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Research Report No. 2018-11
April 2018

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
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Funding SNF: 159940

Uncertainty Quantification for Spectral Fractional Diffusion: Sparsity Analysis of Parametric Solutions*

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Abstract. In bounded, polygonal domains $\mathrm{D} \subset \mathbb{R}^d$, we analyze solution regularity and sparsity for computational uncertainty quantification for spectral fractional diffusion. Two types of uncertainty are considered: i) uncertain, parametric diffusion coefficients, and ii) uncertain physical domains D. For either of these problem classes, we analyze sparsity of countably-parametric solution families. Principal novel technical contribution of the present paper is a sparsity analysis for operator equations with distributed uncertain inputs which, in particular, may be given as a general gpc representation, generalizing earlier results which required an affine-parametric representation. The summability results established here imply best N-term approximation rate bounds as well as dimension-independent convergence rates of numerical approximation methods such as stochastic collocation, Smolyak and Quasi-Monte Carlo integration methods and compressed sensing or least-squares approximations.

Key words. Fractional diffusion, nonlocal operators, uncertainty quantification, sparsity, generalized polynomial chaos.

AMS subject classifications. 26A33, 65N12.

1. Introduction. The mathematical analysis of numerical approximation methods for partial differential equations (PDEs) with uncertain input data has received substantial attention in recent years. In theoretical, mathematical analysis, particular focus has been on distributed uncertain input data taking values in function spaces. Uncertainty parametrization with suitable (countable) representation systems of such inputs renders corresponding responses countably parametric. Numerical approximation of response manifolds is therefore intimately related to approximation on high-dimensional parameter spaces. Sparsity of collections of (parametric) solutions have been found to play a crucial role in convergence rate bounds which are free from the so-called curse of dimensionality. Specific results on uncertainty quantification (UQ) for diffusion problems in heterogeneous media with uncertain constitutive properties were considered in, e.g., [10, 12, 20] and the references there.

In recent years, there has been significant interest in the numerical analysis of fractional diffusion equations; we mention only [4, 5, 3] and the references there. This is due to the widespread appearance of fractional diffusion models in applications, ranging from biology to financial modelling.

The present paper is, to our knowledge, the first mathematical sparsity analysis in UQ for fractional PDEs. We consider two types of uncertainty for spectral fractional diffusion: first, parametric uncertainty in the diffusion coefficient of the spectral fractional diffusion operators and, second, domain uncertainty of the spectral fractional diffusion operator. In either case, our analysis relies on the localization of the spectral fractional diffusion operator which was emphasized by Caffarelli, Stinga and coworkers (see, e.g., [8, 26] and the references there).

*This work was supported in part by the Swiss National Science Foundation (SNSF) under grant SNF 159940.

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This localization of the spectral fractional diffusion (at the expense of admitting one extra "spacial" variable in the problem) allows the mathematical analysis of sensitivities which is necessary for the analysis of efficient computational UQ, such as higher order Quasi-Monte Carlo (QMC) and Smolyak quadrature (see, e.g., [19, 15, 18] for details). We also note that in numerical computations, the extra "spacial" variable introduced through the Caffarelli-Stinga extension is, in fact, not causing undue inflation of the computational work for the numerical solution: as it was shown in [3] and the references there, anisotropic tensor product FE discretizations of the extended (local) parametric problem allow numerical solution of the parametric fractional diffusion problem for any value of the parameter.

The *new contributions of the present paper* are as follows: we prove new sparsity results for solutions of spectral fractional diffusion problems, where the diffusion matrix can be anisotropic and may depend on possibly countably many parameters $(y_j)_{j\in\mathbb{N}}$. We admit affine-parametric dependence as well as analytic dependence of the diffusion coefficient on the parameters. We prove that summability of the coefficients in the parametric expansion of the diffusion coefficients will, in turn, imply corresponding summability in the sequence of gpc ("generalized polynomial chaos") coefficients of the parametric solution.

These gpc summability results for divergence form equations with non-affine, gpc input obtained in the present paper are the first results on sparsity in gpc expansions of fractional diffusion problems. They generalize, in the case of analytic dependence of the diffusion coefficient, known summability results e.g. from [2] even in the diffusion case, see Remark 2.16. In particular, they allow to track possibly local supports of basis functions in representation systems used in uncertainty parametrization also for non-affine, parametric inputs. Summability results of this type were, so far, available only for linear or nonlinear, elliptic and parabolic *differential* operators, i.e., for operators which are local. The present mathematical sparsity analysis of nonlocal fractional diffusion operators is based on their localization by the so-called *Caffarelli-Stinga-Torres* extension of the fractional operators, see [8]. It quantifies sparsity of parametric solutions for fractional differential operators which are defined in parametric families of domains. The present results also allow to infer dimension-independent convergence rates of Quasi-Monte Carlo and Smolyak type quadrature algorithms as analyzed in [27, 16]. This will be developed in [21].

1.1. Spectral fractional diffusion. To prepare the ensuing presentation of the fractional diffusion problem with uncertain input data and the error analysis of sparse discretization schemes, we present the spectral fractional diffusion operator, initially without uncertain inputs.

For some $0 < s < 1$, and in a bounded domain $\mathrm{D} \subset \mathbb{R}^d$, for given $f \in L^2(\mathrm{D})$ we consider the Dirichlet problem of the fractional power $\mathcal{L}^s$ of the linear, elliptic, self-adjoint, second order divergence form differential operator

$$\mathcal{L}w = -\mathrm{div}(A\nabla w), \tag{1.1}$$

In (1.1), the diffusion coefficient $A \in L^\infty(\mathrm{D}; \mathbb{R}^{d\times d})$ is assumed symmetric, uniformly positive definite in the sense that there exists $\mu > 0$ such that

$$\forall \xi \in \mathbb{R}^d : \quad \operatorname*{ess\,inf}_{x\in\mathrm{D}} \xi^\top A(x)\xi \geq \mu|\xi|^2 \, . \tag{1.2}$$

We denote the set of symmetric matrices with real-valued entries by $\mathbb{R}^{d\times d}_{\mathrm{sym}}$. Identifying $\mathcal{L}$ in (1.1) with a bounded, linear operator from $H^1_0(\mathrm{D})$ to $H^{-1}(\mathrm{D})$, the Dirichlet problem for the fractional diffusion in D reads, formally:

Given a fractional order $s \in (0,1)$ and $f \in L^2(\mathrm{D})$, we seek u such that

$$\mathcal{L}^s u = f \quad \text{in D} . \tag{1.3}$$

In (1.3) and throughout the following, the domain $\mathrm{D} \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary $\partial\mathrm{D}$. Further conditions on D will be introduced as needed for the ensuing analysis.

Throughout we adhere to the following notational conventions. For $x \in \mathbb{C}^d$ and $A \in \mathbb{C}^{d\times d}$ we denote by $\|x\|_2$, $\|A\|_2$ the Euclidean norm of x and the spectral norm of A respectively. Moreover, if $A \in L^\infty(\mathrm{D};\mathbb{R}^{d\times d})$, then $\|A\|_{L^\infty(\mathrm{D};\mathbb{R}^{d\times d})} := \operatorname{ess\,sup}_{x'\in\mathrm{D}} \|A(x')\|_2$, i.e. with respect to the spectral norm on $\mathbb{R}^{d\times d}$, and similarly we always assume $\|\cdot\|_2$ as the underlying norm for vector valued functions. If $\psi : \mathrm{D} \to \mathbb{C}^n$ is a vector valued function for some $n \in \mathbb{N}$, then $|\psi| : \mathrm{D} \to [0,\infty)^n$ shall refer to the componentwise modulus of ψ. Finally, for a Banach space X we denote by B^X_r the open ball with center $0 \in X$ and radius $r > 0$ in X. If $\boldsymbol{\gamma} = (\gamma_j)_{j\in I} \in (0,\infty)^I$ is a sequence, then $B^X_{\boldsymbol{\gamma}}$ is understood as $\times_{j\in I} B^X_{\gamma_j} \subseteq X^I$. In case $X = \mathbb{R}$ we omit the superscript $\mathbb{R}$.

1.2. Caffarelli-Stinga extension. For $0 < s < 1$, $\mathcal{L}^s$ in (1.3) is a nonlocal operator [6, 7, 8, 9], which admits several possible interpretations. We consider the so-called *spectral version of the fractional Laplacean*. It was proposed by Caffarelli and Silvestre in [8] to localize the (nonlocal) operator $\mathcal{L}^s$ on the unbounded domain $\mathbb{R}^d$ via a singular elliptic PDE depending on one extra variable. Cabré and Tan [7] and Stinga and Torrea [26] extended this to bounded domains D and more general operators, thereby obtaining an extension posed on the semi-infinite cylinder $\mathcal{C} := \mathrm{D}\times(0,\infty)$. The extension is defined via the *local boundary value problem*

$$\begin{cases} \mathfrak{L}\mathscr{U} = -\mathrm{div}\,(z^\alpha \boldsymbol{A}\nabla\mathscr{U}) = 0 & \text{in } \mathcal{C} , \\ \mathscr{U} = 0 & \text{on } \partial_L\mathcal{C} , \\ \partial_{\nu^\alpha}\mathscr{U} = d_s f & \text{on } \mathrm{D}\times\{0\} , \end{cases} \tag{1.4}$$

where $\boldsymbol{A} = \mathrm{diag}(A,1) \in L^\infty(\mathcal{C};\mathbb{R}^{(d+1)\times(d+1)}_{\mathrm{sym}})$, $\partial_L\mathcal{C} := \partial\mathrm{D}\times(0,\infty)$, $d_s := 2^{1-2s}\Gamma(1-s)/\Gamma(s) > 0$ and where $\alpha = 1-2s \in (-1,1)$ [8, 26]. In (1.4), the so–called *conormal exterior derivative* of $\mathscr{U}$ at $\mathrm{D}\times\{0\}$ is

$$\partial_{\nu^\alpha}\mathscr{U} = -\lim_{y\to 0^+} y^\alpha \mathscr{U}_y. \tag{1.5}$$

The limit in (1.5) is in the distributional sense [7, 8, 26]. Fractional powers of $\mathcal{L}$ in (1.3) and the Dirichlet-to-Neumann operator of problem (1.4) are related by

$$d_s\mathcal{L}^s u = \partial_{\nu^\alpha}\mathscr{U} \quad \text{in D} . \tag{1.6}$$

We write $x = (x',z) \in \mathcal{C}$ and $\nabla = (\nabla_{x'},\partial_z)^\top$ with $x' \in \mathrm{D}$ and $z > 0$. Let us introduce the bilinear form $a_{\mathcal{C}} : \mathring{H}^1(z^\alpha,\mathcal{C}) \times \mathring{H}^1(z^\alpha,\mathcal{C}) \to \mathbb{R}$ defined by

$$a_{\mathcal{C}}(v,w) = \int_{\mathcal{C}} z^\alpha(\boldsymbol{A}\nabla v\cdot\nabla w)\,\mathrm{d}x'\,\mathrm{d}z. \tag{1.7}$$

In $\mathring{H}^1(z^\alpha, \mathcal{C})$ we introduce the following norm (see [24, eq. (2.21)] for definiteness)

$$\|v\|_{\mathring{H}^1(z^\alpha,\mathcal{C})} := \|\nabla v\|_{L^2(z^\alpha,\mathcal{C})},$$

where $\|v\|^2_{L^2(z^\alpha,\mathcal{C})} := \int_0^\infty \int_D z^\alpha |v|^2 \mathrm{d}x\mathrm{d}z$. This norm is equivalent to the quadratic functional defined by

$$\|v\|^2_{\mathcal{C}} := a_{\mathcal{C}}(v,v) \sim \|\nabla v\|^2_{L^2(z^\alpha,\mathcal{C})} = \|v\|^2_{\mathring{H}^1(z^\alpha,\mathcal{C})}\,. \tag{1.8}$$

With these definitions at hand, the weak formulation of (1.4) reads: Find $\mathscr{U} \in \mathring{H}^1(z^\alpha, \mathcal{C})$ such that

$$a_{\mathcal{C}}(\mathscr{U}, v) = d_s\langle f, \mathrm{tr_D}\, v\rangle \qquad \forall v \in \mathring{H}^1(z^\alpha, \mathcal{C}). \tag{1.9}$$

We introduce, for $s > 0$, the domain of $\mathcal{L}^s$ as

$$\mathbb{H}^s_A(\mathrm{D}) = \left\{ w = \sum_{k=1}^\infty w_k\varphi_k \in L^2(\mathrm{D}) : \|w\|^2_{\mathbb{H}^s_A(\mathrm{D})} = \sum_{k=1}^\infty \lambda_k^{2s} w_k^2 < \infty \right\}. \tag{1.10}$$

Here, $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+ \times H^1_0(\mathrm{D})$ is the countable collection of eigenpairs of $\mathcal{L}$ with homogeneous Dirichlet boundary conditions, with φ_k normalized such that $\{\varphi_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathrm{D})$ and an orthogonal basis of $(H^1_0(\mathrm{D}), a_{\mathrm{D}}(\cdot,\cdot))$. Here, as A was assumed symmetric, the "energy" inner product associated with the operator $\mathcal{L}$ is a symmetric, coercive bilinear form $a_{\mathrm{D}}(\cdot,\cdot)$ on $H^1_0(\mathrm{D}) \times H^1_0(\mathrm{D})$, i.e.

$$a_{\mathrm{D}}(w, v) = \int_{\mathrm{D}} (A\nabla w \cdot \nabla v)\; \mathrm{d}x'\,. \tag{1.11}$$

Under the coercivity assumption (1.2), the operator $\mathcal{L}$ in (1.1) induced by this bilinear form is an isomorphism $\mathcal{L} : H^1_0(\mathrm{D}) \to H^{-1}(\mathrm{D})$. The spectral fractional diffusion operator is, for $0 < s \leq 1$, then defined by

$$\mathcal{L}^s w := \sum_{k=1}^\infty \lambda_k^s w_k \varphi_k, \quad \text{for } w = \sum_{k=1}^\infty w_k\varphi_k \in \mathbb{H}^s_A(\mathrm{D})\,.$$

Tacitly, the eigenpairs $\{\lambda_k, \varphi_k\}_{k\in\mathbb{N}}$ also depend on A, which is not explicit in our notation. The ellipticity condition in (1.2) implies that $\mathbb{H}^s_A(\mathrm{D})$ is isomorphic to $\mathbb{H}^s_{\mathrm{Id}}(\mathrm{D})$ with equivalent norms, where Id denotes the indentity matrix on $\mathbb{R}^d$. We shall denote $\mathbb{H}^s(\mathrm{D}) = \mathbb{H}^s_{\mathrm{Id}}(\mathrm{D})$.

The spaces $\mathbb{H}^s(\mathrm{D})$ and $\mathring{H}^1(z^\alpha, \mathcal{C})$ are related by

$$\mathrm{tr_D}\, \mathring{H}^1(z^\alpha, \mathcal{C}) = \mathbb{H}^s(\mathrm{D}), \qquad \|\mathrm{tr_D}\, w\|_{\mathbb{H}^s(\mathrm{D})} \leq C_{\mathrm{tr_D}} \|w\|_{\mathring{H}^1(z^\alpha,\mathcal{C})}. \tag{1.12}$$

The result of Caffarelli and Silvestre [8] (see [7, Prop. 2.2] and [26, Thm. 1.1] for bounded domains and for general elliptic operators) relates the fractional diffusion operator to a certain Dirichlet-to-Neumann map:

Proposition 1.1. *Given* $f \in \mathbb{H}^{-s}(\mathrm{D})$*, let* $u \in \mathbb{H}^{s}(\mathrm{D})$ *solve* (1.3)*. If* $\mathscr{U} \in \mathring{H}^1(z^\alpha, \mathcal{C})$ *solves* (1.9)*, then* $u = \mathrm{tr}_{\mathrm{D}}\, \mathscr{U}$ *and*

$$d_s \mathcal{L}^s u = \partial_{\nu^\alpha} \mathscr{U} = d_s f \quad \text{in} \quad \mathbb{H}^{-s}(\mathrm{D}) \; . \tag{1.13}$$

As a consequence of the Lax–Milgram lemma, (1.12), and (1.2), there holds the a-priori estimate

$$\|u\|_{\mathbb{H}^s(\mathrm{D})} \leq C_{\mathrm{tr}_{\mathrm{D}}} \|\mathscr{U}\|_{\mathring{H}^1(z^\alpha, \mathcal{C})} \leq \frac{C^2_{\mathrm{tr}_{\mathrm{D}}}}{\min\{1, \mu\}} d_s \|f\|_{H^{-s}(\mathrm{D})}, \tag{1.14}$$

where $C_{\mathrm{tr}_{\mathrm{D}}}$ is the constant from the trace estimate (1.12).

1.3. Objectives of the present paper. The objective of this paper is to analyze sparsity of parametric solution famlies to the fractional diffusion problem (1.3) with uncertain, parametrized coefficients $A(\boldsymbol{y})$ and parametric right hand side $f(\boldsymbol{y})$, for parameter sequences $\boldsymbol{y} = (y_j)_{j \in \mathbb{N}}$ of real-valued y_j.

We consider two classes of problems: i) affine-parametric, non-isotropic coefficients $A(\boldsymbol{y})$ with bounded $\boldsymbol{y}$, and ii) parametrized domains $\mathrm{D}_{\boldsymbol{y}}$ with bounded $\boldsymbol{y}$. The latter class will, via suitable parametric domain transformations, reduce to parametric fractional diffusion problems in a fixed, so-called *nominal* domain where, however, coefficients and right-hand side are non-affine, parametric. The analysis of sparsity in gpc expansions of parametric solutions performed in this paper is based on tools which are novel, even for diffusion problems, and which could be of independent interest.

The new gpc coefficient bounds and summability statements which we establish in the present paper imply novel, sufficient conditions on the representation of the uncertain data for dimension-independent convergences rates of sparse collocation, Smolyak-type quadrature algorithms, and Quasi-Monte Carlo quadratures, for the numerical approximation of the parametric solution as well as of certain statistical moments of these. The summability results for gpc expansions of parametric solutions are also relevant for other approximation methods, such as compressed sensing and least squares techniques. We refer to [25, 13] and the references there for details.

2. Uncertainty Quantification. As mentioned, we consider UQ for two types of uncertainty for the fractional diffusion operator: i) affine-parametric uncertainty of the diffusion coefficient $A(x')$ in (1.1), and ii) domain uncertainty in the fractional boundary value problem (1.3).

In each case, the uncertain inputs are parametrized by a sequence $\boldsymbol{y} = (y_j)_{j \in \mathbb{N}}$ of parameters, which implies that the solution of (1.3) becomes, in turn, parametric, which we denote by $u(\boldsymbol{y})$. Placing a probability measure on the set U of all admissible parameter sequences will allow to evaluate, for example, statistical quantities by means of (numerical) integration of (functionals of) the parametric solution $u(\boldsymbol{y})$ over the (in general infinite-dimensional) parameter space U.

2.1. Affine-parametric models. In affine parametric models, the coefficients A in (1.1) are assumed to depend on a sequence $\boldsymbol{y} = (y_j)_{j \in \mathbb{N}} \in U = [-1, 1]^{\mathbb{N}}$ of parameters y_j in an affine fashion. Specifically,

$$A(\boldsymbol{y}) = \overline{A} + \sum_{j\in\mathbb{N}} y_j \Psi_j \in L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}}) \tag{2.1}$$

for a *nominal diffusion coefficient* $\overline{A} \in L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and for a sequence $(\Psi_j)_{j\in\mathbb{N}} \subset L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ of *fluctuations*. We assume that $\overline{A}$ satisfies the coercivity condition (1.2), with coercivity constant $\mu > 0$, and that the fluctuations Ψ_j are small w.r.t. to μ in the sense that for some $\kappa \in (0,\mu)$

$$\forall \boldsymbol{y} \in U \, \forall \xi \in \mathbb{R}^d : \quad \operatorname*{ess\,inf}_{x'\in\mathrm{D}} \xi^\top A(\boldsymbol{y};x')\xi \geq (\mu-\kappa)|\xi|^2 \,. \tag{2.2}$$

The parametric uncertainty quantification for the fractional diffusion problem then takes the following form: for a given parameter $\boldsymbol{y} \in U$ and for a fractional power $0 < s < 1$, define $\mathcal{L}^s(\boldsymbol{y})$ as the fractional s power of the parametric diffusion operator $\mathcal{L}(\boldsymbol{y}) := -\nabla_{x'} \cdot (A(\boldsymbol{y};x')\nabla_{x'}) : H^1_0(\mathrm{D}) \to H^{-1}(\mathrm{D})$. Then, for given $f \in L^2(\mathrm{D})$ and $\boldsymbol{y} \in U$, and for given $0 < s < 1$, the *parametric fractional diffusion problem* is to find $u(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D})$ such that for every $\boldsymbol{y} \in U$

$$\mathcal{L}^s(\boldsymbol{y})u(\boldsymbol{y}) = f \quad \text{in } \mathbb{H}^{-s}(\mathrm{D}). \tag{2.3}$$

Here, the fractional order space $\mathbb{H}^s(\mathrm{D})$ is as in (1.10). The localization result Prop. 1.1 has an immediate parametric analog.

