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Petrov-Galerkin space-time hp-approximation of parabolic equations in $H^{1/2}$

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We analyze a class of variational space-time discretizations for a broad class of initial boundary value problems for linear, parabolic evolution equations.
The space-time variational formulation is based on fractional Sobolev spaces of order $1/2$ and the Riemann-Liouville derivative of order $1/2$ with respect to the temporal variable. It accomodates general, conforming space discretizations and naturally accomodates discretization of infinite horizon evolution problems. We prove an inf-sup condition for hp-time semidiscretizations with an explicit expression of stable test functions given in terms of Hilbert transforms of the corresponding trial functions; inf-sup constants are independent of temporal order and the time-step sequences, allowing quasioptimal, high-order discretizations on graded time step sequences, and also hp-time discretizations. For solutions exhibiting Gevrey regularity in time and taking values in certain weighted Bochner spaces, we establish novel exponential convergence estimates in terms of N_t, the number of (elliptic) spatial problems to be solved. The space-time variational setting allows general space discretizations, and, in particular, for spatial hp-FEM discretizations.
We report numerical tests of the method for model problems in one space dimension with typical singular solutions in the spatial and temporal variable. hp-discretizations in both spatial and temporal variables are used without any loss of stability, resulting in overall exponential convergence of the space-time discretization.

Keywords: Parabolic partial differential equations, space-time approximation, hp-refinements, exponential convergence, a priori error estimates

1. Introduction

We consider linear, parabolic evolution problems which are set in a pair of separable Hilbert spaces $(V,(\cdot,\cdot)_V)$ and $(H,(\cdot,\cdot)_H)$ such that the embedding $V \subset H$ is continuous, dense and compact and assume without loss of generality that $\|\cdot\|_H \leqslant \|\cdot\|_V$. Identifying H with its dual H^*, we have the Gelfand triple $V \subseteq H \subseteq V^*$. Given a linear self-adjoint operator $\mathscr{A} \in \mathscr{L}(V,V^*)$ and a time horizon $0 < T \leqslant \infty$, our goal is to solve the initial-value parabolic problem

$$\mathfrak{B}u := \partial_t u + \mathscr{A}u = f, \tag{1.1a}$$

$$u(0) = 0, \tag{1.1b}$$

in $I = (0,T)$. In (1.1a), $\mathscr{A}$ is meant as a linear, strongly elliptic operator of order $2m$ and V as a closed subspace of the Sobolev space $H^m(D)$ taking into account essential boundary conditions. Here $D \subseteq \mathbb{R}^d$ is a bounded domain for $d = 1,2,3$. These equations can be used for instance to model heat conduction in a possibly heterogeneous material or flow of an incompressible Newtonian fluid through the Stokes equations.

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In order to approximate the solution of (1.1a)-(1.1b), the usual approach is to consider time stepping schemes (Thomée, 2006). The idea is to first discretize the equations in space so as to obtain a system of ordinary differential equations (ODEs). In a second step, a solver for ODEs is used. On the other hand, first discretizing in time and then in space yields the so-called Rothe's method. The algorithm that we introduce is based on a different approach and belongs to the class of space-time methods. In that setting, the problem is considered over the space-time cylinder $Q = I \times D$ and the solution is sought in a suitable Bochner space. This approach has already been well-studied in different contexts. We mention among others the parareal method which aims at time-parallel integration (Gander, 2015). Closer to what we present, several methods based on finite element (FEM) approximation have been derived. For instance, in Schötzau (1999); Schötzau & Schwab (2000); Werder *et al.* (2001), the authors use discontinuous elements for the discretization of the temporal component. In particular, they show that exponential convergence can be obtained in that setting. Another algorithm which was introduced by Schwab & Stevenson (2009) considers adaptive wavelet methods. Based on this, it is possible to recover the optimal convergence rate associated to the underlying elliptic problem for the fully space-time discrete problem. We consider a discretization based on continuous FEM approximation for the time variable. Several algorithms using this idea already exist (Andreev, 2013, 2014; Devaud & Schwab, 2018; Langer *et al.*, 2016; Schwab & Stevenson, 2009).

The first ingredient to derive a space-time approximation is to introduce a space-time weak formulation associated to (1.1a)-(1.1b). In Schwab & Stevenson (2009), the authors present a formulation whose solution belongs to $L^2(I;V) \cap H^1(I;V^*)$ and prove that the problem is well-posed. A different approach was presented by Langer *et al.* (2016) and Devaud & Schwab (2018). There, it is shown that the bilinear form associated to the problem is coercive and continuous with respect to a mesh-dependent norm, yielding that it admits a unique solution. Our approach is based on the weak formulation used in Larsson & Schwab (2015), which was first introduced by Fontes (2009). A similar framework has been presented by Steinbach & Zank (2018) for parabolic equations set on a finite time interval. Considering (partial) integration by parts for the time derivative, we obtain a formulation containing derivatives of fractional order (see Section 2). The main advantage of such approach is that it is possible to prove inf-sup stability of the bilinear form using a continuous linear operator from the trial to the test space (Theorem 2.3). This important feature is then used to build inf-sup stable pairs of discrete spaces. A time regularity result for the solution of (1.1a)-(1.1b) is also discussed. Considering a smooth forcing term, the associated solution exhibits the same regularity in time except for a potential algebraic singularity at the initial time due to an incompatibility between f and the initial condition. This lack of regularity does not allow us to obtain a high-order method based on for instance p-FEM approximation. Instead, an hp-discretization has to be considered and exponential convergence for the approximation of the temporal component can be proven (Theorem 4.2). Combining this result with both low-order elements (Section 4.2.1) and hp-approximation in the case of the two-dimensional heat equation (Section 4.2.2), it is possible to derive convergence rates for the fully discrete scheme. In particular, we show that space-time hp-approximation allows for exponential convergence (Theorem 4.7).

In the next section, well-posedness of the continuous problem is discussed. Then, continuous piecewise polynomial approximation in the fractional order Sobolev space $H^{1/2}_{00}(I)$ is discussed in Section 3. Space-time discretization of parabolic equations is described in the penultimate section and we conclude with numerical results in Section 5.

Throughout the paper, the set of all positive integers $1,2,\dots$ is denoted $\mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$ and two vectors $a,b \in \mathbb{R}^n$, we write $a \geqslant b$ ($a > b$) if $a_i \geqslant b_i$ ($a_i > b_i$) for all $i = 1,\dots,n$. Furthermore, for $k \in \mathbb{N}$, $a+k = (a_1+k,\dots,a_n+k)$. For an interval I and a Banach space $(A, \|\cdot\|_A)$, $\mathscr{C}(I;A)$ ($\mathscr{C}(\bar{I};A)$) denotes the space of continuous functions on I ($\bar{I}$) taking values in A. Given two Banach spaces $(A, \|\cdot\|_A)$

and $(B, \|\cdot\|_B)$, $\mathscr{L}(A,B)$ denotes the space of continuous linear operators from A to B and $\mathscr{B}(A,B)$ the space of bounded bilinear forms from $A \times B$ to $\mathbb{R}$. $\mathscr{L}_{\text{Iso}}(A,B)$ denotes the space of isomorphism between A and B. Finally, $[A,B]_\theta$ refers to the complex interpolation space of order $\theta \in [0,1]$ between A and B (Bergh & Löfström, 1976, Chapter 4).

2. Linear parabolic evolution problems

The framework associated to the parabolic equations (1.1a)-(1.1b) is presented in this section. This formulation was first introduced by Fontes (2009) and is based on fractional order Bochner-Sobolev spaces (Hytönen *et al.*, 2016, Section 2.5). Definitions of these spaces and several different characterizations are presented in Section 2.2. The main advantage of that setting is that it is possible to show an inf-sup condition using a bounded linear operator between the trial and the test space (Theorem 2.3). This allows us then to build explicitly inf-sup stable pairs of discrete spaces as explained in Section 5. The bilinear form associated to the weak formulation of (1.1a)-(1.1b) is obtained using fractional integration by parts on the time variable. This is discussed in the following section.

2.1 *Fractional calculus*

The theory presented here is based on Fontes (2009); Larsson & Schwab (2015); Samko *et al.* (1993). To obtain well-posedness of our problem on $\mathbb{R}_+$, we relate it to the problem stated on $\mathbb{R}$. For a Hilbert space $(W, (\cdot,\cdot)_W)$, let us define

$$\mathscr{F}(\mathbb{R};W) := \left\{ g \in \mathscr{C}^\infty(\mathbb{R};W) \;\middle|\; \|g\|_{H^s(\mathbb{R};W)} < \infty,\ \forall s \in \mathbb{R} \right\},$$
$$\mathscr{F}(\mathbb{R}_+;W) := \left\{ g \in \mathscr{C}^\infty(\mathbb{R}_+;W) \;\middle|\; \exists \tilde{g} \in \mathscr{F}(\mathbb{R};W) : g = \tilde{g}|_{\mathbb{R}_+} \right\},$$

where $H^s(\mathbb{R};W)$ denote the Bessel potential spaces for $s \in \mathbb{R}$ (Hytönen *et al.*, 2016, Definition 5.6.2). The topology on $\mathscr{F}(\mathbb{R};W)$ is the one induced by the family of norms $\{\|\cdot\|_{H^s(\mathbb{R};W)}\}_{s\in\mathbb{N}_0}$. In order to treat initial conditions, we also consider

$$\mathscr{F}_0(\mathbb{R}_+;W) := \{ g \in \mathscr{C}^\infty(\mathbb{R}_+;W) \mid E_0 g \in \mathscr{F}(\mathbb{R};W) \},$$

where E_0 denotes the "extension by zero" operator of functions defined on $\mathbb{R}_+$ to $\mathbb{R}$. We also introduce the "restriction to $\mathbb{R}_+$" operator $R_>$. Furthermore, the dual spaces of $\mathscr{F}(\mathbb{R};W)$, $\mathscr{F}(\mathbb{R}_+;W)$ and $\mathscr{F}_0(\mathbb{R}_+;W)$ are denoted

$$\mathscr{F}'(\mathbb{R};W) := \mathscr{F}(\mathbb{R};W)^*, \qquad \mathscr{F}_0'(\mathbb{R}_+;W) := \mathscr{F}(\mathbb{R}_+;W)^*, \qquad \mathscr{F}'(\mathbb{R}_+;W) := \mathscr{F}_0(\mathbb{R}_+;W)^*,$$

where W has been identified with its dual using Riesz representation theorem. The topologies on $\mathscr{F}(\mathbb{R}_+;W)$ and $\mathscr{F}_0(\mathbb{R}_+;W)$ are the ones defined in (Fontes, 2009, p.8).

The Riemann-Liouville fractional derivatives of order $0 < \alpha < 1$ are then defined for $u \in \mathscr{F}(\mathbb{R};W)$, $v \in \mathscr{F}_0(\mathbb{R}_+;W)$ and $w \in \mathscr{F}(\mathbb{R}_+;W)$ as

$$(D^\alpha_{\mathbb{R},+}u)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-\infty}^t \frac{u(s)}{(t-s)^\alpha}ds, \quad (D^\alpha_{\mathbb{R},-}u)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty \frac{u(s)}{(s-t)^\alpha}ds, \quad t \in \mathbb{R},$$
$$(D^\alpha_{+}v)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{v(s)}{(t-s)^\alpha}ds, \qquad (D^\alpha_{-}w)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^\infty \frac{w(s)}{(s-t)^\alpha}ds, \quad t \in \mathbb{R}_+,$$

where $\Gamma(s)$ denotes the Gamma function for $s > 0$. Due to their convenient treatment of boundary conditions, Caputo fractional derivatives are more widely used in the literature than the Riemann-Liouville ones. However, we point out that both derivatives on $\mathbb{R}_+$ coincide, which is the setting considered in our context since our initial value problem is set over the positive real axis (Kilbas *et al.*, 2006, (2.4.6) and (2.4.7)). The following properties hold for all $0 < \alpha < 1$, $u, v \in \mathscr{F}(\mathbb{R};W)$, $f \in \mathscr{F}_0(\mathbb{R}_+;W)$ and $g \in \mathscr{F}(\mathbb{R}_+;W)$

$$\int_{\mathbb{R}} \left(D^{\alpha}_{\mathbb{R},+}u, v\right)_W dt = \int_{\mathbb{R}} \left(u, D^{\alpha}_{\mathbb{R},-}v\right)_W dt, \qquad \int_{\mathbb{R}_+} \left(D^{\alpha}_{+}f, g\right)_W dt = \int_{\mathbb{R}_+} \left(f, D^{\alpha}_{-}g\right)_W dt.$$

Using this we define the $\mathscr{F}'(\mathbb{R};W)$, $\mathscr{F}_0'(\mathbb{R}_+;W)$ and $\mathscr{F}'(\mathbb{R}_+;W)$ distribution derivatives

$$\begin{aligned}
\left\langle D^{\alpha}_{\mathbb{R},\pm}\Lambda, u\right\rangle_{\mathscr{F}'(\mathbb{R};W)} &:= \left\langle \Lambda, D^{\alpha}_{\mathbb{R},\mp}u\right\rangle_{\mathscr{F}'(\mathbb{R};W)}, && \forall \Lambda \in \mathscr{F}'(\mathbb{R};W),\ u \in \mathscr{F}(\mathbb{R};W),\\
\left\langle D^{\alpha}_{+}\Phi, v\right\rangle_{\mathscr{F}_0'(\mathbb{R}_+;W)} &:= \left\langle \Phi, D^{\alpha}_{-}v\right\rangle_{\mathscr{F}_0'(\mathbb{R}_+;W)}, && \forall \Phi \in \mathscr{F}_0'(\mathbb{R}_+;W),\ v \in \mathscr{F}(\mathbb{R}_+;W),\\
\left\langle D^{\alpha}_{-}\Psi, w\right\rangle_{\mathscr{F}'(\mathbb{R}_+;W)} &:= \left\langle \Psi, D^{\alpha}_{+}w\right\rangle_{\mathscr{F}'(\mathbb{R}_+;W)}, && \forall \Psi \in \mathscr{F}'(\mathbb{R}_+;W),\ w \in \mathscr{F}_0(\mathbb{R}_+;W).
\end{aligned}$$

Hence we have that the following operators are continuous (Fontes, 2009, Lemmas 2.1 and 2.2, (2.24)-(2.28))

$$\begin{aligned}
D^{\alpha}_{\mathbb{R},\pm} &: \mathscr{F}(\mathbb{R};W) \to \mathscr{F}(\mathbb{R};W), & D^{\alpha}_{\mathbb{R},\pm} &: \mathscr{F}'(\mathbb{R}_+;W) \to \mathscr{F}'(\mathbb{R}_+;W),\\
D^{\alpha}_{+} &: \mathscr{F}_0(\mathbb{R}_+;W) \to \mathscr{F}_0(\mathbb{R}_+;W), & D^{\alpha}_{+} &: \mathscr{F}_0'(\mathbb{R}_+;W) \to \mathscr{F}_0'(\mathbb{R}_+;W),\\
D^{\alpha}_{-} &: \mathscr{F}(\mathbb{R}_+;W) \to \mathscr{F}(\mathbb{R}_+;W), & D^{\alpha}_{-} &: \mathscr{F}'(\mathbb{R}_+;W) \to \mathscr{F}'(\mathbb{R}_+;W).
\end{aligned}$$

Furthermore, we will need the Hilbert transform defined for functions $v \in \mathscr{F}(\mathbb{R};W)$ as

$$(\mathfrak{H}v)(t) := \frac{1}{\pi}\,\mathrm{p.v.}\int_{-\infty}^{\infty} \frac{v(s)}{t-s}\,ds := \frac{1}{\pi}\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{t-\varepsilon} \frac{v(s)}{t-s}\,ds + \int_{t+\varepsilon}^{\infty} \frac{v(s)}{t-s}\,ds\right), \qquad t \in \mathbb{R}, \tag{2.2}$$

where p. v. denotes the Cauchy principal value. It holds $\mathfrak{H} \in \mathscr{L}\left(\mathscr{F}(\mathbb{R};W)\right)$ (Fontes, 2009, Lemma 2.1) and

$$\int_{\mathbb{R}} (v, w)_W\, dt = \int_{\mathbb{R}} (\mathfrak{H}v, \mathfrak{H}w)_W\, dt, \qquad \int_{\mathbb{R}} (v, \mathfrak{H}v)_W\, dt = 0, \qquad \forall v, w \in \mathscr{F}(\mathbb{R};W). \tag{2.3}$$

This is a consequence of (Pandey, 1996, (2.34) and (2.36)). It follows from $\mathfrak{H}^{-1} = -\mathfrak{H}$ (Pandey, 1996, Theorem 1 p.76) and (2.3) that

$$\int_{\mathbb{R}} (\mathfrak{H}v, w)_W\, dt = -\int_{\mathbb{R}} (v, \mathfrak{H}w)_W\, dt, \qquad \forall v, w \in \mathscr{F}(\mathbb{R};W). \tag{2.4}$$

2.2 *The spaces $H^{1/2}$ and $H^{1/2}_{00}$*

The weak formulation associated to (1.1a)-(1.1b) is based on the Sobolev spaces of fractional order $H^{1/2}$. The key property to link these spaces to the theory developed in the previous section is that they can be defined equivalently using the Riemann-Liouville fractional derivatives of order $\alpha = 1/2$.

DEFINITION 2.1 (Sobolev-Slobodeckij spaces) Let $-\infty \leqslant a < b \leqslant \infty$ and $I = (a, b)$.

1. $H^{1/2}(I;W)$ is the set of functions $v \in L^2(I;W)$ for which the norm

$$\|v\|^2_{H^{1/2}(I;W)} := \|v\|^2_{L^2(I;W)} + \int_I \int_I \frac{\|v(s)-v(t)\|^2_W}{|s-t|^2} dsdt \tag{2.5}$$

is finite.

2. $H^{1/2}_{00}(I;W)$ is the set of functions $v \in L^2(I;W)$ for which the following norm is finite

$$\|v\|^2_{H^{1/2}_{00}(I;W)} := \|v\|^2_{H^{1/2}(I;W)} + \int_I \frac{\|v(t)\|^2_W}{t-a} dt + \int_I \frac{\|v(t)\|^2_W}{b-t} dt, \tag{2.6}$$

with the convention that the boundary terms do not appear in the case $a = \infty$ and/or $b = \infty$.

The embeddings $\mathscr{F}_0(\mathbb{R}_+;W) \subset H^{1/2}_{00}(\mathbb{R}_+;W)$ and $\mathscr{F}(\mathbb{R}_+;W) \subset H^{1/2}(\mathbb{R}_+;W)$ are dense (Fontes, 2009, Lemma 3.7). The following important integration by parts properties can be found in (Larsson & Schwab, 2015, Lemma 2.7) for the $\mathbb{R}_+$ case and is obtained by fractional integration by parts (using the Fourier characterization of $D^{1/2}_{\mathbb{R},\pm}$) in the $\mathbb{R}$ case.

LEMMA 2.1 The following properties hold:

1. For $v \in H^{1/2}(\mathbb{R};W)$, it holds

$$\langle Dv, w\rangle_{\mathscr{F}'(\mathbb{R};W)} = \int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+}v, D^{1/2}_{\mathbb{R},-}w\right)_W dt, \qquad \forall w \in \mathscr{F}(\mathbb{R};W), \tag{2.7}$$

where the $D^{1/2}_{\mathbb{R},+}$ derivative has to be understood in the $\mathscr{F}'(\mathbb{R};W)$ sense.

2. For $v \in H^{1/2}_{00}(\mathbb{R}_+;W)$, it holds

$$\langle Dv, w\rangle_{\mathscr{F}'_0(\mathbb{R}_+;W)} = \int_{\mathbb{R}_+} \left(D^{1/2}_{+}v, D^{1/2}_{-}w\right)_W dt, \qquad \forall w \in \mathscr{F}(\mathbb{R}_+;W), \tag{2.8}$$

where the $D^{1/2}_{+}$ derivative has to be understood in the $\mathscr{F}'_0(\mathbb{R}_+;W)$ sense.

The following characterizations of $H^{1/2}(\mathbb{R};W)$, $H^{1/2}_{00}(\mathbb{R}_+;W)$ and $H^{1/2}(\mathbb{R}_+;W)$ can be found in (Fontes, 2009, Lemmas 3.2, 3.8 and 3.9). Together with the previous lemma, it allows us to derive the weak formulation of our problem and obtain that the associated bilinear form is continuous.

