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Abstract

The scientific literature contains a number of numerical approximation results for stochastic partial differential equations (SPDEs) with superlinearly growing nonlinearities but, to the best of our knowledge, none of them prove strong or weak convergence rates for full-discrete numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities. In particular, in the scientific literature there exists neither a result which proves strong convergence rates nor a result which proves weak convergence rates for full-discrete numerical approximations of stochastic Allen-Cahn equations. In this article we bridge this gap and establish strong convergence rates for full-discrete numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities such as stochastic Allen-Cahn equations. Moreover, we also establish lower bounds for strong temporal and spatial approximation errors which demonstrate that our strong convergence rates are essentially sharp and can, in general, not be improved.

Contents

1 Introduction

In this article we are interested in strong convergence rates for full-discrete numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities such as stochastic Allen-Cahn equations. The literature contains a number of numerical approximation results for SPDEs with superlinearly growing nonlinearities (cf., e.g., Gyöngy & Millet [8], Gyöngy, Sabanis, & Šiška [9], Jentzen & Pušnik [16], Kovács, Larsson, & Lindgren [19], Becker & Jentzen [3], Hutzenthaler, Jentzen, & Salimova [13], Jentzen & Pušnik [17], Furihata et al. [7], and Blömker & Kamrani [4]). The articles [8, 9, 7, 13] establish strong convergence of numerical approximations for such SPDEs with no information on the speed of strong convergence and the papers [16, 19, 3] prove strong convergence rates for numerical approximations of such SPDEs. To be more specific, the article [3] establishes strong convergence rates for semi-discrete temporal numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities such as stochastic Allen-Cahn equations. The papers [16, 19] prove strong convergence rates for full-discrete (temporal and spatial discrete) numerical approximations for SPDEs with superlinearly growing nonlinearities in the case of the more regular trace class noise. To the best of our knowledge, there exists no result in the scientific literature which establishes strong or weak convergence rates for a full-discrete numerical approximation scheme of a space-time white noise driven SPDE with a superlinearly growing nonlinearity such as the stochastic Allen-Cahn equation. A key difficulty in the case of full-discrete numerical approximations for space-time white noise driven SPDEs with superlinearly growing nonlinearities is to derive appropriate uniform a priori moment bounds for the numerical approximation processes.

In this article we overcome this difficulty (cf. (7)–(8) below for our approach to this challenge) and establish essentially sharp strong convergence rates for full-discrete numerical approximations of space-time white noise driven SPDEs with superlinearly growing nonlinearities such as stochastic Allen-Cahn equations; see Theorem 5.5 in Section 5 below for the main convergence rate result in this work. To illustrate Theorem 5.5, we now present in Theorem 1.1 below the specialization of Theorem 5.5 to the case of stochastic Allen-Cahn equations.

Theorem 1.1. *Let* $T \in (0,\infty)$, $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2((0,1);\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2((0,1);\mathbb{R})}, \|\cdot\|_{L^2((0,1);\mathbb{R})})$, $a_0, a_1, a_2 \in \mathbb{R}$, $a_3 \in (-\infty, 0]$, $(e_n)_{n\in\mathbb{N}} \subseteq H$, $(P_n)_{n\in\mathbb{N}} \subseteq L(H)$, $F\colon L^6((0,1);\mathbb{R}) \to H$ *satisfy for all* $n \in \mathbb{N}$, $v \in L^6((0,1);\mathbb{R})$ *that* $e_n(\cdot) = \sqrt{2}\sin(n\pi(\cdot))$, $F(v) = \sum_{k=0}^{3} a_k v^k$, $P_n(v) = \sum_{k=1}^{n} \langle e_k, v\rangle_H \, e_k$, *and* $a_2 \mathbb{1}_{[0,\infty)}(a_3) = 0$, *let* $A\colon D(A) \subseteq H \to H$ *be the Laplacian with Dirichlet boundary conditions on* H, *let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, let* $(W_t)_{t\in[0,T]}$ *be an* Id_H*-cylindrical Wiener process, let* $\xi \in D((-A)^{1/2})$, $\gamma \in (1/6, 1/4)$, $\chi \in (0, \gamma/3 - 1/18]$, *let* $\mathcal{O}^{M,N}\colon [0,T] \times \Omega \to P_N(H)$, $M, N \in \mathbb{N}$, *and*

$\mathcal{X}^{M,N}\colon [0,T]\times\Omega\to P_N(H)$, $M,N\in\mathbb{N}$, be stochastic processes which satisfy that for all $M,N\in\mathbb{N}$, $m\in\{0,1,2,\dots,M-1\}$, $t\in({}^{mT}/_{M},{}^{(m+1)T}/_{M}]$ we have $\mathbb{P}$-a.s. that

$$\mathcal{O}_0^{M,N}=\mathcal{X}_0^{M,N}=P_N\,\xi,\qquad \mathcal{O}_t^{M,N}=e^{(t-mT/M)A}\left[\mathcal{O}_{mT/M}^{M,N}+\int_{mT/M}^{t}P_N\,dW_s\right],\tag{1}$$

and

$$\begin{aligned}\mathcal{X}_t^{M,N}&=e^{(t-mT/M)A}\,\mathcal{X}_{mT/M}^{M,N}+\mathcal{O}_t^{M,N}-e^{(t-mT/M)A}\,\mathcal{O}_{mT/M}^{M,N}\\&\quad+P_NA^{-1}(e^{(t-mT/M)A}-\mathrm{Id}_H)\,\mathbb{1}_{\{\|(-A)^\gamma\mathcal{X}_{mT/M}^{M,N}\|_H+\|(-A)^\gamma\mathcal{O}_{mT/M}^{M,N}\|_H\le(M/T)^\chi\}}\,F(\mathcal{X}_{mT/M}^{M,N}).\end{aligned}\tag{2}$$

Then

(i) we have that there exists an up to indistinguishability unique stochastic process $X\colon[0,T]\times\Omega\to L^6((0,1);\mathbb{R})$ with continuous sample paths which satisfies for all $t\in[0,T]$, $p\in(0,\infty)$ that $\sup_{s\in[0,T]}\mathbb{E}\big[\|X_s\|_{L^6((0,1);\mathbb{R})}^p\big]<\infty$ and

$$\mathbb{P}\bigg(X_t=e^{tA}\xi+\int_0^t e^{(t-s)A}F(X_s)\,ds+\int_0^t e^{(t-s)A}\,dW_s\bigg)=1,\tag{3}$$

(ii) we have for all $p\in(0,\infty)$ that $\sup_{r\in(-\infty,\gamma]}\sup_{M,N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\|(-A)^r\mathcal{X}_t^{M,N}\|_H^p\big]<\infty$, and

(iii) we have for all $p,\varepsilon\in(0,\infty)$ that there exists a real number $C\in\mathbb{R}$ such that for all $M,N\in\mathbb{N}$ it holds that

$$\sup_{t\in[0,T]}\Big(\mathbb{E}\big[\|X_t-\mathcal{X}_t^{M,N}\|_H^p\big]\Big)^{1/p}\le C(M^{(\varepsilon-1/4)}+N^{(\varepsilon-1/2)}).\tag{4}$$

Theorem 1.1 follows from Corollary 6.11 which, in turn, follows from our main result, Theorem 5.5 below. Theorem 5.5 also proves strong convergence rates for full-discrete numerical approximations of a more general class of SPDEs than Theorem 1.1 above. Next we would like to point out that the numerical approximation scheme (2) has been proposed in Hutzenthaler, Jentzen, & Salimova [13] and has there been referred to as a nonlinearity-truncated approximation scheme (cf. [13, (3) in Section 1] and, e.g., [11, 29, 12, 10, 28, 25, 26, 9, 16, 17] for further research articles on explicit approximation schemes for stochastic differential equations with superlinearly growing nonlinearities). Moreover, note that Theorem 1.1 demonstrates that the full-discrete numerical approximations in (2) converge for every $\varepsilon\in(0,\infty)$ strongly to the solution of the stochastic Allen-Cahn equation (3) with the spatial rate of convergence $1/2-\varepsilon$ and the temporal rate of convergence $1/4-\varepsilon$. We also would like to point out that the strong convergence rates established in Theorem 1.1 can, in general, not essentially be improved. More formally, Corollary 7.7 below proves in the case where $\sum_{i=0}^{3} |a_i| = 0$ and $\xi = 0$ in the framework of Theorem 1.1 that there exist real numbers $c, C \in (0, \infty)$ such that for all $M, N \in \mathbb{N}$ we have that

$$c\, M^{-1/4} \leq \lim_{n \to \infty} \sup_{t \in [0,T]} \Big(\mathbb{E}\big[\| X_t - \mathcal{X}_t^{M,n} \|_H^p \big] \Big)^{1/p} \leq C M^{-1/4} \tag{5}$$

and

$$c\, N^{-1/2} \leq \lim_{m \to \infty} \sup_{t \in [0,T]} \Big(\mathbb{E}\big[\| X_t - \mathcal{X}_t^{m,N} \|_H^p \big] \Big)^{1/p} \leq C N^{-1/2}. \tag{6}$$

Inequalities (5) and (6) thus show that the spatial rate $1/2 - \varepsilon$ and the temporal rate $1/4 - \varepsilon$ established in Theorem 1.1 can essentially not be improved. Further related lower bounds for strong approximation errors in the linear case $\sum_{i=0}^{3} |a_i| = 0$ can, e.g., be found in Müller-Gronbach, Ritter, & Wagner [23, Theorem 1], Müller-Gronbach & Ritter [22, Theorem 1], Müller-Gronbach, Ritter, & Wagner [24, Theorem 4.2], Conus, Jentzen, & Kurniawan [5, Lemma 6.2], and Jentzen & Kurniawan [15, Corollary 9.4].

Finally, we would like to add some comments on the proof of Theorem 1.1 above and Theorem 5.5 below, respectively. The main difficulty to prove Theorem 1.1 is to obtain uniform a priori moment bounds for the space-time discrete numerical approximations (2) (see Section 2 and Section 5.4 below). Once the uniform a priori moment bounds have been established, we exploit the fact that the nonlinearity of the stochastic Allen-Cahn equation satisfies a global monotonicity property to prevent that the local discretization errors accumulate too quickly. It thus remains to sketch our procedure to establish uniform a priori moment bounds for the numerical approximations. We first subtract the noise process from (2) as it is often done in the literature. The key idea that we use to derive uniform a priori bounds for the subtracted equation is then to employ a suitable path-dependent Lyapunov-type function which on the one hand incorporates the dissipative dynamics of the stochastic Allen-Cahn equation (3) and which on the other hand respects the spatial spectral Galerkin approximations used for the spatial discretization of (3). More formally, a key contribution of this work is to reveal that there exists a suitable $\mathcal{B}(\mathcal{C}([0,T], L^\infty((0,1);\mathbb{R})))/\mathcal{B}([0,\infty))$-measurable mapping $\phi\colon \mathcal{C}([0,T], L^\infty((0,1);\mathbb{R})) \to [0,\infty)$ such that for every $N \in \mathbb{N}$ we have that the mapping

$$P_N(H) \times \mathcal{C}([0,T], L^\infty((0,1);\mathbb{R})) \ni (v, w) \mapsto \|(-A)^{1/2} v\|_H^2 + \phi(w) \|v\|_H^2 \in \mathbb{R} \tag{7}$$

is an appropriate path dependent Lyapunov-type function for the system of the N-dimensional spatial spectral Galerkin approximation of the subtracted equation associated to the stochastic Allen-Cahn equation (3) (variable $v \in P_N(H)$) and the N-dimensional spatial spectral Galerkin approximation of the Ornstein-Uhlenbeck process (variable $w \in \mathcal{C}([0,T], L^\infty((0,1);\mathbb{R}))$). It is crucial that the Lyapunov-type function (7) does not only depend on w_T but on the whole path w_t, $t \in [0,T]$, of w. Applying the fundamental theorem of calculus to (7) results, roughly speaking, in the coercivity type condition that there exist real numbers $\epsilon \in [0,1)$, $c \in (0,\infty)$ and

$\mathcal{B}(\mathcal{C}([0,T],L^\infty((0,1);\mathbb{R})))/\mathcal{B}([0,\infty))$-measurable mappings $\phi,\Phi\colon \mathcal{C}([0,T],L^\infty((0,1);\mathbb{R}))\to[0,\infty)$ such that for every $N\in\mathbb{N}$, $v\in P_N(H)$, $w\in\mathcal{C}([0,T],L^\infty((0,1);\mathbb{R}))$ we have that

$$\begin{aligned}&\sup_{t\in[0,T]}\big(\langle(-A)^{1/2}v,(-A)^{1/2}P_NF(v+w_t)\rangle_H+\phi(w)\langle v,F(v+w_t)\rangle_H\big)\\&\leq\epsilon\|Av\|_H^2+(c+\phi(w))\|(-A)^{1/2}v\|_H^2+c\phi(w)\|v\|_H^2+\Phi(w).\end{aligned}\tag{8}$$

Essentially, the coercivity type condition (8) appears as one of our assumptions of Theorem 5.5 below (see (102) in Section 5.1 below for details). Our proposal for this specific Lyapunov-type function is partially inspired by the arguments in Section 4 in Bianchi, Blömker, & Schneider [1] (cf. [1, Theroem 4.1 and Lemma 4.4]).

The remainder of this article is structured as follows. Section 2 establishes suitable a priori bounds for the numerical approximations. In Section 3 the error analysis for the considered nonlinearity-truncated approximation schemes is carried out in the pathwise sense and in Section 4 we perform the error analysis for these numerical schemes in the strong L^p-sense. In Section 5 we combine the results from Section 4 with appropriate uniform a priori moment bounds for the numerical approximation processes (see Section 2) to establish Theorem 5.5 which is the main result of this article. Section 6 makes sure that the assumptions of Theorem 5.5 are satisfied for stochastic Allen-Cahn equations and finally, in Section 7 we prove lower and upper bounds for strong approximation errors of numerical approximations of linear stochastic heat equations.

1.1 Notation

Throughout this article the following notation is used. For every measurable space $(A,\mathcal{A})$ and every measurable space $(B,\mathcal{B})$ we denote by $\mathcal{M}(\mathcal{A},\mathcal{B})$ the set of all $\mathcal{A}/\mathcal{B}$-measurable functions. For every set A we denote by $\#_A\in\{0,1,2,\dots\}\cup\{\infty\}$ the number of elements of A, we denote by $\mathcal{P}(A)$ the power set of A, and we denote by $\mathcal{P}_0(A)$ the set given by $\mathcal{P}_0(A)=\{B\in\mathcal{P}(A)\colon \#_B<\infty\}$. For every set A and every set $\mathcal{A}$ with $\mathcal{A}\subseteq\mathcal{P}(A)$ we denote by $\sigma_A(\mathcal{A})$ the smallest sigma-algebra on A which contains $\mathcal{A}$. For every topological space (X,τ) we denote by $\mathcal{B}(X)$ the set given by $\mathcal{B}(X)=\sigma_X(\tau)$. For every natural number $d\in\mathbb{N}$ and every set $A\in\mathcal{B}(\mathbb{R}^d)$ we denote by $\lambda_A\colon\mathcal{B}(A)\to[0,\infty]$ the Lebesgue-Borel measure on A. We denote by $\lfloor\cdot\rfloor_h\colon\mathbb{R}\to\mathbb{R}$, $h\in(0,\infty)$, the functions which satisfy for all $h\in(0,\infty)$, $t\in\mathbb{R}$ that $\lfloor t\rfloor_h=\max(\{0,h,-h,2h,-2h,\dots\}\cap(-\infty,t])$. For every measure space $(\Omega,\mathcal{F},\nu)$, every measurable space $(S,\mathcal{S})$, every set R, and every function $f\colon\Omega\to R$ we denote by $[f]_{\nu,\mathcal{S}}$ the set given by $[f]_{\nu,\mathcal{S}}=\{g\in\mathcal{M}(\mathcal{F},\mathcal{S})\colon(\exists\,A\in\mathcal{F}\colon\nu(A)=0\text{ and }\{\omega\in\Omega\colon f(\omega)\neq g(\omega)\}\subseteq A)\}$. For every set Ω and every set A we denote by $\mathbb{1}_A^\Omega\colon\Omega\to\mathbb{R}$ the function which satisfies for all $x\in\Omega$ that

$$\mathbb{1}_A^\Omega(x)=\begin{cases}1&:x\in A\\0&:x\notin A.\end{cases}\tag{9}$$

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2 A priori bounds for the numerical approximation

Lemma 2.1. *Consider the notation in Section 1.1, let* $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ *be a separable* $\mathbb{R}$*-Hilbert space, let* $\mathbb{H} \subseteq H$ *be a non-empty orthonormal basis of* H*, let* $T, \varphi, c \in (0,\infty)$*,* $C \in [0,\infty)$*,* $\epsilon, \kappa, \rho \in [0,1)$*,* $\gamma \in (\rho, 1)$*,* $\chi \in (0, {}^{(\gamma-\rho)}/_{(1+\varphi/2)}] \cap (0, {}^{(1-\rho)}/_{(1+\varphi)}]$*,* $M \in \mathbb{N}$*,* $\mu\colon \mathbb{H} \to \mathbb{R}$ *satisfy* $\sup_{h\in\mathbb{H}} \mu_h < 0$*, let* $A\colon D(A) \subseteq H \to H$ *be the linear operator which satisfies* $D(A) = \{v \in H\colon \sum_{h\in\mathbb{H}} |\mu_h \langle h, v\rangle_H|^2 < \infty\}$ *and* $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \mu_h \langle h, v\rangle_H h$*, let* $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$*,* $r \in \mathbb{R}$*, be a family of interpolation spaces associated to* $-A$ *(cf., e.g., [27, Section 3.7]), let* $I \in \mathcal{P}_0(\mathbb{H})$*,* $P \in L(H)$ *satisfy for all* $v \in H$ *that* $P(v) = \sum_{h\in I} \langle h, v\rangle_H h$*, and let* $\mathcal{X}\colon [0,T] \to P(H)$*,* $\mathcal{O} \in \mathcal{C}([0,T], P(H))$*,* $F \in \mathcal{C}(P(H), H)$*,* $\phi, \Phi\colon \mathcal{C}([0,T], P(H)) \to [0,\infty)$ *satisfy for all* $u, v \in P(H)$*,* $w \in \mathcal{C}([0,T], P(H))$*,* $t \in [0,T]$ *that*

$$\|F(u)\|_H^2 \leq C \max\{1, \|u\|_{H_\gamma}^{(2+\varphi)}\}, \tag{10}$$

$$\|F(u) - F(v)\|_H^2 \leq C \max\{1, \|u\|_{H_\gamma}^{\varphi}\} \|u - v\|_{H_\rho}^2 + C\|u - v\|_{H_\rho}^{(2+\varphi)}, \tag{11}$$

$$\begin{aligned} &\langle v, PF(v + w_t)\rangle_{H_{1/2}} + \phi(w)\langle v, F(v + w_t)\rangle_H \\ &\leq \epsilon\|v\|_{H_1}^2 + (c + \phi(w))\|v\|_{H_{1/2}}^2 + \kappa c \phi(w)\|v\|_H^2 + \Phi(w), \end{aligned} \tag{12}$$

and
$$\mathcal{X}_t = \int_0^t P e^{(t-s)A} \mathbb{1}^{\mathbb{R}}_{[0,(M/T)^\chi]}(\|\mathcal{X}_{\lfloor s\rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}_{\lfloor s\rfloor_{T/M}}\|_{H_\gamma})\, F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\, ds + \mathcal{O}_t. \tag{13}$$

Then

(i) we have that the function $[0,T] \ni t \mapsto \mathcal{X}_t - \mathcal{O}_t \in P(H)$ *is continuous and*

(ii) we have that

$$\begin{aligned} &\sup_{t\in[0,T]} \big(\|\mathcal{X}_t - \mathcal{O}_t\|_{H_{1/2}}^2 + \phi(\mathcal{O})\|\mathcal{X}_t - \mathcal{O}_t\|_H^2\big) \\ &\leq \frac{e^{2cT}}{c}\left(\Phi(\mathcal{O}) + \frac{\max\{1,\phi(\mathcal{O})\}C(c+1)}{2(1-\epsilon)(1-\kappa)c}\left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)}\right). \end{aligned} \tag{14}$$

Proof of Lemma 2.1. Throughout this proof assume w.l.o.g. that $I \neq \emptyset$ and let $\bar{\mathcal{X}}\colon [0,T] \to P(H)$ and $Z\colon [0,T] \to \{0,1\}$ be the functions which satisfy for all $s \in [0,T]$ that

$$\bar{\mathcal{X}}_s = \mathcal{X}_s - \mathcal{O}_s \qquad \text{and} \qquad Z_s = \mathbb{1}^{\mathbb{R}}_{[0,(M/T)^\chi]}(\|\mathcal{X}_s\|_{H_\gamma} + \|\mathcal{O}_s\|_{H_\gamma}). \tag{15}$$

Observe that, e.g., Lemma 2.4 in [18] (with $V = P(H)$, $\|\cdot\|_V = P(H) \ni v \mapsto \|v\|_H^2 \in [0,\infty)$, $T = T$, $\eta = 0$, $A = P(H) \ni v \mapsto Av \in P(H)$, $\mathbb{V} = P(H) \ni v \mapsto \|v\|^2_{H_{1/2}} + \phi(\mathcal{O})\|v\|_H^2 \in \mathbb{R}$, $Z = [0,T] \times \Omega \ni (t,\omega) \mapsto Z_t(\omega) \in \mathbb{R}$, $Y = \mathcal{X}$, $O = \mathcal{O}$, $\mathbb{O} = \mathcal{O}$, $F = P(H) \ni v \mapsto PF(v) \in P(H)$, $\phi = V \ni v \mapsto 2c \in \mathbb{R}$, $f = V \ni v \mapsto 0 \in \mathbb{R}$, $h = T/M$ in the notation of Lemma 2.4 in [18]) implies that (i) holds and that for all $t \in [0,T]$ it holds that

$$\begin{aligned} &e^{-2ct}\Big[\|\bar{\mathcal{X}}_t\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|^2_H\Big] \\ &= 2\int_0^t e^{-2cs}\Big[\langle \bar{\mathcal{X}}_s, A\bar{\mathcal{X}}_s + Z_{\lfloor s\rfloor_{T/M}} PF(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_{H_{1/2}} \\ &\quad + \phi(\mathcal{O})\langle \bar{\mathcal{X}}_s, A\bar{\mathcal{X}}_s + Z_{\lfloor s\rfloor_{T/M}} PF(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\Big]\,ds \\ &\quad + 2\int_0^t e^{-2cs}\, Z_{\lfloor s\rfloor_{T/M}}\Big[\langle \bar{\mathcal{X}}_s, P[F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})]\rangle_{H_{1/2}} \\ &\quad + \phi(\mathcal{O})\langle \bar{\mathcal{X}}_s, P[F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})]\rangle_H\Big]\,ds \\ &\quad - 2c\int_0^t e^{-2cs}\Big[\|\bar{\mathcal{X}}_s\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|^2_H\Big]\,ds. \end{aligned} \tag{16}$$

The fact that $P \in L(H)$ is symmetric hence proves for all $t \in [0,T]$ that

$$\begin{aligned} &e^{-2ct}\Big[\|\bar{\mathcal{X}}_t\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|^2_H\Big] \\ &= -2\int_0^t e^{-2cs}\Big[\langle \bar{\mathcal{X}}_s, (-A)\bar{\mathcal{X}}_s\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle \bar{\mathcal{X}}_s, (-A)\bar{\mathcal{X}}_s\rangle_H\Big]\,ds \\ &\quad + 2\int_0^t e^{-2cs}\, Z_{\lfloor s\rfloor_{T/M}}\Big[\langle \bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle P\bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\Big]\,ds \\ &\quad + 2\int_0^t e^{-2cs}\, Z_{\lfloor s\rfloor_{T/M}}\Big[\langle (-A)^{1/2}\bar{\mathcal{X}}_s, (-A)^{1/2}P[F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})]\rangle_H \\ &\quad + \phi(\mathcal{O})\langle P\bar{\mathcal{X}}_s, F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\Big]\,ds \\ &\quad - 2c\int_0^t e^{-2cs}\Big[\|\bar{\mathcal{X}}_s\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|^2_H\Big]\,ds. \end{aligned} \tag{17}$$

The fact that $\forall\, s \in [0,T]\colon \bar{\mathcal{X}}_s \in P(H)$ therefore implies for all $t \in [0,T]$ that

$$
\begin{aligned}
& e^{-2ct}\left[\|\bar{\mathcal{X}}_t\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|^2_H\right] \\
& = -2\int_0^t e^{-2cs}\left[\langle(-A)\bar{\mathcal{X}}_s,(-A)\bar{\mathcal{X}}_s\rangle_H + \phi(\mathcal{O})\langle(-A)^{1/2}\bar{\mathcal{X}}_s,(-A)^{1/2}\bar{\mathcal{X}}_s\rangle_H\right]ds \\
& \quad + 2\int_0^t e^{-2cs}\, Z_{\lfloor s\rfloor_{T/M}}\left[\langle\bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle\bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\right]ds \\
& \quad + 2\int_0^t e^{-2cs}\, Z_{\lfloor s\rfloor_{T/M}}\Big[\langle(-A)\bar{\mathcal{X}}_s, P[F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})]\rangle_H \\
& \quad + \phi(\mathcal{O})\langle\bar{\mathcal{X}}_s, F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\Big]\,ds \\
& \quad - 2c\int_0^t e^{-2cs}\left[\|\bar{\mathcal{X}}_s\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|^2_H\right]ds.
\end{aligned}
\tag{18}
$$

This and the Cauchy-Schwarz inequality ensure for all $t \in [0,T]$ that

$$
\begin{aligned}
& e^{-2ct}\left[\|\bar{\mathcal{X}}_t\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|^2_H\right] \\
& \le -2\int_0^t e^{-2cs}\left[\|\bar{\mathcal{X}}_s\|^2_{H_1} + (c+\phi(\mathcal{O}))\|\bar{\mathcal{X}}_s\|^2_{H_{1/2}} + c\,\phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|^2_H\right]ds \\
& \quad + 2\int_0^t e^{-2cs}\, Z_{\lfloor s\rfloor_{T/M}}\left[\langle\bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle\bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\right]ds \\
& \quad + 2\int_0^t e^{-2cs}\, Z_{\lfloor s\rfloor_{T/M}}\Bigg[\left(\sqrt{2(1-\epsilon)}\,\|\bar{\mathcal{X}}_s\|_{H_1}\right)\left(\tfrac{1}{\sqrt{2(1-\epsilon)}}\,\|P[F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})]\|_H\right) \\
& \quad + \phi(\mathcal{O})\left(\sqrt{2c(1-\kappa)}\,\|\bar{\mathcal{X}}_s\|_H\right)\left(\tfrac{1}{\sqrt{2c(1-\kappa)}}\,\|F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s\rfloor_{T/M}})\|_H\right)\Bigg]\,ds.
\end{aligned}
\tag{19}
$$

The fact that

$$
\forall\, x,y \in \mathbb{R}\colon 2xy \le x^2 + y^2 \tag{20}
$$

hence proves that for all $t \in [0,T]$ we have that

$$
\begin{aligned}
&e^{-2ct}\Big[\|\bar{\mathcal{X}}_t\|_{H_{1/2}}^2 + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|_H^2\Big] \\
&\leq -2\int_0^t e^{-2cs}\Big[\|\bar{\mathcal{X}}_s\|_{H_1}^2 + (c+\phi(\mathcal{O}))\|\bar{\mathcal{X}}_s\|_{H_{1/2}}^2 + c\,\phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|_H^2\Big]\,ds \\
&\quad + 2\int_0^t e^{-2cs}\,Z_{\lfloor s\rfloor_{T/M}}\Big[\langle\bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle\bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\Big]\,ds \\
&\quad + \int_0^t e^{-2cs}\,Z_{\lfloor s\rfloor_{T/M}}\Big[2(1-\epsilon)\,\|\bar{\mathcal{X}}_s\|_{H_1}^2 + \tfrac{1}{2(1-\epsilon)}\,\|P\|_{L(H)}^2\,\|F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\|_H^2 \\
&\quad + 2c(1-\kappa)\,\phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|_H^2 + \tfrac{\phi(\mathcal{O})}{2c(1-\kappa)}\,\|F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\|_H^2\Big]\,ds.
\end{aligned}
\tag{21}
$$

The fact $\|P\|_{L(H)} \leq 1$ therefore shows for all $t \in [0,T]$ that

$$
\begin{aligned}
&e^{-2ct}\Big[\|\bar{\mathcal{X}}_t\|_{H_{1/2}}^2 + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|_H^2\Big] \\
&\leq -2\int_0^t e^{-2cs}\Big[\epsilon Z_{\lfloor s\rfloor_{T/M}}\|\bar{\mathcal{X}}_s\|_{H_1}^2 + (c+\phi(\mathcal{O}))\|\bar{\mathcal{X}}_s\|_{H_{1/2}}^2 + \kappa c\,\phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|_H^2\Big]\,ds \\
&\quad + 2\int_0^t e^{-2cs}\,Z_{\lfloor s\rfloor_{T/M}}\Big[\langle\bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle\bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\Big]\,ds \\
&\quad + \Big[\tfrac{1}{2(1-\epsilon)} + \tfrac{\phi(\mathcal{O})}{2c(1-\kappa)}\Big]\int_0^t e^{-2cs}\,Z_{\lfloor s\rfloor_{T/M}}\|F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\|_H^2\,ds.
\end{aligned}
\tag{22}
$$

Hence, we obtain that for all $t \in [0,T]$ we have that

$$
\begin{aligned}
&\|\bar{\mathcal{X}}_t\|_{H_{1/2}}^2 + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|_H^2 \\
&\leq -2\int_0^t e^{2c(t-s)}\Big[\epsilon Z_{\lfloor s\rfloor_{T/M}}\|\bar{\mathcal{X}}_s\|_{H_1}^2 + (c+\phi(\mathcal{O}))\|\bar{\mathcal{X}}_s\|_{H_{1/2}}^2 + \kappa c\,\phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|_H^2\Big]\,ds \\
&\quad + 2\int_0^t e^{2c(t-s)}\,Z_{\lfloor s\rfloor_{T/M}}\Big[\langle\bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle\bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\rangle_H\Big]\,ds \\
&\quad + \left[\frac{1}{2(1-\epsilon)} + \frac{\phi(\mathcal{O})}{2c(1-\kappa)}\right]\int_0^t e^{2c(t-s)}\,Z_{\lfloor s\rfloor_{T/M}}\|F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s+\mathcal{O}_{\lfloor s\rfloor_{T/M}})\|_H^2\,ds.
\end{aligned}
\tag{23}
$$

Moreover, note that the triangle inequality implies that for all $s \in [0,T]$ we have that

$$\begin{aligned}
&Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_s - \bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}}\|_{H_\rho} \\
&= Z_{\lfloor s \rfloor_{T/M}} \|(\mathcal{X}_s - \mathcal{O}_s) - (\mathcal{X}_{\lfloor s \rfloor_{T/M}} - \mathcal{O}_{\lfloor s \rfloor_{T/M}})\|_{H_\rho} \\
&\leq Z_{\lfloor s \rfloor_{T/M}} \|(e^{(s-\lfloor s \rfloor_{T/M})A} - \mathrm{Id}_H)(\mathcal{X}_{\lfloor s \rfloor_{T/M}} - \mathcal{O}_{\lfloor s \rfloor_{T/M}})\|_{H_\rho} \\
&\quad + Z_{\lfloor s \rfloor_{T/M}} \|(\mathcal{X}_s - \mathcal{O}_s) - e^{(s-\lfloor s \rfloor_{T/M})A}(\mathcal{X}_{\lfloor s \rfloor_{T/M}} - \mathcal{O}_{\lfloor s \rfloor_{T/M}})\|_{H_\rho} \\
&\leq Z_{\lfloor s \rfloor_{T/M}} \|(-A)^{-(\gamma-\rho)}(e^{(s-\lfloor s \rfloor_{T/M})A} - \mathrm{Id}_H)\|_{L(H)} \|\mathcal{X}_{\lfloor s \rfloor_{T/M}} - \mathcal{O}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} \\
&\quad + Z_{\lfloor s \rfloor_{T/M}} \int_{\lfloor s \rfloor_{T/M}}^s \|P e^{(s-u)A} Z_{\lfloor u \rfloor_{T/M}} F(\mathcal{X}_{\lfloor u \rfloor_{T/M}})\|_{H_\rho} \, du.
\end{aligned} \tag{24}$$

The fact that

$$\forall\, s \in (0,\infty), r \in [0,1]\colon \|(-sA)^{-r}(e^{sA} - \mathrm{Id}_H)\|_{L(H)} \leq 1, \tag{25}$$

the triangle inequality, the fact that

$$\forall\, s \in [0,\infty), r \in [0,1]\colon \|(-sA)^r e^{sA}\|_{L(H)} \leq 1, \tag{26}$$

the fact that $\|P\|_{L(H)} \leq 1$, and the assumption that

$$\forall\, v \in P(H)\colon \|F(v)\|_H^2 \leq C \max\{1, \|v\|_{H_\gamma}^{(2+\varphi)}\} \tag{27}$$

hence ensure for all $s \in [0,T]$ that

$$\begin{aligned}
&Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_s - \bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}}\|_{H_\rho} \\
&\leq (s - \lfloor s \rfloor_{T/M})^{(\gamma-\rho)} Z_{\lfloor s \rfloor_{T/M}} \big(\|\mathcal{X}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma}\big) \\
&\quad + Z_{\lfloor s \rfloor_{T/M}} \int_{\lfloor s \rfloor_{T/M}}^s \|P\|_{L(H)} \|(-A)^\rho e^{(s-u)A}\|_{L(H)} \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\|_H \, du \\
&\leq |{}^T\!/\!{}_M|^{(\gamma-\rho)} |{}^M\!/\!{}_T|^\chi + \sqrt{C} \int_{\lfloor s \rfloor_{T/M}}^s (s-u)^{-\rho}\, Z_{\lfloor s \rfloor_{T/M}} \max\Big\{1, \|\mathcal{X}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma}^{(1+\varphi/2)}\Big\} \, du \\
&\leq |{}^T\!/\!{}_M|^{(\gamma-\rho-\chi)} + \sqrt{C} \max\big\{1, |{}^M\!/\!{}_T|^{(1+\varphi/2)\chi}\big\} \int_{\lfloor s \rfloor_{T/M}}^s (s-u)^{-\rho} \, du.
\end{aligned} \tag{28}$$

This shows that for all $s \in [0,T]$ we have that

$$
\begin{aligned}
&Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_s - \bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}}\|_{H_\rho} \\
&\leq |{}^{T}\!/\!_{M}|^{(\gamma-\rho-\chi)} + \sqrt{C} \max\big\{1, |{}^{M}\!/\!_{T}|^{(1+\varphi/2)\chi}\big\} \, \frac{(s - \lfloor s \rfloor_{T/M})^{(1-\rho)}}{(1-\rho)} \\
&\leq \frac{1}{(1-\rho)} \Big[|{}^{T}\!/\!_{M}|^{(\gamma-\rho-\chi)} + \sqrt{C} \max\big\{ |{}^{T}\!/\!_{M}|^{(1-\rho)}, |{}^{T}\!/\!_{M}|^{(1-\rho-(1+\varphi/2)\chi)} \big\} \Big] \\
&\leq \frac{(1+\sqrt{C})}{(1-\rho)} \max\big\{ {}^{T}\!/\!_{M}, |{}^{T}\!/\!_{M}|^{(\gamma-\rho-\chi)}, |{}^{T}\!/\!_{M}|^{(1-\rho-(1+\varphi/2)\chi)} \big\} .
\end{aligned}
\tag{29}
$$

Next observe that the assumption

$$
\forall\, u, v \in P(H) \colon \|F(u) - F(v)\|_H^2 \leq C \max\{1, \|u\|_{H_\gamma}^{\varphi}\} \|u - v\|_{H_\rho}^2 + C \|u - v\|_{H_\rho}^{(2+\varphi)} \tag{30}
$$

ensures for all $s \in [0,T]$ that

$$
\begin{aligned}
&Z_{\lfloor s \rfloor_{T/M}} \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s \rfloor_{T/M}})\|_H^2 \\
&\leq C Z_{\lfloor s \rfloor_{T/M}} \Big[\max\{1, \|\mathcal{X}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma}^{\varphi}\} \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho}^2 + \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho}^{(2+\varphi)} \Big] \\
&\leq C Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho}^2 \Big[\max\{1, |{}^{M}\!/\!_{T}|^{\varphi\chi}\} + Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho}^{\varphi} \Big] \\
&\leq 2C Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho}^2 \Big[\max\big\{1, |{}^{M}\!/\!_{T}|^{\varphi\chi}, Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho}^{\varphi} \big\} \Big] \\
&= 2C Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho}^2 \Big[\max\big\{1, |{}^{M}\!/\!_{T}|^{\chi}, Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_{\lfloor s \rfloor_{T/M}} - \bar{\mathcal{X}}_s\|_{H_\rho} \big\} \Big]^{\varphi} .
\end{aligned}
\tag{31}
$$

This together with (29) proves for all $s \in [0,T]$ that

$$
\begin{aligned}
&Z_{\lfloor s \rfloor_{T/M}} \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s \rfloor_{T/M}})\|_H^2 \\
&\leq \frac{2C(1+\sqrt{C})^{(2+\varphi)}}{(1-\rho)^{(2+\varphi)}} \big| \max\big\{ {}^{T}\!/\!_{M}, |{}^{T}\!/\!_{M}|^{(\gamma-\rho-\chi)}, |{}^{T}\!/\!_{M}|^{(1-\rho-(1+\varphi/2)\chi)} \big\} \big|^2 \\
&\quad \cdot \big| \max\big\{ 1, |{}^{M}\!/\!_{T}|^{\chi}, {}^{T}\!/\!_{M}, |{}^{T}\!/\!_{M}|^{(\gamma-\rho-\chi)}, |{}^{T}\!/\!_{M}|^{(1-\rho-(1+\varphi/2)\chi)} \big\} \big|^{\varphi} \\
&= \frac{2C(1+\sqrt{C})^{(2+\varphi)}}{(1-\rho)^{(2+\varphi)}} \big| \max\big\{ {}^{T}\!/\!_{M}, |{}^{T}\!/\!_{M}|^{(\gamma-\rho-\chi)}, |{}^{T}\!/\!_{M}|^{(1-\rho-(1+\varphi/2)\chi)} \big\} \big|^2 \\
&\quad \cdot \big| \max\big\{ {}^{T}\!/\!_{M}, |{}^{M}\!/\!_{T}|^{\chi}, |{}^{T}\!/\!_{M}|^{(\gamma-\rho-\chi)}, |{}^{T}\!/\!_{M}|^{(1-\rho-(1+\varphi/2)\chi)} \big\} \big|^{\varphi} .
\end{aligned}
\tag{32}
$$

In addition, note that the assumption that $\chi \in (0, {}^{(\gamma-\rho)}\!/\!_{(1+\varphi/2)}] \cap (0, {}^{(1-\rho)}\!/\!_{(1+\varphi)}]$ ensures that

$$
\gamma - \rho - \chi \in (0,1), \qquad 1 - \rho - (1 + \varphi/2)\chi \in (0,1), \tag{33}
$$

and

$$\min\{\gamma - \rho - (1 + \varphi/2)\chi, 1 - \rho - (1 + \varphi)\chi\} \in [0, 1). \tag{34}$$

This implies that for all $h \in (0, 1]$ we have that

$$\begin{aligned} &\left|\max\left\{h, h^{(\gamma-\rho-\chi)}, h^{(1-\rho-(1+\varphi/2)\chi)}\right\}\right|^2 \left|\max\left\{h, h^{-\chi}, h^{(\gamma-\rho-\chi)}, h^{(1-\rho-(1+\varphi/2)\chi)}\right\}\right|^\varphi \\ &= h^{2\min\{\gamma-\rho-\chi, 1-\rho-(1+\varphi/2)\chi\}} h^{-\varphi\chi} = h^{2\min\{\gamma-\rho-(1+\varphi/2)\chi, 1-\rho-(1+\varphi)\chi\}} \leq 1. \end{aligned} \tag{35}$$

Moreover, observe that (33) shows for all $h \in (1, \infty)$ that

$$\left|\max\left\{h, h^{(\gamma-\rho-\chi)}, h^{(1-\rho-(1+\varphi/2)\chi)}\right\}\right|^2 \left|\max\left\{h, h^{-\chi}, h^{(\gamma-\rho-\chi)}, h^{(1-\rho-(1+\varphi/2)\chi)}\right\}\right|^\varphi = h^{(2+\varphi)}. \tag{36}$$

Combining (32) with (35) and (36) yields that for all $s \in [0, T]$ we have that

$$Z_{\lfloor s \rfloor_{T/M}} \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}}) - F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s \rfloor_{T/M}})\|_H^2 \leq 2C \left[\frac{\max\{1, T\}(1 + \sqrt{C})}{(1 - \rho)}\right]^{(2+\varphi)}. \tag{37}$$

Furthermore, note that the assumption that $\forall\, v \in P(H), w \in \mathcal{C}([0, T], P(H)), s \in [0, T]\colon \langle v, PF(v + w_s)\rangle_{H_{1/2}} + \phi(w)\langle v, F(v + w_s)\rangle_H \leq \epsilon\|v\|_{H_1}^2 + (c + \phi(w))\|v\|_{H_{1/2}}^2 + \kappa c\phi(w)\|v\|_H^2 + \Phi(w)$ ensures that for all $v \in P(H)$, $w \in \mathcal{C}([0, T], P(H))$, $s \in [0, T]$ we have that

$$\begin{aligned} &\langle v, PF(v + w_{\lfloor s \rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(w)\langle v, F(v + w_{\lfloor s \rfloor_{T/M}})\rangle_H \\ &\qquad \leq \epsilon\|v\|_{H_1}^2 + (c + \phi(w))\|v\|_{H_{1/2}}^2 + \kappa c\phi(w)\|v\|_H^2 + \Phi(w). \end{aligned} \tag{38}$$

This implies that for all $w \in \mathcal{C}([0, T], P(H))$, $s \in [0, T]$ we have that

$$\begin{aligned} &\langle \bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s + w_{\lfloor s \rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(w)\langle \bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s + w_{\lfloor s \rfloor_{T/M}})\rangle_H \\ &\qquad \leq \epsilon\|\bar{\mathcal{X}}_s\|_{H_1}^2 + (c + \phi(w))\|\bar{\mathcal{X}}_s\|_{H_{1/2}}^2 + \kappa c\phi(w)\|\bar{\mathcal{X}}_s\|_H^2 + \Phi(w). \end{aligned} \tag{39}$$

The assumption that $\mathcal{O} \in \mathcal{C}([0, T], P(H))$ hence guarantees for all $s \in [0, T]$ that

$$\begin{aligned} &\langle \bar{\mathcal{X}}_s, PF(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s \rfloor_{T/M}})\rangle_{H_{1/2}} + \phi(\mathcal{O})\langle \bar{\mathcal{X}}_s, F(\bar{\mathcal{X}}_s + \mathcal{O}_{\lfloor s \rfloor_{T/M}})\rangle_H \\ &\qquad \leq \epsilon\|\bar{\mathcal{X}}_s\|_{H_1}^2 + (c + \phi(\mathcal{O}))\|\bar{\mathcal{X}}_s\|_{H_{1/2}}^2 + \kappa c\phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|_H^2 + \Phi(\mathcal{O}). \end{aligned} \tag{40}$$

Combining (23) with (37) and (40) demonstrates that for all $t \in [0,T]$ we have that

$$
\begin{aligned}
&\|\bar{\mathcal{X}}_t\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|^2_H \\
&\leq -2\int_0^t e^{2c(t-s)} \left[\epsilon Z_{\lfloor s \rfloor_{T/M}} \|\bar{\mathcal{X}}_s\|^2_{H_1} + (c+\phi(\mathcal{O}))\|\bar{\mathcal{X}}_s\|^2_{H_{1/2}} + \kappa c \phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|^2_H\right] ds \\
&\quad + 2\int_0^t e^{2c(t-s)}\, Z_{\lfloor s \rfloor_{T/M}} \left[\epsilon \|\bar{\mathcal{X}}_s\|^2_{H_1} + (c+\phi(\mathcal{O}))\|\bar{\mathcal{X}}_s\|^2_{H_{1/2}} + \kappa c \phi(\mathcal{O})\|\bar{\mathcal{X}}_s\|^2_H + \Phi(\mathcal{O})\right] ds \\
&\quad + 2C\left[\frac{1}{2(1-\epsilon)} + \frac{\phi(\mathcal{O})}{2c(1-\kappa)}\right]\left[\frac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)} \left[\int_0^t e^{2c(t-s)}\, ds\right].
\end{aligned} \tag{41}
$$

This shows that for all $t \in [0,T]$ we have that

$$
\begin{aligned}
&\|\bar{\mathcal{X}}_t\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|^2_H \\
&\leq \frac{\Phi(\mathcal{O})\,[e^{2ct}-1]}{c} + \frac{C\,[e^{2ct}-1]}{c}\left[\frac{1}{2(1-\epsilon)} + \frac{\phi(\mathcal{O})}{2c(1-\kappa)}\right]\left[\frac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)}.
\end{aligned} \tag{42}
$$

Hence, we obtain that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \left[\|\bar{\mathcal{X}}_t\|^2_{H_{1/2}} + \phi(\mathcal{O})\|\bar{\mathcal{X}}_t\|^2_H\right] \\
&\leq \frac{\left(e^{2cT}-1\right)}{c}\left(\Phi(\mathcal{O}) + \frac{C}{2}\left[\frac{1}{(1-\epsilon)} + \frac{\phi(\mathcal{O})}{c(1-\kappa)}\right]\left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)}\right) \\
&\leq \frac{\left(e^{2cT}-1\right)}{c}\left(\Phi(\mathcal{O}) + \frac{C\max\{1,\phi(\mathcal{O})\}}{2(1-\epsilon)(1-\kappa)}\left[1+\frac{1}{c}\right]\left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)}\right) \\
&= \frac{\left(e^{2cT}-1\right)}{c}\left(\Phi(\mathcal{O}) + \frac{\max\{1,\phi(\mathcal{O})\}C(1+c)}{2(1-\epsilon)(1-\kappa)c}\left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)}\right).
\end{aligned} \tag{43}
$$

The proof of Lemma 2.1 is thus completed. □

3 Pathwise error estimates

3.1 Setting

Consider the notation in Section 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $T, c, \varphi \in (0,\infty)$, $C \in [0,\infty)$, $M \in \mathbb{N}$, $\mu\colon \mathbb{H} \to \mathbb{R}$ satisfy $\sup_{h\in\mathbb{H}} \mu_h < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) =$

$\{v \in H\colon \sum_{h\in\mathbb{H}} |\mu_h \langle h, v\rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \mu_h \langle h, v\rangle_H h$, let $(V, \|\cdot\|_V)$ be an $\mathbb{R}$-Banach space with $D(A) \subseteq V \subseteq H$ continuously and densely, and let $O, \mathcal{O}, \mathbf{X}, \mathcal{X}\colon [0,T] \to V$ and $\mathcal{V}\colon V \times V \to [0,\infty)$ be functions, and let $X \in \mathcal{C}([0,T], V)$, $F \in \mathcal{C}(V, H)$, $I \in \mathcal{P}_0(\mathbb{H})$, $P \in L(H)$ satisfy for all $v, w \in D(A)$, $t \in [0,T]$ that

$$P(v) = \textstyle\sum_{h\in I} \langle h, v\rangle_H h, \qquad \langle v - w, Av + F(v) - Aw - F(w)\rangle_H \leq c\|v - w\|_H^2, \tag{44}$$

$$\|F(v) - F(w)\|_H^2 \leq C\|v - w\|_V^2 (1 + \|v\|_V^{\varphi} + \|w\|_V^{\varphi}), \tag{45}$$

$$X_t = \int_0^t e^{(t-s)A} F(X_s)\, ds + O_t, \qquad \mathbf{X}_t = \int_0^t P e^{(t-s)A} F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\, ds + PO_t, \tag{46}$$

$$\text{and} \qquad \mathcal{X}_t = \int_0^t P e^{(t-s)A} \mathbb{1}^{\mathbb{R}}_{[0,M/T]}(\mathcal{V}(\mathcal{X}_{\lfloor s\rfloor_{T/M}}, \mathcal{O}_{\lfloor s\rfloor_{T/M}}))\, F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\, ds + \mathcal{O}_t. \tag{47}$$

3.2 On the separability of a certain Banach space

The next elementary lemma, Lemma 3.1, ensures that the $\mathbb{R}$-Banach space $(V, \|\cdot\|_V)$ in Section 3.1 is separable. Lemma 3.1 is well-known in the literature. The proof of Lemma 3.1 is given for completeness.