Proposition 2.1. *Assume* (2.1), (2.2). *Then, for the affine-parametric diffusion coefficient* $A(\boldsymbol{y}) \in L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ *with* $\boldsymbol{y} \in U$, *define* $\boldsymbol{A}(\boldsymbol{y}) = \operatorname{diag}(A(\boldsymbol{y}),1) \in L^\infty(\mathcal{C},\mathbb{R}^{(d+1)\times(d+1)}_{\mathrm{sym}})$. *For given* $f \in \mathbb{H}^{-s}(\mathrm{D})$ *and for given* $\boldsymbol{y} \in U$, *let* $u(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D})$ *solve* (2.3).

Then there holds the following parametric localization result: if $\mathscr{U}(\boldsymbol{y}) \in \mathring{H}^1(z^\alpha,\mathcal{C})$ *solves*

$$a_{\mathcal{C}}(\boldsymbol{y};\mathscr{U}(\boldsymbol{y}),v) = d_s\langle f, \operatorname{tr}_{\mathrm{D}} v\rangle \qquad \forall v \in \mathring{H}^1(z^\alpha,\mathcal{C}). \tag{2.4}$$

with the affine-parametric bilinear form $a_{\mathcal{C}}(\boldsymbol{y};\cdot,\cdot)$ *defined by*

$$a_{\mathcal{C}}(\boldsymbol{y};v,w) := \int_0^\infty z^\alpha \int_{\mathrm{D}} (\boldsymbol{A}(\boldsymbol{y})\nabla v \cdot \nabla w)\,\mathrm{d}x'\,\mathrm{d}z,$$

(with $\nabla = (\nabla_{x'},\partial_z)^\top$*) then* $u(\boldsymbol{y}) = \operatorname{tr}_{\mathrm{D}} \mathscr{U}(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D})$ *and*

$$d_s\mathcal{L}^s(\boldsymbol{y})u(\boldsymbol{y}) = \partial_{\nu^\alpha}\mathscr{U}(\boldsymbol{y}) \quad \textit{in } \mathbb{H}^{-s}(\mathrm{D}). \tag{2.5}$$

We observe that not only does Prop. 2.1 localize the nonlocal parametric operator $\mathcal{L}^s(\boldsymbol{y})$, but the localized problem in $\mathcal{C}$ also preserves the affine-parametric structure (2.1) of the uncertain diffusion coefficient. We shall exploit this observation in establishing bounds on the derivatives of the parametric solution $U \ni \boldsymbol{y} \to u(\boldsymbol{y})$ of (2.3), and to investigate summability of gpc expansions of $u(\boldsymbol{y})$.

To this end, let $\mathcal{F} := \{\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\nu}| < \infty\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ denote the (countable) set of "finitely supported" multiindices. For every $\boldsymbol{\nu} \in \mathcal{F}$, denote its support by $\operatorname{supp}(\boldsymbol{\nu}) := \{k \in \mathbb{N} : \nu_k \neq 0\}$ and define the parametric partial derivatives by $\partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} := \partial^{|\boldsymbol{\nu}|}/\prod_{j\in\mathbb{N}} \partial y_j^{\nu_j}$.

Prop. 2.1 facilitates the quantitative analysis of the "sensitivities" $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\boldsymbol{y})$ of the parametric solution $u(\boldsymbol{y})$ of the parametric fractional diffusion problem (2.3) with respect to the parameter sequence $\boldsymbol{y}$ via the corresponding sensitivities $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}(\boldsymbol{y})$ of the extended parametric problem (2.4): there holds

$$\forall \boldsymbol{\nu} \in \mathcal{F}: \quad \mathrm{tr}_{\mathrm{D}}\, \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}(\boldsymbol{y}) = \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\boldsymbol{y}) \quad \text{in} \quad \mathbb{H}^s(\mathrm{D}) \,. \tag{2.6}$$

To see (2.6), we observe that by Prop. 2.1

$$\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\boldsymbol{y}) = \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}}\, \mathrm{tr}_{\mathrm{D}}\, \mathscr{U}(\boldsymbol{y}) \,.$$

Then, (2.6) follows from commutation of tr_{D} and $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}}$, i.e., $\partial^{\boldsymbol{\nu}}(\mathrm{tr}_{\mathrm{D}}\, u(\boldsymbol{y})) = \mathrm{tr}_{\mathrm{D}}\, \partial^{\boldsymbol{\nu}} u(\boldsymbol{y})$ follows by an application of the chain rule (with the $\boldsymbol{y}$ independent bounded linear operator tr_{D}), *if* the function $\mathscr{U}(\boldsymbol{y})$ is differentiable in $\boldsymbol{y}$. In particular, with (2.1) the structure of the extended parametric problem (2.5) is analogous to that of affine-parametric diffusion problems considered in [2].

Bounds on the sensitivities $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\boldsymbol{y})$ immediately imply convergence rates of N-term truncated polynomial chaos expansions of the parametric solution map $U \ni \boldsymbol{y} \mapsto u(\boldsymbol{y})$: for example, for the *Taylor polynomial chaos* expansion

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{\nu} \in \mathcal{F}} t_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} \,, \qquad t_{\boldsymbol{\nu}} := \frac{1}{\boldsymbol{\nu}!} (\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\boldsymbol{y}))\Big|_{\boldsymbol{y}=\boldsymbol{0}} \,. \tag{2.7}$$

Via (2.5), we shall deduce unconditional (in $\mathbb{H}^s(\mathrm{D})$) convergence of (2.7) in $\mathbb{H}^s(\mathrm{D})$ for $\boldsymbol{y} \in U$ from the unconditional convergence of the corresponding Taylor gpc expansion of $\mathscr{U}(\boldsymbol{y})$ in $\mathring{H}^1(z^\alpha, \mathcal{C})$. Summability results for the sequence $(\|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})_{\boldsymbol{\nu} \in \mathcal{F}}$ of Taylor gpc coefficients of the parametric solution $u(\boldsymbol{y})$ of (2.3) can now be established along the lines of [11, 12].

A first result on the summability of the Taylor coefficients $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathcal{F}}$ of $u(\boldsymbol{y})$ and sparsity of gpc expansions exploits the affine-parametric nature of the coefficient $A(\boldsymbol{y}; x')$ in (2.1). We work under the following assumptions.

Assumption 2.2. *Let* $\overline{A}$, $\Psi_j \in L^\infty(\mathrm{D}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$, *for every* $j \in \mathbb{N}$.

i) There exists a constant $\overline{A}_{\min} > 0$ *such that for a.e.* $x' \in \mathrm{D}$

$$\overline{A}_{\min} \le \inf_{\xi \neq 0} \frac{\xi^\top \overline{A}(x') \xi}{\xi^\top \xi} \,. \tag{2.8}$$

ii) For some sequence $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ *of positive weights there holds the* weighted uniform ellipticity assumption

$$\delta := \left\| \max\{1, \|\overline{A}^{-1}(\cdot)\|_2\} \sum_{j\in\mathbb{N}} \rho_j \|\Psi_j(\cdot)\|_2 \right\|_{L^\infty(\mathrm{D})} < 1, \tag{2.9}$$

Theorem 2.3. *Consider the affine-parametric coefficient* (2.1) *and suppose that Assumption* 2.2 *holds for some* $\boldsymbol{\rho} \in (1,\infty)^{\mathbb{N}}$. *Then,*

$$(\boldsymbol{\rho}^{\boldsymbol{\nu}} \|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^2(\mathcal{F}) \tag{2.10}$$

and there holds

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} (\boldsymbol{\rho}^{\boldsymbol{\nu}} \|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})^2 \leq \frac{d_s^2 C_{\mathrm{tr}_\mathrm{D}}^4}{(\min\{\overline{A}_{\min}, 1\})^2} \frac{2-\delta}{2-2\delta} \|f\|^2_{\mathbb{H}^{-s}(\mathrm{D})} < \infty.$$

Moreover, if in (2.9) the sequence $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ is such that $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^q(\mathbb{N})$ with $q = 2p/(2-p)$ for some $0 < p < 2$, then $(\|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p(\mathcal{F})$.

Proof. The idea of the proof is to relate bounds on $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\boldsymbol{y})$ to corresponding bounds on $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}(\boldsymbol{y})$ via (2.6). In turn, the bounds for $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}(\boldsymbol{y})$ follow from an analysis of the localized extension, problem (2.4), the structure of which is an affine-parametric, linear second order diffusion problem analogous to those considered in [2]. The degeneracy w.r. to the variable z and the general coefficients do not require fundamentally different arguments. The proof proceeds in several steps.

Step 1: ($\rho_j = 1$) When $\rho_j = 1$ for every j, the bound (2.10) amounts to proving square summability of $(\|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})_{\boldsymbol{\nu}\in\mathcal{F}}$. We start with recursive estimates: from the affine parameter dependence (2.1) follows an analogous, affine-parametric structure of $\boldsymbol{A}(\boldsymbol{y}; x') = \mathrm{diag}(A(\boldsymbol{y}; x'), 1)$ as follows: we denote $\overline{\boldsymbol{A}} := \mathrm{diag}(\overline{A}, 1)$ and $\boldsymbol{\Psi}_j := \mathrm{diag}(\Psi_j, 0)$ so that $\boldsymbol{A}(\boldsymbol{y}; x') = \overline{\boldsymbol{A}} + \sum_{j\in\mathbb{N}} y_j \boldsymbol{\Psi}_j$. The bounds (2.8) remain valid also for $\overline{\boldsymbol{A}}(x')$. Furthermore, we point out that since $\|\overline{\boldsymbol{A}}^{-1}(x')\|_2 = \max\{1, \|\overline{A}^{-1}(x')\|_2\}$ as well as $\|\boldsymbol{\Psi}_j(x')\|_2 = \|\Psi_j(x')\|_2$, (2.9) is also satisfied by the boldface matrices.

For every $\boldsymbol{\nu} \in \mathcal{F}$, define the Taylor coefficient $T_{\boldsymbol{\nu}} := \frac{1}{\boldsymbol{\nu}!} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}(\boldsymbol{y}; x', z)|_{\boldsymbol{y}=\boldsymbol{0}}$, of the parametric solution $\mathscr{U}(\boldsymbol{y})$. Here, $\boldsymbol{\nu}! = \nu_1!\nu_2!...$ is well-defined for $\boldsymbol{\nu} \in \mathcal{F}$ due to the convention $0! := 1$. Then, from (2.4) and (2.1),

$$a_{\mathcal{C}}(\boldsymbol{0}; T_{\boldsymbol{0}}, v) = d_s \langle f, \mathrm{tr}_\mathrm{D}\, v\rangle \qquad \forall v \in \mathring{H}^1(z^\alpha, \mathcal{C}). \tag{2.11}$$

For $\boldsymbol{0} \neq \boldsymbol{\nu} \in \mathcal{F}$, the $T_{\boldsymbol{\nu}} \in \mathring{H}^1(z^\alpha, \mathcal{C})$ satisfy the following recurrence: for every $v \in \mathring{H}^1(z^\alpha, \mathcal{C})$,

$$\int_{\mathcal{C}} z^\alpha \nabla T_{\boldsymbol{\nu}} \cdot \overline{\boldsymbol{A}} \nabla v \,\mathrm{d}x'\,\mathrm{d}z = -\sum_{j:\nu_j\neq 0} \int_{\mathcal{C}} z^\alpha \nabla T_{\boldsymbol{\nu}-e_j} \cdot \boldsymbol{\Psi}_j \nabla v \,\mathrm{d}x'\,\mathrm{d}z\,. \tag{2.12}$$

We introduce the following notation: for $\boldsymbol{0} \neq \boldsymbol{\nu} \in \mathcal{F}$ we define

$$G_{\boldsymbol{\nu}} := a_{\mathcal{C}}(\boldsymbol{0}; T_{\boldsymbol{\nu}}, T_{\boldsymbol{\nu}}) = \int_{\mathcal{C}} z^\alpha \nabla T_{\boldsymbol{\nu}} \cdot \overline{\boldsymbol{A}} \nabla T_{\boldsymbol{\nu}} \,\mathrm{d}x'\,\mathrm{d}z = \|T_{\boldsymbol{\nu}}\|^2_{\overline{\boldsymbol{A}}}\,,$$

where we defined $\|v\|^2_{\overline{\boldsymbol{A}}} := \int_{\mathcal{C}} z^\alpha \nabla v \overline{\boldsymbol{A}} \nabla v \mathrm{d}x'\mathrm{d}z$, $v \in \mathring{H}^1(z^\alpha; \mathcal{C})$. For $j \in \mathrm{supp}(\boldsymbol{\nu}) = \{k : \nu_k \neq 0\}$, we denote

$$G_{\boldsymbol{\nu},j} := \int_{\mathcal{C}} z^\alpha \|\nabla T_{\boldsymbol{\nu}}\|_2^2 \|\boldsymbol{\Psi}_j\|_2 \,\mathrm{d}x'\,\mathrm{d}z\,. \tag{2.13}$$

Then, choosing in (2.12) the testfunction $v = T_{\boldsymbol{\nu}}$ and bounding the right hand side using the Cauchy–Schwarz and Young's inequality, we find the recursive estimate

$$G_{\boldsymbol{\nu}} \leq \sum_{j:\nu_j\neq 0} \int_{\mathcal{C}} z^\alpha \|\nabla T_{\boldsymbol{\nu}-e_j}\|_2 \|\boldsymbol{\Psi}_j\|_2 \|\nabla T_{\boldsymbol{\nu}}\|_2 \,\mathrm{d}x'\,\mathrm{d}z \leq \frac{1}{2} \sum_{j:\nu_j\neq 0} (G_{\boldsymbol{\nu}-e_j,j} + G_{\boldsymbol{\nu},j})\,. \tag{2.14}$$

The uniform ellipticity assumption (2.9) with $\rho_j = 1$ implies for every $0 \neq \xi \in \mathbb{R}^{d+1}$, for every $\boldsymbol{y} \in U$ and for a.e. $x' \in \mathrm{D}$

$$\begin{aligned} \xi^\top \boldsymbol{A}(\boldsymbol{y}; x')\xi &= \xi^\top \left(\overline{\boldsymbol{A}}(x') - \sum_{j\in\mathbb{N}} y_j \boldsymbol{\Psi}_j(x') \right) \xi \\ &\geq \xi^\top \left(\overline{\boldsymbol{A}}(x') \Big(\boldsymbol{I} - \sum_{j\geq 1} \|\overline{\boldsymbol{A}}^{-1}(x')\|_2 \|\boldsymbol{\Psi}(x')\|_2 \Big) \right) \xi \\ &\geq (1-\delta)\xi^\top \overline{\boldsymbol{A}}(x')\xi. \end{aligned}$$

Due to (2.8), we obtain

$$\inf_{\boldsymbol{y}\in U} \operatorname*{ess\,inf}_{x'\in\mathrm{D}} \xi^\top \boldsymbol{A}(\boldsymbol{y}; x')\xi \geq (1-\delta)\min\{\overline{A}_{\min}, 1\}\|\xi\|_2^2.$$

From (2.13) and from (2.9) (with $\rho_j = 1$) and using symmetry of $\overline{\boldsymbol{A}}^{1/2}$, we obtain

$$\begin{aligned} \sum_{j\in\mathbb{N}} G_{\boldsymbol{\nu},j} &= \int_{\mathcal{C}} z^\alpha \|\nabla T_{\boldsymbol{\nu}}\|_2^2 \sum_{j\in\mathbb{N}} \|\boldsymbol{\Psi}_j\|_2 \,\mathrm{d}x' \,\mathrm{d}z \\ &= \int_{\mathcal{C}} z^\alpha \nabla T_{\boldsymbol{\nu}}^\top \overline{\boldsymbol{A}}^{1/2} \left(\overline{\boldsymbol{A}}^{-1} \sum_{j\in\mathbb{N}} \|\boldsymbol{\Psi}_j\|_2 \right) \overline{\boldsymbol{A}}^{1/2} \nabla T_{\boldsymbol{\nu}} \,\mathrm{d}x' \,\mathrm{d}z \\ &\leq \delta G_{\boldsymbol{\nu}} \ . \end{aligned}$$

By (2.14), this implies for every $\boldsymbol{0} \neq \boldsymbol{\nu} \in \mathcal{F}$

$$(2-\delta) G_{\boldsymbol{\nu}} \leq \sum_{j:\nu_j\neq 0} G_{\boldsymbol{\nu}-e_j,j} \ .$$

Summing over all $\boldsymbol{\nu} \in \mathcal{F}$ such that $|\boldsymbol{\nu}| = k \geq 1$, we get

$$(2-\delta) \sum_{|\boldsymbol{\nu}|=k} G_{\boldsymbol{\nu}} \leq \sum_{|\boldsymbol{\nu}|=k} \sum_{j:\nu_j\neq 0} G_{\boldsymbol{\nu}-e_j,j} \leq \delta \sum_{|\boldsymbol{\nu}|=k-1} G_{\boldsymbol{\nu}} \ ,$$

so that

$$\sum_{|\boldsymbol{\nu}|=k} G_{\boldsymbol{\nu}} \leq \frac{\delta}{2-\delta} \sum_{|\boldsymbol{\nu}|=k-1} G_{\boldsymbol{\nu}} \ .$$

With $0 < \delta < 1$ this bound implies, upon summation over $k \geq 1$,

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|T_{\boldsymbol{\nu}}\|_{\boldsymbol{A}}^2 \leq \frac{2-\delta}{2-2\delta} \|T_{\boldsymbol{0}}\|_{\boldsymbol{A}}^2 \ . \tag{2.15}$$

Step 2: We relate the Taylor coefficients $T_{\boldsymbol{\nu}} = \frac{1}{\boldsymbol{\nu}!}\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}|_{\boldsymbol{y}=\boldsymbol{0}}$ to $t_{\boldsymbol{\nu}}$ as follows: from Prop. 2.1, we obtain with (2.6) that

$$\forall \boldsymbol{\nu} \in \mathcal{F} : \quad t_{\boldsymbol{\nu}} = \mathrm{tr}_{\mathrm{D}}\, T_{\boldsymbol{\nu}} \ .$$

Since $\|T_{\mathbf{0}}\|^2_{\boldsymbol{A}} = d_s\langle f, \mathrm{tr}_{\mathrm{D}}\, \mathscr{U}(\mathbf{0})\rangle$, (1.12) and (1.14) imply

$$\|T_{\mathbf{0}}\|^2_{\boldsymbol{A}} \leq \frac{d_s^2 C^2_{\mathrm{tr}_{\mathrm{D}}}}{\min\{\overline{A}_{\min}, 1\}} \|f\|^2_{\mathbb{H}^s(\mathrm{D})}, \tag{2.16}$$

where we used that $T_{\mathbf{0}}$ solves the nominal extended problem in $\mathcal{C}$, i.e., (2.11). From the continuity estimate (1.12), (2.15), and (2.16), we find

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|t_{\boldsymbol{\nu}}\|^2_{\mathbb{H}^s(\mathrm{D})} \leq \frac{C^2_{\mathrm{tr}_{\mathrm{D}}}}{\min\{\overline{A}_{\min}, 1\}} \sum_{\boldsymbol{\nu}\in\mathcal{F}} \|T_{\boldsymbol{\nu}}\|^2_{\boldsymbol{A}} \leq \frac{C^2_{\mathrm{tr}_{\mathrm{D}}}}{\min\{\overline{A}_{\min}, 1\}} \frac{2-\delta}{2-2\delta} \|T_{\mathbf{0}}\|^2_{\boldsymbol{A}} \leq C\|f\|^2_{\mathbb{H}^{-s}(\mathrm{D})},$$

where $C := (d_s^2 C^4_{\mathrm{tr}_{\mathrm{D}}}(2-\delta))/((2-2\delta)(\min\{\overline{A}_{\min}, 1\})^2)$. This proves (2.10) in the case $\rho_j = 1$.

Step 3: Consider now $\rho_j \geq 1$. Then, defining the dilated coefficient $A_{\boldsymbol{\rho}}(\boldsymbol{y}) := A(G_{\boldsymbol{\rho}}\boldsymbol{y})$ with $G_{\boldsymbol{\rho}}\boldsymbol{y} := (\rho_j y_j)_{j\in\mathbb{N}}$, we find that for every $\boldsymbol{\nu} \in \mathcal{F}$

$$t_{\boldsymbol{\rho},\boldsymbol{\nu}} := \frac{1}{\boldsymbol{\nu}!}\partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} u_{\boldsymbol{\rho}}(\boldsymbol{y})\Big|_{\boldsymbol{y}=0}, \quad u_{\boldsymbol{\rho}}(\boldsymbol{y}) = u(G_{\boldsymbol{\rho}}\boldsymbol{y}) . \tag{2.17}$$

We observe that the weighted condition (2.9) for the parametric coefficient $A(\boldsymbol{y})$ is equivalent to the same condition with $\rho_j = 1$ for the coefficient $A_{\boldsymbol{\rho}}(\boldsymbol{y})$. Applying step 2 to $A_{\boldsymbol{\rho}}$ and to $u_{\boldsymbol{\rho}}$, the assertion (2.10) in the case $\rho_j \in \mathbb{N}$ follows.