PROPOSITION 2.2 The following properties hold:

1. A function $v \in L^2(\mathbb{R};W)$ belongs to $H^{1/2}(\mathbb{R};W)$ if and only if its $\mathscr{F}'(\mathbb{R};W)$-derivatives $D^{1/2}_{\mathbb{R},\pm}v$ belong to $L^2(\mathbb{R};W)$. Moreover, it holds

$$|v|_{H^{1/2}(\mathbb{R};W)} = 2\pi \left\|D^{1/2}_{+}v\right\|^2_{L^2(\mathbb{R};W)} = 2\pi \left\|D^{1/2}_{-}v\right\|^2_{L^2(\mathbb{R};W)}, \qquad \forall v \in H^{1/2}(\mathbb{R};W). \tag{2.9}$$

2. A function $v \in L^2(\mathbb{R}_+;W)$ belongs to $H^{1/2}(\mathbb{R}_+;W)$ if and only if its $\mathscr{F}'(\mathbb{R}_+;W)$-derivative $D^{1/2}_{-}v$ belongs to $L^2(\mathbb{R}_+;W)$. Moreover, the norm

$$\|v\|^2_{H^{1/2}_{-}(\mathbb{R}_+;W)} := \|v\|^2_{L^2(\mathbb{R}_+;W)} + \left\|D^{1/2}_{-}v\right\|^2_{L^2(\mathbb{R}_+;W)}, \qquad \forall v \in H^{1/2}(\mathbb{R}_+;W),$$

is equivalent to $\|\cdot\|_{H^{1/2}(\mathbb{R}_+;W)}$ defined in (2.5).

3. A function $v \in L^2(\mathbb{R}_+;W)$ belongs to $H^{1/2}_{00}(\mathbb{R}_+;W)$ if and only if its $\mathscr{F}_0'(\mathbb{R}_+;W)$-derivative $D_+^{1/2}v$ belongs to $L^2(\mathbb{R}_+;W)$. Moreover, the norm

$$\|v\|^2_{H_+^{1/2}(\mathbb{R}_+;W)} := \|v\|^2_{L^2(\mathbb{R}_+;W)} + \left\|D_+^{1/2}v\right\|^2_{L^2(\mathbb{R}_+;W)}, \qquad \forall v \in H^{1/2}_{00}(\mathbb{R}_+;W),$$

is equivalent to $\|\cdot\|_{H^{1/2}_{00}(\mathbb{R}_+;W)}$ defined in (2.6).

2.3 *Initial-value problems*

Associated to $\mathscr{A}$, we define the bilinear form $\mathfrak{a} \in \mathscr{B}(V,V)$ as $\mathfrak{a}(v,w) := \langle \mathscr{A}v, w\rangle_{V^*}$ for all $v,w \in V$ and assume that there exist $0 < \lambda_- \leqslant \lambda_+ < \infty$ such that

$$\mathfrak{a}(v,w) \leqslant \lambda_+ \|v\|_V \|w\|_V, \qquad \forall v,w \in V, \tag{2.10a}$$

$$\mathfrak{a}(v,v) \geqslant \lambda_- \|v\|_V^2, \qquad \forall v \in V, \tag{2.10b}$$

i.e. $\mathfrak{a}$ is continuous and coercive on V, respectively. We see that (2.10a) follows from the assumption that $\mathscr{A} \in \mathscr{L}(V,V^*)$. We introduce the normed vector spaces

$$\mathscr{X} := H^{1/2}_{00}(\mathbb{R}_+;H) \cap L^2(\mathbb{R}_+;V), \qquad \|v\|^2_{\mathscr{X}} := \|v\|^2_{H^{1/2}_{00}(\mathbb{R}_+;H)} + \|v\|^2_{L^2(\mathbb{R}_+;V)}, \qquad \forall v \in \mathscr{X},$$

$$\mathscr{Y} := H^{1/2}(\mathbb{R}_+;H) \cap L^2(\mathbb{R}_+;V), \qquad \|v\|^2_{\mathscr{Y}} := \|v\|^2_{H^{1/2}(\mathbb{R}_+;H)} + \|v\|^2_{L^2(\mathbb{R}_+;V)}, \qquad \forall v \in \mathscr{Y}.$$

Note that $\mathscr{X}$ is isomorphic to $(H^{1/2}_{00}(\mathbb{R}_+) \otimes H) \cap (L^2(\mathbb{R}_+) \otimes V)$ while $\mathscr{Y}$ is isomorphic to $(H^{1/2}(\mathbb{R}_+) \otimes H) \cap (L^2(\mathbb{R}_+) \otimes V)$ (Larsson & Schwab, 2015, (3.1)). This will be used later in the approximation theory to split the error estimation between the spatial and the temporal components. The weak formulation of (1.1a)-(1.1b) is then: given $f \in \mathscr{Y}^*$, find $u \in \mathscr{X}$ such that

$$\mathfrak{b}(u,v) := \int_{\mathbb{R}_+} \left[\left(D_+^{1/2}u, D_-^{1/2}v \right)_H + \mathfrak{a}(u,v) \right] dt = \langle f, v\rangle_{\mathscr{Y}^*}, \qquad \forall v \in \mathscr{Y}. \tag{2.11}$$

This is justified by Lemma 2.1. Due to Proposition 2.2, it is clear that $\mathfrak{b}$ is a continuous bilinear form. In order to show well-posedness of (2.11), we relate it to a similar weak formulation on $\mathbb{R}$.

2.3.1 *Parabolic equations on the real axis.* Let us consider the space $\mathscr{X}_{\mathbb{R}} := H^{1/2}(\mathbb{R};H) \cap L^2(\mathbb{R};V)$ endowed with the norm $\|v\|^2_{\mathscr{X}_{\mathbb{R}}} := \|v\|^2_{H^{1/2}(\mathbb{R};H)} + \|v\|^2_{L^2(\mathbb{R};V)}$. We then consider the problem: given $f \in \mathscr{X}_{\mathbb{R}}{}^*$, find $u \in \mathscr{X}_{\mathbb{R}}$ such that

$$\mathfrak{b}_{\mathbb{R}}(u,v) := \int_{\mathbb{R}} \left[\left(D_{\mathbb{R},+}^{1/2}u, D_{\mathbb{R},-}^{1/2}v \right)_H + \mathfrak{a}(u,v) \right] dt = \langle f, v\rangle_{\mathscr{X}_{\mathbb{R}}{}^*}, \qquad \forall v \in \mathscr{X}_{\mathbb{R}}.$$

Let us define for $\eta \in \mathbb{R}$ the operator

$$\mathscr{H}^\eta_{\mathbb{R}} := \cos(\pi\eta)\mathrm{Id} + \sin(\pi\eta)\mathfrak{H},$$

where $\mathfrak{H}$ is defined in (2.2). It then holds $\mathscr{H}_{\mathbb{R}}^{\eta} \in \mathscr{L}(\mathscr{F}(\mathbb{R};V))$. Furthermore, for $v \in \mathscr{F}(\mathbb{R};V)$ and $\eta \in \left(0, \frac{1}{2}\right)$, we have

$$
\begin{aligned}
\mathfrak{b}_{\mathbb{R}}(v, \mathscr{H}_{\mathbb{R}}^{-\eta} v) &= \int_{\mathbb{R}} \left[\cos(\pi\eta)(D_{\mathbb{R},+}^{1/2} v, D_{\mathbb{R},-}^{1/2} v)_H - \sin(\pi\eta)(D_{\mathbb{R},+}^{1/2} v, D_{\mathbb{R},-}^{1/2} \mathfrak{H} v)_H\right] dt \\
&\quad + \int_{\mathbb{R}} [\cos(\pi\eta)\mathfrak{a}(v,v) - \sin(\pi\eta)\mathfrak{a}(v,\mathfrak{H}v)]\, dt \\
&= \sin(\pi\eta) \left\| D_{\mathbb{R},+}^{1/2} v \right\|_{L^2(\mathbb{R};H)}^2 + \cos(\pi\eta) \int_{\mathbb{R}} \mathfrak{a}(v,v) dt \\
&\geqslant \sin(\pi\eta) \left\| D_{\mathbb{R},+}^{1/2} v \right\|_{L^2(\mathbb{R};H)}^2 + \lambda_- \cos(\pi\eta) \|v\|_{L^2(\mathbb{R};V)}^2 \\
&\geqslant \min\left(\frac{\sin(\pi\eta)}{2\pi} + \frac{\lambda_-}{2}\cos(\pi\eta), \frac{\lambda_-}{2}\cos(\pi\eta) \right) \|v\|_{\mathscr{X}_{\mathbb{R}}}^2 =: \alpha_{\mathbb{R}}(\eta) \|v\|_{\mathscr{X}_{\mathbb{R}}}^2,
\end{aligned} \tag{2.12}
$$

where we have used (2.3), Property 1. of Proposition 2.2, $\langle Dv, v\rangle_{\mathscr{F}'(\mathbb{R},H)} = 0$ and $-D_{\mathbb{R},-}^{1/2}\mathfrak{H} = D_{\mathbb{R},+}^{1/2}$ in $\mathscr{F}(\mathbb{R};V)$ (Fontes, 2009, Lemma 2.1). Since $\mathscr{F}(\mathbb{R};V) \subset \mathscr{X}_{\mathbb{R}}$ is dense, we have that the above development holds for all $v \in \mathscr{X}_{\mathbb{R}}$. Furthermore, $\mathscr{H}_{\mathbb{R}}^{-\eta} \in \mathscr{L}_{\text{Iso}}(\mathscr{X}_{\mathbb{R}})$ for every $\eta \in \mathbb{R}$ (Fontes, 2000). Hence, for $\eta \in \left(0, \frac{1}{2}\right)$, there exists $\beta_{\mathbb{R}}(\eta) > 0$ such that

$$
\mathfrak{b}_{\mathbb{R}}(v, \mathscr{H}_{\mathbb{R}}^{-\eta} v) \geqslant \beta_{\mathbb{R}}(\eta) \|v\|_{\mathscr{X}_{\mathbb{R}}} \left\| \mathscr{H}_{\mathbb{R}}^{-\eta} v \right\|_{\mathscr{X}_{\mathbb{R}}}, \qquad \forall v \in \mathscr{X}_{\mathbb{R}},
$$

which shows that the bilinear form $\mathfrak{b}_{\mathbb{R}}(\cdot,\cdot) : \mathscr{X}_{\mathbb{R}} \times \mathscr{X}_{\mathbb{R}} \to \mathbb{R}$ is inf-sup stable.

2.3.2 *Well-posedness of parabolic equations on* $\mathbb{R}_+$. In order to show well-posedness of (2.11), let us introduce the spaces

$$
\begin{aligned}
H_S^{1/2}(\mathbb{R};H) &:= \left\{ v \in H^{1/2}(\mathbb{R};H) \;\middle|\; v(t) = v(-t) \text{ for a.e. } t \in \mathbb{R} \right\}, \\
H_A^{1/2}(\mathbb{R};H) &:= \left\{ v \in H^{1/2}(\mathbb{R};H) \;\middle|\; v(t) = -v(-t) \text{ for a.e. } t \in \mathbb{R} \right\},
\end{aligned}
$$

together with the operators

$$
(E_S v)(t) := v(|t|), \qquad (E_A v)(t) := \begin{cases} v(t) & t \geqslant 0, \\ -v(-t) & t < 0. \end{cases}
$$

On both spaces, we consider the norm $\|\cdot\|_{H^{1/2}(\mathbb{R};H)}$ defined in (2.5). We point out that $E_A + E_S = 2E_0$. The proof of the following lemmas are omitted here but can be found in the online supplementary material.

LEMMA 2.2 The operator $\mathfrak{H} \in \mathscr{L}_{\text{Iso}}\left(H_A^{1/2}(\mathbb{R};H), H_S^{1/2}(\mathbb{R};H)\right)$ is an isometry with $\mathfrak{H}^{-1} = -\mathfrak{H}$.

LEMMA 2.3 It holds

$$
E_S \in \mathscr{L}_{\text{Iso}}(H^{1/2}(\mathbb{R}_+;H), H_S^{1/2}(\mathbb{R};H)) \text{ and } E_A \in \mathscr{L}_{\text{Iso}}(H_{00}^{1/2}(\mathbb{R}_+;H), H_A^{1/2}(\mathbb{R};H)),
$$

with $E_S^{-1} = R_>|_{H_S^{1/2}(\mathbb{R};H)}$ and $E_A^{-1} = R_>|_{H_A^{1/2}(\mathbb{R};H)}$. For $v \in H_{00}^{1/2}(\mathbb{R}_+;H)$ and $w \in H^{1/2}(\mathbb{R}_+;H)$, it holds $\|(E_A+E_S)v\|^2_{H^{1/2}(\mathbb{R};H)} = \|E_A\|^2_{H^{1/2}(\mathbb{R};H)} + \|E_S v\|^2_{H^{1/2}(\mathbb{R};H)}$ and

$$\sqrt{2}\,\|w\|_{H^{1/2}(\mathbb{R}_+;H)} \leqslant \quad \|E_S w\|_{H^{1/2}(\mathbb{R};H)} \quad \leqslant 2\,\|w\|_{H^{1/2}(\mathbb{R}_+;H)}, \tag{2.13a}$$

$$\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)} \leqslant \quad \|E_A v\|_{H^{1/2}(\mathbb{R};H)} \quad \leqslant 2\sqrt{2}\,\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)}, \tag{2.13b}$$

$$2\,\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)} \leqslant \|(E_A+E_S)v\|_{H^{1/2}(\mathbb{R};H)} \leqslant 2\sqrt{2}\,\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)}. \tag{2.13c}$$

LEMMA 2.4 For $v \in H_{00}^{1/2}(\mathbb{R}_+;H)$ and $w \in H^{1/2}(\mathbb{R}_+;H)$, it holds

$$\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_A v, D^{1/2}_{\mathbb{R},-} E_S w\right)_H dt = 2\int_{\mathbb{R}_+} \left(D^{1/2}_{+} v, D^{1/2}_{-} w\right)_H dt, \tag{2.14a}$$

$$\int_{\mathbb{R}} (E_S v, E_S w)_H \, dt = 2\int_{\mathbb{R}_+} (v,w)_H \, dt, \tag{2.14b}$$

$$\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_A v, D^{1/2}_{\mathbb{R},-} E_S v\right)_H dt = \int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_S v, D^{1/2}_{\mathbb{R},-} E_S w\right)_H dt = \int_{\mathbb{R}} (E_A v, E_S w)_H \, dt = 0. \tag{2.14c}$$

Moreover, if $w \in H_{00}^{1/2}(\mathbb{R}_+;H)$, it holds

$$\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_A v, D^{1/2}_{\mathbb{R},-} E_S w\right)_H dt = \int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_S v, D^{1/2}_{\mathbb{R},-} E_A w\right)_H dt. \tag{2.15}$$

Using Lemma 2.4, we have for every $v \in \mathscr{X}$ and $w \in \mathscr{Y}$

$$\begin{aligned}\mathfrak{b}_{\mathbb{R}}\left((E_A+E_S)\,v, E_S w\right) &= \int_{\mathbb{R}} \left(D_{\mathbb{R},+} E_A v, D_{\mathbb{R},-} E_S w\right)_H dt + \int_{\mathbb{R}} \left(D_{\mathbb{R},+} E_S v, D_{\mathbb{R},-} E_S w\right)_H dt \\ &\quad + \int_{\mathbb{R}} \mathfrak{a}\,(E_A v, E_S w)\, dt + \int_{\mathbb{R}} \mathfrak{a}\,(E_S v, E_S w)\, dt = 2\mathfrak{b}(v,w).\end{aligned}$$

Let us then define for $\eta \in \mathbb{R}$ the operator $\mathscr{H}^\eta : \mathscr{X} \to \mathscr{Y}$ as

$$\mathscr{H}^\eta := \frac{1}{2} R_> \left(\cos(\pi\eta)(E_A+E_S) + \sin(\pi\eta)\mathfrak{H}(E_A+E_S)\right) = R_> \mathscr{H}^\eta E_0.$$

Due to $R_> \in \mathscr{L}_{\mathrm{Iso}}(H_S^{1/2}(\mathbb{R};H), H^{1/2}(\mathbb{R}_+;H)) \cap \mathscr{L}_{\mathrm{Iso}}(H_A^{1/2}(\mathbb{R};H), H_{00}^{1/2}(\mathbb{R}_+;H))$ and the embeddings $H_S^{1/2}(\mathbb{R};H), H_A^{1/2}(\mathbb{R};H) \subset H^{1/2}(\mathbb{R};H)$, we have $R_> \in \mathscr{L}(H^{1/2}(\mathbb{R};H), H^{1/2}(\mathbb{R}_+;H))$ so that indeed $\mathscr{H}^\eta$ maps to $\mathscr{Y}$.

THEOREM 2.3 Assume that (2.10a)-(2.10b) hold. Then (2.11) admits a unique solution. Furthermore for all $\eta \in \left(0, \frac{1}{2}\right)$, there exists $\beta(\eta) > 0$ such that for any $v \in \mathscr{X}$ it holds

$$\mathfrak{b}(v, \mathscr{H}^{-\eta} v) \geqslant \beta(\eta)\, \|v\|_{\mathscr{X}} \left\|\mathscr{H}^{-\eta} v\right\|_{\mathscr{Y}}. \tag{2.16}$$

Proof. For $v \in \mathscr{X}$ and $\eta \in \left(0, \frac{1}{2}\right)$, let $w = \mathscr{H}^{-\eta} v \in \mathscr{Y}$. Since $R_> \in \mathscr{L}_{\mathrm{Iso}}(H_S^{1/2}(\mathbb{R};H), H^{1/2}(\mathbb{R}_+;H))$ with $R_>^{-1} = E_S$ and $\cos(\pi\eta) E_S v, \sin(\pi\eta)\mathfrak{H} E_A v \in H_S^{1/2}(\mathbb{R};H)$, it follows that

$$E_S w = \cos(\pi\eta) E_S v - \frac{\sin(\pi\eta)}{2} \mathfrak{H} E_A v - \frac{\sin(\pi\eta)}{2} E_S R_> \mathfrak{H} E_S v.$$

Then from (2.4), (2.12), (2.13c), (2.14a)-(2.14c) and (2.15), we have

$$\begin{aligned}
4\mathfrak{b}(v,w) &= 2\mathfrak{b}_{\mathbb{R}}((E_S+E_A)v,E_Sw) = 2\int_{\mathbb{R}}\left[\left(D_{\mathbb{R},+}^{1/2}E_Av, D_{\mathbb{R},-}^{1/2}E_Sw\right)_H + \mathfrak{a}(E_Sv,E_Sw)\right]dt \\
&= 2\cos(\pi\eta)\int_{\mathbb{R}}\left(D_{\mathbb{R},+}^{1/2}E_Av, D_{\mathbb{R},-}^{1/2}E_Sv\right)_H dt - \sin(\pi\eta)\int_{\mathbb{R}}\left(D_{\mathbb{R},+}^{1/2}E_Av, D_{\mathbb{R},-}^{1/2}\mathfrak{H}E_Av\right)_H dt \\
&- \sin(\pi\eta)\int_{\mathbb{R}}\left(D_{\mathbb{R},+}^{1/2}E_Av, D_{\mathbb{R},-}^{1/2}E_SR_>\mathfrak{H}E_Sv\right)_H dt + 2\cos(\pi\eta)\int_{\mathbb{R}}\mathfrak{a}(E_Sv,E_Sv)dt \\
&- \sin(\pi\eta)\int_{\mathbb{R}}\mathfrak{a}(E_Sv,\mathfrak{H}E_Av)dt - \sin(\pi\eta)\int_{\mathbb{R}}\mathfrak{a}(E_Sv,E_SR_>\mathfrak{H}E_Sv)dt \\
&= -\sin(\pi\eta)\int_{\mathbb{R}}\left(D_{\mathbb{R},+}^{1/2}E_Av, D_{\mathbb{R},-}^{1/2}\mathfrak{H}E_Av\right)_H dt - \sin(\pi\eta)\int_{\mathbb{R}}\left(D_{\mathbb{R},+}^{1/2}E_Sv, D_{\mathbb{R},-}^{1/2}\mathfrak{H}E_Sv\right)_H dt \\
&+ 2\cos(\pi\eta)\int_{\mathbb{R}}\mathfrak{a}(E_Sv,E_Sv)dt - \sin(\pi\eta)\int_{\mathbb{R}}\mathfrak{a}(E_Sv,\mathfrak{H}E_Av)dt - \sin(\pi\eta)\int_{\mathbb{R}}\mathfrak{a}(E_Av,\mathfrak{H}E_Sv)dt \\
&= \sin(\pi\eta)\left(\left\|D_{\mathbb{R},+}^{1/2}E_Sv\right\|_{L^2(\mathbb{R};H)}^2 + \left\|D_{\mathbb{R},+}^{1/2}E_Av\right\|_{L^2(\mathbb{R};H)}^2\right) \\
&+ \cos(\pi\eta)\left(\int_{\mathbb{R}}\mathfrak{a}(E_Sv,E_Sv)dt + \int_{\mathbb{R}}\mathfrak{a}(E_Av,E_Av)dt\right) \\
&= \mathfrak{b}(E_Av,\mathscr{H}_{\mathbb{R}}^{-\eta}E_Av) + \mathfrak{b}(E_Sv,\mathscr{H}_{\mathbb{R}}^{-\eta}E_Sv) \geqslant \alpha_{\mathbb{R}}(\eta)\left(\|E_Av\|_{\mathscr{X}_{\mathbb{R}}}^2 + \|E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2\right).
\end{aligned}$$

