta^2)\,,
$$

where we use the formula for $(\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1}$ and the facts that $\boldsymbol{f}^{\delta k} = \mathcal{O}(\delta)$ and $\widehat{\boldsymbol{\Gamma}}^{k,\varepsilon}_{+,1}$ is linear. With Proposition 4.19 we split $(\boldsymbol{\mathcal{A}}^{\delta k,\varepsilon})^{-1}[\boldsymbol{f}^{\delta k}]$ into $\boldsymbol{\mu}^{\delta k}_{\star}$ and $\boldsymbol{\mu}^{\delta k}_{\sim}$ and thus we have for $t \in (-\varepsilon\,,\varepsilon)$

$$
\partial_\nu u^{\delta k}\,|_{\,\Lambda_j}\left(\begin{pmatrix} t \pm \xi \\ h \end{pmatrix}\right) = (\boldsymbol{\mu}^{\delta k}_{\star})_j(t) + (\boldsymbol{\mu}^{\delta k}_{\sim})_j(t) + \mathcal{O}(\delta^2)\,.
$$

We also see from Proposition 4.19 that $\partial_\nu u^{\delta k}\,|_{\,\Lambda_j} = \mathcal{O}(\delta)$.

Now we want to calculate the first order expansion term in δ for the solution in the far-field. Let $z \in \mathbb{R}^2$, where $z_2 \gg 1$.

Lemma 4.20 *We have for $z \in \mathbb{R}^2_+$, $z_2 > 1$,*

$$
\begin{aligned}
u^{\delta k}_s(z) = u^{\delta k}_{\mathrm{RHS},p}(z) + u^{\delta k}_{\mathrm{RHS},e}(z) + (\mathcal{S}^{\delta k}_{+,p} + \mathcal{S}^{\delta k}_{+,e})[\partial_\nu u^{\delta k}\,|_{\,\Lambda_1\cup\Lambda_2}](z) \\
+ (\mathcal{T}^{\delta k}_{+,p} + \mathcal{T}^{\delta k}_{+,e})[\partial_\nu u^{\delta k}\,|_{\,\Lambda_1\cup\Lambda_2}](z) + \mathcal{O}(\delta^2)\,, \quad (4.53)
\end{aligned}
$$

where for $\boldsymbol{\mu} \in \boldsymbol{\mathcal{X}}^\varepsilon$

$$
\begin{aligned}
\mathcal{S}^{\delta k}_{+,p}[\boldsymbol{\mu}](z) &:= \int_{\Lambda_1\cup\Lambda_2} \boldsymbol{\mu}(y)\Gamma^{\delta k}_{+,p}(z,y)\mathrm{d}\sigma_y\,, \\
\mathcal{S}^{\delta k}_{+,e}[\boldsymbol{\mu}](z) &:= \int_{\Lambda_1\cup\Lambda_2} \boldsymbol{\mu}(y)\Gamma^{\delta k}_{+,e}(z,y)\mathrm{d}\sigma_y\,, \\
\mathcal{T}^{\delta k}_{+,p}[\boldsymbol{\mu}](z) &:= -\int_{\Lambda_1\cup\Lambda_2}\int_{\partial D_1\cup\partial D_2} \boldsymbol{\mu}(y)\mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,\times}(y,w)\partial_{\nu_w}\Gamma^{\delta k}_{+,p}(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,, \\
\mathcal{T}^{\delta k}_{+,e}[\boldsymbol{\mu}](z) &:= -\int_{\Lambda_1\cup\Lambda_2}\int_{\partial D_1\cup\partial D_2} \boldsymbol{\mu}(y)\mathrm{N}^{\delta k}_{\partial\boldsymbol{\Omega}^1,\times}(y,w)\partial_{\nu_w}\Gamma^{\delta k}_{+,e}(z,w)\mathrm{d}\sigma_w\mathrm{d}\sigma_y\,,
\end{aligned}
$$

and

$$u^{\delta k}_{\mathrm{RHS},p}(z) = e^{i\delta(k_2 z_2 - k_1 z_1)} \left[\int_{\partial D_1 \cup \partial D_2} \partial_\nu (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y) \frac{\sin(\delta k_2 y_2)\, e^{i\delta k_1 y_1}}{\delta k_2 p} \mathrm{d}\sigma_y \right.$$
$$\left. - \int\limits_{\partial D_1 \cup \partial D_2} \int\limits_{\partial D_1 \cup \partial D_2} \partial_\nu (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial \mathbf{\Omega}^1,\times}(y,w) \nu_w \cdot \begin{pmatrix} \frac{ik_1}{pk_2} \sin(\delta k_2 w_2) \\ \frac{1}{p} \cos(\delta k_2 w_2) \end{pmatrix} e^{i\delta k_1 w_1} \mathrm{d}\sigma_w \mathrm{d}\sigma_y \right],$$

$$u^{\delta k}_{\mathrm{RHS},e}(z) = \left[- \int_{\partial D_1 \cup \partial D_2} \partial_\nu (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y) \Gamma^{\delta k}_{+,e}(z,y) \mathrm{d}\sigma_y \right.$$
$$\left. + \int\limits_{\partial D_1 \cup \partial D_2} \int\limits_{\partial D_1 \cup \partial D_2} \partial_\nu (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial \mathbf{\Omega}^1,\times}(y,w)\, \partial_{\nu_w} \Gamma^{\delta k}_{+,e}(z,w) \mathrm{d}\sigma_w \mathrm{d}\sigma_y \right].$$

Proof We define $u^{\delta k}_f(z) := u^{\delta k}(z) + f^{\delta k}_{\partial \mathbf{\Omega}^1}(z) =: u^{\delta k}_s(z) - u^{\delta k}_{\mathrm{RHS}}(z)$. Then we have from Proposition 4.5 that

$$\begin{aligned} u^{\delta k}_f(z) &= \int_{\Lambda_1 \cup \Lambda_2} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y \\ &= \int_{\Lambda_1 \cup \Lambda_2} \partial_\nu u^{\delta k}(y) \Gamma^{\delta k}_+(z,y) \mathrm{d}\sigma_y + \int_{\Lambda_1 \cup \Lambda_2} \partial_\nu u^{\delta k}(y) \mathrm{R}^{\delta k}_{\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y \\ &=: \mathcal{S}^{\delta k}_+ [\partial_\nu u^{\delta k} \,|_{\Lambda_1 \cup \Lambda_2}] + \mathcal{T}^{\delta k}_+ [\partial_\nu u^{\delta k} \,|_{\Lambda_1 \cup \Lambda_2}]. \end{aligned}$$

Using the splitting $\Gamma^{\mathbf{k}}_+ = \Gamma^{\mathbf{k}}_{+,p} + \Gamma^{\mathbf{k}}_{+,e}$, see Proposition 2.4, we already obtain $\mathcal{S}^{\delta k}_{+,p}$ and $\mathcal{S}^{\delta k}_{+,e}$. To study the terms $u^{\delta k}_{\mathrm{RHS}}$ and $\mathcal{T}^{\delta k}_+$, we consider in view of Proposition 2.9 that

$$\mathrm{R}^{\mathbf{k}}_{\Omega,+}(z,y) = - \int_{\partial E} \mathrm{N}^{\mathbf{k}}_{\partial\Omega,\times}(y,w) \partial_{\nu_y} \Gamma^{\mathbf{k}}_+(z,w) \mathrm{d}\sigma_w .$$

Hence

$$\int_{\Lambda_1 \cup \Lambda_2} \partial_\nu u^{\delta k}(y) \mathrm{R}^{\delta k}_{\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y = - \int\limits_{\Lambda_1 \cup \Lambda_2} \int\limits^{\partial D_1 \cup \partial D_2} \partial_\nu u^{\delta k}(y) \mathrm{N}^{\delta k}_{\partial \mathbf{\Omega}^1,\times}(y,w) \partial_{\nu_w} \Gamma^{\delta k}_+(z,w) \mathrm{d}\sigma_w \mathrm{d}\sigma_y ,$$

and

$$\begin{aligned} &\int_{\partial D_1 \cup \partial D_2} \partial_\nu (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y) \mathrm{R}^{\delta k}_{\mathbf{\Omega}^1,+}(z,y) \mathrm{d}\sigma_y \\ &\quad = - \int_{\Lambda_1 \cup \Lambda_2} \int_{\partial D_1 \cup \partial D_2} \partial_\nu (u_0^{\delta k} - u_0^{\delta k} \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial \mathbf{\Omega}^1,\times}(y,w) \partial_{\nu_w} \Gamma^{\delta k}_+(z,w) \mathrm{d}\sigma_w \mathrm{d}\sigma_y . \quad (4.54) \end{aligned}$$

Using again the splitting $\nabla \Gamma^{\mathbf{k}}_+ = \nabla \Gamma^{\mathbf{k}}_{+,p} + \nabla \Gamma^{\mathbf{k}}_{+,e}$ and the explicit formula for $\Gamma^{\mathbf{k}}_{+,p}$ and $\nabla \Gamma^{\mathbf{k}}_{+,p}$ we obtain the formulas stated in Lemma 4.20. ■

Let us approximate $\mathcal{S}_+^{\delta k}$ and $\mathcal{T}_+^{\delta k}$. Let $z_2 > 1$. We define

$$\begin{aligned}
\mathcal{S}_{+,p,0}^{\delta k}[\boldsymbol{\mu}](z) &:= -\, e^{i(\delta k_2 z_2 - \delta k_1 z_1)} \frac{1}{p} \int_{\Lambda_1 \cup \Lambda_2} y_2 \boldsymbol{\mu}(y) \mathrm{d}\sigma_y \,, \\
\mathcal{S}_{+,e,0}^{\delta k}[\boldsymbol{\mu}](z) &:= e^{-i\delta k_1 z_1} \left(\int_{\Lambda_1 \cup \Lambda_2} \boldsymbol{\mu}(y) \Gamma_+^0(z,y) \mathrm{d}\sigma_y + \frac{1}{p} \int_{\Lambda_1 \cup \Lambda_2} y_2 \boldsymbol{\mu}(y) \mathrm{d}\sigma_y \right) \\
&= -\, e^{-i\delta k_1 z_1} \int_{\Lambda_1 \cup \Lambda_2} \boldsymbol{\mu}(y) \sum_{\substack{n \in \mathbb{Z} \setminus \{0\} \\ l := 2\pi n / p}} \frac{1}{p|l|} e^{il(z_1 - y_1)} e^{-|l| z_2} \sinh(|l| y_2) \mathrm{d}\sigma_y \,, \qquad (4.55)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}_{+,p,0}^{\delta k}[\boldsymbol{\mu}](z) &:= e^{i(\delta k_2 z_2 - \delta k_1 z_1)} \frac{1}{p} \int_{\Lambda_1 \cup \Lambda_2} \int_{\partial D_1 \cup \partial D_2} \boldsymbol{\mu}(y) \mathrm{N}_{\partial \boldsymbol{\Omega}^1, \times}^{\delta k}(y,w) \nu_w \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}\sigma_w \mathrm{d}\sigma_y \,, \\
\mathcal{T}_{+,e,0}^{\delta k}[\boldsymbol{\mu}](z) &:= -\, e^{-i\delta k_1 z_1} \left(\int_{\Lambda_1 \cup \Lambda_2} \int_{\partial D_1 \cup \partial D_2} \boldsymbol{\mu}(y) \mathrm{N}_{\partial \boldsymbol{\Omega}^1, \times}^{\delta k}(y,w) \partial_{\nu_w} \Gamma_+^0(z,w) \mathrm{d}\sigma_w \mathrm{d}\sigma_y \right. \\
&\qquad \left. + e^{-i\delta k_2 z_2} \mathcal{T}_{+,p,0}^{\delta k}[\boldsymbol{\mu}](z) \right).
\end{aligned}$$

Lemma 4.21 *There exist $C_{(4.56)} > 0$, $C_{(4.57)} > 0$, $C_{(4.58)} > 0$ and $C_{(4.59)} > 0$ such that, for all $z \in \mathbb{R}_+^2$ with $z_2 > 1$ and all $\boldsymbol{\mu} \in \boldsymbol{\mathcal{X}}^\varepsilon$, it holds that*

$$\left| \left(\mathcal{S}_{+,p}^{\delta k} - \mathcal{S}_{+,p,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| + \frac{1}{\delta} \left| \nabla \left(\mathcal{S}_{+,p}^{\delta k} - \mathcal{S}_{+,p,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| \leq C_{(4.56)} \|\boldsymbol{\mu}\|_{\mathcal{X}^\varepsilon} \delta^2 \,, \qquad (4.56)$$

$$\left| \left(\mathcal{S}_{+,e}^{\delta k} - \mathcal{S}_{+,e,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| + \frac{1}{\delta} \left| \nabla \left(\mathcal{S}_{+,e}^{\delta k} - \mathcal{S}_{+,e,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| \leq C_{(4.57)} \|\boldsymbol{\mu}\|_{\mathcal{X}^\varepsilon} \delta \,, \qquad (4.57)$$

and

$$\left| \left(\mathcal{T}_{+,p}^{\delta k} - \mathcal{T}_{+,p,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| + \frac{1}{\delta} \left| \nabla \left(\mathcal{T}_{+,p}^{\delta k} - \mathcal{T}_{+,p,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| \leq C_{(4.58)} \|\boldsymbol{\mu}\|_{\mathcal{X}^\varepsilon} \delta \,, \qquad (4.58)$$

$$\left| \left(\mathcal{T}_{+,e}^{\delta k} - \mathcal{T}_{+,e,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| + \frac{1}{\delta} \left| \nabla \left(\mathcal{T}_{+,e}^{\delta k} - \mathcal{T}_{+,e,0}^{\delta k} \right) [\boldsymbol{\mu}](z) \right| \leq C_{(4.59)} \|\boldsymbol{\mu}\|_{\mathcal{X}^\varepsilon} \delta \,. \qquad (4.59)$$

Proof Let us consider $\mathcal{S}_+^{\delta k}$ first. Using the following splitting for $\Gamma_+^{\delta k}(z,y)$, which is given in Proposition 2.4, we have

$$\Gamma_+^{\delta k}(z,y) = \Gamma_{+,p}^{\delta k}(z,x) + \Gamma_{+,e}^{\delta k}(z,x) \,,$$

where

$$\Gamma_{+,p}^{\delta k}(z,x) = -\, \frac{\sin(\delta k_2 x_2)}{\delta k_2 p} e^{i\delta(k_2 z_2 - k_1 z_1)} e^{i\delta k_1 x_1} \,, \qquad (4.60)$$

$$\begin{aligned}
\Gamma_{+,e}^{\delta k}(z,x) =& e^{-i\delta k_1 z_1} \sum_{\substack{n \in \mathbb{Z} \setminus \{0\} \\ l := 2\pi n / p}} \left(\frac{-e^{il(z_1 - x_1) + i\delta k_1 x_1}}{p \sqrt{|l - \delta k_1|^2 - \delta^2 k^2}} \right. \\
&\left. \cdot \sinh\left(\sqrt{|l - \delta k_1|^2 - \delta^2 k^2}\, x_2 \right) \right) e^{-\sqrt{|l - \delta k_1|^2 - \delta^2 k^2}\, z_2} \,. \qquad (4.61)
\end{aligned}$$

$\mathcal{S}^{\delta k}_{+,p,0}$ is the zeroth order term of the Taylor expansion with respect to δ of $\Gamma^{\delta k}_{+,p}(z,x)$, but without the $e^{i\delta(k_2z_2-k_1z_1)}$ term, and $\mathcal{S}^{\delta k}_{+,e,0}$ is the zeroth order term of the Taylor expansion with respect to δ of $\Gamma^{\delta k}_{+,e}(z,x)$, but without the $e^{-i\delta k_1 z_1}$ term, which is located before the evanescent sum in (3.65). To see that, write $\mathcal{S}^{\delta k}_{+,e,0}$ using

$$\Gamma^0_+(z,x) = -\frac{x_2}{p} - \sum_{\substack{n\in\mathbb{Z}\setminus\{0\}\\ l:=2\pi n/p}} \frac{1}{p|l|} e^{il(z_1-x_1)} e^{-|l|z_2} \sinh(|l|x_2)\,, \tag{4.62}$$

which is given through (2.2) for $z_2 > 1$, and rewrite $\int_\Lambda \partial_\nu u^{\delta k}(y)\Gamma^0_+(z,y)\mathrm{d}\sigma_y$ with it. Then we see that the sum in (4.62) is exactly the zeroth order term of the Taylor expansion. With that, we obtain the formula in (4.55).

Inserting these exact formulas into the expressions in Lemma 4.21 and using

$$\int_{-\varepsilon}^{\varepsilon} \mu(t)\varphi(t)\mathrm{d}t = \int_{-\varepsilon}^{\varepsilon} \varphi(t)\frac{\sqrt[4]{\varepsilon^2-t^2}}{\sqrt[4]{\varepsilon^2-t^2}}\mu(t)\mathrm{d}t \le \|\varphi\|_{\mathcal{C}^0}\,\|\mu\|_{\mathcal{X}^\varepsilon}\sqrt{\pi}\,,$$

where $\varphi \in \mathcal{C}^0([-\varepsilon,\varepsilon])$ and where we used the L^2-Cauchy-Schwarz inequality, yields the desired estimates for $\mathcal{S}^{\delta k}_+$. For $\mathcal{T}^{\delta k}_+$ it works analogously. ■

We define

$$\begin{aligned}
u^{\delta k}_{\mathcal{S}^\star_p}(z) &:= \mathcal{S}^{\delta k}_{+,p,0}[\boldsymbol{\mu}^{\delta k}_\star](z)\,,\\
u^{\delta k}_{\mathcal{S}^\star_e}(z) &:= \mathcal{S}^{\delta k}_{+,e,0}[\boldsymbol{\mu}^{\delta k}_\star](z)\,,\\
u^{\delta k}_{\mathcal{T}^\star_p}(z) &:= \mathcal{T}^{\delta k}_{+,p,0}[\boldsymbol{\mu}^{\delta k}_\star](z)\,,\\
u^{\delta k}_{\mathcal{T}^\star_e}(z) &:= \mathcal{T}^{\delta k}_{+,e,0}[\boldsymbol{\mu}^{\delta k}_\star](z)\,.
\end{aligned}$$

We then have the following proposition:

Proposition 4.22 *Let $V_r := \{z\in\mathbb{R}^2_+ \,|\, z_2 > r\}$. There exists a constant $C_{(4.63)} > 0$ such that*

$$\begin{aligned}
&\left\| u^{\delta k}_s - (u^{\delta k}_{\mathcal{S}^\star_p} + u^{\delta k}_{\mathcal{S}^\star_e} + u^{\delta k}_{\mathcal{T}^\star_p} + u^{\delta k}_{\mathcal{T}^\star_e} + u^{\delta k}_{\mathrm{RHS},p} + u^{\delta k}_{\mathrm{RHS},e})\right\|_{\mathrm{L}^\infty(V_r)}\\
&\quad + \frac{1}{\delta}\left\|\nabla\left[u^{\delta k}_s - (u^{\delta k}_{\mathcal{S}^\star_p} + u^{\delta k}_{\mathcal{S}^\star_e} + u^{\delta k}_{\mathcal{T}^\star_p} + u^{\delta k}_{\mathcal{T}^\star_e} + u^{\delta k}_{\mathrm{RHS},p} + u^{\delta k}_{\mathrm{RHS},e})\right]\right\|_{\mathrm{L}^\infty(V_r)}\\
&\quad \le C_{(4.63)}\left(\delta c^3_\varepsilon\left(\|(\mathbf{M}^{\delta k}_+\mathbf{M}^{\delta k}_-)^{-1}\|\right) + \delta c^2_\varepsilon + \delta\varepsilon^1 + \delta^2\right)\,,
\end{aligned} \tag{4.63}$$

for ε small enough.