Step 4: We show the ℓ^p summability of the sequence $\|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})}$. Assume that $(\rho_j)_{j\in\mathbb{N}}$ is such that $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^q(\mathbb{N})$ where $q = 2p/(2-p)$ for some $0 < p < 2$. Then Hölder's inequality implies

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|t_{\boldsymbol{\nu}}\|^2_{\mathbb{H}^s(\mathrm{D})} \leq \left(\sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\rho}^{2\boldsymbol{\nu}} \|t_{\boldsymbol{\nu}}\|^2_{\mathbb{H}^s(\mathrm{D})}\right)^{p/2} \left(\sum_{\boldsymbol{\rho}\in\mathcal{F}} \boldsymbol{\rho}^{-\frac{2p}{p-2}\boldsymbol{\nu}}\right)^{(2-p)/2} .$$

Observing that

$$\sum_{\boldsymbol{\rho}\in\mathcal{F}} \boldsymbol{\rho}^{-\frac{2p}{p-2}\boldsymbol{\nu}} = \prod_{j\in\mathbb{N}} \left(\sum_{k=0}^{\infty} \rho_j^{-qk}\right) = \prod_{j\in\mathbb{N}} (1-\rho_j^{-q})^{-1} < \infty$$

if and only if $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^p(\mathbb{N})$ completes the proof. ∎

The preceding result on summability of the Taylor gpc coefficients $(t_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ of the solution family $\{u(\boldsymbol{y}) : \boldsymbol{y} \in U\}$ of the parametric fractional diffusion problem implies corresponding summability results for coefficients in gpc Legendre and, more generally, in gpc Jacobi expansions: Let L_j be the jth Legendre polynomial on $[-1, 1]$, where we assume the normalization $\int_{-1}^1 L_j(x)^2\,\mathrm{d}x = 2$. Let $\mu(\mathrm{d}\boldsymbol{y}) = \bigotimes_{j\in\mathbb{N}} \mathrm{d}y_j/2$ be the probability measure obtained as the inifinite product of the Lebesgue measure weighted by 1/2 on $[-1, 1]$. Then $L_{\boldsymbol{\nu}} := \prod_{j\in\mathbb{N}} L_{\nu_j}(y_j) \in L^2(U, \mu)$ for $\boldsymbol{\nu} \in \mathcal{F}$ is an orthonormal basis of $L^2(U, \mu)$. Reasoning as in the proof of [2, Thm. 3.1, Remark 3.5], one obtains (see also Cor. 2.15 ahead):

Corollary 2.4. *Under the assumptions of Thm. 2.3, with the Legendre coefficients* $l_{\boldsymbol{\nu}} := \int_U u(\boldsymbol{y})L_{\boldsymbol{\nu}}(\boldsymbol{y})\,\mathrm{d}\mu(\boldsymbol{y})$ *and the weights* $w_{\boldsymbol{\nu}} := \prod_{j\in\mathbb{N}} \sqrt{2\nu_j + 1}$ *it holds*

$$(w_{\boldsymbol{\nu}}^{-1}\boldsymbol{\rho}^{\boldsymbol{\nu}}\|l_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^2(\mathcal{F}).$$

2.2. Nonaffine-parametric uncertainty. We now extend the results of the previous subsection to more general diffusion coefficients $A(\boldsymbol{y};x')$. More precisely, rather than affine dependence on y_j, we admit this dependence to be of gpc type, with given summability, as could be implied, for example, by holomorphic dependence on parameters in the uncertainty parametrization. This comprises in particular the case where A is a rational function of y_j, which occurs for instance when the diffusion coefficient arises from pulling back the fractional PDE to some nominal domain via a parametric domain transformation.

This section is structured as follows. In Sec. 2.2.1 we first present an extension of Thm. 2.3 to the case when the uncertain diffusion coefficient $A(\boldsymbol{y};x')$ in the elliptic divergence form operator (1.1) has a non-affine parameter dependence. Again, under suitable summability assumptions on the gpc coefficients of $A(\boldsymbol{y};x')$, we establish corresponding summability of the gpc expansion of the parametric solutions $u(\boldsymbol{y})$ of the fractional diffusion problem. Contrary to results obtained via analytic continuation as for example in [10], the result can account for localized supports of the coefficients in the gpc uncertainty parametrization of the diffusion coefficients. This generalizes, in particular, also [2], [1] to the non-affine parametric setting. In Sec. 2.2.2, we apply the foregoing, general results to the particular case of domain uncertainty quantification where a nonaffine-parametric diffusion coefficient $A(\boldsymbol{y},x')$ naturally arises under pullback into the nominal domain $\mathrm{D}_{\mathbf{0}}$.

2.2.1. Solution sparsity for a non-affine, gpc coefficient. We consider the following setting. Generalizing (2.1), we formally let

$$A(\boldsymbol{y}) = \overline{A} + \sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{y}^{\boldsymbol{\nu}}\Psi_{\boldsymbol{\nu}} \in L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}}) \,, \tag{2.18a}$$

for a nominal diffusion coefficient $\overline{A} \in L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ which is uniformly positive definite in D and where $(\Psi_{\boldsymbol{\nu}})_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \subset L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is again a sequence of fluctuations. In the following it will be convenient to write $\Psi_{\mathbf{0}} := \overline{A}$. With $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ being a sequence of positive numbers, t he weighted uniform ellipticity assumption (2.9) then becomes

$$\delta := \left\| \max\{1, \|\overline{A}^{-1}(\cdot)\|_2\} \sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\rho}^{\boldsymbol{\nu}} \|\Psi_{\boldsymbol{\nu}}(\cdot)\|_2 \right\|_{L^\infty(\mathrm{D})} < 1 \,. \tag{2.18b}$$

Similarly we allow parametric right-hand sides

$$f(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{y}^{\boldsymbol{\nu}} f_{\boldsymbol{\nu}} \in \mathbb{H}^{-s}(\mathrm{D}) \tag{2.19a}$$

where

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} (\boldsymbol{\rho}^{\boldsymbol{\nu}} \|f_{\boldsymbol{\nu}}\|_{\mathbb{H}^{-s}(\mathrm{D})})^2 < \infty. \tag{2.19b}$$

Let us comment on the meaning of the gpc expansions (2.18a), (2.19a). Assuming (2.19b) we note that for some constant $C > 0$ it holds $\|f_{\boldsymbol{\nu}}\|_{\mathbb{H}^{-s}(\mathrm{D})} \leq C\boldsymbol{\rho}^{-\boldsymbol{\nu}}$. An analogous estimate can be deduced for the $\Psi_{\boldsymbol{\nu}}$ based on assumption (2.18b). This entails that the gpc expansions (2.18a), (2.19a) converge uniformly and unconditionally for all $\boldsymbol{y}$ belonging to *some* complex polydisc centered at $\mathbf{0}$ as the next lemma shows.

For a real Banach space X, in the following we denote by $X^{\mathbb{C}}$ its complexification: by this we mean the vector space

$$X^{\mathbb{C}} = \{x_1 + \mathrm{i}x_2 \,:\, x_1, x_2 \in X\}, \tag{2.20}$$

where i is a complex square root of -1, and the space is equipped with the norm $\|x_1+\mathrm{i}x_2\|_{X^{\mathbb{C}}} = \sup_{t\in[0,2\pi)} \|x_1\cos(t) - x_2\sin(t)\|_X$, which generalizes the norm on X, cf. [23].

Lemma 2.5. *Let X be a Banach space over $\mathbb{R}$.*

i) Assume given a sequence $\{t_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathcal{F}\} \subset X$ such that for some constant C it holds $\|t_{\boldsymbol{\nu}}\|_X \le C\boldsymbol{\rho}^{-\boldsymbol{\nu}}$ for all $\boldsymbol{\nu} \in \mathcal{F}$, where $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$. Let $\boldsymbol{\gamma} \in (0,\infty)^{\mathbb{N}}$ be such that $\sum_{j\in\mathbb{N}} \gamma_j\rho_j^{-1} < \infty$ and $\sup_j \gamma_j\rho_j^{-1} < 1$. Then $\sum_{\boldsymbol{\nu}\in\mathcal{F}} t_{\boldsymbol{\nu}}\boldsymbol{z}^{\boldsymbol{\nu}}$ converges uniformly and unconditionally on the polydisc $B_{\boldsymbol{\gamma}}^{\mathbb{C}} = \times_{j\in\mathbb{N}} B_{\gamma_j}^{\mathbb{C}} \subseteq \mathbb{C}^{\mathbb{N}}$ centered at $\mathbf{0}$ to a uniformly bounded function $\boldsymbol{z} \ni B_{\boldsymbol{\gamma}}^{\mathbb{C}} \mapsto u(\boldsymbol{z}) \in X^{\mathbb{C}}$ which is a holomorphic function of each z_j.

ii) Let $\boldsymbol{\rho} \in (0,\infty)^{\mathbb{N}}$. Assume that $\boldsymbol{z} \mapsto u(\boldsymbol{z}) \in X^{\mathbb{C}}$ is a uniformly bounded function on the polydisc $B_{\boldsymbol{\rho}}^{\mathbb{C}} = \times_{j\in\mathbb{N}} B_{\rho_j}^{\mathbb{C}} \subseteq \mathbb{C}^{\mathbb{N}}$ centered at $\mathbf{0}$, such that $u(\boldsymbol{z})$ is holomorphic in each $z_j \in B_{\rho_j}^{\mathbb{C}}$. Let the weight sequence $\boldsymbol{\gamma}$ be as defined in i). Then the Taylor gpc expansion $\sum_{\boldsymbol{\nu}\in\mathcal{F}} t_{\boldsymbol{\nu}}\boldsymbol{z}^{\boldsymbol{\nu}}$, where $t_{\boldsymbol{\nu}} := \frac{1}{\boldsymbol{\nu}!}\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} u(\boldsymbol{z})|_{\boldsymbol{z}=\mathbf{0}}$, converges uniformly and unconditionally in $X^{\mathbb{C}}$ to $u(\boldsymbol{z}) \in X^{\mathbb{C}}$ for all $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}^{\mathbb{C}}$ which satisfy

$$\lim_{N\to\infty} \|u(z_1,\ldots,z_N,0,\ldots) - u(\boldsymbol{z})\|_X = 0. \tag{2.21}$$

Proof. i) We have by assumption of this lemma

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} |\boldsymbol{z}^{\boldsymbol{\nu}}|\|t_{\boldsymbol{\nu}}\|_X \le C\sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\gamma}^{\boldsymbol{\nu}}\boldsymbol{\rho}^{-\boldsymbol{\nu}} < \infty$$

for all $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}^{\mathbb{C}}$, where finiteness holds according to [11, Lemma 7.1] This proves uniform convergence of the series towards some function $u(\boldsymbol{z})$. Fix $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}^{\mathbb{C}}$. The fact that $u(\boldsymbol{z}) \in \mathbb{H}^s(\mathrm{D})$ is holomorphic as a function of $z_j \in B_{\gamma_j}^{\mathbb{C}}$ is a direct consequence of the unconditional convergence in $X^{\mathbb{C}}$ of the Taylor series

$$\sum_{k\in\mathbb{N}_0} z_j^k \left(\sum_{\substack{\boldsymbol{\nu}\in\mathcal{F}\\ \nu_j=k}} \frac{\boldsymbol{z}^{\boldsymbol{\nu}}}{z_j^k} t_{\boldsymbol{\nu}}\right) \tag{2.22}$$

where the kth Taylor coefficient satisfies the bound

$$\left\|\left(\sum_{\substack{\boldsymbol{\nu}\in\mathcal{F}\\ \nu_j=k}} \frac{\boldsymbol{z}^{\boldsymbol{\nu}}}{z_j^k} t_{\boldsymbol{\nu}}\right)\right\|_{X^{\mathbb{C}}} \le C\rho_j^{-k} \sum_{\substack{\boldsymbol{\nu}\in\mathcal{F}\\ \nu_j=k}} \prod_{i\neq k} \gamma^{\nu_i} \prod_{i\neq k} \rho_i^{\nu_i} \le C\rho_j^{-k} \sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\rho}^{-\boldsymbol{\nu}}\boldsymbol{\gamma}^{\boldsymbol{\nu}} \le \tilde{C}\rho_j^{-k}$$

for every fixed $(z_i)_{i\neq j} \in \bigtimes_{i\neq j} B_{\gamma_i}$. Thus the convergence radius of the series (2.22) in z_j is at least $\rho_j \geq \gamma_j$, where we used $\gamma_j\rho_j^{-1} \leq 1$. Moreover, the Taylor series (2.22) converges towards $u(\boldsymbol{z}) \in X^{\mathbb{C}}$ on $B^{\mathbb{C}}_{\gamma_j}$, which is a consequence of the unconditional convergence of the original series $u(\boldsymbol{z}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{z}^{\boldsymbol{\nu}} t_{\boldsymbol{\nu}} \in X^{\mathbb{C}}$, for all $\boldsymbol{z} \in B^{\mathbb{C}}_{\boldsymbol{\gamma}}$.

ii) Since $u(\boldsymbol{z})$ is uniformly bounded in $X^{\mathbb{C}}$ with respect to $\boldsymbol{z} \in B^{\mathbb{C}}_{\boldsymbol{\rho}}$, and moreover holomorphic with respect to each $z_j \in B^{\mathbb{C}}_{\rho_j}$, from the Cauchy integral theorem, one can deduce $\|t_{\boldsymbol{\nu}}\|_X \leq C\boldsymbol{\rho}^{-\boldsymbol{\nu}}$ for some constant C which is independent of both, $\boldsymbol{\rho}$ and $\boldsymbol{\nu}$ (see for example [11, Lemma 2.4]). By the first item, this shows uniform convergence of the Taylor series towards some $\tilde{u}(\boldsymbol{z}) \in X^{\mathbb{C}}$ for all $\boldsymbol{z} \in B^{\mathbb{C}}_{\boldsymbol{\gamma}}$.
Fix $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}$. Due to the assumed holomorphy of $u(\boldsymbol{z})$ in each z_j, $j \in \mathbb{N}$, for every fixed finite $N \in \mathbb{N}$ the Taylor series

$$\sum_{\substack{\boldsymbol{\nu}\in\mathcal{F}\\ \mathrm{supp}\boldsymbol{\nu}\subseteq\{1,\dots,N\}}} t_{\boldsymbol{\nu}}\boldsymbol{z}^{\boldsymbol{\nu}}$$

converges to $u(z_1,\dots,z_N,0,\dots) \in X^{\mathbb{C}}$, see e.g. [22, Thm. 2.1.3]. Letting $N \to \infty$, a diagonal argument proves $\tilde{u}(\boldsymbol{z}) = u(\boldsymbol{z}) \in X^{\mathbb{C}}$ for all $\boldsymbol{z} \in B^{\mathbb{C}}_{\boldsymbol{\gamma}}$ satisfying (2.21). ∎

At this point we do not assume the series (2.18a), (2.19a) to converge for all $\boldsymbol{y} \in U$ w.r.t. the $L^\infty(\mathrm{D})$, $\mathbb{H}^{-s}(\mathrm{D})$ topologies. Nonetheless, due to (2.18b), as long as $\rho_j \geq 1$ for all $j \in \mathbb{N}$, for fixed $\boldsymbol{y} \in U$ the *pointwise limit* of (2.18a) describes a function in $L^\infty(\mathrm{D})$.

Theorem 2.6. *Let A, f admit unconditionally convergent expansions (2.18a), (2.19a) in $\prod_{j\in\mathbb{N}}(-\gamma_j,\gamma_j)$ for some $\boldsymbol{\gamma} \in (0,\infty)^{\mathbb{N}}$ such that $\overline{A} \in L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ in (2.18a) is uniformly positive definite, i.e.*

$$\operatorname*{ess\,inf}_{x'\in\mathrm{D}} \inf_{0\neq\zeta\in\mathbb{R}^d} \frac{\zeta^\top\overline{A}(x')\zeta}{\zeta^\top\zeta} = \overline{A}_{\min} > 0\,. \tag{2.23}$$

Let further $\boldsymbol{\rho} = (\rho_j)_{j\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that (2.18b) and (2.19b) are satisfied. Denote $\mathcal{L}(\boldsymbol{y}) = -\nabla_{x'}\cdot(A(\boldsymbol{y})\nabla_{x'}) : H^1_0(\mathrm{D}) \to H^{-1}(\mathrm{D})$. For some $s \in (0,1)$, let $u : U \to \mathbb{H}^s(\mathrm{D}) : \boldsymbol{y} \mapsto u(\boldsymbol{y})$ denote the weak solution of the parametric fractional diffusion problem $\mathcal{L}^s(\boldsymbol{y})u(\boldsymbol{y}) = f(\boldsymbol{y})$.

Then, for every $\boldsymbol{\nu} \in \mathcal{F}$ the Taylor gpc coefficient $t_{\boldsymbol{\nu}} = \partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} u(\boldsymbol{y})|_{\boldsymbol{y}=\boldsymbol{0}}/\boldsymbol{\nu}! \in \mathbb{H}^s(\mathrm{D})$ is well-defined, and it holds $(\boldsymbol{\rho}^{\boldsymbol{\nu}}\|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^2(\mathcal{F})$ as well as

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}} (\boldsymbol{\rho}^{\boldsymbol{\nu}}\|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})})^2 \leq \frac{C}{1-\delta}\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|f_{\boldsymbol{\nu}}\|^2_{\mathbb{H}^{-s}(\mathrm{D})} \tag{2.24}$$

for a constant C which depends on $\overline{A}_{\min}$ but is independent of f.

The ensuing lemma will be required in the proof.

Lemma 2.7. *Let $(F_k)_{k\in\mathbb{N}}$, $(G_{\boldsymbol{\eta}})_{\boldsymbol{\eta}\in\mathcal{F}}$, $(d_{\boldsymbol{\nu},k})_{\boldsymbol{\nu}\in\mathcal{F},k\in\mathbb{N}}$ be sequences of nonnegative real numbers such that $\sup_{\boldsymbol{\nu}\in\mathcal{F}}\sum_{k\in\mathbb{N}} d_{\boldsymbol{\nu},k} \leq \delta < 1$ and $(F_k)_{k\in\mathbb{N}} \in \ell^1(\mathbb{N})$. Assume $G_{\boldsymbol{0}} < \infty$ and, for all $k \geq 1$,*

$$\sum_{|\boldsymbol{\eta}|=k} G_{\boldsymbol{\eta}} \leq F_k + \sum_{l=0}^{k-1}\sum_{|\boldsymbol{\nu}|=l} G_{\boldsymbol{\nu}} d_{\boldsymbol{\nu},k-l}.$$

Then $(G_{\boldsymbol{\eta}})_{\boldsymbol{\eta}\in\mathcal{F}} \in \ell^1(\mathcal{F})$ *and*

$$\sum_{\boldsymbol{\eta}\in\mathcal{F}} G_{\boldsymbol{\eta}} \leq \frac{\|(F_k)_{k\geq 1}\|_{\ell^1} + G_{\mathbf{0}}}{1-\delta}. \tag{2.25}$$

Proof. By assumption $d_{\boldsymbol{\nu},l} \leq \delta < 1$ for all $\boldsymbol{\nu} \in \mathcal{F}$, $l \in \mathbb{N}$. Since $G_{\mathbf{0}} < \infty$ and since for all $k \in \mathbb{N}$, the series $\sum_{|\boldsymbol{\eta}|=k} G_{\boldsymbol{\eta}}$ can be bounded by the $(\sum_{|\boldsymbol{\eta}|=l} G_{\boldsymbol{\eta}})_{l<k}$ and by $F_k < \infty$, we have $\sum_{|\boldsymbol{\eta}|=k} G_{\boldsymbol{\eta}} < \infty$ for all $k \in \mathbb{N}$. Next, for arbitrary, fixed $n \in \mathbb{N}$

$$\begin{aligned}\sum_{k=0}^{n}\sum_{|\boldsymbol{\eta}|=k} G_{\boldsymbol{\eta}} &\leq G_{\mathbf{0}} + \sum_{k=1}^{n} F_k + \sum_{k=1}^{n}\sum_{l=0}^{k-1}\sum_{|\boldsymbol{\nu}|=l} G_{\boldsymbol{\nu}} d_{\boldsymbol{\nu},k-l}\\ &= G_{\mathbf{0}} + \sum_{k=1}^{n} F_k + \sum_{l=0}^{n-1}\sum_{|\boldsymbol{\nu}|=l} G_{\boldsymbol{\nu}} \sum_{k=l+1}^{n} d_{\boldsymbol{\nu},k-l},\end{aligned}$$

and since $\sum_{k=l+1}^{n} d_{\boldsymbol{\nu},k-l} \leq \delta$, there holds

$$(1-\delta)\sum_{k=0}^{n-1}\sum_{|\boldsymbol{\eta}|=k} G_{\boldsymbol{\eta}} \leq G_{\mathbf{0}} + \sum_{k=1}^{n} F_k - \sum_{|\boldsymbol{\nu}|=n} G_{\boldsymbol{\nu}} \leq G_{\mathbf{0}} + \sum_{k=1}^{n} F_k.$$

Letting $n \to \infty$ we obtain $(G_{\boldsymbol{\eta}})_{\boldsymbol{\eta}\in\mathcal{F}} \in \ell^1(\mathcal{F})$ and (2.25). ∎

Proof of Thm. 2.6. We proceed as in the proof of Thm. 2.3: first the case $\rho_j = 1$ for all $j \in \mathbb{N}$ is considered for the solution of (2.4). Then we deduce the general result by taking the trace and rescaling w.r.t. ρ_j.