Note that for $\eta \in \mathbb{R}$, it holds from (2.13a)-(2.13c) and Lemmas 2.2 and 2.3 that

$$\begin{aligned}
4\|v\|_{\mathscr{X}}^2 &\leqslant \|(E_A+E_S)v\|_{\mathscr{X}_{\mathbb{R}}}^2 = \|E_Av\|_{\mathscr{X}_{\mathbb{R}}}^2 + \|E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2, \\
2\|w\|_{\mathscr{Y}}^2 &\leqslant \|E_Sw\|_{\mathscr{X}_{\mathbb{R}}}^2 = \left\|\cos(\pi\eta)E_Sv - \frac{\sin(\pi\eta)}{2}\mathfrak{H}E_Av - \frac{\sin(\pi\eta)}{2}E_SR_>\mathfrak{H}E_Sv\right\|_{\mathscr{X}_{\mathbb{R}}}^2 \\
&\leqslant 2\cos(\pi\eta)^2\|E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2 + \sin(\pi\eta)^2\|\mathfrak{H}E_Av\|_{\mathscr{X}_{\mathbb{R}}}^2 + \sin(\pi\eta)^2\|E_SR_>\mathfrak{H}E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2 \\
&\leqslant \cos(\pi\eta)^2\|E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2 + \cos(\pi\eta)^2\|E_Av\|_{\mathscr{X}_{\mathbb{R}}}^2 + \sin(\pi\eta)^2\|E_Av\|_{\mathscr{X}_{\mathbb{R}}}^2 + 4\sin(\pi\eta)^2\|E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2 \\
&\leqslant 4\left(\|E_Av\|_{\mathscr{X}_{\mathbb{R}}}^2 + \|E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2\right),
\end{aligned}$$

where we have used $\|E_SR_>\mathfrak{H}E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2 \leqslant 4\|R_>\mathfrak{H}E_Sv\|_{\mathscr{X}}^2 \leqslant 4\|E_AR_>\mathfrak{H}E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2 = 4\|\mathfrak{H}E_Sv\|_{\mathscr{X}_{\mathbb{R}}}^2$, which can be obtained from (2.13a)-(2.13b) and $E_AR_>\mathfrak{H}E_Sv = \mathfrak{H}E_Sv$. It yields

$$\mathfrak{b}(v,w) \geqslant \frac{\alpha_{\mathbb{R}}(\eta)}{4}\|(E_A+E_S)v\|_{\mathscr{X}_{\mathbb{R}}}^2 \geqslant \frac{\alpha_{\mathbb{R}}(\eta)}{2\sqrt{2}}\|v\|_{\mathscr{X}}\|w\|_{\mathscr{Y}} =: \beta(\eta)\|v\|_{\mathscr{X}}\|w\|_{\mathscr{Y}},$$

which is (2.16).

From (Babuška, 1971, Theorem 2.1), we need to show that for all $w \in \mathscr{Y}$, there exists $v \in \mathscr{X}$ such that $\mathfrak{b}(v,w) \neq 0$ in order to obtain that (2.11) admits a unique solution. Let $0 \neq w \in \mathscr{Y}$ and $(w_n)_{n\in\mathbb{N}} \subset \mathscr{F}(\mathbb{R}_+;V)$ such that $w_n \to w$ in $\mathscr{Y}$ as $n \to \infty$. For $\varepsilon > 0$, let further $\chi_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}_+)$ be such that $\chi_\varepsilon' \geqslant 0$ and

$$\chi_\varepsilon(t) = \begin{cases} 1 & t > \varepsilon, \\ 0 & t < \frac{\varepsilon}{2}. \end{cases}$$

Then $\chi_\varepsilon w \in \mathscr{X}$, $(\chi_\varepsilon w_n)_{n\in\mathbb{N}} \subset \mathscr{F}_0(\mathbb{R}_+;V)$, $\chi_\varepsilon w_n \to \chi_\varepsilon w$ in $\mathscr{X}$ as $n\to\infty$ and we have by (2.8)

$$\int_{\mathbb{R}_+} \left(D_+^{1/2}\chi_\varepsilon w_n, D_-^{1/2} w_n\right)_H dt = \int_{\mathbb{R}_+} (\chi_\varepsilon D w_n, w_n)_H\, dt + \int_{\mathbb{R}_+} \chi_\varepsilon' \|w_n\|_H^2\, dt$$
$$= -\int_{\mathbb{R}_+} \left(D_+^{1/2}\chi_\varepsilon w_n, D_-^{1/2} w_n\right)_H dt + \int_{\mathbb{R}_+} \chi_\varepsilon' \|w_n\|_H^2\, dt.$$

Taking the limit as $n\to\infty$ yields

$$\int_{\mathbb{R}_+} \left(D_+^{1/2}\chi_\varepsilon w, D_-^{1/2} w\right)_H dt = \frac{1}{2}\int_{\mathbb{R}_+} \chi_\varepsilon' \|w\|_H^2\, dt,$$

so that

$$\mathfrak{b}(\chi_\varepsilon w, w) \geqslant \frac{1}{2}\int_{\mathbb{R}_+} \chi_\varepsilon' \|w\|_H^2\, dt + \lambda_- \int_{\mathbb{R}_+} \chi_\varepsilon \|w\|_V^2\, dt \geqslant \lambda_- \int_{\mathbb{R}_+} \chi_\varepsilon \|w\|_V^2\, dt.$$

Since $0 \neq w$, there exists $\varepsilon > 0$ small enough such that $\mathfrak{b}(\chi_\varepsilon w, w) > 0$. □

We point out that (2.16) is a stronger condition than the usual inf-sup inequality. It provides us with an explicit way of defining inf-sup stable pairs of discrete spaces. Indeed, considering a discrete trial space, an associated test space can be chosen as the image of the trial space by $\mathscr{H}^{-\eta}$ for some $\eta \in \left(0, \frac{1}{2}\right)$. This is explained in Section 4.

REMARK 2.1 In (Steinbach & Zank, 2018, Sections 2 and 3), the authors define an operator similar to $\mathfrak{H}$ in order to deal with parabolic equations on a finite time interval. They obtain stability of the space-time formulation of the heat equation in Bochner-Sobolev spaces of fractional order together with convergence rates for first order space-time Galerkin approximation. We point out that using the approximation results derived in Section 3 in the setting of Steinbach and Zank, *hp*-approximation results can be derived for parabolic equations considered over finite time interval. Furthermore, the stiffness and mass matrices in this case are the same as the one presented in (Devaud, 2017, Section 6.1).

2.4 *Regularity results*

We present a new regularity result for solutions of linear parabolic equations. Since it is not the main focus of this paper, its proof is omitted here but can be found in the online supplementary material. The usual regularity theory for parabolic equations is based on the assumption that the right-hand side is analytic with respect to its time variable (Schötzau, 1999). In that case, it can be shown that the solution inherits this regularity except for a potential algebraic singularity at the initial time due to an incompatibility between f and the initial condition. We extend this theory to Gevrey classes. Such regularity allows for instance to localize right-hand side functions since they can have compact supports, which is not the case of analytic functions. The technique used to prove this result is the same as in Schötzau (1999), namely it is based on the spectral decomposition of the operator $\mathscr{A}$.

DEFINITION 2.4 (Gevrey class) Let $I = (a,b) \subset \mathbb{R}$ be an interval for some $-\infty < a < b \leqslant \infty$. For $\delta \geqslant 1$ and $\theta \geqslant 0$, we say that v is of *Gevrey type* (δ,θ) and we write $v \in \mathscr{G}^{\delta,\theta}(I;H)$ if $v \in \mathscr{C}(\bar{I};H)$ and there exist $C_v, d_v > 0$ such that

$$\left\| v^{(l)}(t) \right\|_H \leqslant C_v d_v^l \Gamma(l+1)^\delta (t-a)^{\theta-l}, \qquad \forall l \in \mathbb{N},\ t\in I. \tag{2.17}$$

Moreover we say that v is of *Gevrey type* δ and write $v \in \mathscr{G}^{\delta}(I;H)$ if $v \in \mathscr{C}(\bar{I};H)$ and there exist $C_v, d_v > 0$ such that

$$\left\| v^{(l)}(t) \right\|_H \leqslant C_v d_v^l \Gamma(l+1)^{\delta}, \qquad \forall l \in \mathbb{N}_0,\ t \in \bar{I}.$$

If I is bounded, then $\mathscr{G}^{\delta,\bar{\theta}}(I;H) \subset \mathscr{G}^{\delta,\theta}(I;H)$ for all $0 \leqslant \theta \leqslant \bar{\theta} < \infty$. We point out that the bound (2.17) is assumed to hold only for $l \in \mathbb{N}$. Hence we control the derivatives of the function but do not enforce that $v(a) = 0$. This point is crucial since it allows to treat solutions of parabolic equations with non-homogeneous initial value. Condition (2.17) could be equivalently replaced by

$$\left\| v^{(l)}(t) \right\|_H \leqslant C_v d_v^l \Gamma(l+1)^{\delta} (t-a)^{\min(0,\theta-l)}, \qquad \forall l \in \mathbb{N}_0,\ t \in I.$$

PROPOSITION 2.5 Let $f \in L^2(\mathbb{R}_+;H)$ satisfy $\operatorname{supp}(f) \subset \bar{I} = [0,T]$ and $f \in \mathscr{G}^{\delta}(I;H)$ for some $0 < T < \infty$ and $\delta \geqslant 1$. Then for any $T < T_\infty \leqslant \infty$ we have for $I_\infty = (T, T_\infty)$

$$u|_I \in \mathscr{G}^{\delta,1}(I;H) \cap \mathscr{G}^{\delta,1/2}(I;V), \qquad u|_{I_\infty} \in \mathscr{G}^{1,1/2}(I_\infty;H) \cap \mathscr{G}^{1,0}(I_\infty;V),$$

where $u \in \mathscr{X}$ is the solution of (2.11) associated to f.

REMARK 2.2 The previous proposition implies that for every T_∞, there exist $C_u, d_u > 0$ such that for all $l \in \mathbb{N}$ it holds

$$\left\| u^{(l)}(t) \right\|_H \leqslant C_u d_u^l \Gamma(l+1)(t-T)^{1/2-l}, \qquad \left\| u^{(l)}(t) \right\|_V \leqslant C_u d_u^l \Gamma(l+1)(t-T)^{-l}, \qquad \forall t \in I_\infty.$$

It is important to point out that in this case neither C_u nor d_u depend on T_∞.

3. *hp*-approximation in $H^{1/2}(I;H)$

As we will see in Section 4, equation (2.16) allows us to build explicitly inf-sup stable pairs of discrete spaces. Using this approach yields quasi-optimality of the discrete solution of (2.11). Hence to obtain convergence rates for the discretization, it is sufficient to investigate approximation properties of the space $\mathscr{X}$. In particular, we define in the following an interpolation operator and derive associated error estimates in the $H^{1/2}_{00}(I;H)$-norm. In this section, $(H, (\cdot,\cdot)_H)$ denotes a generic Hilbert space.

3.1 *Piecewise polynomial approximation of functions in* $H^{1/2}(I;H)$

For a bounded interval $I = (a,b)$ and $M \in \mathbb{N}$, let $\mathscr{T} = \{I_n\}_{n=1}^M$ be a partition of I, where $I_n = (t_{n-1}, t_n)$ and $a = t_0 < t_1 < \cdots < t_M = b$. Furthermore we denote $h_n = t_n - t_{n-1}$ for $n = 1, \ldots, M$. Associated to $\mathscr{T}$, we consider a vector $\mathbf{p} = \{p_n\}_{n=1}^M \in \mathbb{N}^M$ of polynomial degrees and define for $l \in \mathbb{N}_0$

$$S^{\mathbf{p},l}(I,\mathscr{T};H) := \left\{ v \in H^l(I;H) \;\middle|\; v|_{I_n} \in \mathscr{P}^{p_n}(I_n;H),\ n = 1, \ldots, M \right\}, \tag{3.1}$$

where $\mathscr{P}^{p_n}(I_n;H)$ denotes the space of polynomials of degree at most p_n over I_n taking values in H. In order to analyze piecewise polynomial approximation of functions in $H^{1/2}(I;H)$, we define for $k_n \in \mathbb{N}_0$ and an interval I_n

$$|v|^2_{H^{k_n}_J(I_n;H)} := \int_{I_n} \left\| v^{(k_n)}(t) \right\|_H^2 (t - t_{n-1})^{k_n - 1/2} (t_n - t)^{k_n - 1/2} dt.$$

This quantity is a seminorm for a so-called Jacobi-weighted Sobolev space, whence the subscript J. When taken over the reference interval $\hat{I} := (-1,1)$, the weight in the integral is the one associated with the orthogonal Jacobi polynomials for a specific choice of parameters. This idea is strongly used in the error analysis and was originally introduced by (Babuška & Guo, 2001, 2002; Guo & Heuer, 2004). We then introduce an interpolation operator over $\hat{I}$ and using an affine mapping between every I_n and $\hat{I}$, it yields an operator $\pi_{I_n}^{p_n}$ such that $\left(\pi_{I_n}^{p_n} v\right)(t_{n-1}) = v(t_{n-1})$ and $\left(\pi_{I_n}^{p_n} v\right)(t_n) = v(t_n)$ for every $v \in \mathscr{C}(\overline{I_n};H)$ and $n = 1,\dots,M$. A global interpolation operator is then defined for $v \in \mathscr{C}(\bar{I};H)$ as $\pi_{\mathscr{T}}^{\mathbf{p}} v\big|_{I_n} = \pi_{I_n}^{p_n} v$. The proof of the following result follows essentially the same lines as what is presented in (Schwab, 1998, Section 3.3). There, the analysis is based on the orthogonal Legendre polynomials. The complete development yielding the following proposition can be found in (Devaud, 2017, Chapter 4).

PROPOSITION 3.1 Let I, $\mathscr{T}$ and $\mathbf{p} \in \mathbb{N}^M$ be as above. Let further $v \in \mathscr{C}\left(\bar{I};H\right)$ and assume that for every $n \in \{1,\dots,M\}$ there exists $k_n \in \mathbb{N}$ such that $p_n + 1 \geqslant k_n$ and $|v|_{H_J^{k_n}(I_n;H)} < \infty$. Then it holds

$$\left\|v - \pi_{\mathscr{T}}^{\mathbf{p}} v\right\|^2_{H_{00}^{1/2}(I;H)} \leqslant C \sum_{n=1}^{M} \frac{\Gamma(p_n - k_n + 2)}{\Gamma(p_n + k_n + 1)} |v|^2_{H_J^{k_n}(I_n;H)},$$

$$\left\|v - \pi_{\mathscr{T}}^{\mathbf{p}} v\right\|^2_{L^2(I;H)} \leqslant C \sum_{n=1}^{M} \left(\frac{h_n}{2}\right) \frac{\Gamma(p_n - k_n + 2)\Gamma(p_n+1)}{\Gamma(p_n + k_n + 1)\Gamma(p_n+2)} |v|^2_{H_J^{k_n}(I_n;H)},$$

for a constant $C > 0$. Moreover $v(t_n) = \left(\pi_{\mathscr{T}}^{\mathbf{p}} v\right)(t_n)$ for all $n = 0,\dots,M$.

REMARK 3.1 Let us consider the setting of Proposition 3.1 and assume further that for every $n \in \{2,\dots,M\}$ there exists $k_n \in \mathbb{N}$ such that $p_n \geqslant k_n$ and $v|_{I_n} \in H^{k_n+1}(I_n;H)$. Then using

$$\max_{t \in I_n} (t - t_{n-1})^{k_n - 1/2} (t_n - t)^{k_n - 1/2} = \left(\frac{h_n}{2}\right)^{2k_n - 1},$$

we obtain

$$\left\|u - \pi_{\mathscr{T}}^{\mathbf{p}} u\right\|^2_{H_{00}^{1/2}(I;H)} \leqslant C \left(|u|^2_{H_J^1(I_1;H)} + \sum_{n=2}^{M} \left(\frac{h_n}{2}\right)^{2k_n+1} \frac{\Gamma(p_n - k_n + 1)}{\Gamma(p_n + k_n + 2)} |u|^2_{H^{k_n+1}(I_n;H)} \right),$$

$$\left\|u - \pi_{\mathscr{T}}^{\mathbf{p}} u\right\|^2_{L^2(I;H)} \leqslant C \left(\frac{h_1}{2} |u|^2_{H_J^1(I_1;H)} + \sum_{n=2}^{M} \left(\frac{h_n}{2}\right)^{2k_n+2} \frac{\Gamma(p_n - k_n + 1)\Gamma(p_n+1)}{\Gamma(p_n + k_n + 2)\Gamma(p_n+2)} |u|^2_{H^{k_n+1}(I_n;H)} \right).$$

This estimate is useful if u contains a singularity at the endpoint a so that $u|_{I_1} \notin H^1(I_1;H)$. In particular, it allows us to derive high-order convergence for the approximation of the solution of (2.11). As discussed in Section 2.4, assuming that the right-hand side is smooth in time, then the associated solution will also be smooth except for a potential algebraic singularity at the initial time due to an incompatibility between f and the initial condition. The weighted seminorm allows us then to treat this singularity.

REMARK 3.2 Using the bound (Olver *et al.*, 2010, (5.6.1.)) on the Gamma function, if $v \in H^k(I;H)$ for some $k \in \mathbb{N}$ and choosing $h_n = h$ and $p_n = p \geqslant k-1$ for all $n \in \{1,\dots,M\}$, we obtain that there exists $C > 0$ such that

$$\left\|v - \pi_{\mathscr{T}}^{\mathbf{p}} v\right\|_{H_{00}^{1/2}(I;H)} \leqslant C \left(\frac{he}{2p}\right)^{k-1/2} |v|_{H^k(I;H)}, \qquad \left\|v - \pi_{\mathscr{T}}^{\mathbf{p}} v\right\|_{L^2(I;H)} \leqslant C \left(\frac{he}{2p}\right)^{k} |v|_{H^k(I;H)}.$$

This allows us to derive convergence rates for h- and p-refinements in time; see Remark 4.1.