Lemma 3.1. *Let $(V, \|\cdot\|_V)$ be a separable $\mathbb{R}$-Banach space and let $(W, \|\cdot\|_W)$ be an $\mathbb{R}$-Banach space with $V \subseteq W$ continuously and densely. Then $(W, \|\cdot\|_W)$ is a separable $\mathbb{R}$-Banach space.*

Proof of Lemma 3.1. Throughout this proof let $w \in W$ be a vector and let $\varepsilon \in (0,\infty)$ be a strictly positive real number. Note that the assumption that $V \subseteq W$ continuously and densely ensures that there exists a $c \in \mathbb{R}$ and a $v \in V$ such that

$$c = \sup\!\left(\left\{\frac{\|u\|_W}{\|u\|_V}\colon u \in V \setminus \{0\}\right\} \cup \{1\}\right) \qquad \text{and} \qquad \|v - w\|_W < \frac{\varepsilon}{2}. \tag{48}$$

In addition, observe that the assumption that $(V, \|\cdot\|_V)$ is a separable $\mathbb{R}$-Banach space shows that there exists an at most countable set $A \subseteq V$ such that

$$\overline{A}^V = V. \tag{49}$$

Hence, we obtain that there exists an $a \in A$ such that $\|a - v\|_V < \frac{\varepsilon}{2c}$. Combining this and (48) with the triangle inequality establishes that

$$\begin{split} \|a - w\|_W &\leq \|a - v\|_W + \|v - w\|_W < \|a - v\|_W + \frac{\varepsilon}{2} \\ &\leq c\,\|a - v\|_V + \frac{\varepsilon}{2} < c \cdot \frac{\varepsilon}{2c} + \frac{\varepsilon}{2} = \varepsilon. \end{split} \tag{50}$$

The proof of Lemma 3.1 is thus completed. ∎

3.3 Analysis of the error between the Galerkin projection of the exact solution and the Galerkin projection of the semilinear integrated version of the numerical approximation

In our error analysis in Lemma 3.3 below we employ the following elementary result, Lemma 3.2 below, on mild solutions of certain semilinear evolution equations.

Lemma 3.2. *Consider the notation in Section 1.1, let $(V, \|\cdot\|_V)$ be a separable $\mathbb{R}$-Banach space, and let $A \in L(V)$, $T \in (0, \infty)$, $Y\colon [0,T] \to V$, $Z \in \mathcal{C}([0,T], V)$ satisfy for all $t \in [0,T]$ that $Y_t = \int_0^t e^{(t-s)A} Z_s \, ds$. Then*

(i) we have that Y is continuously differentiable,

(ii) we have for all $t \in [0,T]$ that $Y_t = \int_0^t AY_s + Z_s \, ds$, and

(iii) we have for all $t \in [0,T]$ that $\frac{d}{dt} Y_t = AY_t + Z_t$.

Proof of Lemma 3.2. Note that Lemma 2.2 in [18] (with $V = V$, $A = A$, $T = T$, $Y = Y$, $Z = Z$ in the notation of Lemma 2.2 in [18]) implies that the function $[0,T] \ni t \mapsto Y_t \in V$ is continuous and that (ii) holds. Next observe that for all $t_1, t_2 \in [0,T]$ with $t_1 \leq t \leq t_2$ we have that

$$\begin{aligned}
&Y_{t_2} - Y_{t_1} - AY_t\,(t_2 - t_1) - Z_t\,(t_2 - t_1) \\
&= \int_{t_1}^{t_2} e^{(t_2-s)A} Z_s \, ds + e^{(t_2-t_1)A} \int_0^{t_1} e^{(t_1-s)A} Z_s \, ds - Y_{t_1} - AY_t\,(t_2 - t_1) - Z_t\,(t_2 - t_1) \\
&= \big(e^{(t_2-t_1)A} - \mathrm{Id}_V\big)\, Y_{t_1} - AY_t\,(t_2 - t_1) + \int_{t_1}^{t_2} \big(e^{(t_2-s)A} Z_s - Z_t\big)\, ds \\
&= \big(e^{(t_2-t_1)A} - \mathrm{Id}_V - A(t_2 - t_1)\big)\, Y_t + \big(e^{(t_2-t_1)A} - \mathrm{Id}_V\big)\,(Y_{t_1} - Y_t) \\
&\quad + \int_{t_1}^{t_2} e^{(t_2-s)A}\,(Z_s - Z_t)\, ds + \int_{t_1}^{t_2} \big(e^{(t_2-s)A} - \mathrm{Id}_V\big)\, Z_t \, ds.
\end{aligned} \tag{51}$$

This proves for all $t_1, t_2 \in [0,T]$ with $t_1 \leq t \leq t_2$ that

$$\begin{aligned} &Y_{t_2} - Y_{t_1} - AY_t\,(t_2 - t_1) - Z_t\,(t_2 - t_1) \\ &= \int_0^{(t_2-t_1)} \left(e^{sA} - \mathrm{Id}_V\right) AY_t\,ds + \int_0^{(t_2-t_1)} e^{sA} A(Y_{t_1} - Y_t)\,ds \\ &\quad + \int_{t_1}^{t_2} e^{(t_2-s)A}\,(Z_s - Z_t)\,ds + \int_0^{(t_2-t_1)} \left(e^{sA} - \mathrm{Id}_V\right) Z_t\,ds \\ &= \int_0^{(t_2-t_1)} \int_0^{s} e^{rA} A(AY_t + Z_t)\,dr\,ds + \int_0^{(t_2-t_1)} e^{sA} A(Y_{t_1} - Y_t)\,ds \\ &\quad + \int_{t_1}^{t_2} e^{(t_2-s)A}\,(Z_s - Z_t)\,ds. \end{aligned} \tag{52}$$

Hence, we obtain that for all $t \in [0,T]$, $(t_1,t_2) \in ([0,t] \times [t,T]) \setminus \{(t,t)\}$ we have that

$$\begin{aligned} &\frac{\|Y_{t_2} - Y_{t_1} - AY_t\,(t_2 - t_1) - Z_t\,(t_2 - t_1)\,\|_V}{(t_2 - t_1)} \\ &\leq (t_2 - t_1) \left[\sup_{s\in(0,T)} \|e^{sA}\|_{L(V)}\right] \|A\|_{L(V)}\, \|AY_t + Z_t\|_V \\ &\quad + \left[\sup_{s\in(0,T)} \|e^{sA}\|_{L(V)}\right] \|A\|_{L(V)}\, \|Y_{t_1} - Y_t\|_V \\ &\quad + \left[\sup_{s\in(0,T)} \|e^{sA}\|_{L(V)}\right] \left[\sup_{s\in(t_1,t_2)} \|Z_s - Z_t\|_V\right]. \end{aligned} \tag{53}$$

The fact that $Y \in \mathcal{C}([0,T],V)$ and the assumption that $Z \in \mathcal{C}([0,T],V)$ therefore ensure for all $t \in [0,T]$ that

$$\limsup_{\substack{(t_1,t_2)\to(t,t),\\ (t_1,t_2)\in([0,t]\times[t,T])\setminus\{(t,t)\}}} \left(\frac{\|Y_{t_2} - Y_{t_1} - AY_t\,(t_2 - t_1) - Z_t\,(t_2 - t_1)\,\|_V}{(t_2 - t_1)}\right) = 0. \tag{54}$$

This completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume the setting in Section 3.1 and let $\kappa \in (2,\infty)$, $t \in [0,T]$. Then*

$$\begin{aligned} \|PX_t - \mathbf{X}_t\|_H^2 \leq \frac{Ce^{\kappa cT}}{(\kappa-2)\,c} \int_0^t &\left[\|X_s - PX_s\|_V + \|\mathbf{X}_s - \mathcal{X}_{\lfloor s \rfloor_{T/M}}\|_V\right]^2 \\ &\cdot \left(1 + \|X_s\|_V^{\varphi} + \|PX_s\|_V^{\varphi} + \|\mathbf{X}_s\|_V^{\varphi} + \|\mathcal{X}_{\lfloor s \rfloor_{T/M}}\|_V^{\varphi}\right) ds. \end{aligned} \tag{55}$$

Proof of Lemma 3.3. Throughout this proof assume w.l.o.g. that $t \in (0,T]$ (otherwise the proof is clear). Observe that Lemma 3.2 (with $V = P(H)$, $A = (P(H) \ni v \mapsto Av \in P(H))$, $T = T$, $Y = ([0,T] \ni s \mapsto P(X_s - O_s) \in P(H))$, $Z = ([0,T] \ni s \mapsto PF(X_s) \in P(H))$ in the notation of Lemma 3.2) shows that the function $[0,T] \ni s \mapsto P(X_s - O_s) \in P(H)$ is continuously differentiable and that for all $s \in [0,T]$ we have that

$$P(X_s - O_s) = \int_0^s AP(X_u - O_u) + PF(X_u)\, du. \tag{56}$$

Next note that Lemma 2.1 in Hutzenthaler et al. [13] (with $V = H$, $A = A$, $T = T$, $h = {}^{T}\!/\!{}_{M}$, $Y = ([0,T] \ni s \mapsto \mathbf{X}_s - PO_s \in H)$, $Z = ([0,T] \ni s \mapsto PF(\mathcal{X}_s) \in H)$ in the notation of Lemma 2.1 in Hutzenthaler et al. [13]) implies that for all $s \in [0,T]$ we have that $\mathbf{X}_s - PO_s \in D(A)$, that the function $[0,T] \ni s \mapsto \mathbf{X}_s - PO_s \in D(A)$ is continuous, that the function $[0,T] \setminus \{0, \frac{T}{M}, \frac{2T}{M}, \ldots\} \ni s \mapsto \mathbf{X}_s - PO_s \in H$ is continuously differentiable, and that for all $s \in [0,T]$ we have that

$$\mathbf{X}_s - PO_s = \int_0^s A(\mathbf{X}_u - PO_u) + PF(\mathcal{X}_{\lfloor u \rfloor_{T/M}})\, du. \tag{57}$$

This, (56), the fundamental theorem of calculus, and the fact that

$$\forall\, s \in [0,T] \colon APX_s, A\mathbf{X}_s \in P(H) \tag{58}$$

prove that

$$\begin{aligned}
&e^{-\kappa ct} \|PX_t - \mathbf{X}_t\|_H^2 = e^{-\kappa ct} \|P(X_t - O_t) - (\mathbf{X}_t - PO_t)\|_H^2 \\
&= 2\int_0^t e^{-\kappa cs} \left\langle PX_s - \mathbf{X}_s, AP(X_s - O_s) + PF(X_s) - A(\mathbf{X}_s - PO_s) - PF(\mathcal{X}_{\lfloor s \rfloor_{T/M}}) \right\rangle_H ds \\
&\quad - \kappa c \int_0^t e^{-\kappa cs} \|PX_s - \mathbf{X}_s\|_H^2\, ds \\
&= 2\int_0^t e^{-\kappa cs} \left\langle PX_s - \mathbf{X}_s, APX_s + PF(X_s) - A\mathbf{X}_s - PF(\mathcal{X}_{\lfloor s \rfloor_{T/M}}) \right\rangle_H ds \\
&\quad - \kappa c \int_0^t e^{-\kappa cs} \|PX_s - \mathbf{X}_s\|_H^2\, ds \\
&= 2\int_0^t e^{-\kappa cs} \left\langle PX_s - \mathbf{X}_s, P\big[APX_s + F(X_s) - A\mathbf{X}_s - F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\big] \right\rangle_H ds \\
&\quad - \kappa c \int_0^t e^{-\kappa cs} \|PX_s - \mathbf{X}_s\|_H^2\, ds.
\end{aligned} \tag{59}$$

The fact that $P \in L(H)$ is symmetric together with the fact that

$$\forall\, s \in [0,T] \colon PX_s, \mathbf{X}_s \in P(H) \tag{60}$$

therefore ensures that

$$
\begin{aligned}
&e^{-\kappa ct}\,\|PX_t-\mathbf{X}_t\|_H^2\\
&=2\int_0^t e^{-\kappa cs}\left\langle PX_s-\mathbf{X}_s, APX_s+F(X_s)-A\mathbf{X}_s-F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\right\rangle_H ds\\
&\quad-\kappa c\int_0^t e^{-\kappa cs}\,\|PX_s-\mathbf{X}_s\|_H^2\,ds\\
&=2\int_0^t e^{-\kappa cs}\left\langle PX_s-\mathbf{X}_s, APX_s+F(PX_s)-A\mathbf{X}_s-F(\mathbf{X}_s)\right\rangle_H ds\\
&\quad+2\int_0^t e^{-\kappa cs}\left\langle PX_s-\mathbf{X}_s, F(X_s)-F(PX_s)+F(\mathbf{X}_s)-F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\right\rangle_H ds\\
&\quad-\kappa c\int_0^t e^{-\kappa cs}\,\|PX_s-\mathbf{X}_s\|_H^2\,ds.
\end{aligned}
\tag{61}
$$

The assumption that

$$\forall\, v,w\in D(A)\colon\ \langle v-w, Av+F(v)-Aw-F(w)\rangle_H\le c\|v-w\|_H^2, \tag{62}$$

the Cauchy-Schwarz inequality, and the fact that

$$\forall\, x,y\in\mathbb{R}\colon\ xy\le {}^{x^2}\!/_2+{}^{y^2}\!/_2 \tag{63}$$

hence demonstrate that

$$
\begin{aligned}
&e^{-\kappa ct}\,\|PX_t-\mathbf{X}_t\|_H^2\\
&\le 2c\int_0^t e^{-\kappa cs}\,\|PX_s-\mathbf{X}_s\|_H^2\,ds-\kappa c\int_0^t e^{-\kappa cs}\,\|PX_s-\mathbf{X}_s\|_H^2\,ds\\
&\quad+2\int_0^t e^{-\kappa cs}\,\|PX_s-\mathbf{X}_s\|_H\,\|F(X_s)-F(PX_s)+F(\mathbf{X}_s)-F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\|_H\,ds\\
&=(2-\kappa)\,c\int_0^t e^{-\kappa cs}\,\|PX_s-\mathbf{X}_s\|_H^2\,ds+2\int_0^t e^{-\kappa cs}\left[\sqrt{(\kappa-2)\,c}\,\|PX_s-\mathbf{X}_s\|_H\right]\\
&\quad\cdot\left[\tfrac{1}{\sqrt{(\kappa-2)c}}\,\|F(X_s)-F(PX_s)+F(\mathbf{X}_s)-F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\|_H\right]ds\\
&\le(2-\kappa)\,c\int_0^t e^{-\kappa cs}\,\|PX_s-\mathbf{X}_s\|_H^2\,ds+\int_0^t e^{-\kappa cs}\bigg[(\kappa-2)\,c\,\|PX_s-\mathbf{X}_s\|_H^2\\
&\quad+\tfrac{1}{(\kappa-2)c}\,\|F(X_s)-F(PX_s)+F(\mathbf{X}_s)-F(\mathcal{X}_{\lfloor s\rfloor_{T/M}})\|_H^2\bigg]\,ds.
\end{aligned}
\tag{64}
$$

The triangle inequality and the fact that

$$\forall\, v, w \in V\colon \|F(v) - F(w)\|_H^2 \leq C\|v - w\|_V^2 (1 + \|v\|_V^\varphi + \|w\|_V^\varphi) \tag{65}$$

therefore yield that

$$\begin{aligned} & e^{-\kappa ct} \, \|PX_t - \mathbf{X}_t\|_H^2 \\ & \leq \frac{1}{(\kappa - 2)\, c} \int_0^t e^{-\kappa cs} \left[\big\|F(X_s) - F(PX_s)\big\|_H + \big\|F(\mathbf{X}_s) - F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\big\|_H \right]^2 ds \\ & \leq \frac{C}{(\kappa - 2)\, c} \int_0^t e^{-\kappa cs} \Bigg[\|X_s - PX_s\|_V \sqrt{1 + \|X_s\|_V^\varphi + \|PX_s\|_V^\varphi} \\ & \quad + \|\mathbf{X}_s - \mathcal{X}_{\lfloor s \rfloor_{T/M}}\|_V \sqrt{1 + \|\mathbf{X}_s\|_V^\varphi + \|\mathcal{X}_{\lfloor s \rfloor_{T/M}}\|_V^\varphi} \Bigg]^2 ds. \end{aligned} \tag{66}$$

This completes the proof of Lemma 3.3. □

3.4 Analysis of the error between the numerical approximation and the Galerkin projection of the semilinear integrated version of the numerical approximation

Lemma 3.4. *Assume the setting in Section 3.1 and let $\alpha \in (0, \infty)$, $\rho \in [0, 1)$, $t \in [0, T]$ satisfy $\sup_{s \in (0,T]} s^\rho \|e^{sA}\|_{L(H,V)} < \infty$. Then*

$$\begin{aligned} & \|\mathbf{X}_t - \mathcal{X}_t\|_V \\ & \leq \frac{T^\alpha}{M^\alpha} \left[\sup_{s \in (0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)} \right] \int_0^t (t - s)^{-\rho} \left| \mathcal{V}(\mathcal{X}_{\lfloor s \rfloor_{T/M}}, \mathcal{O}_{\lfloor s \rfloor_{T/M}}) \right|^\alpha \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\|_H \, ds \\ & \quad + \|PO_t - \mathcal{O}_t\|_V. \end{aligned} \tag{67}$$

Proof of Lemma 3.4. Throughout this proof we assume w.l.o.g. that $t \in (0, T]$ (otherwise the proof is clear). Observe that

$$
\begin{aligned}
&\|\mathbf{X}_t - \mathcal{X}_t\|_V \\
&\leq \int_0^t \big\|Pe^{(t-s)A}\big[1 - \mathbb{1}^{\mathbb{R}}_{[0,M/T]}(\mathcal{V}(\mathcal{X}_{\lfloor s \rfloor_{T/M}}, \mathcal{O}_{\lfloor s \rfloor_{T/M}}))\big] F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\big\|_V \, ds + \|PO_t - \mathcal{O}_t\|_V \\
&\leq \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_0^t (t-s)^{-\rho} \left\|\mathbb{1}^{\mathbb{R}}_{(M/T,\infty)}(\mathcal{V}(\mathcal{X}_{\lfloor s \rfloor_{T/M}}, \mathcal{O}_{\lfloor s \rfloor_{T/M}}))\, PF(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\right\|_H ds \\
&\quad + \|PO_t - \mathcal{O}_t\|_V \\
&\leq \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_0^t (t-s)^{-\rho}\, \mathbb{1}^{\mathbb{R}}_{(M/T,\infty)}(\mathcal{V}(\mathcal{X}_{\lfloor s \rfloor_{T/M}}, \mathcal{O}_{\lfloor s \rfloor_{T/M}})) \left[\tfrac{T}{M}\mathcal{V}(\mathcal{X}_{\lfloor s \rfloor_{T/M}}, \mathcal{O}_{\lfloor s \rfloor_{T/M}})\right]^\alpha \\
&\quad \cdot \|P\|_{L(H)} \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\|_H \, ds + \|PO_t - \mathcal{O}_t\|_V.
\end{aligned}
\tag{68}
$$

This and the fact that $\|P\|_{L(H)} \leq 1$ complete the proof of Lemma 3.4. □

3.5 Temporal regularity for the Galerkin projection of the semilinear integrated version of the numerical approximation

Lemma 3.5. *Assume the setting in Section 3.1 and let $\rho \in [0,1)$, $\varrho \in [0, 1-\rho)$, $t_1 \in [0,T)$, $t_2 \in (t_1, T]$ satisfy $\sup_{s\in(0,T]} s^\rho \|e^{sA}\|_{L(H,V)} < \infty$. Then*

$$
\begin{aligned}
\|\mathbf{X}_{t_1} - \mathbf{X}_{t_2}\|_V \leq{}& \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \left[\int_{t_1}^{t_2} (t_2 - s)^{-\rho} \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\|_H \, ds \right. \\
&\left. + 2^{(\rho+\varrho)} (t_2 - t_1)^\varrho \int_0^{t_1} (t_1 - s)^{-(\rho+\varrho)} \|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\|_H \, ds \right] + \|P(O_{t_2} - O_{t_1})\|_V.
\end{aligned}
\tag{69}
$$

Proof of Lemma 3.5. Observe that

$$
\begin{aligned}
&\|\mathbf{X}_{t_2} - \mathbf{X}_{t_1}\|_V \\
&\leq \int_{t_1}^{t_2} \big\|Pe^{(t_2-s)A} F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\big\|_V \, ds + \int_0^{t_1} \big\|P\big(e^{(t_2-s)A} - e^{(t_1-s)A}\big) F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\big\|_V \, ds \\
&\quad + \big\|P(O_{t_2} - O_{t_1})\big\|_V \\
&\leq \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_{t_1}^{t_2} (t_2 - s)^{-\rho} \left\|PF(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\right\|_H ds + \left\|P(O_{t_2} - O_{t_1})\right\|_V \\
&\quad + \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_0^{t_1} \left[\frac{2}{t_1 - s}\right]^\rho \left\|Pe^{\frac{1}{2}(t_1-s)A} \left(e^{(t_2-t_1)A} - \mathrm{Id}_H\right) F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\right\|_H ds.
\end{aligned}
\tag{70}
$$

This, the fact that
$$\forall\, s \in [0,\infty), r \in [0,1]\colon \|(-sA)^r e^{sA}\|_{L(H)} \leq 1, \tag{71}$$
the fact that
$$\forall\, s \in (0,\infty), r \in [0,1]\colon \|(-sA)^{-r}(e^{sA} - \mathrm{Id}_H)\|_{L(H)} \leq 1, \tag{72}$$
and the fact that $\|P\|_{L(H)} \leq 1$ prove that
$$\begin{aligned}
&\|\mathbf{X}_{t_2} - \mathbf{X}_{t_1}\|_V \\
&\leq \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_{t_1}^{t_2} (t_2-s)^{-\rho} \left\|P\right\|_{L(H)} \left\|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\right\|_H ds \\
&\quad + \left\|P(O_{t_2} - O_{t_1})\right\|_V \\
&\quad + 2^\rho \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_0^{t_1} (t_1-s)^{-\rho} \left\|(-A)^\varrho \, e^{\frac{1}{2}(t_1-s)A}\right\|_{L(H)} \\
&\quad \cdot \left\|(-A)^{-\varrho}\left(e^{(t_2-t_1)A} - \mathrm{Id}_H\right)\right\|_{L(H)} \left\|P\right\|_{L(H)} \left\|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\right\|_H ds \\
&\leq \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_{t_1}^{t_2} (t_2-s)^{-\rho} \left\|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\right\|_H ds + \left\|P(O_{t_2} - O_{t_1})\right\|_V \\
&\quad + 2^{(\rho+\varrho)} (t_2-t_1)^\varrho \left[\sup_{s\in(0,T)} s^\rho \left\|e^{sA}\right\|_{L(H,V)}\right] \int_0^{t_1} (t_1-s)^{-(\rho+\varrho)} \left\|F(\mathcal{X}_{\lfloor s \rfloor_{T/M}})\right\|_H ds.
\end{aligned} \tag{73}$$
The proof of Lemma 3.5 is thus completed. □

4 Strong error estimates

4.1 Setting

Consider the notation in Section 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \left\|\cdot\right\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathbb{H} \subseteq H$ be a non-empty orthonormal basis of H, let $T, c, C, \varphi \in (0,\infty)$, $\mathcal{D} \subseteq \mathcal{P}_0(\mathbb{H})$, $\mu\colon \mathbb{H} \to \mathbb{R}$ satisfy $\sup_{h\in\mathbb{H}} \mu_h < 0$, let $A\colon D(A) \subseteq H \to H$ be the linear operator which satisfies $D(A) = \{v \in H\colon \sum_{h\in\mathbb{H}} |\mu_h \langle h, v\rangle_H|^2 < \infty\}$ and $\forall\, v \in D(A)\colon Av = \sum_{h\in\mathbb{H}} \mu_h \langle h, v\rangle_H h$, let $(V, \left\|\cdot\right\|_V)$ be an $\mathbb{R}$-Banach space with $D(A) \subseteq V \subseteq H$ continuously and densely, and let $\mathcal{V} \in \mathcal{M}\big(\mathcal{B}(V\times V), \mathcal{B}([0,\infty))\big)$, $F \in \mathcal{C}(V, H)$, $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$ satisfy for all $v, w \in D(A)$, $I \in \mathcal{P}(\mathbb{H})$ that
$$\langle v - w, Av + F(v) - Aw - F(w)\rangle_H \leq c\|v-w\|_H^2, \tag{74}$$
$$\|F(v) - F(w)\|_H^2 \leq C\|v-w\|_V^2(1 + \|v\|_V^\varphi + \|w\|_V^\varphi), \tag{75}$$

and $P_I(v) = \sum_{h\in I} \langle h, v\rangle_H h$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X\colon [0,T]\times\Omega \to V$ be a stochastic process with continuous sample paths, and let $O\colon [0,T]\times\Omega \to V$ and $\mathbf{X}^{M,I}, \mathcal{X}^{M,I}, \mathcal{O}^{M,I}\colon [0,T]\times\Omega \to V$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, be stochastic processes which satisfy for all $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ that

$$X_t = \int_0^t e^{(t-s)A} F(X_s)\, ds + O_t, \qquad \mathbf{X}_t^{M,I} = \int_0^t P_I e^{(t-s)A} F(\mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}})\, ds + P_I O_t, \tag{76}$$

$$\text{and} \qquad \mathcal{X}_t^{M,I} = \int_0^t P_I e^{(t-s)A} \mathbb{1}^{\Omega}_{\{\mathcal{V}(\mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}}, \mathcal{O}^{M,I}_{\lfloor s\rfloor_{T/M}}) \leq M/T\}} F(\mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}})\, ds + \mathcal{O}_t^{M,I}. \tag{77}$$

4.2 Analysis of the error between the Galerkin projection of the exact solution and the Galerkin projection of the semilinear integrated version of the numerical approximation

Lemma 4.1. *Assume the setting in Section 4.1 and let $\kappa \in (2,\infty)$, $t \in [0,T]$, $p \in [2,\infty)$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ satisfy $\sup_{s\in(0,T)} \mathbb{E}\big[\|X_s\|_V^{p\varphi} + \|P_I X_s\|_V^{p\varphi} + \|\mathbf{X}_s^{M,I}\|_V^{p\varphi} + \|\mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_V^{p\varphi}\big] < \infty$. Then*

$$\begin{aligned} \|P_I X_t - \mathbf{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} &\leq \frac{\sqrt{CTe^{\kappa cT}}}{\sqrt{(\kappa-2)\,c}} \left[\sup_{s\in(0,T)} \Big(\|P_{\mathbb{H}\setminus I} X_s\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|\mathbf{X}_s^{M,I} - \mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \Big)\right] \\ &\quad \cdot \left[1 + \sup_{s\in(0,T)} \Big(\|X_s\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|P_I X_s\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|\mathbf{X}_s^{M,I}\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|\mathcal{X}_s^{M,I}\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \Big)\right]. \end{aligned} \tag{78}$$

Proof of Lemma 4.1. Note that Lemma 3.3 and Hölder's inequality ensure that

$$\begin{aligned} &\|P_I X_t - \mathbf{X}_t^{M,I}\|^2_{\mathcal{L}^p(\mathbb{P};H)} \\ &= \big\| \|P_I X_t - \mathbf{X}_t^{M,I}\|_H^2 \big\|_{\mathcal{L}^{p/2}(\mathbb{P};\mathbb{R})} \\ &\leq \frac{Ce^{\kappa cT}}{(\kappa-2)\,c} \int_0^t \bigg\| \Big[\|X_s - P_I X_s\|_V + \|\mathbf{X}_s^{M,I} - \mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_V \Big]^2 \\ &\quad \cdot \Big(1 + \|X_s\|_V^{\varphi} + \|P_I X_s\|_V^{\varphi} + \|\mathbf{X}_s^{M,I}\|_V^{\varphi} + \|\mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_V^{\varphi} \Big) \bigg\|_{\mathcal{L}^{p/2}(\mathbb{P};\mathbb{R})} ds \\ &\leq \frac{Ce^{\kappa cT}}{(\kappa-2)\,c} \int_0^t \big\| \|X_s - P_I X_s\|_V + \|\mathbf{X}_s^{M,I} - \mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_V \big\|^2_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \\ &\quad \cdot \big\| 1 + \|X_s\|_V^{\varphi} + \|P_I X_s\|_V^{\varphi} + \|\mathbf{X}_s^{M,I}\|_V^{\varphi} + \|\mathcal{X}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_V^{\varphi} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} ds. \end{aligned} \tag{79}$$

This shows that

$$
\begin{aligned}
&\|P_I X_t - \mathbf{X}_t^{M,I}\|^2_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{C e^{\kappa c T}}{(\kappa-2)\, c} \int_0^t \Big[\|X_s - P_I X_s\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|\mathbf{X}_s^{M,I} - \mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \Big]^2 \\
&\quad \cdot \Big[1 + \|X_s\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|P_I X_s\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|\mathbf{X}_s^{M,I}\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}}\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \Big]\, ds \\
&\leq \frac{C T e^{\kappa c T}}{(\kappa-2)\, c} \left[\sup_{s \in (0,T)} \Big(\|P_{\mathbb{H} \setminus I} X_s\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|\mathbf{X}_s^{M,I} - \mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \Big) \right]^2 \\
&\quad \cdot \left[1 + \sup_{s \in (0,T)} \Big(\|X_s\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \|P_I X_s\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|\mathbf{X}_s^{M,I}\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}}\|^{\varphi}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \Big) \right].
\end{aligned}
\tag{80}
$$

Combining this with the fact that

$$
\forall\, n \in \mathbb{N}, x_1, \dots, x_n \in [0,\infty) \colon \sqrt{x_1 + \ldots + x_n} \leq \sqrt{x_1} + \ldots + \sqrt{x_n} \tag{81}
$$

completes the proof of Lemma 4.1. □

4.3 Analysis of the error between the numerical approximation and the Galerkin projection of the semilinear integrated version of the numerical approximation

Lemma 4.2. *Assume the setting in Section 4.1 and let $\alpha \in (0,\infty)$, $\rho \in [0,1)$, $t \in [0,T]$, $p \in [1,\infty)$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ satisfy $\sup_{s \in (0,T]} s^{\rho} \|e^{sA}\|_{L(H,V)} + \sup_{s \in [0,T)} \mathbb{E}\big[|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})|^{2p\alpha} + \|F(\mathcal{X}_s^{M,I})\|_H^{2p} \big] < \infty$. Then*

$$
\begin{aligned}
&\|\mathbf{X}_t^{M,I} - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&\leq \frac{T^{(1+\alpha-\rho)}}{(1-\rho)\, M^{\alpha}} \left[\sup_{s \in (0,T)} s^{\rho} \, \|e^{sA}\|_{L(H,V)} \right] \left[\sup_{s \in [0,T)} \Big(\|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})\|^{\alpha}_{\mathcal{L}^{2p\alpha}(\mathbb{P};\mathbb{R})} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \Big) \right] \\
&\quad + \|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)}.
\end{aligned}
\tag{82}
$$

Proof of Lemma 4.2. Note that Lemma 3.4 and Hölder's inequality imply that

$$
\begin{aligned}
&\|\mathbf{X}_t^{M,I} - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&\leq \frac{T^\alpha}{M^\alpha}\left[\sup_{s\in(0,T)} s^\rho\,\|e^{sA}\|_{L(H,V)}\right] \\
&\quad\cdot\left[\int_0^t (t-s)^{-\rho}\,\big\||\mathcal{V}(\mathcal{X}_{\lfloor s\rfloor_{T/M}}^{M,I},\mathcal{O}_{\lfloor s\rfloor_{T/M}}^{M,I})|^\alpha\,\|F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}^{M,I})\|_H\big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}\,ds\right] \\
&\quad+\|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&\leq \frac{T^\alpha}{M^\alpha}\left[\sup_{s\in(0,T)} s^\rho\,\|e^{sA}\|_{L(H,V)}\right] \\
&\quad\cdot\left[\int_0^t (t-s)^{-\rho}\,\|\mathcal{V}(\mathcal{X}_{\lfloor s\rfloor_{T/M}}^{M,I},\mathcal{O}_{\lfloor s\rfloor_{T/M}}^{M,I})\|_{\mathcal{L}^{2p\alpha}(\mathbb{P};\mathbb{R})}^\alpha\,\|F(\mathcal{X}_{\lfloor s\rfloor_{T/M}}^{M,I})\|_{\mathcal{L}^{2p}(\mathbb{P};H)}\,ds\right] \\
&\quad+\|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)}.
\end{aligned}
\tag{83}
$$

This and the fact that $\int_0^t (t-s)^{-\rho}\,ds = \frac{t^{(1-\rho)}}{(1-\rho)}$ complete the proof of Lemma 4.2. □

4.4 Temporal regularity for the Galerkin projection of the semilinear integrated version of the numerical approximation

Lemma 4.3. *Assume the setting in Section 4.1 and let $\rho\in[0,1)$, $\varrho\in[0,1-\rho)$, $t_1\in[0,T)$, $t_2\in(t_1,T]$, $p\in[1,\infty)$, $M\in\mathbb{N}$, $I\in\mathcal{D}$ satisfy $\sup_{s\in(0,T]} s^\rho\|e^{sA}\|_{L(H,V)}<\infty$. Then*

$$
\begin{aligned}
\|\mathbf{X}_{t_1}^{M,I} - \mathbf{X}_{t_2}^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)} &\leq \|P_I(O_{t_1}-O_{t_2})\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&\quad+\frac{3\,T^{(1-\rho-\varrho)}\,(t_2-t_1)^\varrho}{(1-\rho-\varrho)}\left[\sup_{s\in(0,T)} s^\rho\,\|e^{sA}\|_{L(H,V)}\right]\left[\sup_{s\in[0,T)}\|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^p(\mathbb{P};H)}\right].
\end{aligned}
\tag{84}
$$

Proof of Lemma 4.3. Observe that Lemma 3.5 proves that

$$
\begin{aligned}
&\|\mathbf{X}^{M,I}_{t_1} - \mathbf{X}^{M,I}_{t_2}\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&\leq \left[\sup_{s\in(0,T)} s^\rho \, \|e^{sA}\|_{L(H,V)}\right] \left[\sup_{s\in[0,T)} \|F(\mathcal{X}^{M,I}_s)\|_{\mathcal{L}^p(\mathbb{P};H)}\right] \Bigg[\int_{t_1}^{t_2} (t_2 - s)^{-\rho} \, ds \\
&\quad + 2^{(\rho+\varrho)} \, (t_2 - t_1)^{\varrho} \int_0^{t_1} (t_1 - s)^{-(\rho+\varrho)} \, ds \Bigg] + \|P_I(O_{t_2} - O_{t_1})\|_{\mathcal{L}^p(\mathbb{P};V)} \\
&= \left[\sup_{s\in(0,T)} s^\rho \, \|e^{sA}\|_{L(H,V)}\right] \left[\sup_{s\in[0,T)} \|F(\mathcal{X}^{M,I}_s)\|_{\mathcal{L}^p(\mathbb{P};H)}\right] \\
&\quad \cdot \left[\frac{(t_2 - t_1)^{(1-\rho)}}{(1-\rho)} + \frac{2^{(\rho+\varrho)} \, (t_2 - t_1)^{\varrho} \, |t_1|^{(1-\rho-\varrho)}}{(1-\rho-\varrho)} \right] + \|P_I(O_{t_2} - O_{t_1})\|_{\mathcal{L}^p(\mathbb{P};V)}.
\end{aligned}
\tag{85}
$$

This completes the proof of Lemma 4.3. □

4.5 Analysis of the error between the exact solution and the numerical approximation

Proposition 4.4. *Assume the setting in Section 4.1 and let $\alpha \in (0,\infty)$, $\rho \in [0,1)$, $\varrho \in [0, 1-\rho)$, $\kappa \in (2,\infty)$, $p \in [\max\{2, 1/\varphi\}, \infty)$, $M \in \mathbb{N}$, $I \in \mathcal{D}$. Then*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \mathcal{X}^{M,I}_t\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{4^{(2+\varphi)} \max\{1, T^{(3/2+\alpha-\rho+\varphi/2-\rho\varphi/2)}\} \max\{1, C^{(1+\varphi/4)}\} \sqrt{e^{\kappa c T}}}{\min\{1, \sqrt{c\,(\kappa-2)}\} \, (1-\rho-\varrho)^{(1+\varphi/2)}} \Bigg[\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I} X_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + M^{-\min\{\alpha,\varrho\}} \\
&+ \sup_{t\in(0,T)} \|P_I(O_t - O_{\lfloor t \rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{t\in[0,T]} \|P_I O_t - \mathcal{O}^{M,I}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \Bigg] \max\left\{1, \sup_{v\in V\setminus\{0\}} \frac{\|v\|_H}{\|v\|_V}\right\} \\
&\cdot \left[1 + \sup_{s\in(0,T)} \|X_s\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \sup_{s\in(0,T)} \|P_I X_s\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \sup_{s\in[0,T)} \|\mathcal{X}^{M,I}_s\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \sup_{s\in(0,T)} \|P_I O_s\|^{\varphi/2}_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \right] \\
&\cdot \left[1 + \sup_{s\in[0,T)} \|\mathcal{V}(\mathcal{X}^{M,I}_s, \mathcal{O}^{M,I}_s)\|^{\alpha}_{\mathcal{L}^{4p\alpha}(\mathbb{P};\mathbb{R})} \right] \max\left\{1, \sup_{s\in(0,T)} \left[s^\rho \, \|e^{sA}\|_{L(H,V)}\right]^{(1+\varphi/2)}\right\} \\
&\cdot \left[\max\left\{1, \sup_{s\in[0,T)} \|\mathcal{X}^{M,I}_s\|^{[(1+\varphi/2)^2]}_{\mathcal{L}^{p(1+\varphi/2)\max\{4,\varphi\}}(\mathbb{P};V)}\right\} + \|F(0)\|^{(1+\varphi/2)}_H \right].
\end{aligned}
\tag{86}
$$