Step 1: We start by showing that $t_{\boldsymbol{\nu}}$ is well-defined: By Lemma 2.5 i), we can find $\boldsymbol{\gamma} \in (0,\infty)^{\mathbb{N}}$ such that the expansions (2.18a), (2.19a) converge uniformly for all $\boldsymbol{z}$ in the complex polydisc $B^{\mathbb{C}}_{\boldsymbol{\gamma}} \subseteq \mathbb{C}^{\mathbb{N}}$ centered at $\mathbf{0}$ to elements $A(\boldsymbol{z}) \in (L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}}))^{\mathbb{C}}$, $f(\boldsymbol{z}) \in (\mathbb{H}^{-s}(\mathrm{D}))^{\mathbb{C}}$ respectively (cf. (2.20)). Moreover, the dependence of $A(\boldsymbol{z})$, $f(\boldsymbol{z})$ on $z_j \in B^{\mathbb{C}}_{\gamma_j}$ is holomorphic.

In particular, we have uniform convergence for all $\boldsymbol{y} \in B_{\boldsymbol{\gamma}} := \times_{j\in\mathbb{N}}(-\gamma_j,\gamma_j) \subseteq \mathbb{R}^{\mathbb{N}}$ towards $A(\boldsymbol{y}) \in L^\infty(\mathrm{D},\mathbb{R}^{d\times d}_{\mathrm{sym}})$, $f(\boldsymbol{y}) \in \mathbb{H}^{-s}(\mathrm{D})$. Further decreasing $\gamma_j > 0$ such that $\gamma_j < \min\{1, \rho_j \overline{A}_{\min}/2\}$, $j \in \mathbb{N}$, implies that for all $\boldsymbol{y} \in B_{\boldsymbol{\gamma}}$

$$\begin{aligned}\operatorname*{ess\,inf}_{x'\in\mathrm{D}}\inf_{0\neq\zeta\in\mathbb{R}^d}\frac{\zeta^\top A(\boldsymbol{y};x')\zeta}{\zeta^\top\zeta} &\geq \overline{A}_{\min} - \left\|\sum_{\boldsymbol{\nu}\neq\mathbf{0}} \boldsymbol{\gamma}^{\boldsymbol{\nu}}\|\Psi_{\boldsymbol{\nu}}(x')\|\right\|_{L^\infty(\mathrm{D})}\\ &\geq \overline{A}_{\min} - \sup_{\mathbf{0}\neq\boldsymbol{\nu}}\prod_{j\in\mathbb{N}}\left(\frac{\gamma_j}{\rho_j}\right)^{\nu_j}\left\|\sum_{\boldsymbol{\nu}\neq\mathbf{0}}\boldsymbol{\rho}^{\boldsymbol{\nu}}\|\Psi_{\boldsymbol{\nu}}\|\right\|_{L^\infty(\mathrm{D})} \geq \frac{\overline{A}_{\min}}{2}.\end{aligned}$$

Hence $A(\boldsymbol{y};x')$ is uniformly SPD for all $\boldsymbol{y} \in B_{\boldsymbol{\gamma}}$ and for almost every $x' \in \mathrm{D}$. This implies that $\mathcal{L}^s : \mathbb{H}^s(\mathrm{D}) \to \mathbb{H}^{-s}(\mathrm{D})$ is an isomorphism. Therefore, $u(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D})$ is well-defined for all $\boldsymbol{y} \in B_{\boldsymbol{\gamma}}$.

Let $\{j_i\}_{i=1}^n \subseteq \mathbb{N}$ be an arbitrary finite subset. Fixing $y_k = 0$ for all $k \in \mathbb{N}\backslash\{j_i\}_{i=1}^n$, we claim that $u(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D})$ is real analytic as a function of $(y_{j_1}, \dots, y_{j_n})$ in a neighbourhood of $0 \in \mathbb{R}^n$. To prove the claim, we note that for any multiindex $\boldsymbol{\mu} \in \mathbb{N}_0^n$, (2.18) implies $\|\Psi_{\boldsymbol{\mu}}\|_{L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le C\boldsymbol{\rho}^{-\boldsymbol{\mu}}$. Thus, there holds

$$\left\| \frac{1}{\boldsymbol{\mu}!} \frac{\partial^{|\boldsymbol{\mu}|}}{\partial z_{j_1}^{\mu_1} \cdots \partial z_{j_n}^{\mu_n}} A(\boldsymbol{z})|_{\boldsymbol{z}=\boldsymbol{0}} \right\|_{L^\infty(\mathrm{D};\mathbb{R}^{d\times d}_{\mathrm{sym}})} \le C \prod_{i=1}^n \rho_{j_i}^{-\mu_i}. \tag{2.26}$$

By (2.18a), $A(\boldsymbol{y}) \in L^\infty(\mathrm{D}, \mathbb{R}^{d\times d}_{\mathrm{sym}})$ admits the unconditional Taylor gpc expansion

$$A(\boldsymbol{y}) = \sum_{\boldsymbol{\mu}\in\mathbb{N}_0^n} \frac{1}{\boldsymbol{\mu}!} \frac{\partial^{|\boldsymbol{\mu}|}}{\partial z_{j_1}^{\mu_1} \cdots \partial z_{j_n}^{\mu_n}} A(\boldsymbol{z})|_{\boldsymbol{z}=\boldsymbol{0}} \prod_{j=1}^n y_{j_i}^{\mu_j} \tag{2.27}$$

for $(y_{j_1}, \dots, y_{j_n}) \in \times_{i=1}^n (-\rho_{j_i}, \rho_{j_i})$. Hence, for each $n \in \mathbb{N}$, the matrix function $A(\boldsymbol{y})$ is analytic as a function of $(y_{j_1}, \cdots, y_{j_n}) \in \times_{i=1}^n (-\rho_{j_i}, \rho_{j_i})$. With the same argument we obtain that $f(\boldsymbol{y}) \in \mathbb{H}^{-s}(\mathrm{D})$ is analytic as a function of $(y_{j_1}, \cdots, y_{j_n}) \in \times_{i=1}^n (-\rho_{j_i}, \rho_{j_i})$. Next, for arbitrary, fixed $v \in \mathbb{H}^s(\mathrm{D})$, define $\mathcal{N}(v, \boldsymbol{y}) := \mathcal{L}^s(\boldsymbol{y})v - f(\boldsymbol{y}) \in \mathbb{H}^{-s}(\mathrm{D})$. For every $v \in \mathbb{H}^s(\mathrm{D})$, at every fixed $\boldsymbol{y} \in U$, the map $\boldsymbol{y} \mapsto \mathcal{N}(v, \boldsymbol{y})$ is real analytic, taking values in $\mathbb{H}^{-s}(\mathrm{D})$, when considered as a function of $(y_{j_1}, \dots, y_{j_n}) \in \times_{i=1}^n (-\rho_{j_i}, \rho_{j_i})$. Furthermore, for every $v \in \mathbb{H}^s(\mathrm{D})$, the differential $\partial_u \mathcal{N}(u, \boldsymbol{y})|_{u=v} = \mathcal{L}^s(\boldsymbol{y}) \in L(\mathbb{H}^s(\mathrm{D}), \mathbb{H}^{-s}(\mathrm{D}))$ is an isomorphism. It then follows from the (analytic) implicit function theorem (see for example [14, Thm. 15.3]), that $u(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D})$ is real analytic as a function of $(y_{j_1}, \dots, y_{j_n})$ in a neighbourhood of the origin in $\mathbb{R}^n$. As the selection $(j_i)_{i=1}^n$ was arbitrary, it follows that for every $\boldsymbol{\nu} \in \mathcal{F}$, the Taylor coefficient $t_{\boldsymbol{\nu}} := \frac{1}{\boldsymbol{\nu}!}(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u(\boldsymbol{y}))|_{\boldsymbol{y}=\boldsymbol{0}}$ exists and is well-defined.

Step 2: Assume $\rho_j = 1$ for all $j \in \mathbb{N}$. As before let $\boldsymbol{A}(\boldsymbol{y}; x') = \mathrm{diag}(A(\boldsymbol{y}; x'), 1)$, $\overline{\boldsymbol{A}} = \boldsymbol{\Psi}_{\boldsymbol{0}} = \mathrm{diag}(\overline{A}, 1)$ as well as $\boldsymbol{\Psi}_{\boldsymbol{\nu}}(x') = \mathrm{diag}(\Psi(x'), 0)$ for $\boldsymbol{0} \neq \boldsymbol{\nu} \in \mathcal{F}$. Since the parametric solution $\mathscr{U}(\boldsymbol{y})$ is a weak solution of (2.4), i.e.

$$\int_{\mathcal{C}} z^\alpha \nabla \mathscr{U}(\boldsymbol{y}; x', z)^\top \boldsymbol{A}(\boldsymbol{y}; x') \nabla v(x', z)\, \mathrm{d}x'\, \mathrm{d}z = d_s \langle f(\boldsymbol{y}), \mathrm{tr}_{\mathrm{D}}\, v\rangle \qquad \forall v \in \mathring{H}^1(z^\alpha, \mathcal{C}), \tag{2.28}$$

we get for $\boldsymbol{0} \neq \boldsymbol{\eta} \in \mathcal{F}$ (with the notation $\overline{\boldsymbol{A}} = \boldsymbol{\Psi}_{\boldsymbol{0}}$, and omitting the argument $(x', z) \in \mathcal{C}$ for simplicity)

$$\begin{aligned}
0 &= \partial_{\boldsymbol{y}}^{\boldsymbol{\eta}} \int_{\mathcal{C}} z^\alpha \nabla \mathscr{U}(\boldsymbol{y})^\top \boldsymbol{A}(\boldsymbol{y}) \nabla v\, \mathrm{d}x'\, \mathrm{d}z - \partial_{\boldsymbol{y}}^{\boldsymbol{\eta}} d_s \langle f(\boldsymbol{y}), \mathrm{tr}_{\mathrm{D}}\, v\rangle \\
&= \partial_{\boldsymbol{y}}^{\boldsymbol{\eta}} \int_{\mathcal{C}} z^\alpha \nabla \mathscr{U}(\boldsymbol{y})^\top \left(\sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{y}^{\boldsymbol{\nu}} \boldsymbol{\Psi}_{\boldsymbol{\nu}} \right) \nabla v\, \mathrm{d}x'\, \mathrm{d}z - \partial_{\boldsymbol{y}}^{\boldsymbol{\eta}} d_s \left\langle \sum_{\boldsymbol{\nu}\in\mathcal{F}} f_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}}, \mathrm{tr}_{\mathrm{D}}\, v \right\rangle \\
&= \int_{\mathcal{C}} z^\alpha \sum_{\boldsymbol{\nu}\in\mathcal{F}} \sum_{\substack{\boldsymbol{\gamma}\le\boldsymbol{\eta} \\ \boldsymbol{\gamma}\le\boldsymbol{\nu}}} \binom{\boldsymbol{\eta}}{\boldsymbol{\gamma}} \frac{\boldsymbol{\nu}!}{(\boldsymbol{\nu}-\boldsymbol{\gamma})!} \boldsymbol{y}^{\boldsymbol{\nu}-\boldsymbol{\gamma}} (\partial_{\boldsymbol{y}}^{\boldsymbol{\eta}-\boldsymbol{\gamma}} \nabla \mathscr{U}(\boldsymbol{y}))^\top \boldsymbol{\Psi}_{\boldsymbol{\nu}} \nabla v\, \mathrm{d}x'\, \mathrm{d}z \\
&\quad - d_s \left\langle \sum_{\boldsymbol{\nu}\ge\boldsymbol{\eta}} \frac{\boldsymbol{\nu}!}{(\boldsymbol{\nu}-\boldsymbol{\eta})!} \boldsymbol{y}^{\boldsymbol{\nu}-\boldsymbol{\eta}} f_{\boldsymbol{\nu}}, \mathrm{tr}_{\mathrm{D}}\, v \right\rangle .
\end{aligned}$$

The elementwise differentiation is justified, since these are convergent Taylor series (in a neighbourhood of $\mathbf{0}$, considered as a function of the finitely many parameters y_j for which $\eta_j \neq 0$). Evaluating at $\boldsymbol{y} = \mathbf{0}$ results in the identity

$$0 = \sum_{\boldsymbol{\nu} \leq \boldsymbol{\eta}} \binom{\boldsymbol{\eta}}{\boldsymbol{\nu}} \boldsymbol{\nu}! \int_{\mathcal{C}} z^{\alpha} (\partial_{\boldsymbol{y}}^{\boldsymbol{\eta}-\boldsymbol{\nu}} \nabla \mathscr{U}(\boldsymbol{y})|_{\boldsymbol{y}=\mathbf{0}})^{\top} \boldsymbol{\Psi}_{\boldsymbol{\nu}} \nabla v \, \mathrm{d}x' \, \mathrm{d}z - \boldsymbol{\eta}! d_s \langle f_{\boldsymbol{\eta}}, \mathrm{tr}_{\mathrm{D}} \, v \rangle \qquad \forall v \in \mathring{H}^1(z^{\alpha}, \mathcal{C}).$$

By $\binom{\boldsymbol{\eta}}{\boldsymbol{\nu}} = \frac{\boldsymbol{\eta}!}{\boldsymbol{\nu}!(\boldsymbol{\eta}-\boldsymbol{\nu})!}$ with the Taylor coefficient $T_{\boldsymbol{\nu}} := \frac{1}{\boldsymbol{\nu}!} \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}(\boldsymbol{y})|_{\boldsymbol{y}=\mathbf{0}} \in \mathring{H}^1(z^{\alpha}, \mathcal{C})$, for every $v \in \mathring{H}^1(z^{\alpha}, \mathcal{C})$ there holds

$$\begin{aligned} \int_{\mathcal{C}} z^{\alpha} \nabla T_{\boldsymbol{\eta}}^{\top} \overline{\boldsymbol{A}} \nabla v \, \mathrm{d}x' \, \mathrm{d}z &= \int_{\mathcal{C}} z^{\alpha} \nabla \left(\frac{\partial_{\boldsymbol{y}}^{\boldsymbol{\eta}} \mathscr{U}(\boldsymbol{y})|_{\boldsymbol{y}=\mathbf{0}}}{\boldsymbol{\nu}!} \right)^{\top} \overline{\boldsymbol{A}} \nabla v \, \mathrm{d}x' \, \mathrm{d}z \\ &= d_s \langle f_{\boldsymbol{\eta}}, \mathrm{tr}_{\mathrm{D}} \, v \rangle - \sum_{\boldsymbol{\nu} < \boldsymbol{\eta}} \int_{\mathcal{C}} z^{\alpha} \frac{(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \mathscr{U}(\boldsymbol{y})|_{\boldsymbol{y}=\mathbf{0}})^{\top}}{\boldsymbol{\nu}!} \nabla \boldsymbol{\Psi}_{\boldsymbol{\eta}-\boldsymbol{\nu}} \nabla v \, \mathrm{d}x' \, \mathrm{d}z \\ &= d_s \langle f_{\boldsymbol{\eta}}, \mathrm{tr}_{\mathrm{D}} \, v \rangle - \sum_{\boldsymbol{\nu} < \boldsymbol{\eta}} \int_{\mathcal{C}} z^{\alpha} \nabla T_{\boldsymbol{\nu}}^{\top} \boldsymbol{\Psi}_{\boldsymbol{\eta}-\boldsymbol{\nu}} \nabla v \, \mathrm{d}x' \, \mathrm{d}z. \end{aligned} \tag{2.29}$$

To obtain recursive bounds on the gpc coefficients, we introduce the notation

$$G_{\boldsymbol{\eta}} := \int_{\mathcal{C}} z^{\alpha} \nabla T_{\boldsymbol{\eta}}^{\top} \overline{\boldsymbol{A}} \nabla T_{\boldsymbol{\eta}} \, \mathrm{d}x' \, \mathrm{d}z = a_{\mathcal{C}}(0; T_{\boldsymbol{\eta}}, T_{\boldsymbol{\eta}})$$

and

$$G_{\boldsymbol{\eta},\boldsymbol{\nu}} := \int_{\mathcal{C}} z^{\alpha} \|\nabla T_{\boldsymbol{\eta}}(x')\|_2^2 \|\boldsymbol{\Psi}_{\boldsymbol{\nu}}(x')\|_2 \, \mathrm{d}x' \, \mathrm{d}z.$$

Choosing in (2.29) the test function $v = T_{\boldsymbol{\eta}}$ we arrive at the recursive estimates

$$\begin{aligned} G_{\boldsymbol{\eta}} &\leq d_s \|f_{\boldsymbol{\eta}}\|_{\mathbb{H}^{-s}(\mathrm{D})} \|\mathrm{tr}_{\mathrm{D}} \, T_{\boldsymbol{\eta}}\|_{\mathbb{H}^s(\mathrm{D})} + \sum_{\boldsymbol{\nu} < \boldsymbol{\eta}} \int_{\mathcal{C}} z^{\alpha} \|\nabla T_{\boldsymbol{\nu}}\|_2 \|\boldsymbol{\Psi}_{\boldsymbol{\eta}-\boldsymbol{\nu}}\|_2 \|\nabla T_{\boldsymbol{\eta}}\|_2 \, \mathrm{d}x' \, \mathrm{d}z \\ &\leq \frac{C_{\mathrm{tr}_{\mathrm{D}}} d_s}{\min\{1, \overline{A}_{\min}\}^{1/2}} \|f_{\boldsymbol{\eta}}\|_{\mathbb{H}^{-s}(\mathrm{D})} G_{\boldsymbol{\eta}}^{1/2} \\ &\quad + \sum_{\boldsymbol{\nu} < \boldsymbol{\eta}} \left(\int_{\mathcal{C}} z^{\alpha} \|\nabla T_{\boldsymbol{\nu}}\|_2^2 \|\boldsymbol{\Psi}_{\boldsymbol{\eta}-\boldsymbol{\nu}}\|_2 \, \mathrm{d}x' \, \mathrm{d}z \right)^{1/2} \left(\int_{\mathcal{C}} z^{\alpha} \|\nabla T_{\boldsymbol{\eta}}\|_2^2 \|\boldsymbol{\Psi}_{\boldsymbol{\eta}-\boldsymbol{\nu}}\|_2 \, \mathrm{d}x' \, \mathrm{d}z \right)^{1/2} \\ &\leq \frac{C_{\mathrm{tr}_{\mathrm{D}}}^2 d_s^2}{2(1-\delta) \min\{1, \overline{A}_{\min}\}} \|f_{\boldsymbol{\eta}}\|_{\mathbb{H}^{-s}(\mathrm{D})}^2 + \frac{1-\delta}{2} G_{\boldsymbol{\eta}} + \frac{1}{2} \sum_{\boldsymbol{\nu} < \boldsymbol{\eta}} (G_{\boldsymbol{\nu},\boldsymbol{\eta}-\boldsymbol{\nu}} + G_{\boldsymbol{\eta},\boldsymbol{\eta}-\boldsymbol{\nu}}), \end{aligned} \tag{2.30}$$

where we used Young's inequality $|ab| \leq a^2/(2(1-\delta)) + (1-\delta)b^2/2$ for a, $b \in \mathbb{R}$, and continuity of the trace $\mathrm{tr}_{\mathrm{D}} : \mathring{H}^1(z^{\alpha}, \mathcal{C}) \to \mathbb{H}^s(\mathrm{D})$, i.e. (1.12). In particular $C_0 := C_{\mathrm{tr}_{\mathrm{D}}}^2 d_s^2 / (2(1-\delta) \min\{1, \overline{A}_{\min}\})$ is independent of $\boldsymbol{\eta}$. Now, employing (2.18b) and symmetry of $\overline{\boldsymbol{A}}^{1/2}$, we obtain

$$
\begin{aligned}
\sum_{\boldsymbol{\nu}<\boldsymbol{\eta}} G_{\boldsymbol{\eta},\boldsymbol{\eta}-\boldsymbol{\nu}} \leq \sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{F}} G_{\boldsymbol{\eta},\boldsymbol{\gamma}} &\leq \int_{\mathcal{C}} z^{\alpha} T_{\boldsymbol{\eta}}^{\top} \overline{\boldsymbol{A}}^{\frac{1}{2}} \left(\sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{F}} \|\boldsymbol{\Psi}_{\boldsymbol{\gamma}}\| \overline{\boldsymbol{A}}^{-1} \right) \overline{\boldsymbol{A}}^{\frac{1}{2}} T_{\boldsymbol{\eta}} \,\mathrm{d}x' \,\mathrm{d}z \\
&\leq \delta \int_{\mathcal{C}} z^{\alpha} T_{\boldsymbol{\eta}}^{\top} \overline{\boldsymbol{A}} T_{\boldsymbol{\eta}} \,\mathrm{d}x' \,\mathrm{d}z = \delta G_{\boldsymbol{\eta}} \qquad (2.31)
\end{aligned}
$$

and conclude with (2.30) and with the constant C_0 defined above that

$$
G_{\boldsymbol{\eta}} \leq 2C_0 \|f_{\boldsymbol{\eta}}\|_{\mathbb{H}^{-s}(\mathrm{D})}^2 + \sum_{\boldsymbol{\nu}<\boldsymbol{\eta}} G_{\boldsymbol{\nu},\boldsymbol{\eta}-\boldsymbol{\nu}}.
$$