Proof According to (4.53), we have

$$u^{\delta k}_s - u^{\delta k}_{\mathrm{RHS},p} - u^{\delta k}_{\mathrm{RHS},e} = \left(\mathcal{S}^{\delta k}_{+,e} + \mathcal{S}^{\delta k}_{+,p} + \mathcal{T}^{\delta k}_{+,e} + \mathcal{T}^{\delta k}_{+,p}\right)[u^{\delta k}|_{\Lambda_1\cup\Lambda_2}] + \mathcal{O}(\delta^2)\,.$$

With Lemma 4.21 and the fact that $u^{\delta k}\,|_{\,\Lambda_1\cup\Lambda_2} = \boldsymbol{\mu}^{\delta k}_\star + \boldsymbol{\mu}^{\delta k}_\sim + \mathcal{O}(\delta^2)$, $\|\boldsymbol{\mu}^{\delta k}_\star\|_{\boldsymbol{\mathcal{X}}^\varepsilon} = \mathcal{O}(\delta)$ and $\|\boldsymbol{\mu}^{\delta k}_\sim\|_{\boldsymbol{\mathcal{X}}^\varepsilon} = \mathcal{O}(\delta)$, we readily proof Proposition 4.22. ■

Proof (Theorem 4.2) We see that $u^{\delta k}_{\mathcal{S}^\star_e}$, $u^{\delta k}_{\mathcal{T}^\star_e}$ and $u^{\delta k}_{\mathrm{RHS},e}$ are exponentially decaying in z_2 and are of order $\mathcal{O}(\delta)$, thus their $\mathrm{L}^\infty(V_r)$-norm is of order $\delta e^{-C\,r}$ for some constant $C > 0$. Then with Proposition 4.22 and the change into the macroscopic view with $U^k(x) = u^{\delta k}(x/\delta)$, we obtain Theorem 4.2. ■

4.3.6 Evaluating the Impedance Boundary Condition

We switch back to the macroscopic variable $U^k(x) = u^{\delta k}(x/\delta)$. We approximate our solution in the far-field with the function

$$\begin{aligned} U^k_{\mathrm{app}}(z) :=& (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(z/\delta) + u^{\delta k}_{\mathcal{S}^\star_p}(z/\delta) + u^{\delta k}_{\mathcal{T}^\star_p}(z/\delta) + u^{\delta k}_{\mathrm{RHS},p}(z/\delta) \\ =& -2ia_0 e^{-ik_1 z_1} \sin(k_2 z_2) + e^{i(k_2 z_2 - k_1 z_1)}\, C^{\delta k}_{(4.65)}\,, \end{aligned} \tag{4.64}$$

where

$$\begin{aligned} C^{\delta k}_{(4.65)} :=& -\frac{h}{p} \int_{\Lambda_1 \cup \Lambda_2} \boldsymbol{\mu}^{\delta k}_\star(y) \mathrm{d}\sigma_y + \frac{1}{p} \int_{\Lambda_1 \cup \Lambda_2} \int_{\partial D_1 \cup \partial D_2} \boldsymbol{\mu}^{\delta k}_\star(y) \mathrm{N}^{\delta k}_{\partial \boldsymbol{\Omega}^{1,\times}}(y,w) \nu_w \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}\sigma_w \mathrm{d}\sigma_y \\ &+ \int_{\partial D_1 \cup \partial D_2} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \frac{\sin(\delta k_2 y_2)\, e^{i\delta k_1 y_1}}{\delta k_2 p} \mathrm{d}\sigma_y \\ &- \int\limits_{\partial D_1 \cup \partial D_2} \int\limits_{\partial D_1 \cup \partial D_2} \partial_\nu (u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P})(y) \mathrm{N}^{\delta k}_{\partial \boldsymbol{\Omega}^{1,\times}}(y,w) \nu_w \cdot \begin{pmatrix} \frac{i k_1}{p k_2} \sin(\delta k_2 w_2) \\ \frac{1}{p} \cos(\delta k_2 w_2) \end{pmatrix} e^{i\delta k_1 w_1} \mathrm{d}\sigma_w \mathrm{d}\sigma_y\,. \end{aligned} \tag{4.65}$$

Consider that $C^{\delta k}_{(4.65)} = \mathcal{O}(\delta)$ since $\boldsymbol{\mu}^{\delta k}_\star = \mathcal{O}(\delta)$ and $(u^{\delta k}_0 - u^{\delta k}_0 \circ \mathtt{P}) = \mathcal{O}(\delta)$ and the other factors are of size $\mathcal{O}(1)$.

We want that U^k_{app} satisfies the impedance boundary condition, that is $U^k_{\mathrm{app}}(z) + \delta c_{\mathrm{IBC}} \partial_{z_2} U^k_{\mathrm{app}}(z) = 0$ at $\partial \mathbb{R}^2_+$. By (4.64), we obtain the condition

$$e^{-ik_1 z_1}\, C^{\delta k}_{(4.65)} + \delta c_{\mathrm{IBC}} (-2ia_0 k_2 e^{-ik_1 z_1} + i\, k_2 e^{-ik_1 z_1}\, C^{\delta k}_{(4.65)}) = 0\,, \quad \text{for all } z_2 \in \mathbb{R}\,.$$

After rearranging the terms, we obtain

$$c_{\mathrm{IBC}} = \frac{-C^{\delta k}_{(4.65)}}{i\,\delta k_2 (\, C^{\delta k}_{(4.65)} - 2a_0)}\,.$$

Using that $\frac{1}{1+\mathcal{O}(\delta)} = 1 - \mathcal{O}(\delta)$ we have

$$c_{\mathrm{IBC}} = \frac{C^{\delta k}_{(4.65)}}{2ia_0\, \delta k_2} + \mathcal{O}(\delta)\,.$$

This proves Theorem 4.3.

4.4 Numerical Illustrations

In this subsection we compute the impedance boundary condition constant c_{IBC} with numerical means using Theorem 4.3. We use two geometries, both build up upon rectangles, but with different sizes and for each geometry we have different ranges for the wave vector k and the gap length ε.

We fix $\delta = 0.01$.

Again, we implement $\Gamma_\sharp^{\mathbf{k}}$ using Edwald's Method see [17] or [8, Chapter 7.3.2]. We implement the remainders using Lemmas 2.10, 2.11 and 2.12.

The first geometry has the following set-up. There are two rectangles both with length 0.9 and height 0.9, whose gaps are centered at $(-0.5, 1)^{\mathrm{T}}$ and $(0.5, 1)^{\mathrm{T}}$, respectively. The period is $p = 2$ and $h = 1$. The amplitude of the incident wave is $a_0 = 1$. The number of points, with which we approximate the boundary of each rectangle, is 200. For the wave vector k, we take 341 equidistant points in the interval $[30, 200]$. For ε we pick 5 values, those are $\{0.1, 0.05, 0.03, 0.01, 0.001\}$. The result can be seen in Figure 5.

The second geometry has the following set-up. There are two rectangles both with length 0.2 and height 0.3, whose gaps are centered at $(-0.5, 1)^{\mathrm{T}}$ and $(0.5, 1)^{\mathrm{T}}$, respectively. The period is $p = 2$ and $h = 0.5$. The amplitude of the incident wave is $a_0 = 1$. The number of points, with which we approximate the boundary of each rectangle, is 200. For the wave vector k, we take 301 equidistant points in the interval $[100, 400]$. For ε we pick 5 values, those are $\{0.1, 0.05, 0.03, 0.01, 0.001\}$. Consider that the case $\varepsilon = 0.1$ implies that the whole upper boundary of both rectangles are gaps. The result can be seen in Figure 6.

Consider that the first geometry is the same geometry as in the one resonator case up to a translated origin and thus Figure 5, has the same appearance as Figure 2.

5 Changing a Small Part of the Boundary from Dirichlet to Neumann

Let us consider a bounded domain, we can think of it as a cavity, where there is put up a source point, which emits a wave. On the boundary, we have mounted a device, which we can toggle to act like the other part of the boundary or to act in a reflecting manner. As shown in the previous sections, such a device can be constructed using arrays of Helmholtz resonators. Given a receiving point in the domain, we want to be able to decide, which of the two options for the device give the higher signal at a given receiving point.

After establishing a mathematical set-up for the above described environment, we give the first order expansion term for the difference of the signal between the two option in terms of the size of the device. To this end, we establish the invertiblity of an operator, which emerges from Green's formula, and compute then the inverse of that operator. Most of the results in this section are inspired by [7], where layer potential techniques were first introduced for solving the narrow escape problem of a

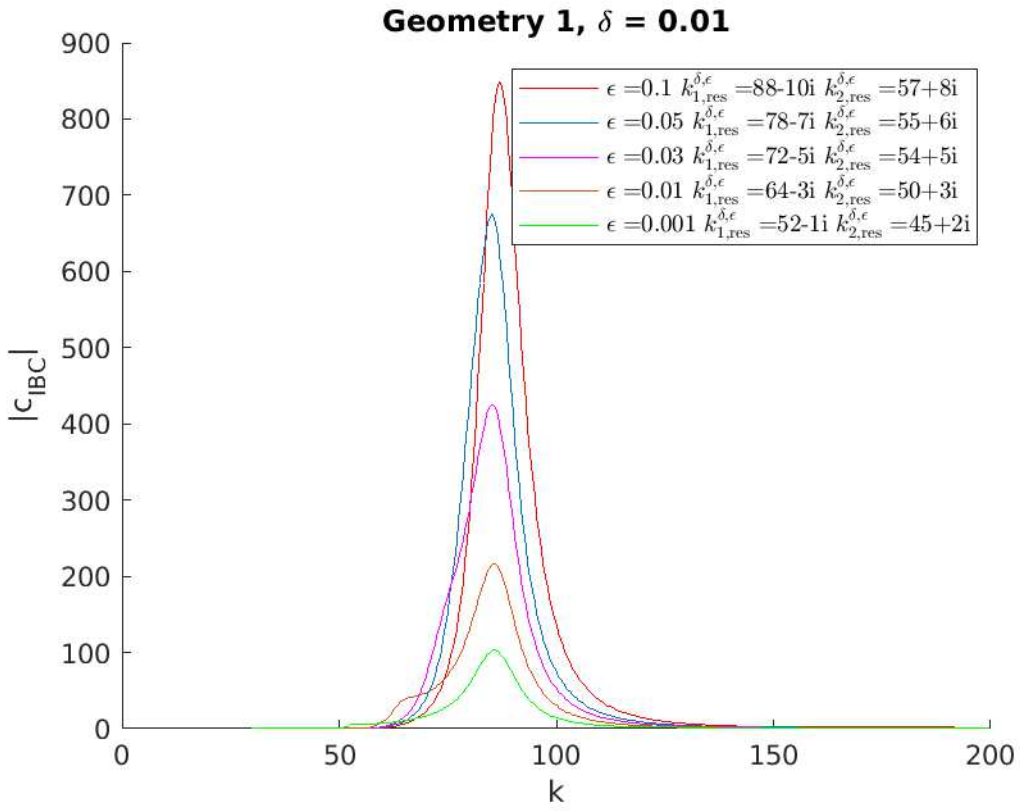


Figure 5: *The plot of the absolute value of the variable c_{IBC} depending on the wave vector k for the first geometry. For every value of ε, the rounded values of the resonance values $k_{1,\mathrm{res}}^{\delta,\varepsilon} := k_{+,1}^{\delta,\varepsilon}$ and $k_{2,\mathrm{res}}^{\delta,\varepsilon} := k_{+,2}^{\delta,\varepsilon}$ are displayed.*

Brownian particle through a small boundary absorbing part. It is worth emphasizing that, in the narrow escape problem, the small part of the boundary is absorbing while the remaining part is reflecting. Because of such a difference, the derivation of an asymptotic formula for the Green's function here is technically more involved than in [7]. We refer the reader to [4, 1, 2, 22, 15] for the analysis of the mixed boundary value problem and the evaluation of the associated eigenvalues and eigenfunctions.

5.1 Preliminaries

5.1.1 Statement of the Problem

Let Ω be an open, bounded, and simply connected subset of $\mathbb{R}^2$ with a $\mathcal{C}^2$-boundary. Let $\partial\Omega$ be partitioned in two open intervals $\partial\Omega_N$ and $\partial\Omega_D$ such that $\partial\Omega_N$ is a line segment with length 2ε, where $\varepsilon > 0$, and with center $(0,0)^{\mathrm{T}} \in \mathbb{R}^2$. For simplicity, we assume that Ω is rotated so that $\partial\Omega_N$ is parallel to the first coordinate axis, all

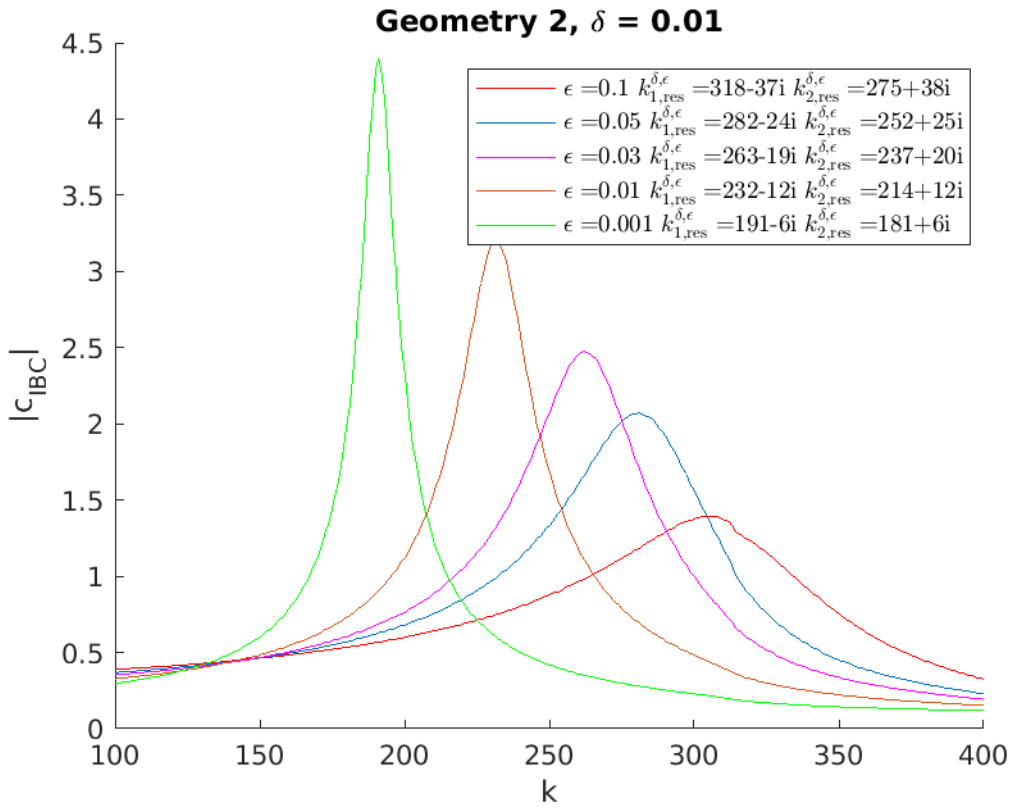


Figure 6: *The plot of the absolute value of the variable c_{IBC} depending on the wave vector k for the second geometry. For every value of ε, the rounded values of the resonance values $k^{\delta,\varepsilon}_{1,\mathrm{res}} := k^{\delta,\varepsilon}_{+,1}$ and $k^{\delta,\varepsilon}_{2,\mathrm{res}} := k^{\delta,\varepsilon}_{+,2}$ are displayed.*

points on $\partial\Omega_N$ have height 0, and the normal on $\partial\Omega_N$ is $(0,1)^{\mathrm{T}}$. Then we fix two points, one is the source point $x_S \in \Omega$ and the other the receiving point $x_R \in \Omega$. We are looking for an asymptotic expansion of the following function in terms of ε and an analytic expression for the first order term. This leading order term gives the topological derivative of the Green's function of the cavity with respect to changes in the boundary conditions. In other terms, it describes the nucleation of a Neumann boundary condition.

Let $u^{k,\varepsilon}_{x_S} : \Omega \setminus \{x_S\} \to \mathbb{C}$ be the solution to

$$\begin{cases} \left(\triangle_y + k^2\right) u^{k,\varepsilon}_{x_S}(y) = \delta_0(x_S - y) & \text{in} \quad \Omega\,, \\ u^{k,\varepsilon}_{x_S}(y) = 0 & \text{on} \quad \partial\Omega_D\,, \\ \partial_{\nu_y} u^{k,\varepsilon}_{x_S}(y) = 0 & \text{on} \quad \partial\Omega_N\,, \end{cases} \tag{5.1}$$

where we assume that $k \in (0,\infty)$ is not an eigenvalue to $-\triangle$ with the above boundary conditions, and thus $u^{k,\varepsilon}_{x_S}$ is uniquely solvable.