3.2 *Exponential convergence of Gevrey functions*

As Proposition 2.5 suggests, solutions of parabolic equations typically contain an algebraic singularity at the initial time. To circumvent the lack of regularity of functions in $\mathscr{G}^{\delta,\theta}(I;H)$, a graded mesh towards the singularity is considered. More specifically, for a grading factor $\sigma \in (0,1)$ and $M \in \mathbb{N}$, we define

$$t_{\sigma,0} = a, \qquad t_{\sigma,m} = a + (b-a)\sigma^{M-m}, \qquad m = 1,\dots,M, \tag{3.2}$$

and $\mathscr{T}_\sigma := \{I_{\sigma,m}\}_{m=1}^{M}$, where $I_{\sigma,m} := (t_{\sigma,m-1}, t_{\sigma,m})$ for $m = 1,\dots,M$. It yields that for $\kappa = \sigma^{-1} - 1$

$$h_{\sigma,m} = t_{\sigma,m} - t_{\sigma,m-1} = \kappa(b-a)\sigma^{M-m+1}. \tag{3.3}$$

Furthermore, given $\mu > 0$ and $\delta \geqslant 1$, we consider the vector of polynomial degrees $\mathbf{p}_{\mu,\delta} := \left\{p_{\mu,\delta,m}\right\}_{m=1}^{M}$ defined through

$$p_{\mu,\delta,1} = 1, \qquad p_{\mu,\delta,m} = \max\left(1, \left\lfloor \mu m^\delta \right\rfloor\right), \qquad m = 2,\dots,M. \tag{3.4}$$

We then denote $N = \dim\left(S^{\mathbf{p}_{\mu,\delta},1}(I,\mathscr{T}_\sigma)\right)$, where $S^{\mathbf{p}_{\mu,\delta},1}(I,\mathscr{T}_\sigma)$ is defined in (3.1). Since $\mathscr{G}^{\delta,\theta}(I;H) \subset \mathscr{C}\left(\bar{I};H\right)$, the interpolation operator $\pi_{\mathscr{T}_\sigma}^{\mathbf{p}_{\mu,\delta}}$ is well-defined for functions in this space. In order to simplify the notations, we write $\pi_{\mu,\sigma,\delta} = \pi_{\mathscr{T}_\sigma}^{\mathbf{p}_{\mu,\delta}}$. We will need the following lemma in order to prove exponential convergence of the *hp*-approximation of Gevrey functions.

LEMMA 3.1 Let $\alpha > 0$. Then there exists $\mu_0 \geqslant 1$ such that for any $M \in \mathbb{N}$ and defining $p_n = \left\lfloor \mu n^\delta \right\rfloor$ for some $\mu > \mu_0$ and $n = 1,\dots,M$, it holds

$$\sum_{n=1}^{M} \alpha^{2n} \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \Gamma(n+2)^{2\delta} \leqslant C,$$

for a constant $C = C(\alpha,\mu,\delta) > 0$ independent of M.

Proof. Using the bound (Olver *et al.*, 2010, (5.6.1.)) on the Gamma function, it is possible to show that

$$\sqrt{2\pi} s^{s+a-\frac{1}{2}} e^{-(s+a)} C_{L,\Gamma}(a) \leqslant \Gamma(s+a) \leqslant \sqrt{2\pi} s^{s+a-\frac{1}{2}} e^{-(s+a)} C_{U,\Gamma}(a), \qquad \forall s \geqslant 1-a.$$

where

$$C_{L,\Gamma}(a) := \begin{cases} 1 & a \geqslant \frac{1}{2}, \\ (1+a)^{a-1/2} & 0 \leqslant a < \frac{1}{2}, \\ e^a & a < 0, \end{cases} \qquad C_{U,\Gamma}(a) := \begin{cases} e^{a+1}(1+a)^{a-1/2} & a \geqslant \frac{1}{2}, \\ e^{a+1} & 0 \leqslant a < \frac{1}{2}, \\ e & a < 0. \end{cases}$$

It follows

$$\Gamma(n+2) < \sqrt{2\pi} e 3^{3/2} n^{n+3/2} e^{-n}, \qquad \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \leqslant \frac{\Gamma(\mu n^\delta - n + 1)}{\Gamma(\mu n^\delta + n + 1)} \leqslant e \left(\frac{\mu n^\delta}{e}\right)^{-2n}.$$

Hence there exists $C > 0$ such that for all $n = 1,\dots,M$

$$\alpha^{2n} \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \Gamma(n+2)^{2\delta} \leqslant C n^{3\delta} \left(\frac{\alpha}{\mu e^{\delta-1}}\right)^{2n}.$$

Defining $\mu_0 = \max\left(1, \frac{\alpha}{e^{\delta-1}}\right)$, it follows that for any $\mu > \mu_0$, we have $\frac{\alpha}{\mu e^{\delta-1}} =: \zeta \in (0,1)$. Hence we get

$$\sum_{n=1}^{M} \alpha^{2n} \frac{\Gamma(p_n - n + 1)}{\Gamma(p_n + n + 2)} \Gamma(n+2)^{2\delta} \leqslant C \sum_{n=1}^{M} n^{3\delta} \zeta^n \leqslant C \sum_{n=1}^{\infty} n^{3\delta} \zeta^n,$$

and it can be shown using for instance a ratio test that the last series converges. □

PROPOSITION 3.2 Let $u \in \mathscr{G}^{\delta,\theta}(I;H)$ for some $\theta \geqslant 0$, $\delta \geqslant 1$ and assume that the constants C_u and d_u in (2.17) are independent of $|I|$. For $\sigma \in (0,1)$ and $M \in \mathbb{N}$, define $\mathscr{T}_\sigma$ through (3.2). Then there exists $\mu_0 \geqslant 1$ such that defining $\mathbf{p}_{\mu,\delta}$ through (3.4) for any $\mu > \mu_0$, it holds

$$\left\| u - \pi_{\mu,\sigma,\delta} u \right\|_{L^2(I;H)} \leqslant C |I|^{1/2} \max\left(|I|^\theta, 1\right) \exp\left(-\tau N^{\frac{1}{\delta+1}}\right), \tag{3.5}$$

for some constants $C, \tau > 0$ independent of $|I|$. Furthermore if $\theta > 1/4$, it holds

$$\left\| u - \pi_{\mu,\sigma,\delta} u \right\|_{H_{00}^{1/2}(I;H)} \leqslant C |I|^\theta \exp\left(-\tau N^{\frac{1}{\delta+1}}\right). \tag{3.6}$$

Proof. We first prove (3.6). On $I_{\sigma,1}$, from the definition of $|\cdot|_{H_J^1(I_{\sigma,1};H)}$ and $u \in \mathscr{G}^{\delta,\theta}(I;H)$, we have that for $\theta > \frac{1}{4}$ it holds

$$\begin{aligned}
|u|^2_{H_J^1(I_{\sigma,1};H)} &\leqslant C_u^2 d_u^2 \int_a^{t_1} (t-a)^{2(\theta-1)+1/2} (t_1 - t)^{1/2} dt \\
&= C_u^2 d_u^2 \left(\frac{t_1 - a}{2}\right)^{2\theta} \int_{-1}^{1} (t+1)^{2(\theta-1)+1/2} (1-t)^{1/2} dt \\
&= C\, (t_1 - a)^{2\theta} = C(b-a)^{2\theta} \sigma^{2\theta(M-1)},
\end{aligned}$$

for a constant $C = C(C_u, d_u, \theta) > 0$. Furthermore, it holds

$$\begin{aligned}
|u|^2_{H^l(I_{\sigma,m};H)} &\leqslant C_u^2 d_u^{2l} \Gamma(l+1)^{2\delta} \int_{t_{\sigma,m-1}}^{t_{\sigma,m}} (t-a)^{2(\theta-l)} dt \\
&= C_u^2 d_u^{2l} \Gamma(l+1)^{2\delta} (t_{\sigma,m-1} - a)^{2(\theta-l)+1} \left(\frac{\sigma^{2(l-\theta)-1} - 1}{2(\theta - l) + 1}\right),
\end{aligned}$$

and it can be shown that the term in brackets is decreasing in $l \in \mathbb{N}_0$, so that

$$\begin{aligned}
|u|^2_{H^l(I_{\sigma,m};H)} &\leqslant C_u^2 \left(\frac{\sigma^{-(2\theta+1)}}{2\theta+1}\right) d_u^{2l} \Gamma(l+1)^{2\delta} (t_{\sigma,m-1} - a)^{2(\theta-l)+1} \\
&= C(\theta,\sigma)(b-a)^{2\theta+1} \left(\frac{d_u}{b-a}\right)^{2l} \Gamma(l+1)^{2\delta} \sigma^{(M-m+1)(2(\theta-l)+1)},
\end{aligned}$$

where we have used $t_{\sigma,m-1} - a = (b-a)\sigma^{M-m+1}$. From (3.3), we then have

$$\left(\frac{h_{\sigma,n}}{2}\right)^{2k_n+1} |u|^2_{H^{k_n+1}(I_{\sigma,n};H)} \leqslant C(\theta,\sigma) d_u (b-a)^{2\theta} \left(\frac{\kappa d_u}{2}\right)^{2k_n+1} \Gamma(k_n+2)^{2\delta} \sigma^{2\theta(M-n+1)},$$

so that Remark 3.1 yields

$$\left\|u-\pi_{\mu,\sigma,\delta}u\right\|^2_{H^{1/2}_{00}(I;H)} \leqslant C(b-a)^{2\theta}\sigma^{2\theta M}\left(1+\sum_{n=2}^{M}\left(\frac{\kappa d_u}{2}\right)^{2k_n}\frac{\Gamma(p_n-k_n+1)}{\Gamma(p_n+k_n+2)}\Gamma(k_n+2)^{2\delta}\sigma^{-2\theta n}\right).$$

We then choose $k_n = n$ in Lemma 3.1 with $\alpha = \frac{\kappa d_u}{2\sigma^\theta}$. Hence there exists $C(\theta,\sigma,\delta,\mu) > 0$ satisfying

$$\left\|u-\pi_{\mu,\sigma,\delta}u\right\|^2_{H^{1/2}_{00}(I;H)} \leqslant C(b-a)^{2\theta}\sigma^{2\theta M} = C(b-a)^{2\theta}\exp\left(-2\theta\left|\log(\sigma)\right|M\right).$$

Since

$$N = 1+\sum_{n=2}^{M}\left\lfloor \mu n^\delta\right\rfloor \leqslant \mu\left(\sum_{n=1}^{M} n^\delta\right) \leqslant \mu M^{\delta+1},$$

the bound (3.6) follows with $\tau = \theta\left|\log(\sigma)\right|\mu^{-\frac{1}{\delta+1}}$ since $|I| = b-a$.

Let us then turn to (3.5). On $I_{\sigma,1}$, we have by definition $\pi_{\mu,\sigma,\delta}u \in \mathscr{P}^1(I_{\sigma,1};H)$, so that

$$\left(\pi_{\mu,\sigma,\delta}u\right)(t) = \frac{1}{t_{\sigma,1}-a}\left(u(t_{\sigma,1})(t-a)+u(a)(t_{\sigma,1}-t)\right), \qquad t\in I_{\sigma,1}.$$

Hence we obtain

$$\left\|u-\pi_{\mu,\sigma,\delta}u\right\|^2_{L^2(I_{\sigma,1};H)} \leqslant C(t_{\sigma,1}-a) = C(b-a)\sigma^{M-1},$$

where we have used $\|u(t)\|_H \leqslant C$ for all $t\in I_{\sigma,1}$ since by assumption $u\in\mathscr{C}\left(\bar{I};H\right)$. Based on this, the proof of (3.5) follows then the same lines as that of (3.6) except that the estimation on $I_{\sigma,1}$ has been replaced by the above inequality. □

4. Space-time Petrov-Galerkin approximation

In this section, we introduce a space-time discretization for the solution of (2.11) and derive associated convergence rates. Let $\left(\Theta^l_{\mathscr{X}}\right)_{l\in\mathbb{N}} \subset H^{1/2}_{00}(\mathbb{R}_+)$ and $\left(\Sigma^m_V\right)_{m\in\mathbb{N}} \subset V$ be two sequences of finite dimensional spaces. Based on this, we define for $l\in\mathbb{N}$ and $\eta\in\left(0,\frac{1}{2}\right)$

$$\mathscr{X}^l := \Theta^l_{\mathscr{X}}\otimes\Sigma^l_V, \qquad \mathscr{Y}^l := \Theta^l_{\mathscr{Y}}\otimes\Sigma^l_V,$$

where $\Theta^l_{\mathscr{Y}} := \mathscr{H}^{-\eta}\left(\Theta^l_{\mathscr{X}}\right) = \left\{\mathscr{H}^{-\eta}\left(v^l\right) \mid v^l\in\Theta^l_{\mathscr{X}}\right\}$. If the sequences $\left(\Theta^l_{\mathscr{X}}\right)_{l\in\mathbb{N}} \subset H^{1/2}_{00}(\mathbb{R}_+)$ and $\left(\Sigma^m_V\right)_{m\in\mathbb{N}} \subset V$ are dense, then $\left(\mathscr{X}^l\right)_{l\in\mathbb{N}}$ is dense in $\mathscr{X}$ (Weidmann, 1980, Theorem 3.12). Furthermore, we have the following result which is a direct consequence of Theorem 2.3. We point out that the choice of η is arbitrary and only influences the inf-sup constant associated to the bilinear form $\mathfrak{b}$.

PROPOSITION 4.1 Assume that (2.10a)-(2.10b) hold and let $\mathscr{X}^l$ be as above, $\eta\in\left(0,\frac{1}{2}\right)$ and $\Theta^l_{\mathscr{Y}} := \mathscr{H}^{-\eta}\left(\Theta^l_{\mathscr{X}}\right)$. Then for any $f\in\mathscr{Y}^*$ and $l\in\mathbb{N}$, there exists a unique $u^l\in\mathscr{X}^l$ such that

$$\mathfrak{b}(u^l,v^l) = \langle f,v^l\rangle_{\mathscr{Y}^*}, \qquad \forall v^l\in\mathscr{Y}^l. \tag{4.1}$$

Moreover, there exists $C = C(\eta) > 0$ such that

$$\left\|u-u^l\right\|_{\mathscr{X}} \leqslant C\inf_{w^l\in\mathscr{X}^l}\left\|u-w^l\right\|_{\mathscr{X}}, \tag{4.2}$$

where $u\in\mathscr{X}$ is the unique solution of (2.11) associated to f.

Proof. We already know that the bilinear form $\mathfrak{b} : \mathscr{X}^l \times \mathscr{Y}^l \to \mathbb{R}$ is continuous. Furthermore, (2.16) and the construction of $\mathscr{Y}^l$ yield that

$$\inf_{0\neq v^l\in\mathscr{X}^l} \sup_{0\neq w^l\in\mathscr{Y}^l} \frac{\mathfrak{b}\left(v^l,w^l\right)}{\|v^l\|_{\mathscr{X}}\|w^l\|_{\mathscr{Y}}} \geqslant \inf_{0\neq v^l\in\mathscr{X}^l} \frac{\mathfrak{b}\left(v^l,\mathscr{H}^{-\eta}\left(v^l\right)\right)}{\|v^l\|_{\mathscr{X}}\|\mathscr{H}^{-\eta}\left(v^l\right)\|_{\mathscr{Y}}} \geqslant \beta(\eta). \tag{4.3}$$

This condition implies that the operator associated to $\mathfrak{b}(\cdot,\cdot)$ defined on the discrete spaces is injective. Since $\dim\left(\mathscr{X}^l\right) = \dim\left(\mathscr{Y}^l\right) < \infty$, we obtain that it is surjective which is equivalent to say that it is non-degenerate. Hence it follows from (Braess, 2007, Theorem 3.6) that (4.1) admits a unique solution for any $f \in \left(\mathscr{Y}^l\right)^*$ and in particular for any $f \in \mathscr{Y}^*$.

It remains to show (4.2). Note that we have the Galerkin orthogonality $\mathfrak{b}\left(u-u^l,v^l\right) = 0$ for all $v^l \in \mathscr{Y}^l$ since $\mathscr{Y}^l \subset \mathscr{Y}$. It follows from (4.3) that for any $w^l \in \mathscr{X}^l$, there exists $v^l \in \mathscr{Y}^l$ such that

$$\beta(\eta)\left\|u^l-w^l\right\|_{\mathscr{X}}\left\|v^l\right\|_{\mathscr{Y}} \leqslant \mathfrak{b}\left(u^l-w^l,v^l\right) = \mathfrak{b}\left(u-w^l,v^l\right) \leqslant \|\mathfrak{b}\|_{\mathscr{B}(\mathscr{X},\mathscr{Y})}\left\|u-w^l\right\|_{\mathscr{X}}\left\|v^l\right\|_{\mathscr{Y}}.$$

Hence, for all $w^l \in \mathscr{X}^l$ it holds

$$\left\|u-u^l\right\|_{\mathscr{X}} \leqslant \left\|u-w^l\right\|_{\mathscr{X}} + \left\|u^l-w^l\right\|_{\mathscr{X}} \leqslant \left(1+\frac{\|\mathfrak{b}\|_{\mathscr{B}(\mathscr{X},\mathscr{Y})}}{\beta(\eta)}\right)\left\|u-w^l\right\|_{\mathscr{X}}.$$

□

Using the tensor structure of $\mathscr{X}$, if we can define two operators $\Pi_{\Theta^l_{\mathscr{X}}} \in \mathscr{L}(L^2(\mathbb{R}_+),\Theta^l_{\mathscr{X}})$ and $\Pi_{\Sigma^l_V} \in \mathscr{L}(H,\Sigma^l_V)$ with bounds uniform in l and m, then we can estimate the error bound of the full operator $\Pi^l_{\mathscr{X}} := \Pi_{\Theta^l_{\mathscr{X}}} \otimes \Pi_{\Sigma^l_V} : \mathscr{X} \to \mathscr{X}^l$ from the relation

$$(\mathrm{Id}\otimes\mathrm{Id}) - (\Pi_{\Theta^l_{\mathscr{X}}} \otimes \Pi_{\Sigma^l_V}) = \left(\left(\mathrm{Id}-\Pi_{\Theta^l_{\mathscr{X}}}\right)\otimes\Pi_{\Sigma^l_V}\right) + \left(\mathrm{Id}\otimes\left(\mathrm{Id}-\Pi_{\Sigma^l_V}\right)\right).$$

Indeed, using a triangular inequality and the continuity of $\Pi_{\Theta^l_{\mathscr{X}}}$ and $\Pi_{\Sigma^l_V}$, we obtain

$$\left\|u-\Pi^l_{\mathscr{X}}u\right\|_{\mathscr{X}} \leqslant C\left(\left\|u-\left(\Pi_{\Theta^l_{\mathscr{X}}}\otimes\mathrm{Id}\right)u\right\|_{\mathscr{X}} + \left\|u-\left(\mathrm{Id}\otimes\Pi_{\Sigma^l_V}\right)u\right\|_{\mathscr{X}}\right). \tag{4.4}$$

It follows that in order to obtain convergence rates for the discrete solution $u^l \in \mathscr{X}^l$, we only need to derive approximation properties for the spaces V and $H^{1/2}_{00}(\mathbb{R}_+)$.

4.1 *hp-approximation for the time variable*

In this section, we discuss approximation for the time variable in the $H^{1/2}_{00}(\mathbb{R}_+)$-norm. This is essentially the same theory as the one developed in Section 3 except that we approximate functions on the unbounded domain $\mathbb{R}_+$. Let us assume that f satisfies the assumption of Proposition 2.5, i.e. there exist $0 < T < \infty$ and $\delta \geqslant 1$ such that $f \in L^2(\mathbb{R}_+;H)$, $\mathrm{supp}(f) \subset [0,T]$ and $f \in \mathscr{G}^{\delta}((0,T);H)$. We split $\mathbb{R}_+$ in three subintervals for the analysis, namely $(0,T)$, (T,T_∞) and (T_∞,∞) for a T_∞ to be defined later.