For every $\boldsymbol{\nu} \in \mathcal{F}$ and $l \in \mathbb{N}$ we define $d_{\boldsymbol{\nu},l} := \sum_{|\boldsymbol{\gamma}|=l} G_{\boldsymbol{\nu},\boldsymbol{\gamma}}/G_{\boldsymbol{\nu}}$ if $G_{\boldsymbol{\nu}} \neq 0$, and $d_{\boldsymbol{\nu},l} := 0$ otherwise. In both cases this entails $\sum_{|\boldsymbol{\gamma}|=l} G_{\boldsymbol{\nu},\boldsymbol{\gamma}} = d_{\boldsymbol{\nu},l} G_{\boldsymbol{\nu}}$ because $G_{\boldsymbol{\nu}} = 0$ implies $T_{\boldsymbol{\nu}} = 0$ and thus $\sum_{|\boldsymbol{\gamma}|=l} G_{\boldsymbol{\nu},\boldsymbol{\gamma}} = 0$. For $k \geq 1$ define $F_k := 2C_0 \sum_{|\boldsymbol{\nu}|=k} \|f_{\boldsymbol{\nu}}\|_{\mathbb{H}^{-s}(\mathrm{D})}^2$. We obtain

$$
\begin{aligned}
\sum_{|\boldsymbol{\eta}|=k} G_{\boldsymbol{\eta}} &\leq F_k + \sum_{|\boldsymbol{\eta}|=k} \sum_{\boldsymbol{\nu}<\boldsymbol{\eta}} G_{\boldsymbol{\nu},\boldsymbol{\eta}-\boldsymbol{\nu}} = F_k + \sum_{|\boldsymbol{\nu}|<k} \sum_{|\boldsymbol{\gamma}|=k-|\boldsymbol{\nu}|} G_{\boldsymbol{\nu},\boldsymbol{\gamma}} \\
&= F_k + \sum_{l=0}^{k-1} \sum_{|\boldsymbol{\nu}|=l} \sum_{|\boldsymbol{\gamma}|=k-l} G_{\boldsymbol{\nu},\boldsymbol{\gamma}} \leq F_k + \sum_{l=0}^{k-1} \sum_{|\boldsymbol{\nu}|=l} d_{\boldsymbol{\nu},k-l} G_{\boldsymbol{\nu}}.
\end{aligned}
$$

For $\boldsymbol{\nu} \in \mathcal{F}$ fixed, as in (2.31) it holds $G_{\boldsymbol{\nu}} \sum_{l\in\mathbb{N}} d_{\boldsymbol{\nu},l} = \sum_{\mathbf{0}\neq\boldsymbol{\gamma}\in\mathcal{F}} G_{\boldsymbol{\nu},\boldsymbol{\gamma}} \leq \delta G_{\boldsymbol{\nu}}$, which shows $\sup_{\boldsymbol{\nu}\in\mathcal{F}} \sum_{l\in\mathbb{N}} d_{l,\boldsymbol{\nu}} \leq \delta < 1$. Furthermore, by assumption $G_{\mathbf{0}} < \infty$. Therefore, Lemma 2.7 gives

$$
\sum_{\boldsymbol{\eta}\in\mathcal{F}} G_{\boldsymbol{\eta}} = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\eta}|=k} G_{\boldsymbol{\eta}} \leq \frac{2C_0 \sum_{\boldsymbol{\nu}\in\mathcal{F}} \|f_{\boldsymbol{\nu}}\|_{\mathbb{H}^{-s}(\mathrm{D})}^2 + G_{\mathbf{0}}}{(1-\delta)}.
$$

This shows $\sum_{\boldsymbol{\eta}\in\mathcal{F}} G_{\boldsymbol{\eta}} = \sum_{\boldsymbol{\nu}\in\mathcal{F}} \|T_{\boldsymbol{\nu}}\|_{\overline{\boldsymbol{A}}}^2 \leq (2C_0 \sum_{\boldsymbol{\nu}\in\mathcal{F}} \|f_{\boldsymbol{\nu}}\|_{\mathbb{H}^{-s}(\mathrm{D})}^2 + \|T_{\mathbf{0}}\|_{\overline{\boldsymbol{A}}}^2)/(1-\delta)$.

Step 3: From Prop. 2.1, we obtain with (2.6) that $t_{\boldsymbol{\nu}} = \mathrm{tr}_{\mathrm{D}}\, T_{\boldsymbol{\nu}}$. The rest of the proof is now completely analogous to the one of Thm. 2.3: From the continuity estimate (1.12), there exists a constant C such that

$$
\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|t_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D})}^2 \leq \frac{C_{\mathrm{tr}_{\mathrm{D}}}^2}{\min\{\overline{A}_{\min}, 1\}} \sum_{\boldsymbol{\nu}\in\mathcal{F}} \|T_{\boldsymbol{\nu}}\|_{\overline{\boldsymbol{A}}}^2 \leq C \frac{\sum_{\boldsymbol{\nu}\in\mathcal{F}} \|f_{\boldsymbol{\nu}}\|_{\mathbb{H}^{-s}(\mathrm{D})}^2 + \|T_{\mathbf{0}}\|_{\overline{\boldsymbol{A}}}^2}{1-\delta} < \infty.
$$

Since $T_{\mathbf{0}}$ solves (2.28), we have the apriori bound $\|T_{\mathbf{0}}\|_{\overline{\boldsymbol{A}}} \leq C_{\mathrm{tr}_{\mathrm{D}}} d_s \min\{1, \overline{A}_{\min}\}^{-1} \|f_{\mathbf{0}}\|_{\mathbb{H}^{-s}(\mathrm{D})}$ due to (1.14).This proves (2.10) in the case $\rho_j = 1$ for all $j \in \mathbb{N}$. As in Step 3 of the proof of Thm. 2.3 we obtain the statement for general $\rho_j \geq 1$ by rescaling the equation. ■

2.2.2. Domain Uncertainty Quantification. Let $\mathrm{D}_{\mathbf{0}} \subseteq \mathbb{R}^d$ be a Lipschitz domain, and let $T : \mathrm{D}_{\mathbf{0}} \to T(\mathrm{D}_{\mathbf{0}}) =: \mathrm{D}_T \subseteq \mathbb{R}^d$ be a bi-Lipschitz transformation such that D_T is also a Lipschitz domain. Denote $\mathcal{C}_{\mathbf{0}} := \mathrm{D}_{\mathbf{0}} \times (0, \infty)$ as well as $\mathcal{C}_T := \mathrm{D}_T \times (0, \infty)$. For some diffusion coefficient $A \in L^\infty(\mathrm{D}_T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ we consider again problem (1.3) on D_T, i.e. $\mathcal{L}^s u = f$ in D_T, with homogeneous Dirichlet boundary condition $u|_{\partial\mathrm{D}} = 0$, fractional exponent $s \in (0,1)$, $f \in L^2(\mathrm{D}_T)$ and the differential operator $\mathcal{L}u = -\nabla_{x'}\cdot(A(x')\nabla_{x'})$. As explained in Sec. 1.2, with $\boldsymbol{A} = \mathrm{diag}(A,1)$ this problem is equivalent to the weak formulation

$$\int_{\mathcal{C}_T} z^\alpha(\nabla \mathscr{U}^\top \boldsymbol{A}\nabla v)\,\mathrm{d}x'\,\mathrm{d}z = d_s\int_{\mathrm{D}_T} f\,\mathrm{tr}_{\mathrm{D}}\, v\,\mathrm{d}x' \qquad \forall v\in \mathring{H}^1(z^\alpha,\mathcal{C}_T), \tag{2.32}$$

for $\mathscr{U} \in \mathring{H}^1(z^\alpha,\mathcal{C}_T)$ in the sense of Prop. 1.1. Denote in the following by $\boldsymbol{T}:\mathcal{C}_{\mathbf{0}}\to\mathcal{C}_T$ the transformation $\boldsymbol{T}(x',z) := (T(x'),z)$. Transforming the integrals in (2.32) to the extended nominal domain $\mathcal{C}_{\mathbf{0}} = \mathrm{D}_{\mathbf{0}}\times(0,\infty)$, the pullback $\hat{\mathscr{U}} := \mathscr{U}\circ\boldsymbol{T}$ is a weak solution of

$$\begin{aligned}\int_{\mathcal{C}_{\mathbf{0}}} & z^\alpha \nabla\hat{\mathscr{U}}^\top(D\boldsymbol{T}^{-1}\boldsymbol{A}\circ\boldsymbol{T}D\boldsymbol{T}^{-\top})\nabla(v\circ\boldsymbol{T}))\det D T\,\mathrm{d}x'\,\mathrm{d}z\\ &= d_s\int_{\mathrm{D}_{\mathbf{0}}} f\circ T\,\mathrm{tr}_{\mathrm{D}}(v\circ\boldsymbol{T})\det DT\,\mathrm{d}x' \qquad \forall v\in\mathring{H}^1(z^\alpha,\mathcal{C}_T)\,.\end{aligned} \tag{2.33}$$

Lemma 2.8. *Let $s\in(0,1)$. Let $f\in L^2(\mathrm{D}_T)$, $A\in L^\infty(\mathrm{D}_T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ be uniformly SPD and denote by $u\in\mathbb{H}^s(\mathrm{D}_T)$ the solution to $\mathcal{L}^s u = f$ where $\mathcal{L} = -\nabla_{x'}\cdot(A\nabla_{x'})$. Then $\hat u := u\circ T\in\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})$ is the unique solution to $\hat{\mathcal{L}}^s\hat u = \hat f$, where $\hat f := f\circ T\det DT\in L^2(\mathrm{D}_{\mathbf{0}})$ and $\hat{\mathcal{L}} := -\nabla_{x'}\cdot((DT^{-1}A\circ T\mathrm{D}_T^{-\top}\det DT)\nabla_{x'})$.*

Proof. By Prop. 1.1, we know that $u = \mathrm{tr}_{\mathrm{D}}\,\mathscr{U}$ where $\mathscr{U}\in\mathring{H}^1(z^\alpha,\mathcal{C}_T)$ is the solution of (2.32). Since $T:\mathrm{D}_{\mathbf{0}}\to\mathrm{D}_T$ is bi-Lipschitz, we observe that $\Phi: v\mapsto v\circ\boldsymbol{T}$ is a bounded linear map from $\mathring{H}^1(z^\alpha,\mathrm{D}_T)$ to $\mathring{H}^1(z^\alpha,\mathrm{D}_{\mathbf{0}})$. Its inverse is clearly given by $v\mapsto v\circ\boldsymbol{T}^{-1}$ with $\boldsymbol{T}^{-1}(x',z) = (T^{-1}(x'),z)$, and consequently $\Phi:\mathring{H}^1(z^\alpha,\mathrm{D}_T)\to\mathring{H}^1(z^\alpha,\mathrm{D}_{\mathbf{0}})$ is an isomorphism. Transforming the weak formulation we obtain (2.33) as a weak formulation on the nominal domain. By construction its solution is given by $\hat{\mathscr{U}} = \mathscr{U}\circ\boldsymbol{T}$. Due to the fact that Φ is an isomorphism we note that $\{v\circ\boldsymbol{T} : v\in\mathring{H}^1(z^\alpha,\mathrm{D}_T)\} = \mathring{H}^1(z^\alpha,\mathrm{D}_{\mathbf{0}})$. Hence we may again employ Prop. 1.1, to observe that $\hat u = \mathrm{tr}_{\mathrm{D}}\,\hat{\mathscr{U}}\in\mathbb{H}^s(\mathrm{D}_T)$ is the solution to $\hat{\mathcal{L}}^s\hat u = \hat f$. Since $DT^{-1}A\circ T\mathrm{D}_T^{-\top}\det DT$ is uniformly SPD, the solution of $\hat{\mathcal{L}}^s\hat u = \hat f$ is unique. ■

We characterize uncertainty in the domain through a parametric family of domain mappings $T_{\boldsymbol{y}}:\mathrm{D}_{\mathbf{0}}\to\mathrm{D}_{\boldsymbol{y}}$ so that $\mathrm{D}_{\boldsymbol{y}} := T_{\boldsymbol{y}}(\mathrm{D}_{\mathbf{0}})$ for all $\boldsymbol{y}\in U$. The fractional diffusion problem pulled back to the nominal domain $\mathrm{D}_{\mathbf{0}}$ then reads: find $\hat u: U\to\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})$ such that for all $\boldsymbol{y}\in U$

$$\hat{\mathcal{L}}^s(\boldsymbol{y})\hat u(\boldsymbol{y}) = \hat f(\boldsymbol{y}) \quad \text{in } \mathbb{H}^{-s}(\mathrm{D}_{\mathbf{0}}), \tag{2.34a}$$

where $0<s<1$ and

$$\begin{aligned}\hat{\mathcal{L}}(\boldsymbol{y}) &:= -\nabla_{x'}\cdot(DT_{\boldsymbol{y}}^{-1}(A\circ T_{\boldsymbol{y}})DT_{\boldsymbol{y}}^{-\top}\det DT_{\boldsymbol{y}}\nabla_{x'})\in L(H^1_0(\mathrm{D}_{\mathbf{0}}),H^{-1}(\mathrm{D}_{\mathbf{0}})),\\ \hat f(\boldsymbol{y}) &:= f\circ T_{\boldsymbol{y}}\det DT_{\boldsymbol{y}}\in\mathbb{H}^{-s}(\mathrm{D}_{\mathbf{0}})\,.\end{aligned} \tag{2.34b}$$

Next, we introduce further assumptions on the admissible domain transformations. The constant $\delta>0$ appearing below will be specified in Thm. 2.10 ahead.

Assumption 2.9. *There exists $\delta > 0$ and $(\psi_j)_{j\in\mathbb{N}} \subseteq W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)$ such that with the Jacobian matrix $D\psi_j \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^{d\times d})$*

$$\left\| \sum_{j\in\mathbb{N}} \rho_j \left(\|\psi_j(x')\|_2 + \|D\psi_j(x')\|_2 \right) \right\|_{L^\infty(\mathrm{D}_{\mathbf{0}})} < \delta, \tag{2.35}$$

where $\rho_j > 1$ for all $j \in \mathbb{N}$ and $\rho_j \to \infty$ as $j \to \infty$. For $\boldsymbol{y} \in U$ let $T_{\boldsymbol{y}} := \mathrm{Id} + \sum_{j\in\mathbb{N}} y_j\psi_j \in W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)$, and set $\mathrm{D}_{\boldsymbol{y}} := T_{\boldsymbol{y}}(\mathrm{D}_{\mathbf{0}}) \subseteq \mathbb{R}^d$. It holds that $\mathrm{D}_{\boldsymbol{y}}$ is a Lipschitz domain such that $T_{\boldsymbol{y}} : \mathrm{D}_{\mathbf{0}} \to \mathrm{D}_{\boldsymbol{y}}$ is bi-Lipschitz for every $\boldsymbol{y} \in U = [-1,1]^{\mathbb{N}}$.

The goal of this section is to prove the following statement on the domain sensitivities.

Theorem 2.10. *Let $s \in (0,1)$, $2 \le d \in \mathbb{N}$ and let $\mathrm{D}_{\mathbf{0}} \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Assume that the* hold-all domain $\mathrm{D}_{\mathrm{H}} \subseteq \mathbb{R}^d$ *is bounded and let $f : \mathrm{D}_{\mathrm{H}} \to \mathbb{R}$ and $A : \mathrm{D}_{\mathrm{H}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ allow real analytic extensions to an open superset $O \subseteq \mathbb{R}^d$ with $\overline{\mathrm{D}_{\mathrm{H}}} \subseteq O$, such that A satisfies the uniform ellipticity assumption*

$$\operatorname*{ess\,inf}_{x'\in\mathrm{D}_{\mathrm{H}}} \inf_{0\neq\zeta\in\mathbb{R}^d} \frac{\zeta^\top A(x')\zeta}{\zeta^\top\zeta} = \underline{A}_{\min} > 0. \tag{2.36}$$

There exists $\delta = \delta(f, A, \mathrm{D}_{\mathbf{0}}) > 0$ depending on (the holomorhpy domains of) f, A and $\mathrm{D}_{\mathbf{0}}$ such that the following holds: If Assumption 2.9 is satisfied (with this δ), and if additionally $\overline{\mathrm{D}_{\boldsymbol{y}}} \subseteq \mathrm{D}_{\mathrm{H}}$ for all $\boldsymbol{y} \in U$, then for the parametric solution $\{\hat{u}(\boldsymbol{y}) : \boldsymbol{y} \in U\} \subset \mathbb{H}^s(\mathrm{D}_{\mathbf{0}})$ of (2.34), we have

i) with $\mathcal{L} = -\nabla_{x'} \cdot (A(x')\nabla_{x'})$ it holds $\mathcal{L}^s(\hat{u}(\boldsymbol{y}) \circ T_{\boldsymbol{y}}^{-1}) = f|_{\mathrm{D}_{\boldsymbol{y}}} \in \mathbb{H}^{-s}(\mathrm{D}_{\boldsymbol{y}})$ for all $\boldsymbol{y} \in U$,
ii) $\hat{u}(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D}_{\mathbf{0}})$ depends continuously on $\boldsymbol{y} \in U$ (with the product topology on U),
iii) the Taylor gpc coefficients $\hat{t}_{\boldsymbol{y};\boldsymbol{\nu}} := \partial^{\boldsymbol{\nu}}_{\boldsymbol{z}} \hat{u}(\boldsymbol{z})|_{\boldsymbol{z}=\boldsymbol{y}}/\boldsymbol{\nu}!$, $\boldsymbol{\nu} \in \mathcal{F}$, fulfill

$$\sup_{\boldsymbol{y}\in U} \sum_{\boldsymbol{\nu}\in\mathcal{F}} (\boldsymbol{\rho}^{\boldsymbol{\nu}} \|\hat{t}_{\boldsymbol{y};\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})})^2 < \infty. \tag{2.37}$$

Item iii) can in particular be used to deduce summability of the Taylor coefficients at $\mathbf{0} \in U$ as in Step 4 of the proof of Thm. 2.3: Applying Hölder's inequality, we obtain $(\|\hat{t}_{\mathbf{0};\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^{2q/(2+q)}(\mathbb{N})$ under the presumption that $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^q(\mathbb{N})$ and $\inf_{j\in\mathbb{N}} \rho_j > 1$. To prove iii), we will employ Thm. 2.6 for the pullback solutions on the nominal domain. The purpose of the next lemma is to verify the assumptions of Thm. 2.6 for the corresponding (pullback) coefficients occuring in (2.34b). After proving the lemma we proceed with the proof of Thm. 2.10.

Lemma 2.11. *Let d, m, $n \in \mathbb{N}$ and let $\mathrm{D}_{\mathbf{0}} \subseteq \mathbb{R}^d$ be bounded. Assume further that $F \in L^\infty(O \times O_{\mathrm{D}};\mathbb{C}^m)$ is holomorphic, where $O \subseteq \mathbb{C}^n$, $O_{\mathrm{D}} \subseteq \mathbb{C}^d$ are open, $0 \in O$ and $\overline{\mathrm{D}_{\mathbf{0}}} \subseteq O_{\mathrm{D}}$.*

Then, for every $\gamma > 0$ there exists $\delta > 0$ (depending on F and γ) such that for every sequence $\{\psi_j\}_{j\in\mathbb{N}} \subseteq L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)$ satisfying $\|\sum_{j\in\mathbb{N}} |\psi_j(\cdot)|\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^n)} < \delta$ there holds

$$\sup_{\boldsymbol{y}\in U} \left\| \sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \frac{1}{\boldsymbol{\nu}!} \left| \partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} \Big(F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot), \cdot \Big) \Big) \right| \right\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^m)} < \gamma, \tag{2.38}$$

where $|\partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} F(\cdots)|$ denotes the componentwise modulus.

Proof. Without loss of generality we assume $m = 1$, since for general $m \in \mathbb{N}$ the statement follows by applying the result to each component of F separately.

Step 1: We show that, if $\delta > 0$ is small enough, then $F(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}})$ is complex differentiable in each y_j at $\boldsymbol{y} \in U$, and all partial derivatives in (2.38) are well-defined as elements of $L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C})$. To this end let $\delta > 0$ be small enough such that the ball of radius 3δ with center $0 \in \mathbb{C}^n$ is contained in O. Then $\Phi(\psi)(\cdot) := F(\psi(\cdot),\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C})$ is well-defined for all $\psi \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)$ with $\|\psi\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)} \le \delta$. Additionally let $h \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)$ with components $h = (h_j)_{j=1}^n$ have sufficiently small $L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)$-norm. For $\boldsymbol{\zeta} \in O \subseteq \mathbb{C}^n$ we write $\frac{\partial}{\partial\zeta_j}F(\boldsymbol{\zeta},x') \in \mathbb{C}$ to denote the partial derivative of $F(\boldsymbol{\zeta},x')$ w.r.t. ζ_j. Then, for a.e. $x' \in \mathrm{D}_{\mathbf{0}}$, there holds

$$F(\psi(x')+h(x'),x') - F(\psi(x'),x') = \left(\sum_{j=1}^n \frac{\partial}{\partial\zeta_j}F(\psi(x'),x')h_j(x')\right) + R(h(x')) \tag{2.39}$$

with a remainder term $R(h(x'))$. To estimate the remainder term we note that if $\|\psi\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)}$, $\|h\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)} \le \delta$, then for a.e. $x' \in \mathrm{D}_{\mathbf{0}}$

$$\begin{aligned} F(\psi(x')+h(x'),x') - F(\psi(x'),x') &= \int_0^1 \sum_{j=1}^n \frac{\partial}{\partial\zeta_j}F(\psi(x')+th(x'),x')h_j(x')\,\mathrm{d}t \\ &= \sum_{j=1}^n \frac{\partial}{\partial\zeta_j}F(\psi(x'),x')h_j(x') + \underbrace{\int_0^1\int_0^t \sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2}{\partial\zeta_i\partial\zeta_j}F(\psi(x')+sh(x'),x')h_i(x')h_j(x')\,\mathrm{d}s\,\mathrm{d}t}_{=R(h(x'))}. \end{aligned} \tag{2.40}$$

We claim that $\|R(h(\cdot))\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C})} = O(\|h\|^2_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)})$ for $\|h\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)} \to 0$. This is true since we have a uniform bound on the second derivatives $\operatorname{ess\,sup}_{x'\in\mathrm{D}_{\mathbf{0}}}\sup_{1\le i,j\le n}|\frac{\partial^2}{\partial\zeta_i\partial\zeta_j}F(\boldsymbol{\zeta},x')|$ for $\boldsymbol{\zeta}$ in the compact set $\{\boldsymbol{\zeta}\in\mathbb{C}^n : \|\boldsymbol{\zeta}\|_2 \le 2\delta\} \subseteq O$ and for x' in the compact set $\overline{\mathrm{D}_{\mathbf{0}}} \subseteq O_{\mathrm{D}}$. This shows that Φ is (complex) differentiable at $\psi \in B_\delta^{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)}$ with differential

$$D\Phi(\psi)(h)(\cdot) = \sum_{j=1}^n \frac{\partial}{\partial\zeta_j}F(\psi(\cdot),\cdot)h_j(\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}) \tag{2.41}$$

for every $h \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)$.