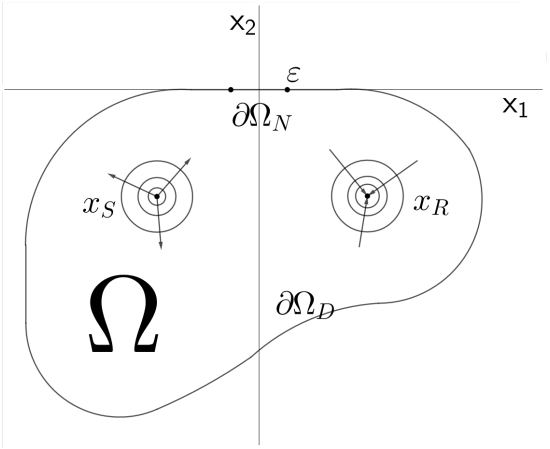


Figure 7: *This picture depicts $\Omega \subset \mathbb{R}^2$ with the two disjoint boundary components $\partial\Omega_D$ and $\partial\Omega_N$. We have a source point point at x_S and a receiving point at x_R.*

Next, we need the function, which satisfies the above partial differential equation but has the Dirichlet condition on the whole boundary. It is often denoted as the Dirichlet function $G^k_\Omega(z,\cdot) : \Omega \setminus \{z\} \to \mathbb{C}$ and satisfies

$$\begin{cases} \left(\triangle_y + k^2\right) G^k_\Omega(z,y) = \delta_0(z-y) & \text{in} \quad \Omega\,, \\ G^k_\Omega(z,y) = 0 & \text{on} \quad \partial\Omega\,, \end{cases} \tag{5.2}$$

for $k \in (0,\infty)$ not an eigenvalue to $-\triangle$ with the above boundary condition.

We will see that we can express $u^{k,\varepsilon}_{x_S}$ as

$$u^{k,\varepsilon}_{x_S}(x_R) = G^k_\Omega(x_S,x_R) + \mathcal{O}(\varepsilon)\,.$$

5.1.2 The Dirichlet Function

We have the following formula for the Dirichlet function:

Proposition 5.1 *Let $z \in \Omega$, $x \in \Omega \setminus \{z\}$ and k not be an eigenvalue to $-\triangle$ with the boundary condition given in PDE (5.2), then we have*

$$G^k_\Omega(z,x) = \Gamma^k(z,x) + R^k_{G,\Omega}(z,x)\,,$$

where $R^k_{G,\Omega}$ is the solution to

$$\begin{cases} \left(\triangle_x + k^2\right) R^k_{G,\Omega}(z,x) = 0 & \textit{in} \quad \Omega\,, \\ R^k_{G,\Omega}(z,x) = -\Gamma^k(z,x) & \textit{on} \quad \partial\Omega\,. \end{cases}$$

The uniqueness and existence of $R^k_{G,\Omega}$ in Proposition 5.1 is a standard result. We are especially interested in a formula for $\partial_{\nu_x} G^k_\Omega(z,x)$, for $z \in \Omega$, with a smooth enough

remainder. To this end, we use $G^k_\Omega(z,x) = \Gamma^k(z,x) + R^k_{G,\Omega}(z,x)$, then from (2.1) we can extract the first two spatial, singular terms, that is $\frac{1}{2\pi}\log(|z-x|)$ and $\frac{-1}{8\pi}k^2\log(k|z-x|)|z-x|^2$, and obtain a formula with a smooth enough remainder:

Proposition 5.2 *Let $z \in \Omega$, $x \in \partial\Omega$ and k not be an eigenvalue to $-\triangle$ with the boundary condition given in PDE (5.2), then we have*

$$\partial_{\nu_x} G^k_\Omega(z,x) = \frac{1}{2\pi}\frac{\nu_x\cdot(x-z)}{|z-x|^2} - \frac{1}{8\pi}k^2\,\nu_x\cdot(x-z)(2\log(k|z-x|)+1) + R^k_{\partial G,\Omega}(z,x)\,,$$

where $R^k_{\partial G,\Omega}(z,\cdot) \in H^{5/2}(\Omega)$.

5.2 Main Results

We define

$$\begin{aligned}
\mathcal{X}^\varepsilon &:= \left\{\mu \in L^2((-\varepsilon,\varepsilon)) \,\middle|\, \int_{-\varepsilon}^{\varepsilon}\sqrt{\varepsilon^2-t^2}|\mu(t)|^2\mathrm{d}t < \infty\right\},\\
\|\mu\|_{\mathcal{X}^\varepsilon} &:= \left(\int_{-\varepsilon}^{\varepsilon}\sqrt{\varepsilon^2-t^2}|\mu(t)|^2\mathrm{d}t\right)^{1/2},\\
\mathcal{Y}^\varepsilon &:= \left\{\mu \in \mathcal{C}^0([-\varepsilon,\varepsilon]) | \mu' \in \mathcal{X}^\varepsilon\right\},\\
\|\mu\|_{\mathcal{Y}^\varepsilon} &:= \left(\|\mu\|^2_{\mathcal{X}^\varepsilon} + \|\mu'\|^2_{\mathcal{X}^\varepsilon}\right)^{1/2},\\
\mathcal{Y}^\varepsilon_\Pi &:= \left\{\mu \in \mathcal{Y}^\varepsilon \mid \mu(-\varepsilon) = \mu(\varepsilon) = 0\right\}.
\end{aligned}$$

We define the operator $\mathcal{L}^\varepsilon : \mathcal{X}^\varepsilon \to \mathcal{Y}^\varepsilon$ and the operator $\mathcal{J}^\varepsilon : \mathcal{Y}^\varepsilon_\Pi \to \mathcal{X}^\varepsilon$,

$$\begin{aligned}
\mathcal{L}^\varepsilon[\mu](\tau) &:= \int_{-\varepsilon}^{\varepsilon}\mu(t)\log(|\tau-t|)\mathrm{d}t\,, \quad \tau \in (-\varepsilon,\varepsilon),\\
\mathcal{J}^\varepsilon[\mu](\tau) &:= \text{H.f.p.}\int_{-\varepsilon}^{\varepsilon}\frac{\mu(t)}{(\tau-t)^2}\mathrm{d}t\,.
\end{aligned}$$

where the 'H.f.p' denotes a Hadamard-finite-part integral. $\mathcal{L}^\varepsilon$ is invertible and the inverse is given in Proposition 3.11. $\mathcal{J}^\varepsilon$ is invertible and the inverse is given in Proposition 5.10. We then have the following result:

Theorem 5.3 *Let $\varepsilon > 0$ be small enough, $\alpha, \beta > 0$. The operator $-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon : \mathcal{Y}^\varepsilon_\Pi \to \mathcal{X}^\varepsilon$ is linear and invertible and the inverse is given in Proposition 5.11. The exact function $(-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon)^{-1}[1]$ is given in Lemma 5.12.*

Theorem 5.4 *Let $\varepsilon > 0$ be small enough and let $k \in (0,\infty)$ not be an eigenvalue to $-\triangle$ with the boundary condition given in PDE (5.1) as well as with the boundary condition given in PDE (5.2). The value $u^{k,\varepsilon}_{x_S}(x_R)$ is determined through*

$$u^{k,\varepsilon}_{x_S}(x_R) = G^k_\Omega(x_S,x_R) + G^{k,\varepsilon}_{(1)}(x_S,x_R) + G^{k,\varepsilon}_{(2)}(x_S,x_R)\,, \tag{5.3}$$

where

$$\mathrm{G}^{k,\varepsilon}_{(1)}(x_S,x_R) := -\int_{-\varepsilon}^{\varepsilon} \partial_{\nu_y}\mathrm{G}^k_\Omega\left(x_R,\begin{pmatrix}t\\0\end{pmatrix}\right)\left(\frac{-1}{2\pi}\mathcal{J}^\varepsilon+\frac{k^2}{4\pi}\mathcal{L}^\varepsilon\right)^{-1}\left[\partial_{\nu_x}\mathrm{G}^k_\Omega\left(x_S,\begin{pmatrix}\tau\\0\end{pmatrix}\right)\right](t)\mathrm{d}t\,,$$

where $\mathrm{G}^{k,\varepsilon}_{(1)}(x_S,x_R)=\mathcal{O}\Big(\frac{\varepsilon}{|\log(\varepsilon/2)|}\Big)$ *and* $\mathrm{G}^{k,\varepsilon}_{(2)}(x_S,x_R)=\mathcal{O}\left(\frac{\varepsilon^2}{|\log(\varepsilon/2)|^2}\right)$.

5.3 Proof of the Main Results

The idea of the proof is inspired by [7] and is as follows. Using Green's formula we readily establish that $u^{k,\varepsilon}_{x_S}$ is a small perturbation of $\mathrm{G}^k_\Omega(x_S,\cdot)$. We see that the difference $v^{k,\varepsilon}_z := u^{k,\varepsilon}_z - \mathrm{G}^k_\Omega(z,\cdot)$ satisfies the following two conditions:

$$\begin{aligned} v^{k,\varepsilon}_z(x) &= \int_{\partial\Omega_N}\partial_{\nu_y}\mathrm{G}^k_\Omega(x,y)\,v^{k,\varepsilon}_z(y)\mathrm{d}\sigma_y\,, \quad \text{for } x\in\Omega\,,\\ \partial_{\nu_x}v^{k,\varepsilon}_z(x) &= -\partial_{\nu_x}\mathrm{G}^k_\Omega(z,x)\,, \quad \text{for } x\in\partial\Omega_N\,, \end{aligned}$$

where the first one comes from Green's formula and the second one from the partial differential equation for $u^{k,\varepsilon}_z$ and $\mathrm{G}^k_\Omega(z,\cdot)$. Combining both leads us to the condition

$$\begin{aligned} -\partial_{\nu_x}\mathrm{G}^k_\Omega\left(z,\begin{pmatrix}\tau\\0\end{pmatrix}\right) =& -\frac{1}{2\pi}\mathcal{J}^\varepsilon[v^{k,\varepsilon}_z]+\frac{k^2}{4\pi}\mathcal{L}^\varepsilon[v^{k,\varepsilon}_z]\\ &+\frac{k^2}{8\pi}(2\log(k)+1)\int_{-\varepsilon}^{\varepsilon}v^{k,\varepsilon}_z(t)\mathrm{d}t+\int_{-\varepsilon}^{\varepsilon}v^{k,\varepsilon}_z(t)\partial_{x_2}\mathrm{R}^k_{\partial\mathrm{G},\Omega}\left(\begin{pmatrix}\tau\\0\end{pmatrix},\begin{pmatrix}t\\0\end{pmatrix}\right)\mathrm{d}t\,. \end{aligned}$$

The key now is to invert the operator $\frac{-1}{2\pi}\mathcal{J}^\varepsilon+\frac{k^2}{4\pi}\mathcal{L}^\varepsilon[v^{k,\varepsilon}_z]$ and proving that the integrals over $(-\varepsilon,\varepsilon)$ with integrand $v^{k,\varepsilon}_z$ are then of lower order. The proof for invertiblity uses a result given in [26, Chapter 11]. For finding the inverse, we use that $\mathcal{J}^\varepsilon$ is of the form $\partial_t\mathcal{H}^\varepsilon$, where $\mathcal{H}^\varepsilon$ is the finite Hilbert transform, and that $(\mathcal{L}^\varepsilon)^{-1}$ is of the form $(\mathcal{H}^\varepsilon)^\dagger[\partial_t]+C$, where $(\mathcal{H}^\varepsilon)^\dagger$ is the inverse of $\mathcal{H}^\varepsilon$ on $\ker(\mathcal{H}^\varepsilon)^\perp$. This, together with

$$(-\alpha\mathcal{J}^\varepsilon+\beta\mathcal{L}^\varepsilon)^{-1}=(\mathcal{L}^\varepsilon)^{-1}(\beta\mathcal{I}-\alpha\mathcal{J}^\varepsilon(\mathcal{L}^\varepsilon)^{-1})^{-1}\,,$$

leads us to the inverse. For the estimates we adapt the technique used in [8, Lemma 5.4]. To this end, we have to compute some integrals. To determine them, we use the mathematics tool Mathematica [19].

5.3.1 Condition on the Gap

Proposition 5.5 *Let* $z\in\Omega$ *and* $x\in\Omega\setminus\{z\}$ *then*

$$u^{k,\varepsilon}_z(x)=\mathrm{G}^k_\Omega(z,x)+\int_{\partial\Omega_N}\partial_{\nu_y}\mathrm{G}^k_\Omega(x,y)\left(u^{k,\varepsilon}_z(y)-\mathrm{G}^k_\Omega(z,y)\right)\mathrm{d}\sigma_y\,.$$

Proof With Green's formula we have

$$\begin{aligned} u_z^{k,\varepsilon}(x) &= \int_\Omega \left(\triangle + k^2\right) \mathrm{G}_\Omega^k(x,y)\, u_z^{k,\varepsilon}(y)\mathrm{d}y \\ &= \int_\Omega \mathrm{G}_\Omega^k(x,y)\ \left(\triangle + k^2\right) u_z^{k,\varepsilon}(y)\mathrm{d}y \\ &\qquad + \int_{\partial\Omega} \partial_{\nu_y}\mathrm{G}_\Omega^k(x,y)\, u_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y - \int_{\partial\Omega} \mathrm{G}_\Omega^k(x,y)\, \partial_{\nu_y} u_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y \\ &= \mathrm{G}_\Omega^k(x,z) + \int_{\partial\Omega_N} \partial_{\nu_y}\mathrm{G}_\Omega^k(x,y)\, u_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y - 0\,. \end{aligned}$$

We claim that for $z, x \in \Omega$, $z \neq x$, we have that $\mathrm{G}_\Omega^k(z,x) = \mathrm{G}_\Omega^k(x,z)$. With that claim we conclude then

$$\begin{aligned} u_z^{k,\varepsilon}(x) =& \mathrm{G}_\Omega^k(z,x) + \int_{\partial\Omega_N} \partial_{\nu_y}\mathrm{G}_\Omega^k(x,y)\, u_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y \\ =& \mathrm{G}_\Omega^k(z,x) + \int_{\partial\Omega_N} \partial_{\nu_y}\mathrm{G}_\Omega^k(x,y)\, u_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y - \int_{\partial\Omega_N} \partial_{\nu_y}\mathrm{G}_\Omega^k(x,y)\, \mathrm{G}_\Omega^k(z,y)\mathrm{d}\sigma_y \\ =& \mathrm{G}_\Omega^k(z,x) + \int_{\partial\Omega_N} \partial_{\nu_y}\mathrm{G}_\Omega^k(x,y)\, \left(u_z^{k,\varepsilon}(y) - \mathrm{G}_\Omega^k(z,y)\right) \mathrm{d}\sigma_y\,. \end{aligned}$$

Thus the proof follows. To prove the claim, consider that with Green's formula

$$\begin{aligned} \mathrm{G}_\Omega^k(z,x) &= \int_\Omega \left(\triangle + k^2\right) \mathrm{G}_\Omega^k(x,y)\, \mathrm{G}_\Omega^k(z,y)\mathrm{d}y\,, \\ &= \int_\Omega \mathrm{G}_\Omega^k(x,y)\ \left(\triangle + k^2\right) \mathrm{G}_\Omega^k(z,y)\mathrm{d}y + 0 + 0\,, \\ &= \mathrm{G}_\Omega^k(x,z)\,, \end{aligned}$$

which is exactly what we wanted. ■

Let us define $v_z^{k,\varepsilon}(x) \coloneqq u_z^{k,\varepsilon}(x) - \mathrm{G}_\Omega^k(z,x)$. Thus we see that $v_z^{k,\varepsilon}$ satisfies the partial differential equation

$$\begin{cases} \left(\triangle_x + k^2\right) v_z^{k,\varepsilon}(x) = 0 & \text{in} \quad \Omega\,, \\ v_z^{k,\varepsilon}(x) = 0 & \text{on} \quad \partial\Omega_D\,, \\ \partial_{\nu_x} v_z^{k,\varepsilon}(x) = -\partial_{\nu_x}\mathrm{G}_\Omega^k(z,x) & \text{on} \quad \partial\Omega_N\,, \end{cases} \tag{5.4}$$

and from Proposition 5.5 we have for $x \in \Omega \setminus \{z\}$

$$v_z^{k,\varepsilon}(x) = \int_{\partial\Omega_N} \partial_{\nu_y}\mathrm{G}_\Omega^k(x,y)\, v_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y\,. \tag{5.5}$$

Combining (5.4) and (5.5) we obtain the following condition for $v_z^{k,\varepsilon}$:

Lemma 5.6 *Let $z \in \Omega$ and $x \in \partial\Omega_N$ then*

$$-\partial_{\nu_x}\mathrm{G}_\Omega^k(z,x) = \nu_x \cdot \lim_{\substack{\widetilde{x}\to x \\ \widetilde{x}\in\Omega}} \nabla_{\widetilde{x}} \Big[\int_{\partial\Omega_N} \partial_{\nu_y}\mathrm{G}_\Omega^k(\widetilde{x},y)\, v_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y \Big]\,,$$

where ν_x is the outside normal at x.

