Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics

Multilevel quasi-Monte Carlo integration with product weights for elliptic PDEs with lognormal coefficients

L. Herrmann and Ch. Schwab

Research Report No. 2017-19
April 2017

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Funding SNF: 159940/1

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Lukas Herrmann and Christoph Schwab

April 10, 2017

Abstract

We analyze the convergence rate of a multilevel quasi-Monte Carlo (MLQMC) Finite Element Method for a scalar diffusion equation with log-Gaussian, isotropic coefficients in a bounded, polytopal domain $D \subset \mathbb{R}^d$. The multilevel algorithm Q_L^* which is investigated here was first proposed in [Frances Y. Kuo, Christoph Schwab, and Ian H. Sloan: Multi-level quasi-Monte Carlo finite element methods for a class of elliptic PDEs with random coefficients, Journ. Found. Comp. Math.**15** (2015) pp. 411–449]. The random coefficient is assumed to admit a representation with locally supported coefficient functions, such as indicator functions or multiresolution representations. The present analysis builds on and generalizes the single-level analysis in [Lukas Herrmann and Christoph Schwab: QMC integration for lognormal-parametric, elliptic PDEs: local supports imply product weights, Report 2016-39, Seminar for Applied Mathematics, ETH Zürich] and also extends the MLQMC error analysis in [Frances Y. Kuo, Robert Scheichl, Christoph Schwab, Ian H. Sloan, and Elisabeth Ullmann: Multilevel quasi-Monte Carlo methods for lognormal diffusion problems (to appear in Math. Comp. 2017)], to locally supported basis functions in the representation of the Gaussian random field (GRF) in D, and to product weights. In particular, in polytopal domains $D \subset \mathbb{R}^d$, $d = 2, 3$, our analysis is based on weighted function spaces that allow GRFs and solutions whose realizations become singular at edges and vertices of D. This is natural for covariance operators whose associated precision operator is a fractional power of the Dirichlet Laplacean in D. In these weighted Sobolev spaces in D, first order, Lagrangean Finite Elements on regular, simplicial triangulations of D with suitable mesh refinement yield optimal asymptotic convergence rates. Our analysis yields also bounds for the ε-complexity of the MLQMC algorithm, uniformly with respect to the dimension of the parameter space.

*This work was supported in part by the Swiss National Science Foundation (SNSF) under grant SNF 200021_159940/1.

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1 Introduction

The numerical analysis of solution methods for partial differential equations (PDEs for short) and more general operator equations with random input data has received increasing attention in recent years, in particular with the development of computational uncertainty quantification and computational science and engineering. There, particular models of randomness in the PDEs' input parameters entail particular requirements to efficient computational uncertainty quantification algorithms. A basic case arises when there are only a finite number of random variables whose densities have bounded support and which parametrize the uncertain input in the forward PDE model: computation of statistical moments of responses and also Bayesian inversion then amounts to numerical integration over a bounded domain of finite dimension s. Statistical independence and scaling implies numerical integration over the unit cube $[0,1]^s$, against a product probability measure. In the context of PDEs, so-called *distributed random inputs* such as spatially heterogeneous diffusion coefficients, uncertain physical domains, etc. imply, via *uncertainty parametrizations* (such as Fourier- , B-spline or wavelet expansions) in physical domains D, a countably-infinite number of random parameters (being, for example, Fourier- or wavelet coefficients). This, in turn, renders the problem of estimation of response statistics of solutions a problem of infinite-dimensional numerical integration. Assuming again statistical independence of the system of (countably many) random input parameters results in the problem of numerical integration against a product measure. In the case of the uncertain PDE input being a Gaussian random field (GRF for short) considered here, in addition the domain Ω of integration is the countable product of real lines $\mathbb{R}^{\mathbb{N}}$, endowed with the Gaussian product measure (GM for short) μ and with the product sigma algebra obtained by completing the finite dimensional cylinders of Borel sets on $\mathbb{R}$ (we refer to [8] for details on GMs on $\mathbb{R}^{\mathbb{N}}$).