Proof of Proposition 4.4. Throughout this proof assume w.l.o.g. that $\sup_{s\in(0,T)} \big(s^\rho \|e^{sA}\|_{L(H,V)} + \mathbb{E}\big[\|P_I O_s\|_V^{p\varphi} + \|P_I X_s\|_V^{p\varphi} + \|X_s\|_V^{p\varphi}\big]\big) + \sup_{s\in[0,T)} \mathbb{E}\big[|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})|^{4p\alpha} + \|\mathcal{X}_s^{M,I}\|_V^{p(1+\varphi/2)\max\{4,\varphi\}}\big] < \infty$. Note that the fact that

$$\forall\, v,w \in V\colon \|F(v) - F(w)\|_H^2 \leq C\|v-w\|_V^2(1 + \|v\|_V^\varphi + \|w\|_V^\varphi) \tag{87}$$

and the fact that

$$\forall\, x,y \in [0,\infty)\colon \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \tag{88}$$

imply for all $s \in [0,T]$ that

$$\begin{aligned} \|F(\mathcal{X}_s^{M,I})\|_H &\leq \|F(\mathcal{X}_s^{M,I}) - F(0)\|_H + \|F(0)\|_H \\ &\leq \sqrt{C\, \|\mathcal{X}_s^{M,I}\|_V^2 \big(1 + \|\mathcal{X}_s^{M,I}\|_V^\varphi\big)} + \|F(0)\|_H \\ &\leq \sqrt{C}\, \|\mathcal{X}_s^{M,I}\|_V \big(1 + \|\mathcal{X}_s^{M,I}\|_V^{\varphi/2}\big) + \|F(0)\|_H \\ &= \sqrt{C}\, \Big(\|\mathcal{X}_s^{M,I}\|_V + \|\mathcal{X}_s^{M,I}\|_V^{(1+\varphi/2)}\Big) + \|F(0)\|_H. \end{aligned} \tag{89}$$

Hence, we obtain for all $q \in [1,\infty)$ that

$$\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^q(\mathbb{P};H)} \leq 2\,\sqrt{C} \max\left\{1, \sup_{s\in[0,T)} \|\mathcal{X}_s^{M,I}\|_{\mathcal{L}^{q(1+\varphi/2)}(\mathbb{P};V)}^{(1+\varphi/2)}\right\} + \|F(0)\|_H. \tag{90}$$

Next observe that the triangle inequality, Lemma 4.2, Lemma 4.3, and Hölder's inequality imply that

$$
\begin{aligned}
&\sup_{s\in(0,T)} \|\mathbf{X}_s^{M,I} - \mathcal{X}_{\lfloor s \rfloor_{T/M}}^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\leq \sup_{s\in(0,T)} \|\mathbf{X}_s^{M,I} - \mathbf{X}_{\lfloor s \rfloor_{T/M}}^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{s\in[0,T)} \|\mathbf{X}_s^{M,I} - \mathcal{X}_s^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\leq \frac{3\,T^{(1-\rho)}}{(1-\rho-\varrho)\,M^{\varrho}} \left[\sup_{s\in(0,T)} s^{\rho}\,\|e^{sA}\|_{L(H,V)}\right] \left[\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{2p}(\mathbb{P};H)}\right] \\
&+ \frac{T^{(1+\alpha-\rho)}}{(1-\rho)\,M^{\alpha}} \left[\sup_{s\in(0,T)} s^{\rho}\,\|e^{sA}\|_{L(H,V)}\right] \\
&\cdot \left[\sup_{s\in[0,T)} \|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})\|_{\mathcal{L}^{4p\alpha}(\mathbb{P};\mathbb{R})}^{\alpha}\right] \left[\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{4p}(\mathbb{P};H)}\right] \\
&+ \sup_{s\in(0,T)} \|P_I(O_s - O_{\lfloor s \rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{s\in[0,T)} \|P_I O_s - \mathcal{O}_s^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\leq \frac{3\max\{1, T^{(1+\alpha-\rho)}\}}{(1-\rho-\varrho)\,M^{\min\{\alpha,\varrho\}}} \left[\sup_{s\in(0,T)} s^{\rho}\,\|e^{sA}\|_{L(H,V)}\right] \left[1 + \sup_{s\in[0,T)} \|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})\|_{\mathcal{L}^{4p\alpha}(\mathbb{P};\mathbb{R})}^{\alpha}\right] \\
&\cdot \left[\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{4p}(\mathbb{P};H)}\right] + \sup_{s\in(0,T)} \|P_I(O_s - O_{\lfloor s \rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&+ \sup_{s\in[0,T)} \|P_I O_s - \mathcal{O}_s^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)}.
\end{aligned}
\tag{91}
$$

Moreover, note that (90) and the fact that $\|P_I\|_{L(H)} \leq 1$ yields that

$$\begin{aligned}
&\sup_{s\in(0,T)} \|\mathbf{X}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \\
&\leq \sup_{s\in(0,T)} \left\| \int_0^s P_I e^{(s-u)A} F(\mathcal{X}^{M,I}_{\lfloor u \rfloor_{T/M}})\, du \right\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \\
&\leq \sup_{s\in(0,T)} \int_0^s \|e^{(s-u)A}\|_{L(H,V)} \|P_I\|_{L(H)} \|F(\mathcal{X}^{M,I}_{\lfloor u \rfloor_{T/M}})\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H)}\, du + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \\
&\leq \left[\sup_{s\in(0,T)} s^\rho \, \|e^{sA}\|_{L(H,V)} \right] \left[\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H)} \right] \left[\sup_{s\in(0,T)} \int_0^s (s-u)^{-\rho}\, du \right] \qquad (92) \\
&\quad + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \\
&\leq \frac{T^{(1-\rho)}}{(1-\rho)} \left[\sup_{s\in(0,T)} s^\rho \, \|e^{sA}\|_{L(H,V)} \right] \left[\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H)} \right] \\
&\quad + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} < \infty.
\end{aligned}$$

Combining this, (91), and the fact that

$$\forall\, x, y \in [0,\infty)\colon (x+y)^{\varphi/2} \leq 2^{\max\{0,\varphi/2-1\}} (x^{\varphi/2} + y^{\varphi/2}) \qquad (93)$$

with Lemma 4.1 proves that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|P_I X_t - \mathbf{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{2^{\max\{0,\varphi/2-1\}}\sqrt{CTe^{\kappa cT}}}{\sqrt{(\kappa-2)\,c}} \Bigg(\sup_{s\in(0,T)} \|P_{\mathbb{H}\setminus I} X_s\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{s\in(0,T)} \|P_I(O_s - O_{\lfloor s\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&+ \sup_{s\in[0,T)} \|P_I O_s - \mathcal{O}_s^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \frac{3\max\{1, T^{(1+\alpha-\rho)}\}}{(1-\rho-\varrho)\,M^{\min\{\alpha,\varrho\}}} \left[\sup_{s\in(0,T)} s^\rho \,\|e^{sA}\|_{L(H,V)}\right] \\
&\cdot \left[1 + \sup_{s\in[0,T)} \|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})\|_{\mathcal{L}^{4p\alpha}(\mathbb{P};\mathbb{R})}^{\alpha}\right] \left[\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{4p}(\mathbb{P};H)}\right] \Bigg) \\
&\cdot \Bigg(1 + \sup_{s\in(0,T)} \|X_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I X_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in[0,T)} \|\mathcal{X}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} \\
&+ \frac{T^{(1-\rho)\varphi/2}}{(1-\rho)^{\varphi/2}} \left[\sup_{s\in(0,T)} s^\rho \,\|e^{sA}\|_{L(H,V)}\right]^{\varphi/2} \left[\sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H)}\right]^{\varphi/2} \\
&+ \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} \Bigg).
\end{aligned}
\tag{94}
$$

Hence, we obtain that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|P_I X_t - \mathbf{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{3\cdot 2^{\max\{0,\varphi/2-1\}} \max\{1, T^{(3/2+\alpha-\rho+\varphi/2-\rho\varphi/2)}\}\sqrt{Ce^{\kappa cT}}}{(1-\rho-\varrho)^{(1+\varphi/2)}\sqrt{(\kappa-2)\,c}} \Bigg[\sup_{s\in(0,T)} \|P_{\mathbb{H}\setminus I} X_s\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\quad + \sup_{s\in(0,T)} \|P_I(O_s - O_{\lfloor s\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{s\in[0,T)} \|P_I O_s - \mathcal{O}_s^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + M^{-\min\{\alpha,\varrho\}} \Bigg] \\
&\cdot \Bigg[2 + \sup_{s\in(0,T)} \|X_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I X_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in[0,T)} \|\mathcal{X}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} \\
&\quad + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} \Bigg] \left[1 + \sup_{s\in[0,T)} \|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})\|_{\mathcal{L}^{4p\alpha}(\mathbb{P};\mathbb{R})}^{\alpha} \right] \\
&\quad \cdot \max\left\{1, \sup_{s\in(0,T)} \left[s^\rho \,\|e^{sA}\|_{L(H,V)}\right]^{(1+\varphi/2)}\right\} \max\left\{1, \sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{p\max\{4,\varphi\}}(\mathbb{P};H)}^{(1+\varphi/2)}\right\}.
\end{aligned}
\tag{95}
$$

In the next step observe that the triangle inequality implies that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \sup_{t\in[0,T]} \Big[\|X_t - P_I X_t\|_{\mathcal{L}^p(\mathbb{P};H)} + \|P_I X_t - \mathbf{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} + \|\mathbf{X}_t^{M,I} - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)}\Big] \\
&\leq \sup_{t\in[0,T]} \|P_I X_t - \mathbf{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\quad + \left[\sup_{v\in V\setminus\{0\}} \frac{\|v\|_H}{\|v\|_V}\right] \sup_{t\in[0,T]} \Big[\|P_{\mathbb{H}\setminus I} X_t\|_{\mathcal{L}^p(\mathbb{P};V)} + \|\mathbf{X}_t^{M,I} - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)}\Big] \\
&\leq \left[\sup_{t\in[0,T]} \|P_I X_t - \mathbf{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} + \sup_{t\in[0,T]} \Big[\|P_{\mathbb{H}\setminus I} X_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|\mathbf{X}_t^{M,I} - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};V)}\Big]\right] \\
&\quad \cdot \max\left\{1, \sup_{v\in V\setminus\{0\}} \frac{\|v\|_H}{\|v\|_V}\right\}.
\end{aligned} \tag{96}
$$

This, (95), and Lemma 4.2 prove that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{\big[3 \cdot 2^{\max\{1,\varphi/2\}} + 1\big] \max\{1, T^{(3/2+\alpha-\rho+\varphi/2-\rho\varphi/2)}\} \max\{1, \sqrt{Ce^{\kappa cT}}\}}{\min\{1, \sqrt{(\kappa-2)\,c}\}\,(1-\rho-\varrho)^{(1+\varphi/2)}} \Bigg[\sup_{t\in[0,T]} \|P_{\mathbb{H}\setminus I} X_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&+ M^{-\min\{\alpha,\varrho\}} + \sup_{t\in(0,T)} \|P_I(O_t - O_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{t\in[0,T]} \|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)}\Bigg] \\
&\cdot \Bigg[1 + \sup_{s\in(0,T)} \|X_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I X_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} \\
&+ \sup_{s\in[0,T)} \|\mathcal{X}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2}\Bigg] \\
&\cdot \left[1 + \sup_{s\in[0,T)} \|\mathcal{V}(\mathcal{X}_s^{M,I}, \mathcal{O}_s^{M,I})\|_{\mathcal{L}^{4p\alpha}(\mathbb{P};\mathbb{R})}^{\alpha}\right] \max\left\{1, \sup_{s\in(0,T)} \big[s^\rho \, \|e^{sA}\|_{L(H,V)}\big]^{(1+\varphi/2)}\right\} \\
&\cdot \max\left\{1, \sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{p\max\{4,\varphi\}}(\mathbb{P};H)}^{(1+\varphi/2)}\right\} \max\left\{1, \sup_{v\in V\setminus\{0\}} \frac{\|v\|_H}{\|v\|_V}\right\}.
\end{aligned} \tag{97}
$$

Next note that (90) and the fact that

$$\forall\, x, y \in [0,\infty)\colon (x+y)^{(1+\varphi/2)} \leq 2^{\varphi/2}(x^{(1+\varphi/2)} + y^{(1+\varphi/2)}) \tag{98}$$

ensure that

$$\begin{aligned}
&\max\left\{1, \sup_{s\in[0,T)} \|F(\mathcal{X}_s^{M,I})\|_{\mathcal{L}^{p\max\{4,\varphi\}}(\mathbb{P};H)}^{(1+\varphi/2)}\right\} \\
&\leq \left[2\max\{1,\sqrt{C}\}\max\left\{1, \sup_{s\in[0,T)} \|\mathcal{X}_s^{M,I}\|_{\mathcal{L}^{p(1+\varphi/2)\max\{4,\varphi\}}(\mathbb{P};V)}^{(1+\varphi/2)}\right\} + \|F(0)\|_H\right]^{(1+\varphi/2)} \\
&\leq 2^{(1+\varphi)}\max\{1, C^{(1/2+\varphi/4)}\}\left[\max\left\{1, \sup_{s\in[0,T)} \|\mathcal{X}_s^{M,I}\|_{\mathcal{L}^{p(1+\varphi/2)\max\{4,\varphi\}}(\mathbb{P};V)}^{[(1+\varphi/2)^2]}\right\} + \|F(0)\|_H^{(1+\varphi/2)}\right] \\
&< \infty.
\end{aligned} \tag{99}$$

Combining this with (97) and the fact that

$$\begin{aligned}
\left[3\cdot 2^{\max\{1,\varphi/2\}}+1\right]\cdot 2^{(1+\varphi)} &\leq 3\cdot 2^{\max\{1,\varphi/2\}}\left[1+1/6\right]\cdot 2^{(1+\varphi)} \\
&= 7\cdot 2^{\max\{1+\varphi,3\varphi/2\}} \leq 4^{(2+\varphi)}
\end{aligned} \tag{100}$$

completes the proof of Proposition 4.4. ◻

5 Main result

5.1 Setting

Consider the notation in Section 1.1, let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space, let $\mathbb{H}\subseteq H$ be a non-empty orthonormal basis of H, let $T, c, \varphi \in (0,\infty)$, $\epsilon\in[0,1)$, $\rho\in[0,1/2)$, $\gamma\in(\rho,1/2]$, $\chi\in(0,(\gamma-\rho)/(1+\varphi/2)]\cap(0,(1-\rho)/(1+\varphi)]$, $\mathcal{D}\subseteq\mathcal{P}_0(\mathbb{H})\setminus\{\emptyset\}$, $\mu\colon\mathbb{H}\to\mathbb{R}$ satisfy $\sup_{h\in\mathbb{H}}\mu_h<0$, let $A\colon D(A)\subseteq H\to H$ be the linear operator which satisfies that $D(A)=\{v\in H\colon \sum_{h\in\mathbb{H}}|\mu_h\langle h,v\rangle_H|^2<\infty\}$ and $\forall\, v\in D(A)\colon Av=\sum_{h\in\mathbb{H}}\mu_h\langle h,v\rangle_H\, h$, let $(H_r,\langle\cdot,\cdot\rangle_{H_r},\|\cdot\|_{H_r})$, $r\in\mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [27, Section 3.7]), let $(V,\|\cdot\|_V)$ be an $\mathbb{R}$-Banach space with $H_\rho\subseteq V\subseteq H$ continuously and densely, let $\phi,\Phi\colon\mathcal{C}([0,T],H_1)\to[0,\infty)$ be $\mathcal{B}(\mathcal{C}([0,T],H_1))/\mathcal{B}([0,\infty))$-measurable functions, let $F\in\mathcal{C}(V,H)$, $(P_I)_{I\in\mathcal{P}_0(\mathbb{H})}\subseteq L(H)$ satisfy for all $I\in\mathcal{P}_0(\mathbb{H})$, $u\in H$, $v,w\in P_I(H)$, $x\in\mathcal{C}([0,T],H_1)$ that

$$P_I(u)=\textstyle\sum_{h\in I}\langle h,u\rangle_H\, h, \qquad \langle v-w, Av+F(v)-Aw-F(w)\rangle_H\leq c\|v-w\|_H^2, \tag{101}$$

$$\begin{aligned}
&\sup_{t\in[0,T]}\big(\langle v, P_I F(v+x_t)\rangle_{H_{1/2}}+\phi(x)\langle v, F(v+x_t)\rangle_H\big) \\
&\leq \epsilon\|v\|_{H_1}^2+(c+\phi(x))\|v\|_{H_{1/2}}^2+c\phi(x)\|v\|_H^2+\Phi(x),
\end{aligned} \tag{102}$$

$$\text{and} \qquad \|F(v) - F(w)\|_H^2 \leq c\|v - w\|_V^2(1 + \|v\|_V^\varphi + \|w\|_V^\varphi), \tag{103}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X, O\colon [0,T] \times \Omega \to V$ and $\mathcal{O}^{M,I}\colon [0,T] \times \Omega \to H_1$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, be stochastic processes with continuous sample paths, let $\mathcal{X}^{M,I}\colon [0,T] \times \Omega \to H_\gamma$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, be stochastic processes, and assume for all $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ that $\mathcal{O}^{M,I}([0,T] \times \Omega) \subseteq P_I(H)$ and

$$\begin{aligned} &\mathbb{P}\bigg(X_t = \int_0^t e^{(t-s)A} F(X_s)\, ds + O_t\bigg) \\ &= \mathbb{P}\bigg(\mathcal{X}_t^{M,I} = \int_0^t P_I\, e^{(t-s)A}\, \mathbb{1}^{\Omega}_{\{\|\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} \leq (M/T)^\chi\}}\, F(\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}})\, ds + \mathcal{O}_t^{M,I}\bigg) = 1. \end{aligned} \tag{104}$$

5.2 Comments on the setting

In the next two results, Lemma 5.1 and Lemma 5.2 below, we establish a few elementary consequences of the framework in Section 5.1.

Lemma 5.1. *Assume the setting in Section 5.1 and let $r \in [0, \infty)$. Then $\overline{\operatorname{span}(\mathbb{H})}^{H_r} = H_r$.*

Lemma 5.1. Throughout this proof let $u \in H_r$, let $\mathbb{H}_n \subseteq \mathbb{H}$, $n \in \mathbb{N}$, be a non-decreasing sequence of finite subsets of $\mathbb{H}$ which satisfies $\cup_{n \in \mathbb{N}} \mathbb{H}_n = \mathbb{H}$, and let $(u_n)_{n \in \mathbb{N}} \subseteq \operatorname{span}(\mathbb{H})$ satisfy for all $n \in \mathbb{N}$ that

$$u_n = \sum_{h \in \mathbb{H}_n} \langle h, u \rangle_H \, h. \tag{105}$$

Note that the fact that $\mathbb{H} \subseteq H$ is orthogonal in H_r and the fact that

$$\limsup_{n \to \infty} \sum_{h \in \mathbb{H}_n} |\langle h, u \rangle_H|^2 \, |\mu_h|^{2r} = \sum_{h \in \mathbb{H}} |\langle h, u \rangle_H|^2 \, |\mu_h|^{2r} < \infty \tag{106}$$

show that

$$
\begin{aligned}
&\inf_{N\in\mathbb{N}} \sup_{\substack{m,n\in\mathbb{N},\\ m\geq n\geq N}} \|u_m - u_n\|_{H_r}^2 \\
&= \inf_{N\in\mathbb{N}} \sup_{\substack{m,n\in\mathbb{N},\\ m\geq n\geq N}} \left\| \sum_{h\in\mathbb{H}_m} \langle h, u\rangle_H \, h - \sum_{h\in\mathbb{H}_n} \langle h, u\rangle_H \, h \right\|_{H_r}^2 \\
&= \inf_{N\in\mathbb{N}} \sup_{\substack{m,n\in\mathbb{N},\\ m\geq n\geq N}} \left\| \sum_{h\in\mathbb{H}_m\setminus\mathbb{H}_n} \langle h, u\rangle_H \, h \right\|_{H_r}^2 \\
&= \inf_{N\in\mathbb{N}} \sup_{\substack{m,n\in\mathbb{N},\\ m\geq n\geq N}} \left[\sum_{h\in\mathbb{H}_m\setminus\mathbb{H}_n} \|\langle h, u\rangle_H \, h\|_{H_r}^2 \right] = \inf_{N\in\mathbb{N}} \sup_{\substack{m,n\in\mathbb{N},\\ m\geq n\geq N}} \left[\sum_{h\in\mathbb{H}_m\setminus\mathbb{H}_n} |\langle h, u\rangle_H|^2 \, \|(-A)^r h\|_H^2 \right] \\
&= \inf_{N\in\mathbb{N}} \sup_{\substack{m,n\in\mathbb{N},\\ m\geq n\geq N}} \left[\sum_{h\in\mathbb{H}_m\setminus\mathbb{H}_n} |\langle h, u\rangle_H|^2 \, |\mu_h|^{2r} \right] \leq \inf_{N\in\mathbb{N}} \sup_{\substack{m,n\in\mathbb{N},\\ m\geq n\geq N}} \left[\sum_{h\in\mathbb{H}\setminus\mathbb{H}_n} |\langle h, u\rangle_H|^2 \, |\mu_h|^{2r} \right] \\
&= \inf_{N\in\mathbb{N}} \sup_{\substack{n\in\mathbb{N},\\ n\geq N}} \left[\sum_{h\in\mathbb{H}\setminus\mathbb{H}_n} |\langle h, u\rangle_H|^2 \, |\mu_h|^{2r} \right] = \limsup_{n\to\infty} \left[\sum_{h\in\mathbb{H}\setminus\mathbb{H}_n} |\langle h, u\rangle_H|^2 \, |\mu_h|^{2r} \right] = 0.
\end{aligned} \tag{107}
$$

Hence, we obtain that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in H_r. This together with the fact that H_r is complete implies that there exists a vector $\tilde{u} \in H_r$ such that

$$
\limsup_{n\to\infty} \|\tilde{u} - u_n\|_{H_r} = 0. \tag{108}
$$

Moreover, observe that the fact that $\mathbb{H} \subseteq H$ is an orthonormal basis of H shows that

$$
\limsup_{n\to\infty} \|u - u_n\|_H = \limsup_{n\to\infty} \left\| u - \sum_{h\in\mathbb{H}_n} \langle h, u\rangle_H \, h \right\|_H = 0. \tag{109}
$$

Combining (108) with (109) and the fact that $H_r \subseteq H$ continuously proves that $u = \tilde{u}$. This completes the proof of Lemma 5.1. □

Lemma 5.2. *Assume the setting in Section 5.1. Then*

(i) we have that $\operatorname{span}(\mathbb{H}) = \cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H)$,

(ii) we have for all $r \in [0,\infty)$ *that* $\overline{\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H)}^{H_r} = H_r$,

(iii) we have for all $v, w \in H_1$ *that* $\langle v - w, Av + F(v) - Aw - F(w)\rangle_H \leq c\|v - w\|_H^2$*, and*

(iv) we have for all $v, w \in V$ *that* $\|F(v) - F(w)\|_H^2 \leq c\|v - w\|_V^2(1 + \|v\|_V^\varphi + \|w\|_V^\varphi)$.

Proof of Lemma 5.2. Note for all $I, J \in \mathcal{P}_0(\mathbb{H})$, $v \in P_I(H)$, $w \in P_J(H)$, $\alpha, \beta \in \mathbb{R}$ that

$$\alpha v + \beta w \in P_{I\cup J}(H) \subseteq (\cup_{K\in\mathcal{P}_0(\mathbb{H})} P_K(H)). \tag{110}$$

This implies that $\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H)$ is an $\mathbb{R}$-vector space. Moreover, observe that for all $h \in \mathbb{H}$ we have that

$$h \in P_{\{h\}}(H) \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H)). \tag{111}$$

Hence, we obtain that $\mathbb{H} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$. This together with the fact that $\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H)$ is an $\mathbb{R}$-vector space proves that

$$\operatorname{span}(\mathbb{H}) \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H)). \tag{112}$$

In addition, note that the fact that $\forall\, I \in \mathcal{P}_0(\mathbb{H}),\, v \in P_I(H)\colon v \in \operatorname{span}(\mathbb{H})$ implies that

$$(\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H)) \subseteq \operatorname{span}(\mathbb{H}). \tag{113}$$

Combining this with (112) establishes (i). Furthermore, observe that (ii) is an immediate consequence of (i) and Lemma 5.1. In the next step note that the assumption that

$$\forall\, I \in \mathcal{P}_0(\mathbb{H}), v, w \in P_I(H)\colon \langle v - w, Av + F(v) - Aw - F(w)\rangle_H \leq c\|v - w\|_H^2 \tag{114}$$

ensures that for all $(v_k)_{k\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$, $(w_k)_{k\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$, $n \in \mathbb{N}$ we have that

$$\langle v_n - w_n, Av_n + F(v_n) - Aw_n - F(w_n)\rangle_H \leq c\|v_n - w_n\|_H^2. \tag{115}$$

Combining this and the fact that $H_1 \ni v \mapsto Av \in H$ is continuous with the fact that $F_{|H_1} \in \mathcal{C}(H_1, H)$ proves that for all $v_0, w_0 \in H_1$, $(v_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$, $(w_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$ with

$$\limsup_{n\to\infty} \|v_n - v_0\|_{H_1} = \limsup_{n\to\infty} \|w_n - w_0\|_{H_1} = 0 \tag{116}$$

we have that

$$\begin{aligned}
&\langle v_0 - w_0, Av_0 + F(v_0) - Aw_0 - F(w_0)\rangle_H \\
&= \limsup_{n\to\infty} \langle v_n - w_n, Av_n + F(v_n) - Aw_n - F(w_n)\rangle_H \\
&\leq c \limsup_{n\to\infty} \|v_n - w_n\|_H^2 = c\|v_0 - w_0\|_H^2.
\end{aligned} \tag{117}$$

Moreover, observe that (ii) ensures that for every $v_0, w_0 \in H_1$ there exist sequences $(v_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$ and $(w_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$ which satisfy that

$$\limsup_{n\to\infty} \|v_n - v_0\|_{H_1} = \limsup_{n\to\infty} \|w_n - w_0\|_{H_1} = 0. \tag{118}$$

This and (117) establish (iii). Next note that the assumption that

$$\forall\, I \in \mathcal{P}_0(\mathbb{H}), v, w \in P_I(H)\colon \|F(v) - F(w)\|_H^2 \leq c\|v - w\|_V^2(1 + \|v\|_V^\varphi + \|w\|_V^\varphi) \tag{119}$$

ensures that for all $(v_k)_{k\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$, $(w_k)_{k\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$, $n \in \mathbb{N}$ we have that

$$\|F(v_n) - F(w_n)\|_H^2 \leq c\|v_n - w_n\|_V^2(1 + \|v_n\|_V^\varphi + \|w_n\|_V^\varphi). \tag{120}$$

This and the assumption that $F \in \mathcal{C}(V, H)$ imply for all $v_0, w_0 \in V$, $(v_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$, $(w_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$ with

$$\limsup_{n\to\infty} \|v_n - v_0\|_V = \limsup_{n\to\infty} \|w_n - w_0\|_V = 0 \tag{121}$$

that

$$\begin{aligned}
&\|F(v_0) - F(w_0)\|_H^2 \\
&= \limsup_{n\to\infty} \|F(v_n) - F(w_n)\|_H^2 \leq c \limsup_{n\to\infty} \Big[\|v_n - w_n\|_V^2(1 + \|v_n\|_V^\varphi + \|w_n\|_V^\varphi)\Big] \\
&= c\|v_0 - w_0\|_V^2(1 + \|v_0\|_V^\varphi + \|w_0\|_V^\varphi).
\end{aligned} \tag{122}$$

In addition, observe that (ii) together with the assumption that $H_\rho \subseteq V$ continuously and densely guarantees that for every $v_0, w_0 \in V$ there exist sequences $(v_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$ and $(w_n)_{n\in\mathbb{N}} \subseteq (\cup_{I\in\mathcal{P}_0(\mathbb{H})} P_I(H))$ which satisfy that

$$\limsup_{n\to\infty} \|v_n - v_0\|_V = \limsup_{n\to\infty} \|w_n - w_0\|_V = 0. \tag{123}$$

Combining this with (122) establishes (iv). The proof of Lemma 5.2 is thus completed. □

5.3 On the measurability of a certain function

In our proof of Theorem 5.5 (the main result of this article) we employ the following well-known result.

Lemma 5.3. *Consider the notation in Section 1.1, let $(V, \|\cdot\|_V)$ be a separable $\mathbb{R}$-Banach space, let $(W, \|\cdot\|_W)$ be an $\mathbb{R}$-Banach space with $V \subseteq W$ continuously and densely, let $(S, \mathcal{S})$ be a measurable* *space, let $s \in S$, let $\psi\colon V \to \mathcal{S}$ be a $\mathcal{B}(V)/\mathcal{S}$-measurable function, and let $\Psi\colon W \to S$ be the function which satisfies for all $v \in W$ that*

$$\Psi(v) = \begin{cases} \psi(v) & : v \in V \\ s & : v \in W \setminus V. \end{cases} \tag{124}$$

Then we have that $\Psi\colon W \to \mathcal{S}$ is a $\mathcal{B}(W)/\mathcal{S}$-measurable function.

Proof of Lemma 5.3. First, observe that Lemma 3.1 ensures that $(W, \|\cdot\|_W)$ is a separable $\mathbb{R}$-Banach space. This and, e.g., Lemma 2.2 in Andersson et al. [2] (with $V_0 = W$, $V_1 = V$ in the notation of Lemma 2.2 in Andersson et al. [2]) ensure that

$$V \in \mathcal{B}(V) \subseteq \mathcal{B}(W). \tag{125}$$

The assumption that $\psi\colon V \to \mathcal{S}$ is a $\mathcal{B}(V)/\mathcal{S}$-measurable function hence ensures that for all $A \in \mathcal{S}$ with $s \notin A$ we have that

$$\begin{aligned} \Psi^{-1}(A) &= \{v \in W\colon \Psi(v) \in A\} = \{v \in V\colon \Psi(v) \in A\} \cup \{v \in (W \setminus V)\colon \Psi(v) \in A\} \\ &= \{v \in V\colon \Psi(v) \in A\} = \{v \in V\colon \psi(v) \in A\} = \psi^{-1}(A) \in \mathcal{B}(V) \subseteq \mathcal{B}(W). \end{aligned} \tag{126}$$

Next note that (125) and the assumption that $\psi\colon V \to \mathcal{S}$ is a $\mathcal{B}(V)/\mathcal{S}$-measurable function prove that for all $A \in \mathcal{S}$ with $s \in A$ we have that

$$\begin{aligned} \Psi^{-1}(A) &= \{v \in W\colon \Psi(v) \in A\} = \{v \in V\colon \Psi(v) \in A\} \cup (W \setminus V) \\ &= \{v \in V\colon \psi(v) \in A\} \cup (W \setminus V) = \underbrace{\psi^{-1}(A)}_{\in \mathcal{B}(V) \subseteq \mathcal{B}(W)} \cup \underbrace{(W \setminus V)}_{\in \mathcal{B}(W)} \in \mathcal{B}(W). \end{aligned} \tag{127}$$

Combining (126) and (127) demonstrates that for all $A \in \mathcal{S}$ we have that $\Psi^{-1}(A) \in \mathcal{B}(W)$. This completes the proof of Lemma 5.3. ☐

5.4 A priori moment bounds for the numerical approximation

Lemma 5.4. *Assume the setting in Section 5.1, let $p \in [1, \infty)$, $\sigma \in [0, \gamma]$, and assume that*

$$\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0,T]} \mathbb{E}\big[\|\mathcal{O}_t^{M,I}\|_{H_\sigma}^{2p} + |\Phi(\mathcal{O}^{M,I})|^p + |\phi(\mathcal{O}^{M,I})|^p\big] < \infty. \tag{128}$$

Then

$$\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0,T]} \mathbb{E}\big[\|\mathcal{X}_t^{M,I}\|_{H_\sigma}^{2p}\big] < \infty. \tag{129}$$

Proof of Lemma 5.4. Throughout this proof let $\kappa \in (0,1)$ be a real number, let $\tilde{\mathcal{X}}^{M,I}\colon [0,T]\times\Omega \to P_I(H)$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, be the functions which satisfy for all $M \in \mathbb{N}$, $I \in \mathcal{D}$, $t \in [0,T]$ that

$$\tilde{\mathcal{X}}_t^{M,I} = \int_0^t P_I\, e^{(t-s)A}\, \mathbb{1}^{\Omega}_{\{\|\tilde{\mathcal{X}}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} \leq (M/T)^{\chi}\}}\, F(\tilde{\mathcal{X}}^{M,I}_{\lfloor s \rfloor_{T/M}})\, ds + \mathcal{O}_t^{M,I}, \tag{130}$$

and let $C, K \in [0,\infty]$ be the extended real numbers given by

$$K = \sqrt{c}\max\left\{1, \sup_{v \in (V \cap H_\rho)\setminus\{0\}} \frac{\|v\|_V^{(1+\varphi/2)}}{\|v\|_{H_\rho}^{(1+\varphi/2)}}\right\} \tag{131}$$

and

$$\begin{split} C = \max\Bigg\{ & 3K^2(1+2^{\max\{0,\varphi-1\}}) \left[1 + \sup_{v \in H_\gamma\setminus\{0\}} \frac{\|v\|_{H_\rho}^{\varphi}}{\|v\|_{H_\gamma}^{\varphi}}\right], \\ & (8K^2 + 2\|F(0)\|_H^2)\max\left\{1, \sup_{v \in H_\gamma\setminus\{0\}} \frac{\|v\|_{H_\rho}^{(2+\varphi)}}{\|v\|_{H_\gamma}^{(2+\varphi)}}\right\}\Bigg\}. \end{split} \tag{132}$$

Note that the fact that $H_\gamma \subseteq H_\rho \subseteq V$ continuously ensures that $C, K \in [0,\infty)$. In the next step observe that Lemma 5.2 (iv) and the fact that

$$\forall\, x,y,z \in [0,\infty)\colon \sqrt{x+y+z} \leq \sqrt{x} + \sqrt{y} + \sqrt{z} \tag{133}$$

imply for all $v, w \in H_\gamma$ that

$$\begin{split} \|F(v) - F(w)\|_H &\leq \sqrt{c\|v-w\|_V^2\,(1 + \|v\|_V^{\varphi} + \|w\|_V^{\varphi})} \\ &\leq \sqrt{c}\left[\sup_{u \in H_\rho\setminus\{0\}} \frac{\|u\|_V}{\|u\|_{H_\rho}}\right]\|v-w\|_{H_\rho}\left(1 + \left[\sup_{u \in H_\rho\setminus\{0\}} \frac{\|u\|_V^{\varphi}}{\|u\|_{H_\rho}^{\varphi}}\right]\left[\|v\|_{H_\rho}^{\varphi} + \|w\|_{H_\rho}^{\varphi}\right]\right)^{1/2} \\ &\leq K\|v-w\|_{H_\rho}\left(1 + \|v\|_{H_\rho}^{\varphi/2} + \|w\|_{H_\rho}^{\varphi/2}\right). \end{split} \tag{134}$$

Combining this with Lemma 2.4 in Hutzenthaler et al. [13] (with $(V, \|\cdot\|_V) = (H_\gamma, \|\cdot\|_{H_\gamma})$, $(\mathcal{V}, \|\cdot\|_{\mathcal{V}}) = (H_\rho, \|\cdot\|_{H_\rho})$, $(W, \|\cdot\|_W) = (H, \|\cdot\|_H)$, $(\mathcal{W}, \|\cdot\|_{\mathcal{W}}) = (H, \|\cdot\|_H)$, $\epsilon = K$, $\theta = C$, $\varepsilon = \varphi/2$, $\vartheta = \varphi$, $F = H_\gamma \ni v \mapsto F(v) \in H$ in the notation of Lemma 2.4 in Hutzenthaler et al. [13]) implies that for all $u, v \in H_\gamma$ we have that

$$\|F(u)\|_H^2 \leq C\max\{1, \|u\|_{H_\gamma}^{(2+\varphi)}\} \tag{135}$$

and

$$\|F(u) - F(v)\|_H^2 \leq C \max\{1, \|u\|_{H_\gamma}^\varphi\} \|u - v\|_{H_\rho}^2 + C\|u - v\|_{H_\rho}^{(2+\varphi)}. \tag{136}$$

Moreover, observe that the assumption that for all $I \in \mathcal{P}_0(\mathbb{H})$, $v \in P_I(H)$, $w \in \mathcal{C}([0,T], H_1)$ we have that

$$\begin{aligned} &\sup_{t\in[0,T]} \big(\langle v, P_I F(v + w_t)\rangle_{H_{1/2}} + \phi(w)\langle v, F(v + w_t)\rangle_H\big) \\ &\leq \epsilon\|v\|_{H_1}^2 + (c + \phi(w))\|v\|_{H_{1/2}}^2 + c\phi(w)\|v\|_H^2 + \Phi(w) \end{aligned} \tag{137}$$

ensures that for all $I \in \mathcal{D}$, $v \in P_I(H)$, $w \in \mathcal{C}([0,T], P_I(H))$, $t \in [0,T]$ we have that

$$\begin{aligned} &\langle v, P_I F(v + w_t)\rangle_{H_{1/2}} + \phi([0,T] \ni s \mapsto w_s \in H_1)\langle v, F(v + w_t)\rangle_H \\ &\leq \epsilon\|v\|_{H_1}^2 + (\tfrac{c}{\kappa} + \phi([0,T] \ni s \mapsto w_s \in H_1))\|v\|_{H_{1/2}}^2 + c\phi([0,T] \ni s \mapsto w_s \in H_1)\|v\|_H^2 \\ &\quad + \Phi([0,T] \ni s \mapsto w_s \in H_1). \end{aligned} \tag{138}$$

This together with (130), (135), and (136) allows us to apply Lemma 2.1 (with $H = H$, $\mathbb{H} = \mathbb{H}$, $T = T$, $\varphi = \varphi$, $c = c/\kappa$, $C = C$, $\epsilon = \epsilon$, $\kappa = \kappa$, $\rho = \rho$, $\gamma = \gamma$, $\chi = \chi$, $M = M$, $\mu = \mu$, $A = A$, $I = I$, $P = P_I$, $\mathcal{X} = [0,T] \ni t \mapsto \tilde{\mathcal{X}}_t^{M,I}(\omega) \in P_I(H)$, $\mathcal{O} = [0,T] \ni t \mapsto \mathcal{O}_t^{M,I}(\omega) \in P_I(H)$, $F = P_I(H) \ni v \mapsto F(v) \in H$, $\phi = \mathcal{C}([0,T], P_I(H)) \ni v \mapsto \phi([0,T] \ni t \mapsto v(t) \in H_1) \in [0,\infty)$, $\Phi = \mathcal{C}([0,T], P_I(H)) \ni v \mapsto \Phi([0,T] \ni t \mapsto v(t) \in H_1) \in [0,\infty)$ for $I \in \mathcal{D}$, $M \in \mathbb{N}$, $\omega \in \Omega$ in the notation of Lemma 2.1) to obtain that for every $M \in \mathbb{N}$, $I \in \mathcal{D}$, $\omega \in \Omega$ we have that the function $[0,T] \ni t \mapsto \tilde{\mathcal{X}}_t^{M,I}(\omega) - \mathcal{O}_t^{M,I}(\omega) \in P_I(H)$ is continuous and that

$$\begin{aligned} &\sup_{t\in[0,T]} \big(\|\tilde{\mathcal{X}}_t^{M,I}(\omega) - \mathcal{O}_t^{M,I}(\omega)\|_{H_{1/2}}^2 + \|\tilde{\mathcal{X}}_t^{M,I}(\omega) - \mathcal{O}_t^{M,I}(\omega)\|_H^2\big) \\ &\leq \frac{\kappa}{c}\exp\left(\frac{2cT}{\kappa}\right)\left(\Phi(\mathcal{O}^{M,I}(\omega)) + \frac{\max\{1, \phi(\mathcal{O}^{M,I}(\omega))\}C(c+\kappa)}{2(1-\epsilon)(1-\kappa)c}\left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)}\right). \end{aligned} \tag{139}$$

This, in particular, implies for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ that

$$\Big(\Omega \ni \omega \mapsto \sup\nolimits_{t\in[0,T]} \|\tilde{\mathcal{X}}_t^{M,I}(\omega) - \mathcal{O}_t^{M,I}(\omega)\|_{H_{1/2}} \in \mathbb{R}\Big) \in \mathcal{M}(\mathcal{F}, \mathcal{B}(\mathbb{R})). \tag{140}$$

The assumption that $p \geq 1$, the fact that

$$\forall\, x, y \in [0,\infty)\colon \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}, \tag{141}$$

and (139) hence ensure for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ that

$$\begin{aligned}
&\big\| \sup\nolimits_{t\in[0,T]} \|\tilde{\mathcal{X}}_t^{M,I} - \mathcal{O}_t^{M,I}\|_{H_{1/2}} \big\|_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})} \\
&\leq \left\| \frac{\kappa}{c} \exp\!\left(\frac{2cT}{\kappa}\right) \left(\Phi(\mathcal{O}^{M,I}) + \frac{\max\{1, \phi(\mathcal{O}^{M,I})\} C(c+\kappa)}{2(1-\epsilon)(1-\kappa)c} \left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)} \right) \right\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}^{1/2} \\
&\leq \frac{\sqrt{\kappa}}{\sqrt{c}} \exp\!\left(\frac{cT}{\kappa}\right) \\
&\quad \cdot \left(\|\Phi(\mathcal{O}^{M,I})\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} + \frac{\big(\|\phi(\mathcal{O}^{M,I})\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} + 1\big)\, C(c+\kappa)}{2(1-\epsilon)(1-\kappa)c} \left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(2+\varphi)} \right)^{1/2} \\
&\leq \frac{\sqrt{\kappa}}{\sqrt{c}} \exp\!\left(\frac{cT}{\kappa}\right) \|\Phi(\mathcal{O}^{M,I})\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}^{1/2} \\
&\quad + \left(\|\phi(\mathcal{O}^{M,I})\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}^{1/2} + 1 \right) \frac{\sqrt{C(c+\kappa)}\exp\!\left(\frac{cT}{\kappa}\right)}{c\sqrt{2(1-\epsilon)(1/\kappa - 1)}} \left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(1+\varphi/2)}.
\end{aligned} \tag{142}$$

In addition, observe that the fact that for every $r \in \mathbb{R}$, $I \in \mathcal{D}$ we have that $P_I(H) \subseteq H_r$ continuously and, e.g., Andersson et al. [2, Lemma 2.2] (with $V_0 = H_r$, $\|\cdot\|_{V_0} = \|\cdot\|_{H_r}$, $V_1 = P_I(H)$, $\|\cdot\|_{V_1} = P_I(H) \ni v \mapsto \|v\|_H \in [0,\infty)$ for $I \in \mathcal{D}$, $r \in \mathbb{R}$ in the notation of Andersson et al. [2, Lemma 2.2]) prove that for all $I \in \mathcal{D}$ we have that

$$\mathcal{B}(P_I(H)) = \big\{ S \in \mathcal{P}(P_I(H)) \colon \big(\exists\, B \in \mathcal{B}(H_r) \colon S = B \cap P_I(H)\big) \big\} \subseteq \mathcal{B}(H_r). \tag{143}$$