Using Proposition 5.2, we have

$$\nabla_x \int_{\partial\Omega_N} \partial_{\nu_y} \mathrm{G}^k_\Omega(x,y)\, v_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y = \nabla_x \int_{\partial\Omega_N} \left[\frac{1}{2\pi} \frac{y_2 - x_2}{|y-x|^2} \right.$$
$$\left. - \frac{1}{8\pi} k^2 \,(y_2 - x_2)(2\log(k|y-x|) + 1) + \mathrm{R}^k_{\partial\mathrm{G},\Omega}(x,y) \right] v_z^{k,\varepsilon}(y)\mathrm{d}\sigma_y \,. \quad (5.6)$$

Using that $\partial\Omega_N$ is a line segment of length 2ε with center $(0,0)^\mathrm{T}$, we further compute that the right-hand-side in the last equation is

$$\nabla_x \int_{-\varepsilon}^{\varepsilon} \left[\frac{1}{2\pi} \frac{-x_2}{(t-x_1)^2 + x_2^2} - \frac{1}{8\pi} k^2 \,(-x_2) \left(2\log\left(k\sqrt{(t-x_1)^2 + x_2^2} \right) + 1 \right) \right.$$
$$\left. + \mathrm{R}^k_{\partial\mathrm{G},\Omega}(x,(t,0)^\mathrm{T}) \right] v_z^{k,\varepsilon}((t,0)^\mathrm{T})\mathrm{d}t \,. \quad (5.7)$$

Pulling the ∇-operator inside the integral, then pulling the limes in Lemma 5.6 inside the integral, wherever possible, and considering that $\partial_{\nu_x} = \partial_{x_2}$, we obtain

$$\lim_{\substack{h\to 0 \\ h>0}} \int_{-\varepsilon}^{\varepsilon} \left[\frac{-1}{2\pi} \frac{((t-x_1)^2 - h) v_z^{k,\varepsilon}(t)}{((t-x_1)^2 + h)^2} + \frac{k^2\, v_z^{k,\varepsilon}(t)}{8\pi} \,(2\log(k|t-x_1|) + 1) \right.$$
$$\left. + v_z^{k,\varepsilon}(t) \partial_{x_2} \mathrm{R}^k_{\partial\mathrm{G},\Omega}(x,(t,0)^\mathrm{T}) \right] \mathrm{d}t \,, \quad (5.8)$$

where we identified $v_z^{k,\varepsilon}(t)$ with $v_z^{k,\varepsilon}((t,0)^\mathrm{T})$. This leads us to

$$-\partial_{\nu_x} \mathrm{G}^k_\Omega(z,x) = \frac{-1}{2\pi} \lim_{h\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{((t-x_1)^2 - h) v_z^{k,\varepsilon}(t)}{((t-x_1)^2 + h)^2} \mathrm{d}t + \frac{k^2}{4\pi} \int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t) \log|t-x_1| \,\mathrm{d}t \quad (5.9)$$
$$+ \frac{k^2}{8\pi} (2\log(k) + 1) \int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t)\mathrm{d}t + \int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t) \partial_{x_2} \mathrm{R}^k_{\partial\mathrm{G},\Omega}(x,(t,0)^\mathrm{T})\mathrm{d}t \,.$$

Consider that for $h > 0$

$$\int_{-\varepsilon}^{\varepsilon} \frac{((t-x_1)^2 - h)\, v_z^{k,\varepsilon}(t)}{((t-x_1)^2 + h)^2} \mathrm{d}t = \left[v_z^{k,\varepsilon}(t) \frac{x_1 - t}{h + (t-x_1)^2} \right]_{t=-\varepsilon}^{\varepsilon} + \int_{-\varepsilon}^{\varepsilon} \frac{(t-x_1)\, \partial_t v_z^{k,\varepsilon}(t)}{(t-x_1)^2 + h} \mathrm{d}t \,.$$

Using that $v_z^{k,\varepsilon}(-\varepsilon) = v_z^{k,\varepsilon}(\varepsilon) = 0$, because of the Dirichlet boundary, we can readily compute that

$$\lim_{h\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{((t-x_1)^2 - h) v_z^{k,\varepsilon}(t)}{((t-x_1)^2 + h)^2} \mathrm{d}t = -\, \mathop{\mathrm{p.v.}}\!\!\int_{-\varepsilon}^{\varepsilon} \frac{\partial_t v_z^{k,\varepsilon}(t)}{x_1 - t} \mathrm{d}t = \mathop{\mathrm{H.f.p.}}\!\!\int_{-\varepsilon}^{\varepsilon} \frac{v_z^{k,\varepsilon}(t)}{(x_1 - t)^2} \mathrm{d}t \,,$$

where the last integral is the Hadamard-finite-part integral.

Definition 5.7 *We define the operators $\mathcal{H}^\varepsilon : \mathcal{X}^\varepsilon \to \mathcal{X}^\varepsilon$ and $\mathcal{J}^\varepsilon : \mathcal{Y}^\varepsilon_\Pi \to \mathcal{X}^\varepsilon$ as*

$$\mathcal{H}^\varepsilon[\mu](\tau) := \text{p.v.} \int_{-\varepsilon}^{\varepsilon} \frac{\mu(t)}{\tau - t} \mathrm{d}t\,,$$
$$\mathcal{J}^\varepsilon[\mu](\tau) := \text{H.f.p.} \int_{-\varepsilon}^{\varepsilon} \frac{\mu(t)}{(\tau - t)^2} \mathrm{d}t\,.$$

Remark 5.8 *From the discussion above we especially obtain formulas for the operators $\mathcal{H}^\varepsilon$ and $\mathcal{J}^\varepsilon$ for $\mu \in \mathcal{X}^\varepsilon$ and $\mu \in \mathcal{Y}^\varepsilon_\Pi$, respectively. These are:*

$$\mathcal{H}^\varepsilon[\mu] = \text{p.v.} \int_{-\varepsilon}^{\varepsilon} \frac{\mu(t)}{\tau - t} \mathrm{d}t = \lim_{\substack{h \to 0 \\ h > 0}} \int_{-\varepsilon}^{\varepsilon} \frac{(\tau - t)\, \mu(t)}{(\tau - t)^2 + h} \mathrm{d}t\,,$$
$$\mathcal{J}^\varepsilon[\mu] = \text{H.f.p.} \int_{-\varepsilon}^{\varepsilon} \frac{\mu(t)}{(\tau - t)^2} \mathrm{d}t = -\mathcal{H}^\varepsilon[\partial_\tau \mu] = \lim_{\substack{h \to 0 \\ h > 0}} \int_{-\varepsilon}^{\varepsilon} \frac{((\tau - t)^2 - h)\mu(t)}{((\tau - t)^2 + h)^2} \mathrm{d}t\,.$$

With Definition 5.7 and (5.9) we then obtain the following proposition:

Proposition 5.9 *Let $z \in \Omega$ and $\tau \in (-\varepsilon, \varepsilon)$, then*

$$-\partial_{\nu_x} \mathrm{G}^k_\Omega \left(z, \begin{pmatrix} \tau \\ 0 \end{pmatrix} \right) = -\frac{1}{2\pi} \mathcal{J}^\varepsilon[v_z^{k,\varepsilon}] + \frac{k^2}{4\pi} \mathcal{L}^\varepsilon[v_z^{k,\varepsilon}]$$
$$+ \frac{k^2}{8\pi}(2\log(k) + 1) \int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t) \mathrm{d}t + \int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t) \partial_{x_2} \mathrm{R}^k_{\partial\mathrm{G},\Omega} \left(\begin{pmatrix} \tau \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix} \right) \mathrm{d}t\,,$$

where $\mathcal{L}^\varepsilon$ is defined in Definition 3.9 and ν_x denotes the outward normal at $(\tau, 0)^\mathrm{T}$.

5.3.2 Hypersingular Operator Analysis

We know that $\mathcal{L}^\varepsilon : \mathcal{X}^\varepsilon \to \mathcal{Y}^\varepsilon$ is an isomorphism, where the inverse is given in Proposition 3.11. From [26, Chapter 11.5] we get that $\mathcal{J}^\varepsilon : \mathcal{Y}^\varepsilon_\Pi \to \mathcal{X}^\varepsilon$ is an isomorphism. Moreover, we have the following formula for the inverse.

Proposition 5.10 *Let $0 < \varepsilon < 2$. The operator $\mathcal{J}^\varepsilon : \mathcal{Y}^\varepsilon_\Pi \to \mathcal{X}^\varepsilon$ is linear and invertible and has the inverse*

$$(\mathcal{J}^\varepsilon)^{-1}[\eta](t) = -\frac{1}{\pi^2 \sqrt{\varepsilon^2 - t^2}} \left(\text{p.v.} \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2} \int_{-\varepsilon}^{\tau} \eta(\tilde{\tau}) \mathrm{d}\tilde{\tau}}{t - \tau} \mathrm{d}\tau + t\, C_{\mathcal{J},1} + C_{\mathcal{J},2} \right),$$

where

$$C_{\mathcal{J},1} = -\int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{\varepsilon^2-\tau^2}} \left(\int_{-\varepsilon}^{\tau} \eta(\tilde{\tau}) \mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau \,,$$

$$C_{\mathcal{J},2} = -\int_{-\varepsilon}^{\varepsilon} \frac{\tau}{\sqrt{\varepsilon^2-\tau^2}} \left(\int_{-\varepsilon}^{\tau} \eta(\tilde{\tau}) \mathrm{d}\tilde{\tau} \right) \mathrm{d}\tau \,.$$

are constants depending on η and they are linear in η.

Proof The proof for invertibility is given in [26, Chapter 11.5]. Thus for every $\mu \in \mathcal{Y}_{\Pi}^{\varepsilon}$ there exists exactly one $\eta \in \mathcal{X}^{\varepsilon}$ such that $\mathcal{J}^{\varepsilon}[\mu] = \eta$. Using the fact that the Hadamard-finite-part integral can be expressed as $\mathcal{J}^{\varepsilon}[\mu] = \partial_\tau \mathcal{H}^{\varepsilon}[\mu]$, and that $\mathcal{H}^{\varepsilon}$ is isomorphic up to a one dimensional kernel of the form $\ker(\mathcal{H}^{\varepsilon}) = \operatorname{span}\{(\varepsilon^2 - t^2)^{-1/2}\}$, and the inverse on $\ker(\mathcal{H}^{\varepsilon})^{\perp}$, which we call $(\mathcal{H}^{\varepsilon})^{\dagger}$, is of the following form (see for instance in [23]; it is also used in [8, Chapter 5.2.3])

$$(\mathcal{H}^{\varepsilon})^{\dagger}[\eta](t) = -\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}} \operatorname{p\!f\!v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\eta(\tau)}{t-\tau} \mathrm{d}\tau \,,$$

we can rewrite $\mathcal{J}^{\varepsilon}[\mu] = \eta$ as $\mathcal{H}^{\varepsilon}[\mu] = \int_{-\varepsilon}^{\tau} \eta + C_{\int}$ and then write

$$\begin{aligned}
\mu(t) =& (\mathcal{H}^{\varepsilon})^{\dagger} \left[\int_{-\varepsilon}^{\tau} \eta + C_{\int} \right] + \frac{C_\eta}{\sqrt{\varepsilon^2-t^2}} \,, \\
=& -\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}} \left(\operatorname{p\!f\!v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\int_{-\varepsilon}^{\tau}\eta}{t-\tau}\mathrm{d}\tau + \operatorname{p\!f\!v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,C_{\int}}{t-\tau}\mathrm{d}\tau - \pi^2 C_\eta \right) , \quad (5.10) \\
=& -\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}} \left(\operatorname{p\!f\!v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\int_{-\varepsilon}^{\tau}\eta}{t-\tau}\mathrm{d}\tau + \pi\, t\, C_{\int} - \pi^2 C_\eta \right) , \quad (5.11)
\end{aligned}$$

where we computed $\operatorname{p\!f\!v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}}{t-\tau}\mathrm{d}\tau = \pi\, t$, see [8, Chapter 5.2.3].

Let us find explicit expressions for the constants C_η and $C_{\int}$. Consider that the part in between the brackets in (5.11) has to be zero for the values $t = \varepsilon$ and $t = -\varepsilon$, so that we can satisfy the condition $\mu \in \mathcal{Y}_{\Pi}^{\varepsilon}$. This leads us to the system of equations

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}}{\sqrt{\varepsilon-\tau}} \left(\int_{-\varepsilon}^{\tau} \eta \right) \mathrm{d}\tau + \pi\varepsilon\, C_{\int} - \pi^2 C_\eta = 0 \,,$$

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon-\tau}}{-\sqrt{\varepsilon+\tau}} \left(\int_{-\varepsilon}^{\tau} \eta \right) \mathrm{d}\tau - \pi\varepsilon\, C_{\int} - \pi^2 C_\eta = 0 \,.$$

After solving this, we obtain

$$C_{\int} = -\frac{1}{2\varepsilon\pi} \operatorname{p\!f\!v.}\int_{-\varepsilon}^{\varepsilon} \frac{2\varepsilon}{\sqrt{\varepsilon^2-\tau^2}} \left(\int_{-\varepsilon}^{\tau} \eta\right) \mathrm{d}\tau\,,$$

$$C_{\eta} = \frac{1}{2\pi^2} \operatorname{p\!f\!v.}\int_{-\varepsilon}^{\varepsilon} \frac{2\tau}{\sqrt{\varepsilon^2-\tau^2}} \left(\int_{-\varepsilon}^{\tau} \eta\right) \mathrm{d}\tau\,.$$

This proves Proposition 5.10. ■

Let us consider the operator $-\frac{1}{2\pi}\mathcal{J}^{\varepsilon} + \frac{k^2}{4\pi}\mathcal{L}^{\varepsilon}$.

Proposition 5.11 *Let ε be small enough, $\alpha, \beta > 0$. The operator $-\alpha\mathcal{J}^{\varepsilon} + \beta\mathcal{L}^{\varepsilon} : \mathcal{Y}_{\Pi}^{\varepsilon} \to \mathcal{X}^{\varepsilon}$ is linear and invertible and for $\eta \in \mathcal{X}^{\varepsilon}$, the inverse is given by*

$$(-\alpha\mathcal{J}^{\varepsilon} + \beta\mathcal{L}^{\varepsilon})^{-1}[\eta](t) = (\mathcal{L}^{\varepsilon})^{-1}[\mu_{\iota}](t)\,, \tag{5.12}$$

where

$$\begin{aligned}\mu_{\iota}(\tau) =& C_{(5.13),1}\, e^{\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}} + C_{(5.13),2}\, e^{-\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}} \\ &+ \frac{1}{2\sqrt{\alpha\beta}} e^{-\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}} \int_{-\varepsilon}^{\tau} \eta(s)\, e^{\frac{\sqrt{\beta}s}{\sqrt{\alpha}}}\,\mathrm{d}s - \frac{1}{2\sqrt{\alpha\beta}} e^{\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}} \int_{-\varepsilon}^{\tau} \eta(s)\, e^{-\frac{\sqrt{\beta}s}{\sqrt{\alpha}}}\,\mathrm{d}s\,,\end{aligned} \tag{5.13}$$

where $C_{(5.13),1} \in \mathbb{C}$ and $C_{(5.13),2} \in \mathbb{C}$ are given through solving the system of equations

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}\,\partial_{\tau}\mu_{\iota}(\tau)}{\sqrt{\varepsilon-\tau}}\mathrm{d}\tau - \frac{\pi}{\log(\varepsilon/2)}\, C_{\mathcal{L}^{\varepsilon}[\mu_{\iota}]} = 0\,, \tag{5.14}$$

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon-\tau}\,\partial_{\tau}\mu_{\iota}(\tau)}{-\sqrt{\varepsilon+\tau}}\mathrm{d}\tau - \frac{\pi}{\log(\varepsilon/2)}\, C_{\mathcal{L}^{\varepsilon}[\mu_{\iota}]} = 0\,. \tag{5.15}$$

Proof From [26, Chapter 11.1] we have that $-\alpha\mathcal{J}^{\varepsilon} + \beta\mathcal{L}^{\varepsilon}$ is a Fredholm operator with index 0. Thus we only have to show that it is injective.
Let us show that $-\alpha\mathcal{J}^{\varepsilon} + \beta\mathcal{L}^{\varepsilon}$ is injective. To this end, consider that with Fubini's theorem we have

$$\mathcal{L}^{\varepsilon}[\mu](\tau) = \int_{-\varepsilon}^{\varepsilon} \log(|\tau - t|)\mu(t)\mathrm{d}t = \int_{-\varepsilon}^{\tau} \mathcal{H}^{\varepsilon}[\mu](\tilde{\tau})\mathrm{d}\tilde{\tau} - C_{(5.16)}\,, \tag{5.16}$$

where $C_{(5.16)} = -\int_{-\varepsilon}^{\varepsilon} \log(|\varepsilon + t|)\mu(t)\mathrm{d}t$. Then we get that

$$(-\alpha\mathcal{J}^{\varepsilon} + \beta\mathcal{L}^{\varepsilon})[\mu](\tau) = -\alpha\,\partial_{\tau}\mathcal{H}^{\varepsilon}[\mu](\tau) + \beta \int_{-\varepsilon}^{\tau} \mathcal{H}^{\varepsilon}[\mu](\tilde{\tau})\mathrm{d}\tilde{\tau} - \beta C_{(5.16)}\,.$$

Now, let $\mu \in \ker(-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon)$, it follows that μ satisfies

$$-\frac{\alpha}{\beta}\,\partial_\tau \mathcal{H}^\varepsilon[\mu](\tau) + \int_{-\varepsilon}^{\tau} \mathcal{H}^\varepsilon[\mu](\tilde{\tau})\mathrm{d}\tilde{\tau} = C_{(5.16)}\,. \tag{5.17}$$

Deriving both sides, we obtain

$$-\frac{\alpha}{\beta}\,\partial_\tau^2 \mathcal{H}^\varepsilon[\mu](\tau) + \mathcal{H}^\varepsilon[\mu](\tau) = 0\,.$$

With the substitution $\mathcal{M} := \mathcal{H}^\varepsilon[\mu]$ we obtain a second-order linear ordinary differential equation with the solution

$$\mathcal{M}(\tau) = C_{(5.18),1}\,\exp\left(\frac{\sqrt{\beta}\,\tau}{\sqrt{\alpha}}\right) + C_{(5.18),2}\,\exp\left(-\frac{\sqrt{\beta}\,\tau}{\sqrt{\alpha}}\right)\,. \tag{5.18}$$

Inserting $\mathcal{M}$ into (5.17), we obtain

$$-C_{(5.18),1}\,e^{-\varepsilon\frac{\sqrt{\alpha}}{\sqrt{\beta}}} + C_{(5.18),2}\,e^{\varepsilon\frac{\sqrt{\alpha}}{\sqrt{\beta}}} = \frac{\sqrt{\beta}}{\sqrt{\alpha}} C_{(5.16)}\,. \tag{5.19}$$

Equation $\mathcal{M} = \mathcal{H}^\varepsilon[\mu]$ has the general solution

$$\mu(t) = -\frac{1}{\pi^2\sqrt{\varepsilon^2 - t^2}}\left(\mathrm{p.v.}\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\mathcal{M}(\tau)}{t-\tau}\mathrm{d}\tau - \pi^2\,C_{\mathcal{M}}\right)\,, \tag{5.20}$$

compare the proof of Proposition 5.10. We insert this expression into $C_{(5.16)}$ and obtain

$$\begin{aligned}
C_{(5.16)} =& C_{(5.18),1}\,\frac{1}{\pi^2}\int_{-\varepsilon}^{\varepsilon} \log(|\varepsilon+t|)\frac{1}{\sqrt{\varepsilon^2-t^2}}\left(\mathrm{p.v.}\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon^2-\tau^2}}{t-\tau}\exp\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\tau\right)\mathrm{d}\tau\right)\mathrm{d}t \\
&+ C_{(5.18),2}\,\frac{1}{\pi^2}\int_{-\varepsilon}^{\varepsilon} \log(|\varepsilon+t|)\frac{1}{\sqrt{\varepsilon^2-t^2}}\left(\mathrm{p.v.}\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon^2-\tau^2}}{t-\tau}\exp\left(-\frac{\sqrt{\beta}}{\sqrt{\alpha}}\tau\right)\mathrm{d}\tau\right)\mathrm{d}t \\
&- C_{\mathcal{M}}\int_{-\varepsilon}^{\varepsilon}\log(|\varepsilon+t|)\frac{1}{\sqrt{\varepsilon^2-t^2}}\mathrm{d}t\,.
\end{aligned}$$

Consider that $\mu \in \mathcal{Y}^\varepsilon_{\Pi}$, this implies that the expression inside the brackets in (5.20) has to be zero for $t = \varepsilon$ and $t = -\varepsilon$. This leads us to the system of equations

$$C_{(5.18),1}\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon+\tau}\,e^{\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}}}{\sqrt{\varepsilon-\tau}}\mathrm{d}\tau + C_{(5.18),2}\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon+\tau}\,e^{-\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}}}{\sqrt{\varepsilon-\tau}}\mathrm{d}\tau - \pi^2\,C_{\mathcal{M}} = 0\,, \tag{5.21}$$

$$C_{(5.18),1}\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon-\tau}\,e^{\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}}}{-\sqrt{\varepsilon+\tau}}\mathrm{d}\tau + C_{(5.18),2}\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon-\tau}\,e^{-\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}}}{-\sqrt{\varepsilon+\tau}}\mathrm{d}\tau - \pi^2\,C_{\mathcal{M}} = 0\,. \tag{5.22}$$

Using the power series of the exponential function, we have

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}\, e^{\pm\frac{\sqrt{\beta}\tau}{\sqrt{\alpha}}}}{\sqrt{\varepsilon-\tau}} \mathrm{d}\tau = \varepsilon\pi \pm \frac{\sqrt{\beta}}{\sqrt{\alpha}} \frac{\varepsilon^2\pi}{2} + \mathcal{O}(\varepsilon^3)\,,$$

We can compute that

$$\int_{-\varepsilon}^{\varepsilon} \log(|\varepsilon+t|) \frac{1}{\sqrt{\varepsilon^2-t^2}} \mathrm{d}t = \pi \log\left(\frac{\varepsilon}{2}\right)\,.$$

With the mathematics tool Mathematica [19] we can further compute for $c \in \mathbb{R}$

$$\int_{-\varepsilon}^{\varepsilon} \log(|\varepsilon+t|) \frac{1}{\sqrt{\varepsilon^2-t^2}} \left(\mathrm{p.v.}\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}}{t-\tau} \exp(c\tau) \mathrm{d}\tau \right) \mathrm{d}t = \varepsilon\pi^2 - c\frac{\varepsilon^2\pi^2}{4} + \mathcal{O}(\varepsilon^3)\,.$$

Using (5.19), (5.21), (5.22), we get a 3×3 system of equations, whose only solution is $C_{(5.18),1} = C_{(5.18),2} = C_{\mathcal{M}} = 0$, for ε small enough. We conclude $\mu = 0$ and that $-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon$ is an injective Fredholm operator of index 0, hence it is invertible.