Here, as in [16, 21] and the references there, we analyze the combined discretization by quasi-Monte Carlo (QMC for short) quadratures and the Finite Element (FE for short) solution of linear, second order elliptic PDEs in a bounded, polygonal domain D, with isotropic, log-Gaussian diffusion coefficient $a = \exp(Z)$, where Z is a GRF in D. As in [16, 21], we confine the analysis to first order, randomly shifted lattice rules proposed originally in [26], and to continuous, piecewise linear "Courant" FE methods in D. We adopt the setting of our analysis [19] of the single-level QMC-FE algorithm: in a bounded, polytopal domain $D \subset \mathbb{R}^d$, $d = 2, 3$ we consider the model Dirichlet problem

$$-\nabla \cdot (a \nabla u) = f, \quad u\Big|_{\partial D} = 0 \, . \tag{1}$$

As in [19], the Gaussian random field $Z = \log(a) : \Omega \to L^\infty(D)$ is (formally) represented as

$$Z := \sum_{j \geq 1} y_j \psi_j, \tag{2}$$

where $(\psi_j)_{j\geq1}$ is a sequence of real-valued, bounded, and measurable functions in D. In particular, with respect to the GM μ the sequence $\boldsymbol{y} = (y_j)_{j\geq1}$ has independent and identically distributed (i.i.d. for short) components and for every $j \geq 1$, y_j is standard normally distributed. That is to say, $y_j \sim \mathcal{N}(0,1)$, i.i.d. for $j \in \mathbb{N}$. The lognormal coefficient a in (1) is given by

$$a := \exp(Z) \, . \tag{3}$$

The series (2) converges in $L^q(\Omega; L^\infty(D))$, $q \in [1,\infty)$, under the assumption that there exists a positive sequence $(b_j)_{j\geq1} \in \ell^p(\mathbb{N})$ for some $p \in (0,\infty)$ such that

$$K := \left\| \sum_{j\geq1} \frac{|\psi_j|}{b_j} \right\|_{L^\infty(D)} < \infty \, . \tag{\textbf{A1}}$$

In the setting of (2) and (3), the expectation with respect to the GM μ of the solution to (1) can be computed with QMC by randomly shifted lattice rules and product weights with dimension-independend convergence rates under the assumption (**A1**) with $p < 2$, cp. [19]. The assumption in (**A1**) can account for locality in the support of the function ψ_j. An assumption of the type of (**A1**) in the case of so called affine-parametric coefficients in conjunction with the application of QMC with product weights was already discussed in [12]. In the present work, we extend the analysis of [19] to a multilevel QMC algorithm with log-Gaussian inputs to reduce the overall work. The perspective of multilevel QMC integration with product weights for random inputs ψ_j with localized supports was originally introduced in [11] for the case of so-called affine-parametric coefficients. Multilevel QMC for elliptic PDEs with affine coefficients was first introduced in [22] (there for globally supported Karhúnen-Loève eigenfunctions and with so-called "POD" weights). As we showed there, localization of supports allows to obtain in certain cases estimates for the work of the evaluation of the multilevel QMC quadrature, which are asymptotically equal to the work to solve one instance of the corresponding deterministic PDE with the same error tolerance also in the case that the FE convergence rate is higher than $1/d$ with respect to the FE degrees of freedom. The FE convergence rate of first order FE is higher than $1/d$ if, for example, the spatial error is considered in a weaker Sobolev norm. In the present paper, the cost of generating the QMC points using the fast CBC construction of [27, 28] is included into the overall complexity estimate. This is due to product weights affording construction cost of QMC rules which is linear scaling in terms of the dimension of the integration domain.