The hypothesis that $\mathcal{O}^{M,I} \colon [0,T] \times \Omega \to H_1$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, are stochastic processes therefore demonstrates that for every $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that $\tilde{\mathcal{X}}_{:}^{M,I}[0,T] \times \Omega \to P_I(H)$ is a stochastic process. Combining this with (143) shows that for all $B \in \mathcal{B}(H_\gamma)$, $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned}
&\Big(\Omega \ni \omega \mapsto \tilde{\mathcal{X}}_t^{M,I}(\omega) \in H_\gamma\Big)^{-1}(B) = \Big(\Omega \ni \omega \mapsto \tilde{\mathcal{X}}_t^{M,I}(\omega) \in H_\gamma\Big)^{-1}\big(B \cap P_I(H)\big) \\
&= \Big(\Omega \ni \omega \mapsto \tilde{\mathcal{X}}_t^{M,I}(\omega) \in P_I(H)\Big)^{-1}\big(B \cap P_I(H)\big) = \Big(\tilde{\mathcal{X}}_t^{M,I}\Big)^{-1}\big(\underbrace{B \cap P_I(H)}_{\in \mathcal{B}(P_I(H))}\big) \in \mathcal{F}.
\end{aligned} \tag{144}$$

Hence, we obtain for all $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ that

$$\big\{\omega \in \Omega \colon \mathcal{X}_t^{M,I}(\omega) = \tilde{\mathcal{X}}_t^{M,I}(\omega)\big\} \in \mathcal{F}. \tag{145}$$

The assumption that for every $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\mathbb{P}\bigg(\mathcal{X}_t^{M,I} = \int_0^t P_I\, e^{(t-s)A}\, \mathbb{1}^{\Omega}_{\{\|\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} \leq (M/T)^{\chi}\}}\, F(\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}})\, ds + \mathcal{O}_t^{M,I}\bigg) = 1 \tag{146}$$

therefore implies that for all $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\mathbb{P}\big(\mathcal{X}_t^{M,I} = \tilde{\mathcal{X}}_t^{M,I}\big) = 1. \tag{147}$$

This and the triangle inequality assure that

$$\begin{aligned}
&\sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \|\mathcal{X}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};H_\sigma)} = \sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \|\tilde{\mathcal{X}}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};H_\sigma)} \\
&\leq \sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \Big[\|\tilde{\mathcal{X}}_t^{M,I} - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};H_\sigma)} + \|\mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};H_\sigma)}\Big] \\
&\leq \left[\sup_{v\in H_{1/2}\setminus\{0\}} \frac{\|v\|_{H_\sigma}}{\|v\|_{H_{1/2}}}\right]\left[\sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}} \big\|\sup\nolimits_{t\in[0,T]}\|\tilde{\mathcal{X}}_t^{M,I} - \mathcal{O}_t^{M,I}\|_{H_{1/2}}\big\|_{\mathcal{L}^{2p}(\mathbb{P};\mathbb{R})}\right] \\
&\quad + \sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \|\mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};H_\sigma)}.
\end{aligned} \tag{148}$$

The assumption that

$$\sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \mathbb{E}\big[\|\mathcal{O}_t^{M,I}\|_{H_\sigma}^{2p} + |\Phi(\mathcal{O}^{M,I})|^p + |\phi(\mathcal{O}^{M,I})|^p\big] < \infty, \tag{149}$$

the fact that $H_{1/2} \subseteq H_\sigma$ continuously, and (142) hence prove that

$$\begin{aligned}
&\sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \|\mathcal{X}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};H_\sigma)} \\
&\leq \left[\sup_{v\in H_{1/2}\setminus\{0\}} \frac{\|v\|_{H_\sigma}}{\|v\|_{H_{1/2}}}\right]\Bigg[\frac{\sqrt{\kappa}}{\sqrt{c}}\exp\!\left(\frac{cT}{\kappa}\right)\sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\|\Phi(\mathcal{O}^{M,I})\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}^{1/2} \\
&\quad + \left(\sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\|\phi(\mathcal{O}^{M,I})\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}^{1/2} + 1\right)\frac{\sqrt{C(c+\kappa)}\exp\!\left(\frac{cT}{\kappa}\right)}{c\sqrt{2(1-\epsilon)(1/\kappa - 1)}}\left[\tfrac{\max\{1,T\}(1+\sqrt{C})}{(1-\rho)}\right]^{(1+\varphi/2)}\Bigg] \\
&\quad + \sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \|\mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};H_\sigma)} < \infty.
\end{aligned} \tag{150}$$

This completes the proof of Lemma 5.4. ☐

5.5 Main result

Theorem 5.5. *Assume the setting in Section 5.1, let $\vartheta \in (0,\infty)$, $p \in [\max\{2, 1/\varphi\}, \infty)$, $\varrho \in [0, 1-\rho)$, and assume that*

$$\begin{aligned}
&\sup_{t\in[0,T]}\sup_{I\in\mathcal{D}} \mathbb{E}\big[\|X_t\|_V^{p(1+\varphi/2)\max\{2,\varphi\}} + \|P_I O_t\|_V^{p\varphi}\big] \\
&+ \sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]} \mathbb{E}\Big[\big|\|\mathcal{O}_t^{M,I}\|_{H_\gamma}^2 + \Phi(\mathcal{O}^{M,I}) + \phi(\mathcal{O}^{M,I})\big|^{p\max\{2\vartheta, 2+\varphi, (1+\varphi/2)\varphi/2\}}\Big] < \infty.
\end{aligned} \tag{151}$$

Then we have

(i) *that* $\sup_{M\in\mathbb{N}}\sup_{I\in\mathcal{D}}\sup_{t\in[0,T]}\mathbb{E}\big[\|\mathcal{X}_t^{M,I}\|_{H_\gamma}^{p\max\{4\vartheta,4+2\varphi,\varphi(1+\varphi/2)\}}\big]<\infty$ *and*

(ii) *that there exists a real number* $C\in(0,\infty)$ *such that for all* $M\in\mathbb{N}$*,* $I\in\mathcal{D}$ *it holds that*

$$\begin{aligned}&\sup_{t\in[0,T]}\Big(\mathbb{E}\big[\|X_t-\mathcal{X}_t^{M,I}\|_H^p\big]\Big)^{1/p}\leq C\Big[M^{-\min\{\vartheta\chi,\varrho\}}+\|(-A)^{-\varrho}(\mathrm{Id}_H-P_I)\|_{L(H)}\\&\quad+\sup_{t\in[0,T]}\Big(\mathbb{E}\big[\|(\mathrm{Id}_H-P_I)O_t\|_V^{2p}+\|P_IO_t-\mathcal{O}_t^{M,I}\|_V^{2p}+\|P_I(O_t-O_{\lfloor t\rfloor_{T/M}})\|_V^{2p}\big]\Big)^{1/2p}\Big].\end{aligned}\tag{152}$$

Proof of Theorem 5.5. Throughout this proof let $\kappa\in(2,\infty)$ be a real number, let $(\mathfrak{P}_I)_{I\in\mathcal{P}(\mathbb{H})}\subseteq L(H)$ be the linear operators which satisfy for all $I\in\mathcal{P}(\mathbb{H})$, $v\in H$ that

$$\mathfrak{P}_I(v)=\sum_{h\in I}\langle h,v\rangle_H h,\tag{153}$$

let $\mathcal{V}\colon V\times V\to[0,\infty)$ be the function which satisfies for all $v,w\in V$ that

$$\mathcal{V}(v,w)=\begin{cases}(\|v\|_{H_\gamma}+\|w\|_{H_\gamma})^{1/\chi} & :(v,w)\in H_\gamma\times H_\gamma\\ 0 & :(v,w)\in(V\times V)\setminus(H_\gamma\times H_\gamma),\end{cases}\tag{154}$$

let $\tilde{\mathcal{X}}^{M,I}\colon[0,T]\times\Omega\to H_\gamma$, $M\in\mathbb{N}$, $I\in\mathcal{D}$, be the functions which satisfy for all $M\in\mathbb{N}$, $I\in\mathcal{D}$, $t\in[0,T]$ that

$$\tilde{\mathcal{X}}_t^{M,I}=\int_0^t P_I\,e^{(t-s)A}\,\mathbb{1}^\Omega_{\{\|\tilde{\mathcal{X}}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_{H_\gamma}+\|\mathcal{O}^{M,I}_{\lfloor s\rfloor_{T/M}}\|_{H_\gamma}\leq(M/T)^\chi\}}\,F(\tilde{\mathcal{X}}^{M,I}_{\lfloor s\rfloor_{T/M}})\,ds+\mathcal{O}_t^{M,I},\tag{155}$$

let $\tilde\Omega\subseteq\Omega$ be the set given by

$$\begin{aligned}\tilde\Omega&=\big\{\forall\,t\in[0,T]\colon X_t=\textstyle\int_0^t e^{(t-s)A}F(X_s)\,ds+O_t\big\}\\&=\big\{\forall\,t\in[0,T]\cap\mathbb{Q}\colon X_t=\textstyle\int_0^t e^{(t-s)A}F(X_s)\,ds+O_t\big\},\end{aligned}\tag{156}$$

and let $\tilde X\colon[0,T]\times\Omega\to V$ and $\tilde O\colon[0,T]\times\Omega\to V$ be the functions which satisfy for all $t\in[0,T]$ that $\tilde X_t=X_t\mathbb{1}^\Omega_{\tilde\Omega}$ and

$$\tilde O_t=O_t\mathbb{1}^\Omega_{\tilde\Omega}-\bigg[\int_0^t e^{(t-s)A}F(0)\,ds\bigg]\mathbb{1}^\Omega_{\Omega\setminus\tilde\Omega}=O_t\mathbb{1}^\Omega_{\tilde\Omega}+A^{-1}(\mathrm{Id}_H-e^{tA})F(0)\mathbb{1}^\Omega_{\Omega\setminus\tilde\Omega}.\tag{157}$$

Observe that for all $I\in\mathcal{P}_0(\mathbb{H})\subseteq\mathcal{P}(\mathbb{H})$ we have that

$$\mathfrak{P}_I=P_I\qquad\text{and}\qquad\mathfrak{P}_{\mathbb{H}\setminus I}=\mathrm{Id}_H-P_I.\tag{158}$$

Next note that the fact that $H_1 \subseteq H_\gamma$ continuously and, e.g., Andersson et al. [2, Lemma 2.2] (with $V_0 = H_\gamma$, $\|\cdot\|_{V_0} = \|\cdot\|_{H_\gamma}$, $V_1 = H_1$, $\|\cdot\|_{V_1} = \|\cdot\|_{H_1}$ in the notation of Andersson et al. [2, Lemma 2.2]) ensure that

$$\mathcal{B}(H_1) = \big\{ S \in \mathcal{P}(H_1) \colon \big(\exists\, B \in \mathcal{B}(H_\gamma) \colon S = B \cap H_1 \big) \big\} \subseteq \mathcal{B}(H_\gamma). \tag{159}$$

The hypothesis that $\mathcal{O}^{M,I} \colon [0,T] \times \Omega \to H_1$, $M \in \mathbb{N}$, $I \in \mathcal{D}$, are stochastic processes therefore demonstrates that for every $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that $\tilde{\mathcal{X}}^{M,I} \colon [0,T] \times \Omega \to H_\gamma$ is a stochastic process. Hence, we obtain for all $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ that

$$\big\{ \omega \in \Omega \colon \mathcal{X}_t^{M,I}(\omega) = \tilde{\mathcal{X}}_t^{M,I}(\omega) \big\} \in \mathcal{F}. \tag{160}$$

The assumption that for every $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\mathbb{P}\bigg(\mathcal{X}_t^{M,I} = \int_0^t P_I\, e^{(t-s)A}\, \mathbb{1}^{\Omega}_{\{\|\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}^{M,I}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} \leq (M/T)^\chi\}}\, F(\mathcal{X}^{M,I}_{\lfloor s \rfloor_{T/M}})\, ds + \mathcal{O}_t^{M,I} \bigg) = 1 \tag{161}$$

therefore implies that for all $t \in [0,T]$, $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\mathbb{P}\big(\mathcal{X}_t^{M,I} = \tilde{\mathcal{X}}_t^{M,I} \big) = 1. \tag{162}$$

Combining this and the assumption that

$$\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0,T]} \mathbb{E}\big[|\|\mathcal{O}_t^{M,I}\|_{H_\gamma}^2 + \Phi(\mathcal{O}^{M,I}) + \phi(\mathcal{O}^{M,I})|^{p \max\{2\vartheta, 2+\varphi, (1+\varphi/2)\varphi/2\}} \big] < \infty \tag{163}$$

with Lemma 5.4 demonstrates that

$$\begin{aligned} &\sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0,T]} \|\tilde{\mathcal{X}}_t^{M,I}\|_{\mathcal{L}^{p \max\{4\vartheta, 4+2\varphi, \varphi(1+\varphi/2)\}}(\mathbb{P}; H_\gamma)} \\ &= \sup_{M \in \mathbb{N}} \sup_{I \in \mathcal{D}} \sup_{t \in [0,T]} \|\mathcal{X}_t^{M,I}\|_{\mathcal{L}^{p \max\{4\vartheta, 4+2\varphi, \varphi(1+\varphi/2)\}}(\mathbb{P}; H_\gamma)} < \infty. \end{aligned} \tag{164}$$

This establishes (i). Moreover, note that the assumption that $\forall\, t \in [0,T] \colon \mathbb{P}(X_t = \int_0^t e^{(t-s)A} F(X_s)\, ds + O_t) = 1$ yields that $\mathbb{P}(\tilde{\Omega}) = 1$. Hence, we obtain for all $t \in [0,T]$ that $\mathbb{P}(\tilde{X}_t = X_t) \geq \mathbb{P}(\tilde{\Omega}) = 1$. Combining this with the assumption that $\sup_{t \in [0,T]} \mathbb{E}\big[\|X_t\|_V^{p(1+\varphi/2)\max\{2,\varphi\}}\big] < \infty$ ensures that

$$\sup_{t \in [0,T]} \|\tilde{X}_t\|_{\mathcal{L}^{p(1+\varphi/2)\max\{2,\varphi\}}(\mathbb{P}; V)} = \sup_{t \in [0,T]} \|X_t\|_{\mathcal{L}^{p(1+\varphi/2)\max\{2,\varphi\}}(\mathbb{P}; V)} < \infty. \tag{165}$$

Furthermore, observe that the fact that

$$\forall\, t \in [0,\infty), r \in [0,1] \colon \|(-tA)^r\, e^{tA}\|_{L(H)} \leq 1, \tag{166}$$

the triangle inequality, and Lemma 5.2 (iv) show for all $t \in (0,T]$ that

$$\begin{aligned}
&\int_0^t \|(-A)^{(\rho+\varrho)} e^{(t-s)A} F(\tilde{X}_s)\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};H)}\,ds \\
&\leq \int_0^t \|(-A)^{(\rho+\varrho)} e^{(t-s)A}\|_{L(H)}\, \|F(\tilde{X}_s)\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};H)}\,ds \\
&\leq \int_0^t (t-s)^{-(\rho+\varrho)} \Big[\|F(\tilde{X}_s)-F(0)\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};H)} + \|F(0)\|_H\Big]\,ds \\
&\leq \int_0^t (t-s)^{-(\rho+\varrho)} \bigg[\Big\|\sqrt{c}\,\|\tilde{X}_s\|_V\big(1+\|\tilde{X}_s\|_V^{\varphi/2}\big)\Big\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};\mathbb{R})} + \|F(0)\|_H\bigg]\,ds \\
&\leq \int_0^t (t-s)^{-(\rho+\varrho)} \Big[\sqrt{c}\,\Big(\|\tilde{X}_s\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};V)} + \|\tilde{X}_s\|^{(1+\varphi/2)}_{\mathcal{L}^{p(1+\varphi/2)\max\{2,\varphi\}}(\mathbb{P};V)}\Big) + \|F(0)\|_H\Big]\,ds.
\end{aligned} \tag{167}$$

Inequality (165) hence implies for all $t \in (0,T]$ that

$$\begin{aligned}
&\int_0^t \|(-A)^{(\rho+\varrho)} e^{(t-s)A} F(\tilde{X}_s)\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};H)}\,ds \\
&\leq \int_0^t (t-s)^{-(\rho+\varrho)} \Big[2\sqrt{c}\max\Big\{1, \|\tilde{X}_s\|^{(1+\varphi/2)}_{\mathcal{L}^{p(1+\varphi/2)\max\{2,\varphi\}}(\mathbb{P};V)}\Big\} + \|F(0)\|_H\Big]\,ds \\
&\leq \Bigg[2\sqrt{c}\max\Bigg\{1, \sup_{s\in(0,t)} \|\tilde{X}_s\|^{(1+\varphi/2)}_{\mathcal{L}^{p(1+\varphi/2)\max\{2,\varphi\}}(\mathbb{P};V)}\Bigg\} + \|F(0)\|_H\Bigg] \left[\int_0^t (t-s)^{-(\rho+\varrho)}\,ds\right] \\
&\leq \Bigg[2\sqrt{c}\max\Bigg\{1, \sup_{s\in(0,T)} \|\tilde{X}_s\|^{(1+\varphi/2)}_{\mathcal{L}^{p(1+\varphi/2)\max\{2,\varphi\}}(\mathbb{P};V)}\Bigg\} + \|F(0)\|_H\Bigg] \frac{T^{(1-\rho-\varrho)}}{(1-\rho-\varrho)} < \infty.
\end{aligned} \tag{168}$$

This and the fact that

$$\forall\, t \in [0,T]\colon \tilde{X}_t = \int_0^t e^{(t-s)A} F(\tilde{X}_s)\,ds + \tilde{O}_t \tag{169}$$

prove that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\tilde{X}_t - \tilde{O}_t\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};H_{(\rho+\varrho)})} \\
&\leq \sup_{t\in[0,T]} \left(\int_0^t \|e^{(t-s)A} F(\tilde{X}_s)\|_{\mathcal{L}^{p\max\{2,\varphi\}}(\mathbb{P};H_{(\rho+\varrho)})}\,ds\right) < \infty.
\end{aligned} \tag{170}$$

In addition, observe that the triangle inequality assures for all $I \in \mathcal{D}$ that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I} \tilde{X}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \leq \sup_{t\in[0,T]} \Big[\|\mathfrak{P}_{\mathbb{H}\setminus I}(\tilde{X}_t - \tilde{O}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|\mathfrak{P}_{\mathbb{H}\setminus I}\tilde{O}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \Big] \\
&\leq \left[\sup_{v\in H_\rho\setminus\{0\}} \frac{\|v\|_V}{\|v\|_{H_\rho}} \right] \left[\sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}(\tilde{X}_t - \tilde{O}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H_\rho)} \right] + \sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}\tilde{O}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&= \left[\sup_{v\in H_\rho\setminus\{0\}} \frac{\|v\|_V}{\|v\|_{H_\rho}} \right] \left[\sup_{t\in[0,T]} \|(-A)^{-\varrho}\mathfrak{P}_{\mathbb{H}\setminus I}(-A)^{(\rho+\varrho)}(\tilde{X}_t - \tilde{O}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H)} \right] \\
&\quad + \sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}\tilde{O}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)}.
\end{aligned}
\tag{171}
$$

The fact that

$$
\forall\, I \in \mathcal{P}(\mathbb{H})\colon \|\mathfrak{P}_{\mathbb{H}\setminus I}\|_{L(H)} \leq 1 \tag{172}
$$

hence guarantees for all $I \in \mathcal{D}$ that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I} \tilde{X}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\leq \left[\sup_{v\in H_\rho\setminus\{0\}} \frac{\|v\|_V}{\|v\|_{H_\rho}} \right] \left[\sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}(\tilde{X}_t - \tilde{O}_t)\|_{\mathcal{L}^{2p}(\mathbb{P};H_{(\rho+\varrho)})} \right] \|(-A)^{-\varrho}\mathfrak{P}_{\mathbb{H}\setminus I}\|_{L(H)} \\
&\quad + \sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}\tilde{O}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\leq \left[\sup_{v\in H_\rho\setminus\{0\}} \frac{\|v\|_V}{\|v\|_{H_\rho}} \right] \left[\sup_{t\in[0,T]} \|\tilde{X}_t - \tilde{O}_t\|_{\mathcal{L}^{2p}(\mathbb{P};H_{(\rho+\varrho)})} \right] \|(-A)^{-\varrho}\mathfrak{P}_{\mathbb{H}\setminus I}\|_{L(H)} \\
&\quad + \sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}\tilde{O}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)}.
\end{aligned}
\tag{173}
$$

Furthermore, observe that the triangle inequality, the fact that

$$
\forall\, I \in \mathcal{P}_0(\mathbb{H})\colon \|P_I\|_{L(H)} \leq 1, \tag{174}
$$

the fact that $H_{(\rho+\varrho)} \subseteq V$ continuously, the fact that

$$
\forall\, t \in [0,T]\colon \mathbb{P}(O_t = \tilde{O}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1, \tag{175}
$$

the assumption that

$$
\sup_{t\in[0,T]} \sup_{I\in\mathcal{D}} \mathbb{E}\big[\|P_I O_t\|_V^{p\varphi}\big] < \infty, \tag{176}
$$

and (170) imply that

$$\begin{aligned}
&\sup_{t\in[0,T]}\sup_{I\in\mathcal{D}}\|P_I\tilde{X}_t\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}\\
&\leq \sup_{t\in[0,T]}\sup_{I\in\mathcal{D}}\|P_I\tilde{X}_t - P_I\tilde{O}_t\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \sup_{t\in[0,T]}\sup_{I\in\mathcal{D}}\|P_I\tilde{O}_t\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}\\
&\leq \left[\sup_{v\in H_{(\rho+\varrho)}\setminus\{0\}}\frac{\|v\|_V}{\|v\|_{H_{(\rho+\varrho)}}}\right]\left[\sup_{t\in[0,T]}\sup_{I\in\mathcal{D}}\|P_I(\tilde{X}_t-\tilde{O}_t)\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H_{(\rho+\varrho)})}\right] + \sup_{t\in[0,T]}\sup_{I\in\mathcal{D}}\|P_I\tilde{O}_t\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}\\
&\leq \left[\sup_{v\in H_{(\rho+\varrho)}\setminus\{0\}}\frac{\|v\|_V}{\|v\|_{H_{(\rho+\varrho)}}}\right]\left[\sup_{t\in[0,T]}\|\tilde{X}_t-\tilde{O}_t\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H_{(\rho+\varrho)})}\right] + \sup_{t\in[0,T]}\sup_{I\in\mathcal{D}}\|P_IO_t\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} < \infty.
\end{aligned} \tag{177}$$

In the next step we note that the hypothesis that $H_\rho \subseteq V$ continuously and the fact that

$$\forall\, t\in[0,\infty), r\in[0,1]\colon \|(-tA)^r e^{tA}\|_{L(H)} \leq 1 \tag{178}$$

ensure that

$$\begin{aligned}
\sup_{t\in(0,T)} t^\rho\|e^{tA}\|_{L(H,V)} &= \sup_{t\in(0,T)}\sup_{v\in H\setminus\{0\}}\left[\frac{t^\rho\|e^{tA}v\|_V}{\|v\|_H}\right]\\
&\leq \left[\sup_{v\in H_\rho\setminus\{0\}}\frac{\|v\|_V}{\|v\|_{H_\rho}}\right]\left[\sup_{t\in(0,T)}\sup_{v\in H\setminus\{0\}}\frac{\|(-tA)^\rho e^{tA}v\|_H}{\|v\|_H}\right]\\
&= \left[\sup_{v\in H_\rho\setminus\{0\}}\frac{\|v\|_V}{\|v\|_{H_\rho}}\right]\left[\sup_{t\in(0,T)}\|(-tA)^\rho e^{tA}\|_{L(H)}\right]\\
&\leq \left[\sup_{v\in H_\rho\setminus\{0\}}\frac{\|v\|_V}{\|v\|_{H_\rho}}\right] < \infty.
\end{aligned} \tag{179}$$

Moreover, observe that, e.g., Lemma 5.3 (with $V = H_\gamma\times H_\gamma$, $W = V\times V$, $S = [0,\infty)$, $\mathcal{S} = \mathcal{B}([0,\infty))$, $s = 0$, $\psi = H_\gamma\times H_\gamma \ni (v,w)\mapsto (\|v\|_{H_\gamma}+\|w\|_{H_\gamma})^{1/\chi}\in[0,\infty)$, $\Psi = \mathcal{V}$ in the notation of Lemma 5.3) establishes that

$$\mathcal{V}\in\mathcal{M}\big(\mathcal{B}(V\times V),\mathcal{B}([0,\infty))\big). \tag{180}$$

The fact that $\forall\, t\in[0,T]\colon \tilde{X}_t = \int_0^t e^{(t-s)A}F(\tilde{X}_s)\,ds + \tilde{O}_t$, (155), (179), Lemma 5.2 (iii), Lemma 5.2 (iv), and, e.g., Andersson et al. [2, Lemma 2.2] hence allow us to apply Proposition 4.4 (with $H = H$, $\mathbb{H} = \mathbb{H}$, $T = T$, $c = c$, $\varphi = \varphi$, $C = c$, $\mathcal{D} = \mathcal{D}$, $\mu = \mu$, $A = A$, $V = V$, $\mathcal{V} = \mathcal{V}$, $F = F$, $(P_J)_{J\in\mathcal{P}(\mathbb{H})} \subseteq L(H) = (\mathfrak{P}_J)_{J\in\mathcal{P}(\mathbb{H})} \subseteq L(H)$, $(\Omega,\mathcal{F},\mathbb{P}) = (\Omega,\mathcal{F},\mathbb{P})$, $X_t(\omega) = \tilde{X}_t(\omega)$, $O_t(\omega) = \tilde{O}_t(\omega)$, $\mathbf{X}_t^{M,I}(\omega) = \int_0^t P_I e^{(t-s)A}F(\tilde{\mathcal{X}}^{M,I}_{\lfloor s\rfloor_{T/M}}(\omega))\,ds + P_I(\tilde{O}_t(\omega))$, $\mathcal{X}_t^{M,I}(\omega) = \tilde{\mathcal{X}}_t^{M,I}(\omega)$, $\mathcal{O}_t^{M,I}(\omega) = \mathcal{O}_t^{M,I}(\omega)$,

$\alpha = \vartheta\chi$, $\rho = \rho$, $\varrho = \varrho$, $\kappa = \kappa$, $p = p$, $M = M$, $I = I$ for $M \in \mathbb{N}$, $I \in \mathcal{D}$, $t \in [0,T]$, $\omega \in \Omega$ in the notation of Proposition 4.4) to obtain that for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\tilde{X}_t - \tilde{\mathcal{X}}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{4^{(2+\varphi)} \max\{1, T^{(3/2+\vartheta\chi-\rho+\varphi/2-\rho\varphi/2)}\} \max\{1, c^{(1+\varphi/4)}\}\sqrt{e^{\kappa c T}}}{\min\{1, \sqrt{c\,(\kappa-2)}\}\,(1-\rho-\varrho)^{(1+\varphi/2)}} \Bigg[\sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}\tilde{X}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + M^{-\min\{\vartheta\chi,\varrho\}} \\
&+ \sup_{t\in(0,T)} \|P_I(\tilde{O}_t - \tilde{O}_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{t\in[0,T]} \|P_I\tilde{O}_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)}\Bigg] \max\left\{1, \sup_{v\in V\setminus\{0\}} \frac{\|v\|_H}{\|v\|_V}\right\} \\
&\cdot \left[1 + \sup_{s\in(0,T)} \|\tilde{X}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I\tilde{X}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in[0,T)} \|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I\tilde{O}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2}\right] \\
&\cdot \left[1 + \sup_{s\in[0,T)} \|\mathcal{V}(\tilde{\mathcal{X}}_s^{M,I}, \mathcal{O}_s^{M,I})\|_{\mathcal{L}^{4p\vartheta\chi}(\mathbb{P};\mathbb{R})}^{\vartheta\chi}\right] \max\left\{1, \sup_{s\in(0,T)} \left[s^\rho \|e^{sA}\|_{L(H,V)}\right]^{(1+\varphi/2)}\right\} \\
&\cdot \left[\max\left\{1, \sup_{s\in[0,T)} \|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{p(1+\varphi/2)\max\{4,\varphi\}}(\mathbb{P};V)}^{[(1+\varphi/2)^2]}\right\} + \|F(0)\|_H^{(1+\varphi/2)}\right].
\end{aligned} \tag{181}$$

The fact that $\forall\, t \in [0,T]\colon \mathbb{P}(X_t = \tilde{X}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1$, the fact that $\forall\, t \in [0,T]\colon \mathbb{P}(O_t = \tilde{O}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1$, and (154) hence prove that for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \tilde{\mathcal{X}}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} = \sup_{t\in[0,T]} \|\tilde{X}_t - \tilde{\mathcal{X}}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{4^{(2+\varphi)} \max\{1, T^{(3/2+\vartheta\chi-\rho+\varphi/2-\rho\varphi/2)}\} \max\{1, c^{(1+\varphi/4)}\}\sqrt{e^{\kappa c T}}}{\min\{1, \sqrt{c\,(\kappa-2)}\}\,(1-\rho-\varrho)^{(1+\varphi/2)}} \Bigg[\sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I}\tilde{X}_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + M^{-\min\{\vartheta\chi,\varrho\}} \\
&+ \sup_{t\in(0,T)} \|P_I(O_t - O_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{t\in[0,T]} \|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)}\Bigg] \max\left\{1, \sup_{v\in V\setminus\{0\}} \frac{\|v\|_H}{\|v\|_V}\right\} \\
&\cdot \left[1 + \sup_{s\in(0,T)} \|\tilde{X}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I\tilde{X}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in[0,T)} \|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2}\right] \\
&\cdot \left[1 + \sup_{s\in[0,T)} \left\{\|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{4p\vartheta}(\mathbb{P};H_\gamma)} + \|\mathcal{O}_s^{M,I}\|_{\mathcal{L}^{4p\vartheta}(\mathbb{P};H_\gamma)}\right\}^{\vartheta}\right] \max\left\{1, \sup_{s\in(0,T)} \left[s^\rho \|e^{sA}\|_{L(H,V)}\right]^{(1+\varphi/2)}\right\} \\
&\cdot \left[\max\left\{1, \sup_{s\in[0,T)} \|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{p(1+\varphi/2)\max\{4,\varphi\}}(\mathbb{P};V)}^{[(1+\varphi/2)^2]}\right\} + \|F(0)\|_H^{(1+\varphi/2)}\right].
\end{aligned} \tag{182}$$

The fact that

$$\forall\, M \in \mathbb{N}, I \in \mathcal{D}, t \in [0,T] \colon \mathbb{P}(\mathcal{X}_t^{M,I} = \tilde{\mathcal{X}}_t^{M,I}) = 1, \tag{183}$$

the fact that

$$\forall\, t \in [0,T] \colon \mathbb{P}(O_t = \tilde{O}_t) \geq \mathbb{P}(\tilde{\Omega}) = 1, \tag{184}$$

the fact that $\forall\, I \in \mathcal{D} \colon \|P_I\|_{L(H)} \leq 1$, (173), and (179) therefore assure that for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} = \sup_{t\in[0,T]} \|X_t - \tilde{\mathcal{X}}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq \frac{4^{(2+\varphi)} \max\{1, T^{(3/2+\vartheta\chi-\rho+\varphi/2-\rho\varphi/2)}\} \max\{1, c^{(1+\varphi/4)}\} \sqrt{e^{\kappa c T}}}{\min\{1, \sqrt{c\,(\kappa-2)}\}\,(1-\rho-\varrho)^{(1+\varphi/2)}} \Bigg[\sup_{t\in[0,T]} \|\mathfrak{P}_{\mathbb{H}\setminus I} O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&+ \sup_{t\in[0,T]} \|\tilde{X}_t - \tilde{O}_t\|_{\mathcal{L}^{2p}(\mathbb{P};H_{(\rho+\varrho)})} \|(-A)^{-\varrho}\, \mathfrak{P}_{\mathbb{H}\setminus I}\|_{L(H)} + M^{-\min\{\vartheta\chi,\varrho\}} \\
&+ \sup_{t\in(0,T)} \|P_I(O_t - O_{\lfloor t \rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \sup_{t\in[0,T]} \|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \Bigg] \max\left\{1, \sup_{v\in V\setminus\{0\}} \frac{\|v\|_H}{\|v\|_V}\right\} \\
&\cdot \left[1 + \sup_{s\in(0,T)} \|\tilde{X}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I \tilde{X}_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2} + \sup_{s\in[0,T)} \|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H_\gamma)}^{\varphi/2} + \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}^{\varphi/2}\right] \\
&\cdot \left[1 + \sup_{s\in[0,T)} \left\{\|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{4p\vartheta}(\mathbb{P};H_\gamma)} + \|\mathcal{O}_s^{M,I}\|_{\mathcal{L}^{4p\vartheta}(\mathbb{P};H_\gamma)}\right\}^{\vartheta}\right] \max\left\{1, \sup_{v\in H_\gamma\setminus\{0\}} \frac{\|v\|_V^{[(1+\varphi/2)^2+\varphi/2]}}{\|v\|_{H_\gamma}^{[(1+\varphi/2)^2+\varphi/2]}}\right\} \\
&\cdot \left[\max\left\{1, \sup_{s\in[0,T)} \|\tilde{\mathcal{X}}_s^{M,I}\|_{\mathcal{L}^{p(1+\varphi/2)\max\{4,\varphi\}}(\mathbb{P};H_\gamma)}^{[(1+\varphi/2)^2]}\right\} + \|F(0)\|_H^{(1+\varphi/2)}\right] \max\left\{1, \sup_{v\in H_\rho\setminus\{0\}} \frac{\|v\|_V^{(2+\varphi/2)}}{\|v\|_{H_\rho}^{(2+\varphi/2)}}\right\}.
\end{aligned} \tag{185}$$

The fact that $H_\gamma \subseteq H_\rho \subseteq V \subseteq H$ continuously, (164), (165), (170), (177), and the fact that

$$\sup_{M\in\mathbb{N}} \sup_{I\in\mathcal{D}} \sup_{t\in[0,T]} \left[\|\mathcal{O}_t^{M,I}\|_{\mathcal{L}^{4p\vartheta}(\mathbb{P};H_\gamma)} + \|P_I O_t\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}\right] < \infty \tag{186}$$

therefore establish (ii). The proof of Theorem 5.5 is thus completed. ☐

Corollary 5.6. *Assume the setting in Section 5.1, let $\theta \in [0,\infty)$, $\vartheta \in (0,\infty)$, $p \in [\max\{2, 1/\varphi\}, \infty)$,*

$\varrho \in [0, 1-\rho)$, *and assume that*

$$\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|_V^{p(1+\varphi/2)\max\{2,\varphi\}}\big]$$
$$+ \sup_{M\in\mathbb{N}} \sup_{I\in\mathcal{D}} \sup_{t\in[0,T]} \bigg(M^\theta \bigg\{ \Big(\mathbb{E}\big[\|P_I(O_t - O_{\lfloor t\rfloor_{T/M}})\|_V^{2p}\big]\Big)^{\frac{1}{2p}} + \Big(\mathbb{E}\big[\|P_I O_t - \mathcal{O}_t^{M,I}\|_V^{p\max\{2,\varphi\}}\big]\Big)^{\frac{1}{p\max\{2,\varphi\}}} \bigg\}\bigg)$$
$$+ \sup_{M\in\mathbb{N}} \sup_{I\in\mathcal{D}} \sup_{t\in[0,T]} \mathbb{E}\Big[\big|\|\mathcal{O}_t^{M,I}\|_{H_\gamma}^2 + \Phi(\mathcal{O}^{M,I}) + \phi(\mathcal{O}^{M,I})\big|^{p\max\{2\vartheta,2+\varphi,(1+\varphi/2)\varphi/2\}}\Big] < \infty. \tag{187}$$

Then we have

(i) that $\sup_{M\in\mathbb{N}} \sup_{I\in\mathcal{D}} \sup_{t\in[0,T]} \mathbb{E}\big[\|\mathcal{X}_t^{M,I}\|_{H_\gamma}^{p\max\{4\vartheta,4+2\varphi,\varphi(1+\varphi/2)\}}\big] < \infty$ *and*

(ii) that there exists a real number $C \in (0,\infty)$ *such that for all* $M \in \mathbb{N}$, $I \in \mathcal{D}$ *it holds that*

$$\begin{aligned} &\sup_{t\in[0,T]} \Big(\mathbb{E}\big[\|X_t - \mathcal{X}_t^{M,I}\|_H^p\big]\Big)^{1/p} \\ &\le C\left[M^{-\min\{\vartheta\chi,\varrho,\theta\}} + \|(-A)^{-\varrho}(\mathrm{Id}_H - P_I)\|_{L(H)} + \sup_{t\in[0,T]} \|(\mathrm{Id}_H - P_I)O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)}\right]. \end{aligned} \tag{188}$$

Proof of Corollary 5.6. Note that the triangle inequality, (187), and the fact that $H_\gamma \subseteq V$ continuously yield that

$$\begin{aligned} \sup_{I\in\mathcal{D}} \sup_{s\in(0,T)} \|P_I O_s\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} &\le \sup_{M\in\mathbb{N}} \sup_{I\in\mathcal{D}} \sup_{s\in(0,T)} \big[\|P_I O_s - \mathcal{O}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} + \|\mathcal{O}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)}\big] \\ &\le \sup_{M\in\mathbb{N}} \sup_{I\in\mathcal{D}} \sup_{s\in(0,T)} \|P_I O_s - \mathcal{O}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};V)} \\ &\quad + \left[\sup_{v\in H_\gamma\setminus\{0\}} \frac{\|v\|_V}{\|v\|_{H_\gamma}}\right]\left[\sup_{M\in\mathbb{N}} \sup_{I\in\mathcal{D}} \sup_{s\in(0,T)} \|\mathcal{O}_s^{M,I}\|_{\mathcal{L}^{p\varphi}(\mathbb{P};H_\gamma)}\right] < \infty. \end{aligned} \tag{189}$$

This together with (187) allows us to apply Theorem 5.5 to obtain that (i) holds and that there exists a real number $K \in (0,\infty)$ such that for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned} &\sup_{t\in[0,T]} \|X_t - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \le K\bigg(M^{-\min\{\vartheta\chi,\varrho\}} + \|(-A)^{-\varrho}(\mathrm{Id}_H - P_I)\|_{L(H)} \\ &\quad + \sup_{t\in[0,T]} \Big[\|(\mathrm{Id}_H - P_I)O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|P_I(O_t - O_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)}\Big]\bigg). \end{aligned} \tag{190}$$

Hence, we obtain that for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq K\Bigg(M^{-\min\{\vartheta\chi,\varrho\}} + \|(-A)^{-\varrho}(\mathrm{Id}_H - P_I)\|_{L(H)} + \sup_{t\in[0,T]} \|(\mathrm{Id}_H - P_I)O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\quad + M^{-\theta}\left\{ \sup_{t\in[0,T]} \left(M^{\theta}\left[\|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|P_I(O_t - O_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right]\right)\right\}\Bigg).
\end{aligned} \tag{191}$$

This implies that for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq K\Bigg(M^{-\min\{\vartheta\chi,\varrho,\theta\}} + \|(-A)^{-\varrho}(\mathrm{Id}_H - P_I)\|_{L(H)} + \sup_{t\in[0,T]} \|(\mathrm{Id}_H - P_I)O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \\
&\quad + M^{-\min\{\vartheta\chi,\varrho,\theta\}}\left\{ \sup_{t\in[0,T]} \left(M^{\theta}\left[\|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|P_I(O_t - O_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right]\right)\right\}\Bigg) \\
&= K\Bigg(\|(-A)^{-\varrho}(\mathrm{Id}_H - P_I)\|_{L(H)} + \sup_{t\in[0,T]} \|(\mathrm{Id}_H - P_I)O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + M^{-\min\{\vartheta\chi,\varrho,\theta\}} \\
&\quad \cdot \left\{ 1 + \sup_{t\in[0,T]} \left(M^{\theta}\left[\|P_I O_t - \mathcal{O}_t^{M,I}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|P_I(O_t - O_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right]\right)\right\}\Bigg).
\end{aligned} \tag{192}$$

Therefore, we obtain that for all $M \in \mathbb{N}$, $I \in \mathcal{D}$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \mathcal{X}_t^{M,I}\|_{\mathcal{L}^p(\mathbb{P};H)} \\
&\leq K\Bigg(\|(-A)^{-\varrho}(\mathrm{Id}_H - P_I)\|_{L(H)} + \sup_{t\in(0,T)} \|(\mathrm{Id}_H - P_I)O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + M^{-\min\{\vartheta\chi,\varrho,\theta\}} \\
&\quad \cdot \left[1 + \sup_{N\in\mathbb{N}} \sup_{J\in\mathcal{D}} \sup_{t\in[0,T]} \left(N^{\theta}\left[\|P_J(O_t - O_{\lfloor t\rfloor_{T/N}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|P_J O_t - \mathcal{O}_t^{N,J}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right]\right)\right]\Bigg) \\
&\leq K\left[1 + \sup_{N\in\mathbb{N}} \sup_{J\in\mathcal{D}} \sup_{t\in[0,T]} \left(N^{\theta}\left[\|P_J(O_t - O_{\lfloor t\rfloor_{T/N}})\|_{\mathcal{L}^{2p}(\mathbb{P};V)} + \|P_J O_t - \mathcal{O}_t^{N,J}\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right]\right)\right] \\
&\quad \cdot \left[M^{-\min\{\vartheta\chi,\varrho,\theta\}} + \|(-A)^{-\varrho}(\mathrm{Id}_H - P_I)\|_{L(H)} + \sup_{t\in(0,T)} \|(\mathrm{Id}_H - P_I)O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right].
\end{aligned} \tag{193}$$

Combining this with (187) establishes (ii). The proof of Corollary 5.6 is thus completed. □

6 Stochastic Allen-Cahn equations

6.1 Setting

Consider the notation in Section 1.1, let $T, \nu \in (0,\infty)$, $a_0, a_1, a_2 \in \mathbb{R}$, $a_3 \in (-\infty, 0]$, $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H) = (L^2(\lambda_{(0,1)}; \mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})})$, $(e_n)_{n\in\mathbb{N}} \subseteq H$, $F\colon L^6(\lambda_{(0,1)}; \mathbb{R}) \to H$, $(P_n)_{n\in\mathbb{N}} \subseteq L(H)$ satisfy for all $n \in \mathbb{N}$, $v \in L^6(\lambda_{(0,1)}; \mathbb{R})$ that $a_2 \, \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3) = 0$, $e_n = [(\sqrt{2}\sin(n\pi x))_{x\in(0,1)}]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}$, $F(v) = \sum_{k=0}^{3} a_k v^k$, $P_n(v) = \sum_{k=1}^{n} \langle e_k, v\rangle_H \, e_k$, let $A\colon D(A) \subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H times the real number ν, let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., [27, Section 3.7]), let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an Id_H-cylindrical Wiener process, and let $\underline{(\cdot)}\colon \big\{[v]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})} \in H \colon (v \in \mathcal{C}((0,1),\mathbb{R}) \text{ is a uniformly continuous function})\big\} \to \mathcal{C}((0,1),\mathbb{R})$ be the function which satisfies for all uniformly continuous functions $v\colon (0,1) \to \mathbb{R}$ that $\underline{[v]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}} = v$.