Given two grading factors $\sigma_1,\sigma_2 \in (0,1)$ and $M \in \mathbb{N}$, we consider a mesh $\mathscr{T}_\sigma = \{I_{\sigma,n}\}_{n=1}^{2M+1}$, where $I_{\sigma,n} = (t_{\sigma,n-1},t_{\sigma,n})$ for

$$t_{\sigma,0} = 0, \qquad t_{\sigma,m} = T\sigma_1^{M-m}, \qquad t_{\sigma,M+m} = T+M\sigma_2^{M-m}, \qquad m = 1,\ldots,M, \tag{4.5}$$

where $\sigma = (\sigma_1, \sigma_2)$. Moreover we choose $t_{\sigma,2M+1} = T + M + \sigma_2$ and $T_\infty = T + M$. The choice of $t_{\sigma,2M+1}$ is arbitrary and any constant instead of σ_2 will lead to the result presented below. Furthermore, for slope parameters $\mu_1, \mu_2 > 0$ and $\delta \geqslant 1$, we choose for $m = 2, \dots, M$

$$p_{\mu,\delta,1} = p_{\mu,\delta,M+1} = p_{\mu,\delta,2M+1} = 1, \qquad p_{\mu,\delta,m} = \max\left(1, \left\lfloor \mu_1 m^\delta \right\rfloor\right), \qquad p_{\mu,\delta,M+m} = \max\left(1, \lfloor \mu_2 m \rfloor\right), \tag{4.6}$$

and denote $\mathbf{p}_{\mu,\delta} := \left\{p_{\mu,\delta,m}\right\}_{m=1}^{2M+1}$ for $\mu := (\mu_1, \mu_2)$. Let $N_t := \dim\left(S_0^{\mathbf{p}_{\mu,\delta},1}(\mathbb{R}_+, \mathscr{T}_\sigma)\right)$, where

$$S_0^{\mathbf{p}_{\mu,\delta},1}(\mathbb{R}_+, \mathscr{T}_\sigma) := \left\{ v \in H_0^1(\mathbb{R}_+) \;\middle|\; v|_{I_{\sigma,n}} \in \mathscr{P}^{p_{\mu,\delta,n}}(I_{\sigma,n}),\ n = 1, \dots, 2M+1,\ v|_{(t_{\sigma,2M+1},\infty)} = 0 \right\}.$$

We then define

$$\left(\pi_{N_t}^{\sigma,\mu,\delta} u\right)(t) := \begin{cases} \left(\pi_{I_{\sigma,n}}^{p_{\mu,\delta,n}} u\right)(t) & t \in I_{\sigma,n},\ n \in \{1, \dots, 2M\}, \\ \left(\frac{t_{\sigma,2M+1} - t}{h_{\sigma,2M+1}}\right) u(T_\infty) & t \in I_{\sigma,2M+1}, \\ 0 & t > t_{\sigma,2M+1}, \end{cases} \tag{4.7}$$

where $\pi_{I_{\sigma,n}}^{p_{\mu,\delta,n}}$ is the interpolation operator defined in Section 3.1 for $n \in \{1, \dots, 2M\}$. In other words, we use the interpolation operator built in Section 3.1 on the intervals $(0,T)$ and (T, T_∞) while on $(T_\infty, t_{\sigma,2M+1})$ we consider a linear interpolation of the function. The reason for choosing a linear approximation on the last interval and not a constant approximation is to ensure that we obtain a continuous element. On the remaining of the interval, since the solution u associated to f decays exponentially, we approximate the function by zero. In particular, we have the following lemma. The proof is based on the spectral decomposition of the operator $\mathscr{A}$ and the idea that for $t > T$, the solution decays exponentially since $\operatorname{supp}(f) \subset [0,T]$. It is omitted here and we refer the reader to the online supplementary material.

LEMMA 4.1 Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\operatorname{supp}(f) \subset [0,T]$ for some $0 < T < \infty$. Then there exist $C, \tau > 0$ such that for any $T < T_\infty < \infty$ it holds

$$\left\| u - \pi_{N_t}^{\sigma,\mu,\delta} u \right\|_{L^2((T_\infty,\infty);V)} \leqslant C e^{-\tau(T_\infty - T)},$$
$$\left\| u - \pi_{N_t}^{\sigma,\mu,\delta} u \right\|_{H_{00}^{1/2}((T_\infty,\infty);H)} \leqslant C e^{-\tau(T_\infty - T)}.$$

We point out that this result is independent of the choice of the parameters σ, μ and δ. In fact, it only depends on T_∞ and $t_{\sigma,2M+1}$. It can hence also be used in the case of h- or p-approximation to obtain convergence results. We then obtain the following important result.

THEOREM 4.2 Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\operatorname{supp}(f) \subset [0,T]$ and $f \in \mathscr{G}^\delta([0,T]; H)$ for some $\delta \geqslant 1$ and $0 < T < \infty$. Furthermore let $\mathscr{T}_\sigma$ be defined according to (4.5) for given $M \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in (0,1)$. Then there exists $\mu_0 = (\mu_{1,0}, \mu_{2,0}) > 0$ such that defining $\mathbf{p}_{\mu,\delta}$ as (4.6) for any $\mu > \mu_0$, there exist $C, \tau > 0$ with

$$\left\| u - \pi_{N_t}^{\sigma,\mu,\delta} u \right\|_{\mathscr{X}} \leqslant C \exp\left(-\tau N_t^{\frac{1}{\delta+1}}\right).$$

Proof. To simplify the notations, we write $I_1 = (0,T)$, $I_2 = (T, T_\infty)$ and $I_3 = (T_\infty, \infty)$. From (Devaud, 2017, Remark A.2.5), there exists $C > 0$ such that

$$\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{\mathscr{X}}^2 \leqslant C\Bigg(\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{L^2(I_1;V)}^2+\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{L^2(I_2;V)}^2+\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{L^2(I_3;V)}^2 \\ +\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{H_{00}^{1/2}(I_1;H)}^2+\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{H_{00}^{1/2}(I_2;H)}^2+\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{H_{00}^{1/2}(I_3;H)}^2\Bigg).$$

Note that

$$N_t = 2+\sum_{m=2}^{M}\left(\left\lfloor \mu_1 m^\delta\right\rfloor+\lfloor\mu_2 m\rfloor\right) \leqslant 2\max(1,\mu_1,\mu_2)\sum_{m=1}^{M} m^\delta \leqslant 2\max(1,\mu_1,\mu_2)M^{\delta+1}.$$

From Lemma 4.1 and since $T_\infty = T+M$, there exists $\tilde{\tau}_3 > 0$ such that

$$\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{L^2(I_3;V)} \leqslant C_3\exp\left(-\tilde{\tau}_3(T_\infty-T)\right) = C_3\exp\left(-\tilde{\tau}_3 M\right) \leqslant C_3\exp\left(-\tau_3 N_t^{\frac{1}{\delta+1}}\right),$$
$$\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{H_{00}^{1/2}(I_3;H)} \leqslant C_3\exp\left(-\tilde{\tau}_3(T_\infty-T)\right) = C_3\exp\left(-\tilde{\tau}_3 M\right) \leqslant C_3\exp\left(-\tau_3 N_t^{\frac{1}{\delta+1}}\right),$$

where $\tau_3 = \tilde{\tau}_3\left(2\max(1,\mu_1,\mu_2)\right)^{-\frac{1}{\delta+1}}$. From Proposition 2.5, it holds

$$u|_{I_1} \in \mathscr{G}^{\delta,1}(I_1;H)\cap\mathscr{G}^{\delta,1/2}(I_1;V), \qquad u|_{I_2} \in \mathscr{G}^{1,1/2}(I_2;H)\cap\mathscr{G}^{1,0}(I_2;V),$$

and Remark 2.2 yields that the constants C_u and d_u in (2.17) on the interval I_2 are independent of T_∞. From Proposition 3.2, there exist $\mu_{1,H},\mu_{1,V} > 0$ such that for $\mu_1 > \max(\mu_{1,H},\mu_{1,V}) =: \mu_{1,0}$ it holds

$$\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{L^2(I_1;V)} \leqslant C_1\exp\left(-\tau_1 N_t^{\frac{1}{\delta+1}}\right), \qquad \left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{H_{00}^{1/2}(I_1;H)} \leqslant C_1\exp\left(-\tau_1 N_t^{\frac{1}{\delta+1}}\right).$$

Using the same argument together with the explicit dependence on $|I_2|$ in (3.5) and (3.6), there exist $\mu_{2,H},\mu_{2,V} > 0$ such that for $\mu_2 > \max(\mu_{2,H},\mu_{2,V}) =: \mu_{2,0}$ it holds

$$\left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{L^2(I_2;V)} \leqslant C_2\exp\left(-\tau_2 N_t^{\frac{1}{2}}\right), \qquad \left\|u-\pi_{N_t}^{\sigma,\mu,\delta}u\right\|_{H_{00}^{1/2}(I_2;H)} \leqslant C_2\exp\left(-\tau_2 N_t^{\frac{1}{2}}\right).$$

Wrapping up everything, we obtain the desired result since $N_t^{\frac{1}{2}} \geqslant N_t^{\frac{1}{\delta+1}}$. □

REMARK 4.1 Using Remark 3.2, we can also obtain convergence results for h- and p-refinements considering a quasi-uniform mesh and a fixed polynomial degree. However in the case of h-approximation, due to the fact that the solution is approximated over an unbounded domain, the convergence rate is decreased by a logarithmic factor. More precisely, assume that $f \in L^2(\mathbb{R}_+;H)$ satisfy $\operatorname{supp}(f) \subset [0,T]$ for some $0 < T < \infty$ and that there exists $r \geqslant 0$ such that

$$u|_{(0,T)} \in H^{r+1}((0,T);H)\cap H^{r+1/2}((0,T);V) \text{ and } u|_{(T,\infty)} \in H^{r+1}((T,\infty);H)\cap H^{r+1/2}((T,\infty);V).$$

Then for $p,M \in \mathbb{N}$, $h_1 = \frac{T}{M}$ and $h_2 = \frac{2\min(p,r)+1}{2M}$, we define

$$t_0 = 0, \qquad t_j = jh_1, \qquad t_{j+M} = T+jh_2, \qquad p_j = p, \qquad j = 1,\dots,M. \tag{4.8}$$

It is then possible to show that there exists a constant $C > 0$ (depending on r) such that for all p

$$\left\|u - \pi_{N_t}^p u\right\|_{\mathscr{X}} \leqslant C \left(\frac{\log(N_t)}{N_t}\right)^{\min(p,r)+1/2}, \tag{4.9}$$

where $N_t = Mp$ and $\pi_{N_t}^p$ is the interpolation operator defined in (4.7) associated to the mesh defined through (4.8) and the vector of fixed polynomial degrees equal to p.

Turning to the p-refinement case, let f be as above and assume that there exist $C, d > 0$ such that

$$|u|_{H^r((0,T);V)} \leqslant Cd^r\Gamma(r+1), \qquad |u|_{H^r((T,\infty);V)} \leqslant Cd^r, \qquad r \in \mathbb{N}_0.$$

For $p, M \in \mathbb{N}$, $h_1 = \frac{T}{M}$ and $h_2 = \frac{p}{M}$, the underlying fixed mesh and the vector of polynomial degrees are defined as in (4.8). It is then possible to show that

$$\left\|u - \pi_{N_t}^p u\right\|_{\mathscr{X}} \leqslant Ce^{-\tau p},$$

for some constants $C, \tau > 0$.

4.2 *Space-time approximation*

So far, we only derived error estimates for the approximation of the temporal variable. We now turn to the full space-time discretization. First, we derive error estimates in an abstract setting and apply this theory to space-time h-refinement. In a second step, the two-dimensional heat equation in a polygonal domain is considered. We show that using an hp-approximation in both time and space it is possible to obtain exponential convergence with respect to the total number of degrees of freedom. Since we consider in this section both space and time approximations, we use t and x as subscripts associated to the temporal and spatial components, respectively.

4.2.1 *An abstract result.* Let us consider a scale of Hilbert spaces $\{X_s\}_{s\geqslant 0}$ such that $X_0 = H$, $X_\varepsilon = V$ for some $\varepsilon > 0$ and $X_{s_2} \subset X_{s_1}$ for all $0 \leqslant s_1 \leqslant s_2 < \infty$. We also assume that the norm on X_s is equivalent to the interpolation norm on $[X_{s_1}, X_{s_2}]_\theta$ for any $\varepsilon \leqslant s_1 < s < s_2 < \infty$ and $\theta = \frac{s-s_1}{s_2-s_1}$. The spaces X_s are then called *smoothness spaces* associated to (H, V).

DEFINITION 4.3 (Approximation of order $\bar{s}$) Let $\{X_s\}_{s\geqslant 0}$ be a sequence of *smoothness spaces* associated to (H, V) and $\varepsilon \leqslant \bar{s} < \infty$. Then a sequence of finite dimensional spaces $\{V_{N_x}\}_{N_x\in\mathbb{N}} \subset V$ is said to be an approximation of order $\bar{s}$ of the pair (H, V) (or equivalently (X_0, X_ε)) if for every $N_x \in \mathbb{N}$ there exists an operator $\varPi_{N_x} : V \to V_{N_x}$ such that for $\varepsilon \leqslant s \leqslant \bar{s}$ and $v \in X_s$ it holds

$$\|v - \varPi_{N_x} v\|_V \leqslant C_s N_x^{-(s-\varepsilon)} \|v\|_{X_s}, \qquad \|v - \varPi_{N_x} v\|_H \leqslant C_s N_x^{-s} \|v\|_{X_s},$$

for a constant $C_s > 0$ independent of N_x.

The mesh defined through (4.8) will be denoted $\mathscr{T}_{t,h}$. Furthermore, we write p_t for the vector of fixed polynomial degrees.

PROPOSITION 4.4 Let $\{V_{N_x}\}_{N_x\in\mathbb{N}} \subset V$ be an approximation of order $\bar{s}$ of the pair $(H, V) = (X_0, X_\varepsilon)$ and $f \in L^2(\mathbb{R}_+; H)$ satisfy $\operatorname{supp}(f) \subset [0, T]$ for some $0 < T < \infty$. Given $p_t, N_t, N_x \in \mathbb{N}$ such that $\frac{N_t}{p_t} \in \mathbb{N}$, we denote u_h the unique solution of (4.1) in the discrete space $\mathscr{X}_h := S_0^{p_t,1}(\mathbb{R}_+,\mathscr{T}_{t,h})\otimes V_{N_x}$. Let us assume that

$$u|_{(0,T)} \in H^{r+1}((0,T);H)\cap H^{r+1/2}((0,T);V) \text{ and } u|_{(T,\infty)} \in H^{r+1}((T,\infty);H)\cap H^{r+1/2}((T,\infty);V)$$

for some $r \geqslant 0$ and

$$u \in H_{00}^{1/2}(\mathbb{R}_+;X_s) \text{ and } u \in L^2(\mathbb{R}_+;X_{s+\varepsilon})$$

for some $\varepsilon \leqslant s \leqslant \bar{s}-\varepsilon$. Then there exists $C>0$ such that

$$\|u-u_h\|_{\mathscr{X}} \leqslant C\left(N_x^{-s}+\left(\frac{N_t}{\log(N_t)}\right)^{-(\min(p_t,r)+1/2)}\right).$$

Proof. From (4.2) and (4.4), we have

$$\|u-u_h\|_{\mathscr{X}} \leqslant C\left(\left\|u-\left(\pi_{N_t}^{p_t}\otimes \mathrm{Id}\right)u\right\|_{\mathscr{X}}+\|u-(\mathrm{Id}\otimes \Pi_{N_x})u\|_{\mathscr{X}}\right).$$

From (4.9), we have

$$\left\|u-\left(\pi_{N_t}^{p_t}\otimes \mathrm{Id}\right)u\right\|_{\mathscr{X}} \leqslant C\left(\frac{N_t}{\log(N_t)}\right)^{-(\min(p_t,r)+1/2)}.$$

Since $\{V_{N_x}\}_{N_x\in\mathbb{N}}$ is an approximation of order $\bar{s}$ of (H,V), we have

$$\|\mathrm{Id}_V-\Pi_{N_x}\|_{\mathscr{L}(X_{s+\varepsilon},V)} \leqslant C_s N_x^{-s}, \qquad \|\mathrm{Id}_H-\Pi_{N_x}\|_{\mathscr{L}(X_s,H)} \leqslant C_s N_x^{-s}.$$

Since $H_{00}^{1/2}(\mathbb{R}_+;H)$ and $L^2(\mathbb{R}_+;V)$ are isomorphic to $H_{00}^{1/2}(\mathbb{R}_+)\otimes H$ and $L^2(\mathbb{R}_+)\otimes V$, respectively, the result follows by interpolation (Weidmann, 1980, Theorem 8.32(b)). □

EXAMPLE 4.5 Let $D\subset\mathbb{R}^d$ with $d=1,2$ be an interval or a bounded polygonal domain in the case $d=2$ and define $H=L^2(D)$ and $V=H_0^1(D)$. Then for every $s\geqslant 0$, we define $X_s=H^{ds}(D)\cap H_0^1(D)$ and choose $\varepsilon=1/d$. In that case, the interpolation property follows from (Triebel, 1995, Theorem 2 p.317). Let us consider a quasi-uniform mesh $\mathscr{T}_{x,h}$ of D with meshwidth h_x. Given a polynomial degree $p_x\in\mathbb{N}$, we consider the space of continuous, piecewise polynomial functions

$$V_{N_x} := S_0^{p_x,1}(D,\mathscr{T}_{x,h}) := \left\{v\in H_0^1(D)\;\middle|\; v|_T\in\mathscr{P}^{p_x}(T),\ \forall T\in\mathscr{T}_{x,h}\right\}.$$

Then, there exist $c_1,c_2>0$ such that $c_1 N_x^{-1/d}\leqslant h_x\leqslant c_2 N_x^{-1/d}$. From (Scott & Zhang, 1990, Theorem 4.1) (for $d=2$) and (Schwab, 1998, Theorem 3.17) (for $d=1$), we obtain that the sequence of spaces $\{V_{N_x}\}_{N_x\in\mathbb{N}}$ is an approximation of order $\frac{p_x+1}{d}$ of the pair $(L^2(D),H_0^1(D))$. Hence if

$$u|_{(0,T)} \in H^{p_t+1}((0,T);L^2(D))\cap H^{p_t+1/2}((0,T);H_0^1(D)),$$
$$u|_{(T,\infty)} \in H^{p_t+1}((T,\infty);L^2(D))\cap H^{p_t+1/2}((T,\infty);H_0^1(D)),$$

and

$$u\in H_{00}^{1/2}(\mathbb{R}_+;H^{p_x}(D)\cap H_0^1(D)) \text{ and } u\in L^2(\mathbb{R}_+;H^{p_x+1}(D)\cap H_0^1(D)),$$

then

$$\|u-u_h\|_{\mathscr{X}} \leqslant C\left(N_x^{-p_x/d}+\left(\frac{N_t}{\log(N_t)}\right)^{-(p_t+1/2)}\right).$$

Moreover if u has support in $(0,T)$, the above estimate reduces to

$$\|u-u_h\|_{\mathscr{X}} \leqslant C\left(N_x^{-p_x/d}+N_t^{-(p_t+1/2)}\right). \tag{4.10}$$

4.2.2 *Space-time hp-approximation of the two-dimensional heat equation.* Let $D \subset \mathbb{R}^2$ be a (possibly non-convex) bounded polygonal domain and $\mathscr{A} v := -\operatorname{div}_x (A \nabla_x v)$ for some $A \in \left(W^{2,\infty}(D)\right)^{2\times 2}$ symmetric positive definite uniformly in $x \in \mathbb{R}^2$. In that case $H = L^2(D)$, $V = H_0^1(D)$ and $\mathfrak{a}(v,w) = \int_D (A\nabla_x v) \cdot \nabla_x w$.

In order to account for the spatial regularity, we introduce weighted Sobolev spaces which are used in the regularity theory of elliptic equations. We follow here (Schwab, 1998, Chapter 4). Let $A_1, \dots, A_L$ denote the vertices of D, $\gamma = (\gamma_1, \dots, \gamma_L) \in [0,1)^L$ and

$$\Phi_\gamma(x) := \prod_{j=1}^{L} (r_j(x))^{\gamma_j}, \qquad r_i(x) := \min(1, |x - A_i|), \qquad \forall x \in \overline{D},\ i = 1, \dots, L. \tag{4.11}$$

Then for a Hilbert space $(H, (\cdot,\cdot)_H)$ and $m \geqslant l \geqslant 1$, we define the weighted Bochner-Sobolev space $H_\gamma^{m,l}(D;H)$ as the space of functions $v \in L^2(D;H)$ for which

$$\|v\|^2_{H_\gamma^{m,l}(D;H)} := \|v\|^2_{H^{l-1}(D;H)} + |v|^2_{H_\gamma^{m,l}(D;H)} < \infty,$$

where

$$|v|^2_{H_\gamma^{m,l}(D;H)} := \sum_{k=l}^{m} \sum_{|\alpha|=k} \left\| \Phi_{\gamma+k-l} \partial_x^\alpha v \right\|^2_{L^2(D;H)}.$$

DEFINITION 4.6 Let $D \subset \mathbb{R}^2$ be a bounded polygonal domain, $(H, (\cdot,\cdot)_H)$ a Hilbert space and $\gamma = (\gamma_1, \dots, \gamma_L) \in [0,1)^L$. For $l \geqslant 1$, we say that $v \in \mathscr{B}_\gamma^l(D;H)$ if $v \in H_\gamma^{m,l}(D;H)$ for all $m \geqslant l$ and there exist $C_v > 0$ and $d_v \geqslant 1$ such that for all $k \geqslant l$ it holds

$$\sum_{|\alpha|=k} \left\| \Phi_{\gamma+k-l} \partial_x^\alpha v \right\|_{L^2(D;H)} \leqslant C_v d_v^{k-l} \Gamma(k-l+1),$$

where Φ_γ is defined in (4.11).