The outline of this paper is as follows. In Section 2, we recapitulate known results on the well-posedness of problem (1) - (3) under assumption (**A1**), and on the integrability of random solution with respect to the GM. We also present bounds on the error incurred in the random solution when the expansion (2) is truncated to a finite number s of terms. As we combine QMC quadrature approximation of the GM with continuous, piecewise linear FE discretization of (1) of the random solution in polytopic domains $D \subset \mathbb{R}^d$, $d = 2, 3$, we also review in Section 2 elements of elliptic regularity theory and FE approximation theory in D; notably, handling corner and (in space dimension $d = 3$) edge singularities induced by D we review weighted Sobolev spaces in D in which (1) admits a full regularity shift. Corresponding weighted spaces also appear in our convergence rate analysis of the expansion (2) of the GRF. In Section 3, we review QMC convergence theory from [26, 19]. Suitable (weighted) spaces on $\mathbb{R}^s$ of integrand functions with mixed first derivatives which ensure (nearly) first order convergence with dimension-independent constants are introduced. Section 4 presents the key mathematical results: parametric regularity analysis for the integrand functions which arise from the dimensionally truncated, FE discretized problem, generalizing the single level QMC analysis in [19] by admitting locally supported functions ψ_j in the representation (2) of the GRF; while similar in spirit to the multilevel analysis in [16], there are significant technical differences due to accounting for local supports of ψ_j, analogous to the recent gpc N-term approximation rate analysis in [6]. The error bounds are then combined in Section 5 to a novel, MLQMC convergence rate bound in terms of the (sequences of) truncation dimensions $\{s_\ell\}_{\ell\geq 0}$, numbers $\{M_\ell\}_{\ell\geq 0}$ of FE degrees of freedom and of QMC sample numbers $\{N_\ell\}_{\ell\geq 0}$. Judicious choices of these parameters for concrete MLQMC-FE algorithms are derived in Section 6 by the "usual" error vs. work analysis through optimization, of the error bounds in Section 5, derived analogously to [22, 16]. Numerical experiments of this multilevel QMC algorithm in one spatial dimension are presented in [20, Section 5].

2 Well-posedness and spatial approximation

2.1 Well-posedness

We consider the variational formulation of the PDE (1) with lognormal coefficient $a = \exp(Z)$, i.e., to find $u : \Omega \to V$ such that

$$\int_D a\nabla u \cdot \nabla v \mathrm{d}x = f(v), \quad v \in V. \tag{4}$$

Under the assumption that for some $p_0 \in (0,\infty)$, $(b_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ it holds that $Z \in L^q(\Omega, L^\infty(D))$ for every $q \in [1,\infty)$, cp. [19, Theorem 2]. Hence, $0 < \operatorname{ess\,inf}_{x\in D}\{a(x)\} \leq \|a\|_{L^\infty(D)} < \infty$, μ-a.s. . As in previous works [19, 21, 16], in the ensuing error analysis, the quantities

$$a_{\min} := \operatorname*{ess\,inf}_{x\in D}\{a(x)\} \quad \text{and} \quad a_{\max} := \|a\|_{L^\infty(D)}$$

will play an important role. Under Assumption (**A1**), $a_{\min}$ and $a_{\max}$ are random variables on the probability space $(\Omega, \bigotimes_{j\geq 1}\mathcal{B}(\mathbb{R}), \mu)$ (see, for example, [8, Example 2.3.5]). Therefore, continuity and coercivity of the random bilinear form $(w, v) \mapsto \int_D a\nabla w \cdot \nabla v \mathrm{d}x$ in (4) on $V \times V$ holds with coercivity constant $a_{\min}$ and continuity constant $a_{\max}$, μ-a.s. By the Lax-Milgram Lemma, a unique solution u to (4) exists μ-a.s. and solves (4) uniquely by the Lax–Milgram lemma. By [19, Proposition 3], for every $q \in [1,\infty)$,

$$\|u\|_{L^q(\Omega;V)} \leq \|1/a_{\min}\|_{L^q(\Omega)}\|f\|_{V^*} < \infty,$$

where the strong measurability of $u : \Omega \to V$ follows, since the V-valued solution u depends continuously on the $L^\infty(D)$-valued coefficent a (by the second Strang lemma).

Numerical approximation of (functionals of) the random solution by QMC quadratures requires a finite dimensional domain of integration. To this end, the expansion of the Gaussian random field Z in (2) is truncated to a finite number s of terms: the *s-term truncated lognormal random field* a^s is defined by $a^s := \exp(Z^s) = \exp(\sum_{j=1}^s y_j\psi_j)$, for every $s \in \mathbb{N}$. With a^s, we associate the random variables

$$a^s_{\min} := \operatorname*{ess\,inf}_{x\in D}\{a^s(x)\} \quad \text{and} \quad a^s_{\max} := \|a^s\|_{L^\infty(D)}.$$

By u^s we denote the solution of the variational problem (4) with the s-term truncated, parametric coefficient a^s in place of a, i.e.,

$$u^s : \Omega \to V \text{ s.t.} \quad \int_D a^s\nabla u^s \cdot \nabla v \mathrm{d}x = f(v), \quad v \in V\ . \tag{5}$$