6.2 Properties of the nonlinearities of stochastic Allen-Cahn equations

Lemma 6.1. *Assume the setting in Section 6.1 and let* $\epsilon \in (0,1)$, $c \in [\frac{32}{\epsilon}\max\{\frac{|a_2|^2}{|a_3|+\mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3)}, |a_3|\}, \infty)$. *Then there exists a real number* $C \in (0,\infty)$ *such that for all* $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ *we have that*

$$\begin{aligned}
&\langle v, P_N F(v + w_t)\rangle_{H_{1/2}} + c\big[\sup\nolimits_{s\in[0,T]} \|w_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\big]\langle v, F(v+w_t)\rangle_H \\
&\leq \epsilon\|v\|^2_{H_1} + \Big(|a_1| + \tfrac{|a_2|^2}{3|a_3|+\mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3)}\Big)\|v\|^2_{H_{1/2}} \\
&\quad + c\big[\sup\nolimits_{s\in[0,T]} \|w_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\big]\Big(|a_0| + \tfrac{3|a_1|}{2}\Big)\|v\|^2_H \\
&\quad + C\big[\sup\nolimits_{s\in[0,T]} \|w_s\|^8_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\big].
\end{aligned} \tag{194}$$

Proof of Lemma 6.1. Throughout this proof let $\eta\colon \mathbb{N}_0 \times \mathbb{N}_0 \to (0,\infty)$ be a function which satisfies that

$$\mathbb{1}^{\mathbb{R}}_{(-\infty,0)}(a_3)\left(a_3 + \frac{1}{2}\sum_{k=0}^{3}\sum_{j=0}^{\min\{k,2\}}\binom{k}{j}(j+1)\,|a_k|\,|\eta(k,j)|^{\frac{4}{(j+1)}}\right) \leq 0. \tag{195}$$

Observe that the fact that for every $N \in \mathbb{N}$ we have that P_N is symmetric implies that for all $N \in \mathbb{N}$,

$v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^{3} a_k P_N [v + w_t]^k \rangle_{H_{1/2}} \\
&= \sum_{k=0}^{3} a_k \langle (-A)^{1/2} v, (-A)^{1/2} P_N [v + w_t]^k \rangle_H = \sum_{k=0}^{3} a_k \langle (-A) v, P_N [v + w_t]^k \rangle_H \\
&= -\sum_{k=0}^{3} a_k \langle A P_N v, [v + w_t]^k \rangle_H = -\sum_{k=0}^{3} a_k \langle A v, [v + w_t]^k \rangle_H .
\end{aligned} \tag{196}$$

This shows that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^{3} a_k P_N [v + w_t]^k \rangle_{H_{1/2}} \\
&= -\nu \sum_{k=0}^{3} a_k \langle v'', [v + w_t]^k \rangle_H = -\nu \sum_{k=0}^{3} \sum_{j=0}^{k} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
&= -\nu \sum_{k=1}^{3} \binom{k}{k} a_k \langle v'', v^k \rangle_H - \nu \sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H .
\end{aligned} \tag{197}$$

Moreover, note that integration by parts and the fact that

$$\forall\, x, y \in \mathbb{R}, r \in (0,\infty) \colon x\sqrt{2r} \cdot \tfrac{y}{\sqrt{2r}} \leq r x^2 + \tfrac{y^2}{4r} \tag{198}$$

prove that for all $N \in \mathbb{N}$, $v \in P_N(H)$ we have that

$$\begin{aligned}
&-\nu \sum_{k=1}^{3} a_k \langle v'', v^k \rangle_H = \nu \sum_{k=1}^{3} k a_k \langle v', v^{(k-1)} v' \rangle_H \\
&\leq 3\nu a_3 \int_0^1 [\underline{v}'(x)]^2 \, [\underline{v}(x)]^2 \, dx + 2\nu |a_2| \int_0^1 [\underline{v}'(x)]^2 \, |\underline{v}(x)| \, dx + \nu |a_1| \, \|v'\|_H^2 \\
&\leq 3\nu a_3 \int_0^1 [\underline{v}'(x)]^2 \, [\underline{v}(x)]^2 \, dx + \nu \int_0^1 [\underline{v}'(x)]^2 \left(3|a_3| [\underline{v}(x)]^2 + \frac{4|a_2|^2}{4\big(3|a_3| + \mathbb{1}_{\{0\}}^{\mathbb{R}}(a_3)\big)} \right) dx \\
&\quad + \nu |a_1| \|v'\|_H^2 \\
&= \nu \left(|a_1| + \frac{|a_2|^2}{3|a_3| + \mathbb{1}_{\{0\}}^{\mathbb{R}}(a_3)} \right) \|v'\|_H^2 = \left(|a_1| + \frac{|a_2|^2}{3|a_3| + \mathbb{1}_{\{0\}}^{\mathbb{R}}(a_3)} \right) \|v\|_{H_{1/2}}^2 .
\end{aligned} \tag{199}$$

Furthermore, observe that the fact that

$$\forall\, x, y \in \mathbb{R} \colon xy \leq \epsilon x^2 + \tfrac{y^2}{4\epsilon} \tag{200}$$

shows that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$\begin{aligned}
&- \nu \sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
&\leq \nu \sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} |a_k| \int_0^1 |\underline{v}''(x)| \, |\underline{v}(x)|^j \, |\underline{w_t}(x)|^{(k-j)} \, dx \\
&= \int_0^1 \nu \, |\underline{v}''(x)| \left[\sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} |a_k| \, |\underline{v}(x)|^j \, |\underline{w_t}(x)|^{(k-j)} \right] dx \\
&\leq \epsilon \nu^2 \|v''\|_H^2 + \frac{1}{4\epsilon} \int_0^1 \left[\sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} |a_k| \, |\underline{v}(x)|^j \, |\underline{w_t}(x)|^{(k-j)} \right]^2 dx.
\end{aligned} \tag{201}$$

The fact that

$$\forall \, x_1, x_2, \ldots, x_7 \in \mathbb{R} \colon [x_1 + x_2 + \ldots + x_7]^2 \leq 7([x_1]^2 + [x_2]^2 + \ldots + [x_7]^2) \tag{202}$$

hence assures that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$\begin{aligned}
&- \nu \sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
&\leq \epsilon \nu^2 \|v''\|_H^2 + \frac{7}{4\epsilon} \sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \left[\tbinom{k}{j} |a_k| \right]^2 \int_0^1 |\underline{v}(x)|^{2j} \, |\underline{w_t}(x)|^{2(k-j)} \, dx \\
&= \epsilon \|v\|_{H_1}^2 + \frac{7}{4\epsilon} \sum_{k=0}^{3} |a_k|^2 \int_0^1 |\underline{w_t}(x)|^{2k} \, dx \\
&\quad + \frac{7}{4\epsilon} \sum_{k=2}^{3} \sum_{j=1}^{k-1} \left[\tbinom{k}{j} |a_k| \right]^2 \int_0^1 |\underline{v}(x)|^{2j} \, |\underline{w_t}(x)|^{2(k-j)} \, dx.
\end{aligned} \tag{203}$$

This and the fact that

$$\forall \, x, y \in \mathbb{R} \colon xy \leq \tfrac{x^2}{2} + \tfrac{y^2}{2} \tag{204}$$

imply that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$
\begin{aligned}
&- \nu \sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
&\le \epsilon \|v\|_{H_1}^2 + \frac{7}{\epsilon} \left[\max_{k \in \{0,1,2,3\}} |a_k|^2 \right] \int_0^1 \max\{|\underline{w_t}(x)|^6, 1\}\, dx \\
&+ \frac{7}{4\epsilon} \left[\max_{k \in \{2,3\}, j \in \{1,2\}} \tbinom{k}{j} |a_k| \right]^2 \int_0^1 \left[|\underline{v}(x)|^2 \, |\underline{w_t}(x)|^2 + |\underline{v}(x)|^2 \, |\underline{w_t}(x)|^4 + |\underline{v}(x)|^4 \, |\underline{w_t}(x)|^2 \right] dx \\
&\le \epsilon \|v\|_{H_1}^2 + \frac{7}{\epsilon} \left[\max_{k \in \{0,1,2,3\}} |a_k|^2 \right] \left(\|w_t\|_{L^6(\lambda_{(0,1)};\mathbb{R})}^6 + 1 \right) \\
&+ \frac{7}{4\epsilon} \left[\max_{k \in \{2,3\}, j \in \{1,2\}} \tbinom{k}{j} |a_k| \right]^2 \int_0^1 \left[|\underline{v}(x)|^4 + |\underline{w_t}(x)|^4 + |\underline{v}(x)|^4 |\underline{w_t}(x)|^4 \right] dx.
\end{aligned}
\tag{205}
$$

Hölder's inequality therefore ensures that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$
\begin{aligned}
&- \nu \sum_{k=0}^{3} \sum_{j=0}^{\max\{0,k-1\}} \binom{k}{j} a_k \langle v'', v^j (w_t)^{(k-j)} \rangle_H \\
&\le \epsilon \|v\|_{H_1}^2 + \frac{7}{\epsilon} \left[\max_{k \in \{0,1,2,3\}} |a_k|^2 \right] \left(\|w_t\|_{L^6(\lambda_{(0,1)};\mathbb{R})}^6 + 1 \right) \\
&\quad + \frac{63}{4\epsilon} \left[\max_{k \in \{2,3\}} |a_k|^2 \right] \int_0^1 |\underline{w_t}(x)|^4 \, dx \\
&\quad + \frac{63}{4\epsilon} \left[\max_{k \in \{2,3\}} |a_k|^2 \right] \int_0^1 |\underline{v}(x)|^4 \left(1 + |\underline{w_t}(x)|^4 \right) dx \\
&\le \epsilon \|v\|_{H_1}^2 + \frac{23}{\epsilon} \left[\max_{k \in \{0,1,2,3\}} |a_k|^2 \right] \left(\|w_t\|_{L^6(\lambda_{(0,1)};\mathbb{R})}^6 + 1 \right) \\
&\quad + \frac{16}{\epsilon} \left[\max_{k \in \{2,3\}} |a_k|^2 \right] \|v\|_{L^4(\lambda_{(0,1)};\mathbb{R})}^4 \left(\|w_t\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}^4 + 1 \right).
\end{aligned}
\tag{206}
$$

In the next step we combine (197) with (199) and (206) to obtain that for all $N \in \mathbb{N}$, $v \in P_N(H)$,

$w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^3 a_k P_N [v+w_t]^k \rangle_{H_{1/2}} \\
&\leq \epsilon \|v\|_{H_1}^2 + \left(|a_1| + \frac{|a_2|^2}{3|a_3| + \mathbb{1}_{\{0\}}^{\mathbb{R}}(a_3)} \right) \|v\|_{H_{1/2}}^2 + \left[\max_{k \in \{0,1,2,3\}} \tfrac{5|a_k|}{\sqrt{\epsilon}} \right]^2 \left[\|w_t\|_{L^6(\lambda_{(0,1)};\mathbb{R})}^6 + 1 \right] \\
&\quad + \frac{c|a_3|}{2} \left[\|w_t\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}^4 + 1 \right] \|v\|_{L^4(\lambda_{(0,1)};\mathbb{R})}^4 .
\end{aligned} \tag{207}$$

In addition, note that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^3 a_k [v+w_t]^k \rangle_H \\
&= \sum_{k=0}^3 a_k \langle v, [v+w_t]^k \rangle_H = \sum_{k=0}^3 \sum_{j=0}^k \binom{k}{j} a_k \langle v, v^j (w_t)^{(k-j)} \rangle_H \\
&= a_3 \|v\|_{L^4(\lambda_{(0,1)};\mathbb{R})}^4 + \sum_{k=0}^3 \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} a_k \langle v, v^j (w_t)^{(k-j)} \rangle_H \\
&\leq a_3 \|v\|_{L^4(\lambda_{(0,1)};\mathbb{R})}^4 + \sum_{k=0}^3 \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} |a_k| \int_0^1 |\underline{v}(x)|^{(j+1)} \, |\underline{w_t}(x)|^{(k-j)} \, dx.
\end{aligned} \tag{208}$$

Young's inequality hence demonstrates that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$, $r \in [0,\infty)$ we have that

$$\begin{aligned}
&r \langle v, \textstyle\sum_{k=0}^3 a_k [v+w_t]^k \rangle_H \\
&\leq r a_3 \|v\|_{L^4(\lambda_{(0,1)};\mathbb{R})}^4 + r \sum_{k=0}^3 \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} |a_k| \int_0^1 |\eta(k,j)| \, |\underline{v}(x)|^{(j+1)} \, \frac{|\underline{w_t}(x)|^{(k-j)}}{|\eta(k,j)|} \, dx \\
&\leq r a_3 \|v\|_{L^4(\lambda_{(0,1)};\mathbb{R})}^4 \\
&\quad + r \sum_{k=0}^3 \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} |a_k| \int_0^1 \left[\tfrac{(j+1)}{4} |\eta(k,j)|^{\frac{4}{(j+1)}} |\underline{v}(x)|^4 + \frac{(3-j) |\underline{w_t}(x)|^{\frac{4(k-j)}{(3-j)}}}{4 |\eta(k,j)|^{\frac{4}{(3-j)}}} \right] dx.
\end{aligned} \tag{209}$$

Therefore, we obtain that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$, $r \in [0,\infty)$ we have that

$$
\begin{aligned}
&\mathbb{1}^{\mathbb{R}}_{(-\infty,0)}(a_3)\, r\langle v, \textstyle\sum_{k=0}^{3} a_k[v+w_t]^k\rangle_H \\
&\leq r\|v\|^4_{L^4(\lambda_{(0,1)};\mathbb{R})}\mathbb{1}^{\mathbb{R}}_{(-\infty,0)}(a_3)\left[a_3+\frac{1}{4}\sum_{k=0}^{3}\sum_{j=0}^{\min\{k,2\}}\binom{k}{j}(j+1)\,|a_k|\,|\eta(k,j)|^{\frac{4}{(j+1)}}\right] \\
&\quad + r\sum_{k=0}^{3}\sum_{j=0}^{\min\{k,2\}}\binom{k}{j}\frac{(3-j)|a_k|}{4|\eta(k,j)|^{\frac{4}{(3-j)}}}\int_0^1 \max\{1,|\underline{w_t}(x)|^4\}\,dx \\
&\leq \frac{ra_3}{2}\|v\|^4_{L^4(\lambda_{(0,1)};\mathbb{R})} + r\sum_{k=0}^{3}\sum_{j=0}^{\min\{k,2\}}\binom{k}{j}\frac{(3-j)|a_k|}{4|\eta(k,j)|^{\frac{4}{(3-j)}}}\left[\|w_t\|^4_{L^4(\lambda_{(0,1)};\mathbb{R})}+1\right].
\end{aligned}
\tag{210}
$$

Moreover, note that the fact that

$$
\forall\, x,y\in[0,\infty)\colon xy \leq \tfrac{x^2}{2}+\tfrac{y^2}{2} \tag{211}
$$

ensures that for all $N\in\mathbb{N}$, $v\in P_N(H)$, $w\in\mathcal{C}([0,T],H_1)$, $t\in[0,T]$, $r\in[0,\infty)$ we have that

$$
\begin{aligned}
&r\langle v, \textstyle\sum_{k=0}^{1} a_k[v+w_t]^k\rangle_H \\
&\leq r\int_0^1 |a_0|\,|\underline{v}(x)| + |a_1|\,|\underline{v}(x)|^2 + |a_1|\,|\underline{v}(x)|\,|\underline{w_t}(x)|\,dx \\
&\leq r\int_0^1 |a_0|\left(1+|\underline{v}(x)|^2\right) + |a_1|\,|\underline{v}(x)|^2 + \frac{|a_1|}{2}\left(|\underline{v}(x)|^2+|\underline{w_t}(x)|^2\right)dx \\
&= r\left(|a_0|+\tfrac{3|a_1|}{2}\right)\|v\|^2_H + r\left(|a_0|+\tfrac{|a_1|}{2}\|w_t\|^2_H\right) \\
&\leq r\left(|a_0|+\tfrac{3|a_1|}{2}\right)\|v\|^2_H + r\left(|a_0|+\tfrac{|a_1|}{2}\right)\left[\|w_t\|^4_{L^4(\lambda_{(0,1)};\mathbb{R})}+1\right].
\end{aligned}
\tag{212}
$$

Hence, we obtain that for all $N\in\mathbb{N}$, $v\in P_N(H)$, $w\in\mathcal{C}([0,T],H_1)$, $t\in[0,T]$, $r\in[\|w_t\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})}+1,\infty)$ we have that

$$
\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^{3} a_k P_N[v+w_t]^k\rangle_{H_{1/2}} + cr\langle v, \textstyle\sum_{k=0}^{3} a_k[v+w_t]^k\rangle_H \\
&= \langle v, \textstyle\sum_{k=0}^{3} a_k P_N[v+w_t]^k\rangle_{H_{1/2}} + cr\langle v, \textstyle\sum_{k=0}^{3} a_k[v+w_t]^k\rangle_H \mathbb{1}^{\mathbb{R}}_{(-\infty,0)}(a_3) \\
&\quad + cr\langle v, \textstyle\sum_{k=0}^{3} a_k[v+w_t]^k\rangle_H \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3) \\
&= \langle v, \textstyle\sum_{k=0}^{3} a_k P_N[v+w_t]^k\rangle_{H_{1/2}} + cr\langle v, \textstyle\sum_{k=0}^{3} a_k[v+w_t]^k\rangle_H \mathbb{1}^{\mathbb{R}}_{(-\infty,0)}(a_3) \\
&\quad + cr\langle v, \textstyle\sum_{k=0}^{1} a_k[v+w_t]^k\rangle_H \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3) \\
&\leq \langle v, \textstyle\sum_{k=0}^{3} a_k P_N[v+w_t]^k\rangle_{H_{1/2}} + cr\langle v, \textstyle\sum_{k=0}^{3} a_k[v+w_t]^k\rangle_H \mathbb{1}^{\mathbb{R}}_{(-\infty,0)}(a_3) \\
&\quad + cr\left(|a_0|+\tfrac{3|a_1|}{2}\right)\|v\|^2_H + cr\left(|a_0|+\tfrac{|a_1|}{2}\right)\left[\|w_t\|^4_{L^4(\lambda_{(0,1)};\mathbb{R})}+1\right].
\end{aligned}
\tag{213}
$$

Combining this with (207) and (210) assures that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$, $r \in [\|w_t\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1, \infty)$ we have that

$$
\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^3 a_k P_N[v+w_t]^k \rangle_{H_{1/2}} + cr \langle v, \textstyle\sum_{k=0}^3 a_k [v+w_t]^k \rangle_H \\
&\leq \epsilon \|v\|^2_{H_1} + \left(|a_1| + \frac{|a_2|^2}{3|a_3| + \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3)} \right) \|v\|^2_{H_{1/2}} + cr \left(|a_0| + \tfrac{3|a_1|}{2} \right) \|v\|^2_H \\
&\quad + \frac{cr}{2} \|v\|^4_{L^4(\lambda_{(0,1)};\mathbb{R})} (a_3 + |a_3|) + \left[\max_{k \in \{0,1,2,3\}} \tfrac{5|a_k|}{\sqrt{\epsilon}} \right]^2 \left[\|w_t\|^6_{L^6(\lambda_{(0,1)};\mathbb{R})} + 1 \right] \\
&\quad + cr \left[|a_0| + \frac{|a_1|}{2} + \sum_{k=0}^{3} \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} \frac{(3-j)|a_k|}{4|\eta(k,j)|^{\frac{4}{(3-j)}}} \right] \left[\|w_t\|^4_{L^4(\lambda_{(0,1)};\mathbb{R})} + 1 \right].
\end{aligned}
\tag{214}
$$

Hölder's inequality and the fact that

$$
a_3 + |a_3| = a_3 - a_3 = 0 \tag{215}
$$

therefore prove that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$, $r \in [\|w_t\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1, \infty)$ we have that

$$
\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^3 a_k P_N[v+w_t]^k \rangle_{H_{1/2}} + cr \langle v, \textstyle\sum_{k=0}^3 a_k [v+w_t]^k \rangle_H \\
&\leq \epsilon \|v\|^2_{H_1} + \left(|a_1| + \frac{|a_2|^2}{3|a_3| + \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3)} \right) \|v\|^2_{H_{1/2}} + cr \left(|a_0| + \tfrac{3|a_1|}{2} \right) \|v\|^2_H \\
&\quad + \left[\max_{k \in \{0,1,2,3\}} \tfrac{5|a_k|}{\sqrt{\epsilon}} \right]^2 \left[\|w_t\|^6_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1 \right] \\
&\quad + cr^2 \left[|a_0| + \frac{|a_1|}{2} + \sum_{k=0}^{3} \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} \frac{(3-j)|a_k|}{4|\eta(k,j)|^{\frac{4}{(3-j)}}} \right].
\end{aligned}
\tag{216}
$$

The fact that

$$
\forall\, x, y \in (0,\infty) \colon (x+y)^2 \leq 2(x^2 + y^2) \tag{217}
$$

hence implies that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$
\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^{3} a_k P_N[v+w_t]^k\rangle_{H_{1/2}} + c\left[\sup_{s\in[0,T]} \|w_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\right] \langle v, \textstyle\sum_{k=0}^{3} a_k [v+w_t]^k\rangle_H \\
&\leq \epsilon \|v\|^2_{H_1} + \left(|a_1| + \frac{|a_2|^2}{3|a_3| + \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3)}\right) \|v\|^2_{H_{1/2}} \\
&+ c\left[\sup_{s\in[0,T]} \|w_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\right] \left(|a_0| + \tfrac{3|a_1|}{2}\right) \|v\|^2_H \\
&+ \left[\max_{k\in\{0,1,2,3\}} \tfrac{5|a_k|}{\sqrt{\epsilon}}\right]^2 \left[\|w_t\|^8_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 2\right] \\
&+ c\left[2|a_0| + |a_1| + \sum_{k=0}^{3} \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} \frac{(3-j)|a_k|}{2|\eta(k,j)|^{\frac{4}{(3-j)}}}\right] \left[\sup_{s\in[0,T]} \|w_s\|^8_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\right].
\end{aligned}
\tag{218}
$$

Hence, we obtain that for all $N \in \mathbb{N}$, $v \in P_N(H)$, $w \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$
\begin{aligned}
&\langle v, \textstyle\sum_{k=0}^{3} a_k P_N[v+w_t]^k\rangle_{H_{1/2}} + c\left[\sup_{s\in[0,T]} \|w_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\right] \langle v, \textstyle\sum_{k=0}^{3} a_k [v+w_t]^k\rangle_H \\
&\leq \epsilon \|v\|^2_{H_1} + \left(|a_1| + \frac{|a_2|^2}{3|a_3| + \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3)}\right) \|v\|^2_{H_{1/2}} \\
&\quad + c\left[\sup_{s\in[0,T]} \|w_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\right] \left(|a_0| + \frac{3|a_1|}{2}\right) \|v\|^2_H \\
&\quad + \left[\left[\max_{k\in\{0,1,2,3\}} \tfrac{8|a_k|}{\sqrt{\epsilon}}\right]^2 + 2c|a_0| + c|a_1| + c\sum_{k=0}^{3} \sum_{j=0}^{\min\{k,2\}} \binom{k}{j} \frac{(3-j)|a_k|}{2|\eta(k,j)|^{\frac{4}{(3-j)}}}\right] \\
&\quad \cdot \left[\sup_{s\in[0,T]} \|w_s\|^8_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1\right].
\end{aligned}
\tag{219}
$$

The proof of Lemma 6.1 is thus completed. □

The next elementary lemma, Lemma 6.2 below, establishes a local Lipschitz estimate for the nonlinearity F in Section 6.1. Lemma 6.2 is a slightly modified version of Lemma 6.8 in [3].

Lemma 6.2. *Assume the setting in Section 6.1 and let $q \in [6, \infty)$, $v, w \in L^q(\lambda_{(0,1)}; \mathbb{R})$. Then*

$$
\|F(v) - F(w)\|^2_H \leq 36 \left[\max_{j\in\{1,2,3\}} |a_j|\right]^2 \|v-w\|^2_{L^q(\lambda_{(0,1)};\mathbb{R})} \left(1 + \|v\|^4_{L^q(\lambda_{(0,1)};\mathbb{R})} + \|w\|^4_{L^q(\lambda_{(0,1)};\mathbb{R})}\right). \tag{220}
$$

Proof of Lemma 6.2. Observe that the fundamental theorem of calculus and Jensen's inequality ensure for all $k \in \mathbb{N}$, $x, y \in \mathbb{R}$ that

$$\begin{aligned} |x^k - y^k| &= \left| \int_0^1 k\,(y + r\,(x-y))^{(k-1)}\,(x-y)\,dr \right| \\ &\leq k\,|x-y| \int_0^1 |rx + (1-r)\,y|^{(k-1)}\,dr \\ &\leq k\,|x-y| \int_0^1 \left(r\,|x|^{(k-1)} + (1-r)\,|y|^{(k-1)} \right) dr. \end{aligned} \tag{221}$$

Combining this and Hölder's inequality implies that

$$\begin{aligned} &\|F(v) - F(w)\|_H = \left\| \textstyle\sum_{k=0}^{3} a_k \left(v^k - w^k\right) \right\|_H \leq \sum_{k=1}^{3} |a_k|\,\|v^k - w^k\|_H \\ &\leq \sum_{k=1}^{3} k\,|a_k| \int_0^1 \left\| |v-w| \left(r\,|v|^{(k-1)} + (1-r)\,|w|^{(k-1)} \right) \right\|_H dr \\ &\leq \sum_{k=1}^{3} k\,|a_k| \int_0^1 \|v-w\|_{L^q(\lambda_{(0,1)};\mathbb{R})} \left\| r\,|v|^{(k-1)} + (1-r)\,|w|^{(k-1)} \right\|_{L^{2q/(q-2)}(\lambda_{(0,1)};\mathbb{R})} dr \\ &\leq \|v-w\|_{L^q(\lambda_{(0,1)};\mathbb{R})} \Bigg[|a_1| \\ &\quad + \sum_{k=2}^{3} k\,|a_k| \int_0^1 \left(r\,\|v\|^{(k-1)}_{L^{2q(k-1)/(q-2)}(\lambda_{(0,1)};\mathbb{R})} + (1-r)\,\|w\|^{(k-1)}_{L^{2q(k-1)/(q-2)}(\lambda_{(0,1)};\mathbb{R})} \right) dr \Bigg]. \end{aligned} \tag{222}$$

Again Hölder's inequality therefore demonstrates that

$$\begin{aligned}
&\|F(v)-F(w)\|_H \\
&\leq \|v-w\|_{L^q(\lambda_{(0,1)};\mathbb{R})}\left[|a_1|+\frac{1}{2}\sum_{k=2}^{3}k\,|a_k|\left(\|v\|^{(k-1)}_{L^{2q(k-1)/(q-2)}(\lambda_{(0,1)};\mathbb{R})}+\|w\|^{(k-1)}_{L^{2q(k-1)/(q-2)}(\lambda_{(0,1)};\mathbb{R})}\right)\right] \\
&\leq \|v-w\|_{L^q(\lambda_{(0,1)};\mathbb{R})}\left[|a_1|+\frac{1}{2}\sum_{k=2}^{3}k\,|a_k|\left(\|v\|^{(k-1)}_{L^q(\lambda_{(0,1)};\mathbb{R})}+\|w\|^{(k-1)}_{L^q(\lambda_{(0,1)};\mathbb{R})}\right)\right] \\
&\leq \frac{1}{2}\|v-w\|_{L^q(\lambda_{(0,1)};\mathbb{R})}\left[\max_{j\in\{1,2,3\}}|a_j|\right]\sum_{k=1}^{3}k\left(\|v\|^{(k-1)}_{L^q(\lambda_{(0,1)};\mathbb{R})}+\|w\|^{(k-1)}_{L^q(\lambda_{(0,1)};\mathbb{R})}\right) \\
&\leq 3\|v-w\|_{L^q(\lambda_{(0,1)};\mathbb{R})}\left[\max_{j\in\{1,2,3\}}|a_j|\right]\left(\max\Big\{1,\|v\|^2_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big\}+\max\Big\{1,\|w\|^2_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big\}\right) \\
&\leq 6\|v-w\|_{L^q(\lambda_{(0,1)};\mathbb{R})}\left[\max_{j\in\{1,2,3\}}|a_j|\right]\max\Big\{1,\|v\|^2_{L^q(\lambda_{(0,1)};\mathbb{R})},\|w\|^2_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big\}.
\end{aligned} \tag{223}$$

This completes the proof of Lemma 6.2. □

6.3 Properties of linear stochastic heat equations

In this subsection we present a few elementary regularity and approximation results for linear stochastic heat equations; see Lemmas 6.4–6.8 and Corollary 6.9 below. Similar regularity and approximation results for linear stochastic heat equations can, e.g., be found in Hutzenthaler et al. [13, Lemma 5.6, Corollary 5.8, and Lemma 5.9]. The next lemma, Lemma 6.3 below, presents a well-known fact on centered and normally distributed random variables. Lemma 6.3 is used in the proofs of Lemma 6.5, Lemma 6.7, and Lemma 6.8 below.

Lemma 6.3. *Let $p \in [0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $Y\colon\Omega\to\mathbb{R}$ be a standard normal random variable, and let $Z\colon\Omega\to\mathbb{R}$ be a centered and normally distributed random variable. Then*

$$\mathbb{E}\big[|Z|^p\big]=\mathbb{E}\big[|Y|^p\big]\big(\mathbb{E}\big[|Z|^2\big]\big)^{p/2}. \tag{224}$$

Proof of Lemma 6.3. Throughout this proof assume w.l.o.g. that $\mathbb{E}[|Z|^2]>0$ (otherwise the proof is clear). Note that

$$\begin{aligned}
&\mathbb{E}\big[|Z|^p\big] \\
&=\mathbb{E}\left[\left|\frac{Z}{\big(\mathbb{E}\big[|Z|^2\big]\big)^{1/2}}\big(\mathbb{E}\big[|Z|^2\big]\big)^{1/2}\right|^p\right]=\mathbb{E}\left[\left|\frac{Z}{\big(\mathbb{E}\big[|Z|^2\big]\big)^{1/2}}\right|^p\right]\big(\mathbb{E}\big[|Z|^2\big]\big)^{p/2} \\
&=\mathbb{E}\big[|Y|^p\big]\big(\mathbb{E}\big[|Z|^2\big]\big)^{p/2}.
\end{aligned} \tag{225}$$

This completes the proof of Lemma 6.3. ☐

Lemma 6.4. *Assume the setting in Section 6.1, let $\gamma \in [0, 1/4)$, $\beta \in (1/4, 1/2-\gamma)$, $B \in HS(H, H_{-\beta})$, and let $\varphi\colon [0,T] \to [0,T]$ be a $\mathcal{B}([0,T])/\mathcal{B}([0,T])$-measurable function which satisfies for all $t \in [0,T]$ that $\varphi(t) \leq t$. Then there exists an up to indistinguishability unique stochastic process $O\colon [0,T] \times \Omega \to H_\gamma$ with continuous sample paths which satisfies for all $t \in [0,T]$ that*

$$[O_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-\varphi(s))A} B \, dW_s. \tag{226}$$

Proof of Lemma 6.4. Throughout this proof let $\varepsilon \in (0, 1/2 - \gamma - \beta)$, $p \in (1/\varepsilon, \infty)$ be real numbers. Note for all $s \in [0,T)$, $t \in (s,T]$ that

$$\begin{aligned}
&\int_0^t \big\| e^{(t-\varphi(u))A} B - \mathbb{1}^{\mathbb{R}}_{(-\infty,s)}(u) \, e^{(s-\varphi(u))A} B \big\|^2_{HS(H,H_\gamma)} \, du \\
&= \int_s^t \big\| e^{(t-\varphi(u))A} B \big\|^2_{HS(H,H_\gamma)} \, du + \int_0^s \big\| \big(e^{(t-\varphi(u))A} - e^{(s-\varphi(u))A} \big) B \big\|^2_{HS(H,H_\gamma)} \, du \\
&\leq \|B\|^2_{HS(H,H_{-\beta})} \bigg[\int_s^t \big\| e^{(t-\varphi(u))A} \big\|^2_{L(H_{-\beta},H_\gamma)} \, du \\
&+ \int_0^s \big\| e^{(s-\varphi(u))A} \, \big(e^{(t-s)A} - \mathrm{Id}_H \big) \big\|^2_{L(H_{-\beta},H_\gamma)} \, du \bigg] \\
&\leq \|B\|^2_{HS(H,H_{-\beta})} \bigg[\int_s^t \big\| (-A)^{(\gamma+\beta)} e^{(t-u)A} \big\|^2_{L(H)} \big\| e^{(u-\varphi(u))A} \big\|^2_{L(H)} \, du \\
&+ \int_0^s \big\| (-A)^{(\gamma+\beta+\varepsilon)} e^{(s-u)A} \big\|^2_{L(H)} \big\| e^{(u-\varphi(u))A} \big\|^2_{L(H)} \big\| (-A)^{-\varepsilon} (e^{(t-s)A} - \mathrm{Id}_H) \big\|^2_{L(H)} \, du \bigg].
\end{aligned} \tag{227}$$

The fact that $\forall \, t \in [0,\infty)$, $r \in [0,1]\colon \|(-tA)^r e^{tA}\|_{L(H)} \leq 1$ and the fact that $\forall \, t \in (0,\infty)$, $r \in [0,1]\colon \|(-tA)^{-r}(e^{tA} - \mathrm{Id}_H)\|_{L(H)} \leq 1$ hence prove for all $s \in [0,T)$, $t \in (s,T]$ that

$$\begin{aligned}
&\int_0^t \big\| e^{(t-\varphi(u))A} B - \mathbb{1}^{\mathbb{R}}_{(-\infty,s)}(u) \, e^{(s-\varphi(u))A} B \big\|^2_{HS(H,H_\gamma)} \, du \\
&\leq \|B\|^2_{HS(H,H_{-\beta})} \bigg[\int_s^t (t-u)^{-2(\gamma+\beta)} \, du + \int_0^s (s-u)^{-2(\gamma+\beta+\varepsilon)} \, (t-s)^{2\varepsilon} \, du \bigg] \\
&= \|B\|^2_{HS(H,H_{-\beta})} \bigg[\frac{(t-s)^{(1-2\gamma-2\beta)}}{(1-2\gamma-2\beta)} + \frac{s^{(1-2\gamma-2\beta-2\varepsilon)}(t-s)^{2\varepsilon}}{(1-2\gamma-2\beta-2\varepsilon)} \bigg] \\
&\leq \frac{2 \, (t-s)^{2\varepsilon} \max\{1,T\}}{(1-2\gamma-2\beta-2\varepsilon)} \, \|B\|^2_{HS(H,H_{-\beta})} < \infty.
\end{aligned} \tag{228}$$

This implies that for all $t \in [0,T]$ we have that $\int_0^t \|e^{(t-\varphi(u))A} B\|^2_{HS(H,H_\gamma)} \, du < \infty$. Hence, there exits a stochastic process $\tilde{O} \colon [0,T] \times \Omega \to H_\gamma$ which satisfies for all $t \in [0,T]$ that

$$[\tilde{O}_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-\varphi(s))A} B \, dW_s. \tag{229}$$

Moreover, note that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabcyk [6] and (228) demonstrate that for all $s \in [0,T)$, $t \in (s,T]$ we have that

$$\begin{aligned}
&\|\tilde{O}_t - \tilde{O}_s\|^2_{\mathcal{L}^p(\mathbb{P};H_\gamma)} \\
&= \left\| \int_0^t \left[e^{(t-\varphi(u))A} - \mathbb{1}^{\mathbb{R}}_{(-\infty,s)}(u) \, e^{(s-\varphi(u))A} \right] B \, dW_u \right\|^2_{L^p(\mathbb{P};H_\gamma)} \\
&\leq \frac{p(p-1)}{2} \int_0^t \left\| e^{(t-\varphi(u))A} B - \mathbb{1}^{\mathbb{R}}_{(-\infty,s)}(u) \, e^{(s-\varphi(u))A} B \right\|^2_{HS(H,H_\gamma)} du \\
&\leq \frac{p(p-1) \, (t-s)^{2\varepsilon} \max\{1,T\}}{(1-2\gamma-2\beta-2\varepsilon)} \, \|B\|^2_{HS(H,H_{-\beta})} < \infty.
\end{aligned} \tag{230}$$

The Kolmogorov-Chentsov theorem and the fact that $p\varepsilon > 1$ therefore imply that there exists an up to indistinguishability unique stochastic process $O \colon [0,T] \times \Omega \to H_\gamma$ with continuous sample paths which satisfies for all $t \in [0,T]$ that

$$[O_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-\varphi(s))A} B \, dW_s. \tag{231}$$

The proof of Lemma 6.4 is thus completed. □

Lemma 6.5. *Assume the setting in Section 6.1 and let* $p,q \in [2,\infty)$, $\theta \in [1/4 - 1/2q, 1/4)$, $\xi \in \mathcal{L}^p(\mathbb{P}; H_{2\theta})$. *Then there exists a stochastic process* $O \colon [0,T] \times \Omega \to L^q(\lambda_{(0,1)};\mathbb{R})$ *with continuous sample paths which satisfies*

(i) that for all $t \in [0,T]$ *we have that* $[O_t - e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A} \, dW_s$ *and*

(ii) that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{0\leq s<t\leq T} \left(\frac{\|P_N(O_t - O_s)\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta} \right) \\
&+ \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \left(N^{2\theta} \, \|O_t - P_N O_t\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))} \right) < \infty.
\end{aligned} \tag{232}$$

Proof of Lemma 6.5. Throughout this proof let $Y\colon \Omega \to \mathbb{R}$ be a standard normal random variable and let $\tilde{p} = \max\{p, q\}$, $\beta \in (1/4, 1/2 - \theta)$, $(\mu_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $k \in \mathbb{N}$ that $\mu_k = \nu k^2 \pi^2$. Note that the fact that

$$\forall\, t \in [0,\infty), r \in [0,1]\colon \|(-tA)^r e^{tA}\|_{L(H)} \leq 1, \tag{233}$$

the fact that

$$\forall\, t \in (0,\infty), r \in [0,1]\colon \|(-tA)^{-r}(e^{tA} - \mathrm{Id}_H)\|_{L(H)} \leq 1, \tag{234}$$

and the assumption that $\xi \in \mathcal{L}^p(\mathbb{P}; H_{2\theta})$ assure that for all $s \in [0,T)$, $t \in (s,T]$ we have that

$$\begin{aligned} \|e^{tA}\xi - e^{sA}\xi\|_{\mathcal{L}^p(\mathbb{P};H_\theta)} &\leq \|e^{sA}\|_{L(H)}\, \|(e^{(t-s)A} - \mathrm{Id}_H)\xi\|_{\mathcal{L}^p(\mathbb{P};H_\theta)} \\ &\leq \|(-A)^{-\theta}(e^{(t-s)A} - \mathrm{Id}_H)\|_{L(H)}\, \|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{2\theta})} \\ &\leq (t-s)^\theta\, \|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{2\theta})} < \infty. \end{aligned} \tag{235}$$

In addition, observe that the fact that $A\colon D(A) \subseteq H \to H$ is the generator of a strongly continuous semigroup and the fact that

$$\forall\, t \in [0,\infty)\colon \|e^{tA}\|_{L(H)} \leq 1 \tag{236}$$

prove that for all $\omega \in \Omega$, $t \in [0,T]$ we have that

$$\begin{aligned} &\limsup_{\substack{(t_1,t_2)\to(t,t),\\ (t_1,t_2)\in[0,t]\times[t,T]}} \|e^{t_2 A}\xi(\omega) - e^{t_1 A}\xi(\omega)\|_{H_\theta} \\ &\leq \limsup_{\substack{(t_1,t_2)\to(t,t),\\ (t_1,t_2)\in[0,t]\times[t,T]}} \left[\|e^{t_1 A}\|_{L(H)}\, \|(e^{(t_2-t_1)A} - \mathrm{Id}_H)\xi(\omega)\|_{H_\theta}\right] \\ &\leq \limsup_{\substack{(t_1,t_2)\to(t,t),\\ (t_1,t_2)\in[0,t]\times[t,T]}} \|(e^{(t_2-t_1)A} - \mathrm{Id}_H)(-A)^\theta \xi(\omega)\|_H = 0. \end{aligned} \tag{237}$$

Moreover, observe that the fact that $4\beta > 1$ shows that

$$\begin{aligned} \sum_{k=1}^\infty \|e_k\|_{H_{-\beta}}^2 &= \sum_{k=1}^\infty \|(-A)^{-\beta} e_k\|_H^2 = \sum_{k=1}^\infty |(\nu\pi^2 k^2)^{-\beta}|^2 = \sum_{k=1}^\infty \frac{1}{(\sqrt{\nu} k\pi)^{4\beta}} \\ &= \frac{1}{(\pi\sqrt{\nu})^{4\beta}} \left[\sum_{k=1}^\infty \frac{1}{k^{4\beta}}\right] < \infty. \end{aligned} \tag{238}$$

This allows us to apply Lemma 6.4 (with $\gamma = \theta$, $\beta = \beta$, $B = (H \ni v \mapsto v \in H_{-\beta})$, $\varphi = [0,T] \ni t \mapsto t \in [0,T]$ in the notation of Lemma 6.4) to obtain that there exists an up to indistinguishability unique stochastic process $\tilde{O}\colon [0,T] \times \Omega \to H_\theta$ with continuous sample paths which satisfies for all $t \in [0,T]$ that

$$[\tilde{O}_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\, dW_s. \tag{239}$$

Next note that (237) and the fact that

$$H_\theta \subseteq W^{2\theta,2}((0,1),\mathbb{R}) \subseteq W^{0,q}((0,1),\mathbb{R}) = L^q(\lambda_{(0,1)};\mathbb{R}) \tag{240}$$

continuously (cf., e.g., Da Prato & Zabcyk [6, (A.46) in Section A.5.2] and Lunardi [20]) ensure that there exists a stochastic process $O\colon [0,T]\times\Omega \to L^q(\lambda_{(0,1)};\mathbb{R})$ with continuous sample paths which satisfies for all $t\in[0,T]$, $\omega\in\Omega$ that

$$O_t(\omega) = e^{tA}\xi(\omega) + \tilde{O}_t(\omega). \tag{241}$$

Combining (241) with (239) demonstrates that for all $t\in[0,T]$ we have that

$$[O_t - e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\,dW_s. \tag{242}$$