Functions in $\mathscr{B}_\gamma^2(D;H)$ are globally continuous, analytic inside D and contain potential algebraic singularities at the vertices $A_1, \dots, A_L$.

For $\delta \geqslant 1$, $\sigma_{t,1}, \sigma_{t,2} \in (0,1)$ and $\mu_{t,1}, \mu_{t,2} > 0$, let $\mathscr{T}_{\sigma_t}$ and $\mathbf{p}_{\mu_t,\delta}$ be defined through (4.5) and (4.6), respectively. Here we have used the notation $\sigma_t = (\sigma_{t,1}, \sigma_{t,2})$ and $\mu_t = (\mu_{t,1}, \mu_{t,2})$. Furthermore, let $\mathscr{T}_{\sigma_x}$ be a proper graded mesh of D for a grading factor $\sigma_x \in (0,1)$. Given a slope parameter $\mu_x > 0$, $\mathbf{p}_{\mu_x}$ denotes a polynomial distribution associated to $\mathscr{T}_{\sigma_x}$ whose elements are increasing linearly away from the vertices $A_1, \dots, A_L$. We refer the reader to (Schwab, 1998, Chapter 4) for the construction of $\mathscr{T}_{\sigma_x}$ and $\mathbf{p}_{\mu_x}$. Let then

$$\mathscr{X}^N_{\sigma,\mu,\delta} := S_0^{\mathbf{p}_{\mu_t,\delta},1}(\mathbb{R}_+, \mathscr{T}_{\sigma_t}) \otimes S_0^{\mathbf{p}_{\mu_x},1}(D, \mathscr{T}_{\sigma_x}),$$

where $\sigma = (\sigma_{t,1}, \sigma_{t,2}, \sigma_x)$, $\mu = (\mu_{t,1}, \mu_{t,2}, \mu_x)$ and

$$S_0^{\mathbf{p}_{\mu_x},1}(D, \mathscr{T}_{\sigma_x}) := \left\{ v \in H_0^1(D) \;\middle|\; v|_T \in \mathscr{P}^{p_{\mu_x,T}}(T),\ \forall T \in \mathscr{T}_{\sigma_x} \right\} \subset \mathscr{C}(D).$$

In the following theorem, $N_x := \dim\left(S_0^{\mathbf{p}_{\mu_x},1}(D, \mathscr{T}_{\sigma_x})\right)$ and $N_t = \dim\left(S_0^{\mathbf{p}_{\mu_t,\delta},1}(\mathbb{R}_+, \mathscr{T}_{\sigma_t})\right)$ denote the number of degrees of freedom associated to the spatial and temporal approximations, respectively. Furthermore, $N = N_x N_t$.

THEOREM 4.7 Let $f \in L^2(\mathbb{R}_+;H)$ satisfy $\operatorname{supp}(f) \subset \bar{I} = [0,T]$ and $f \in \mathscr{G}^\delta(I;H)$ for some $0 < T < \infty$ and $\delta \geqslant 1$. Furthermore assume that $u \in H_{00}^{1/2}(\mathbb{R}_+;\mathscr{B}_\gamma^2(D) \cap H_0^1(D))$. Then for grading factors $\sigma_1, \sigma_2, \sigma_x \in (0,1)$, there exist $\mu_0 = (\mu_{t,1,0}, \mu_{t,2,0}, \mu_{x,0}) > 0$ such that for all $\mu > \mu_0$, there exist $C, \tau > 0$ satisfying

$$\|u - u_N\|_{\mathscr{X}} \leqslant C\left(\exp\left(-\tau N_t^{\frac{1}{\delta+1}}\right) + \exp\left(-\tau N_x^{\frac{1}{3}}\right)\right),$$

where u_N denotes the solution of (4.1) in $\mathscr{X}^N_{\sigma,\mu,\delta}$.

Proof. In (Schwab, 1998, Chapter 4), an operator $\pi_{N_x}^{\sigma_x,\mu_x} : \mathscr{B}_\gamma^2(D) \cap H_0^1(D) \to S_0^{\mathbf{p}_{\mu_x},1}(D, \mathscr{T}_{\sigma_x})$ is introduced. It is shown that there exists $\mu_{x,0} > 0$ such that for all $\mu_x > \mu_{x,0}$, there exist $C, \tau > 0$ satisfying for all $v \in \mathscr{B}_\gamma^2(D) \cap H_0^1(D)$

$$\left\|v - \pi_{N_x}^{\sigma_x,\mu_x} v\right\|_{H^1(D)} \leqslant C \exp\left(-\tau N_x^{\frac{1}{3}}\right).$$

We point out that this result remains valid for functions taking values in a Hilbert space. It follows that for $u \in H_{00}^{1/2}(\mathbb{R}_+;\mathscr{B}_\gamma^2(D) \cap H_0^1(D))$, we have

$$\left\|u - \left(\mathrm{Id} \otimes \pi_{N_x}^{\sigma_x,\mu_x}\right) u\right\|_{\mathscr{X}} \leqslant C \exp\left(-\tau N_x^{\frac{1}{3}}\right).$$

The result then follows from (4.4) and Theorem 4.2. □

REMARK 4.2 The argument used above can be simplified in the one-dimensional case in order to obtain similar results. Indeed, a theory can be developed based on (Schwab, 1998, Chapter 3) for the spatial *hp*-refinement. In that setting, using graded meshes and linearly increasing polynomial degrees in a similar fashion as for the temporal discretization, we obtain

$$\|u - u_N\|_{\mathscr{X}} \leqslant C\left(\exp\left(-\tau N_t^{\frac{1}{\delta+1}}\right) + \exp\left(-\tau N_x^{\frac{1}{2}}\right)\right),$$

assuming that f satisfies the assumption of Theorem 4.7 and that u exhibits more regularity in space.

5. Numerical results

In order to provide numerical evidence of the above results, we consider the one-dimensional heat equation with $D = (0,1)$ and $A = 1$. We present convergence results for *h*- and *hp*-refinements. All the computations have been performed using the Python programming language. A complete explanation on how to build the time mass and stiffness matrices is given in (Devaud, 2017, Section 6.1).

5.1 *h-refinement*

Let us consider the solutions

$$u_1(t,x) = \sin(\pi t)\sin(\pi x)\chi_{(0,1)}(t), \qquad u_2(t,x) = \cos\left(\frac{5\pi}{2}t\right)\sin(\pi x)\chi_{(0,1)}(t).$$

We use the same mesh in time and space with $M = M_x = M_t$ elements, so that $N = N_x N_t \approx N_t^2$. Furthermore the functions are piecewise analytic and have compact support in time, so that (4.10) yields

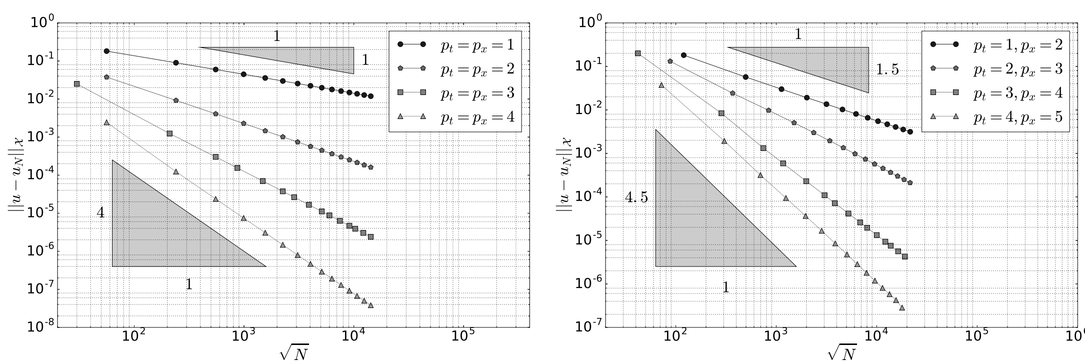
$$\|u_i - u_{i,N}\|_{\mathscr{X}} \leqslant C N^{-\frac{1}{2}\min(p_x, p_t+1/2)}, \qquad i = 1,2.$$

We consider $p = p_t = p_x$ in the case $i = 1$ and $p = p_t = p_x - 1$ for $i = 2$ in order to see that the exponents are sharp. It gives the existence of a constant $C > 0$ such that for all N

$$\|u_1 - u_{1,N}\|_{\mathscr{X}}^2 \leqslant CN^{-p}, \qquad \|u_2 - u_{2,N}\|_{\mathscr{X}}^2 \leqslant CN^{-(p+1/2)}.$$

The experimental convergence rates for the error in the $\mathscr{X}$-norm with respect to $\sqrt{N}$ for both cases and different values of p are presented in the following table. The estimated slopes are based on the last 5 points of the experiment. Convergence plots associated to the experimental rates obtained for u_1 and u_2 are depicted in Figure 1. We see that the expected convergence rates are indeed obtained.

p	u_1 - Expected	u_1 - Observed	u_2 - Expected	u_2 - Observed
1	1	0.996	1.5	1.517
2	2	1.994	2.5	2.442
3	3	2.987	3.5	3.517
4	4	3.928	4.5	4.493



(a) (b)

Figure 1: Convergence of the h-approximation in time and space with respect to $\sqrt{N}$ for several polynomial degrees. The results for u_1 are presented in (a) while (b) corresponds to u_2.

5.2 *hp-refinement in space and time*

Given $\rho > -2$ and $\omega > 0$, we consider $f(t,x) = x^{\rho}(1-x)e^{-\omega t}$. The solution of (4.1) associated to f contains a singularity at $t = x = 0$ and can be computed using a Fourier expansion based on the one-dimensional Laplace operator. The singularity at $x = 0$ is due to the x-dependence of f while the one at $t = 0$ follows from an incompatibility between the t-dependence of f and the initial condition. Assuming that $\omega \neq \pi k$ for all $k \in \mathbb{N}$, the solution is then given by

$$u(t,x) = 2\sum_{k=1}^{\infty} \left(f_{\rho,k} - f_{\rho+1,k}\right) \sin(\pi k x) \left(\frac{e^{-\omega t} - e^{-\pi^2 k^2 t}}{\pi k - \omega}\right), \tag{5.1}$$

where

$$f_{\rho,k} := \int_0^1 x^\rho \sin(\pi k x)dx = \frac{\pi k}{\rho+2}\,{}_1F_2\left(\frac{\rho}{2}+1;\frac{3}{2},\frac{\rho}{2}+2;-\frac{(\pi k)^2}{4}\right), \qquad \forall k \in \mathbb{N},$$

and ${}_1F_2$ denotes a generalized hypergeometric function (Olver *et al.*, 2010, Section 16.2). We consider geometric meshes in time and space and use the same grading factors and slope parameters for the spatial and the temporal discretization. Since $M = M_x = M_t$, we have $N = N_x N_t \approx N_x^2 \approx N_t^2$, so that using Remark 4.2, we have

$$\|u - u_N\|_{\mathscr{X}} \leqslant C\left(\exp\left(-\tau\sqrt{N_t}\right) + \exp\left(-\tau\sqrt{N_x}\right)\right) \leqslant C\exp\left(-\tau N^{\frac{1}{4}}\right).$$

In Figure 2, we present convergence results for *hp*-discretization in time and space for $\rho = 3/4$, $\omega = 5$ and different values of $\sigma = \sigma_x = \sigma_t$ and $\mu = \mu_x = \mu_t$. We see that that for smaller values of σ, larger values of μ need to be considered. Furthermore, exponential convergence can indeed be observed.

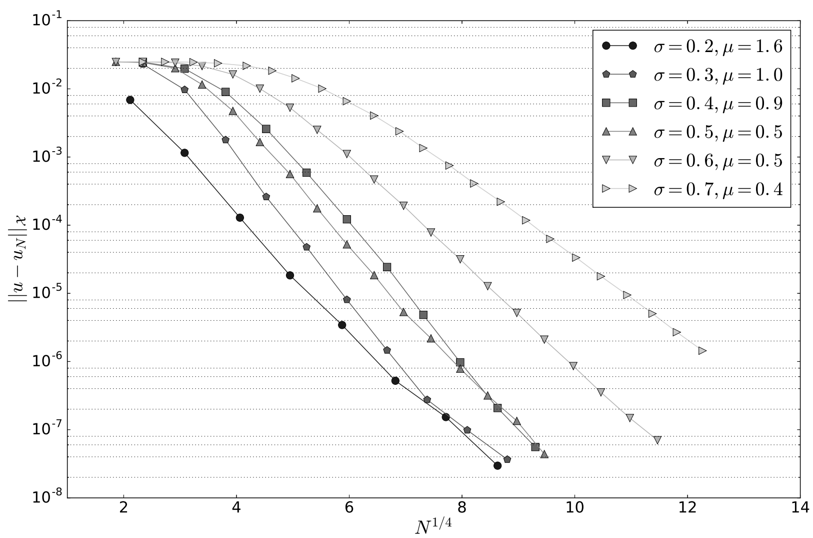


Figure 2: Convergence of the *hp*-approximation in time and space of u given by (5.1) for $\rho = 3/4$ and $\omega = 5$. The grading factors and slopes are the same for the temporal and spatial discretizations.

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Supplementary material to: “Petrov-Galerkin space-time *hp*-approximation of parabolic equations in $H^{1/2}$”

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In this report, we provide additionial material associated to the paper “Petrov-Galerkin space-time *hp*-approximation of parabolic equations in $H^{1/2}$”. As explained in the paper, several technical results are presented without proof since they are in our opinion not essential to the comprehension of the method. We provide these proofs here. The notations and definitions introduced in the associated paper are used in the sequel without repeating them. Furthermore all references starting with $hp-$ correspond to the number used in “Petrov-Galerkin space-time *hp*-approximation of parabolic equations in $H^{1/2}$”. All other references are internal to this manuscript. Most of the following results were presented in [2].

1 Proof of Lemmas $hp-$2.2, $hp-$2.3 and $hp-$2.4

Lemma (Lemma $hp-$2.2)**.** *The operator* $\mathfrak{H} \in \mathcal{L}_{\mathrm{Iso}}\left(H_A^{1/2}(\mathbb{R};H), H_S^{1/2}(\mathbb{R};H)\right)$ *is an isometry with* $\mathfrak{H}^{-1} = -\mathfrak{H}$.

Lemma (Lemma $hp-$2.3)**.** *It holds*

$$E_S \in \mathcal{L}_{\mathrm{Iso}}(H^{1/2}(\mathbb{R}_+;H), H_S^{1/2}(\mathbb{R};H)) \text{ and } E_A \in \mathcal{L}_{\mathrm{Iso}}(H_{00}^{1/2}(\mathbb{R}_+;H), H_A^{1/2}(\mathbb{R};H)), \tag{1.1}$$

with $E_S^{-1} = R_>|_{H_S^{1/2}(\mathbb{R};H)}$ *and* $E_A^{-1} = R_>|_{H_A^{1/2}(\mathbb{R};H)}$. *Moreover, for* $v \in H_{00}^{1/2}(\mathbb{R}_+;H)$ *and* $w \in H^{1/2}(\mathbb{R}_+;H)$, *it holds* $\|(E_A+E_S)v\|^2_{H^{1/2}(\mathbb{R};H)} = \|E_A\|^2_{H^{1/2}(\mathbb{R};H)} + \|E_S v\|^2_{H^{1/2}(\mathbb{R};H)}$ *and*

$$\sqrt{2}\,\|w\|_{H^{1/2}(\mathbb{R}_+;H)} \leq \quad \|E_S w\|_{H^{1/2}(\mathbb{R};H)} \quad \leq 2\,\|w\|_{H^{1/2}(\mathbb{R}_+;H)}\,, \tag{1.2a}$$

$$\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)} \leq \quad \|E_A v\|_{H^{1/2}(\mathbb{R};H)} \quad \leq 2\sqrt{2}\,\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)}\,, \tag{1.2b}$$

$$2\,\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)} \leq \|(E_A+E_S)v\|_{H^{1/2}(\mathbb{R};H)} \leq 2\sqrt{2}\,\|v\|_{H_{00}^{1/2}(\mathbb{R}_+;H)}\,. \tag{1.2c}$$

Lemma (Lemma $hp-$2.4)**.** *For* $v \in H_{00}^{1/2}(\mathbb{R}_+;H)$ *and* $w \in H^{1/2}(\mathbb{R}_+;H)$, *it holds*

$$\int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S w\right)_H dt = 2\int_{\mathbb{R}_+} \left(D_+^{1/2} v, D_-^{1/2} w\right)_H dt, \tag{1.3a}$$

$$\int_{\mathbb{R}} (E_S v, E_S w)_H \, dt = 2\int_{\mathbb{R}_+} (v,w)_H \, dt, \tag{1.3b}$$

$$\int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_A v, D_{\mathbb{R},-}^{1/2} E_S v\right)_H dt = \int_{\mathbb{R}} \left(D_{\mathbb{R},+}^{1/2} E_S v, D_{\mathbb{R},-}^{1/2} E_S w\right)_H dt = \int_{\mathbb{R}} (E_A v, E_S w)_H \, dt = 0. \tag{1.3c}$$

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Moreover, if $w \in H^{1/2}_{00}(\mathbb{R}_+; H)$, it holds

$$\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_A v, D^{1/2}_{\mathbb{R},-} E_S w \right)_H dt = \int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_S v, D^{1/2}_{\mathbb{R},-} E_A w \right)_H dt. \tag{1.4}$$

In order to prove it, we will need some preliminary results. Let us consider the spaces

$$\mathcal{F}_S(\mathbb{R}; H) := \{ g \in \mathcal{F}(\mathbb{R}; H) \mid g(t) = g(-t) \text{ for } t \in \mathbb{R} \},$$
$$\mathcal{F}_A(\mathbb{R}; H) := \{ g \in \mathcal{F}(\mathbb{R}; H) \mid g(t) = -g(-t) \text{ for } t \in \mathbb{R} \},$$

and define for a function $v \in \mathcal{F}(\mathbb{R}; H)$

$$v_S(t) := \frac{1}{2} \left(v(t) + v(-t) \right), \qquad v_A(t) := \frac{1}{2} \left(v(t) - v(-t) \right), \qquad t \in \mathbb{R}. \tag{1.5}$$

Hence v can be decomposed into a symmetric and an antisymmetric part as $v = v_S + v_A$ with $v_S \in \mathcal{F}_S(\mathbb{R}; H)$ and $v_A \in \mathcal{F}_A(\mathbb{R}; H)$. Since $\mathcal{F}_S(\mathbb{R}; H) \cap \mathcal{F}_A(\mathbb{R}; H) = \{0\}$, it follows that $\mathcal{F}(\mathbb{R}; H) = \mathcal{F}_S(\mathbb{R}; H) \oplus \mathcal{F}_A(\mathbb{R}; H)$. It is clear that, for any $v_A, w_A \in \mathcal{F}_A(\mathbb{R}; H)$ and $v_S, w_S \in \mathcal{F}_S(\mathbb{R}; H)$, it holds

$$\int_{\mathbb{R}} (D v_A, w_A)_H \, dt = \int_{\mathbb{R}} (D v_S, w_S)_H \, dt = \int_{\mathbb{R}} (v_A, w_S)_H \, dt = \int_{\mathbb{R}} (v_S, w_A)_H \, dt = 0. \tag{1.6}$$

Lemma 1.1. *It holds*

$$H^{1/2}(\mathbb{R}; H) = H^{1/2}_S(\mathbb{R}; H) \oplus H^{1/2}_A(\mathbb{R}; H).$$

Furthermore, for $v \in H^{1/2}(\mathbb{R}; H)$, we have

$$\|v\|^2_{H^{1/2}(\mathbb{R};H)} = \|v_S\|^2_{H^{1/2}(\mathbb{R};H)} + \|v_A\|^2_{H^{1/2}(\mathbb{R};H)},$$

where $v = v_A + v_S$ with $v_S \in H^{1/2}_S(\mathbb{R}; H)$ and $v_A \in H^{1/2}_A(\mathbb{R}; H)$