The *truncation error* can be controlled if the sequence $(b_j)_{j\geq 1}$ is p-summable. Specifically, if $(b_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ for some $p_0 \in (0,\infty)$, [19, Proposition 7] implies that for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that for every $G(\cdot) \in V^*$ and for every $s \in \mathbb{N}$

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s))| \leq C_\varepsilon \|G(\cdot)\|_{V^*}\|f\|_{V^*} \max_{j>s}\{b_j^{1-\varepsilon}\}. \tag{6}$$

2.2 Elliptic Regularity in D

Approximations of second order, elliptic PDEs with regular, simplicial Finite Elements in a polytope $D \subset \mathbb{R}^d$, $d = 2, 3$, on regular, simplicial families of uniformly refined triangulations may produce suboptimal convergence rates, due to the occurence of singularities in the parametric solutions u and u^s at vertices and, in space dimension $d = 3$, also at edges. In such domains, linear elliptic PDEs admit regularity shifts in certain weighted Sobolev spaces, cp. [5, 25] which we now recapitulate as we require the precise definition of the weighted norms in D in the ensuing QMC error analysis. We assume the polyhedron resp. polygon D to have straight edges and plane faces and J corners $\mathcal{C} := \{c_1, \dots, c_J\} \subset \partial D$.

For $d = 2$, let $\boldsymbol{\beta} = (\beta_1, \dots, \beta_J)$ be a J-tuple of weight exponents, we define the *corner weight function*

$$\Phi_{\boldsymbol{\beta}}(x) := \prod_{i=1}^{J} |x - c_i|^{\beta_i}, \quad x \in D,$$

where $\beta_i \in [0, 1)$, $i = 1, \dots, J$. Here and in the following, the Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. The weighted function spaces $L^q_{\boldsymbol{\beta}}(D)$ and $H^2_{\boldsymbol{\beta}}(D)$ are defined as closures of $C^\infty(\overline{D})$ with respect to the norms

$$\|v\|_{L^q_{\boldsymbol{\beta}}(D)} := \|v\Phi_{\boldsymbol{\beta}}\|_{L^q(D)}, \quad q \in [1, \infty],$$

and

$$\|v\|^2_{H^2_{\boldsymbol{\beta}}(D)} := \|v\|^2_{H^1(D)} + \sum_{|\boldsymbol{\alpha}|=2} \||\partial_x^{\boldsymbol{\alpha}} v|\Phi_{\boldsymbol{\beta}}\|^2_{L^2(D)}.$$

For $d = 3$, let the polyhedron D have J' straight edges $\mathcal{E} := \{e_1, \dots, e_{J'}\} \subset \partial D$ and define $\mathcal{X}_j := \{k : c_j \in \overline{e_k}\}$ as the index set of edges that meet at corner c_j, $j = 1, \dots, J$. Let r_k denote the distance to the edge e_k and let ρ_j denote the distance to the corner c_j. Let $(V_j : j = 1, \dots, J)$ be a finite, open covering of D such that

$$\overline{D} \subset \bigcup_{j=1}^{J} V_j, \quad c_i \notin \overline{V}_j, \text{ if } i \neq j, \quad \text{and} \quad \overline{V}_j \cap \overline{e}_k = \emptyset \text{ if } k \notin \mathcal{X}_j.$$

For a real-valued J-tuple $\boldsymbol{\beta} \in [0, 1)^J$ and a real-valued J'-tuple $\boldsymbol{\delta} \in [0, 1)^{J'}$, define the corner-edge weight function

$$\Phi_{(\boldsymbol{\beta},\boldsymbol{\delta})}(x) := \sum_{j=1}^{J} \rho_j^{\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{\delta_k} \mathbb{1}_{V_j}(x), \quad x \in D. \tag{7}$$