In addition, note that Hölder's inequality, Fubini's theorem, and, e.g., Lemma 6.3 show for all $s\in[0,T)$, $t\in(s,T]$, $N\in\mathbb{N}$ that

$$\begin{aligned}
\mathbb{E}\Big[\|P_N(\tilde{O}_t-\tilde{O}_s)\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] &= \mathbb{E}\bigg[\bigg|\int_0^1 \big|\underline{P_N\tilde{O}_t}(x) - \underline{P_N\tilde{O}_s}(x)\big|^q\,dx\bigg|^{\tilde{p}/q}\bigg] \\
&\le \mathbb{E}\bigg[\int_0^1 \big|\underline{P_N\tilde{O}_t}(x) - \underline{P_N\tilde{O}_s}(x)\big|^{\tilde{p}}\,dx\bigg] \\
&= \int_0^1 \mathbb{E}\Big[\big|\underline{P_N\tilde{O}_t}(x) - \underline{P_N\tilde{O}_s}(x)\big|^{\tilde{p}}\Big]\,dx \\
&= \mathbb{E}\Big[|Y|^{\tilde{p}}\Big]\int_0^1 \bigg(\mathbb{E}\Big[\big|\underline{P_N\tilde{O}_t}(x) - \underline{P_N\tilde{O}_s}(x)\big|^2\Big]\bigg)^{\tilde{p}/2}\,dx.
\end{aligned} \tag{243}$$

This implies for all $s\in[0,T)$, $t\in(s,T]$, $N\in\mathbb{N}$ that

$$\begin{aligned}
&\mathbb{E}\Big[\|P_N(\tilde{O}_t-\tilde{O}_s)\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] \\
&\le \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \\
&\cdot \int_0^1 \Bigg(\mathbb{E}\Bigg[\bigg|\sum_{k=1}^N \underline{e_k}(x)\bigg(\int_0^t e^{-\mu_k(t-u)}\,\langle e_k, dW_u\rangle_H - \int_0^s e^{-\mu_k(s-u)}\,\langle e_k, dW_u\rangle_H\bigg)\bigg|^2\Bigg]\Bigg)^{\tilde{p}/2} dx \\
&= \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \\
&\cdot \int_0^1 \Bigg(\sum_{k=1}^N \big|\underline{e_k}(x)\big|^2\,\mathbb{E}\bigg[\bigg|\int_0^t e^{-\mu_k(t-u)}\,\langle e_k, dW_u\rangle_H - \int_0^s e^{-\mu_k(s-u)}\,\langle e_k, dW_u\rangle_H\bigg|^2\bigg]\Bigg)^{\tilde{p}/2} dx.
\end{aligned} \tag{244}$$

Moreover, note that Itô's isometry yields for all $s \in [0, T)$, $t \in (s, T]$, $k \in \mathbb{N}$ that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t e^{-\mu_k(t-u)} \left\langle e_k, dW_u \right\rangle_H - \int_0^s e^{-\mu_k(s-u)} \left\langle e_k, dW_u \right\rangle_H\right|^2\right] \\
&= \mathbb{E}\left[\left|\int_s^t e^{-\mu_k(t-u)} \left\langle e_k, dW_u \right\rangle_H + \left(e^{-\mu_k(t-s)} - 1\right) \int_0^s e^{-\mu_k(s-u)} \left\langle e_k, dW_u \right\rangle_H\right|^2\right] \\
&= \mathbb{E}\left[\left|\int_s^t e^{-\mu_k(t-u)} \left\langle e_k, dW_u \right\rangle_H\right|^2\right] \\
&\quad + \left(e^{-\mu_k(t-s)} - 1\right)^2 \mathbb{E}\left[\left|\int_0^s e^{-\mu_k(s-u)} \left\langle e_k, dW_u \right\rangle_H\right|^2\right] \\
&= \int_s^t e^{-2\mu_k(t-u)} \, du + \left(e^{-\mu_k(t-s)} - 1\right)^2 \int_0^s e^{-2\mu_k(s-u)} \, du.
\end{aligned}
\tag{245}
$$

The fact that

$$
\begin{aligned}
\sup_{x \in (0,\infty)} \left(x^{-1}\left(1 - e^{-x}\right)\right) &= \sup_{x \in (0,\infty)} \left(x^{-1} \int_0^x e^{-s} \, ds\right) \\
&\leq \sup_{x \in (0,\infty)} \left(x^{-1} \int_0^x ds\right) = 1
\end{aligned}
\tag{246}
$$

hence implies for all $s \in [0, T)$, $t \in (s, T]$, $k \in \mathbb{N}$ that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t e^{-\mu_k(t-u)} \left\langle e_k, dW_u \right\rangle_H - \int_0^s e^{-\mu_k(s-u)} \left\langle e_k, dW_u \right\rangle_H\right|^2\right] \\
&= \frac{\left(1 - e^{-2\mu_k(t-s)}\right)}{2\mu_k} + \left(1 - e^{-\mu_k(t-s)}\right)^2 \frac{\left(1 - e^{-2\mu_k s}\right)}{2\mu_k} \\
&\leq \frac{\left(1 - e^{-2\mu_k(t-s)}\right)}{2\mu_k} + \frac{\left(1 - e^{-\mu_k(t-s)}\right)}{2\mu_k} \leq \frac{\left(1 - e^{-2\mu_k(t-s)}\right)}{\mu_k} \\
&= 2^{2\theta} \left(t - s\right)^{2\theta} \left[\frac{\left(1 - e^{-2\mu_k(t-s)}\right)}{2\mu_k(t-s)}\right]^{2\theta} \left[\frac{\left(1 - e^{-2\mu_k(t-s)}\right)}{\mu_k}\right]^{(1-2\theta)} \\
&\leq \sqrt{2} \left(t - s\right)^{2\theta} \left[\sup_{x \in (0,\infty)} \frac{\left(1 - e^{-x}\right)}{x}\right]^{2\theta} \left(\mu_k\right)^{(2\theta-1)} \leq \sqrt{2} \left(t - s\right)^{2\theta} \left(\mu_k\right)^{(2\theta-1)}.
\end{aligned}
\tag{247}
$$

Combining this with (244) proves that for all $s \in [0,T)$, $t \in (s,T]$, $N \in \mathbb{N}$ we have that

$$\begin{aligned}
&\mathbb{E}\Big[\|P_N(\tilde{O}_t - \tilde{O}_s)\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] \\
&\leq \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \int_0^1 \left[\sum_{k=1}^{N} \big|\underline{e_k}(x)\big|^2 \sqrt{2}\,(t-s)^{2\theta}\,\mu_k^{(2\theta-1)}\right]^{\tilde{p}/2} dx \\
&\leq (t-s)^{\tilde{p}\theta}\,\mathbb{E}\Big[|Y|^{\tilde{p}}\Big]\left[\sqrt{8}\sum_{k=1}^{N}\big(\sqrt{\nu}k\pi\big)^{-(2-4\theta)}\right]^{\tilde{p}/2}.
\end{aligned} \tag{248}$$

Furthermore, observe that the triangle inequality, Hölder's inequality, and (240) show that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}}\sup_{0\leq s<t\leq T}\left(\frac{\|P_N(O_t - O_s)\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta}\right) \\
&\leq \sup_{N\in\mathbb{N}}\sup_{0\leq s<t\leq T}\left[\frac{\|P_N(e^{tA}-e^{sA})\xi\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta} + \frac{\|P_N(\tilde{O}_t - \tilde{O}_s)\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta}\right] \\
&\leq \left[\sup_{v\in H_\theta\setminus\{0\}}\frac{\|v\|_{L^q(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_\theta}}\right]\left[\sup_{N\in\mathbb{N}}\sup_{0\leq s<t\leq T}\frac{\|P_N\|_{L(H)}\,\|(e^{tA}-e^{sA})\xi\|_{\mathcal{L}^p(\mathbb{P};H_\theta)}}{(t-s)^\theta}\right] \\
&\quad + \sup_{N\in\mathbb{N}}\sup_{0\leq s<t\leq T}\frac{\|P_N(\tilde{O}_t - \tilde{O}_s)\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta}.
\end{aligned} \tag{249}$$

Combining this with (235), (240), (248), the fact that $\forall\, N \in \mathbb{N}\colon \|P_N\|_{L(H)} \leq 1$, the assumption that $\xi \in \mathcal{L}^p(\mathbb{P}; H_{2\theta})$, and the fact that $2 - 4\theta > 1$ implies that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}}\sup_{0\leq s<t\leq T}\left(\frac{\|P_N(O_t - O_s)\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta}\right) \\
&\leq \left[\sup_{v\in H_\theta\setminus\{0\}}\frac{\|v\|_{L^q(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_\theta}}\right]\|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{2\theta})} \\
&\quad + \|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})}\left[\sqrt{8}\sum_{k=1}^{\infty}\big(\sqrt{\nu}k\pi\big)^{-(2-4\theta)}\right]^{1/2} < \infty.
\end{aligned} \tag{250}$$

In the next step observe that the fact that $\forall\, t \in [0,\infty)\colon \|e^{tA}\|_{L(H)} \leq 1$ and the assumption that

$\xi \in \mathcal{L}^p(\mathbb{P}; H_{2\theta})$ yield that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \big(N^{2\theta}\,\|e^{tA}\xi - P_N e^{tA}\xi\|_{\mathcal{L}^p(\mathbb{P};H_\theta)}\big) \\
&\le \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \big(N^{2\theta}\,\|e^{tA}\|_{L(H)}\,\|(-A)^{\theta}(\mathrm{Id}_H - P_N)\xi\|_{\mathcal{L}^p(\mathbb{P};H)}\big) \\
&\le \sup_{N\in\mathbb{N}} \big(N^{2\theta}\,\|(-A)^{-\theta}(\mathrm{Id}_H - P_N)\|_{L(H)}\,\|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{2\theta})}\big) \\
&= \left[\sup_{N\in\mathbb{N}} \frac{N^{2\theta}}{\nu^\theta \pi^{2\theta}(N+1)^{2\theta}}\right] \|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{2\theta})} \\
&\le \nu^{-\theta}\|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{2\theta})} < \infty.
\end{aligned} \tag{251}$$

In addition, note that Hölder's inequality, Fubini's theorem, and, e.g., Lemma 6.3 guarantee that for all $t \in [0,T]$, $M, N \in \mathbb{N}$ with $M \ge N$ we have that

$$\begin{aligned}
&\mathbb{E}\Big[\|P_M\tilde{O}_t - P_N\tilde{O}_t\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] \\
&= \mathbb{E}\left[\left|\int_0^1 \big|\underline{P_M\tilde{O}_t}(x) - \underline{P_N\tilde{O}_t}(x)\big|^q \, dx\right|^{\tilde{p}/q}\right] \\
&\le \mathbb{E}\left[\int_0^1 \big|\underline{P_M\tilde{O}_t}(x) - \underline{P_N\tilde{O}_t}(x)\big|^{\tilde{p}} \, dx\right] \\
&= \int_0^1 \mathbb{E}\Big[\Big|\underline{P_M\tilde{O}_t}(x) - \underline{P_N\tilde{O}_t}(x)\Big|^{\tilde{p}}\Big] \, dx \\
&= \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \int_0^1 \left(\mathbb{E}\Big[\Big|\underline{P_M\tilde{O}_t}(x) - \underline{P_N\tilde{O}_t}(x)\Big|^2\Big]\right)^{\tilde{p}/2} dx \\
&= \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \int_0^1 \left(\mathbb{E}\left[\left|\sum_{k=N+1}^{M} \underline{e_k}(x) \int_0^t e^{-\mu_k(t-s)}\,\langle e_k, dW_s\rangle_H\right|^2\right]\right)^{\tilde{p}/2} dx.
\end{aligned} \tag{252}$$

Itô's isometry hence ensures for all $t \in [0,T]$, $M, N \in \mathbb{N}$ with $M \geq N$ that

$$\begin{aligned} &\mathbb{E}\Big[\|P_M\tilde{O}_t - P_N\tilde{O}_t\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] \\ &\leq \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \int_0^1 \left(\sum_{k=N+1}^{M} \big|\underline{e_k}(x)\big|^2 \,\mathbb{E}\bigg[\bigg|\int_0^t e^{-\mu_k(t-s)}\, \langle e_k, dW_s\rangle_H\bigg|^2\bigg]\right)^{\tilde{p}/2} dx \\ &= \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \int_0^1 \left(\sum_{k=N+1}^{M} \big|\underline{e_k}(x)\big|^2 \int_0^t e^{-2\mu_k(t-s)}\, ds\right)^{\tilde{p}/2} dx \\ &= \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \int_0^1 \left(\sum_{k=N+1}^{M} \big|\underline{e_k}(x)\big|^2 \frac{(1-e^{-2\mu_k t})}{2\mu_k}\right)^{\tilde{p}/2} dx. \end{aligned} \tag{253}$$

Therefore, we obtain for all $t \in [0,t]$, $M, N \in \mathbb{N}$ with $M \geq N$ that

$$\begin{aligned} &\mathbb{E}\Big[\|P_M\tilde{O}_t - P_N\tilde{O}_t\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] \\ &\leq \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \left(\sum_{k=N+1}^{M} \frac{(1-e^{-2\mu_k t})}{\mu_k}\right)^{\tilde{p}/2} \\ &\leq \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \left(\sum_{k=N+1}^{\infty} \frac{1}{\nu\pi^2k^2}\right)^{\tilde{p}/2}. \end{aligned} \tag{254}$$

This implies that for all $t \in [0,T]$, $N \in \mathbb{N}$ we have that

$$\begin{aligned} &\mathbb{E}\Big[\|\tilde{O}_t - P_N\tilde{O}_t\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] \\ &= \limsup_{M\to\infty} \mathbb{E}\Big[\|P_M\tilde{O}_t - P_N\tilde{O}_t\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] \\ &\leq \mathbb{E}\Big[|Y|^{\tilde{p}}\Big] \left(\sum_{k=N+1}^{\infty} \frac{1}{\nu\pi^2k^2}\right)^{\tilde{p}/2}. \end{aligned} \tag{255}$$

Hence, we obtain for all $t \in [0,T]$, $N \in \mathbb{N}$ that

$$\begin{aligned} &\|\tilde{O}_t - P_N\tilde{O}_t\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))} \\ &\leq \|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})} \left[\sum_{k=N+1}^{\infty} \frac{1}{\nu\pi^2k^2}\right]^{1/2} = \frac{\|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})}}{\pi\sqrt{\nu}} \left[\sum_{k=N+1}^{\infty} \frac{1}{k^{4\theta}k^{(2-4\theta)}}\right]^{1/2} \\ &\leq \frac{\|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})}}{N^{2\theta}\pi\sqrt{\nu}} \left[\sum_{k=N+1}^{\infty} \frac{1}{k^{(2-4\theta)}}\right]^{1/2} \leq \frac{\|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})}}{N^{2\theta}\pi\sqrt{\nu}} \left[\sum_{k=1}^{\infty} \frac{1}{k^{(2-4\theta)}}\right]^{1/2}. \end{aligned} \tag{256}$$

Combining this with the fact that $2 - 4\theta > 1$ demonstrates that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \big(N^{2\theta} \, \|\tilde{O}_t - P_N \tilde{O}_t\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\big) \\
&\leq \frac{\|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})}}{\pi\sqrt{\nu}} \left[\sum_{k=1}^{\infty} \frac{1}{k^{(2-4\theta)}}\right]^{1/2} < \infty.
\end{aligned} \tag{257}$$

The hypothesis that $\xi \in \mathcal{L}^p(\mathbb{P}; H_{2\theta})$, (240), (251) the triangle inequality, and the Hölder inequality hence assure that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \big(N^{2\theta} \, \|O_t - P_N O_t\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\big) \\
&\leq \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \Big(N^{2\theta} \Big[\|e^{tA}\xi - P_N e^{tA}\xi\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))} + \|\tilde{O}_t - P_N\tilde{O}_t\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\Big]\Big) \\
&\leq \left[\sup_{v\in H_\theta\setminus\{0\}} \frac{\|v\|_{L^q(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_\theta}}\right] \left[\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \big(N^{2\theta} \, \|e^{tA}\xi - P_N e^{tA}\xi\|_{\mathcal{L}^p(\mathbb{P};H_\theta)}\big)\right] \\
&\quad + \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \big(N^{2\theta} \, \|\tilde{O}_t - P_N\tilde{O}_t\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\big) \\
&\leq \left[\sup_{v\in H_\theta\setminus\{0\}} \frac{\|v\|_{L^q(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_\theta}}\right] \frac{\|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{2\theta})}}{\nu^\theta} \\
&\quad + \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \big(N^{2\theta} \, \|\tilde{O}_t - P_N\tilde{O}_t\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\big) \\
&< \infty.
\end{aligned} \tag{258}$$

Combining this, (250), and (242) with the fact that $O\colon [0,T]\times\Omega \to L^q(\lambda_{(0,1)};\mathbb{R})$ is a stochastic process with continuous sample paths completes the proof of Lemma 6.5. ☐

Lemma 6.6. *Assume the setting in Section 6.1 and let $p \in [2,\infty)$, $\theta \in [0, 1/4)$, $\xi \in \mathcal{L}^p(\mathbb{P}; H_\theta)$. Then there exist stochastic processes $\mathcal{O}^{M,N}\colon [0,T]\times\Omega \to P_N(H)$, $M,N\in\mathbb{N}$, with continuous sample paths which satisfy*

(i) that for all $t\in[0,T]$, $M,N\in\mathbb{N}$ we have that $[\mathcal{O}^{M,N}_t - P_N e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s\rfloor_{T/M})A}\, dW_s$ and

(ii) that $\sup_{\gamma\in[0,\theta]} \sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big[\|\mathcal{O}^{M,N}_t\|^p_{H_\gamma}\big] < \infty$.

Proof of Lemma 6.6. Throughout this proof let $\beta \in (1/4, 1/2 - \theta)$. Observe that the fact that

$$[0,\infty) \ni t \mapsto (H_\theta \ni v \mapsto e^{tA}v \in H_\theta) \in L(H_\theta) \tag{259}$$

is a strongly continuous semigroup, the fact that $\forall\, N \in \mathbb{N}\colon \|P_N\|_{L(H)} \leq 1$, and the fact that $\forall\, t \in [0,\infty)\colon \|e^{tA}\|_{L(H)} \leq 1$ prove that for all $\omega \in \Omega$, $t \in [0,T]$, $N \in \mathbb{N}$ we have that

$$\begin{aligned}
&\limsup_{\substack{(t_1,t_2)\to(t,t)\\(t_1,t_2)\in[0,t]\times[t,T]}} \|P_N e^{t_2 A}\xi(\omega) - P_N e^{t_1 A}\xi(\omega)\|_{H_\theta} \\
&\leq \limsup_{\substack{(t_1,t_2)\to(t,t)\\(t_1,t_2)\in[0,t]\times[t,T]}} \big(\|P_N\|_{L(H)}\, \|e^{t_1 A}\|_{L(H)}\, \|(e^{(t_2-t_1)A} - \mathrm{Id}_H)\xi(\omega)\|_{H_\theta}\big) \\
&\leq \limsup_{\substack{(t_1,t_2)\to(t,t)\\(t_1,t_2)\in[0,t]\times[t,T]}} \|(e^{(t_2-t_1)A} - \mathrm{Id}_H)\xi(\omega)\|_{H_\theta} = 0.
\end{aligned} \tag{260}$$

In addition, note that the fact that $4\beta > 1$ shows that

$$\begin{aligned}
\sup_{N\in\mathbb{N}} \|P_N\|^2_{HS(H,H_{-\beta})} &= \sup_{N\in\mathbb{N}} \left[\sum_{k=1}^{N} \|e_k\|^2_{H_{-\beta}}\right] = \sum_{k=1}^{\infty} \|(-A)^{-\beta} e_k\|^2_H \\
&= \sum_{k=1}^{\infty} |(\nu\pi^2 k^2)^{-\beta}|^2 = \sum_{k=1}^{\infty} \frac{1}{(\sqrt{\nu}\pi k)^{4\beta}} < \infty.
\end{aligned} \tag{261}$$

We can hence apply Lemma 6.4 (with $\gamma = \theta$, $\beta = \beta$, $B = H \ni v \mapsto P_N(v) \in H_{-\beta}$, $\varphi = [0,T] \ni t \mapsto \lfloor t \rfloor_{T/M} \in [0,T]$ for $M, N \in \mathbb{N}$ in the notation of Lemma 6.4) to obtain that there exist up to modifications unique stochastic processes $\tilde{\mathcal{O}}^{M,N}\colon [0,T]\times\Omega \to H_\theta$, $M, N \in \mathbb{N}$, with continuous sample paths which satisfy for all $t \in [0,T]$, $M, N \in \mathbb{N}$ that

$$[\tilde{\mathcal{O}}^{M,N}_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s \rfloor_{T/M})A}\, dW_s. \tag{262}$$

Inequality (260) therefore shows that there exists stochastic processes $\mathcal{O}^{M,N}\colon [0,T]\times\Omega \to P_N(H)$, $M, N \in \mathbb{N}$, with continuous sample paths which satisfy for all $t \in [0,T]$, $N, M \in \mathbb{N}$ that

$$\mathcal{O}^{M,N}_t = P_N e^{tA}\xi + P_N \tilde{\mathcal{O}}^{M,N}_t. \tag{263}$$

Next note that (262) ensures that for all $t \in [0,T]$, $N, M \in \mathbb{N}$ we have that

$$[P_N \tilde{\mathcal{O}}^{M,N}_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s \rfloor_{T/M})A}\, dW_s. \tag{264}$$

Combining this with (263) demonstrates that for all $t \in [0,T]$, $N, M \in \mathbb{N}$ we have that

$$[\mathcal{O}^{M,N}_t - P_N e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s \rfloor_{T/M})A}\, dW_s. \tag{265}$$

In the next step observe that, e.g., the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabcyk [6], the fact that

$$\forall\, t \in [0,\infty), r \in [0,1] \colon \|(-tA)^r e^{tA}\|_{L(H)} \leq 1, \tag{266}$$

and (261) imply that for all $t \in [0,T]$, $N, M \in \mathbb{N}$ we have that

$$\begin{aligned}
&\left\| \int_0^t P_N e^{(t-\lfloor s \rfloor_{T/M})A} \, dW_s \right\|^2_{L^p(\mathbb{P};H_\theta)} \leq \frac{p(p-1)}{2} \int_0^t \|P_N e^{(t-\lfloor s \rfloor_{T/M})A}\|^2_{HS(H,H_\theta)} \, ds \\
&\leq \frac{p(p-1)}{2} \|(-A)^{-\beta} P_N\|^2_{HS(H)} \int_0^t \|(-A)^{\beta} e^{(t-\lfloor s \rfloor_{T/M})A}\|^2_{L(H,H_\theta)} \, ds \\
&\leq \frac{p(p-1)}{2} \|P_N\|^2_{HS(H,H_{-\beta})} \int_0^t \|(-A)^{(\theta+\beta)} e^{(t-s)A}\|^2_{L(H)} \, \|e^{(s-\lfloor s \rfloor_{T/M})A}\|^2_{L(H)} \, ds \\
&\leq \frac{p(p-1)}{2} \|P_N\|^2_{HS(H,H_{-\beta})} \int_0^t (t-s)^{-2(\theta+\beta)} \, ds \\
&\leq \frac{p(p-1)\, T^{(1-2\theta-2\beta)}}{2\,(1-2\theta-2\beta)} \|P_N\|^2_{HS(H,H_{-\beta})} < \infty.
\end{aligned} \tag{267}$$

The triangle inequality, the fact that $\forall\, N \in \mathbb{N} \colon \|P_N\|_{L(H)} \leq 1$, the fact that

$$\forall\, t \in [0,\infty) \colon \|e^{tA}\|_{L(H)} \leq 1, \tag{268}$$

the assumption that $\xi \in \mathcal{L}^p(\mathbb{P}; H_\theta)$, and (261) hence assure that

$$\begin{aligned}
&\sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \left\|\mathcal{O}_t^{M,N}\right\|_{\mathcal{L}^p(\mathbb{P};H_\theta)} \\
&\leq \sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \left[\|P_N e^{tA} \xi\|_{\mathcal{L}^p(\mathbb{P};H_\theta)} + \|P_N\|_{L(H)} \left\| \int_0^t P_N e^{(t-\lfloor u \rfloor_{T/M})A} \, dW_u \right\|_{L^p(\mathbb{P};H_\theta)} \right] \\
&\leq \left[\sup_{N\in\mathbb{N}} \|P_N\|_{L(H)} \right] \left[\sup_{t\in[0,T]} \|e^{tA}\|_{L(H)} \right] \|\xi\|_{\mathcal{L}^p(\mathbb{P};H_\theta)} \\
&\quad + \left[\sup_{N\in\mathbb{N}} \|P_N\|_{HS(H,H_{-\beta})} \right] \frac{\sqrt{p(p-1)}\, T^{(1/2-\theta-\beta)}}{\sqrt{2\,(1-2\theta-2\beta)}} < \infty.
\end{aligned} \tag{269}$$

Combining this with (265) and the fact that

$$\begin{aligned}
\sup_{\gamma\in[0,\theta]} \|(-A)^{(\gamma-\theta)}\|_{L(H)} &= \sup_{\gamma\in[0,\theta]} \|(-A)^{-\gamma}\|_{L(H)} \\
&= \sup_{\gamma\in[0,\theta]} \left[(\nu\pi^2)^{-\gamma} \right] = \max\left\{ \frac{1}{(\nu\pi^2)^\theta}, 1 \right\} < \infty
\end{aligned} \tag{270}$$

completes the proof of Lemma 6.6. ☐

Lemma 6.7. *Assume the setting in Section 6.1, let* $p \in [1, \infty)$, $\xi \in \cup_{r \in (1/4, \infty)} \mathcal{L}^p(\mathbb{P}; H_r)$, *and let* $\mathcal{O}^{M,N} \colon [0,T] \times \Omega \to P_N(H)$, $M, N \in \mathbb{N}$, *be stochastic processes with continuous sample paths which satisfy for all* $M, N \in \mathbb{N}$, $t \in [0,T]$ *that*

$$[\mathcal{O}_t^{M,N} - P_N e^{tA} \xi]_{\mathbb{P}, \mathcal{B}(H)} = \int_0^t P_N e^{(t - \lfloor s \rfloor_{T/M})A} \, dW_s. \tag{271}$$

Then

$$\sup_{M,N \in \mathbb{N}} \mathbb{E}\big[\sup_{t \in [0,T]} \|\mathcal{O}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}^p \big] < \infty. \tag{272}$$

Proof of Lemma 6.7. Throughout this proof let $Y \colon \Omega \to \mathbb{R}$ be a standard normal random variable, let $\alpha \in (0, 1/8)$, $q \in (2/\alpha, \infty) \cap [p, \infty)$, $(\mu_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $k \in \mathbb{N}$ that $\mu_k = \nu k^2 \pi^2$, let $\tilde{\mathcal{O}}^{M,N} \colon [0,T] \times \Omega \to P_N(H)$, $M, N \in \mathbb{N}$, be the stochastic processes which satisfy for all $M, N \in \mathbb{N}$, $t \in [0,T]$ that

$$\tilde{\mathcal{O}}_t^{M,N} = \mathcal{O}_t^{M,N} - P_N e^{tA} \xi, \tag{273}$$

let $|\!|\!|\cdot|\!|\!| : \mathcal{C}((0,1) \times [0,T], \mathbb{R}) \to [0, \infty]$ be the function which satisfies for all $v \in \mathcal{C}((0,1) \times [0,T], \mathbb{R})$ that

$$\begin{aligned} |\!|\!|v|\!|\!| = \bigg(&\int_{(0,1)\times[0,T]} |v(x,t)|^q \, \lambda_{\mathbb{R}^2}(dx, dt) \\ &+ \int_{(0,1)\times[0,T]} \int_{((0,1)\times[0,T])\setminus\{(x_2,t_2)\}} \frac{|v(x_1,t_1) - v(x_2,t_2)|^q}{\|(x_1,t_1) - (x_2,t_2)\|_{\mathbb{R}^2}^{(2+\alpha q)}} \, \lambda_{\mathbb{R}^2}(dx_1, dt_1) \, \lambda_{\mathbb{R}^2}(dx_2, dt_2) \bigg)^{1/q}, \end{aligned} \tag{274}$$

and let $C \in [0, \infty]$ be the extended real number given by

$$C = \sup\left(\left\{ \sup_{x \in (0,1)} \sup_{t \in [0,T]} |v(x,t)| \colon \Big[v \in \mathcal{C}((0,1) \times [0,T], \mathbb{R}) \text{ and } |\!|\!|v|\!|\!| \leq 1 \Big] \right\} \right). \tag{275}$$

Observe that the Sobolev embedding theorem and the fact that $\alpha q > 2$ imply that

$$C < \infty. \tag{276}$$

Next note that for all $M, N \in \mathbb{N}$ we have that

$$\begin{aligned} &\mathbb{E}\Big[\sup_{t \in [0,T]} \|\tilde{\mathcal{O}}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}^q \Big] = \mathbb{E}\Big[\sup_{(x,t) \in (0,1)\times[0,T]} |\underline{(\tilde{\mathcal{O}}_t^{M,N})}(x)|^q \Big] \\ &\leq C^q \, \mathbb{E}\bigg[\int_{(0,1)\times[0,T]} |\underline{(\tilde{\mathcal{O}}_t^{M,N})}(x)|^q \, \lambda_{\mathbb{R}^2}(dx, dt) \bigg] \\ &+ C^q \, \mathbb{E}\bigg[\int_{(0,1)\times[0,T]} \int_{((0,1)\times[0,T])\setminus\{(x_2,t_2)\}} \frac{|\underline{(\tilde{\mathcal{O}}_{t_1}^{M,N})}(x_1) - \underline{(\tilde{\mathcal{O}}_{t_2}^{M,N})}(x_2)|^q}{\|(x_1,t_1) - (x_2,t_2)\|_{\mathbb{R}^2}^{(2+\alpha q)}} \, \lambda_{\mathbb{R}^2}(dx_1, dt_1) \, \lambda_{\mathbb{R}^2}(dx_2, dt_2) \bigg]. \end{aligned} \tag{277}$$

This and, e.g., Lemma 6.3 show that for all $M, N \in \mathbb{N}$ we have that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup\nolimits_{t\in[0,T]}\|\tilde{\mathcal{O}}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}^q\Big] \\
&\leq C^q\,\mathbb{E}\big[|Y|^q\big]\int_{(0,1)\times[0,T]}\Big(\mathbb{E}\Big[|\underline{(\tilde{\mathcal{O}}_t^{M,N})}(x)|^2\Big]\Big)^{q/2}\,\lambda_{\mathbb{R}^2}(dx,dt) \\
&+ C^q\,\mathbb{E}\big[|Y|^q\big] \\
&\cdot\int_{(0,1)\times[0,T]}\int_{((0,1)\times[0,T])\setminus\{(x_2,t_2)\}}\frac{\Big(\mathbb{E}\Big[|\underline{(\tilde{\mathcal{O}}_{t_1}^{M,N})}(x_1)-\underline{(\tilde{\mathcal{O}}_{t_2}^{M,N})}(x_2)|^2\Big]\Big)^{q/2}}{\|(x_1,t_1)-(x_2,t_2)\|_{\mathbb{R}^2}^{(2+\alpha q)}}\,\lambda_{\mathbb{R}^2}(dx_1,dt_1)\,\lambda_{\mathbb{R}^2}(dx_2,dt_2).
\end{aligned}
\tag{278}
$$

Moreover, note that Itô's isometry proves that for all $M, N \in \mathbb{N}$, $t \in [0,T]$, $x \in (0,1)$ we have that

$$
\begin{aligned}
&\mathbb{E}\Big[|\underline{(\tilde{\mathcal{O}}_t^{M,N})}(x)|^2\Big] = \mathbb{E}\left[\left|\sum_{k=1}^N \underline{e_k}(x)\int_0^t e^{-\mu_k(t-\lfloor s\rfloor_{T/M})}\,\langle e_k, dW_s\rangle_H\right|^2\right] \\
&= \sum_{k=1}^N |\underline{e_k}(x)|^2\,\mathbb{E}\left[\left|\int_0^t e^{-\mu_k(t-\lfloor s\rfloor_{T/M})}\,\langle e_k, dW_s\rangle_H\right|^2\right] \\
&= \sum_{k=1}^N |\underline{e_k}(x)|^2\int_0^t e^{-2\mu_k(t-\lfloor s\rfloor_{T/M})}\,ds \\
&= \sum_{k=1}^N |\underline{e_k}(x)|^2\int_0^t e^{-2\mu_k(t-s)}\,e^{-2\mu_k(s-\lfloor s\rfloor_{T/M})}\,ds \leq 2\sum_{k=1}^N\int_0^t e^{-2\mu_k(t-s)}\,ds \\
&= 2\sum_{k=1}^N\int_0^t e^{-2\mu_k s}\,ds = 2\sum_{k=1}^N\frac{(1-e^{-2\mu_k t})}{2\mu_k} \leq \sum_{k=1}^N\frac{1}{\mu_k} = \frac{1}{\nu\pi^2}\left(\sum_{k=1}^N\frac{1}{k^2}\right) \\
&= \frac{1}{\nu\pi^2}\left(1+\sum_{k=2}^N\int_{k-1}^k\frac{1}{k^2}\,ds\right) \leq \frac{1}{\nu\pi^2}\left(1+\int_1^N s^{-2}\,ds\right) \\
&= \frac{1}{\nu\pi^2}\left(1-\big[1/s\big]_{s=1}^{s=N}\right) = \frac{1}{\nu\pi^2}\left[2-\frac{1}{N}\right] \leq \frac{2}{\nu\pi^2}.
\end{aligned}
\tag{279}
$$

In the next step note that for all $M, N \in \mathbb{N}$, $s, t \in [0, T]$, $x, y \in (0, 1)$ we have that

$$\begin{aligned}
&\mathbb{E}\Big[|\underline{(\tilde{\mathcal{O}}^{M,N}_t)}(x) - \underline{(\tilde{\mathcal{O}}^{M,N}_s)}(y)|^2\Big] \\
&= \mathbb{E}\left[\left|\sum_{k=1}^{N}\left(\underline{e_k}(x)\int_0^t e^{-\mu_k(t-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H - \underline{e_k}(y)\int_0^s e^{-\mu_k(s-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H\right)\right|^2\right] \\
&= \sum_{k=1}^{N}\mathbb{E}\left[\left|\underline{e_k}(x)\int_0^t e^{-\mu_k(t-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H - \underline{e_k}(y)\int_0^s e^{-\mu_k(s-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H\right|^2\right].
\end{aligned} \tag{280}$$

The fact that

$$\forall\, a, b \in \mathbb{R}\colon |a+b|^2 \leq 2a^2 + 2b^2 \tag{281}$$

hence implies for all $M, N \in \mathbb{N}$, $s, t \in [0, T]$, $x, y \in (0, 1)$ that

$$\begin{aligned}
&\mathbb{E}\Big[|\underline{(\tilde{\mathcal{O}}^{M,N}_t)}(x) - \underline{(\tilde{\mathcal{O}}^{M,N}_s)}(y)|^2\Big] \\
&= \sum_{k=1}^{N}\mathbb{E}\Bigg[\bigg|\big(\underline{e_k}(x) - \underline{e_k}(y)\big)\int_0^t e^{-\mu_k(t-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H \\
&\quad + \underline{e_k}(y)\left(\int_0^t e^{-\mu_k(t-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H - \int_0^s e^{-\mu_k(s-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H\right)\bigg|^2\Bigg] \\
&\leq 2\sum_{k=1}^{N}\big|\underline{e_k}(x) - \underline{e_k}(y)\big|^2\,\mathbb{E}\left[\left|\int_0^t e^{-\mu_k(t-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H\right|^2\right] \\
&\quad + 2\sum_{k=1}^{N}\big|\underline{e_k}(y)\big|^2\,\mathbb{E}\left[\left|\int_0^t e^{-\mu_k(t-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H - \int_0^s e^{-\mu_k(s-\lfloor u\rfloor_{T/M})}\langle e_k, dW_u\rangle_H\right|^2\right].
\end{aligned} \tag{282}$$

In addition, observe that Itô's isometry, the fact that

$$\forall\, a, b \in \mathbb{R}\colon |\sin(a) - \sin(b)| \leq 2, \tag{283}$$

and the fact that

$$\forall\, a, b \in \mathbb{R}\colon |\sin(a) - \sin(b)| \leq |a - b| \tag{284}$$

show for all $M, N \in \mathbb{N}$, $t \in [0,T]$, $x, y \in (0,1)$, $r \in [0,2]$ that

$$
\begin{aligned}
&\sum_{k=1}^{N} \big|\underline{e_k}(x) - \underline{e_k}(y)\big|^2 \, \mathbb{E}\left[\left|\int_0^t e^{-\mu_k (t - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \right|^2\right] \\
&= 2 \sum_{k=1}^{N} |\sin(k\pi x) - \sin(k\pi y)|^{(2-r)} \, |\sin(k\pi x) - \sin(k\pi y)|^{r} \int_0^t e^{-2\mu_k (t-u)} \, e^{-2\mu_k (u - \lfloor u \rfloor_{T/M})} \, du \\
&\leq 2^{(3-r)} |x-y|^r \left[\sum_{k=1}^{N} (k\pi)^r \int_0^t e^{-2\mu_k (t-u)} \, du\right] = 2^{(3-r)} |x-y|^r \left[\sum_{k=1}^{N} \frac{(k\pi)^r (1 - e^{-2\mu_k t})}{2\mu_k}\right].
\end{aligned}
\tag{285}
$$

This yields that for all $M, N \in \mathbb{N}$, $t \in [0,T]$, $x, y \in (0,1)$, $r \in [0,1)$ we have that

$$
\begin{aligned}
&\sum_{k=1}^{N} \big|\underline{e_k}(x) - \underline{e_k}(y)\big|^2 \, \mathbb{E}\left[\left|\int_0^t e^{-\mu_k (t - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \right|^2\right] \\
&\leq 2^{(2-r)} |x-y|^r \left[\sum_{k=1}^{N} \frac{k^r \pi^r}{\nu k^2 \pi^2}\right] = \frac{2^{(2-r)} |x-y|^r}{\nu \pi^{(2-r)}} \left[\sum_{k=1}^{N} k^{(r-2)}\right] \\
&\leq \frac{2^{(2-r)} |x-y|^r}{\nu \pi^{(2-r)}} \left(1 + \int_1^N s^{(r-2)} \, ds\right) = \frac{2^{(2-r)} |x-y|^r}{\nu \pi^{(2-r)}} \left(1 - \left[\frac{s^{(r-1)}}{(1-r)}\right]_{s=1}^{s=N}\right) \\
&= \frac{2^{(2-r)} |x-y|^r}{\nu \pi^{(2-r)}} \left(1 - \frac{(N^{(r-1)} - 1)}{(1-r)}\right) \leq \frac{2^{(2-r)} |x-y|^r}{\nu \pi^{(2-r)}} \left(\frac{2}{(1-r)} - \frac{N^{(r-1)}}{(1-r)}\right) \\
&= \frac{2^{(2-r)} (2 - N^{(r-1)}) |x-y|^r}{\nu \pi^{(2-r)} (1-r)} \leq \frac{2^{(3-r)} |x-y|^r}{\nu \pi^{(2-r)} (1-r)}.
\end{aligned}
\tag{286}
$$

Furthermore, note that for all $M, N \in \mathbb{N}$, $s \in [0,T)$, $t \in (s,T]$ we have that

$$
\begin{aligned}
&\sum_{k=1}^{N} \mathbb{E}\left[\left|\int_0^t e^{-\mu_k (t - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H - \int_0^s e^{-\mu_k (s - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \right|^2\right] \\
&= \sum_{k=1}^{N} \mathbb{E}\left[\left|\int_s^t e^{-\mu_k (t - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H + \left(e^{-\mu_k (t-s)} - 1\right) \int_0^s e^{-\mu_k (s - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \right|^2\right] \\
&= \sum_{k=1}^{N} \mathbb{E}\left[\left|\int_s^t e^{-\mu_k (t - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \right|^2\right] \\
&\quad + \sum_{k=1}^{N} \left(e^{-\mu_k (t-s)} - 1\right)^2 \mathbb{E}\left[\left|\int_0^s e^{-\mu_k (s - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \right|^2\right].
\end{aligned}
\tag{287}
$$

Itô's isometry hence ensures for all $M, N \in \mathbb{N}$, $s \in [0, T)$, $t \in (s, T]$ that

$$
\begin{aligned}
&\sum_{k=1}^{N} \mathbb{E}\left[\left|\int_0^t e^{-\mu_k(t-\lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H - \int_0^s e^{-\mu_k(s-\lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H\right|^2\right] \\
&= \sum_{k=1}^{N} \int_s^t e^{-2\mu_k(t-u)}\, e^{-2\mu_k(u-\lfloor u \rfloor_{T/M})}\, du \\
&\quad + \sum_{k=1}^{N} \left(e^{-\mu_k(t-s)} - 1\right)^2 \int_0^s e^{-2\mu_k(s-u)}\, e^{-2\mu_k(u-\lfloor u \rfloor_{T/M})}\, du \\
&\leq \sum_{k=1}^{N} \int_s^t e^{-2\mu_k(t-u)}\, du + \sum_{k=1}^{N} \left(e^{-\mu_k(t-s)} - 1\right)^2 \int_0^s e^{-2\mu_k(s-u)}\, du \\
&= \sum_{k=1}^{N} \int_0^{(t-s)} e^{-2\mu_k u}\, du + \sum_{k=1}^{N} \left(e^{-\mu_k(t-s)} - 1\right)^2 \int_0^s e^{-2\mu_k u}\, du \\
&= \sum_{k=1}^{N} \frac{\left(1 - e^{-2\mu_k(t-s)}\right)}{2\mu_k} + \sum_{k=1}^{N} \left(e^{-\mu_k(t-s)} - 1\right)^2 \frac{\left(1 - e^{-2\mu_k s}\right)}{2\mu_k}.
\end{aligned}
\tag{288}
$$

The fact that

$$
\begin{aligned}
\sup_{x \in (0,\infty)} \left(x^{-1}(1 - e^{-x})\right) &= \sup_{x \in (0,\infty)} \left(x^{-1}\left(\int_0^x e^{-s}\, ds\right)\right) \\
&\leq \sup_{x \in (0,\infty)} \left(x^{-1}\left(\int_0^x ds\right)\right) = 1
\end{aligned}
\tag{289}
$$

therefore implies that for all $M, N \in \mathbb{N}$, $s \in [0, T)$, $t \in (s, T]$, $r \in [0, 1]$ we have that

$$
\begin{aligned}
&\sum_{k=1}^{N} \mathbb{E}\Bigg[\bigg|\int_0^t e^{-\mu_k(t - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H - \int_0^s e^{-\mu_k(s - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \bigg|^2\Bigg] \\
&\leq \sum_{k=1}^{N} \frac{(1 - e^{-2\mu_k(t-s)})}{2\mu_k} + \sum_{k=1}^{N} \frac{(1 - e^{-\mu_k(t-s)})}{2\mu_k} \leq \sum_{k=1}^{N} \frac{(1 - e^{-2\mu_k(t-s)})}{\mu_k} \\
&= 2^r (t-s)^r \sum_{k=1}^{N} \left[\frac{(1 - e^{-2\mu_k(t-s)})}{2\mu_k(t-s)}\right]^r \left[\frac{(1 - e^{-2\mu_k(t-s)})}{\mu_k}\right]^{(1-r)} \\
&\leq 2^r (t-s)^r \left[\sup_{x \in (0,\infty)} \frac{(1 - e^{-x})}{x}\right]^r \left[\sum_{k=1}^{N} (\mu_k)^{(r-1)}\right] \\
&\leq 2^r (t-s)^r \left[\sum_{k=1}^{N} \frac{1}{(\mu_k)^{(1-r)}}\right] = \frac{2^r (t-s)^r}{\nu^{(1-r)} \pi^{2(1-r)}} \left[\sum_{k=1}^{N} k^{2(r-1)}\right].
\end{aligned}
\tag{290}
$$