Let us find the inverse of $-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon$.

Now that we know that $-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon$ is invertible, we can reformulate the inverse of the operator as

$$(-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon)^{-1} = ((-\alpha\mathcal{J}^\varepsilon(\mathcal{L}^\varepsilon)^{-1} + \beta\mathcal{I})\mathcal{L}^\varepsilon)^{-1} = (\mathcal{L}^\varepsilon)^{-1}(\beta\mathcal{I} - \alpha\mathcal{J}^\varepsilon(\mathcal{L}^\varepsilon)^{-1})^{-1}\,.$$

where $\beta\mathcal{I} - \alpha\mathcal{J}^\varepsilon(\mathcal{L}^\varepsilon)^{-1}$ is an operator from $\mathcal{L}^\varepsilon(\mathcal{Y}^\varepsilon_\Pi)$ to $\mathcal{X}^\varepsilon$, and it is invertible, because $-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon$ and $\mathcal{L}^\varepsilon$ are, and where $\mathcal{I}$ denotes the identity operator on $\mathcal{X}^\varepsilon$. Consider that

$$\begin{aligned}\mathcal{J}^\varepsilon(\mathcal{L}^\varepsilon)^{-1}[\eta] =& \partial_\tau \mathcal{H}^\varepsilon \left[(\mathcal{H}^\varepsilon)^\dagger[\partial_\tau\eta] + \frac{C_{\mathcal{L}}[\eta]}{\pi\log(\varepsilon/2)\sqrt{\varepsilon^2-t^2}} \right] \\ =& \partial_\tau^2\eta + 0\,,\end{aligned}$$

where $(\mathcal{H}^\varepsilon)^\dagger$ is discussed in the proof of Proposition 5.10. Now the general form of the solution to $(\beta\mathcal{I} - \alpha\partial_\tau^2)[\mu_\iota] = \eta$ is

$$\begin{aligned}\mu_\iota(t) =& C_{(5.23),1}\, e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} + C_{(5.23),2}\, e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} \\ &+ \frac{1}{2\sqrt{\alpha\beta}} e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} \int_{-\varepsilon}^{t} \eta(s)\, e^{\frac{\sqrt{\beta}s}{\sqrt{\alpha}}} \mathrm{d}s - \frac{1}{2\sqrt{\alpha\beta}} e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} \int_{-\varepsilon}^{t} \eta(s)\, e^{-\frac{\sqrt{\beta}s}{\sqrt{\alpha}}} \mathrm{d}s\,. \end{aligned} \tag{5.23}$$

Then the solution of $(\beta\mathcal{L}^\varepsilon - \alpha\mathcal{J}^\varepsilon)[\mu] = \eta$ is given through $\mu = (\mathcal{L}^\varepsilon)^{-1}[\mu_\iota]$, where the constant $C_{(5.23),1}$ and $C_{(5.23),2}$ are chosen such that $(\mathcal{L}^\varepsilon)^{-1}[\mu_\iota] \in \mathcal{Y}^\varepsilon_\Pi$, which results in solving a 2×2 matrix. ■

Lemma 5.12 *Let ε be small enough, and $\alpha, \beta > 0$. We have that*

$$(-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon)^{-1}[1](t) = \frac{1}{\beta}(\mathcal{L}^\varepsilon)^{-1}[1] - C_{(5.24)}(\mathcal{L}^\varepsilon)^{-1}\left[\cosh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\,t\right)\right], \tag{5.24}$$

where

$$C_{(5.24)} = \frac{1}{\beta}\frac{1}{\left(1 - \frac{\beta}{2\alpha}\varepsilon^2(\log(\varepsilon/2) - \frac{1}{2})) + \mathcal{O}(\varepsilon^3)\right)},$$

Using the power series of cosh, the difference between $C_{(5.24)}(\mathcal{L}^\varepsilon)^{-1}\left[\cosh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\,t\right)\right]$ and $\frac{1}{\beta}(\mathcal{L}^\varepsilon)^{-1}[1]$ yields a term in $\mathcal{O}(\varepsilon/|\log(\varepsilon)|)$.

Proof Using the notation in Proposition 5.11, we have $\eta = 1$, thus

$$\begin{aligned}\mu_\iota(t) =& C_{(5.25),1}\, e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} + C_{(5.25),2}\, e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} \\ &+ \frac{1}{2\sqrt{\alpha\beta}}e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}}\int_{-\varepsilon}^{t} e^{\frac{\sqrt{\beta}s}{\sqrt{\alpha}}}\,\mathrm{d}s - \frac{1}{2\sqrt{\alpha\beta}}e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}}\int_{-\varepsilon}^{t} e^{-\frac{\sqrt{\beta}s}{\sqrt{\alpha}}}\,\mathrm{d}s \qquad (5.25)\\ =& C_{(5.26),1}\, e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} + C_{(5.26),2}\, e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} + \frac{1}{\beta}. \qquad (5.26)\end{aligned}$$

Thus

$$\partial_t\mu_\iota(t) = \frac{\sqrt{\beta}}{\sqrt{\alpha}}C_{(5.26),1}\, e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} - \frac{\sqrt{\beta}}{\sqrt{\alpha}}C_{(5.26),2}\, e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}}.$$

Let us solve the system of equations (5.14), (5.15). We readily see that

$$\int_{-\varepsilon}^{\varepsilon}\frac{2\varepsilon\,\partial_\tau\mu_\iota(\tau)}{\sqrt{\varepsilon^2-\tau^2}}\mathrm{d}\tau = 0.$$

Using the mathematics tool Mathematica [19], we obtain that $\int_{-\varepsilon}^{\varepsilon}\frac{\exp(c\,t)}{\sqrt{\varepsilon^2-t^2}}\mathrm{d}t = \pi\,\mathrm{I}_0(\varepsilon\,c)$, for $c \in \mathbb{R}$, where I_n is the modified Bessel function of the first kind. This leads us to

$$0 = C_{(5.26),1}\;\mathrm{I}_0(\varepsilon\,\frac{\sqrt{\beta}}{\sqrt{\alpha}}) - C_{(5.26),2}\;\mathrm{I}_0(-\varepsilon\,\frac{\sqrt{\beta}}{\sqrt{\alpha}}).$$

Since I_0 is even we have $C_{(5.26),1} = C_{(5.26),2}$. Thus

$$\partial_t\mu_\iota(t) = \frac{\sqrt{\beta}}{\sqrt{\alpha}}C_{(5.27)}\;\sinh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}t\right). \tag{5.27}$$

Now we solve

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}\,\partial_\tau \mu_\iota(\tau)}{\sqrt{\varepsilon-\tau}} \mathrm{d}\tau - \frac{\pi}{\log(\varepsilon/2)}\, C_{\mathcal{L}^\varepsilon[\mu_\iota]} = 0\,.$$

Then, we obtain

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}\,\partial_\tau \mu_\iota(\tau)}{\sqrt{\varepsilon-\tau}} \mathrm{d}\tau = \frac{\sqrt{\beta}}{\sqrt{\alpha}} C_{(5.27)} \varepsilon\, \pi\, \mathrm{I}_1\left(\varepsilon \frac{\sqrt{\beta}}{\sqrt{\alpha}}\right)\,.$$

and

$$\begin{aligned} C_{\mathcal{L}^\varepsilon[\mu_\iota]} =& \mu_\iota(0) - \mathcal{L}^\varepsilon\Big[- \frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}} \,\mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau \mu_\iota(\tau)}{t-\tau}\mathrm{d}\tau\Big](0)\,,\\ =& C_{(5.27)} + \frac{1}{\beta} + \frac{1}{\pi^2}\int_{-\varepsilon}^{\varepsilon} \frac{\log|t|}{\sqrt{\varepsilon^2-t^2}} \,\mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau \mu_\iota(\tau)}{t-\tau}\mathrm{d}\tau\mathrm{d}t\,,\\ =& C_{(5.27)}\left(1 + \frac{\sqrt{\beta}}{\sqrt{\alpha}}\frac{1}{\pi^2}\int_{-\varepsilon}^{\varepsilon} \frac{\log|t|}{\sqrt{\varepsilon^2-t^2}} \,\mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\sinh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\tau\right)}{t-\tau}\mathrm{d}\tau\mathrm{d}t\right) + \frac{1}{\beta}\,. \end{aligned}$$

Using the power series for the sinus hyperbolicus, we have

$$\int_{-\varepsilon}^{\varepsilon} \frac{\log|t|}{\sqrt{\varepsilon^2-t^2}} \,\mathrm{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\sinh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\tau\right)}{t-\tau}\mathrm{d}\tau\mathrm{d}t = \frac{\sqrt{\beta}}{\sqrt{\alpha}}\frac{\varepsilon^2\pi^2}{4} + \left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\right)^3 \frac{\varepsilon^4\pi^2}{64} + \mathcal{O}(\varepsilon^6)\,,$$

and

$$\mathrm{I}_1\left(\varepsilon\frac{\sqrt{\beta}}{\sqrt{\alpha}}\right) = \frac{\sqrt{\beta}}{\sqrt{\alpha}}\frac{\varepsilon}{2} + \left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\right)^3 \frac{\varepsilon^3}{16} + \mathcal{O}(\varepsilon^5)\,.$$

We infer that

$$C_{(5.27)}\frac{\sqrt{\beta}}{\sqrt{\alpha}}\pi\,\varepsilon\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^3)\right) - \frac{\pi}{\log(\varepsilon/2)} C_{(5.27)}\Big(1 + \frac{\beta}{\alpha}\frac{\varepsilon^2}{4} + \mathcal{O}(\varepsilon^4)\Big) - \frac{\pi}{\log(\varepsilon/2)}\frac{1}{\beta} = 0\,.$$

This leads us to

$$C_{(5.27)}\Big(- \frac{\pi}{\log(\varepsilon/2)} + \frac{\beta}{\alpha}\frac{\pi}{2}\varepsilon^2\Big(1 - \frac{1}{2\log(\varepsilon/2)}\Big) + \mathcal{O}(\varepsilon^3)\Big) = \frac{\pi}{\log(\varepsilon/2)}\frac{1}{\beta}\,.$$

Thus

$$C_{(5.27)}\Big(1 + \mathcal{O}(\varepsilon)\Big) = -\frac{1}{\beta}\,.$$

Hence, we proved Lemma 5.12. ■

Lemma 5.13 *Let ε be small enough, and $\alpha, \beta > 0$. Let $\mathcal{R}$ be the integral operator defined from $\mathcal{Y}^{\varepsilon}_{\Pi}$ into $\mathcal{X}^{\varepsilon}$ by*

$$\mathcal{R}[\mu](\tau) \;=\; \int_{-\varepsilon}^{\varepsilon} R(\tau,t)\,\mu(t)\mathrm{d}t\,,$$

where $R(\tau,t)$ is of class $\mathcal{C}^{0,1/2}$ in τ and t. For ε small enough, there exists a positive constant $C_{(5.28)}$, independent of ε, such that for all $\mu \in \mathcal{Y}^{\varepsilon}_{\Pi}$, we have

$$\left\|(-\alpha\mathcal{J}^{\varepsilon}+\beta\mathcal{L}^{\varepsilon})^{-1}\mathcal{R}[\mu]\right\|_{\mathcal{X}^{\varepsilon}} \leq \frac{\varepsilon\, C_{(5.28)}}{|\log(\varepsilon)|}\,\|R\|_{\mathcal{C}^{0,1/2}}\,\|\mu\|_{\mathcal{X}^{\varepsilon}}\,. \tag{5.28}$$

Proof We define $\eta := \mathcal{R}[\mu]$ and use then the notation in Proposition 5.11. It follows with Fubini's Theorem that

$$\begin{aligned}
\mu_{\iota}(t) =& C_{(5.29),1}\,e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} + C_{(5.29),2}\,e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} \\
&+ \frac{1}{2\sqrt{\alpha\beta}}e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}}\int_{-\varepsilon}^{t}\eta(s)\,e^{\frac{\sqrt{\beta}s}{\sqrt{\alpha}}}\,\mathrm{d}s - \frac{1}{2\sqrt{\alpha\beta}}e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}}\int_{-\varepsilon}^{t}\eta(s)\,e^{-\frac{\sqrt{\beta}s}{\sqrt{\alpha}}}\,\mathrm{d}s \\
=& C_{(5.29),1}\,e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} + C_{(5.29),2}\,e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}} + \frac{1}{2\sqrt{\alpha\beta}}\mathcal{R}_{\star}[\mu](t)\,,
\end{aligned} \tag{5.29}$$

where

$$\begin{aligned}
\mathcal{R}_{\star}[\mu](t) :=& \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{t} R(s,q)\,e^{\frac{\sqrt{\beta}(s-t)}{\sqrt{\alpha}}}\,\mu(q)\,\mathrm{d}s\,\mathrm{d}q - \int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{t} R(s,q)\,e^{-\frac{\sqrt{\beta}(s-t)}{\sqrt{\alpha}}}\,\mu(q)\,\mathrm{d}s\,\mathrm{d}q \\
=& \int_{-\varepsilon}^{\varepsilon}\left(\int_{-\varepsilon}^{t} 2\,R(s,q)\,\sinh\left(\frac{\sqrt{\beta}\,(s-t)}{\sqrt{\alpha}}\right)\mathrm{d}s\right)\,\mu(q)\mathrm{d}q \\
=:& \int_{-\varepsilon}^{\varepsilon} R_{\star}(t,q)\,\mu(q)\mathrm{d}q\,.
\end{aligned}$$

Let us examine those constants. They are given through solving the following system:

$$\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon+\tau}\,\partial_{\tau}\mu_{\iota}(\tau)}{\sqrt{\varepsilon-\tau}}\mathrm{d}\tau - \frac{\pi}{\log(\varepsilon/2)}\,C_{\mathcal{L}^{\varepsilon}[\mu_{\iota}]} = 0\,, \tag{5.30}$$

$$\int_{-\varepsilon}^{\varepsilon}\frac{2\varepsilon\,\partial_{\tau}\mu_{\iota}(\tau)}{\sqrt{\varepsilon^2-\tau^2}}\mathrm{d}\tau = 0\,. \tag{5.31}$$

We compute using $\mathcal{L}^\varepsilon[(\mathcal{L}^\varepsilon)^{-1}] = \mathcal{I}$ and $\mathcal{L}^\varepsilon[1/\sqrt{\varepsilon^2 - t^2}](0) = -\pi \log(\varepsilon/2)$ that

$$\begin{aligned}
C_{\mathcal{L}^\varepsilon[\mu_\iota]} =& \mu_\iota(0) + \mathcal{L}^\varepsilon\left[\frac{1}{\pi^2\sqrt{\varepsilon^2 - t^2}} \operatorname*{p.v.}_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2}\, \partial_\tau \mu_\iota(\tau)}{t - \tau} \mathrm{d}\tau\right](0)\,, \\
=& C_{(5.29),1} + C_{(5.29),2} + \frac{1}{2\sqrt{\alpha\beta}} C_{\mathcal{L}^\varepsilon[\mathcal{R}_\star[\mu]]} \\
&+ \frac{1}{\pi^2}\mathcal{L}^\varepsilon\left[\operatorname*{p.v.}_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2}}{\sqrt{\varepsilon^2 - t^2}} \left(\frac{\sqrt{\beta}}{\sqrt{\alpha}} C_{(5.29),1} \frac{e^{\frac{\sqrt{\beta}}{\sqrt{\alpha}}\tau}}{t - \tau} - \frac{\sqrt{\beta}}{\sqrt{\alpha}} C_{(5.29),2} \frac{e^{-\frac{\sqrt{\beta}}{\sqrt{\alpha}}\tau}}{t - \tau}\right)\mathrm{d}\tau\right](0)\,, \\
=& C_{(5.29),1} + C_{(5.29),2} + \frac{1}{2\sqrt{\alpha\beta}} C_{\mathcal{L}^\varepsilon[\mathcal{R}_\star[\mu]]} \\
&+ \frac{\sqrt{\beta}}{\sqrt{\alpha}} C_{(5.29),1} \left(\frac{\sqrt{\beta}}{\sqrt{\alpha}} \frac{\varepsilon^2}{4} + \mathcal{O}(\varepsilon^4)\right) - \frac{\sqrt{\beta}}{\sqrt{\alpha}} C_{(5.29),2} \left(-\frac{\sqrt{\beta}}{\sqrt{\alpha}} \frac{\varepsilon^2}{4} + \mathcal{O}(\varepsilon^4)\right)\,, \\
=& C_{(5.29),1}\left(1 + \frac{\beta}{4\alpha}\varepsilon^2 + \mathcal{O}(\varepsilon^4)\right) + C_{(5.29),2}\left(1 + \frac{\beta}{4\alpha}\varepsilon^2 + \mathcal{O}(\varepsilon^4)\right) \\
&+ \frac{1}{2\sqrt{\alpha\beta}} C_{\mathcal{L}^\varepsilon[\mathcal{R}_\star[\mu]]}\,.
\end{aligned}$$

and for $c \in \mathbb{R}$

$$\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon + \tau}\, \exp(c\,\tau)}{\sqrt{\varepsilon - \tau}} \mathrm{d}\tau = \pi\varepsilon + \pi\, c\, \frac{\varepsilon^2}{2} + \pi\, c^2\, \frac{\varepsilon^3}{2} + \mathcal{O}(\varepsilon^4)\,,$$

and

$$\int_{-\varepsilon}^{\varepsilon} \frac{\exp(c\,\tau)}{\sqrt{\varepsilon^2 - \tau^2}} \mathrm{d}\tau = \pi + \pi\, c^2 \frac{\varepsilon^2}{2} + \mathcal{O}(\varepsilon^4)\,.$$