Next we show existence of the partial derivatives $\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}}\Phi(\sum_{j\in\mathbb{N}} y_j\psi_j) \in L^\infty(\mathrm{D}_{\boldsymbol{\nu}};\mathbb{C})$ for every $\boldsymbol{\nu} \in \mathcal{F}$ and $\boldsymbol{y} \in U$. Fix $\boldsymbol{y} \in U$. For $i \in \mathbb{N}$ denote by $\boldsymbol{e}_i = (e_{i;j})_{j\in\mathbb{N}} \in \mathbb{N}_0^{\mathbb{N}}$ the multiindex with $e_{i;j} = 1$ if $j = i$ and $e_{i;j} = 0$ otherwise. By assumption it holds

$$\left\|\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot)\right\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)} \le \left\|\sum_{j\in\mathbb{N}} |\psi_j(\cdot)|\right\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^n)} \le \delta\,. \tag{2.42}$$

Therefore, Φ is differentiable at $\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot) \in B_\delta^{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^n)}$, and with the chain rule we conclude for arbitrary $i \in \mathbb{N}$

$$\begin{aligned} \partial_{\boldsymbol{y}}^{\boldsymbol{e}_i}\Phi\Big(\sum_{j\in\mathbb{N}} y_j\psi_j\Big) &= \partial_{\boldsymbol{y}}^{\boldsymbol{e}_i} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot\Big) \\ &= \sum_{k=1}^{n} \frac{\partial}{\partial\zeta_k} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot\Big)\psi_{i;k}(\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}), \end{aligned} \tag{2.43}$$

where we use the notation $\psi_i = (\psi_{i;k})_{k=1}^n$ for the components of ψ_i. The last term is a finite sum of functions of the type $G(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot)\eta(\cdot)$, where $G \in L^\infty(O\times O_{\mathrm{D}};\mathbb{C})$ is holomorphic on $O \times O_{\mathrm{D}}$ and $\eta \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C})$. Note that $G(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C})$ is differentiable w.r.t. y_i iff $G(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot)\eta(\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C})$ is differentiable w.r.t. y_i, and in this case

$$\left(\partial_{\boldsymbol{y}}^{\boldsymbol{e}_i} G\left(\sum_{j\in\mathbb{N}} h y_j\psi_j(\cdot),\cdot\right)\right)\eta(\cdot) = \partial_{\boldsymbol{y}}^{\boldsymbol{e}_i}\left(G\left(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot\right)\eta(\cdot)\right) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}). \tag{2.44}$$

Therefore, applying again the above argument we find for i_1, $i_2 \in \mathbb{N}$ arbitrary,

$$\partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_1}}\partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_2}}\Phi\left(\sum_{j\in\mathbb{N}} y_j\psi_j\right) = \sum_{k_1=1}^{n}\sum_{k_2=1}^{n} \frac{\partial^2}{\partial\zeta_{k_1}\partial\zeta_{k_2}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot\Big)\psi_{i_1;k_1}(\cdot)\psi_{i_2;k_2}(\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}).$$

By further repeated application of the previous arguments, for arbitrary $(i_1,\dots,i_m) \in \mathbb{N}^m$ with finite $m \in \mathbb{N}$, we get

$$\begin{aligned} &\partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_1}}\cdots\partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_m}}\Phi\left(\sum_{j\in\mathbb{N}} y_j\psi_j\right) \\ &= \sum_{k_1=1}^{n}\cdots\sum_{k_m=1}^{n} \frac{\partial^m}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_m}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot),\cdot\Big)\psi_{i_1;k_1}(\cdot)\cdots\psi_{i_m;k_m}(\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}). \end{aligned} \tag{2.45}$$

Finally, since $F : O \times O_{\mathrm{D}} \to \mathbb{C}$ is holomorphic, for any permutation $\pi : \{1,\dots,m\} \to \{1,\dots,m\}$, $\boldsymbol{\zeta} \in O$ and $x' \in O_{\mathrm{D}}$ it holds

$$\frac{\partial^m}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_m}} F\Big(\boldsymbol{\zeta},x'\Big) = \frac{\partial^m}{\partial\zeta_{k_{\pi(1)}}\cdots\partial\zeta_{k_{\pi(m)}}} F\Big(\boldsymbol{\zeta},x'\Big). \tag{2.46}$$

Therefore

$$
\begin{aligned}
&\partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_1}} \cdots \partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_m}} \Phi \left(\sum_{j\in\mathbb{N}} y_j \psi_j \right) \\
&\quad = \sum_{k_1=1}^{n} \cdots \sum_{k_m=1}^{n} \frac{\partial^m}{\partial \zeta_{k_{\pi(1)}} \cdots \partial \zeta_{k_{\pi(m)}}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot), \cdot \Big) \psi_{i_1;k_1}(\cdot) \cdots \psi_{i_m;k_m}(\cdot) \\
&\quad = \sum_{k_{\pi(1)}=1}^{n} \cdots \sum_{k_{\pi(m)}=1}^{n} \frac{\partial^m}{\partial \zeta_{k_{\pi(1)}} \cdots \partial \zeta_{k_{\pi(m)}}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot), \cdot \Big) \psi_{i_{\pi(1)};k_{\pi(1)}}(\cdot) \cdots \psi_{i_{\pi(m)};k_{\pi(m)}}(\cdot) \\
&\quad = \partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_{\pi(1)}}} \cdots \partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_{\pi(m)}}} \Phi \left(\sum_{j\in\mathbb{N}} y_j\psi_j \right) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}).
\end{aligned} \tag{2.47}
$$

For an arbitrary multiindex $\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{F}$, let now $(i_{\boldsymbol{\nu};1}, \dots, i_{\boldsymbol{\nu};|\boldsymbol{\nu}|}) \in \mathbb{N}^{|\boldsymbol{\nu}|}$ be arbitrary such that $|\{l \in \mathbb{N} : i_{\boldsymbol{\nu};l} = j\}| = \nu_j$ for all $j \in \mathbb{N}$. With (2.47) we conclude that

$$
\begin{aligned}
&\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F \left(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot), \cdot \right) = \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \Phi \left(\sum_{j\in\mathbb{N}} y_j\psi_j \right) = \partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_{\boldsymbol{\nu};1}}} \cdots \partial_{\boldsymbol{y}}^{\boldsymbol{e}_{i_{\boldsymbol{\nu};|\boldsymbol{\nu}|}}} \Phi \left(\sum_{j\in\mathbb{N}} y_j\psi_j \right) \\
&\quad = \sum_{k_1=1}^{n} \cdots \sum_{k_{|\boldsymbol{\nu}|}=1}^{n} \frac{\partial^{|\boldsymbol{\nu}|}}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_{|\boldsymbol{\nu}|}}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(\cdot), \cdot \Big) \psi_{i_{\boldsymbol{\nu};1};k_1}(\cdot) \cdots \psi_{i_{\boldsymbol{\nu};|\boldsymbol{\nu}|};k_{|\boldsymbol{\nu}|}}(\cdot) \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C})
\end{aligned} \tag{2.48}
$$

is well-defined and any permutation of $(i_{\boldsymbol{\nu};1}, \dots, i_{\boldsymbol{\nu};|\boldsymbol{\nu}|})$ in (2.48) gives the same result. Therefore, in what follows for every $\mathbf{0} \neq \boldsymbol{\nu} \in \mathcal{F}$ we always assume $(i_{\boldsymbol{\nu};1}, \dots, i_{\boldsymbol{\nu};|\boldsymbol{\nu}|}) \in \mathbb{N}^{|\boldsymbol{\nu}|}$ to be an arbitrarily fixed choice with the above stated property.

Step 2: We prove the assertion of the lemma. Recall, that in the first step we chose $\delta > 0$ to be so small that the ball $B_{3\delta}^{\mathbb{C}^n}$ with radius $3\delta > 0$ and center $0 \in \mathbb{C}^n$ is contained in O. Let $\varepsilon > 0$ be so small that for all $\boldsymbol{\zeta} = (\zeta_j)_{j=1}^n \in B_\delta^{\mathbb{C}^n}$ we have $B_\varepsilon^{\mathbb{C}}(\zeta_1) \times \cdots \times B_\varepsilon^{\mathbb{C}}(\zeta_n) \subseteq O$.

Before proving (2.38), we give an estimate on the partial derivatives of $F(\boldsymbol{\zeta}, x')$ w.r.t. $\boldsymbol{\zeta} \in O$. Fix $l \in \mathbb{N}$ and let $(k_1, \dots, k_l) \in \{1, \dots, n\}^l$ be arbitrary. Moreover, let $\boldsymbol{m} = (m_i)_{i=1}^n \in \mathbb{N}_0^n$ be such that $|\{i : k_i = j\}| = m_j$ for all $j = 1, \dots, n$. This implies $|\boldsymbol{m}| = l$. By (2.45), (2.46) and repeated application of Cauchy's integral formula, we get for $\boldsymbol{\zeta} \in B_\delta^{\mathbb{C}^n}$ and $x' \in \mathrm{D}_{\mathbf{0}}$

$$
\begin{aligned}
&\left| \frac{\partial^l}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_l}} F(\boldsymbol{\zeta}, x') \right| = \left| \partial_{\boldsymbol{\zeta}}^{\boldsymbol{m}} F(\boldsymbol{\zeta}, x') \right| \\
&\quad = \left| \frac{\boldsymbol{m}!}{(2\pi\mathrm{i})^n} \int_{\{z_1\in\mathbb{C}\,:\,|z_1|=\varepsilon\}} \cdots \int_{\{z_n\in\mathbb{C}\,:\,|z_n|=\varepsilon\}} \frac{F(\boldsymbol{z}, x')}{(z_1-\zeta_1)^{m_1+1}\cdots(z_n-\zeta_n)^{m_n+1}} \,\mathrm{d}z_1\cdots\mathrm{d}z_n \right| \\
&\quad \le \frac{\boldsymbol{m}!\|F\|_{L^\infty(O\times O_{\mathrm{D}})}}{\varepsilon^{|\boldsymbol{m}|}} \le \frac{l!\|F\|_{L^\infty(O\times O_{\mathrm{D}})}}{\varepsilon^l},
\end{aligned} \tag{2.49}
$$

where $\{z \in \mathbb{C} : |z| = \varepsilon\}$ in the line integral is oriented positively. In particular, due to the assumption $\| \sum_{j\in\mathbb{N}} |\psi_j(\cdot)| \|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^n)} < \delta$, there exists a null set $\mathcal{N} \subseteq \mathrm{D}_{\mathbf{0}} \subseteq \mathbb{R}^d$, such that

$$\sup_{x'\in\mathrm{D}_{\mathbf{0}}\backslash\mathcal{N}} \sup_{\boldsymbol{y}\in U} \left\| \sum_{j\in\mathbb{N}} y_j\psi_j(x') \right\|_2 \le \sup_{x'\in\mathrm{D}_{\mathbf{0}}\backslash\mathcal{N}} \sup_{\boldsymbol{y}\in U} \left\| \sum_{j\in\mathbb{N}} |y_j||\psi_j(x')| \right\|_2 \le \sup_{x'\in\mathrm{D}_{\mathbf{0}}\backslash\mathcal{N}} \left\| \sum_{j\in\mathbb{N}} |\psi_j(x')| \right\|_2 \le \delta.$$

Therefore (2.49) holds for every $x' \in \mathrm{D}_{\mathbf{0}}$ and for every

$$\boldsymbol{\zeta} \in \left\{ \sum_{j\in\mathbb{N}} y_j\psi_j(x') \,:\, \boldsymbol{y} \in U,\ x' \in D_{\mathbf{0}}\backslash\mathcal{N} \right\} \subseteq B_\delta^{\mathbb{C}^n} \subseteq O. \tag{2.50}$$

With (2.48) we obtain for a.e. $x' \in \mathrm{D}_{\mathbf{0}}$ and for all $\boldsymbol{y} \in U$

$$\begin{aligned}
&\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \frac{1}{\boldsymbol{\nu}!} \Big| \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(x'), x'\Big) \Big| \\
&\quad\le \sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \frac{1}{\boldsymbol{\nu}!} \sum_{k_1,\dots,k_{|\boldsymbol{\nu}|}=1}^{n} \left| \frac{\partial^{|\boldsymbol{\nu}|}}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_{|\boldsymbol{\nu}|}}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(x'), x'\Big) \right| \prod_{r=1}^{|\boldsymbol{\nu}|} \left|\psi_{i_{\boldsymbol{\nu};r};k_r}(x')\right| \\
&\quad= \sum_{l=1}^{\infty} \sum_{|\boldsymbol{\nu}|=l} \frac{1}{\boldsymbol{\nu}!} \sum_{k_1,\dots,k_l=1}^{n} \left| \frac{\partial^{l}}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_l}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(x'), x'\Big) \right| \prod_{r=1}^{l} \left|\psi_{i_{\boldsymbol{\nu};r};k_r}(x')\right| \\
&\quad= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{k_1,\dots,k_l=1}^{n} \left| \frac{\partial^{l}}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_l}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(x'), x'\Big) \right| \sum_{|\boldsymbol{\nu}|=l} \frac{l!}{\boldsymbol{\nu}!} \prod_{r=1}^{l} \left|\psi_{i_{\boldsymbol{\nu};r};k_r}(x')\right| \\
&\quad\le \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{k_1,\dots,k_l=1}^{n} \left| \frac{\partial^{l}}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_l}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(x'), x'\Big) \right| \sum_{|\boldsymbol{\nu}|=l} \frac{l!}{\boldsymbol{\nu}!} \prod_{r=1}^{l} \left| \max_{k\in\{1,\dots,n\}} \psi_{i_{\boldsymbol{\nu};r};k}(x')\right| \\
&\quad= \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{k_1,\dots,k_l=1}^{n} \left| \frac{\partial^{l}}{\partial\zeta_{k_1}\cdots\partial\zeta_{k_l}} F\Big(\sum_{j\in\mathbb{N}} y_j\psi_j(x'), x'\Big) \right| \prod_{r=1}^{l} \left(\sum_{j\in\mathbb{N}} \left| \max_{k\in\{1,\dots,n\}} \psi_{j;k}(x')\right| \right) \\
&\quad\le \sum_{l=1}^{\infty} \sum_{k_1,\dots,k_l=1}^{n} \frac{\|F\|_{L^\infty(O\times O_{\mathrm{D}})}}{\varepsilon^l} (n\delta)^l \le \|F\|_{L^\infty(O\times O_{\mathrm{D}})} \sum_{l=1}^{\infty} \left(\frac{n^2\delta}{\varepsilon}\right)^l.
\end{aligned} \tag{2.51}$$

For the second to last inequality we have used the bound (2.49) for the partial derivatives of $F(\sum_{j\in\mathbb{N}} y_j\psi_j(x'), x')$ with respect to $\boldsymbol{\zeta}$ (due to (2.50) the bound holds for every $\boldsymbol{y} \in U$, and for a.e. $x' \in \mathrm{D}_{\mathbf{0}}$), as well as the assumption $\| \sum_{j\in\mathbb{N}} |\psi_j(\cdot)| \|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^n)} < \delta$, which implies for a.e. $x' \in \mathrm{D}_{\mathbf{0}}$

$$\prod_{r=1}^{l} \left(\sum_{j\in\mathbb{N}} \left| \max_{k\in\{1,\dots,n\}} \psi_{j;k}(x')\right| \right) \le \prod_{r=1}^{l} \sum_{k=1}^{n} \sum_{j\in\mathbb{N}} |\psi_{j;k}(x')| \le \prod_{r=1}^{l} \sum_{k=1}^{n} \left\| \sum_{j\in\mathbb{N}} |\psi_j(x')| \right\|_2 \le (n\delta)^l.$$

Choosing $\delta > 0$ so that $n^2\delta/\varepsilon < \gamma/(\gamma + \|F\|_{L^\infty(O\times O_{\mathrm{D}})})$, we obtain

$$\|F\|_{L^\infty(O\times O_{\mathrm{D}})} n^2\delta/(\varepsilon - n^2\delta) < \gamma$$

as a bound (for a.e. $x' \in \mathrm{D}_{\mathbf{0}}$) for the sum in (2.51). ■

Proof of Thm. 2.10. The proof proceeds in six steps. In the first two steps, we verify i) and ii). Steps 3-6 serve the purpose of proving iii) with the help of Thm. 2.6 and Lemma 2.11. Up to step 3, the constant δ, appearing in the formulation of the theorem, is treated as some fixed positive number in the interval $(0, 1/2)$, that is still at our disposal. The arguments in those steps do not depend on the concrete value of δ. In step 3 we will make use of the upper bound $\delta < 1/2$. In step 4 we shall finally give conditions on $\delta > 0$ depending on the holomorphy domains of A and f, such that the assertion of the theorem is satisfied.

Step 1: We start with i). Consider (2.34). For every $\boldsymbol{y} \in U$, this equation has the parametric diffusion coefficient and right-hand side given by

$$\begin{aligned} \hat{A}(\boldsymbol{y}) &= DT_{\boldsymbol{y}}^{-1}(A \circ T_{\boldsymbol{y}})DT_{\boldsymbol{y}}^{-\top} \det DT_{\boldsymbol{y}} \in L^\infty(\mathrm{D}_{\mathbf{0}}, \mathbb{R}^{d\times d}_{\mathrm{sym}}), \\ \hat{f}(\boldsymbol{y}) &= f \circ T_{\boldsymbol{y}} \det DT_{\boldsymbol{y}} \in L^2(\mathrm{D}_{\mathbf{0}}) \hookrightarrow \mathbb{H}^{-s}(\mathrm{D}_{\mathbf{0}}), \end{aligned} \tag{2.52}$$

respectively. In Step 3 below we shall see that $\hat{A}(\boldsymbol{y}) \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is uniformly elliptic on $\mathrm{D}_{\mathbf{0}}$ for every $\boldsymbol{y} \in U$, and thus $\hat{u}(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D}_{\mathbf{0}})$ exists and is well-defined. The connection to the initial problem on the physical domain stated in item i) follows from Lemma 2.8.

Step 2: We verify continuity as stated in ii). As a consequence of the Strang Lemma, see for example [17, Lemma 2.27], the solution $\tilde{u} \in \mathbb{H}^s(\mathrm{D}_{\mathbf{0}})$ of $\tilde{\mathcal{L}}^s\tilde{u} = \tilde{f}$ with $\tilde{\mathcal{L}} = -\nabla_{x'}\cdot(\tilde{A}(x')\nabla_{x'})$, locally depends continuously on the diffusion coefficient $\tilde{A} \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^{d\times d})$ and the right-hand side $\tilde{f} \in \mathbb{H}^{-s}(\mathrm{D}_{\mathbf{0}})$, as long as $\tilde{A}$ is uniformly SPD on $\mathrm{D}_{\mathbf{0}}$. Furthermore, $\hat{A}(\boldsymbol{y}) \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^{d\times d})$ and $\hat{f}(\boldsymbol{y}) \in L^2(\mathrm{D}_{\mathbf{0}})$ in (2.52) depend continuously on $T_{\boldsymbol{y}} \in W^{1,\infty}(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^d)$. The transformation $T_{\boldsymbol{y}}$ in turn depends continuously on the quantity $\Phi(\boldsymbol{y}) := \sum_{j\in\mathbb{N}} y_j\psi_j \in W^{1,\infty}(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^d)$, since $T_{\boldsymbol{y}}(x') = x' + \sum_{j\in\mathbb{N}} y_j\psi_j(x')$ by definition of $T_{\boldsymbol{y}}$ (cf. Assumption 2.9). To show continuity of $\hat{u}(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D}_{\mathbf{0}})$ as a function of $\boldsymbol{y} \in U$, it suffices to verify that $\Phi(\boldsymbol{y}) \in W^{1,\infty}(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^d)$ depends continuously on $\boldsymbol{y} \in U$.