Proof. It is clear that $H^{1/2}_A(\mathbb{R}; H) \cap H^{1/2}_S(\mathbb{R}; H) = \{0\}$. For $v \in H^{1/2}(\mathbb{R}; H)$, let us consider $v_A \in H^{1/2}_A(\mathbb{R}; H)$ and $v_S \in H^{1/2}_S(\mathbb{R}; H)$ define for μ-a.e. $t \in \mathbb{R}$ through (1.5). It holds

$$\begin{aligned} |v|^2_{H^{1/2}(\mathbb{R};H)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|(v_S(s) - v_S(t)) + (v_A(s) - v_A(t))\|^2_H}{(s-t)^2} dsdt \\ &= |v_S|^2_{H^{1/2}(\mathbb{R};H)} + |v_A|^2_{H^{1/2}(\mathbb{R};H)} + 2 \underbrace{\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(v_S(s) - v_S(t), v_A(s) - v_A(t))_H}{(s-t)^2} dsdt}_{=:I}. \end{aligned}$$

Let us write

$$\begin{aligned} I &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{(v_S(s) - v_S(t), v_A(s) - v_A(t))_H}{(s-t)^2} dsdt + \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} \frac{(v_S(s) - v_S(t), v_A(s) - v_A(t))_H}{(s-t)^2} dsdt \\ &+ \int_{\mathbb{R}_+} \int_{\mathbb{R}_-} \frac{(v_S(s) - v_S(t), v_A(s) - v_A(t))_H}{(s-t)^2} dsdt + \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \frac{(v_S(s) - v_S(t), v_A(s) - v_A(t))_H}{(s-t)^2} dsdt \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Then

$$\begin{aligned} I_1 &= \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} \frac{(v_S(-s) - v_S(-t), v_A(-s) - v_A(-t))_H}{(s-t)^2} dsdt \\ &= - \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} \frac{(v_S(s) - v_S(t), v_A(s) - v_A(t))_H}{(s-t)^2} dsdt = -I_2, \end{aligned}$$

and

$$I_3 = \int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \frac{(v_S(-s) - v_S(-t), v_A(-s) - v_A(-t))_H}{(s-t)^2} dsdt$$
$$= -\int_{\mathbb{R}_-} \int_{\mathbb{R}_+} \frac{(v_S(s) - v_S(t), v_A(s) - v_A(t))_H}{(s-t)^2} dsdt = -I_4,$$

so that $|v|^2_{H^{1/2}(\mathbb{R};H)} = |v_S|^2_{H^{1/2}(\mathbb{R};H)} + |v_A|^2_{H^{1/2}(\mathbb{R};H)}$. Similarly from (1.6), we obtain

$$\|v\|^2_{L^2(\mathbb{R};H)} = \|v_S\|^2_{L^2(\mathbb{R};H)} + \|v_A\|^2_{L^2(\mathbb{R};H)}.$$

□

Lemma 1.2. *The embeddings $\mathcal{F}_S(\mathbb{R};H) \subset H^{1/2}_S(\mathbb{R};H)$ and $\mathcal{F}_A(\mathbb{R};H) \subset H^{1/2}_A(\mathbb{R};H)$ are dense.*

Proof. The result follows from the splitting $\mathcal{F}(\mathbb{R};H) = \mathcal{F}_S(\mathbb{R};H) \oplus \mathcal{F}_A(\mathbb{R};H)$, Lemma 1.1 and the fact that the embedding $\mathcal{F}(\mathbb{R};H) \subset H^{1/2}(\mathbb{R};H)$ is dense. □

Lemma 1.3. *For $v_S, w_S \in H^{1/2}_S(\mathbb{R};H)$ and $v_A, w_A \in H^{1/2}_A(\mathbb{R};H)$, it holds*

$$\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} v_S, D^{1/2}_{\mathbb{R},-} w_S\right)_H dt = \int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} v_A, D^{1/2}_{\mathbb{R},-} w_A\right)_H dt = \int_{\mathbb{R}} (v_A, v_S)_H \, dt = 0.$$

Proof. By Lemma 1.2, we only need to prove the identities for $v_S, w_S \in \mathcal{F}_S(\mathbb{R};H)$ and $v_A, w_A \in \mathcal{F}_A(\mathbb{R};H)$. The results follow using (hp−2.7) and (1.6). □

We also consider

$$\mathcal{F}_S(\mathbb{R}_+;H) := \{g \in \mathcal{C}^\infty(\mathbb{R}_+;H) \mid E_S g \in \mathcal{F}(\mathbb{R};H)\},$$
$$\mathcal{F}_A(\mathbb{R}_+;H) := \{g \in \mathcal{C}^\infty(\mathbb{R}_+;H) \mid E_A g \in \mathcal{F}(\mathbb{R};H)\}.$$

Lemma 1.4. *It holds*

$$\mathcal{F}_S(\mathbb{R}_+;H) \subset \mathcal{F}(\mathbb{R}_+;H), \qquad \mathcal{F}_0(\mathbb{R}_+;H) \subset \mathcal{F}_A(\mathbb{R}_+;H).$$

Proof. For $v \in \mathcal{F}_S(\mathbb{R}_+;H)$ it holds by definition $\tilde{v} = E_S v \in \mathcal{F}(\mathbb{R};H)$ and $\tilde{v}|_{\mathbb{R}_+} = v$, so that $v \in \mathcal{F}(\mathbb{R}_+;H)$. Hence $\mathcal{F}_S(\mathbb{R}_+;H) \subset \mathcal{F}(\mathbb{R}_+;H)$.

We then turn to the proof of the second identity. For $v \in \mathcal{F}_0(\mathbb{R}_+;H)$, we have $E_0 v \in \mathcal{C}^\infty(\mathbb{R};H)$. Hence for any $k \in \mathbb{N}$, it holds by continuity at $t = 0$

$$\lim_{\substack{t \to 0 \\ >}} v^{(k)}(t) = \lim_{\substack{t \to 0 \\ <}} D^k(E_0 v)(t) = 0,$$

so that

$$\lim_{\substack{t \to 0 \\ <}} D^k(E_A v)(t) = (-1)^{k+1} \lim_{\substack{t \to 0 \\ <}} v^{(k)}(-t) = (-1)^{k+1} \lim_{\substack{t \to 0 \\ >}} v^{(k)}(t) = \lim_{\substack{t \to 0 \\ >}} D^k(E_A v)(t).$$

It yields $E_A v \in \mathcal{C}^\infty(\mathbb{R};H)$. Moreover from the definition of E_A, we obtain by explicit computation that $|E_A g|^2_{H^k(\mathbb{R};H)} = 2\,|g|^2_{H^k(\mathbb{R}_+;H)} < \infty$ for every $k \in \mathbb{N}$, whence $E_A v \in \mathcal{F}(\mathbb{R};H)$. □

Lemma 1.5. *The embeddings $\mathcal{F}_S(\mathbb{R}_+;H) \subset H^{1/2}(\mathbb{R}_+;H)$ and $\mathcal{F}_A(\mathbb{R}_+;H) \subset H^{1/2}_{00}(\mathbb{R}_+;H)$ are dense.*

Proof. For $v \in \mathcal{F}_A(\mathbb{R}_+; H)$, we have $\|v\|_{H^{1/2}_{00}(\mathbb{R}_+;H)} \leq \|E_A v\|_{H^{1/2}(\mathbb{R};H)} < \infty$ due to (1.2b). Hence it holds $\mathcal{F}_A(\mathbb{R}_+; H) \subset H^{1/2}_{00}(\mathbb{R}_+; H)$ and the second result follows from $\mathcal{F}_0(\mathbb{R}_+; H) \subset \mathcal{F}_A(\mathbb{R}_+; H)$ and the fact that the embedding $\mathcal{F}_0(\mathbb{R}_+; H) \subset H^{1/2}_{00}(\mathbb{R}_+; H)$ is dense.

On the other hand, for $v \in H^{1/2}(\mathbb{R}_+; H)$, we have $\tilde{v} := E_S v \in H^{1/2}_S(\mathbb{R}; H)$. Hence from Lemma 1.2, there exists a sequence $(\tilde{v}_{S,k})_{k\in\mathbb{N}} \subset \mathcal{F}_S(\mathbb{R}; H)$ such that $\tilde{v}_{S,k} \to \tilde{v}$ in $H^{1/2}_S(\mathbb{R}; H)$ as $k \to \infty$. Since $R_> : \mathcal{F}_S(\mathbb{R}; H) \to \mathcal{F}_S(\mathbb{R}_+; H)$ is bijective, we define $v_{S,k} := R_> \tilde{v}_{S,k} \in \mathcal{F}_S(\mathbb{R}_+; H)$ for $k \in \mathbb{N}$. Moreover from (1.2a), we have

$$\|v - v_{S,k}\|_{H^{1/2}(\mathbb{R}_+;H)} \leq \frac{1}{\sqrt{2}} \|\tilde{v} - \tilde{v}_{S,k}\|_{H^{1/2}(\mathbb{R};H)} \underset{k\to\infty}{\longrightarrow} 0.$$

□

Proof. (Lemma hp−2.2) From $D^{1/2}_{\mathbb{R},+}\mathfrak{H} = D^{1/2}_{\mathbb{R},-}$ [4, Lemma 2.2] and (hp−2.9), it holds $\mathfrak{H} \in \mathcal{L}(H^{1/2}(\mathbb{R}; H))$. Furthermore, by definition, $\mathfrak{H} \in \mathcal{L}(H^{1/2}_A(\mathbb{R}; H), H^{1/2}_S(\mathbb{R}; H))$ and $\mathfrak{H} \in \mathcal{L}(H^{1/2}_S(\mathbb{R}; H), H^{1/2}_A(\mathbb{R}; H))$. From $\mathfrak{H}^{-1} = -\mathfrak{H}$ [7, Theorem 1 p.76], we obtain that the above operators are isomorphism. Furthermore, for any $v \in H^{1/2}(\mathbb{R}; H))$, we have from [7, (2.34) p.76] that $\|\mathfrak{H}v\|_{L^2(\mathbb{R};H)} = \|v\|_{L^2(\mathbb{R};H)}$ and

$$|\mathfrak{H}v|^2_{H^{1/2}(\mathbb{R};H)} = 2\pi \left\| D^{1/2}_{\mathbb{R},+}\mathfrak{H}v \right\|^2_{L^2(\mathbb{R};H)} = 2\pi \left\| D^{1/2}_{\mathbb{R},-}v \right\|^2_{L^2(\mathbb{R};H)} = |v|^2_{H^{1/2}(\mathbb{R};H)},$$

and the result follows. □

Proof. (Lemma hp−2.3) Let us first prove the result for E_S. A direct computation yields for $v \in \mathcal{F}_S(\mathbb{R}_+; H)$

$$|E_S v|^2_{H^{1/2}(\mathbb{R};H)} = 2 \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|v(s) - v(t)\|^2_H}{|s-t|^2} dsdt + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|v(s) - v(t)\|^2_H}{|s+t|^2} dsdt \right) \leq 4 |v|^2_{H^{1/2}(\mathbb{R}_+;H)},$$

where we have used the identity $(s+t)^2 \geq (s-t)^2$ for all $s, t \in \mathbb{R}_+$. Hence (1.2a) follows from $\|E_S v\|^2_{L^2(\mathbb{R};H)} = 2\|v\|^2_{L^2(\mathbb{R}_+;H)}$. For E_A, it holds $\|E_A v\|^2_{L^2(\mathbb{R};H)} = 2\|v\|^2_{L^2(\mathbb{R}_+;H)}$ and

$$\begin{aligned}
|E_A v|^2_{H^{1/2}(\mathbb{R};H)} &= 2 \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|v(s) - v(t)\|^2_H}{|s-t|^2} dsdt + \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|v(s) + v(t)\|^2_H}{|s+t|^2} dsdt \right) \\
&\leq 2 \left(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|v(s) - v(t)\|^2_H}{|s-t|^2} dsdt + 2\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|v(s)\|^2_H + \|v(t)\|^2_H}{|s+t|^2} dsdt \right) \\
&= 2 \left(|v|^2_{H^{1/2}(\mathbb{R}_+;H)} + 4\int_{\mathbb{R}_+} \frac{\|v(s)\|^2_H}{s} ds \right) \leq 8 |v|^2_{H^{1/2}_{00}(\mathbb{R}_+;H)},
\end{aligned}$$

which yields the second inequality of (1.2b). From (1.3a)-(1.3c), we have

$$\|(E_A + E_S)v\|^2_{H^{1/2}(\mathbb{R};H)} = \|E_S v\|^2_{H^{1/2}(\mathbb{R};H)} + \|E_A v\|^2_{H^{1/2}(\mathbb{R};H)}.$$

For (1.2c), we have that

$$\begin{aligned}
|E_S v|^2_{H^{1/2}(\mathbb{R};H)} + |E_A v|^2_{H^{1/2}(\mathbb{R};H)} &= 2 \left(2|v|^2_{H^{1/2}(\mathbb{R}_+;H)} + 2\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \frac{\|v(s)\|^2_H + \|v(t)\|^2_H}{|s+t|^2} dsdt \right) \\
&= 4 \left(|v|^2_{H^{1/2}(\mathbb{R}_+;H)} + 2\int_{\mathbb{R}_+} \frac{\|v(s)\|^2_H}{s} ds \right),
\end{aligned}$$

which gives the desired result since $\|E_S v\|^2_{L^2(\mathbb{R};H)} + \|E_A v\|^2_{L^2(\mathbb{R};H)} = 4\|v\|^2_{L^2(\mathbb{R}_+;H)}$. Using (1.2a) and (1.2c), we have

$$\begin{aligned}\|E_A v\|^2_{H^{1/2}(\mathbb{R};H)} &\geq 4\|v\|^2_{H^{1/2}_{00}(\mathbb{R}_+;H)} - \|E_S v\|^2_{H^{1/2}(\mathbb{R};H)}\\ &\geq 4\left(\|v\|^2_{H^{1/2}_{00}(\mathbb{R}_+;H)} - \|v\|^2_{H^{1/2}(\mathbb{R}_+;H)}\right) = 4\int_{\mathbb{R}_+} \frac{\|v(s)\|^2_H}{s} ds,\end{aligned}$$

so that

$$\begin{aligned}\|E_A v\|^2_{H^{1/2}(\mathbb{R};H)} &= \frac{1}{2}\|E_A v\|^2_{L^2(\mathbb{R};H)} + \frac{1}{2}|E_A v|^2_{H^{1/2}(\mathbb{R};H)} + \frac{1}{2}\|E_A v\|^2_{H^{1/2}(\mathbb{R};H)}\\ &\geq \|v\|^2_{L^2(\mathbb{R}_+;H)} + |v|^2_{H^{1/2}(\mathbb{R}_+;H)} + 2\int_{\mathbb{R}_+} \frac{\|v(s)\|^2_H}{s} ds \geq \|v\|^2_{H^{1/2}_{00}(\mathbb{R}_+;H)},\end{aligned}$$

which proves the first inequality of (1.2b).

Let us turn to the proof of (1.1). By definition, we have

$$E_S \in \mathcal{L}_{\text{Iso}}(\mathcal{F}_S(\mathbb{R}_+;H), \mathcal{F}_S(\mathbb{R};H)) \text{ and } E_A \in \mathcal{L}_{\text{Iso}}(\mathcal{F}_A(\mathbb{R}_+;H), \mathcal{F}_A(\mathbb{R};H)).$$

Moreover $E_S \in \mathcal{L}(H^{1/2}(\mathbb{R}_+;H), H^{1/2}_S(\mathbb{R};H))$ and $E_A \in \mathcal{L}(H^{1/2}_{00}(\mathbb{R}_+;H), H^{1/2}_A(\mathbb{R};H))$ follow from (1.2a)-(1.2b). The results then follow from Lemmas 1.2 and 1.5. □

Proof. (Lemma *hp*–2.4) Since the embedding $\mathcal{F}_S(\mathbb{R}_+;H) \subset H^{1/2}(\mathbb{R}_+;H)$ is dense, we prove the identity for $w \in \mathcal{F}_S(\mathbb{R}_+;H)$. From (*hp*–2.7) and (*hp*–2.8) we have

$$\begin{aligned}-\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_A v, D^{1/2}_{\mathbb{R},-} E_S w\right)_H dt &= \int_{\mathbb{R}} (E_A v, D E_S w)_H \, dt\\ &= \int_{\mathbb{R}_+} (v(t), Dw(t))_H \, dt - \int_{\mathbb{R}_-} (v(-t), D\,(w(-t)))_H \, dt\\ &= \int_{\mathbb{R}_+} (v(t), Dw(t))_H \, dt + \int_{\mathbb{R}_-} (v(-t), Dw(-t))_H \, dt\\ &= 2\int_{\mathbb{R}_+} (v, Dw)_H \, dt\\ &= -2\int_{\mathbb{R}_+} \left(D^{1/2}_+ v, D^{1/2}_- w\right)_H dt,\end{aligned}$$

which is (1.3a). Furthermore, the identity

$$\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_A v, D^{1/2}_{\mathbb{R},-} E_S v\right)_H dt = 0$$

is a consequence of $\int_{\mathbb{R}_+} (Dv, v)_H \, dt = 0$ and the above computation. From Lemma 1.3, we obtain

$$\int_{\mathbb{R}} \left(D^{1/2}_{\mathbb{R},+} E_S v, D^{1/2}_{\mathbb{R},-} E_S w\right)_H dt = 0.$$

Furthermore $\int_{\mathbb{R}} (E_A v, E_S w) \, dt = 0$ and (1.3b) follow from (1.6). Finally, using a similar development as for the proof of (1.3a), we obtain (1.4). □

2 Proof of Proposition hp–2.5

Proposition (Proposition hp–2.5)**.** *Let $f \in L^2(\mathbb{R}_+;H)$ satisfy* $\operatorname{supp}(f) \subset \overline{I} = [0,T]$ *and $f \in \mathscr{G}^\delta(I;H)$ for some $0 < T < \infty$ and $\delta \geq 1$. Then for any $T < T_\infty \leq \infty$ we have for $I_\infty = (T, T_\infty)$*

$$u|_I \in \mathscr{G}^{\delta,1}(I;H) \cap \mathscr{G}^{\delta,1/2}(I;V), \qquad u|_{I_\infty} \in \mathscr{G}^{1,1/2}(I_\infty;H) \cap \mathscr{G}^{1,0}(I_\infty;V),$$

where $u \in \mathcal{X}$ is the solution of (hp–2.11) associated to f.

In order to prove this result, we consider a spectral decomposition of the operator $\mathcal{A}$. The technique used here follows the same lines as the one used in [8, Chapter 2], where a similar result is presented for analytic right-hand side.

Under the assumption that $\mathcal{A} \in \mathcal{L}(V, V^*)$ is a self-adjoint operator satisfying (hp–2.10a)-(hp–2.10b), we know that [9, Theorem 1 p.325] there exists sequences $(w_n)_{n\in\mathbb{N}} \subset V$ and $(\lambda_n)_{n\in\mathbb{N}}$ such that $\lambda_{n+1} > \lambda_n$ and

$$\mathcal{A}v = \sum_{n\in\mathbb{N}} \lambda_n (v, w_n)_H \, w_n, \qquad v \in V.$$

The sequence $(w_n)_{n\in\mathbb{N}}$ forms an orthonormal basis of H and an orthogonal basis of V. For $\theta \in [0,1]$, we define

$$\mathcal{A}^\theta v = \sum_{n\in\mathbb{N}} \lambda_n^\theta (v, w_n)_H \, w_n,$$

and $D(\mathcal{A}^\theta) := \left\{ u \in V \;\middle|\; \mathcal{A}^\theta u \in H \right\}$ induced with the norm

$$\|v\|_{D(\mathcal{A}^\theta)} := \left(\sum_{n\in\mathbb{N}} \lambda_n^{2\theta} \left| (w_n, v)_H \right|^2 \right)^{1/2}.$$

From (hp–2.10a)-(hp–2.10b), it holds

$$\lambda_- \|v\|_V^2 \leq \|v\|_{D(\mathcal{A}^{1/2})}^2 \leq \lambda_+ \|v\|_V^2 .$$

We then have the following result.