This leads us to the 2×2 system of equations

$$\begin{aligned}
&\begin{pmatrix} -\frac{\pi}{\log(\varepsilon/2)} + \mathcal{O}(\varepsilon) & -\frac{\pi}{\log(\varepsilon/2)} + \mathcal{O}(\varepsilon) \\ \frac{\sqrt{\beta}}{\sqrt{\alpha}}\pi + \mathcal{O}(\varepsilon^2) & -\frac{\sqrt{\beta}}{\sqrt{\alpha}}\pi + \mathcal{O}(\varepsilon^2) \end{pmatrix} \begin{pmatrix} C_{(5.29),1} \\ C_{(5.29),2} \end{pmatrix} \\
&\qquad = \frac{-1}{2\sqrt{\alpha\beta}} \begin{pmatrix} \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}}{\sqrt{\varepsilon-\tau}} \partial_\tau \mathcal{R}_\star[\mu](\tau)\mathrm{d}\tau + C_{\mathcal{L}^\varepsilon[\mathcal{R}_\star[\mu]]} \\ \int_{-\varepsilon}^{\varepsilon} \frac{\partial_\tau \mathcal{R}_\star[\mu](\tau)}{\sqrt{\varepsilon^2 - \tau^2}} \mathrm{d}\tau \end{pmatrix}. \qquad (5.32)
\end{aligned}$$

Let us give an expansion in ε for the right-hand-side. According to see Equation (3.18),

we have that $\int_{-\varepsilon}^{\varepsilon} \mu(q)\mathrm{d}q \leq \sqrt{\pi}\,\|\mu\|_{\mathcal{X}^\varepsilon}$. Hence, we establish that

$$\begin{aligned}\partial_t \mathcal{R}_\star[\mu](t) =& -\int_{-\varepsilon}^{\varepsilon} \left(\int_{-\varepsilon}^{t} 2\,R(s,q)\,\cosh\left(\frac{\sqrt{\beta}\,(s-t)}{\sqrt{\alpha}}\right)\mathrm{d}s \right)\,\mu(q)\mathrm{d}q \\ &\leq \|R\|_\infty \frac{2\sqrt{\alpha}}{\sqrt{\beta}} \left|\sinh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}(\varepsilon+t)\right)\right| \sqrt{\pi}\,\|\mu\|_{\mathcal{X}^\varepsilon}\,,\end{aligned}$$

Then we have

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}}{\sqrt{\varepsilon-\tau}}\partial_\tau \mathcal{R}_\star[\mu](\tau)\mathrm{d}\tau \leq& \|R\|_\infty \frac{2\sqrt{\alpha}}{\sqrt{\beta}} \int_{-\varepsilon}^{\varepsilon} |\mu(q)|\mathrm{d}q \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon+\tau}}{\sqrt{\varepsilon-\tau}} \sinh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}(\varepsilon+\tau)\right)\mathrm{d}\tau \\ \leq& \frac{2\sqrt{\alpha}}{\sqrt{\beta}}\,\|R\|_\infty\,\pi\,\|\mu\|_{\mathcal{X}^\varepsilon}\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}\frac{3\varepsilon^2\pi}{2}+\mathcal{O}(\varepsilon^4)\right) \\ =& \pi\,\|R\|_\infty\,\|\mu\|_{\mathcal{X}^\varepsilon}\left(3\varepsilon^2\pi+\mathcal{O}(\varepsilon^4)\right)\,.\end{aligned}$$

Next we have that

$$\begin{aligned}\int_{-\varepsilon}^{\varepsilon} \frac{\partial_\tau \mathcal{R}_\star[\mu](\tau)}{\sqrt{\varepsilon^2-\tau^2}}\mathrm{d}\tau \leq& \pi\frac{2\sqrt{\alpha}}{\sqrt{\beta}}\,\|R\|_\infty\,\|\mu\|_{\mathcal{X}^\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{\sinh\left(\frac{\sqrt{\beta}}{\sqrt{\alpha}}(\varepsilon+t)\right)}{\sqrt{\varepsilon^2-\tau^2}}\mathrm{d}\tau \\ \leq& \pi^2\frac{2\sqrt{\alpha}}{\sqrt{\beta}}\,\|R\|_\infty\,\|\mu\|_{\mathcal{X}^\varepsilon}\left(\varepsilon\frac{\sqrt{\beta}}{\sqrt{\alpha}}+\mathcal{O}(\varepsilon^3)\right)\,.\end{aligned}$$

Finally,

$$\begin{aligned}C_{\mathcal{L}^\varepsilon[\mathcal{R}_\star[\mu]]} =& 0+\mathcal{L}^\varepsilon\left[\frac{1/\pi^2}{\sqrt{\varepsilon^2-t^2}}\,\mathrm{p\!\!\int\!\!\!\!\!\!\diagup_{-\varepsilon}^{\varepsilon}}\,\frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau\mathcal{R}_\star[\mu](\tau)}{t-\tau}\mathrm{d}\tau\right](-\varepsilon) \\ =& \int_{-\varepsilon}^{\varepsilon}\frac{\log(\varepsilon+t)/\pi^2}{\sqrt{\varepsilon^2-t^2}}\int_{-\varepsilon}^{\varepsilon}\sqrt{\varepsilon^2-\tau^2}\,\frac{\partial_\tau\mathcal{R}_\star[\mu](\tau)-\partial_\tau\mathcal{R}_\star[\mu](t)}{t-\tau}\mathrm{d}\tau\mathrm{d}t \\ &+\int_{-\varepsilon}^{\varepsilon}\frac{\log(\varepsilon+t)/\pi^2}{\sqrt{\varepsilon^2-t^2}}\,\mathrm{p\!\!\int\!\!\!\!\!\!\diagup_{-\varepsilon}^{\varepsilon}}\,\sqrt{\varepsilon^2-\tau^2}\frac{\partial_\tau\mathcal{R}_\star[\mu](t)}{t-\tau}\mathrm{d}\tau\mathrm{d}t \\ \leq& \int_{-\varepsilon}^{\varepsilon}\frac{\log(\varepsilon+t)/\pi^2}{\sqrt{\varepsilon^2-t^2}}C_{(5.33)}\pi\varepsilon\sqrt{\varepsilon}\,\mathrm{d}t\,|\partial_t\mathcal{R}_\star[\mu]|_{\mathcal{C}^{0,1/2}} \qquad (5.33) \\ &+\int_{-\varepsilon}^{\varepsilon}\frac{\log(\varepsilon+t)/\pi^2}{\sqrt{\varepsilon^2-t^2}}\,t\,\pi\mathrm{d}t\,\|\partial_\tau\mathcal{R}_\star[\mu]\|_{\mathcal{C}^0} \\ =& C_{(5.33)}\log(\varepsilon/2)\,\varepsilon\sqrt{\varepsilon}\,\,|\partial_t\mathcal{R}_\star[\mu]|_{\mathcal{C}^{0,1/2}}+\varepsilon\,\|\partial_\tau\mathcal{R}_\star[\mu]\|_{\mathcal{C}^0}\,, \qquad (5.34)\end{aligned}$$

where we used that $\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}}{\sqrt{|t-\tau|}} \mathrm{d}\tau \leq C_{(5.33)}\, \pi\, \varepsilon \sqrt{\varepsilon}$. We readily see that

$$|\partial_t \mathcal{R}_\star[\mu]|_{\mathcal{C}^{0,1/2}} \leq C_{(5.35)}\, |R|_{\mathcal{C}^{0,1/2}}\, \|\mu\|_{\mathcal{X}^\varepsilon}\,, \tag{5.35}$$

$$\|\partial_\tau \mathcal{R}_\star[\mu]\|_{\mathcal{C}^0} \leq \varepsilon\, C_{(5.36)}\, \|R\|_{\mathcal{C}^0}\, \|\mu\|_{\mathcal{X}^\varepsilon}\,, \tag{5.36}$$

where $C_{(5.35)} = \mathcal{O}(1)$, and $C_{(5.36)} = \mathcal{O}(1)$, for $\varepsilon \to 0$. Now we can solve (5.32), and obtain that

$$C_{(5.29),1} \leq C_{(5.37)}\, \varepsilon\, \|R\|_{\mathcal{C}^{0,1/2}}\, \|\mu\|_{\mathcal{X}^\varepsilon}\,, \tag{5.37}$$

$$C_{(5.29),2} \leq C_{(5.38)}\, \varepsilon\, \|R\|_{\mathcal{C}^{0,1/2}}\, \|\mu\|_{\mathcal{X}^\varepsilon}\,, \tag{5.38}$$

for ε small enough, where $C_{(5.37)} = \mathcal{O}(1)$ and $C_{(5.38)} = \mathcal{O}(1)$, for $\varepsilon \to 0$.
We have examined the constant. Now we estimate $\left\|(\mathcal{L}^\varepsilon)^{-1}[\mu_\iota]\right\|_{\mathcal{X}^\varepsilon}$. We have

$$\begin{aligned}(\mathcal{L}^\varepsilon)^{-1}[\mu_\iota](t) = C_{(5.29),1} \left\|(\mathcal{L}^\varepsilon)^{-1}\left[e^{\frac{\sqrt{\beta}t}{\sqrt{\alpha}}}\right]\right\|_{\mathcal{X}^\varepsilon} + C_{(5.29),2} \left\|(\mathcal{L}^\varepsilon)^{-1}\left[e^{-\frac{\sqrt{\beta}t}{\sqrt{\alpha}}}\right]\right\|_{\mathcal{X}^\varepsilon} \\ + \frac{1}{2\sqrt{\alpha\beta}} \left\|(\mathcal{L}^\varepsilon)^{-1}[\mathcal{R}_\star[\mu]]\right\|_{\mathcal{X}^\varepsilon}\,.\end{aligned} \tag{5.39}$$

Consider that for $c \in \mathbb{R}$, we have that

$$\left\|(\mathcal{L}^\varepsilon)^{-1}\left[e^{c\,t}\right]\right\|_{\mathcal{X}^\varepsilon} = \left\|(\mathcal{L}^\varepsilon)^{-1}\left[\int_{-\varepsilon}^{\varepsilon} e^{c\,t} \frac{1}{\pi\sqrt{\varepsilon^2-s^2}}\mathrm{d}s\right]\right\|_{\mathcal{X}^\varepsilon} \tag{5.40}$$

$$\leq \frac{C_{(5.41)}}{|\log(\varepsilon)|} \left\|\frac{1}{\pi\sqrt{\varepsilon^2-s^2}}\right\|_{\mathcal{X}^\varepsilon} \tag{5.41}$$

$$= \frac{C_{(5.42)}}{|\log(\varepsilon)|}\,. \tag{5.42}$$

where we used Lemma 3.14. Then

$$(\mathcal{L}^\varepsilon)^{-1}[\mathcal{R}_\star[\mu]] = -\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}} \fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\, \partial_\tau \mathcal{R}_\star[\mu](\tau)}{t-\tau}\mathrm{d}\tau + \frac{C_{\mathcal{L}^\varepsilon[\mathcal{R}_\star[\eta]]}}{\pi \log(\varepsilon/2)\sqrt{\varepsilon^2-t^2}}\,.$$

In (5.34) we already found out that $C_{\mathcal{L}^\varepsilon[\mathcal{R}_\star[\eta]]} = \mathcal{O}(\frac{\varepsilon}{-\log(\varepsilon/2)})$. For the other term, consider that

$$\begin{aligned}&-\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}} \fint_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\, \partial_\tau \mathcal{R}_\star[\mu](\tau)}{t-\tau}\mathrm{d}\tau \\ &\quad = \frac{-1/\pi^2}{\sqrt{\varepsilon^2-t^2}} \int_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2-\tau^2}\, \frac{\partial_\tau \mathcal{R}_\star[\mu](\tau) - \partial_\tau \mathcal{R}_\star[\mu](t)}{t-\tau}\mathrm{d}\tau \\ &\quad + \frac{-1/\pi^2}{\sqrt{\varepsilon^2-t^2}} \fint_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2-\tau^2} \frac{\partial_\tau \mathcal{R}_\star[\mu](t)}{t-\tau}\mathrm{d}\tau\,,\end{aligned}$$

thus

$$\begin{aligned}
&\left\| \frac{-1}{\pi^2\sqrt{\varepsilon^2-t^2}} \operatorname{p\!\!\int\!\!\!\!\!\!\diagup}_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}\,\partial_\tau \mathcal{R}_\star[\mu](\tau)}{t-\tau} \mathrm{d}\tau \right\|_{\mathcal{X}^\varepsilon}^2 \\
&\quad \leq \int_{-\varepsilon}^{\varepsilon} \frac{1/\pi^4}{\sqrt{\varepsilon^2-t^2}} \left| \int_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2-\tau^2}\, \frac{\partial_\tau \mathcal{R}_\star[\mu](\tau) - \partial_\tau \mathcal{R}_\star[\mu](t)}{t-\tau} \mathrm{d}\tau \right|^2 \mathrm{d}t \\
&\quad + \int_{-\varepsilon}^{\varepsilon} \frac{1/\pi^4}{\sqrt{\varepsilon^2-t^2}} \left| \operatorname{p\!\!\int\!\!\!\!\!\!\diagup}_{-\varepsilon}^{\varepsilon} \sqrt{\varepsilon^2-\tau^2} \frac{\partial_\tau \mathcal{R}_\star[\mu](t)}{t-\tau} \mathrm{d}\tau \right|^2 \mathrm{d}t \\
&\quad \leq \int_{-\varepsilon}^{\varepsilon} \frac{1/\pi^4}{\sqrt{\varepsilon^2-t^2}} \left| \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}}{\sqrt{|t-\tau|}} \mathrm{d}\tau \right|^2 \mathrm{d}t \, \|\partial_t \mathcal{R}_\star[\mu]\|_{\mathcal{C}^{0,1/2}}^2 \\
&\quad + \int_{-\varepsilon}^{\varepsilon} \frac{1/\pi^4}{\sqrt{\varepsilon^2-t^2}} \left| \operatorname{p\!\!\int\!\!\!\!\!\!\diagup}_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2-\tau^2}}{t-\tau} \mathrm{d}\tau \right|^2 \mathrm{d}t \, \|\partial_t \mathcal{R}_\star[\mu]\|_{\mathcal{C}^0}^2 \\
&\quad \leq C_{(5.33)}^2 \frac{\varepsilon^3}{\pi} |\partial_t \mathcal{R}_\star[\mu]|_{\mathcal{C}^{0,1/2}}^2 + \frac{\varepsilon^2}{2\pi} \|\partial_t \mathcal{R}_\star[\mu]\|_{\mathcal{C}^0}^2 \,.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\left\| (\mathcal{L}^\varepsilon)^{-1}[\mu_\iota] \right\|_{\mathcal{X}^\varepsilon} &= \|R\|_{\mathcal{C}^{0,1/2}} \left(\mathcal{O}\Big(\frac{\varepsilon}{|\log(\varepsilon)|}\Big) + \mathcal{O}\Big(\frac{\varepsilon}{|\log(\varepsilon)|}\Big) + \mathcal{O}(\varepsilon^{3/2}) + \mathcal{O}\Big(\frac{\varepsilon}{(\log(\varepsilon/2))^2}\Big) \right) \\
&= \|R\|_{\mathcal{C}^{0,1/2}} \, \mathcal{O}\Big(\frac{\varepsilon}{|\log(\varepsilon)|}\Big) \,.
\end{aligned}$$

This proves Lemma 5.13. ■

5.3.3 Solution to the Main Problem

We know from Proposition 5.9, that

$$\begin{aligned}
-\partial_{\nu_x} \mathrm{G}_\Omega^k \left(z, \begin{pmatrix} \tau \\ 0 \end{pmatrix} \right) &= \Big(-\frac{1}{2\pi} \mathcal{J}^\varepsilon + \frac{k^2}{4\pi} \mathcal{L}^\varepsilon \Big)[v_z^{k,\varepsilon}] \\
&\quad + \int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t) \left(\frac{k^2}{8\pi}(2\log(k)+1) + \partial_{x_2} \mathrm{R}_{\partial\mathrm{G},\Omega}^k \left(\begin{pmatrix} \tau \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix} \right) \right) \mathrm{d}t \,.
\end{aligned}$$

In the last subsection we saw that the operator $(-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon)$, with $\beta = \frac{k^2}{4\pi}$ and $\alpha = \frac{1}{2\pi}$, is invertible, thus we have

$$\begin{aligned}
v_z^{k,\varepsilon}(t) = &\Big(-\frac{1}{2\pi} \mathcal{J}^\varepsilon + \frac{k^2}{4\pi} \mathcal{L}^\varepsilon \Big)^{-1} \left[-\partial_{\nu_x} \mathrm{G}_\Omega^k \left(z, \begin{pmatrix} \tau \\ 0 \end{pmatrix} \right) \right](t) \\
- &\Big(-\frac{1}{2\pi} \mathcal{J}^\varepsilon + \frac{k^2}{4\pi} \mathcal{L}^\varepsilon \Big)^{-1} \left[\int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t) \left(\frac{k^2}{8\pi}(2\log(k)+1) + \partial_{x_2} \mathrm{R}_{\partial\mathrm{G},\Omega}^k \left(\begin{pmatrix} \tau \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix} \right) \right) \mathrm{d}t \right] .
\end{aligned}$$

Then Proposition 5.5 yields

$$
\begin{aligned}
u_z^{k,\varepsilon}(x) &= \mathrm{G}_\Omega^k(z,x) + \int_{\partial\Omega_N} \partial_{\nu_y} \mathrm{G}_\Omega^k(x,y) v_z^{k,\varepsilon}(y) \mathrm{d}\sigma_y \\
&= \mathrm{G}_\Omega^k(z,x) - \int_{\partial\Omega_N} \partial_{\nu_y} \mathrm{G}_\Omega^k(x,y) \Big(-\frac{1}{2\pi} \mathcal{J}^\varepsilon + \frac{k^2}{4\pi} \mathcal{L}^\varepsilon \Big)^{-1} \left[\partial_{\nu_x} \mathrm{G}_\Omega^k \left(z, \begin{pmatrix} \tau \\ 0 \end{pmatrix} \right) \right] (y) \mathrm{d}\sigma_y \\
&\quad + \int_{\partial\Omega_N} \partial_{\nu_y} \mathrm{G}_\Omega^k(x,y) \Big(\frac{-1}{2\pi} \mathcal{J}^\varepsilon + \frac{k^2}{4\pi} \mathcal{L}^\varepsilon \Big)^{-1} \Bigg[\int_{-\varepsilon}^{\varepsilon} v_z^{k,\varepsilon}(t) \Bigg(\frac{k^2}{8\pi} (2\log(k) + 1) \\
&\quad + \partial_{x_2} \mathrm{R}^k_{\partial\mathrm{G},\Omega} \left(\begin{pmatrix} \tau \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ 0 \end{pmatrix} \right) \Bigg) \mathrm{d}t \Bigg] (y) \mathrm{d}\sigma_y
\end{aligned}
$$

from where we obtain $\mathrm{G}_{(1)}^{k,\varepsilon}$ and $\mathrm{G}_{(2)}^{k,\varepsilon}$ in Theorem 5.4.