To this end fix $\boldsymbol{y}_0 = (y_{0,j})_{j\in\mathbb{N}} \in U$ and let at first $N_\Phi \subseteq W^{1,\infty}(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^d)$ be an arbitrary neighbourhood of $\Phi(\boldsymbol{y}_0) \in W^{1,\infty}(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^d)$. We need to find a neighbourhood $N_{\boldsymbol{y}_0} \subseteq U$ of $\boldsymbol{y}_0$ such that

$$\{\Phi(\boldsymbol{y}) \,:\, \boldsymbol{y} \in N_{\boldsymbol{y}_0}\} \subseteq N_\Phi. \tag{2.53}$$

Since N_Φ was arbitrary, this then implies continuity of $U \ni \boldsymbol{y} \mapsto \Phi(\boldsymbol{y}) \in W^{1,\infty}(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^d)$. Now, by Assumption 2.9,

$$\begin{aligned} \lim_{n\to\infty} \sup_{\boldsymbol{y}\in U} \left\| \sum_{j>n} y_j\psi_j(\cdot) \right\|_{W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)} &\le \lim_{n\to\infty} \left\| \sum_{j>n} \rho_j^{-1}\rho_j \left(\|\psi_j(\cdot)\|_2 + \|D\psi_j(\cdot)\|_2\right) \right\|_{L^\infty(\mathrm{D}_{\mathbf{0}})} \\ &\le C \lim_{n\to\infty} \sup_{j\ge n} \rho_j^{-1} = 0, \end{aligned} \tag{2.54}$$

where the limit is 0 since $\rho_j \to \infty$ by assumption. Let $\varepsilon \in (0, 1/2)$ be so small that the ball $B_\varepsilon^{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)}(\Phi(\boldsymbol{y}))$ of radius ε with center $\Phi(\boldsymbol{y}) \in W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)$ lies in N_Φ. Due to (2.54), we can find $n \in \mathbb{N}$ such that

$$\sup_{\boldsymbol{y}\in U} \left\| \sum_{j>n} y_j \psi_j \right\|_{W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)} < \frac{\varepsilon}{2}. \tag{2.55}$$

Additionally choose $\gamma := \varepsilon/(2\delta n)$, with $\delta > 0$ sufficiently small as in Assumption 2.9 (and to be specified in step 4), define

$$N_{\boldsymbol{y}_0} := \bigtimes_{j=1}^{n} ((y_{0;j} - \gamma, y_{0;j} + \gamma) \cap [-1,1]) \bigtimes_{j>n} [-1,1] \subseteq U. \tag{2.56}$$

Then $N_{\boldsymbol{y}_0}$ is open w.r.t. the product topology on U. Obviously, $\boldsymbol{y}_0 \in N_{\boldsymbol{y}_0}$ so that $N_{\boldsymbol{y}_0}$ is indeed an open neighbourhood of $\boldsymbol{y}_0$. From (2.35) we deduce $\sup_{j\in\mathbb{N}} \|\psi_j\|_{W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)} \le \sup_{j\in\mathbb{N}} \delta/\rho_j \le \delta$. Therefore by (2.55)

$$\sup_{\boldsymbol{y}\in N_{\boldsymbol{y}_0}} \|\Phi(\boldsymbol{y}) - \Phi(\boldsymbol{y}_0)\|_{W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)} < \gamma \sum_{j=1}^{n} \|\psi_j\|_{W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{R}^d)} + \frac{\varepsilon}{2} \le \gamma n \delta + \frac{\varepsilon}{2} \le \varepsilon. \tag{2.57}$$

This proves (2.53), which overall implies ii).

Step 3: We now begin with the proof of iii). In this step we show that $\hat{A}(\boldsymbol{y})$ in (2.52) is uniformly elliptic on $\mathrm{D}_{\mathbf{0}}$ for all $\boldsymbol{y} \in U$.

For $\boldsymbol{y} \in U$, with $I \in \mathbb{R}^{d\times d}$ denoting the identity matrix, by definition of $T_{\boldsymbol{y}}$ there holds

$$\forall \boldsymbol{y} \in U : \quad x' \mapsto DT_{\boldsymbol{y}}(x') = I + \sum_{j\in\mathbb{N}} y_j D\psi_j(x') \in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^{d\times d}) \,.$$

At the beginning of the proof we imposed the restriction $\delta < 1/2$ on $\delta > 0$. By Assumption 2.9 this implies

$$\sup_{\boldsymbol{y}\in U} \operatorname*{ess\,sup}_{x'\in\mathrm{D}_{\mathbf{0}}} \left\| \sum_{j\in\mathbb{N}} y_j D\psi_j(x') \right\|_2 \le \operatorname*{ess\,sup}_{x'\in\mathrm{D}_{\mathbf{0}}} \sum_{j\in\mathbb{N}} \|D\psi_j(x')\|_2 < \frac{1}{2}. \tag{2.58}$$

For a matrix $M \in \mathbb{R}^{d\times d}$ with $\|M\|_2 \le 1/2$, by a Neumann series argument $I+M$ is nonsingular and it holds $\|(I+M)^{-1}\|_2 \le 2$. Thus the minimal singular value of $I+M$ (which is the reciprocal of the maximal singular value of $(I+M)^{-1}$, which in turn equals $\|(I+M)^{-1}\|_2 \le 2$) is bounded from below by $1/2$. Then, with $(\sigma_j)_{j=1}^d$ denoting the singular values of $I+M$, we get

$$\det(I+M) = \prod_{j=1}^{d} \sigma_j \ge \left(\min_{j\in\{1,\ldots,d\}} \sigma_j \right)^d = \frac{1}{\|(I+M)^{-1}\|_2^d} \ge 2^{-d}. \tag{2.59}$$

Therefore with (2.58)

$$\inf_{\boldsymbol{y}\in U}\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\det DT_{\boldsymbol{y}}(x')\ge 2^{-d}\qquad\text{and}\qquad \sup_{\boldsymbol{y}\in U}\operatorname*{ess\,sup}_{x'\in \mathrm{D}_{\mathbf{0}}}\|DT_{\boldsymbol{y}}(x')\|_2\le \|I\|_2+\frac{1}{2}\le\frac{3}{2}. \tag{2.60}$$

Then for every $\boldsymbol{y}\in U$

$$\begin{aligned}\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\inf_{0\neq\zeta\in\mathbb{R}^d}\frac{\zeta^\top\hat{A}(\boldsymbol{y})\zeta}{\zeta^\top\zeta}&=\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\inf_{0\neq\zeta\in\mathbb{R}^d}\det DT_{\boldsymbol{y}}(x')\frac{\zeta^\top DT_{\boldsymbol{y}}^{-1}(x')A(T_{\boldsymbol{y}}(x'))DT_{\boldsymbol{y}}^{-\top}\zeta}{\zeta^\top\zeta}\\ &\ge \overline{A}_{\min}\left(\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\det DT_{\boldsymbol{y}}(x')\right)\left(\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\inf_{0\neq\zeta\in\mathbb{R}^d}\frac{\|DT_{\boldsymbol{y}}^{-\top}(x')\zeta\|_2^2}{\|\zeta\|_2^2}\right).\end{aligned}\tag{2.61}$$

Here we used $\overline{\mathrm{D}_{\boldsymbol{y}}}\subseteq \mathrm{D}_{\mathrm{H}}$, i.e. $T_{\boldsymbol{y}}(x')\in \mathrm{D}_{\mathrm{H}}$ for all $x'\in \mathrm{D}_{\mathbf{0}}$, so that (2.36) could be applied. By (2.60), $\|\zeta\|_2=\|DT_{\boldsymbol{y}}^\top(x')DT_{\boldsymbol{y}}^{-\top}(x')\zeta\|_2\le(3/2)\|DT_{\boldsymbol{y}}^{-\top}(x')\zeta\|_2$, for a.e. $x'\in \mathrm{D}_{\mathbf{0}}$ and therefore

$$\inf_{\boldsymbol{y}\in U}\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\inf_{0\neq\zeta\in\mathbb{R}^d}\frac{\|DT_{\boldsymbol{y}}^{-\top}(x')\zeta\|_2^2}{\|\zeta\|_2^2}\ge\frac{4}{9}. \tag{2.62}$$

Thus, for every $\boldsymbol{y}\in U$ the right-hand side of (2.61) is bounded from below by

$$\overline{A}_{\min}2^{-d}\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\inf_{0\neq\zeta\in\mathbb{R}^d}\frac{\|DT_{\boldsymbol{y}}^{-\top}(x')\zeta\|_2^2}{\|\zeta\|_2^2}\ge\overline{A}_{\min}\frac{2^{2-d}}{9}=:\tilde{A}_{\min}>0,$$

which shows

$$\inf_{\boldsymbol{y}\in U}\operatorname*{ess\,inf}_{x'\in \mathrm{D}_{\mathbf{0}}}\inf_{0\neq\zeta\in\mathbb{R}^d}\frac{\zeta^\top\hat{A}(\boldsymbol{y})\zeta}{\zeta^\top\zeta}\ge\tilde{A}_{\min}. \tag{2.63}$$

Step 4: The idea for the proof of iii) is to apply Thm. 2.6. In order to do so, in this step we establish some preliminaries, which will be used to verify the prerequisites of Thm. 2.6 stated in (2.18) and (2.19). Lemma 2.11 plays a key role here.

Due to the assumed analyticity of A and f on an open superset of $\overline{\mathrm{D}_{\mathrm{H}}}$, we may holomorphically extend A and f to some open set $O_{\mathrm{H}}\subseteq\mathbb{C}^d$ containing the compact set $\overline{\mathrm{D}_{\mathrm{H}}}$. Throughout the rest of the proof, let $\varepsilon>0$ be so small that $\mathrm{D}_{\mathrm{H}}+B_{3\varepsilon}^{\mathbb{C}^d}\subseteq O_{\mathrm{H}}$. As a further condition on $\delta\in(0,1/2)$ in Assumption 2.9, we impose that

$$\delta<\frac{\varepsilon}{2}. \tag{2.64}$$

For $x_1\in B_\varepsilon^{\mathbb{C}^d}$, $X_2\in\mathbb{C}^{d\times d}$ with $\|X_2\|_2<1/2$ and $x'\in \mathrm{D}_{\mathrm{H}}+B_\varepsilon^{\mathbb{C}^d}$ define

$$G((x_1,X_2),x'):=(\mathrm{I}+X_2)^{-1}A(x'+x_1)(\mathrm{I}+X_2)^{-\top}\det(\mathrm{I}+X_2)\in\mathbb{C}^{d\times d}. \tag{2.65}$$

For $\boldsymbol{y}\in U$ and $x'\in \mathrm{D}_{\mathbf{0}}$ with $\hat{A}(\boldsymbol{y})\in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^{d\times d}_{\mathrm{sym}})$ as in (2.52) we have

$$\hat{A}(\boldsymbol{y};x')=G\Big(\sum_{j\in\mathbb{N}}y_j(\psi_j(x'),D\psi_j(x')),x'\Big) \tag{2.66}$$

and thus

$$\frac{1}{\boldsymbol{\nu}!}\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} G\Big(\sum_{j\in\mathbb{N}} y_j(\psi_j(x'), D\psi_j(x')), x'\Big) = \frac{\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} \hat{A}(\boldsymbol{z};x')|_{\boldsymbol{z}=\boldsymbol{y}}}{\boldsymbol{\nu}!}.$$

We claim that G is uniformly bounded and holomorphic as a function of

$$((x_1, X_2), x') \in ((B_\varepsilon^{\mathbb{C}^d} \times B_{1/2}^{\mathbb{C}^{d\times d}}) \times (\mathrm{D_H} + B_\varepsilon^{\mathbb{C}^d})) =: S. \tag{2.67}$$

To see this, note at first that each component of $(I + X_2)^{-\top} \in \mathbb{C}^{d\times d}$ is a rational function in the entries of X_2, for all $X_2 \in \mathbb{C}^{d\times d}$ such that $I + X_2$ is contained in the open set of invertible matrices in $\mathbb{C}^{d\times d}$. Since $I + X_2$ is regular if $\|X_2\|_2 < 1/2$, we obtain that $(I + X_2)^{-1} \in \mathbb{C}^{d\times d}$ and $(I + X_2)^{-\top} \in \mathbb{C}^{d\times d}$ are holomorphic as a function of $X_2 \in B_{1/2}^{\mathbb{C}^{d\times d}}$. Next, we observe that the map $X_2 \mapsto \det(I + X_2) \in \mathbb{C}$ is holomorphic as a function of $X_2 \in B_{1/2}^{\mathbb{C}^{d\times d}}$ (being a multivariate polynomial in the components of X_2). Finally, $A(t)$ is holomorphic as a function of $t \in \mathrm{D_H} + B_{2\varepsilon}^{\mathbb{C}^d}$ by definition of ε. Thus $A(x' + x_1)$ is holomorphic as a function of $(x_1, x') \in B_\varepsilon^{\mathbb{C}^d} \times (\mathrm{D_H} + B_\varepsilon^{\mathbb{C}^d})$. We conclude that G in (2.65) must be jointly holomorphic as a function of $((x_1, X_2), x') \in S$. By continuity, G is uniformly bounded on S: as S is a bounded set, its closure is compact, and G is holomorphic (and therefore continuous) on an open superset of $\overline{S}$.

Applying Lemma 2.11 to G and using (2.35), we obtain that if $\delta > 0$ in Assumption 2.9 is small enough, then it holds

$$\sup_{\boldsymbol{y}\in U}\Big\|\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\rho}^{\boldsymbol{\nu}} \Big\| \underbrace{\frac{1}{\boldsymbol{\nu}!}\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} G\Big(\sum_{j\in\mathbb{N}} y_j(\psi_j(x'), D\psi_j(x')), x'\Big)}_{=\frac{\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} \hat{A}(\boldsymbol{z};x')|_{\boldsymbol{z}=\boldsymbol{y}}}{\boldsymbol{\nu}!}} \Big\|_2 \Big\|_{L^\infty(\mathrm{D}_\mathbf{0})} < \tilde{A}_{\min}, \tag{2.68}$$

with $\tilde{A}_{\min} > 0$ as in (2.63). Observe that $G((x_1, X_2), x')$ in (2.65) is in fact symmetric if $x' + x_1 \in \mathrm{D_H}$, since $A : \mathrm{D_H} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Moreover, if additionally $x_1 \in B_\varepsilon^{\mathbb{R}^d}$, $X_2 \in B_{1/2}^{\mathbb{R}^{d\times d}}$, then $G((x_1, X_2), x')$ is also real valued i.e. $G((x_1, X_2), x') \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Hence for any $\boldsymbol{y} \in U$, the partial derivative of G in (2.68) must be in $L^\infty(\mathrm{D}_\mathbf{0}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$.

We proceed similary for the right-hand side f. Define

$$F((x_1, X_2), x') := f(x' + x_1)\det(\mathrm{I} + X_2) \tag{2.69}$$

for $((x_1, X_2), x') \in S$ and observe that for $\hat{f}$ as in (2.52)

$$\hat{f}(\boldsymbol{y}; x') = F\Big(\sum_{j\in\mathbb{N}} y_j(\psi_j(x'), D\psi_j(x')), x'\Big). \tag{2.70}$$

As for G, we note that F is well-defined, holomorphic and bounded on S defined in (2.67). If $\delta > 0$ in Assumption 2.9 is small enough, then Lemma 2.11 gives

$$\sup_{\boldsymbol{y}\in U}\Big\|\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\rho}^{\boldsymbol{\nu}} \Big\| \underbrace{\frac{1}{\boldsymbol{\nu}!}\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F\Big(\sum_{j\in\mathbb{N}} y_j(\psi_j(x'), D\psi_j(x')), x'\Big)}_{=\frac{\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} \hat{f}(\boldsymbol{z})|_{\boldsymbol{z}=\boldsymbol{y}}}{\boldsymbol{\nu}!}} \Big\|_2 \Big\|_{L^\infty(\mathrm{D}_\mathbf{0})} < \infty \tag{2.71}$$

analogous to (2.68). Throughout the rest of the proof, $\delta \in (0,1/2)$ in Assumption 2.9 is fixed and small enough such that (2.64), (2.68) and (2.71) hold.

Step 5: In this step we show that for every fixed $\boldsymbol{y}_0 \in U$, $\hat{A}(\boldsymbol{y}_0+\boldsymbol{y})$ and $\hat{f}(\boldsymbol{y}_0+\boldsymbol{y})$ in (2.52) allow Taylor chaos expansions as functions of $\boldsymbol{y}$ (for $\boldsymbol{y}$ in some suitable set defined below).

Fix $\boldsymbol{y}_0 = (y_{0;j})_{j\in\mathbb{N}} \in U$ arbitrary. With $(\rho_j)_{j\in\mathbb{N}} \in (1,\infty)^{\mathbb{N}}$ as in Assumption 2.9, and δ, $\varepsilon > 0$ as in step 4, let $\boldsymbol{\gamma} \in (0,\infty)^{\mathbb{N}}$ be such that

$$\sup_{j\in\mathbb{N}} \frac{\gamma_j}{\rho_j} < \min\left\{1, \frac{\varepsilon}{\delta}\right\} \qquad \text{and} \qquad \frac{1+\gamma_j}{\rho_j} \to 0, \tag{2.72}$$

which is possible because $\rho_j \to \infty$ as $j \to \infty$. Next, for $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}^{\mathbb{C}}$ and with G and F as in (2.65) and (2.69), we claim that the quantities

$$\begin{aligned} \hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z}) &:= G\Big(\sum_{j\in\mathbb{N}} (y_{0;j}+z_j)(\psi_j(x'), D\psi_j(x')), x'\Big) \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{C}^{d\times d}),\\ \hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z}) &:= F\Big(\sum_{j\in\mathbb{N}} (y_{0;j}+z_j)(\psi_j(x'), D\psi_j(x')), x'\Big) \in L^2(\mathrm{D}_{\mathbf{0}}; \mathbb{C}), \end{aligned} \tag{2.73}$$

are well-defined. First note that by (2.66) and (2.70), we have

$$\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{y}) = \hat{A}(\boldsymbol{y}_0+\boldsymbol{y}) \in L^\infty(\mathrm{D}_{\boldsymbol{\nu}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}) \qquad \text{and} \qquad \hat{f}_{\boldsymbol{y}_0}(\boldsymbol{y}) = \hat{f}(\boldsymbol{y}_0+\boldsymbol{y}) \in L^2(\mathrm{D}_{\mathbf{0}}) \tag{2.74}$$

whenever $\boldsymbol{y}_0 + \boldsymbol{y} \in U$. Next, for $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}^{\mathbb{C}}$ it holds by (2.35) and because $\sup_{j\in\mathbb{N}} \gamma_j/\rho_j < 1$ as well as $\rho_j \geq 1$ for all $j \in \mathbb{N}$

$$\begin{aligned} \left\| \sum_{j\in\mathbb{N}} (y_{0;j}+z_j)\psi_j(\cdot) \right\|_{W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{C}^d)} &\leq \left\| \sum_{j\in\mathbb{N}} \frac{\gamma_j+1}{\rho_j} \rho_j(\|\psi_j(\cdot)\|_2 + \|D\psi_j(\cdot)\|_2) \right\|_{L^\infty(\mathrm{D}_{\mathbf{0}})}\\ &\leq \sup_{j\in\mathbb{N}} \frac{1+\gamma_j}{\rho_j}\delta < 2\delta < \varepsilon, \end{aligned} \tag{2.75}$$

where we have also employed (2.64). This proves that for a.e. $x' \in \mathrm{D}_{\mathbf{0}}$

$$\sum_{j\in\mathbb{N}} (y_{0;j}+z_j)\psi_j(x') \in B_\varepsilon^{\mathbb{C}^d} \qquad \text{and} \qquad \sum_{j\in\mathbb{N}} (y_{0;j}+z_j) D\psi_j(x') \in B_\varepsilon^{\mathbb{C}^{d\times d}}. \tag{2.76}$$

In step 4 we showed that G and F are holomorphic and uniformly bounded on S in (2.67), and thus $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})$, $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z})$ in (2.73) are well-defined for $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}^{\mathbb{C}}$ due to (2.76) (and because $\varepsilon \leq 1/2$ so that $B_\varepsilon^{\mathbb{C}^{d\times d}} \subseteq B_{1/2}^{\mathbb{C}^{d\times d}}$).