This yields that for all $M, N \in \mathbb{N}$, $s \in [0, T)$, $t \in (s, T]$, $r \in [0, 1/2)$ we have that

$$
\begin{aligned}
&\sum_{k=1}^{N} \mathbb{E}\Bigg[\bigg|\int_0^t e^{-\mu_k(t - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H - \int_0^s e^{-\mu_k(s - \lfloor u \rfloor_{T/M})} \langle e_k, dW_u \rangle_H \bigg|^2\Bigg] \\
&\leq \frac{2^r (t-s)^r}{\nu^{(1-r)} \pi^{2(1-r)}} \left(1 + \int_1^N s^{2(r-1)} \, ds\right) = \frac{2^r (t-s)^r}{\nu^{(1-r)} \pi^{2(1-r)}} \left(1 - \left[\frac{s^{(2r-1)}}{(1-2r)}\right]_{s=1}^{s=N}\right) \\
&= \frac{2^r (t-s)^r}{\nu^{(1-r)} \pi^{2(1-r)}} \left(1 - \frac{(N^{(2r-1)} - 1)}{(1-2r)}\right) \leq \frac{2^{(1+r)} (t-s)^r}{\nu^{(1-r)} \pi^{2(1-r)} (1-2r)}.
\end{aligned}
\tag{291}
$$

Moreover, observe that Jensen's inequality proves that for all $s, t, x, y \in [0, \infty)$, $r \in (0, 2]$ we have that

$$
\begin{aligned}
\|(x,t) - (y,s)\|_{\mathbb{R}^2}^r &= \Big[|x-y|^2 + |t-s|^2\Big]^{r/2} = 2^{r/2} \Big[\tfrac{1}{2}|x-y|^2 + \tfrac{1}{2}|t-s|^2\Big]^{r/2} \\
&= 2^{r/2} \Big[\tfrac{1}{2}\{|x-y|^r\}^{2/r} + \tfrac{1}{2}\{|t-s|^r\}^{2/r}\Big]^{r/2} \\
&\geq 2^{r/2} \Big[\tfrac{1}{2}|x-y|^r + \tfrac{1}{2}|t-s|^r\Big] = 2^{(r/2-1)} \Big[|x-y|^r + |t-s|^r\Big].
\end{aligned}
\tag{292}
$$

Combining (282) with (286), (291), and (292) yields that for all $M, N \in \mathbb{N}$, $s, t \in [0, T]$, $x, y \in (0, 1)$

we have that

$$\begin{aligned}
&\mathbb{E}\Big[|\underline{(\tilde{\mathcal{O}}_t^{M,N})}(x) - \underline{(\tilde{\mathcal{O}}_s^{M,N})}(y)|^2\Big] \\
&\leq \frac{2^{(4-4\alpha)}|x-y|^{4\alpha}}{\nu\pi^{(2-4\alpha)}(1-4\alpha)} + \frac{2^{(3+4\alpha)}|t-s|^{4\alpha}}{\nu^{(1-4\alpha)}\pi^{2(1-4\alpha)}(1-8\alpha)} \\
&\leq \frac{2^{(4-4\alpha)}}{\min\{1,\nu\}\pi^{(2-8\alpha)}(1-8\alpha)}\Big[|x-y|^{4\alpha} + |t-s|^{4\alpha}\Big] \\
&\leq \frac{2^{(5-6\alpha)}\|(x,t)-(y,s)\|_{\mathbb{R}^2}^{4\alpha}}{\min\{1,\nu\}\pi^{(2-8\alpha)}(1-8\alpha)} \\
&\leq \frac{2^3\pi^{2\alpha}\|(x,t)-(y,s)\|_{\mathbb{R}^2}^{4\alpha}}{\min\{1,\nu\}(1-8\alpha)} = \frac{\pi^{2\alpha}\|(x,t)-(y,s)\|_{\mathbb{R}^2}^{4\alpha}}{\min\{1,\nu\}(1/8-\alpha)}.
\end{aligned} \tag{293}$$

In the next step we note that (278) together with (279), (293), and the fact that $\alpha q > 2$ assures that for all $M, N \in \mathbb{N}$ we have that

$$\begin{aligned}
&\mathbb{E}\Big[\sup\nolimits_{t\in[0,T]}\|\tilde{\mathcal{O}}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}^q\Big] \\
&\leq \frac{2^{q/2}C^q\,\mathbb{E}\big[|Y|^q\big]}{\nu^{q/2}\pi^q}\int_{(0,1)\times[0,T]}\lambda_{\mathbb{R}^2}(dx,dt) \\
&+ \frac{\pi^{\alpha q}C^q\,\mathbb{E}\big[|Y|^q\big]}{\min\{1,\nu^{q/2}\}(1/8-\alpha)^{q/2}} \\
&\cdot \int_{(0,1)\times[0,T]}\int_{(0,1)\times[0,T]}\|(x_1,t_1)-(x_2,t_2)\|_{\mathbb{R}^2}^{(\alpha q-2)}\,\lambda_{\mathbb{R}^2}(dx_1,dt_1)\,\lambda_{\mathbb{R}^2}(dx_2,dt_2) \\
&\leq \frac{C^qT\,\mathbb{E}\big[|Y|^q\big]}{\nu^{q/2}} \\
&+ \frac{\pi^{\alpha q}C^qT^2\,\mathbb{E}\big[|Y|^q\big]}{\min\{1,\nu^{q/2}\}(1/8-\alpha)^{q/2}}\left[\sup_{x_1,x_2\in(0,1)}\sup_{t_1,t_2\in[0,T]}\|(x_1,t_1)-(x_2,t_2)\|_{\mathbb{R}^2}^{(\alpha q-2)}\right] \\
&= \frac{C^qT\,\mathbb{E}\big[|Y|^q\big]}{\nu^{q/2}} \\
&+ \frac{\pi^{\alpha q}C^qT^2\,\mathbb{E}\big[|Y|^q\big]}{\min\{1,\nu^{q/2}\}(1/8-\alpha)^{q/2}}\left[\sup_{x_1,x_2\in(0,1)}\sup_{t_1,t_2\in[0,T]}\big(|x_1-x_2|^2+|t_1-t_2|^2\big)\right]^{(\frac{\alpha q}{2}-1)} \\
&\leq \frac{C^qT\,\mathbb{E}\big[|Y|^q\big]}{\nu^{q/2}} + \frac{\pi^{\alpha q}C^qT^2(1+T^2)^{(\alpha q/2-1)}\mathbb{E}\big[|Y|^q\big]}{\min\{1,\nu^{q/2}\}(1/8-\alpha)^{q/2}} \\
&\leq \frac{\pi^{\alpha q}C^q\,\mathbb{E}\big[|Y|^q\big]\big[T+T^2(1+T^2)^{(\alpha q/2-1)}\big]}{\min\{1,\nu^{q/2}\}(1/8-\alpha)^{q/2}}.
\end{aligned} \tag{294}$$

In addition, observe that the assumption that $\xi \in \cup_{r\in(1/4,\infty)}\mathcal{L}^p(\mathbb{P}; H_r)$ implies that there exists a real number $\varepsilon \in (0,\infty)$ such that

$$\xi \in \mathcal{L}^p(\mathbb{P}; H_{(1/4+\varepsilon)}). \tag{295}$$

The triangle inequality and Hölder's inequality hence show that

$$\begin{aligned}
&\sup_{N,M\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|\mathcal{O}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \\
&\leq \sup_{N,M\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|P_N e^{tA}\xi\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \\
&\quad + \sup_{N,M\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|\tilde{\mathcal{O}}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \\
&\leq \left[\sup_{v\in H_{(1/4+\varepsilon)}\setminus\{0\}} \frac{\|v\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_{(1/4+\varepsilon)}}} \right] \left[\sup_{N\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|P_N e^{tA}\xi\|_{H_{(1/4+\varepsilon)}} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \right] \\
&\quad + \sup_{N,M\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|\tilde{\mathcal{O}}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \big\|_{\mathcal{L}^q(\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{296}$$

The fact that $\forall\, N \in \mathbb{N}\colon \|P_N\|_{L(H)} \leq 1$, the fact that

$$\forall\, t \in [0,\infty)\colon \|e^{tA}\|_{L(H)} \leq 1, \tag{297}$$

the Sobolev embedding theorem, (294), and (295) therefore prove that

$$\begin{aligned}
&\sup_{N,M\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|\mathcal{O}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \\
&\leq \left[\sup_{v\in H_{(1/4+\varepsilon)}\setminus\{0\}} \frac{\|v\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_{(1/4+\varepsilon)}}} \right] \left[\sup_{N\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|P_N\|_{L(H)} \|e^{tA}\|_{L(H)} \|\xi\|_{H_{(1/4+\varepsilon)}} \big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})} \right] \\
&\quad + \sup_{N,M\in\mathbb{N}} \Big(\mathbb{E}\Big[\sup\nolimits_{t\in[0,T]} \|\tilde{\mathcal{O}}_t^{M,N}\|^q_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \Big] \Big)^{1/q} \\
&\leq \left[\sup_{v\in H_{(1/4+\varepsilon)}\setminus\{0\}} \frac{\|v\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_{(1/4+\varepsilon)}}} \right] \|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{(1/4+\varepsilon)})} \\
&\quad + \frac{\pi^\alpha C\, \|Y\|_{\mathcal{L}^q(\mathbb{P};\mathbb{R})} \left[T + T^2(1+T^2)^{(\alpha q/2-1)} \right]^{1/q}}{\min\{1,\sqrt{\nu}\}\sqrt{1/8-\alpha}} < \infty.
\end{aligned} \tag{298}$$

This completes the proof of Lemma 6.7. ◻

Lemma 6.8. *Assume the setting in Section 6.1, let $p, q \in [2,\infty)$, $\xi \in \mathcal{L}^p(\mathbb{P}; L^q(\lambda_{(0,1)};\mathbb{R}))$, $\theta \in [0, 1/4)$, and let $O\colon [0,T]\times\Omega \to L^q(\lambda_{(0,1)};\mathbb{R})$ and $\mathcal{O}^{M,N}\colon [0,T]\times\Omega \to P_N(H)$, $M,N \in \mathbb{N}$, be* *stochastic processes which satisfy for all $t \in [0,T]$, $M, N \in \mathbb{N}$ that $[O_t - e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\, dW_s$ and $[\mathcal{O}_t^{M,N} - P_N e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s \rfloor_{T/M})A}\, dW_s$. Then we have that*

$$\sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \left[M^\theta \left(\mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|_{L^q(\lambda_{(0,1)};\mathbb{R})}^p \Big] \right)^{1/p} \right] < \infty. \tag{299}$$

Proof of Lemma 6.8. Throughout this proof let $Y\colon \Omega \to \mathbb{R}$ be a standard normal random variable and let $\tilde{p} = \max\{p,q\}$, $\varepsilon \in (1/2 + 2\theta, 1)$, $(\mu_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $k \in \mathbb{N}$ that $\mu_k = \nu\pi^2 k^2$. Next note that Hölder's inequality, Fubini's theorem, and, e.g., Lemma 6.3 imply that for all $t \in [0,T]$, $M, N \in \mathbb{N}$ we have that

$$\begin{aligned}
&\mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|_{L^q(\lambda_{(0,1)};\mathbb{R})}^{\tilde{p}} \Big] \\
&= \mathbb{E}\left[\left| \int_0^1 \big| \underline{P_N O_t}(x) - \underline{\mathcal{O}_t^{M,N}}(x) \big|^q \, dx \right|^{\tilde{p}/q} \right] \\
&\le \mathbb{E}\left[\int_0^1 \big| \underline{P_N O_t}(x) - \underline{\mathcal{O}_t^{M,N}}(x) \big|^{\tilde{p}} \, dx \right] \\
&= \int_0^1 \mathbb{E}\Big[\big| \underline{P_N O_t}(x) - \underline{\mathcal{O}_t^{M,N}}(x) \big|^{\tilde{p}} \Big] \, dx \\
&= \mathbb{E}\big[|Y|^{\tilde{p}}\big] \int_0^1 \left(\mathbb{E}\Big[\big| \underline{P_N O_t}(x) - \underline{\mathcal{O}_t^{M,N}}(x) \big|^2 \Big] \right)^{\tilde{p}/2} dx.
\end{aligned} \tag{300}$$

This shows that for all $t \in [0,T]$, $M, N \in \mathbb{N}$ we have that

$$\begin{aligned}
&\mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|_{L^q(\lambda_{(0,1)};\mathbb{R})}^{\tilde{p}} \Big] \\
&= \mathbb{E}\big[|Y|^{\tilde{p}}\big] \int_0^1 \left(\mathbb{E}\left[\left| \sum_{k=1}^N \underline{e_k}(x) \int_0^t \big(e^{-\mu_k(t-s)} - e^{-\mu_k(t-\lfloor s \rfloor_{T/M})} \big) \langle e_k, dW_s \rangle_H \right|^2 \right] \right)^{\tilde{p}/2} dx \\
&= \mathbb{E}\big[|Y|^{\tilde{p}}\big] \int_0^1 \left(\sum_{k=1}^N \big| \underline{e_k}(x) \big|^2 \, \mathbb{E}\left[\left| \int_0^t \big(e^{-\mu_k(t-s)} - e^{-\mu_k(t-\lfloor s \rfloor_{T/M})} \big) \langle e_k, dW_s \rangle_H \right|^2 \right] \right)^{\tilde{p}/2} dx \\
&\le \mathbb{E}\big[|Y|^{\tilde{p}}\big] \left(2 \sum_{k=1}^N \mathbb{E}\left[\left| \int_0^t \big(e^{-\mu_k(t-s)} - e^{-\mu_k(t-\lfloor s \rfloor_{T/M})} \big) \langle e_k, dW_s \rangle_H \right|^2 \right] \right)^{\tilde{p}/2}.
\end{aligned} \tag{301}$$

Moreover, Itô's isometry, the fact that

$$\forall\, x \in [0,\infty), r \in [0,1]\colon x^r e^{-x} \le 1, \tag{302}$$

and the fact that

$$\forall\, x \in (0,\infty), r \in [0,1]\colon (1-e^{-x})/x^r \leq 1 \tag{303}$$

show that for all $t \in (0,T]$, $k, M \in \mathbb{N}$ we have that

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \left(e^{-\mu_k(t-s)} - e^{-\mu_k(t-\lfloor s\rfloor_{T/M})}\right)\langle e_k, dW_s\rangle_H\right|^2\right] = \int_0^t \left|e^{-\mu_k(t-s)} - e^{-\mu_k(t-\lfloor s\rfloor_{T/M})}\right|^2 ds \\
&= \int_0^t \left[e^{-\mu_k(t-s)}(e^{-\mu_k(t-s)} - e^{-\mu_k(t-\lfloor s\rfloor_{T/M})}) + e^{-\mu_k(t-\lfloor s\rfloor_{T/M})}(e^{-\mu_k(t-\lfloor s\rfloor_{T/M})} - e^{-\mu_k(t-s)})\right] ds \\
&\leq 2\int_0^t \left|e^{-\mu_k(t-s)} - e^{-\mu_k(t-\lfloor s\rfloor_{T/M})}\right| ds = 2\int_0^t e^{-\mu_k(t-s)}\left(1 - e^{-\mu_k(s-\lfloor s\rfloor_{T/M})}\right) ds \\
&\leq 2\int_0^t [\mu_k(t-s)]^{-\varepsilon}\left[\mu_k(s-\lfloor s\rfloor_{T/M})\right]^{2\theta} ds \leq \frac{2T^{2\theta}}{(\mu_k)^{(\varepsilon-2\theta)}M^{2\theta}}\int_0^t (t-s)^{-\varepsilon}\, ds \\
&\leq \frac{2T^{(1+2\theta-\varepsilon)}}{(\mu_k)^{(\varepsilon-2\theta)}(1-\varepsilon)M^{2\theta}}.
\end{aligned} \tag{304}$$

Combining this and (301) demonstrates that for all $t \in [0,T]$, $M, N \in \mathbb{N}$ we have that

$$\begin{aligned}
\mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|^{\tilde{p}}_{L^q(\lambda_{(0,1)};\mathbb{R})}\Big] &\leq \mathbb{E}\big[|Y|^{\tilde{p}}\big]\left(2\sum_{k=1}^N \frac{2T^{(1+2\theta-\varepsilon)}}{(\mu_k)^{(\varepsilon-2\theta)}(1-\varepsilon)M^{2\theta}}\right)^{\tilde{p}/2} \\
&= E\big[|Y|^{\tilde{p}}\big]\left(\frac{4T^{(1+2\theta-\varepsilon)}}{(1-\varepsilon)M^{2\theta}}\sum_{k=1}^N \frac{1}{(\mu_k)^{(\varepsilon-2\theta)}}\right)^{\tilde{p}/2}.
\end{aligned} \tag{305}$$

Hence, we obtain that for all $t \in [0,T]$, $M, N \in \mathbb{N}$ we have that

$$\|P_N O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))} \leq \|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})}\left[\frac{4T^{(1+2\theta-\varepsilon)}}{(1-\varepsilon)M^{2\theta}}\sum_{k=1}^N \frac{1}{(\mu_k)^{(\varepsilon-2\theta)}}\right]^{1/2}. \tag{306}$$

Hölder's inequality and the fact that $2(\varepsilon - 2\theta) > 1$ therefore prove that

$$
\begin{aligned}
&\sup_{M,N\in\mathbb{N}} \sup_{t\in(0,T]} \Big(M^\theta \, \| P_N O_t - \mathcal{O}_t^{M,N} \|_{\mathcal{L}^p(\mathbb{P}; L^q(\lambda_{(0,1)};\mathbb{R}))} \Big) \\
&\leq \sup_{M,N\in\mathbb{N}} \sup_{t\in(0,T]} \Big(M^\theta \, \| P_N O_t - \mathcal{O}_t^{M,N} \|_{\mathcal{L}^{\tilde{p}}(\mathbb{P}; L^q(\lambda_{(0,1)};\mathbb{R}))} \Big) \\
&\leq \sup_{M,N\in\mathbb{N}} \left(M^\theta \, \|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})} \left[\frac{4T^{(1+2\theta-\varepsilon)}}{(1-\varepsilon)M^{2\theta}} \sum_{k=1}^{N} \frac{1}{(\mu_k)^{(\varepsilon-2\theta)}} \right]^{1/2} \right) \\
&= \frac{2T^{(1/2+\theta-\varepsilon/2)} \|Y\|_{\mathcal{L}^{\tilde{p}}(\mathbb{P};\mathbb{R})}}{|\sqrt{\nu\pi}|^{(\varepsilon-2\theta)}\sqrt{1-\varepsilon}} \left[\sum_{k=1}^{\infty} \frac{1}{k^{2(\varepsilon-2\theta)}} \right]^{1/2} < \infty.
\end{aligned}
\tag{307}
$$

The proof of Lemma 6.8 is thus completed. □

Corollary 6.9. *Assume the setting in Section 6.1 and let $p, q \in [2,\infty)$, $\theta \in [1/4 - 1/2q, 1/4)$, $\xi \in \cup_{r\in(1/4,\infty)\cap[2\theta,\infty)} \mathcal{L}^p(\mathbb{P}; H_r)$. Then there exist stochastic processes $O\colon [0,T]\times\Omega \to L^q(\lambda_{(0,1)};\mathbb{R})$ and $\mathcal{O}^{M,N}\colon [0,T]\times\Omega \to P_N(H)$, $M,N\in\mathbb{N}$, with continuous sample paths which satisfy*

(i) that for all $t\in[0,T]$ we have that $[O_t - e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\, dW_s$,

(ii) that for all $M,N\in\mathbb{N}$, $t\in[0,T]$ we have that $[\mathcal{O}_t^{M,N} - P_N e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s\rfloor_{T/M})A}\, dW_s$, and

(iii) that for all $\gamma\in[0,2\theta]\cap[0,1/4)$ we have that

$$
\begin{aligned}
&\sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \left[M^\theta \Big(\mathbb{E}\big[\|P_N(O_t - O_{\lfloor t\rfloor_{T/M}})\|^p_{L^q(\lambda_{(0,1)};\mathbb{R})} + \|P_N O_t - \mathcal{O}_t^{M,N}\|^p_{L^q(\lambda_{(0,1)};\mathbb{R})} \big] \Big)^{1/p} \right] \\
&+ \sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big[\|\mathcal{O}_t^{M,N}\|^p_{H_\gamma}\big] + \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} N^{2\theta} \Big(\mathbb{E}\big[\|O_t - P_N O_t\|^p_{L^q(\lambda_{(0,1)};\mathbb{R})} \big] \Big)^{1/p} \\
&+ \sup_{M,N\in\mathbb{N}} \mathbb{E}\big[\sup\nolimits_{t\in[0,T]} \|\mathcal{O}_t^{M,N}\|^p_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \big] < \infty.
\end{aligned}
\tag{308}
$$

Proof of Corollary 6.9. First of all, note that Lemma 6.5 implies that there exists a stochastic process $O\colon [0,T]\times\Omega \to L^q(\lambda_{(0,1)};\mathbb{R})$ with continuous sample paths which satisfies for all $t\in[0,T]$ that $[O_t - e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\, dW_s$ and

$$
\begin{aligned}
&\sup_{N\in\mathbb{N}} \sup_{0\leq s<t\leq T} \frac{\|P_N(O_t - O_s)\|_{\mathcal{L}^p(\mathbb{P}; L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta} \\
&\quad + \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \Big(N^{2\theta} \, \|O_t - P_N O_t\|_{\mathcal{L}^p(\mathbb{P}; L^q(\lambda_{(0,1)};\mathbb{R}))} \Big) < \infty.
\end{aligned}
\tag{309}
$$

Hence, we obtain that

$$\begin{aligned}
&\sup_{M,N\in\mathbb{N}}\sup_{t\in[0,T]}\Big[M^\theta \,\|P_N(O_t-O_{\lfloor t\rfloor_{T/M}})\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\Big]\\
&\leq\left[\sup_{M\in\mathbb{N}}\sup_{t\in[0,T]} M^\theta\,(t-\lfloor t\rfloor_{T/M})^\theta\right]\left[\sup_{N\in\mathbb{N}}\sup_{0\leq s<t\leq T}\frac{\|P_N(O_t-O_s)\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta}\right]\\
&\leq T^\theta\left[\sup_{N\in\mathbb{N}}\sup_{0\leq s<t\leq T}\frac{\|P_N(O_t-O_s)\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}}{(t-s)^\theta}\right]<\infty.
\end{aligned}\tag{310}$$

Next note that the assumption that $\xi\in\cup_{r\in(1/4,\infty)\cap[2\theta,\infty)}\mathcal{L}^p(\mathbb{P};H_r)$ and Lemma 6.6 (with $p=p$, $\theta=\vartheta$, $\xi=\Omega\ni\omega\mapsto\xi(\omega)\in H_\vartheta$ for $\vartheta\in[0,2\theta]\cap[0,1/4)$ in the notation of Lemma 6.6) yield that there exist stochastic processes $\mathcal{O}^{M,N}\colon[0,T]\times\Omega\to P_N(H)$, $M,N\in\mathbb{N}$, with continuous sample paths which satisfy for all $t\in[0,T]$, $M,N\in\mathbb{N}$ that

$$[\mathcal{O}_t^{M,N}-P_Ne^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)}=\int_0^t P_Ne^{(t-\lfloor s\rfloor_{T/M})A}\,dW_s\tag{311}$$

and which satisfy for all $\vartheta\in[0,2\theta]\cap[0,1/4)$ that

$$\sup_{M,N\in\mathbb{N}}\sup_{t\in[0,T]}\|\mathcal{O}_t^{M,N}\|_{\mathcal{L}^p(\mathbb{P};H_\vartheta)}<\infty.\tag{312}$$

Next observe that (311) together with the assumption that $\xi\in\cup_{r\in(1/4,\infty)\cap[2\theta,\infty)}\mathcal{L}^p(\mathbb{P};H_r)$ together with (ii) enables us to apply Lemma 6.7 to obtain that

$$\sup_{M,N\in\mathbb{N}}\big\|\sup\nolimits_{t\in[0,T]}\|\mathcal{O}_t^{M,N}\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})}\big\|_{\mathcal{L}^p(\mathbb{P};\mathbb{R})}<\infty.\tag{313}$$

Moreover, note that the fact that $\mathbb{E}\big[\|\xi\|^p_{H_{(1/4-1/2q)}}\big]<\infty$ and the fact that

$$H_{(1/4-1/2q)}\subseteq W^{1/2-1/q,2}((0,1),\mathbb{R})\subseteq W^{0,q}((0,1),\mathbb{R})=L^q(\lambda_{(0,1)};\mathbb{R})\tag{314}$$

continuously (cf., e.g., Da Prato & Zabcyk [6, (A.46) in Section A.5.2] and Lunardi [20]) prove that

$$\begin{aligned}
&\|\xi\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\\
&\leq\left[\sup_{v\in H_{(1/4-1/2q)}\setminus\{0\}}\frac{\|v\|_{L^q(\lambda_{(0,1)};\mathbb{R})}}{\|v\|_{H_{(1/4-1/2q)}}}\right]\|\xi\|_{\mathcal{L}^p(\mathbb{P};H_{(1/4-1/2q)})}<\infty.
\end{aligned}\tag{315}$$

This together with (309) and (311) allows us to apply Lemma 6.8 to obtain that

$$\sup_{M,N\in\mathbb{N}}\sup_{t\in[0,T]}\Big(M^\theta\|P_NO_t-\mathcal{O}_t^{M,N}\|_{\mathcal{L}^p(\mathbb{P};L^q(\lambda_{(0,1)};\mathbb{R}))}\Big)<\infty.\tag{316}$$

Combining this with (309)–(313) and (315) completes the proof of Corollary 6.9. □

6.4 Strong convergence rates for numerical approximations of stochastic Allen-Cahn equations

Lemma 6.10. *Assume the setting in Section 6.1, let* $p \in [2, \infty)$, $\vartheta \in (0, \infty)$, $\theta \in [1/6, 1/4)$, $\xi \in \cup_{r \in (1/4, \infty) \cap [2\theta, \infty)} \mathcal{L}^{16p \max\{3, \vartheta\}}(\mathbb{P}; H_r)$, $\gamma \in (1/6, 1/4)$, $\chi \in (0, \gamma/3 - 1/18]$, *let* $X\colon [0,T] \times \Omega \to L^6(\lambda_{(0,1)}; \mathbb{R})$ *be a stochastic process with continuous sample paths which satisfies for all* $t \in [0,T]$ *that* $[X_t - e^{tA}\xi - \int_0^t e^{(t-s)A} F(X_s)\, ds]_{\mathbb{P}, \mathcal{B}(H)} = \int_0^t e^{(t-s)A}\, dW_s$ *and* $\sup_{s \in [0,T]} \mathbb{E}\big[\|X_s\|^{12p}_{L^6(\lambda_{(0,1)}; \mathbb{R})}\big] < \infty$, *and let* $\mathcal{X}^{M,N}\colon [0,T] \times \Omega \to H_\gamma$, $M, N \in \mathbb{N}$, *and* $\mathcal{O}^{M,N}\colon [0,T] \times \Omega \to P_N(H)$, $M, N \in \mathbb{N}$, *be stochastic processes which satisfy for all* $M, N \in \mathbb{N}$, $t \in [0,T]$ *that* $[\mathcal{O}^{M,N}_t - P_N e^{tA}\xi]_{\mathbb{P}, \mathcal{B}(H)} = \int_0^t P_N e^{(t - \lfloor s \rfloor_{T/M})A}\, dW_s$ *and*

$$\mathbb{P}\Big(\mathcal{X}^{M,N}_t = \int_0^t P_N\, e^{(t-s)A}\, \mathbb{1}^{\Omega}_{\{\|\mathcal{X}^{M,N}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}^{M,N}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} \leq (M/T)^\chi\}}\, F(\mathcal{X}^{M,N}_{\lfloor s \rfloor_{T/M}})\, ds + \mathcal{O}^{M,N}_t\Big) = 1. \tag{317}$$

Then we have

(i) that $\sup_{M,N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\big[\|\mathcal{X}^{M,N}_t\|^{4p \max\{3, \vartheta\}}_{H_\gamma}\big] < \infty$ *and*

(ii) that there exists a real number $C \in \mathbb{R}$ *such that for all* $M, N \in \mathbb{N}$ *we have that*

$$\sup_{t \in [0,T]} \Big(\mathbb{E}\big[\|X_t - \mathcal{X}^{M,N}_t\|^p_H\big]\Big)^{1/p} \leq C(M^{-\min\{\vartheta\chi, \theta\}} + N^{-2\theta}). \tag{318}$$

Proof of Lemma 6.10. Throughout this proof let $(V, \|\cdot\|_V) = (L^6(\lambda_{(0,1)}; \mathbb{R}), \|\cdot\|_{L^6(\lambda_{(0,1)}; \mathbb{R})})$ and let $\epsilon \in (0,1)$, $C \in [32/\epsilon \max\{|a_2|^2/(|a_3| + \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3)), |a_3|\}, \infty)$ be real numbers. Note that, e.g., [3, Lemma 6.7] proves that for all $v, w \in L^{18}(\lambda_{(0,1)}; \mathbb{R})$ with $v - w \in H_1$ we have that

$$\begin{aligned} &\langle v - w, A(v-w) + F(v) - F(w)\rangle_H \\ &\leq 2 \max\{1, 1/(|a_3| + \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3))\} \max\Big\{1, \max_{k \in \{1,2\}} \big[k|a_k|\big]^2\Big\} \|v - w\|^2_H. \end{aligned} \tag{319}$$

The fact that $H_1 \subseteq L^{18}(\lambda_{(0,1)}; \mathbb{R})$ therefore implies that for all $v, w \in H_1$ we have that

$$\begin{aligned} &\langle v - w, Av + F(v) - Aw - F(w)\rangle_H \\ &\leq 2 \max\{1, 1/(|a_3| + \mathbb{1}^{\mathbb{R}}_{\{0\}}(a_3))\} \max\Big\{1, \max_{k \in \{1,2\}} \big[k|a_k|\big]^2\Big\} \|v - w\|^2_H. \end{aligned} \tag{320}$$

Combining this, Lemma 6.1 (with $\epsilon = \epsilon$, $c = C$ in the notation of Lemma 6.1) and Lemma 6.2 ensures that there exit a real number $c \in (0, \infty)$ such that for all $N \in \mathbb{N}$, $v, w \in P_N(H)$, $x \in \mathcal{C}([0,T], H_1)$, $t \in [0,T]$ we have that

$$\langle v - w, Av + F(v) - Aw - F(w)\rangle_H \leq c\|v - w\|^2_H, \tag{321}$$

$$
\begin{aligned}
&\langle v, P_N F(v + x_t)\rangle_{H_{1/2}} + C \left[\sup_{s\in[0,T]} \|x_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1 \right] \langle v, F(v + x_t)\rangle_H \\
&\leq \epsilon \|v\|^2_{H_1} + c\|v\|^2_{H_{1/2}} + c\,C \left[\sup_{s\in[0,T]} \|x_s\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1 \right] \|v\|^2_H \\
&\quad + c \left[\sup_{s\in[0,T]} \|x_s\|^8_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1 \right],
\end{aligned} \tag{322}
$$

and

$$
\|F(v) - F(w)\|^2_H \leq c\|v - w\|^2_V \left(1 + \|v\|^4_V + \|w\|^4_V\right). \tag{323}
$$

Moreover, note that Corollary 6.9 (with $p = 16p\max\{3, \vartheta\}$, $q = 6$, $\theta = \theta$, $\xi = \xi$ in the notation of Corollary 6.9) and the fact that $2\theta > \gamma$ imply that there exist stochastic processes $O\colon [0,T]\times\Omega \to V$ and $\tilde{\mathcal{O}}^{M,N}\colon [0,T]\times\Omega \to P_N(H)$, $M, N \in \mathbb{N}$, with continuous sample paths which satisfy that

$$
\begin{aligned}
&\sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \Big(M^\theta \big\| \|P_N(O_t - O_{\lfloor t\rfloor_{T/M}})\|_V + \|P_N O_t - \tilde{\mathcal{O}}^{M,N}_t\|_V \big\|_{\mathcal{L}^{16p\max\{3,\vartheta\}}(\mathbb{P};\mathbb{R})} \Big) \\
&+ \sup_{M,N\in\mathbb{N}} \sup_{t\in[0,T]} \big\| \|\tilde{\mathcal{O}}^{M,N}_t\|_{H_\gamma} + N^{2\theta}\|O_t - P_N O_t\|_V \big\|_{\mathcal{L}^{16p\max\{3,\vartheta\}}(\mathbb{P};\mathbb{R})} \\
&+ \sup_{M,N\in\mathbb{N}} \big\| \sup\nolimits_{t\in[0,T]} \|\tilde{\mathcal{O}}^{M,N}_t\|_{L^\infty(\lambda_{(0,1)};\mathbb{R})} \big\|_{\mathcal{L}^{16p\max\{3,\vartheta\}}(\mathbb{P};\mathbb{R})} < \infty,
\end{aligned} \tag{324}
$$

that for all $t \in [0,T]$ we have that

$$
[O_t - e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \textstyle\int_0^t e^{(t-s)A}\, dW_s, \tag{325}
$$

and that for all $t \in [0,T]$, $M, N \in \mathbb{N}$ we have that

$$
[\tilde{\mathcal{O}}^{M,N}_t - P_N\, e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \textstyle\int_0^t P_N\, e^{(t-\lfloor s\rfloor_{T/M})A}\, dW_s. \tag{326}
$$

Observe that (325) and the fact that $\forall\, t \in [0,T], \forall\, M, N \in \mathbb{N}\colon \mathbb{P}(\mathcal{O}^{M,N}_t = \tilde{\mathcal{O}}^{M,N}_t) = 1$ guarantee for all $t \in [0,T]$, $M, N \in \mathbb{N}$ that

$$
\begin{aligned}
&\mathbb{P}\Big(X_t = \textstyle\int_0^t e^{(t-s)A} F(X_s)\, ds + O_t \Big) \\
&= \mathbb{P}\Big(\mathcal{X}^{M,N}_t = \textstyle\int_0^t P_N\, e^{(t-s)A}\, \mathbb{1}^{\Omega}_{\{\|\mathcal{X}^{M,N}_{\lfloor s\rfloor_{T/M}}\|_{H_\gamma} + \|\tilde{\mathcal{O}}^{M,N}_{\lfloor s\rfloor_{T/M}}\|_{H_\gamma} \leq (M/T)^\chi\}}\, F(\mathcal{X}^{M,N}_{\lfloor s\rfloor_{T/M}})\, ds + \tilde{\mathcal{O}}^{M,N}_t \Big) \\
&= 1.
\end{aligned} \tag{327}
$$

Combining (321)–(323) and (324)–(327) allows us to apply Corollary 5.6 (with $H = H$, $\mathbb{H} = \{e_k\colon k \in \mathbb{N}\}$, $T = T$, $c = c$, $\varphi = 4$, $\epsilon = \epsilon$, $\rho = 1/6$, $\gamma = \gamma$, $\chi = \chi$, $\mathcal{D} = \{\{e_1\}, \{e_1, e_2\}, \{e_1, e_2, e_3\}, \ldots\}$,

$\mu(e_N) = -\nu\pi^2 N^2$, $A = A$, $H_r = H_r$, $V = V$, $F = F$, $\phi = C([0,T], H_1) \ni w \mapsto C[\sup_{t\in[0,T]} \|w_t\|^4_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1] \in [0,\infty)$, $\Phi = C([0,T], H_1) \ni w \mapsto c[\sup_{t\in[0,T]} \|w_t\|^8_{L^\infty(\lambda_{(0,1)};\mathbb{R})} + 1] \in [0,\infty)$, $P_{\{e_1,e_2,\dots,e_N\}}(v) = \sum_{k=1}^N \langle e_k, v\rangle_H \, e_k$, $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P})$, $X = X$, $O = O$, $\mathcal{O}^{M,\{e_1,e_2,\dots,e_N\}} = [0,T] \times \Omega \ni (\omega, t) \mapsto \tilde{\mathcal{O}}^{M,N}_t(\omega) \in H_1$, $\mathcal{X}^{M,\{e_1,e_2,\dots,e_N\}} = \mathcal{X}^{M,N}$, $\theta = \theta$, $\vartheta = \vartheta$, $p = p$, $\varrho = \theta$ for $N \in \mathbb{N}$, $r \in \mathbb{R}$, $v \in H$ in the notation of Corollary 5.6) to obtain that (i) holds and that there exists a real number $K \in (0,\infty)$ such that for all $M, N \in \mathbb{N}$ we have that

$$\begin{aligned} &\sup_{t\in[0,T]} \|X_t - \mathcal{X}^{M,N}_t\|_{\mathcal{L}^p(\mathbb{P};H)} \\ &\leq K \left[M^{-\min\{\vartheta\chi,\theta\}} + \|(-A)^{-\theta}(\mathrm{Id}_H - P_N)\|_{L(H)} + \sup_{t\in[0,T]} \|O_t - P_N O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right]. \end{aligned} \tag{328}$$

The fact that

$$\forall\, N \in \mathbb{N}\colon \|(-A)^{-\theta}(\mathrm{Id}_H - P_N)\|_{L(H)} = (\nu\pi^2(N+1)^2)^{-\theta} \tag{329}$$

hence yields that for all $M, N \in \mathbb{N}$ we have that

$$\begin{aligned} &\sup_{t\in[0,T]} \|X_t - \mathcal{X}^{M,N}_t\|_{\mathcal{L}^p(\mathbb{P};H)} \\ &\leq K \left[M^{-\min\{\vartheta\chi,\theta\}} + N^{-2\theta} \left(\frac{N^{2\theta}}{\nu^\theta \pi^{2\theta}(N+1)^{2\theta}} + \sup_{t\in[0,T]} \left[N^{2\theta} \|O_t - P_N O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right] \right) \right] \\ &\leq K \left[M^{-\min\{\vartheta\chi,\theta\}} + N^{-2\theta} \left(\frac{1}{\nu^\theta \pi^{2\theta}} + \sup_{t\in[0,T]} \left[N^{2\theta} \|O_t - P_N O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right] \right) \right] \\ &\leq K \max\!\left\{ 1, \frac{1}{\nu^\theta} + \sup_{t\in[0,T]} \left[N^{2\theta} \|O_t - P_N O_t\|_{\mathcal{L}^{2p}(\mathbb{P};V)} \right] \right\} \left[M^{-\min\{\vartheta\chi,\theta\}} + N^{-2\theta} \right]. \end{aligned} \tag{330}$$

Combining this with (324) completes the proof of Lemma 6.10. □

Corollary 6.11. *Assume the setting in Section 6.1, let* $\xi \in \cap_{p\in[1,\infty)} \mathcal{L}^p(\mathbb{P}; H_{1/2})$, $\gamma \in (1/6, 1/4)$, $\chi \in (0, \gamma/3 - 1/18]$, *and let* $\mathcal{X}^{M,N}\colon [0,T] \times \Omega \to P_N(H)$, $M, N \in \mathbb{N}$, *and* $\mathcal{O}^{M,N}\colon [0,T] \times \Omega \to P_N(H)$, $M, N \in \mathbb{N}$, *be stochastic processes which satisfy for all* $M, N \in \mathbb{N}$, $t \in [0,T]$ *that* $[\mathcal{O}^{M,N}_t - P_N e^{tA}\xi]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t - \lfloor s \rfloor_{T/M})A} \, dW_s$ *and*

$$\mathbb{P}\Big(\mathcal{X}^{M,N}_t = \int_0^t P_N \, e^{(t-s)A} \, \mathbb{1}^{\Omega}_{\{\|\mathcal{X}^{M,N}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} + \|\mathcal{O}^{M,N}_{\lfloor s \rfloor_{T/M}}\|_{H_\gamma} \leq (M/T)^\chi\}} \, F(\mathcal{X}^{M,N}_{\lfloor s \rfloor_{T/M}}) \, ds + \mathcal{O}^{M,N}_t \Big) = 1. \tag{331}$$

Then

(i) we have that there exists an up to indistinguishability unique stochastic process $X\colon [0,T]\times\Omega \to L^6(\lambda_{(0,1)};\mathbb{R})$ *with continuous sample paths which satisfies for all* $t\in[0,T]$, $p\in(0,\infty)$ *that* $\sup_{s\in[0,T]}\mathbb{E}\big[\|X_s\|^p_{L^6(\lambda_{(0,1)};\mathbb{R})}\big]<\infty$ *and*

$$\big[X_t - e^{tA}\xi - \textstyle\int_0^t e^{(t-s)A}F(X_s)\,ds\big]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\,dW_s, \tag{332}$$

(ii) we have for all $p\in(0,\infty)$ *that* $\sup_{r\in(-\infty,\gamma]}\sup_{M,N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\|\mathcal{X}^{M,N}_t\|^p_{H_r}\big]<\infty$, *and*

(iii) we have for all $p\in(0,\infty)$, $r\in[0,1/4)$ *that there exists a real number* $C\in\mathbb{R}$ *such that for all* $M,N\in\mathbb{N}$ *it holds that*

$$\sup_{t\in[0,T]}\Big(\mathbb{E}\big[\|X_t-\mathcal{X}^{M,N}_t\|^p_H\big]\Big)^{1/p} \leq C(M^{-r}+N^{-2r}). \tag{333}$$

Proof of Corollary 6.11. Note that under the assumptions of Corollary 6.11 it is well known (cf., e.g., [21, Theorem 3.4.1 (ii) in Section 3.4, Lemma 2.4.2 in Section 2, and Definition 2.7 in Section 2], [14, Lemma 28 in Section 3.2], Lemma 6.2, the hypothesis that $\xi\in\cap_{p\in[1,\infty)}\mathcal{L}^p(\mathbb{P};H_{1/2})$, and [6, (A.46) in Section A.5.2]) that there exist stochastic processes $\tilde{X}_q\colon[0,T]\times\Omega\to L^q(\lambda_{(0,1)};\mathbb{R})$, $q\in\{6,7,8,\ldots\}$, with continuous sample paths which satisfy for all $t\in[0,T]$, $q\in\{6,7,8,\ldots\}$ that $\sup_{s\in[0,T]}\mathbb{E}\big[\|\tilde{X}_{q,s}\|^q_{L^q(\lambda_{(0,1)};\mathbb{R})}\big]<\infty$ and

$$\big[\tilde{X}_{q,t} - e^{tA}\xi - \textstyle\int_0^t e^{(t-s)A}F(\tilde{X}_{q,s})\,ds\big]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\,dW_s. \tag{334}$$