With Lemma 5.13, that is $\left\| (-\alpha \mathcal{J}^\varepsilon + \beta \mathcal{L}^\varepsilon)^{-1} \mathcal{R}[\mu] \right\|_{\mathcal{X}^\varepsilon} \leq \frac{\varepsilon\, C_{(5.28)}}{|\log(\varepsilon)|} \|R\|_{\mathcal{C}^{0,1/2}} \|\mu\|_{\mathcal{X}^\varepsilon}$, and with the simple reformulation

$$
-\partial_{\nu_x} \mathrm{G}_\Omega^k \left(z, \begin{pmatrix} \tau \\ 0 \end{pmatrix} \right) = \int_{-\varepsilon}^{\varepsilon} -\partial_{\nu_x} \mathrm{G}_\Omega^k \left(z, \begin{pmatrix} \tau \\ 0 \end{pmatrix} \right) \frac{1}{\pi \sqrt{\varepsilon^2 - t^2}} \mathrm{d}t \,,
$$

we can infer

$$
\left\| v_z^{k,\varepsilon} \right\|_{\mathcal{X}^\varepsilon} \leq \frac{\varepsilon\, C_{(5.43),1}}{|\log(\varepsilon)|} \left\| \partial_{\nu_x} \mathrm{G}_\Omega^k \left(z, \begin{pmatrix} \tau \\ 0 \end{pmatrix} \right) \right\|_{\mathcal{C}^{0,1/2}} + \frac{\varepsilon\, C_{(5.43),2}}{|\log(\varepsilon)|} \left\| v_z^{k,\varepsilon} \right\|_{\mathcal{X}^\varepsilon} . \tag{5.43}
$$

Consider that $\int_{-\varepsilon}^{\varepsilon} \mu(t) \mathrm{d}t \leq \sqrt{\pi} \|\mu\|_{\mathcal{X}^\varepsilon}$, see Equation (3.18). Hence we conclude for $z, x \in \Omega, z \neq x$ that

$$
|\mathrm{G}_{(1)}^{k,\varepsilon}(z,x)| \leq \frac{\varepsilon\, C_{(5.44)}}{|\log(\varepsilon)|} \,, \tag{5.44}
$$

$$
|\mathrm{G}_{(2)}^{k,\varepsilon}(z,x)| \leq \frac{\varepsilon^2\, C_{(5.45)}}{|\log(\varepsilon)|^2} \,. \tag{5.45}
$$

6 Nucleation of the Neumann Boundary Condition

In this section, based on Theorem 5.4, we derive a simple procedure to maximize the norm of the Green's function. The main idea is to nucleate the Neumann boundary conditions in order to increase the transmission between the point source and the receiver. By considering a disk shaped cavity, we illustrate by some numerical experiments the applicability of the proposed approach.

6.1 The Disk Case

Let Ω be the unit disk and let the source and the receiver be respectively $x_S = (x, y)$ and $x_R = (\xi, \eta)$. Suppose that the opening $\partial\Omega_N$ is an arc centered at $(1, 0)$ with length 2ε.

Denote by $\rho = \sqrt{x^2 + y^2}$ the distance between x_S and the origin, and by $\tilde{\rho} = \sqrt{\xi^2 + \eta^2}$ the distance between x_R and the origin. Define $\theta = \arctan(y/x)$, and $\tilde{\theta} = \arctan(\eta/\xi)$. It is well known that the Green's function in the unit disk is given by

$$G_\Omega^k(x_S, x_R) = -\frac{i}{4} H_0^{(1)}(k|x_S - x_R|) + \frac{i}{4} \sum_{n=-\infty}^{+\infty} A_n(k, \rho) J_n(k\tilde{\rho}) e^{in(\theta - \tilde{\theta})}. \tag{6.1}$$

Recall that the cylindrical wave expansion of the free-boundary Green's function is

$$-\frac{i}{4} H_0^{(1)}(k|x_S - x_R|) = -\frac{i}{4} \sum_{n=-\infty}^{+\infty} J_n(k\rho_<) H_n^{(1)}(k\rho_>) e^{in(\theta - \tilde{\theta})}, \tag{6.2}$$

where $\rho_< = \min(\rho, \tilde{\rho})$, $\rho_> = \max(\rho, \tilde{\rho})$. Substituting (6.2) into (6.1) yields

$$G_\Omega^k(x_S, x_R) = \frac{i}{4} \sum_{n=-\infty}^{\infty} (A_n J_n(k\rho) - J_n(k\rho_<) H_n^{(1)}(k\rho_>)) e^{in(\theta - \tilde{\theta})}. \tag{6.3}$$

Imposing the Dirichlet boundary condition on (6.3) gives

$$A_n(k, \rho) = \frac{J_n(k\rho) H_n^{(1)}(k)}{J_n(k)},$$

where J_n is the Bessel function of first kind and order n.

Hence the Green's function is

$$G_\Omega^k(x_S, x_R) = -\frac{i}{4} H_0^{(1)}(k|x_S - x_R|) + \frac{i}{4} \sum_{n=-\infty}^{\infty} \frac{J_n(k\rho) H_n^{(1)}(k)}{J_n(k)} J_n(k\tilde{\rho}), \tag{6.4}$$

and its normal derivative on $\partial\Omega_N$ is

$$\begin{aligned} \frac{\partial G_\Omega^k}{\partial \tilde{\rho}} \left(x_S, \begin{pmatrix} t \\ 1 \end{pmatrix} \right) = &- \frac{i}{4} k H_0^{(1)'}(k\sqrt{\rho^2 + 1 - 2\rho\cos(\theta - \theta_0)}) \frac{1 - \rho\cos(\theta - \theta_0)}{\sqrt{\rho^2 + 1 - 2\rho\cos(\theta - \theta_0)}} \\ &+ \frac{i}{4} \sum_{n=-\infty}^{\infty} \frac{J_n(k\rho) H_n^{(1)}(k)}{J_n(k)} k J_n'(k), \end{aligned} \tag{6.5}$$

with $\arctan\theta_0 = 1/t$.

Define

$$I_1 := \int_{-\varepsilon}^{\varepsilon} \partial_{\nu_y} G_\Omega^k \left(x_R, \begin{pmatrix} t \\ 1 \end{pmatrix} \right) (-\frac{1}{2\pi} \mathcal{J}^\varepsilon + \frac{k^2}{4\pi} \mathcal{L}^\varepsilon)^{-1} \left[\partial_{\nu_x} G_\Omega^k \left(x_S, \begin{pmatrix} s \\ 1 \end{pmatrix} \right) \right] (t) \mathrm{d}t.$$

By Proposition 5.11, we have

$$(-\alpha\mathcal{J}^\varepsilon + \beta\mathcal{L}^\varepsilon)^{-1}[f](t) = (\mathcal{L}^\varepsilon)^{-1}[u_l](t),$$

where

$$u_l(t) = C_1 e^{\sqrt{\frac{\beta}{\alpha}}t} + C_2 e^{-\sqrt{\frac{\beta}{\alpha}}t} + \frac{1}{2\sqrt{\alpha\beta}} e^{-\sqrt{\frac{\beta}{\alpha}}t} \int_{-\varepsilon}^{t} f(s) e^{\sqrt{\frac{\beta}{\alpha}}s} \mathrm{d}s - \frac{1}{2\sqrt{\alpha\beta}} e^{\sqrt{\frac{\beta}{\alpha}}t} \int_{-\varepsilon}^{t} f(s) e^{-\sqrt{\frac{\beta}{\alpha}}s} \mathrm{d}s,$$

and C_1, C_2 are constants determined by (5.14) and (5.15). Therefore,

$$u_l'(\tau) = \sqrt{\frac{\beta}{\alpha}} \left(C_1 e^{\sqrt{\frac{\beta}{\alpha}}\tau} - C_2 e^{-\sqrt{\frac{\beta}{\alpha}}\tau} - \frac{1}{2\sqrt{\alpha\beta}} \int_{-\varepsilon}^{\tau} f(s) 2\cosh(\sqrt{\frac{\beta}{\alpha}}(s-\tau)) \mathrm{d}s \right).$$

By Taylor expansion,

$$\begin{aligned}
&u_l'(\tau) - u_l'(0) \\
=&\sqrt{\frac{\beta}{\alpha}} (C_1 e^{\sqrt{\frac{\beta}{\alpha}}\tau} - C_2 e^{-\sqrt{\frac{\beta}{\alpha}}\tau} - (C_1 - C_2)) - \frac{1}{2\alpha} \int_0^{\tau} f(s) 2\cosh(\sqrt{\frac{\beta}{\alpha}}(s-\tau)) \mathrm{d}s \\
&- \frac{1}{2\alpha} \int_{-\varepsilon}^{0} f(s) 2 \left(\cosh(\sqrt{\frac{\beta}{\alpha}}(s-\tau)) - \cosh(\sqrt{\frac{\beta}{\alpha}}s) \right) \mathrm{d}s \\
=&\sqrt{\frac{\beta}{\alpha}} \left(C_1 \sqrt{\frac{\beta}{\alpha}} + C_2 \sqrt{\frac{\beta}{\alpha}} \right) \tau + \frac{1}{2} \left(\frac{\beta}{\alpha} \right)^{3/2} (C_1 - C_2) \tau^2 \\
&- \frac{1}{2\alpha} \left(2f(0)\tau + f'(0)\tau^2 + \frac{\beta}{\alpha} f(0)\tau^2\varepsilon - \frac{\beta}{\alpha} f(0)\tau\varepsilon^2 + \mathcal{O}(\tau^3) \right) \\
=&\left(\frac{\beta}{\alpha}(C_1 + C_2)\tau - \frac{1}{\alpha} f(0) + \frac{\beta}{2\alpha^2} f(0)\varepsilon^2 \right) \tau \\
&+ \left(\frac{1}{2} \left(\frac{\beta}{\alpha} \right)^{3/2} (C_1 - C_2) - \frac{1}{2\alpha} f'(0) - \frac{\beta}{2\alpha^2} f(0)\varepsilon \right) \tau^2 + \mathcal{O}(\varepsilon, \tau, 3),
\end{aligned} \tag{6.6}$$

where $\mathcal{O}(\varepsilon, \tau, 3)$ denotes the infinite sum of terms of the form $C\,\varepsilon^n\,\tau^m$, where $n + m \geq 3$. Denote by

$$C_{u_l}^1 := \frac{\beta}{\alpha}(C_1 + C_2) - \frac{1}{\alpha} f(0) + \frac{\beta}{2\alpha^2} f(0)\varepsilon^2 \tag{6.7}$$

and by

$$C_{u_l}^2 := \frac{1}{2} \left(\frac{\beta}{\alpha} \right)^{3/2} (C_1 - C_2) - \frac{1}{2\alpha} f'(0) - \frac{\beta}{2\alpha^2} f(0)\varepsilon. \tag{6.8}$$

For $u_l'(0)$ we have

$$\begin{aligned}
u_l'(0) =&\sqrt{\frac{\beta}{\alpha}}\left(C_1 - C_2 - \frac{1}{2\sqrt{\alpha\beta}}\int_{-\varepsilon}^{0} f(s)2\cosh(\sqrt{\frac{\beta}{\alpha}}s)\mathrm{d}s\right)\\
=&\sqrt{\frac{\beta}{\alpha}}(C_1 - C_2)\\
&-\frac{1}{\sqrt{\alpha\beta}}\left(f(0)\sqrt{\frac{\alpha}{\beta}}\sinh(\sqrt{\frac{\beta}{\alpha}}\varepsilon) + f'(0)\frac{\alpha}{\beta}\left(-1+\cosh(\sqrt{\frac{\beta}{\alpha}}\varepsilon) - \varepsilon\sqrt{\frac{\beta}{\alpha}}\sinh(\sqrt{\frac{\beta}{\alpha}}\varepsilon)\right) + O(\varepsilon^3)\right)\\
=&\sqrt{\frac{\beta}{\alpha}}(C_1 - C_2) - \frac{1}{\alpha}f(0)\varepsilon + \frac{1}{2\alpha}f'(0)\varepsilon^2 + O(\varepsilon^3),
\end{aligned} \tag{6.9}$$

while for $u_l(0)$ it holds that

$$\begin{aligned}
u_l(0) &= C_1 + C_2 + \frac{1}{2\sqrt{\alpha\beta}}\int_{-\varepsilon}^{0} f(s)(e^{\sqrt{\frac{\beta}{\alpha}}s} - e^{-\sqrt{\frac{\beta}{\alpha}}s})\mathrm{d}s\\
&= C_1 + C_2 + \frac{1}{\alpha}f(0)(-\frac{1}{2}\varepsilon^2) + O(\varepsilon^3)
\end{aligned} \tag{6.10}$$

Recall the formula for $(\mathcal{L}^\varepsilon)^{-1}$ and $C_{\mathcal{L}^\varepsilon}[u_l]$ in Proposition 3.11. Since $C_{\mathcal{L}^\varepsilon}[u_l]$ is a constant, we can simply evaluate it at $x = 0$. We find that

$$\begin{aligned}
C_{\mathcal{L}^\varepsilon}[u_l] &= u_l(0) - \mathcal{L}^\varepsilon[-\frac{1}{\pi^2\sqrt{\varepsilon^2 - t^2}}\,\mathrm{p}\!\!\int_{-\varepsilon}^{\varepsilon}\frac{\sqrt{\varepsilon^2-\tau^2}u_l'(\tau)}{t-\tau}\mathrm{d}\tau](0)\\
&= u_l(0) + \int_{-\varepsilon}^{\varepsilon}\mathrm{p}\!\!\int_{-\varepsilon}^{\varepsilon}\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}}\frac{\sqrt{\varepsilon^2-\tau^2}u_l'(\tau)}{t-\tau}\log|t|\mathrm{d}\tau\mathrm{d}t.
\end{aligned}$$

From (6.6), we obtain

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon}\mathrm{p}\!\!\int_{-\varepsilon}^{\varepsilon}\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}}\frac{\sqrt{\varepsilon^2-\tau^2}u_l'(\tau)}{t-\tau}\log|t|\mathrm{d}\tau\mathrm{d}t\\
=&\int_{-\varepsilon}^{\varepsilon}\mathrm{p}\!\!\int_{-\varepsilon}^{\varepsilon}\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}}\frac{\sqrt{\varepsilon^2-\tau^2}(u_l'(0) + C^1_{u_l}\tau + C^2_{u_l}\tau^2 + O(\varepsilon^3))}{t-\tau}\log|t|\mathrm{d}\tau\mathrm{d}t\\
=&u_l'(0)\int_{-\varepsilon}^{\varepsilon}\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}}\pi t\log|t|\mathrm{d}t + C^1_{u_l}\int_{-\varepsilon}^{\varepsilon}\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}}(-\frac{\pi\varepsilon^2}{2} + \pi t^2)\log|t|\mathrm{d}t\\
&+C^2_{u_l}\int_{-\varepsilon}^{\varepsilon}\frac{1}{\pi^2\sqrt{\varepsilon^2-t^2}}(-\frac{1}{2}\varepsilon^2\pi t + \pi t^3)\log|t|\mathrm{d}t + O(\varepsilon^4)\\
=&\frac{C^1_{u_l}}{\pi}\left(-\frac{\varepsilon^2}{2}(-\pi\log(\frac{2}{\varepsilon})) - \frac{1}{4}\varepsilon^2\pi(-1+2\log(\frac{2}{\varepsilon}))\right) + 0 + O(\varepsilon^4)\\
=&\frac{C^1_{u_l}}{4}\varepsilon^2 + O(\varepsilon^4).
\end{aligned}$$

Here, we have used the fact that

$$\text{p.v.}\int_{-\varepsilon}^{\varepsilon} \frac{\tau\sqrt{\varepsilon^2-\tau^2}}{t-\tau}\mathrm{d}\tau = -\frac{\pi\varepsilon^2}{2} + \pi t^2,$$
$$\int_{-\varepsilon}^{\varepsilon} \frac{\log|t|}{\sqrt{\varepsilon^2-t^2}}\mathrm{d}t = -\pi\log\left(\frac{2}{\varepsilon}\right),$$

and

$$\int_{-\varepsilon}^{\varepsilon} \frac{t^2\log|t|}{\sqrt{\varepsilon^2-t^2}}\mathrm{d}t = -\frac{1}{4}\varepsilon^2\pi\left(-1+2\log\left(\frac{2}{\varepsilon}\right)\right).$$