Lemma 2.1. *It holds $\mathfrak{B} \in \mathcal{L}_{\mathrm{Iso}}(H_0^1(\mathbb{R}_+;H) \cap L^2(\mathbb{R}_+;D(\mathcal{A})), L^2(\mathbb{R}_+;H))$, where $\mathfrak{B}$ is defined in (hp–1.1a). Moreover, if u denotes the unique solution of (hp–2.11) for $f \in L^2(\mathbb{R}_+;H)$, then $u \in \mathcal{C}(\mathbb{R}_+;H)$ admits the following representation in H for $t \in \mathbb{R}_+$*

$$u(t) = \sum_{i=1}^{\infty} \left[\int_0^t (f(s), w_i)_H \, e^{-\lambda_i (t-s)} ds \right] w_i. \tag{2.1}$$

Proof. The lemma can be proven following the lines of [3, Section 7.1.2]; see also [8, Chapter 2]. □

Based on the representation (2.1), we introduce for $t \geq 0$ the operator

$$S(t)v := \sum_{i=1}^{\infty} e^{-\lambda_i t} (v, w_i)_H \, w_i, \qquad \forall v \in H.$$

We then have the semigroup property for $s, t \geq 0$

$$S(t+s) = S(t)S(s), \qquad S(0) = \mathrm{Id},$$

and we can rewrite (2.1) as

$$u(t) = \int_0^t S(t-s) f(s) ds = \int_0^t S(s) f(t-s) ds.$$

Lemma 2.2. *For $t > 0$, $l \in \mathbb{N}_0$ and $r, s \in [0,1]$ such that $l + s - r \geq 0$, we have*

$$\left\| S^{(l)}(t) \right\|_{\mathcal{L}(D(\mathcal{A}^r), D(\mathcal{A}^s))} \leq \Gamma\left(l + s - r + \frac{1}{2}\right) t^{-(l+s-r)}.$$

Proof. For $v \in H$, we define $v_i := (v, w_i)_H$ and it holds for $l \in \mathbb{N}_0$

$$S^{(l)}(t)v = \sum_{i=1}^{\infty} (-\lambda_i)^l e^{-\lambda_i t} v_i w_i.$$

Hence it follows

$$\left\| S^{(l)}(t)v \right\|_{D(\mathcal{A}^s)}^2 = \sum_{i=1}^{\infty} \lambda_i^{2(l+s)} e^{-2\lambda_i t} v_i^2. \tag{2.2}$$

Given $p \geq 0$, let us consider the function $g(\lambda) := \lambda^{2p} e^{-2\lambda t}$. Then

$$\lambda_{\max} := \arg\max_{\lambda \in \mathbb{R}_+} g(\lambda) = \frac{p}{t}, \qquad g(\lambda_{\max}) = \left(\frac{p}{t}\right)^{2p} e^{-2p},$$

so that for $l + s - r \geq 0$

$$\begin{aligned} \left\| S^{(l)}(t)v \right\|_{D(\mathcal{A}^s)}^2 &\leq \left(\frac{l+s-r}{t}\right)^{2(l+s-r)} e^{-2(l+s-r)} \sum_{i=1}^{\infty} \lambda_i^{2r} v_i^2 \\ &= \left(\frac{l+s-r}{t}\right)^{2(l+s-r)} e^{-2(l+s-r)} \left\| v \right\|_{D(\mathcal{A}^r)}^2. \end{aligned}$$

Let us first assume that $0 \leq l + s - r < 1/2$. Then the result follows from $x^{2x} e^{-2x} \leq 1 \leq \Gamma\left(x + \frac{1}{2}\right)$ for $x \in [0, 1/2)$. Moreover if $l + s - r \geq 1/2$, [6, 5.6.1] yields

$$(2(l+s-r))^{2(l+s-r)} e^{-2(l+s-r)} \leq \sqrt{\frac{2(l+s-r)}{2\pi}} \Gamma(2(l+s-r)).$$

From [5, (1.08) p.35], it can be shown that for $s \geq 1/2$ it holds

$$\Gamma(2s) \sqrt{\frac{s}{\pi}} \left(\frac{1}{2}\right)^{2s} < \Gamma\left(s + \frac{1}{2}\right)^2.$$

Hence

$$\begin{aligned} \left\| S^{(l)}(t)v \right\|_{D(\mathcal{A}^s)}^2 &\leq \Gamma(2(l+s-r)) \sqrt{\frac{(l+s-r)}{\pi}} \left(\frac{1}{2}\right)^{2(l+s-r)} t^{-2(l+s-r)} \left\| v \right\|_{D(\mathcal{A}^r)}^2 \\ &\leq \Gamma\left(l + s - r + \frac{1}{2}\right)^2 t^{-2(l+s-r)} \left\| v \right\|_{D(\mathcal{A}^r)}^2. \end{aligned}$$

□

Following the lines of the proof we see that in the case $l + s - r < 0$ it holds

$$\left\| S^{(l)}(t) \right\|_{\mathcal{L}(D(\mathcal{A}^r), D(\mathcal{A}^s))} \leq \lambda_1^{l+s-r}.$$

This is used to derive the bound for $\| S(t) \|_{\mathcal{L}(V,H)}$ in the following corollary.

Corollary 2.2.1. *For $t > 0$ we have*

$$\left\|S^{(l)}(t)\right\|_{\mathcal{L}(H,V)} \leq \frac{1}{\sqrt{\lambda_-}}\Gamma\left(l+1\right)t^{-\left(l+\frac{1}{2}\right)}, \qquad l \in \mathbb{N}_0, \tag{2.3a}$$

$$\left\|S^{(l)}(t)\right\|_{\mathcal{L}(V)} \leq \sqrt{\frac{\lambda_+}{\lambda_-}}\Gamma\left(l+\frac{1}{2}\right)t^{-l}, \qquad l \in \mathbb{N}_0, \tag{2.3b}$$

$$\left\|S^{(l)}(t)\right\|_{\mathcal{L}(H)} \leq \Gamma\left(l+\frac{1}{2}\right)t^{-l}, \qquad l \in \mathbb{N}_0, \tag{2.3c}$$

$$\left\|S^{(l)}(t)\right\|_{\mathcal{L}(V,H)} \leq \sqrt{\lambda_+}\Gamma\left(l\right)t^{-\left(l-\frac{1}{2}\right)}, \qquad l \in \mathbb{N}, \tag{2.3d}$$

and

$$\|S(t)\|_{\mathcal{L}(V,H)} \leq \sqrt{\frac{\lambda_+}{\lambda_1}}. \tag{2.4}$$

Lemma 2.3. *Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\operatorname{supp}(f) \subset \overline{I} = [0,T]$ for some $0 < T < \infty$. Then for $t > T$ we have for $l \in \mathbb{N}_0$*

$$u^{(l)}(t) = S^{(l)}(t-T)u(T). \tag{2.5}$$

Moreover if there exists $m \in \mathbb{N}_0$ such that $f|_I \in H^m(I;H)$, it holds

$$u^{(m)}(t) = \sum_{i=0}^{m-1} S^{(i)}(t)f^{(m-1-i)}(0) + \int_0^t S(s)f^{(m)}(t-s)ds, \qquad t \in I, \tag{2.6}$$

where the sum is empty for $m = 0$.

Proof. The proof of (2.6) can be found in [8, Lemma 2.8]. Note that by the fact that the embedding $H^l(I;H) \subset \mathcal{C}^{l-1}(\overline{I};H)$ is continuous and the bounds from Corollary 2.2.1, the right-hand side of (2.6) is well-defined.

Let us turn to (2.5). First note that the right-hand side is well-defined by Lemma 2.1. We then have that for any $t > T$

$$\begin{aligned} u(t) &= \int_0^t S(t-s)f(s)ds = \int_0^T S(t-s)f(s)ds + \int_T^t S(t-s)f(s)ds \\ &= S(t-T)\int_0^T S(T-s)f(s)ds = S(t-T)u(T), \end{aligned}$$

and the result follows. ☐

Proof. (Proposition *hp*−2.5) From [1, Theorem 6], it holds

$$B(a,b) \leq \frac{1}{ab}, \qquad a,b > 1,$$

where $B(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}dt$ denotes the *Beta function.* Using the identity $\Gamma(a)\Gamma(b) = B(a,b)\Gamma(a+b)$ [6, (5.12.1)], we have

$$\Gamma(a)\Gamma(b) = B(a,b)\Gamma(a+b) = B(a,b)\Gamma(a+b-1)(a+b-1) \leq \Gamma(a+b-1), \qquad a,b \geq 1,$$

where we have used $ab \geq a + b - 1$. Hence from Lemma 2.3 and (2.3c), we have for $t \in I$ and $l \in \mathbb{N}$

$$
\begin{aligned}
\left\| u^{(l)}(t) \right\|_H &\leq \left\| S(t) f^{(l-1)}(0) \right\|_H + \sum_{i=1}^{l-1} \left\| S^{(i)}(t) f^{(l-1-i)}(0) \right\|_H + \int_0^t \left\| S(s) f^{(l)}(t-s) \right\|_H ds \\
&\leq C_f \left(d_f^{l-1} \Gamma(l)^\delta + \sum_{i=1}^{l-1} \Gamma\left(i + \frac{1}{2}\right) d_f^{l-i-1} \Gamma(l-i)^\delta t^{-i} + \int_0^t d_f^l \Gamma(l+1)^\delta ds \right) \\
&\leq C_f \left(\min_{s \in \mathbb{R}} \Gamma(s) \right)^{1-\delta} \left(d_f^{l-1} \Gamma(l)^\delta + d_f^{l-1} \Gamma\left(l - \frac{1}{2}\right)^\delta \sum_{i=1}^{l-1} (dt)^{-i} + d_f^l \Gamma(l+1)^\delta t \right) \\
&\leq \widetilde{C}_f d_f^{l-1} \Gamma(l+1)^\delta \left(\frac{1}{l} \sum_{i=0}^{l-1} (dt)^{-i} + dt \right) \\
&= \widetilde{C}_f d_f^{l-1} \Gamma(l+1)^\delta \left(\frac{1}{l} \sum_{i=0}^{l-1} (dT)^{-i} \left(\frac{T}{t}\right)^i + dt \right) \\
&\leq \widetilde{C}_f d_f^{l-1} \Gamma(l+1)^\delta \left(\max\left(1, (dT)^{1-l}\right) \left(\frac{T}{t}\right)^{l-1} + dt \right) \\
&\leq C d_f^{l-1} \Gamma(l+1)^\delta \max\left(1, (d_f T)^{1-l}\right) \left(\frac{T}{t}\right)^{l-1} \\
&\leq C \Gamma(l+1)^\delta \max\left(1, (d_f T)^l\right) t^{1-l},
\end{aligned}
$$

where $C = 2C_f \max(1, d_f T) \left(\min_{s \in \mathbb{R}} \Gamma(s)\right)^{1-\delta}$. Note that we have used $\|S(t)\|_{\mathcal{L}(H)} \leq 1$, which can be obtained from (2.2). Hence $u|_I \in \mathscr{G}^{\delta,1}(I; H)$. Similarly we prove $u|_I \in \mathscr{G}^{\delta,1/2}(I; V)$ using (2.3a). The results for the interval I_∞ follow from Lemma 2.3, (2.3b), (2.3d), (2.4) and the bound

$$
\|u(T)\|_V \leq \int_0^T \|S(t-s) f(s)\|_V ds \leq \frac{C_f}{\sqrt{\lambda_-}} \int_0^T t^{-1/2} dt = \frac{2C_f}{\sqrt{\lambda_-}} T^{1/2}. \tag{2.7}
$$

□

3 Proof of Lemma hp–4.1

Lemma (Lemma hp–4.1)**.** *Let $f \in L^2(\mathbb{R}_+; H)$ satisfy $\operatorname{supp}(f) \subset [0, T]$ for some $0 < T < \infty$. Then there exist $C, \tau > 0$ such that for any $T < T_\infty < \infty$ it holds*

$$
\begin{aligned}
\left\| u - \pi_{N_t}^{\sigma,\mu,\delta} u \right\|_{L^2((T_\infty,\infty); V)} &\leq C e^{-\tau(T_\infty - T)}, \\
\left\| u - \pi_{N_t}^{\sigma,\mu,\delta} u \right\|_{H_{00}^{1/2}((T_\infty,\infty); H)} &\leq C e^{-\tau(T_\infty - T)}.
\end{aligned}
$$

Proof. We estimate the errors as

$$
\begin{aligned}
\left\| u - \pi_{N_t}^{\sigma,\mu,\delta} u \right\|_{L^2((T_\infty,\infty); H)}^2 &\leq 2 \left(\|u\|_{L^2((T_\infty,\infty); H)}^2 + \left\| \pi_{N_t}^{\sigma,\mu,\delta} u \right\|_{L^2((T_\infty,\infty); H)}^2 \right), \\
\left| u - \pi_{N_t}^{\sigma,\mu,\delta} u \right|_{H^{1/2}((T_\infty,\infty); H)}^2 &\leq 2 \left(|u|_{H^{1/2}((T_\infty,\infty); H)}^2 + \left| \pi_{N_t}^{\sigma,\mu,\delta} u \right|_{H^{1/2}((T_\infty,\infty); H)}^2 \right),
\end{aligned}
$$

and treat the boundary term of the $\|\cdot\|_{H_{00}^{1/2}((T_\infty,\infty); H)}$-norm separately. From (2.5) and the fact that $(\lambda_n)_{n \in \mathbb{N}}$

is an increasing sequence, we have

$$\begin{aligned}
\|u\|^2_{L^2((T_\infty,\infty);H)} &= \sum_{i=1}^{\infty} (u(T),w_i)_H^2 \int_{T_\infty}^{\infty} e^{-2\lambda_i(t-T)}dt = \sum_{i=1}^{\infty} (u(T),w_i)_H^2 \,\frac{e^{-2\lambda_i(T_\infty-T)}}{2\lambda_i} \\
&\le \frac{e^{-2\lambda_1(T_\infty-T)}}{2\lambda_1}\,\|u(T)\|_H^2\,, \\
\lambda_-\,\|u\|^2_{L^2((T_\infty,\infty);V)} &\le \sum_{i=1}^{\infty} \lambda_i\,(u(T),w_i)_H^2 \int_{T_\infty}^{\infty} e^{-2\lambda_i(t-T)}dt = \sum_{i=1}^{\infty} (u(T),w_i)_H^2 \,\frac{e^{-2\lambda_i(T_\infty-T)}}{2} \\
&\le \frac{e^{-2\lambda_1(T_\infty-T)}}{2}\,\|u(T)\|_H^2\,, \\
|u|^2_{H^{1/2}((T_\infty,\infty);H)} &= \sum_{i=1}^{\infty} e^{2\lambda_i T}(u(T),w_i)_H^2 \int_{T_\infty}^{\infty}\int_{T_\infty}^{\infty} \left(\frac{e^{-\lambda_i s}-e^{-\lambda_i t}}{s-t}\right)^2 dsdt \\
&= \left|e^{-s}\right|^2_{H^{1/2}(\mathbb{R}_+)} \sum_{i=1}^{\infty} e^{-2\lambda_i(T_\infty-T)}(u(T),w_i)_H^2 \le Ce^{-2\lambda_1(T_\infty-T)}\,\|u(T)\|_H^2\,,
\end{aligned}$$

where $C=|e^{-s}|^2_{H^{1/2}(\mathbb{R}_+)}$ is independent of T_∞, T and λ_i for all $i\in\mathbb{N}$. Furthermore

$$\|u(T_\infty)\|_H^2 \le \|u(T_\infty)\|_V^2 \le e^{-2\lambda_1(T_\infty-T)}\,\|u(T)\|_V^2\,,$$

and

$$\begin{aligned}
\left\|\pi_{N_t}^{\sigma,\mu,\delta}u\right\|^2_{L^2((T_\infty,\infty);H)} &= \frac{\|u(T_\infty)\|_H^2}{h^2_{M_1+M_2+1}}\int_{I_{M_1+M_2+1}}(t_{M_1+M_2+1}-t)^2dt \le \frac{\|u(T)\|_V^2}{3}h_{M_1+M_2+1}e^{-2\lambda_1(T_\infty-T)}, \\
\left\|\pi_{N_t}^{\sigma,\mu,\delta}u\right\|^2_{L^2((T_\infty,\infty);V)} &= \frac{\|u(T_\infty)\|_V^2}{h^2_{M_1+M_2+1}}\int_{I_{M_1+M_2+1}}(t_{M_1+M_2+1}-t)^2dt \le \frac{\|u(T)\|_V^2}{3}h_{M_1+M_2+1}e^{-2\lambda_1(T_\infty-T)}, \\
\left|\pi_{N_t}^{\sigma,\mu,\delta}u\right|^2_{H^{1/2}((T_\infty,\infty);H)} &= \frac{\|u(T_\infty)\|_H^2}{h^2_{M_1+M_2+1}}\left|(t_{M_1+M_2+1}-t)\chi_{I_{M_1+M_2+1}}\right|^2_{H^{1/2}(T_\infty,\infty)} = 2\,\|u(T_\infty)\|_H^2 \\
&\le 2\,\|u(T)\|_V^2\,e^{-2\lambda_1(T_\infty-T)}.
\end{aligned}$$

Finally for the boundary term, it holds

$$\int_{T_\infty}^{\infty}\frac{\left\|\left(u-\pi_{N_t}^{\sigma,\mu,\delta}u\right)(t)\right\|_H^2}{t-T_\infty}dt = \int_{T_\infty}^{t_{M_1+M_2+1}}\frac{\left\|\left(u-\pi_{N_t}^{\sigma,\mu,\delta}u\right)(t)\right\|_H^2}{t-T_\infty}dt + \int_{t_{M_1+M_2+1}}^{\infty}\frac{\|u(t)\|_H^2}{t-T_\infty}dt.$$

The second term can be bounded as

$$\int_{t_{M_1+M_2+1}}^{\infty}\frac{\|u(t)\|_H^2}{t-T_\infty}dt \le \frac{1}{h_{M_1+M_2+1}}\int_{t_{M_1+M_2+1}}^{\infty}\|u(t)\|_H^2\,dt \le \frac{e^{-2\lambda_1(T_\infty-T)}}{2\lambda_1 h_{M_1+M_2+1}}\,\|u(T)\|_H^2\,.$$

Denoting $\delta = h_{M_1+M_2+1}$, we obtain for the first term

$$\int_{T_\infty}^{T_\infty+\delta}\frac{\left\|\left(u-\pi_{N_t}^{\sigma,\mu,\delta}u\right)(t)\right\|_H^2}{t-T_\infty}dt = \sum_{i=1}^{\infty}\frac{(u(T_\infty),w_i)_H^2}{\delta^2}\int_{T_\infty}^{T_\infty+\delta}\frac{\left|\delta e^{-\lambda_i(t-T_\infty)}-(T_\infty+\delta-t)\right|^2}{t-T_\infty}dt.$$

Using that for $\xi>0$ it holds

$$\max_{t\ge 0}\frac{\left(1-e^{-\xi t}\right)^2}{t} = \xi\max_{x\ge 0}\frac{\left(1-e^{-x}\right)^2}{x} =: C\xi, \qquad \operatorname*{arg\,min}_{x\in\mathbb{R}_+}\frac{\left(1-e^{-x}\right)^2}{x} = -W\left(-\frac{1}{2\sqrt{e}}\right)-\frac{1}{2},$$

we have

$$\begin{aligned}\frac{1}{\delta^2}\int_{T_\infty}^{T_\infty+\delta}\frac{\left|\delta e^{-\lambda_i(t-T_\infty)}-(T_\infty+\delta-t)\right|^2}{t-T_\infty}dt &= \frac{1}{\delta^2}\int_0^\delta\frac{\left|\delta e^{-\lambda_i t}-(\delta-t)\right|^2}{t}dt \\ &= \int_0^1\frac{\left|e^{-\delta\lambda_i t}-(1-t)\right|^2}{t}dt \\ &\le 2\left(\int_0^1\frac{\left|e^{-\delta\lambda_i t}-1\right|^2}{t}dt+\int_0^1 tdt\right) \\ &\le 2\left(C\delta\lambda_i+\frac{1}{2}\right),\end{aligned}$$

where $W(\cdot)$ denotes the Lambert W-function [6, 4.13 p.111]. Hence

$$\begin{aligned}\int_{T_\infty}^{t_{M_1+M_2+1}}\frac{\left\|\left(u-\pi_{N_t}^{\sigma,\mu,\delta}u\right)(t)\right\|_H^2}{t-T_\infty}dt &\le (2Ch_{M_1+M_2+1}+1)\left\|u(T_\infty)\right\|_V^2 \\ &\le (2Ch_{M_1+M_2+1}+1)\left\|u(T)\right\|_V^2 e^{-2\lambda_1(T_\infty-T)}.\end{aligned}$$

The result then follows from (2.7) and $\|\cdot\|_H \le \|\cdot\|_V$. □

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