Combining this with (331), the assumption that $\xi\in\cap_{p\in[1,\infty)}\mathcal{L}^p(\mathbb{P};H_{1/2})$, and, e.g., Lemma 2.2 in Andersson et al. [2] allows us to apply Lemma 6.10 (with $p=p$, $\vartheta=\vartheta$, $\theta=r$, $\xi=\xi$, $\gamma=\gamma$, $\chi=\chi$, $X=[0,T]\times\Omega\ni(t,\omega)\mapsto\tilde{X}_{p,t}(\omega)\in L^6(\lambda_{(0,1)};\mathbb{R})$, $\mathcal{X}^{M,N}=[0,T]\times\Omega\ni(t,\omega)\mapsto\mathcal{X}^{M,N}_t(\omega)\in H_\gamma$, $\mathcal{O}^{M,N}=\mathcal{O}^{M,N}$ for $p\in\{6,7,8,\ldots\}$, $\vartheta\in[r/\chi,\infty)$, $r\in[1/6,1/4)$, $M,N\in\mathbb{N}$ in the notation of Lemma 6.10) to obtain that there exists a function $C\colon[2,\infty)\times[1/6,1/4)\to\mathbb{R}$ such that for all $p\in\{6,7,8,\ldots\}$, $r\in[1/6,1/4)$, $M,N\in\mathbb{N}$ we have that

$$\sup_{t\in[0,T]}\|\tilde{X}_{p,t}-\mathcal{X}^{M,N}_t\|_{\mathcal{L}^p(\mathbb{P};H)}\leq C_{p,r}(M^{-r}+N^{-2r}). \tag{335}$$

Next observe that the triangle inequality ensures that for all $p_1,p_2\in\{6,7,8,\ldots\}$ with $p_1\leq p_2$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\tilde{X}_{p_1,t}-\tilde{X}_{p_2,t}\|_{\mathcal{L}^{p_1}(\mathbb{P};H)}\\
&=\limsup_{M\to\infty}\limsup_{N\to\infty}\sup_{t\in[0,T]}\|\tilde{X}_{p_1,t}-\mathcal{X}^{M,N}_t+\mathcal{X}^{M,N}_t-\tilde{X}_{p_2,t}\|_{\mathcal{L}^{p_1}(\mathbb{P};H)}\\
&\leq\limsup_{M\to\infty}\limsup_{N\to\infty}\sup_{t\in[0,T]}\Big[\|\tilde{X}_{p_1,t}-\mathcal{X}^{M,N}_t\|_{\mathcal{L}^{p_1}(\mathbb{P};H)}+\|\tilde{X}_{p_2,t}-\mathcal{X}^{M,N}_t\|_{\mathcal{L}^{p_1}(\mathbb{P};H)}\Big].
\end{aligned} \tag{336}$$

Hölder's inequality and (335) hence prove that for all $p_1, p_2 \in \{6, 7, 8, \ldots\}$, $r \in (1/6, 1/4)$ with $p_1 \leq p_2$ we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \|\tilde{X}_{p_1,t} - \tilde{X}_{p_2,t}\|_{\mathcal{L}^{p_1}(\mathbb{P};H)} \\
&\leq \limsup_{M\to\infty} \limsup_{N\to\infty} \sup_{t\in[0,T]} \Big[\|\tilde{X}_{p_1,t} - \mathcal{X}_t^{M,N}\|_{\mathcal{L}^{p_1}(\mathbb{P};H)} + \|\tilde{X}_{p_2,t} - \mathcal{X}_t^{M,N}\|_{\mathcal{L}^{p_2}(\mathbb{P};H)}\Big] \\
&\leq C_{p_1,r} \limsup_{M\to\infty} \limsup_{N\to\infty} \big[M^{-r} + N^{-2r}\big] + C_{p_2,r} \limsup_{M\to\infty} \limsup_{N\to\infty} \big[M^{-r} + N^{-2r}\big] = 0.
\end{aligned} \tag{337}$$

This implies that for all $q_1, q_2 \in \{6, 7, 8, \ldots\}$, $t \in [0, T]$ we have that

$$\mathbb{P}\big(\tilde{X}_{q_1,t} = \tilde{X}_{q_2,t}\big) = 1. \tag{338}$$

In the next step let $\tilde{\Omega} \subseteq \Omega$ be the set given by

$$\tilde{\Omega} = \big\{\omega \in \Omega \colon \big(\forall\, q_1, q_2 \in \{6, 7, 8, \ldots\}, t \in [0, T] \colon \tilde{X}_{q_1,t}(\omega) = \tilde{X}_{q_2,t}(\omega)\big)\big\} \tag{339}$$

and let $X \colon [0, T] \times \Omega \to L^6(\lambda_{(0,1)}; \mathbb{R})$ be the function which satisfies for all $t \in [0, T], \omega \in \Omega$ that

$$X_t(\omega) = \begin{cases} \tilde{X}_{6,t}(\omega) & \colon \omega \in \tilde{\Omega} \\ 0 & \colon \omega \in \Omega \setminus \tilde{\Omega}. \end{cases} \tag{340}$$

Note that the fact that every $q \in \{6, 7, 8, \ldots\}$ we have that $\tilde{X}_q \colon [0, T] \times \Omega \to L^q(\lambda_{(0,1)}; \mathbb{R})$ has continuous sample paths shows that

$$\begin{aligned}
\tilde{\Omega} &= \big\{\omega \in \Omega \colon \big(\forall\, q_1, q_2 \in \{6, 7, 8, \ldots\}, t \in [0, T] \cap \mathbb{Q} \colon \tilde{X}_{q_1,t}(\omega) = \tilde{X}_{q_2,t}(\omega)\big)\big\} \\
&= \cap_{q_1,q_2\in\{6,7,8,\ldots\}} \cap_{t\in[0,T]\cap\mathbb{Q}} \big\{\tilde{X}_{q_1,t}(\omega) = \tilde{X}_{q_2,t}(\omega)\big\}.
\end{aligned} \tag{341}$$

Combining this with (338) ensures that that

$$\tilde{\Omega} \in \mathcal{F} \qquad \text{and} \qquad \mathbb{P}(\tilde{\Omega}) = 1. \tag{342}$$

Next observe that the fact that $\tilde{X} \colon [0, T] \times \Omega \to L^6(\lambda_{(0,1)}; \mathbb{R})$ has continuous sample paths demonstrates that X has continuous sample paths. Moreover, note that (339), (340), and (342) ensure that for all $q \in \{6, 7, 8, \ldots\}$ we have that

$$\mathbb{P}(\forall\, t \in [0, T] \colon \tilde{X}_{q,t} = X_t) = 1. \tag{343}$$

Combining this with (334) demonstrates that for all $t \in [0, T]$, $p \in (0, \infty)$ we have that

$$\sup_{s\in[0,T]} \mathbb{E}\big[\|X_s\|^p_{L^6(\lambda_{(0,1)};\mathbb{R})}\big] < \infty \tag{344}$$

and

$$\big[X_t - e^{tA}\xi - \textstyle\int_0^t e^{(t-s)A}F(X_s)\,ds\big]_{\mathbb{P},\mathcal{B}(H)} = \textstyle\int_0^t e^{(t-s)A}\,dW_s. \tag{345}$$

This, the fact that $X\colon [0,T]\times\Omega \to L^6(\lambda_{(0,1)};\mathbb{R})$ has continuous sample paths, and again Lemma 6.10 (with $p = p$, $\vartheta = \vartheta$, $\theta = \theta$, $\xi = \xi$, $\gamma = \gamma$, $\chi = \chi$, $X = X$, $\mathcal{X}^{M,N} = [0,T]\times\Omega \ni (t,\omega) \mapsto \mathcal{X}_t^{M,N}(\omega) \in H_\gamma$, $\mathcal{O}^{M,N} = \mathcal{O}^{M,N}$ for $p \in [2,\infty)$, $\vartheta \in [\theta/\chi,\infty)$, $\theta \in [1/6, 1/4)$, $M, N \in \mathbb{N}$ in the notation of Lemma 6.10) complete the proof of Corollary 6.11. □

7 Lower and upper bounds for strong approximation errors of numerical approximations of linear stochastic heat equations

7.1 Setting

Consider the notation in Section 1.1, let $T,\nu \in (0,\infty)$, $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (L^2(\lambda_{(0,1)};\mathbb{R}), \langle\cdot,\cdot\rangle_{L^2(\lambda_{(0,1)};\mathbb{R})}, \|\cdot\|_{L^2(\lambda_{(0,1)};\mathbb{R})})$, $(e_n)_{n\in\mathbb{N}} \subseteq H$, $(P_n)_{n\in\mathbb{N}\cup\{\infty\}} \subseteq L(H)$ satisfy for all $m\in\mathbb{N}$, $n \in \mathbb{N}\cup\{\infty\}$, $v \in H$ that $e_m = [(\sqrt{2}\sin(m\pi x))_{x\in(0,1)}]_{\lambda_{(0,1)},\mathcal{B}(\mathbb{R})}$ and $P_n(v) = \sum_{k=1}^n \langle e_k, v\rangle_H\, e_k$, let $A\colon D(A)\subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H times the real number ν, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(W_t)_{t\in[0,T]}$ be an Id_H-cylindrical Wiener process, and let $O\colon [0,T]\times\Omega\to H$ and $\mathcal{O}^{M,N}\colon [0,T]\times\Omega \to H$, $M,N\in\mathbb{N}$, be stochastic processes which satisfy for all $t\in[0,T]$, $M\in\mathbb{N}$, $N\in\mathbb{N}\cup\{\infty\}$ that $[O_t]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t e^{(t-s)A}\,dW_s$ and $[\mathcal{O}_t^{M,N}]_{\mathbb{P},\mathcal{B}(H)} = \int_0^t P_N e^{(t-\lfloor s\rfloor_{T/M})A}\,dW_s$.

7.2 Lower and upper bounds for Hilbert-Schmidt norms of Hilbert-Schmidt operators

Lemma 7.1. *Assume the setting in Section 7.1 and let $N\in\mathbb{N}\cup\{\infty\}$, $s_1, s_2, t\in[0,\infty)$ with $s_1 \le s_2$. Then*

(i) we have that

$$\left(\sum_{n=1}^\infty \|P_N e^{s_1 A}(\mathrm{Id}_H - e^{tA})\,e_n\|_H^2\right)^{1/2} \ge \left(\sum_{n=1}^\infty \|P_N e^{s_2 A}(\mathrm{Id}_H - e^{tA})\,e_n\|_H^2\right)^{1/2} \tag{346}$$

and

(ii) we have that

$$\left(\sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_1 A})\, e_n\|_H^2\right)^{1/2} \leq \left(\sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_2 A})\, e_n\|_H^2\right)^{1/2}. \tag{347}$$

Proof of Lemma 7.1. Throughout this proof let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that $\mu_n = \nu\pi^2 n^2$. Next observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \|P_N e^{s_1 A}(\mathrm{Id}_H - e^{tA})\, e_n\|_H^2 \\
&= \sum_{n=1}^{N} \|e^{s_1 A}(\mathrm{Id}_H - e^{tA})\, e_n\|_H^2 = \sum_{n=1}^{N} \|e^{-\mu_n s_1}(1 - e^{-\mu_n t})\, e_n\|_H^2 \\
&= \sum_{n=1}^{N} |e^{-\mu_n s_1}(1 - e^{-\mu_n t})|^2 \geq \sum_{n=1}^{N} |e^{-\mu_n s_2}(1 - e^{-\mu_n t})|^2 \\
&= \sum_{n=1}^{N} \|e^{s_2 A}(\mathrm{Id}_H - e^{tA})\, e_n\|_H^2 = \sum_{n=1}^{\infty} \|P_N e^{s_2 A}(\mathrm{Id}_H - e^{tA})\, e_n\|_H^2.
\end{aligned} \tag{348}$$

This establishes (i). Moreover, note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_1 A})\, e_n\|_H^2 \\
&= \sum_{n=1}^{N} \|e^{tA}(\mathrm{Id}_H - e^{s_1 A})\, e_n\|_H^2 = \sum_{n=1}^{N} \|e^{-\mu_n t}(1 - e^{-\mu_n s_1})\, e_n\|_H^2 \\
&= \sum_{n=1}^{N} |e^{-\mu_n t}(1 - e^{-\mu_n s_1})|^2 \leq \sum_{n=1}^{N} |e^{-\mu_n t}(1 - e^{-\mu_n s_2})|^2 \\
&= \sum_{n=1}^{N} \|e^{tA}(\mathrm{Id}_H - e^{s_2 A})\, e_n\|_H^2 = \sum_{n=1}^{\infty} \|P_N e^{tA}(\mathrm{Id}_H - e^{s_2 A})\, e_n\|_H^2.
\end{aligned} \tag{349}$$

The proof of Lemma 7.1 is thus completed. ∎

Lemma 7.2. *Assume the setting in Section 7.1 and let $N \in \mathbb{N} \cup \{\infty\}$, $t \in (0, T]$. Then*

$$\begin{aligned}
&\left[\int_0^{\max\{0, t(N+1)^2 - (1+\sqrt{t})^2\}} \frac{(1 - e^{-\nu\pi^2 \min\{1, tN^2\}})^2}{2\nu\pi^2 (x + [1 + \sqrt{T}]^2)^{3/2}}\, dx\right]^{1/2} \\
&\qquad \leq \|P_N(-\sqrt{t}A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|_{HS(H)} \leq \left[\tfrac{1}{\pi\sqrt{\nu}} + \tfrac{1}{\nu\pi^2} + 4\pi\sqrt{\nu}\right]^{1/2}.
\end{aligned} \tag{350}$$

Proof of Lemma 7.2. Observe that

$$
\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|^2_{HS(H)} \\
&= \tfrac{1}{\sqrt{t}}\sum_{k=1}^{N}\|(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})e_k\|^2_H = \tfrac{1}{\sqrt{t}}\sum_{k=1}^{N}\|(\nu\pi^2k^2)^{-1/2}(1-e^{-\nu\pi^2k^2t})e_k\|^2_H \\
&= \sum_{k=1}^{N}\frac{(1-e^{-\nu\pi^2k^2t})^2}{\nu\pi^2k^2\sqrt{t}} = \sum_{k=1}^{N}\int_k^{k+1}\frac{(1-e^{-\nu\pi^2k^2t})^2}{\nu\pi^2k^2\sqrt{t}}\,dx \\
&\geq \sum_{k=1}^{N}\int_k^{k+1}\frac{(1-e^{-\nu\pi^2(x-1)^2t})^2}{\nu\pi^2x^2\sqrt{t}}\,dx \\
&= \int_1^{N+1}\frac{(1-e^{-\nu\pi^2(x-1)^2t})^2}{\nu\pi^2x^2\sqrt{t}}\,dx \geq \int_{1+\min\{1/\sqrt{t},N\}}^{N+1}\frac{(1-e^{-\nu\pi^2(x-1)^2t})^2}{\nu\pi^2x^2\sqrt{t}}\,dx.
\end{aligned}
\tag{351}
$$

This and the integral transformation theorem imply that

$$
\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|^2_{HS(H)} \\
&\geq \int_{1+\min\{1/\sqrt{t},N\}}^{N+1}\frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{\nu\pi^2x^2\sqrt{t}}\,dx \\
&= \int_{(1+\min\{1/\sqrt{t},N\})^2}^{(N+1)^2}\frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2x\sqrt{xt}}\,dx \\
&= \int_{t(1+\min\{1/\sqrt{t},N\})^2}^{t(N+1)^2}\frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2x\sqrt{x}}\,dx \\
&= \int_{\min\{(1+\sqrt{t})^2,t(N+1)^2\}}^{t(N+1)^2}\frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2x\sqrt{x}}\,dx \\
&= \int_0^{t(N+1)^2-\min\{(1+\sqrt{t})^2,t(N+1)^2\}}\frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2(x+\min\{(1+\sqrt{t})^2,t(N+1)^2\})^{3/2}}\,dx \\
&\geq \int_0^{\max\{0,t(N+1)^2-(1+\sqrt{t})^2\}}\frac{(1-e^{-\nu\pi^2\min\{1,tN^2\}})^2}{2\nu\pi^2(x+[1+\sqrt{T}]^2)^{3/2}}\,dx.
\end{aligned}
\tag{352}
$$

Moreover, note that

$$\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|_{HS(H)}^2 \\
&= \tfrac{1}{\sqrt{t}}\sum_{k=1}^{N}\|(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})e_k\|_H^2 = \tfrac{1}{\sqrt{t}}\sum_{k=1}^{N}\|(\nu\pi^2k^2)^{-1/2}(1-e^{-\nu\pi^2k^2t})e_k\|_H^2 \\
&= \sum_{k=1}^{N}\frac{(1-e^{-\nu\pi^2k^2t})^2}{\nu\pi^2k^2\sqrt{t}} = \frac{(1-e^{-\nu\pi^2t})^2}{\nu\pi^2\sqrt{t}} + \sum_{k=2}^{N}\int_{k-1}^{k}\frac{(1-e^{-\nu\pi^2k^2t})^2}{\nu\pi^2k^2\sqrt{t}}\,dx.
\end{aligned} \tag{353}$$

The fact that

$$\forall\, x \in (0,\infty), r \in [0,1]\colon x^{-r}(1-e^{-x}) \leq 1, \tag{354}$$

the fact that

$$\forall\, x \in [1,\infty)\colon (x+1)^2 \leq 4x^2, \tag{355}$$

and the integral transformation theorem hence yield that

$$\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|_{HS(H)}^2 \\
&\leq \frac{(1-e^{-\nu\pi^2t})^{3/2}}{\pi\sqrt{\nu}} + \sum_{k=2}^{N}\int_{k-1}^{k}\frac{(1-e^{-\nu\pi^2(x+1)^2t})^2}{\nu\pi^2x^2\sqrt{t}}\,dx \\
&\leq \frac{1}{\pi\sqrt{\nu}} + \int_1^N \frac{(1-e^{-4\nu\pi^2x^2t})^2}{\nu\pi^2x^2\sqrt{t}}\,dx \\
&= \frac{1}{\pi\sqrt{\nu}} + \int_1^{N^2}\frac{(1-e^{-4\nu\pi^2xt})^2}{2\nu\pi^2x\sqrt{xt}}\,dx = \frac{1}{\pi\sqrt{\nu}} + \int_t^{tN^2}\frac{(1-e^{-4\nu\pi^2x})^2}{2\nu\pi^2x\sqrt{x}}\,dx.
\end{aligned} \tag{356}$$

Again the fact that

$$\forall\, x \in (0,\infty), r \in [0,1]\colon x^{-r}(1-e^{-x}) \leq 1 \tag{357}$$

therefore ensures that

$$\begin{aligned}
&\tfrac{1}{\sqrt{t}}\|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{tA})\|_{HS(H)}^2 \\
&\leq \frac{1}{\pi\sqrt{\nu}} + \int_0^\infty \frac{(1-e^{-4\nu\pi^2x})^2}{2\nu\pi^2x\sqrt{x}}\,dx \\
&\leq \frac{1}{\pi\sqrt{\nu}} + 2\int_0^1 \frac{(1-e^{-4\nu\pi^2x})}{\sqrt{x}}\,dx + \int_1^\infty \frac{1}{2\nu\pi^2x\sqrt{x}}\,dx \\
&\leq \frac{1}{\pi\sqrt{\nu}} + 4\pi\sqrt{\nu}\int_0^1\sqrt{1-e^{-4\nu\pi^2x}}\,dx + \left[\frac{-1}{\nu\pi^2\sqrt{x}}\right]_{x=1}^{x=\infty} \\
&\leq \frac{1}{\pi\sqrt{\nu}} + 4\pi\sqrt{\nu} + \frac{1}{\nu\pi^2}.
\end{aligned} \tag{358}$$

Combining this and (352) completes the proof of Lemma 7.2. ☐

7.3 Lower and upper bounds for strong approximation errors of temporal discretizations of linear stochastic heat equations

Lemma 7.3. *Assume the setting in Section 7.1 and let $M \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$. Then*

$$\begin{aligned}
&\frac{1}{M^{1/4}} \left[\int_0^{\max\left\{0, \frac{T(N+1)^2}{2M} - \left[1+\frac{\sqrt{T}}{\sqrt{2M}}\right]^2\right\}} \frac{\sqrt{T}\left[1-e^{-\nu\pi^2 T}\right]\left[1-\exp(-\nu\pi^2 \min\{1, \frac{TN^2}{2M}\})\right]^2}{8\nu\pi^2\sqrt{2}(x+[1+\sqrt{T}]^2)^{3/2}}\, dx \right]^{1/2} \\
&\leq \|P_N O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sup_{t\in[0,T]} \left[\int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s\rfloor_{T/M})A})\|^2_{HS(H)}\, ds \right]^{1/2} \\
&\leq \frac{1}{M^{1/4}} \left[\frac{\sqrt{T}}{2} \left(\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu} \right) \right]^{1/2}.
\end{aligned} \tag{359}$$

Proof of Lemma 7.3. Throughout this proof let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that $\mu_n = \nu\pi^2 n^2$ and let $\lceil\cdot\rceil_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, be the functions which satisfy for all $h \in (0,\infty)$, $t \in \mathbb{R}$ that $\lceil t\rceil_h = \min(\{0, h, -h, 2h, -2h, \ldots\} \cap [t,\infty))$. Observe that Lemma 7.1 (i) ensures for all $t \in [0,T)$ that

$$\begin{aligned}
&2\int_{\lfloor t\rfloor_{T/M}}^{\lfloor t\rfloor_{T/M}+\frac{T}{M}} \mathbb{1}^{\mathbb{R}}_{[\lfloor s\rfloor_{T/M}, \lfloor s\rfloor_{T/M}+\frac{T}{2M}]}(s)\, \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds \\
&= 2\int_{\lfloor t\rfloor_{T/M}}^{\lfloor t\rfloor_{T/M}+\frac{T}{2M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds \\
&\geq \int_{\lfloor t\rfloor_{T/M}}^{\lfloor t\rfloor_{T/M}+\frac{T}{2M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds + \int_{\lfloor t\rfloor_{T/M}+\frac{T}{2M}}^{\lfloor t\rfloor_{T/M}+\frac{T}{M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds \\
&= \int_{\lfloor t\rfloor_{T/M}}^{\lfloor t\rfloor_{T/M}+\frac{T}{M}} \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds.
\end{aligned} \tag{360}$$

Therefore, we obtain that

$$\begin{aligned}
&2\int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s\rfloor_{T/M}, \lfloor s\rfloor_{T/M}+\frac{T}{2M}]}(s)\, \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds \\
&\geq \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds.
\end{aligned} \tag{361}$$

Next note that Itô's isometry implies for all $t \in [0,T]$ that

$$
\begin{aligned}
&\|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|^2_H\Big] = \mathbb{E}\left[\left\|\int_0^t P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\, dW_s\right\|_H^2\right] \\
&= \int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|^2_{HS(H)}\, ds.
\end{aligned}
\tag{362}
$$

This, the fact that $\forall\, s \in [0,T]\colon T - \lfloor T-s \rfloor_{T/M} = \lceil s \rceil_{T/M}$, and Lemma 7.1 (ii) ensure that

$$
\begin{aligned}
&\|P_N O_T - \mathcal{O}_T^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{(T-s-\lfloor T-s \rfloor_{T/M})A})\|^2_{HS(H)}\, ds \\
&= \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{(\lceil s \rceil_{T/M}-s)A})\|^2_{HS(H)}\, ds \\
&\geq \int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s \rfloor_{T/M}, \lfloor s \rfloor_{T/M} + \frac{T}{2M}]}(s)\, \|P_N e^{sA}(\mathrm{Id}_H - e^{(\lceil s \rceil_{T/M}-s)A})\|^2_{HS(H)}\, ds \\
&\geq \int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s \rfloor_{T/M}, \lfloor s \rfloor_{T/M} + \frac{T}{2M}]}(s)\, \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds.
\end{aligned}
\tag{363}
$$

Inequality (361) hence proves that

$$
\begin{aligned}
&\|P_N O_T - \mathcal{O}_T^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&\geq \frac{1}{2}\left[2\int_0^T \mathbb{1}^{\mathbb{R}}_{[\lfloor s \rfloor_{T/M}, \lfloor s \rfloor_{T/M} + \frac{T}{2M}]}(s)\, \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds\right] \\
&\geq \frac{1}{2}\int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)}\, ds = \frac{1}{2}\int_0^T \sum_{k=1}^N \|e^{sA}(\mathrm{Id}_H - e^{\frac{T}{2M}A})e_k\|^2_H\, ds \\
&= \frac{1}{2}\int_0^T \sum_{k=1}^N |e^{-\mu_k s}(1 - e^{-\mu_k \frac{T}{2M}})|^2\, ds = \frac{1}{2}\sum_{k=1}^N \frac{(1 - e^{-2\mu_k T})}{2\mu_k}\, |1 - e^{-\mu_k \frac{T}{2M}}|^2.
\end{aligned}
\tag{364}
$$

Lemma 7.2 therefore implies that

$$
\begin{aligned}
&\|P_N O_T - \mathcal{O}_T^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&\geq \frac{1}{4}(1-e^{-\mu_1 T}) \sum_{k=1}^{N} \left| \frac{(1-e^{-\mu_k \frac{T}{2M}})}{\sqrt{\mu_k}} \right|^2 = \frac{1}{4}(1-e^{-\mu_1 T}) \sum_{k=1}^{N} \|(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{2M}A})e_k\|_H^2 \\
&= \frac{1}{4}(1-e^{-\mu_1 T}) \|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{2M}A})\|^2_{HS(H)} \\
&\geq \frac{\sqrt{T}(1-e^{-\mu_1 T})}{4\sqrt{2M}} \left[\int_0^{\max\left\{0, \frac{T(N+1)^2}{2M} - \left[1+\frac{\sqrt{T}}{\sqrt{2M}}\right]^2\right\}} \frac{\left[1-\exp(-\nu\pi^2 \min\{1, \frac{TN^2}{2M}\})\right]^2}{2\nu\pi^2(x+[1+\sqrt{T}]^2)^{3/2}} \, dx \right].
\end{aligned}
\tag{365}
$$

In the next step observe that (362) and Lemma 7.1 (ii) assure that

$$
\begin{aligned}
\sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} &\leq \sup_{t\in[0,T]} \int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)} \, ds \\
&= \sup_{t\in[0,T]} \int_0^t \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)} \, ds \\
&= \int_0^T \|P_N e^{sA}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)} \, ds \\
&= \int_0^T \sum_{k=1}^{N} \|e^{sA}(\mathrm{Id}_H - e^{\frac{T}{M}A})e_k\|_H^2 \, ds.
\end{aligned}
\tag{366}
$$

Lemma 7.2 hence yields that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&\leq \int_0^T \sum_{k=1}^{N} |e^{-\mu_k s}(1-e^{-\mu_k \frac{T}{M}})|^2 \, ds = \sum_{k=1}^{N} \frac{(1-e^{-2\mu_k T})}{2\mu_k} |1-e^{-\mu_k \frac{T}{M}}|^2 \\
&\leq \frac{1}{2} \sum_{k=1}^{N} \left| \frac{(1-e^{-\mu_k \frac{T}{M}})}{\sqrt{\mu_k}} \right|^2 = \frac{1}{2} \sum_{k=1}^{N} \|(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{M}A})e_k\|_H^2 \\
&= \frac{1}{2} \|P_N(-A)^{-1/2}(\mathrm{Id}_H - e^{\frac{T}{M}A})\|^2_{HS(H)} \leq \frac{\sqrt{T}}{2\sqrt{M}} \left[\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu} \right].
\end{aligned}
\tag{367}
$$

Combining this with (362) and (365) completes the proof of Lemma 7.3. □

In the next result, Corollary 7.4, we specialize Lemma 7.3 to the case $N = \infty$ where no spatial discretization is applied to the stochastic process $O\colon [0,T] \times \Omega \to H$.

Corollary 7.4. *Assume the setting in Section 7.1 and let $M \in \mathbb{N}$. Then*

$$\begin{aligned}
&\frac{1}{M^{1/4}} \left[\int_0^\infty \frac{\sqrt{T}(1-e^{-\nu\pi^2 T})(1-e^{-\nu\pi^2})^2}{8\nu\pi^2\sqrt{2}(x+[1+\sqrt{T}]^2)^{3/2}}\, dx \right]^{1/2} \\
&\le \liminf_{N\to\infty} \|P_N O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \limsup_{N\to\infty} \|P_N O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \|O_T - \mathcal{O}_T^{M,\infty}\|_{\mathcal{L}^2(\mathbb{P};H)} = \liminf_{N\to\infty} \sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \limsup_{N\to\infty} \sup_{t\in[0,T]} \|P_N O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \sup_{t\in[0,T]} \|O_t - \mathcal{O}_t^{M,\infty}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&\le \frac{1}{M^{1/4}} \left[\frac{\sqrt{T}}{2} \left(\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu} \right) \right]^{1/2}.
\end{aligned} \tag{368}$$

7.4 Lower and upper bounds for strong approximation errors of spatial discretizations of linear stochastic heat equations

Lemma 7.5. *Assume the setting in Section 7.1. Then*

$$\limsup_{M\to\infty} \sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \left\| \int_0^t \left(P_N e^{(t-s)A} - P_N e^{(t-\lfloor s \rfloor_{T/M})A} \right) dW_s \right\|_{L^2(\mathbb{P};H)} = 0. \tag{369}$$

Proof of Lemma 7.5. Throughout this proof let $\alpha \in (0, 1/4)$ and let $\beta \in (1/4, 1/2 - \alpha)$. Note that the fact that $4\beta > 1$ shows that

$$\begin{aligned}
&\sup_{N\in\mathbb{N}} \|P_N\|^2_{HS(H,H_{-\beta})} \\
&= \sup_{N\in\mathbb{N}} \left[\sum_{k=1}^N \|e_k\|^2_{H_{-\beta}} \right] = \sum_{k=1}^\infty \|(-A)^{-\beta} e_k\|^2_H \\
&= \sum_{k=1}^\infty |(\nu\pi^2 k^2)^{-\beta}|^2 = \sum_{k=1}^\infty \frac{1}{(\sqrt{\nu}\pi k)^{4\beta}} < \infty.
\end{aligned} \tag{370}$$

Next observe that for all $M, N \in \mathbb{N}$, $t \in [0,T]$ we have that

$$
\begin{aligned}
&\int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|_{HS(H)}^2 \, ds \\
&\leq \|(-A)^{-\beta} P_N\|_{HS(H)}^2 \int_0^t \|(-A)^{\beta} e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|_{L(H)}^2 \, ds \\
&= \|P_N\|_{HS(H,H_{-\beta})}^2 \int_0^t \|(-A)^{(\alpha+\beta)} e^{(t-s)A}(-A)^{-\alpha}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|_{L(H)}^2 \, ds \\
&\leq \|P_N\|_{HS(H,H_{-\beta})}^2 \int_0^t \|(-A)^{(\alpha+\beta)} e^{(t-s)A}\|_{L(H)}^2 \|(-A)^{-\alpha}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|_{L(H)}^2 \, ds.
\end{aligned}
\tag{371}
$$

The fact that

$$
\forall\, s \in [0,\infty), r \in [0,1] \colon \|(-sA)^r e^{sA}\|_{L(H)} \leq 1 \tag{372}
$$

and the fact that

$$
\forall\, s \in (0,\infty), r \in [0,1] \colon \|(-sA)^{-r}(\mathrm{Id}_H - e^{sA})\|_{L(H)} \leq 1 \tag{373}
$$

hence prove for all $M, N \in \mathbb{N}$, $t \in [0,T]$ that

$$
\begin{aligned}
&\int_0^t \|P_N e^{(t-s)A}(\mathrm{Id}_H - e^{(s-\lfloor s \rfloor_{T/M})A})\|_{HS(H)}^2 \, ds \\
&\leq \|P_N\|_{HS(H,H_{-\beta})}^2 \int_0^t (t-s)^{-2(\alpha+\beta)} (s - \lfloor s \rfloor_{T/M})^{2\alpha} \, ds \\
&\leq \frac{T^{2\alpha}}{M^{2\alpha}} \|P_N\|_{HS(H,H_{-\beta})}^2 \int_0^t (t-s)^{-2(\alpha+\beta)} \, ds \\
&= \frac{t^{(1-2\alpha-2\beta)} T^{2\alpha}}{(1-2\alpha-2\beta) M^{2\alpha}} \|P_N\|_{HS(H,H_{-\beta})}^2.
\end{aligned}
\tag{374}
$$

Itô's isometry therefore ensures for all $M \in \mathbb{N}$ that

$$
\begin{aligned}
&\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \left\| \int_0^t \left(P_N e^{(t-s)A} - P_N e^{(t-\lfloor s \rfloor_{T/M})A} \right) dW_s \right\|_{L^2(\mathbb{P};H)}^2 \\
&= \sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \int_0^t \|P_N e^{(t-s)A} - P_N e^{(t-\lfloor s \rfloor_{T/M})A}\|_{HS(H)}^2 \, ds \\
&\leq \frac{T^{(1-2\beta)}}{(1-2\alpha-2\beta) M^{2\alpha}} \left[\sup_{N \in \mathbb{N}} \|P_N\|_{HS(H,H_{-\beta})}^2 \right].
\end{aligned}
\tag{375}
$$

Combining this with (370) completes the proof of Lemma 7.5. □

Lemma 7.6. *Assume the setting in Section 7.1 and let $N \in \mathbb{N}$. Then*

$$\begin{aligned}
&\left[\frac{\sqrt{1-e^{-\nu T}}}{2\pi\sqrt{\nu}}\right]\frac{1}{\sqrt{N}} \leq \liminf_{M\to\infty} \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \limsup_{M\to\infty} \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \liminf_{M\to\infty} \sup_{t\in[0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \limsup_{M\to\infty} \sup_{t\in[0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \|O_T - P_N O_T\|_{\mathcal{L}^2(\mathbb{P};H)} = \sup_{t\in[0,T]} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P};H)} \leq \left[\frac{1}{\pi\sqrt{2\nu}}\right]\frac{1}{\sqrt{N}}.
\end{aligned} \tag{376}$$

Proof of Lemma 7.6. Throughout this proof let $(\mu_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $n \in \mathbb{N}$ that

$$\mu_n = \nu\pi^2 n^2. \tag{377}$$

Note that Parseval's identity shows that for all $t \in [0,T]$ we have that

$$\begin{aligned}
&\|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \mathbb{E}\big[\|O_t - P_N O_t\|_H^2\big] = \mathbb{E}\left[\sum_{k=N+1}^{\infty} |\langle e_k, O_t\rangle_H|^2\right] = \sum_{k=N+1}^{\infty} \mathbb{E}\big[|\langle e_k, O_t\rangle_H|^2\big] \\
&= \sum_{k=N+1}^{\infty} \mathbb{E}\left[\left|\int_0^t \langle e_k, e^{(t-s)A} dW_s\rangle_H\right|^2\right] = \sum_{k=N+1}^{\infty} \mathbb{E}\left[\left|\int_0^t \langle e^{(t-s)A} e_k, dW_s\rangle_H\right|^2\right].
\end{aligned} \tag{378}$$

Itô's isometry hence proves for all $t \in [0,T]$ that

$$\begin{aligned}
&\|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sum_{k=N+1}^{\infty} \mathbb{E}\left[\left|\int_0^t e^{-\mu_k(t-s)} \langle e_k, dW_s\rangle_H\right|^2\right] \\
&= \sum_{k=N+1}^{\infty} \int_0^t e^{-2\mu_k(t-s)}\, ds = \sum_{k=N+1}^{\infty} \int_0^t e^{-2\mu_k s}\, ds = \sum_{k=N+1}^{\infty} \frac{(1-e^{-2\mu_k t})}{2\mu_k}.
\end{aligned} \tag{379}$$

This shows that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} = \|O_T - P_N O_T\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sum_{k=N+1}^{\infty} \frac{\big(1-e^{-2\mu_k T}\big)}{2\mu_k} = \sum_{k=N+1}^{\infty} \frac{\big(1-e^{-2\nu\pi^2 k^2 T}\big)}{2\nu\pi^2 k^2} \geq \left[\frac{1-e^{-\nu T}}{2\nu\pi^2}\right]\left[\sum_{k=N+1}^{\infty}\frac{1}{k^2}\right] \\
&\geq \left[\frac{1-e^{-\nu T}}{2\nu\pi^2}\right]\left[\sum_{k=N+1}^{\infty}\int_k^{k+1}\frac{1}{x^2}\,dx\right] = \left[\frac{1-e^{-\nu T}}{2\nu\pi^2}\right]\left[\int_{N+1}^{\infty}\frac{1}{x^2}\,dx\right] = \left[\frac{1-e^{-\nu T}}{2\nu\pi^2}\right]\left[-\frac{1}{x}\right]_{x=N+1}^{x=\infty} \\
&= \left[\frac{1-e^{-\nu T}}{2\nu\pi^2}\right]\frac{1}{(N+1)} \geq \left[\frac{1-e^{-\nu T}}{2\nu\pi^2}\right]\frac{1}{(N+N)} = \left[\frac{1-e^{-\nu T}}{4\nu\pi^2}\right]\frac{1}{N}.
\end{aligned} \tag{380}
$$

This implies that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sum_{k=N+1}^{\infty} \frac{\big(1-e^{-2\nu\pi^2 k^2 T}\big)}{2\nu\pi^2 k^2} \leq \left[\frac{1}{2\nu\pi^2}\right]\left[\sum_{k=N+1}^{\infty}\frac{1}{k^2}\right] \leq \left[\frac{1}{2\nu\pi^2}\right]\left[\sum_{k=N+1}^{\infty}\int_{k-1}^{k}\frac{1}{x^2}\,dx\right] \\
&= \left[\frac{1}{2\nu\pi^2}\right]\left[\int_N^{\infty}\frac{1}{x^2}\,dx\right] = \left[\frac{1}{2\nu\pi^2}\right]\left[-\frac{1}{x}\right]_{x=N}^{x=\infty} = \left[\frac{1}{2\nu\pi^2}\right]\frac{1}{N}.
\end{aligned} \tag{381}
$$

In addition, note that the triangle inequality and Lemma 7.5 prove that

$$
\begin{aligned}
&\limsup_{M\to\infty} \sup_{t\in[0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \limsup_{M\to\infty} \sup_{t\in[0,T]} \left\| \int_0^t (e^{(t-s)A} - P_N e^{(t-\lfloor s\rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&= \limsup_{M\to\infty} \sup_{t\in[0,T]} \left\| \int_0^t (e^{(t-s)A} - P_N e^{(t-s)A})\, dW_s + \int_0^t (P_N e^{(t-s)A} - P_N e^{(t-\lfloor s\rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&\leq \limsup_{M\to\infty} \sup_{t\in[0,T]} \left\| \int_0^t (e^{(t-s)A} - P_N e^{(t-s)A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&\quad + \limsup_{M\to\infty} \sup_{t\in[0,T]} \left\| \int_0^t (P_N e^{(t-s)A} - P_N e^{(t-\lfloor s\rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&= \sup_{t\in[0,T]} \left\| \int_0^t (e^{(t-s)A} - P_N e^{(t-s)A})\, dW_s \right\|_{L^2(\mathbb{P};H)} = \sup_{t\in[0,T]} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P};H)}.
\end{aligned} \tag{382}
$$

Furthermore, observe that the triangle inequality, Lemma 7.5, and (380) ensure that

$$
\begin{aligned}
&\liminf_{M\to\infty} \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} = \liminf_{M\to\infty} \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-\lfloor s \rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&= \liminf_{M\to\infty} \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-s)A})\, dW_s + \int_0^T (P_N e^{(T-s)A} - P_N e^{(T-\lfloor s \rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&\geq \liminf_{M\to\infty} \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-s)A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&\quad - \liminf_{M\to\infty} \left\| \int_0^T (P_N e^{(T-s)A} - P_N e^{(T-\lfloor s \rfloor_{T/M})A})\, dW_s \right\|_{L^2(\mathbb{P};H)} \\
&= \left\| \int_0^T (e^{(T-s)A} - P_N e^{(T-s)A})\, dW_s \right\|_{L^2(\mathbb{P};H)} = \|O_T - P_N O_T\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \sup_{t\in[0,T]} \|O_t - P_N O_t\|_{\mathcal{L}^2(\mathbb{P};H)}.
\end{aligned}
\tag{383}
$$

Combining this with (380)–(382) completes the proof of Lemma 7.6. □

7.5 Lower and upper bounds for strong approximation errors of full discretizations of linear stochastic heat equations

Corollary 7.7. *Assume the setting in Section 7.1 and let $M, N \in \mathbb{N}$. Then*

$$
\begin{aligned}
&\frac{1}{M^{1/2}} \left[\int_0^{\max\left\{0, \frac{T(N+1)^2}{2M} - \left[1+\frac{\sqrt{T}}{\sqrt{2M}}\right]^2\right\}} \frac{\sqrt{T}\left[1 - e^{-\nu\pi^2 T}\right]\left[1 - \exp(-\nu\pi^2 \min\{1, \frac{TN^2}{2M}\})\right]^2}{32\nu\pi^2\sqrt{2}(x + [1+\sqrt{T}]^2)^{3/2}}\, dx \right]^{1/2} \\
&\quad + \frac{1}{N^{1/2}} \left[\frac{\sqrt{1-e^{-\nu T}}}{4\pi\sqrt{\nu}} \right] \\
&\leq \|O_T - \mathcal{O}_T^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \leq \sup_{t\in[0,T]} \|O_t - \mathcal{O}_t^{M,N}\|_{\mathcal{L}^2(\mathbb{P};H)} \\
&\leq \frac{1}{M^{1/4}} \left[\frac{\sqrt{T}}{2} \left(\frac{1}{\pi\sqrt{\nu}} + \frac{1}{\nu\pi^2} + 4\pi\sqrt{\nu} \right) \right]^{1/2} + \frac{1}{N^{1/2}} \left[\frac{1}{\pi\sqrt{2\nu}} \right].
\end{aligned}
\tag{384}
$$

Proof of Corollary 7.7. Observe that the fact that P_N is self-adjoint ensures for all $x \in H$, $y \in$

$P_N(H)$ that

$$\begin{aligned}
&\langle x - P_N(x), P_N(x) - y \rangle_H \\
&= \langle x - P_N(x), P_N(x) - P_N(y) \rangle_H = \langle x - P_N(x), P_N(x-y) \rangle_H \\
&= \langle P_N(x - P_N(x)), x - y \rangle_H = \langle P_N(x) - P_N(x), x - y \rangle_H \\
&= \langle 0, x-y \rangle_H = 0.
\end{aligned} \tag{385}$$

This implies for all $t \in [0,T]$ that

$$\begin{aligned}
&\|O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)} \\
&= \mathbb{E}\Big[\|O_t - \mathcal{O}_t^{M,N}\|^2_H\Big] = \mathbb{E}\Big[\|O_t - P_N O_t + P_N O_t - \mathcal{O}_t^{M,N}\|^2_H\Big] \\
&= \mathbb{E}\big[\|O_t - P_N O_t\|^2_H\big] + 2\,\mathbb{E}\Big[\langle O_t - P_N O_t, P_N O_t - \mathcal{O}_t^{M,N} \rangle_H\Big] + \mathbb{E}\Big[\|P_N O_t - \mathcal{O}_t^{M,N}\|^2_H\Big] \\
&= \|O_t - P_N O_t\|^2_{\mathcal{L}^2(\mathbb{P};H)} + \|P_N O_t - \mathcal{O}_t^{M,N}\|^2_{\mathcal{L}^2(\mathbb{P};H)}.
\end{aligned} \tag{386}$$

Combining this with Lemma 7.3, Lemma 7.6, and the fact that

$$\forall\, x, y \in [0,\infty) \colon \sqrt{x}/2 + \sqrt{y}/2 \leq \max\{\sqrt{x}, \sqrt{y}\} \leq \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \tag{387}$$

completes the proof of Corollary 7.7. □

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