Next we prove that $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z}) \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{C}^{d\times d})$ and $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z}) \in L^2(\mathrm{D}_{\mathbf{0}}; \mathbb{C})$ allow convergent Taylor expansions for $\boldsymbol{z} \in B_{\boldsymbol{\gamma}}^{\mathbb{C}}$ as in (2.18a), (2.19a) and the corresponding Taylor coefficients satisfy (2.18b), (2.19b) respectively. We begin with (2.18) for $\hat{A}_{\boldsymbol{y}_0}$. For a multiindex $\boldsymbol{\nu} \in \mathcal{F}$, denote the corresponding Taylor coefficient of $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})$ at $\boldsymbol{z} = \mathbf{0}$ by

$$\hat{A}_{\boldsymbol{y}_0;\boldsymbol{\nu}} := \frac{\partial_{\boldsymbol{z}}^{\boldsymbol{\nu}} \hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})|_{\boldsymbol{z}=\mathbf{0}}}{\boldsymbol{\nu}!} \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^{d\times d}_{\mathrm{sym}}). \tag{2.77}$$

Using (2.72) and (2.35) we have

$$\lim_{n\to\infty}\sup_{\boldsymbol{z}\in B^{\mathbb{C}}_{\boldsymbol{\gamma}}}\left\|\sum_{j>n}(y_{0;j}+z_j)\psi_j(\cdot)\right\|_{W^{1,\infty}(\mathrm{D}_{\mathbf{0}},\mathbb{C}^{d\times d})} \tag{2.78}$$

$$\leq \lim_{n\to\infty}\left\|\sum_{j>n}\frac{1+\gamma_j}{\rho_j}\rho_j\left(\|\psi_j(\cdot)\|_2+\|D\psi_j(\cdot)\|_2\right)\right\|_{L^\infty(\mathrm{D}_{\mathbf{0}})} \leq \lim_{n\to\infty}\sup_{j\geq n}\frac{1+\gamma_j}{\rho_j}=0. \tag{2.79}$$

Due to the continuous dependence of $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})\in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^{d\times d})$ in (2.73) on the quantity $\sum_{j\in\mathbb{N}}(y_{0;j}+z_j)\psi_j\in W^{1,\infty}(\mathrm{D}_{\mathbf{0}};\mathbb{C}^d)$, we conclude

$$\lim_{n\to\infty}\sup_{\boldsymbol{z}\in B^{\mathbb{C}}_{\boldsymbol{\gamma}}}\|\hat{A}_{\boldsymbol{y}_0}(z_1,\dots,z_n,0,\dots)-\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})\|_{L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^{d\times d})}=0. \tag{2.80}$$

By item ii) of Lemma 2.5, our choice of $\boldsymbol{\gamma}$ in (2.72), and due to (2.78), the Taylor chaos expansion

$$\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})=\sum_{\boldsymbol{\nu}\in\mathcal{F}}\hat{A}_{\boldsymbol{y}_0;\boldsymbol{\nu}}\boldsymbol{z}^{\boldsymbol{\nu}}\in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{C}^{d\times d}) \tag{2.81}$$

converges (uniformly) for all $\boldsymbol{z}\in B^{\mathbb{C}}_{\boldsymbol{\gamma}}$. To apply Lemma 2.5, we have also used the fact that $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})$ is holomorphic in each $z_j\in B^{\mathbb{C}}_{\gamma_j}$ for fixed $\boldsymbol{z}\in B^{\mathbb{C}}_{\boldsymbol{\gamma}}$, which can be shown using its definition (2.73) and the same arguments as in step 1 of the proof of Lemma 2.5. This shows (2.18a). Next, the Taylor coefficient of $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})$ at $\boldsymbol{z}=\mathbf{0}$ corresponding to the multiindex $\mathbf{0}$ is given by $\hat{A}_{\boldsymbol{y}_0;\mathbf{0}}=\hat{A}_{\boldsymbol{y}_0}(\mathbf{0})=\hat{A}(\boldsymbol{y}_0)\in L^\infty(\mathrm{D}_{\mathbf{0}};\mathbb{R}^{d\times d}_{\mathrm{sym}})$. Moreover, (2.63) gives a lower bound on the minimal singular value of $\hat{A}(\boldsymbol{y}_0;x')$ for a.e. $x'\in\mathrm{D}_{\mathbf{0}}$. Its reciprocal $\tilde{A}^{-1}_{\mathrm{min}}<\infty$ is thus an upper bound of $\|\hat{A}(\boldsymbol{y}_0;x')^{-1}\|_2$ for a.e. $x'\in\mathrm{D}_{\mathbf{0}}$. Therefore, (2.68) implies (2.18b) for $\hat{A}_{\boldsymbol{y}_0}$, since (cf. (2.77))

$$\left\|\|\hat{A}^{-1}_{\boldsymbol{y}_0;\mathbf{0}}(\cdot)\|_2\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}}\boldsymbol{\rho}^{\boldsymbol{\nu}}\|\hat{A}_{\boldsymbol{y}_0;\boldsymbol{\nu}}(\cdot)\|_2\right\|_{L^\infty(\mathrm{D}_{\mathbf{0}})}\leq\frac{1}{\tilde{A}_{\mathrm{min}}}\left\|\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\mathcal{F}}\boldsymbol{\rho}^{\boldsymbol{\nu}}\|\hat{A}_{\boldsymbol{y}_0;\boldsymbol{\nu}}(\cdot)\|_2\right\|_{L^\infty(\mathrm{D}_{\mathbf{0}})}<1\,.$$

We have shown (2.18) for the parametric diffusion coefficient $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})$ for arbitrary $\boldsymbol{y}_0\in U$.

To prove (2.19) for $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z})$ we proceed analogously: With the Taylor coefficient $\hat{f}_{\boldsymbol{y}_0;\boldsymbol{\nu}}=\partial^{\boldsymbol{\nu}}_{\boldsymbol{z}}\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z})|_{\boldsymbol{z}=\mathbf{0}}$, (2.71) implies

$$\sup_{\boldsymbol{y}\in U}\sum_{\boldsymbol{\nu}\in\mathcal{F}}(\boldsymbol{\rho}^{\boldsymbol{\nu}}\|\hat{f}_{\boldsymbol{y};\boldsymbol{\nu}}\|_{L^2(\mathrm{D}_{\mathbf{0}})})^2\leq\sup_{\boldsymbol{y}\in U}\int_{\mathrm{D}_{\mathbf{0}}}\left(\sum_{\boldsymbol{\nu}\in\mathcal{F}}\boldsymbol{\rho}^{\boldsymbol{\nu}}|\hat{f}_{\boldsymbol{y};\boldsymbol{\nu}}(x')|\right)^2\mathrm{d}x'<\infty, \tag{2.82}$$

showing (2.19b) for $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z})=\sum_{\boldsymbol{\nu}\in\mathcal{F}}\hat{f}_{\boldsymbol{y}_0;\boldsymbol{\nu}}\boldsymbol{z}^{\boldsymbol{\nu}}$. Convergence of the series for $\boldsymbol{z}\in B^{\mathbb{C}}_{\boldsymbol{\gamma}}$ towards $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z})$, i.e. (2.19a), follows by similar arguments as for $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})$ with Lemma 2.5.

Step 6: Ultimately, as previously announced, we now employ Thm. 2.6 to conclude that item iii) is satisfied. Let again $\boldsymbol{y}_0 \in U$. Recall that $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{y}) = \hat{A}(\boldsymbol{y}_0 + \boldsymbol{y}) \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{y}) = \hat{f}(\boldsymbol{y}_0 + \boldsymbol{y}) \in L^2(\mathrm{D}_{\mathbf{0}}) \hookrightarrow \mathbb{H}^{-s}(\mathrm{D}_{\mathbf{0}})$ whenever $\boldsymbol{y}_0 + \boldsymbol{y} \in U$. Denote the parametric solution corresponding to the parametric diffusion coefficient $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{y})$ and parametric right-hand side $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{y})$ by $\hat{u}_{\boldsymbol{y}_0}(\boldsymbol{y}) \in \mathbb{H}^s(\mathrm{D})$, i.e. $\hat{u}_{\boldsymbol{y}_0}(\boldsymbol{y}) = \hat{u}(\boldsymbol{y}_0 + \boldsymbol{y})$. Observe $\partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} \hat{u}_{\boldsymbol{y}_0}(\boldsymbol{y})|_{\boldsymbol{y}=\mathbf{0}} = \partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} \hat{u}(\boldsymbol{y})|_{\boldsymbol{y}=\boldsymbol{y}_0}$.

Therefore, iii) follows from Thm. 2.6: according to (2.63), $\hat{A}_{\boldsymbol{y}_0;\mathbf{0}} \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ is uniformly elliptic, i.e. (2.23) holds. Moreover, by step 5, $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z}) \in L^\infty(\mathrm{D}_{\mathbf{0}}; \mathbb{C}^{d\times d})$ allows a uniformly convergent expansion of the type (2.18a) for $\boldsymbol{z}$ in the complex polydisc $B^{\mathbb{C}}_{\boldsymbol{\gamma}}$ with a sequence $\boldsymbol{\gamma} \in (0,\infty)^{\mathbb{N}}$ that is independent of $\boldsymbol{y}_0 \in U$. Also by step 5, the Taylor expansion of $\hat{A}_{\boldsymbol{y}_0}(\boldsymbol{z})$ satisfies (2.18b) and finally, $\hat{f}_{\boldsymbol{y}_0}(\boldsymbol{z})$ satisfies (2.19). In particular the estimate (2.19b) on the summability of the Taylor coefficients $(\hat{f}_{\boldsymbol{y}_0;\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$, holds with a uniform bound independent of $\boldsymbol{y}_0 \in U$, cf. (2.82). Since all occurring constants were independent of $\boldsymbol{y}_0 \in U$, we conclude that

$$\sup_{\boldsymbol{y}_0\in U} \sum_{\boldsymbol{\nu}\in\mathcal{F}} \boldsymbol{\rho}^{\boldsymbol{\nu}} \left\| \frac{\partial^{\boldsymbol{\nu}}_{\boldsymbol{y}} \hat{u}_{\boldsymbol{y}_0}(\boldsymbol{y})|_{\boldsymbol{y}=\mathbf{0}}}{\boldsymbol{\nu}!} \right\|_{\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})} < \infty, \tag{2.83}$$

which is (2.37). ■

Remark 2.12. *Item ii) of Thm. 2.10 implies* $\lim_{N\to\infty} \|\hat{u}(y_1,\dots,y_N,0,\dots) - \hat{u}(\boldsymbol{y})\|_{\mathbb{H}^s(\mathrm{D})} = 0$ *for all* $\boldsymbol{y} \in U$ *(cf. (2.54)). Moreoever, exploiting the implicit function theorem as we did in Step 1 of the proof of Thm. 2.6, one can show that* $u(\boldsymbol{y})$ *is holomorphic as a function of each* y_j *on some complex polydisc centered at* $\mathbf{0} \in \mathbb{C}^{\mathbb{N}}$ *and containing* U*. Thus Lemma 2.5 ii) implies uniform convergence of the Taylor expansion of* $\hat{u}(\boldsymbol{y})$ *on* U*. The same remark applies to the setting of Thm. 2.3.*

Example 2.13. *Consider the unit ball* $\mathrm{D}_{\mathbf{0}} := \{x \in \mathbb{R}^2 : \|x\|_2 < 1\}$ *in* $\mathbb{R}^2$*. Using polar coordinates* $x = (x_1,x_2) = r(\cos(\varphi),\sin(\varphi))$ *we define the transformation*

$$T_{\boldsymbol{y}}(x_1,x_2) := r\left(1 + \sum_{j\in\mathbb{N}} y_j \xi_j(\varphi)\right)\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix},$$

where $\xi_j : [0,2\pi] \to \mathbb{R}$ *are suitable* 2π*-periodic Lipschitz continuous functions. In order for this transformation to be well-defined such that* $\mathrm{D}_{\boldsymbol{y}} := T_{\boldsymbol{y}}(\mathrm{D}_{\mathbf{0}})$ *is a Lipschitz domain, it suffices to assume that* $\sum_{j\in\mathbb{N}} y_j\xi_j(\varphi)$ *is in* $W^{1,\infty}(0,2\pi)$ *allowing a periodic extension to* $W^{1,\infty}(\mathbb{R})$ *for every* $\boldsymbol{y} \in U$ *and such that* $\inf_\varphi \sum_{j\in\mathbb{N}} y_j\xi_j(\varphi) > 0$*. We may then write* $T_{\boldsymbol{y}} = \mathrm{Id} + \sum_{j\in\mathbb{N}} y_j\psi_j$ *where* $\psi_j(x_1',x_2') = r\xi_j(\varphi)(\cos(\varphi),\sin(\varphi))$*, which fits the setting of Assumption 2.9.*

One possibility is to use a Fourier type expansion of the boundary, e.g. by setting $\xi_j := \theta j^{-\alpha}\sin(j\varphi)$ *for some* $\theta > 0$*,* $\alpha > 1$*. Observe that for* $\alpha > 2$ *and* $\theta > 0$ *small enough it indeed holds that* $\sum_{j\in\mathbb{N}} y_j\xi_j(\varphi) \in W^{1,\infty}(0,2\pi)$ *with periodic boundary conditions and such that* $T_{\boldsymbol{y}}$ *is bi-Lipschitz. Next, computing the Jacobian* $D\psi_j(x_1',x_2')$ *at* $x' = (x_1',x_2') = r(\cos(\varphi),\sin(\varphi))$ *we obtain*

$$\begin{pmatrix}\xi_j(\varphi) - \xi_j'(\varphi)\cos(\varphi)\sin(\varphi) & \xi_j'(\varphi)\cos^2(\varphi)\\ -\xi_j'(\varphi)\sin^2(\varphi) & \xi_j(\varphi) + \xi_j'(\varphi)\cos(\varphi)\sin(\varphi)\end{pmatrix}. \tag{2.84}$$

Thus, if $(c_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$*, we get with* $\rho_j := c_j/(\theta j^{-\alpha+1})$

$$\operatorname*{ess\,sup}_{x'\in \mathrm{D}} \sum_{j\in\mathbb{N}} \rho_j(\|\psi_j(x')\| + \|D\psi_j(x')\|) \leq C \sum_{j\in\mathbb{N}} \sup_{\varphi\in[0,2\pi]} \rho_j(|\xi_j(\varphi)| + |\xi_j'(\varphi)|)$$
$$\leq C \sum_{j\in\mathbb{N}} c_j j^{\alpha-1} j^{-\alpha+1} < \infty.$$

Defining $c_j := j^{-1-\varepsilon}$ *we obtain* $(\rho_j^{-1})_{j\in\mathbb{N}} \in \ell^{1/(\alpha-2)+\varepsilon}(\mathbb{N})$ *for any* $\varepsilon > 0$*. In the setting of Thm. 2.10 (for some appropriate* A *and* f*) this yields* $(\|\hat{u}_{\boldsymbol{\nu}}\|_{\mathbb{H}^{-s}(\mathrm{D}_{\mathbf{0}})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p(\mathcal{F})$ *with* $p = 2/(2\alpha-3)+\varepsilon$ *and* $\varepsilon > 0$ *arbitrary.*

Example 2.14. *Consider again the setting of Example 2.13. Another possible choice for the* ξ_j *comprises wavelet bases, which have the advantage of allowing to exploit the locality of supports: For a generating wavelet* $\xi \in W^{1,\infty}(\mathbb{R})$ *such that* $\mathrm{supp}\xi \subseteq [0,2\pi]$*, at level* $l \in \mathbb{N}_0$ *we consider the* 2^l *functions*

$$\xi_\lambda(\varphi) := \theta 2^{-\alpha l}\xi(2^l\varphi + k2\pi) \qquad k = 0,\dots,2^l-1,$$

defined for $\varphi \in [0,2\pi]$ *with* $\lambda = (l,k)$ *and some fixed* $\theta > 0$*,* $\alpha > 1$*. By construction for* $\varphi \in [0,2\pi]$ *and* $l \in \mathbb{N}$ *fixed, there is at most one* $\lambda = (l,k)$ *such that* $\xi_\lambda(\varphi) \neq 0$*. Hence, for any* $\beta < \alpha - 1$ *and with* $\rho_\lambda := 2^{l\beta}$ *we get*

$$\sup_{\varphi\in[0,2\pi]} \sum_\lambda \rho_\lambda(|\xi_\lambda(\varphi)| + |\xi_\lambda'(\varphi)|) \leq \theta\|\xi\|_{W^{1,\infty}(0,2\pi)} \sum_{l\in\mathbb{N}_0} 2^{l\beta}(2^{-l\alpha} + 2^{-l(\alpha-1)}) < \infty$$

since $\alpha - 1 - \beta > 0$*. More precisely, the last quantity behaves like* $O(\theta)$ *as* $\theta \to 0$*. Let* $\psi_j(x) = \xi_j r(\cos(\varphi), \sin(\varphi))$*. Using (2.84), as in Example 2.13 we conclude that*

$$\sup_{x'\in \mathrm{D}_{\mathbf{0}}} \sum_\lambda \rho_\lambda(\|\psi_\lambda(x')\|_2 + \|D\psi_\lambda(x')\|_2) = O(\theta) \qquad \text{as } \theta \to 0.$$

Denote now by $(\xi_j)_{j\in\mathbb{N}}$ *some rearrangement of* $(\xi_\lambda)_\lambda$*. For* $\theta > 0$ *small enough, we observe that the assumptions of Thm. 2.10 will be satisfied (if additionally* A*,* f *are as stated there). Finally, we remark that* $(\rho_j^{-1})_{j\in\mathbb{N}}$ *constitutes an* $\ell^{1/\beta+\varepsilon}$ *sequence for any* $\varepsilon > 0$*. Hence Thm. 2.10 gives* $(\|\hat{t}_{\boldsymbol{y};\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^p(\mathcal{F})$ *with* $p = 2/(2\beta+1)+\varepsilon$ *for any* $\varepsilon > 0$ *in this case.*

We now show summability of the Legendre coefficients: adopting the notation of Cor. 2.4, we have the following result.

Corollary 2.15. *Under the assumptions of Thm. 2.10, denote by* $\hat{l}_{\boldsymbol{\nu}} := \int_U \hat{u}(\boldsymbol{y})L_{\boldsymbol{\nu}}(\boldsymbol{y})\,\mathrm{d}\mu(\boldsymbol{y})$ *the Legendre coefficients of* $\hat{u}(\boldsymbol{y})$*. Then*

$$(\boldsymbol{\rho}^{\boldsymbol{\nu}}\|\hat{l}_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})})_{\boldsymbol{\nu}\in\mathcal{F}} \in \ell^2(\mathcal{F}).$$

Proof. We use the Rodriguez formula as was done in [2]: The multivariate Legendre polynomials allow the representation

$$L_{\boldsymbol{\nu}}(\boldsymbol{y}) = \partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} \prod_{j\geq 1} \frac{\sqrt{2\nu_j+1}}{\nu_j! 2^{\nu_j}}(1-y_j^2)^{\nu_j}.$$

By repeated integration by parts in the definition of $\hat{l}_{\boldsymbol{\nu}}$ we obtain

$$\hat{l}_{\boldsymbol{\nu}} = \int_U \hat{u}_{\boldsymbol{y}} L_{\boldsymbol{\nu}}(\boldsymbol{y})\,\mathrm{d}\mu(\boldsymbol{y}) = (-1)^{|\boldsymbol{\nu}|}\int_U \hat{u}(\boldsymbol{y}) \prod_{j\in\mathrm{supp}\boldsymbol{\nu}} \partial_{y_j}^{\nu_j}\left(\frac{\sqrt{2\nu_j+1}}{\nu_j!2^{\nu_j}}(1-y_j^2)^{\nu_j}\right)\mathrm{d}\mu(\boldsymbol{y})$$
$$= (-1)^{|\boldsymbol{\nu}|}\int_U \frac{\partial^{\boldsymbol{\nu}}\hat{u}(\boldsymbol{y})}{\boldsymbol{\nu}!}(1-\boldsymbol{y}^2)^{\boldsymbol{\nu}}\,\mathrm{d}\mu(\boldsymbol{y}) \prod_{j\in\mathrm{supp}\boldsymbol{\nu}}\frac{\sqrt{2\nu_j+1}}{2^{\nu_j}}.$$

Thus, by (2.37) and since $\mu(U)=1$ and $\prod_{j\in\mathrm{supp}\boldsymbol{\nu}}\frac{\sqrt{2\nu_j+1}}{2^{\nu_j}}\leq 1$,

$$\sum_{\boldsymbol{\nu}\in\mathcal{F}}(\boldsymbol{\rho}^{\boldsymbol{\nu}}\|\hat{l}_{\boldsymbol{\nu}}\|_{\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})})^2 \leq \sup_{\boldsymbol{y}\in U}\sum_{\boldsymbol{\nu}\in\mathcal{F}}\boldsymbol{\rho}^{\boldsymbol{\nu}}\left\|\frac{\partial^{\boldsymbol{\nu}}u(\boldsymbol{y})}{\boldsymbol{\nu}!}(1-\boldsymbol{y}^2)^{\boldsymbol{\nu}}\right\|^2_{\mathbb{H}^s(\mathrm{D}_{\mathbf{0}})} < \infty. \qquad \blacksquare$$

Remark 2.16. *The proofs of Thm. 2.6, Thm. 2.10, Cor. 2.15 do not restrict to fractional operators. Verbatim arguments apply in the case $s = 1$ (where no extension w.r.t. the z variable is needed), thus generalizing the results of [2], and also of [20] to fully anisotropic, as well as to non-affine parametric dependence.*

3. Conclusions.

We performed a mathematical analysis of solution sparsity of countably parametric solutions of the spectral fractional Laplacean.

Specifially, we obtained summability results of parametric solution families of the fractional Laplace equation for parametric inputs of either the coefficients or of the domain of definition, for bounded parameter domains, constituting the first UQ analysis of nonlocal operator equations with parametric inputs. We considered fractional powers of affine-parametric operators, as well as also non-affine, parametric inputs of gpc type. The present results generalize previous analyses even in the local case, i.e., for UQ of the second order diffusion operator. Our analysis allows to exploit, in particular, also uncertainty parametrizations of the distributed input data with locally supported representation systems, such as splines, wavelets, etc. As our main result, Theorem 2.6, admits gpc-structured parametric input data (rather than merely affine-parametric data), this result opens also an avenue for the sparsity analysis in uncertainty propagation. The corresponding sparsity result in Theorem 2.6 is to our knowledge new even in the case $s = 1$, i.e. for local diffusion PDEs with gpc-parametric uncertain diffusion coefficients. The tools developed in proving our results on parametric regularity and sparsity are of independent interest beyond the presently considered problem classes. The present sparsity results also imply dimension independent convergence rates of several constructive numerical approximations, such as sparse grid interpolation, Smolyak approximation and higher order Quasi-Monte Carlo quadrature. These will be developed in detail in our forthcoming work [21].

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