Therefore, the following estimation holds:

$$C_{\mathcal{L}^\varepsilon}[u_l] = u_l(0) + \frac{C^1_{u_l}}{4}\varepsilon^2 + O(\varepsilon^4). \tag{6.11}$$

We now compute C_1 and C_2 from (5.14) and (5.15). Applying (6.6) to (5.14), we get

$$\int_{-\varepsilon}^{\varepsilon}\sqrt{\frac{\varepsilon+\tau}{\varepsilon-\tau}}(u_l'(0)+C^1_{u_l}\tau+C^2_{u_l}\tau^2+O(\tau^3))\mathrm{d}\tau - \frac{\pi}{\log(\varepsilon/2)}C_{\mathcal{L}^\varepsilon}[u_l] = 0,$$

which implies that

$$u_l'(0)\pi\varepsilon + C^1_{u_l}\frac{\pi\varepsilon^2}{2} + C^2_{u_l}\frac{\pi\varepsilon^3}{2} - \frac{\pi}{\log(\varepsilon/2)}(u_l(0)+\frac{C^1_{u_l}}{4}\varepsilon^2) + O(\varepsilon^4) = 0.$$

Combining the last equation together with (6.10) and (6.9), it follows that

$$\begin{aligned}&\left(\sqrt{\frac{\beta}{\alpha}}(C_1-C_2) - \frac{1}{\alpha}f(0)\varepsilon + O(\varepsilon^2)\right)\pi\varepsilon + C^1_{u_l}\frac{\pi\varepsilon^2}{2} + C^2_{u_l}\frac{\pi\varepsilon^3}{2} + O(\varepsilon^4)\\ &- \frac{\pi}{\log(\varepsilon/2)}\left(C_1+C_2 - \frac{1}{2\alpha}f(0)\varepsilon^2 + O(\varepsilon^4) + \frac{C^1_{u_l}}{4}\varepsilon^2\right) = 0.\end{aligned} \tag{6.12}$$

Thus,

$$\begin{aligned}&\sqrt{\frac{\beta}{\alpha}}\varepsilon(C_1-C_2) - \frac{1}{\log(\varepsilon/2)}(C_1+C_2)\\ &\qquad = \frac{1}{\alpha}f(0)\varepsilon^2 - \frac{C^1_{u_l}}{2}\varepsilon^2 - \frac{C^2_{u_l}}{2}\varepsilon^3 - \frac{1}{\log(\varepsilon/2)}\left(\frac{1}{2\alpha}f(0)\varepsilon^2 - \frac{C^1_{u_l}}{4}\varepsilon^2\right) + O(\varepsilon^4).\end{aligned} \tag{6.13}$$

Similarly, from (5.15) we obtain

$$\begin{aligned}&-\sqrt{\frac{\beta}{\alpha}}\varepsilon(C_1-C_2) - \frac{1}{\log(\varepsilon/2)}(C_1+C_2)\\ &\qquad = -\frac{1}{\alpha}f(0)\varepsilon^2 - \frac{C^1_{u_l}}{2}\varepsilon^2 + \frac{C^2_{u_l}}{2}\varepsilon^3 - \frac{1}{\log(\varepsilon/2)}\left(\frac{1}{2\alpha}f(0)\varepsilon^2 - \frac{C^1_{u_l}}{4}\varepsilon^2\right) + O(\varepsilon^4).\end{aligned} \tag{6.14}$$

Therefore,

$$C_1 + C_2 = \left(\frac{1}{2\alpha} f(0) - \frac{C^1_{u_l}}{4} \right) \varepsilon^2 + \frac{C^1_{u_l}}{2} \log(\varepsilon/2)\varepsilon^2 + O(\varepsilon^3), \tag{6.15}$$

and

$$C_1 - C_2 = \frac{1}{\sqrt{\alpha\beta}} f(0)\varepsilon - \frac{1}{2}\sqrt{\frac{\alpha}{\beta}} C^2_{u_l} \varepsilon^2 + O(\varepsilon^2). \tag{6.16}$$

Combining together (6.7) with (6.15) gives

$$\begin{aligned} C_1 + C_2 &= \frac{\frac{-1}{2\alpha} f(0) \log(\varepsilon/2)\varepsilon^2 + \frac{3}{4\alpha} f(0)\varepsilon^2}{1 - \frac{\beta}{2\alpha}\log(\varepsilon/2)\varepsilon^2 + \frac{\beta}{4\alpha}\varepsilon^2} + O(\varepsilon^3 \log \varepsilon) \\ &= -\frac{1}{2\alpha} f(0) \log(\varepsilon/2)\varepsilon^2 + \frac{3}{4\alpha} f(0)\varepsilon^2 + O(\varepsilon^2 \log \varepsilon) \end{aligned} \tag{6.17}$$

and

$$C^1_{u_l} = -\frac{1}{\alpha} f(0) + \frac{\beta}{\alpha} \left(-\frac{1}{2\alpha} f(0) \log(\varepsilon/2)\varepsilon^2 + \frac{3}{4\alpha} f(0)\varepsilon^2 \right) + O(\varepsilon^2 \log \varepsilon), \tag{6.18}$$

while combining (6.8) and (6.16) together leads to

$$C_1 - C_2 = \frac{1}{\sqrt{\alpha\beta}} f(0)\varepsilon + \frac{1}{4\sqrt{\alpha\beta}} f'(0)\varepsilon^2 + O(\varepsilon^3), \tag{6.19}$$

and

$$C^2_{u_l} = -\frac{1}{2\alpha} f'(0) + \frac{1}{8\alpha^2} f'(0)\varepsilon^2 + O(\varepsilon^3). \tag{6.20}$$

Now, we are ready to estimate $(\mathcal{L}^\varepsilon)^{-1}[u_l](t)$. From Proposition 3.11, we know that

$$\begin{aligned} (\mathcal{L}^\varepsilon)^{-1}[u_l](t) &= -\frac{1}{\pi^2\sqrt{\varepsilon^2 - t^2}} \, \text{p.v.}\!\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2} u_l'(\tau)}{t - \tau} \mathrm{d}\tau + \frac{C_{\mathcal{L}^\varepsilon}[u_l]}{\pi \log(\varepsilon/2)\sqrt{\varepsilon^2 - t^2}} \\ &= -\frac{1}{\pi^2\sqrt{\varepsilon^2 - t^2}} \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - \tau^2}(u_l'(0) + C^1_{u_l}\tau + C^2_{u_l}\tau^2 + O(\tau^3))}{t - \tau} \mathrm{d}\tau \\ &\quad + \frac{C_{\mathcal{L}^\varepsilon}[u_l]}{\pi \log(\varepsilon/2)\sqrt{\varepsilon^2 - t^2}} \\ &= -\frac{1}{\pi\sqrt{\varepsilon^2 - t^2}} u_l'(0)t - \frac{C^1_{u_l}}{\pi\sqrt{\varepsilon^2 - t^2}} (-\frac{1}{2}\varepsilon^2 + \frac{1}{2}t^2) \\ &\quad - \frac{C^2_{u_l}}{\pi\sqrt{\varepsilon^2 - t^2}} (-\frac{1}{2}\varepsilon^2 \pi t + \pi t^3) + O(\varepsilon^4). \end{aligned} \tag{6.21}$$

Plugging $\alpha = 1/(2\pi)$ and $\beta = k^2/(4\pi)$ into (6.9), we get

$$\begin{aligned} &\left(-\frac{1}{2\pi}\mathcal{J}^\varepsilon + \frac{k^2}{4\pi}\mathcal{L}^\varepsilon \right)^{-1} [f](t) \\ &= -\frac{1}{\sqrt{\varepsilon^2 - t^2}} \left(-f(0)t^2 - f'(0)\pi t^3 + f(0)\varepsilon^2 + \frac{1}{2} f'(0)\varepsilon^2 \pi t + \frac{3}{2} f'(0)\varepsilon^2 t + O(\varepsilon^3) \right). \end{aligned} \tag{6.22}$$

Therefore,

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} \partial_{\nu_y} G_\Omega^k(x_R, \begin{pmatrix} t \\ 1 \end{pmatrix})(-\frac{1}{2\pi}\mathcal{J}^\varepsilon + \frac{k^2}{4\pi}\mathcal{L}^\varepsilon)^{-1}[f](t)\mathrm{d}t \\
&= \left(\partial_{\nu_y} G_\Omega^k \left(x_R, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + O(\varepsilon)\right) \\
&\int_{-\varepsilon}^{\varepsilon} -\frac{1}{\sqrt{\varepsilon^2 - t^2}} \left(-f(0)t^2 - f'(0)\pi t^3 + f(0)\varepsilon^2 + \frac{1}{2}f'(0)\varepsilon^2\pi t + \frac{3}{2}f'(0)\varepsilon^2 t + O(\varepsilon^3)\right) \mathrm{d}t \\
&= \left(\partial_{\nu_y} G_\Omega^k \left(x_R, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + O(\varepsilon^1)\right) \left(f(0)\frac{\varepsilon^2\pi}{2} - f(0)\varepsilon^2\pi + O(\varepsilon^3)\right) \\
&= -f(0)\frac{\pi}{2}\varepsilon^2 \partial_{\nu_y} G_\Omega^k \left(x_R, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + O(\varepsilon^3).
\end{aligned} \tag{6.23}$$

Recall in our setting that $f(s) := \partial_{\nu_x} G_\Omega^k \left(x_S, \begin{pmatrix} s \\ 1 \end{pmatrix}\right)$. Thus,

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} \partial_{\nu_y} G_\Omega^k \left(x_R, \begin{pmatrix} t \\ 1 \end{pmatrix}\right) \left(-\frac{1}{2\pi}\mathcal{J}^\varepsilon + \frac{k^2}{4\pi}\mathcal{L}^\varepsilon\right)^{-1} \left[\partial_{\nu_x} G_\Omega^k \left(x_S, \begin{pmatrix} s \\ 1 \end{pmatrix}\right)\right](t)\mathrm{d}t \\
&= -\frac{\pi}{2}\varepsilon^2 \partial_{\nu_x} G_\Omega^k \left(x_S, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \partial_{\nu_y} G_\Omega^k \left(x_R, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) + O(\varepsilon^3).
\end{aligned} \tag{6.24}$$

6.2 Numerical Illustrations

Now, two numerical experiments are presented in order to verify the applicability of the proposed methodology. In each one, the topological derivative is evaluated to detect the parts of the boundary where a Neumann boundary condition should be nucleated.

Denote by

$$F_S(\theta) := \frac{\partial G_\Omega^k}{\partial \tilde{\rho}} \left(x_S, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right), \tag{6.25}$$

and by

$$F_R(\theta) := \frac{\partial G_\Omega^k}{\partial \tilde{\rho}} \left(x_R, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right). \tag{6.26}$$

Our goal is to maximize $|u_{x_S}^{k,\varepsilon}|^2$, i.e., the norm of Green function at the receiver. We plot $y(\theta) := \Re(\pi \partial_{\nu_x} G_\Omega^k \left(x_S, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \partial_{\nu_y} G_\Omega^k \left(x_R, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right))$ as a function of θ.

Set the wave number to be $k = 200$, the distance between the source and the origin to be $\rho = 0.4$, the distance between the receiver and the origin to be $\tilde{\rho} = 0.2$, the angle difference from the receiver to the source to be $\pi/3$. We divide the whole boundary into $N = 10000$ parts, and set $N/1000 = 10$ parts left and right of each local maximal point of $y(\theta)$ to be a Neumann part. Figure 8 presents $y(\theta)$. Figure 9 gives

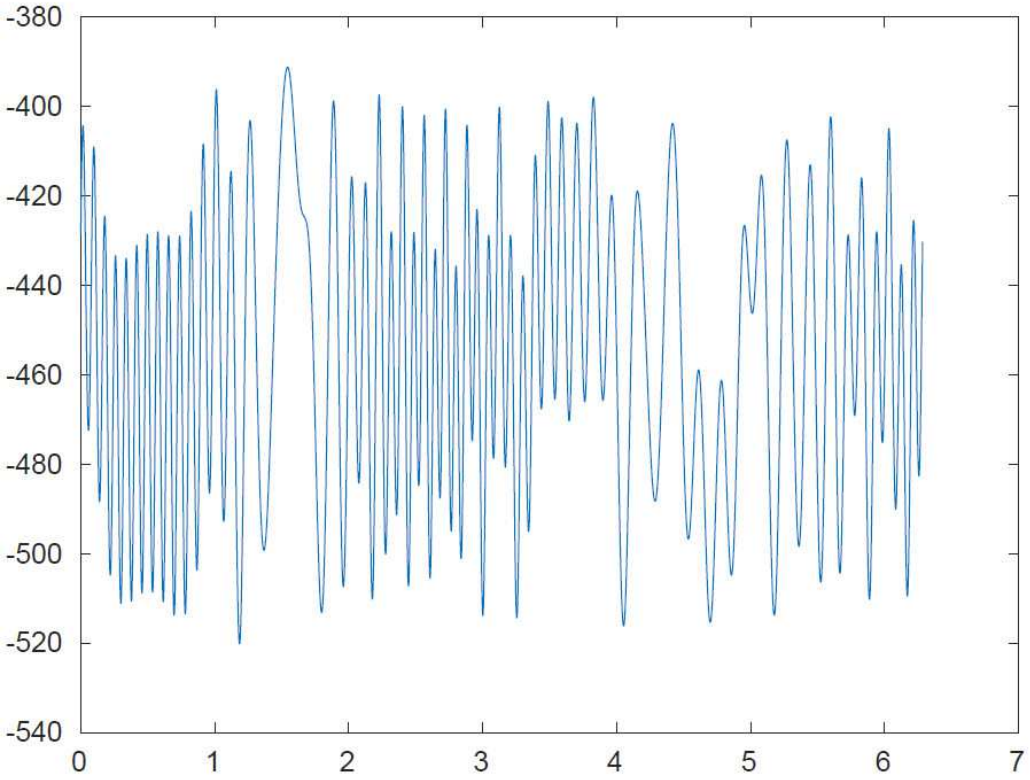


Figure 8: *Plot of $y(\theta)$ as a function of θ.*

the corresponding configuration of the boundary, where the red part corresponds to a Neumann boundary condition and the blue part corresponds to a Dirichlet boundary condition. The implementation shows that the norm of the Green function G^k_Ω with Dirichlet boundary at x_R is 0.067875, while the norm of the Green function $u^{k,\varepsilon}_{x_S}$ with modified boundary at x_R is 32.403610.

7 Conclusion

In this paper, we have established a mathematical theory of micro-scaled periodically arranged Helmholtz resonators and derived expansions of the scattered fields at the subwavelength resonances in terms of the size of the gap opening. We have highlighted the mechanism of the Neumann and Dirichlet functions to exploit the intrinsic properties of the wave behaviour near and away from the gaps. With this knowledge we were able to answer both question; how can we model an array of Helmholtz resonators and how can we enhance the signal at a receiving point inside the cavity by switching the boundary conditions from Dirichlet to Neumann on specific parts of the cavity boundary.

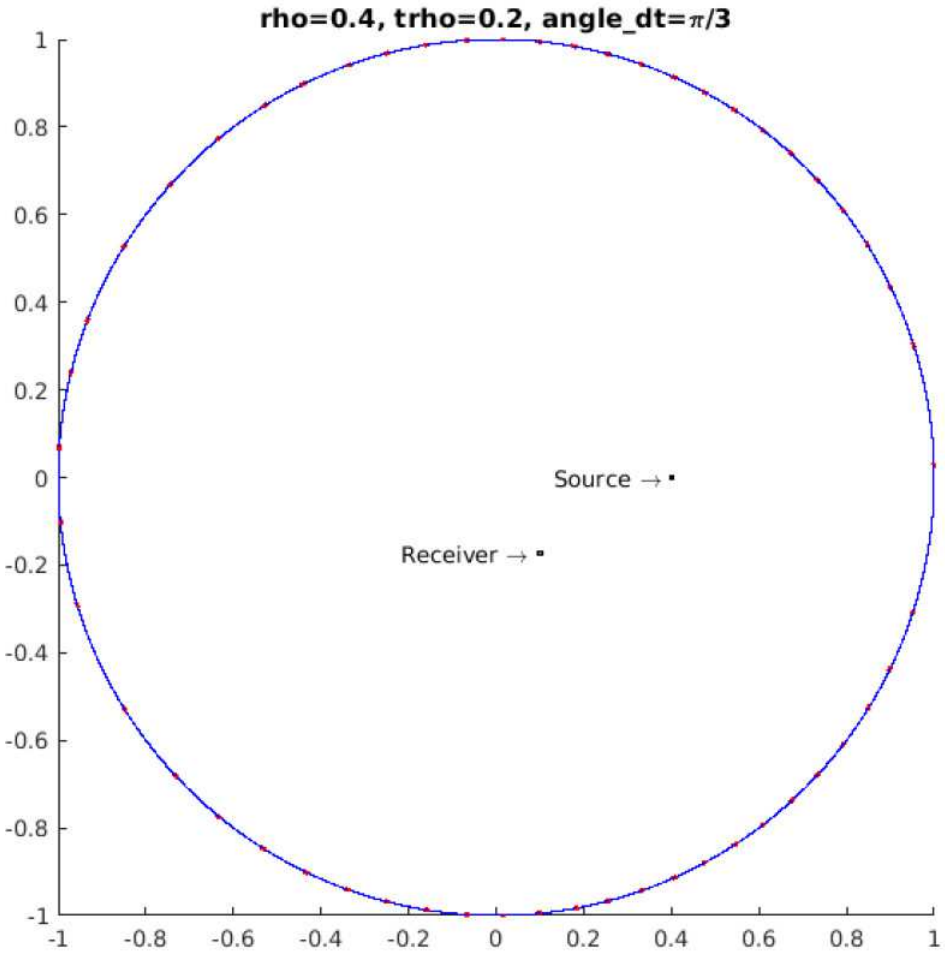


Figure 9: *Nucleation of the Neumann boundary condition.*

Our approach opens many new avenues for mathematical imaging and focusing of waves in complex media. Whereas the results in Sections 3 and 4 are important for industrial objectives, the results in Sections 5 and 6 can lead to enhanced communication between devices, like cell phones by improving the transmission between a source and a receiver through specific eigenmodes of the cavity. However, many challenging problems are still to be solved. For instance, how to optimize some specific cavity eigenmodes or how to design broadband metasurfaces which allow for broadband shaping and controlling of waves in complex media. These challenging problems would be the subject of forthcoming works.

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