With this weight, we associate the weighted Sobolev spaces $L^2_{\boldsymbol{\beta},\boldsymbol{\delta}}(D)$ and $H^2_{\boldsymbol{\beta},\boldsymbol{\delta}}(D)$, cp. [25, Section 4.1.2] as closures of $C_0^\infty(\overline{D} \backslash (\mathcal{C} \cup \mathcal{E}))$ with respect to the norms

$$\|v\|_{L^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)} := \|v\Phi_{\boldsymbol{\beta},\boldsymbol{\delta}}\|_{L^2(D)}$$

and for $\iota = 0, 1, 2$,

$$\|v\|_{H^\iota_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)} := \left(\sum_{j=1}^{J} \sum_{|\boldsymbol{\alpha}| \le \iota} \int_{D \cap V_j} \rho_j^{2(\beta_j - \iota + |\boldsymbol{\alpha}|)}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{2(\delta_j - \iota + |\boldsymbol{\alpha}|)} |\partial^{\boldsymbol{\alpha}} v|^2 \mathrm{d}x\right)^{1/2}.$$

We note that the spaces $L^2_{\boldsymbol{\beta},\boldsymbol{\delta}}(D)$ and $H^0_{\boldsymbol{\beta},\boldsymbol{\delta}}(D)$ are isomorphic with equivalent norms: for every $x \in D$,

$$\sum_{j=1}^{J} \rho_j^{2\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{2\delta_k} \mathbb{1}_{V_j}(x) \le (\Phi_{(\boldsymbol{\beta},\boldsymbol{\delta})}(x))^2 \le J \sum_{j=1}^{J} \rho_j^{2\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{2\delta_k} \mathbb{1}_{V_j}(x),$$

Also, we define the weighted seminorm

$$|v|_{H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)} := \left(\sum_{j=1}^{J} \sum_{|\boldsymbol{\alpha}|=2} \int_{D \cap V_j} \rho_j^{2\beta_j}(x) \prod_{k \in \mathcal{X}_j} \left(\frac{r_k(x)}{\rho_j(x)}\right)^{2\delta_j} |\partial^{\boldsymbol{\alpha}} v|^2 \mathrm{d}x\right)^{1/2}.$$

Lemma 2.1 *For a polygon D (i.e. in spatial dimension $d = 2$), there exists a constant $C > 0$ such that for every $f \in L^2_{\boldsymbol{\beta}}(D)$,*

$$\|f\|_{V^*} \le C\|f\|_{L^2_{\boldsymbol{\beta}}(D)}.$$

For a polyhedron D (i.e. in spatial dimension $d = 3$), there exists a constant $C > 0$ such that for every $f \in L^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$,

$$\|f\|_{V^*} \le C\|f\|_{L^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)}.$$

Proof. The case $d = 2$ is proven in [20, Lemma 1]. The case $d = 3$ follows by [25, Lemma 4.1.4]. Specifically, in the notation of [25] the assertion of this lemma reads that the embedding $V^{0,2}_{\boldsymbol{\beta},\boldsymbol{\delta}}(D) \subset V^{-1,2}_{\mathbf{0},\mathbf{0}}(D)$ is continiuous, if $\beta_j < 1$ and $\delta_k < 1$, $j = 1,\dots,J$, $k = 1,\dots,J'$. We note that here the space $V^{0,2}_{\boldsymbol{\beta},\boldsymbol{\delta}}(D)$ of [25] coincides with our spaces $H^0_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D) = L^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$ and the space $V^{-1,2}_{\mathbf{0},\mathbf{0}}(D)$ is isomorphic to V^*. In the definition of the weighted space $L^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D) = H^0_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$, it has been assumed that $\beta_j < 1$ and $\delta_k < 1$, $j = 1,\dots,J$, $k = 1,\dots,J'$. □

In polygons D in space dimension $d = 2$ and for functions in $H^2_{\boldsymbol{\beta}}(D)$, a full regularity shift for the Laplacean is available, cp. eg. [5, Theorem 3.2]: there exists a constant $C > 0$ such that for every $w \in V$ with $\Delta w \in L^2_{\boldsymbol{\beta}}(D)$,

$$\|w\|_{H^2_{\boldsymbol{\beta}}(D)} \le C\|\Delta w\|_{L^2_{\boldsymbol{\beta}}(D)}, \tag{8}$$

where we assume that the weight exponent sequence $\boldsymbol{\beta}$ satisfies $\max\{0, 1 - \pi/\omega_i\} < \beta_i < 1$, $i = 1,\dots,J$. Here, ω_i denotes the interior angle of the polygon D at corner c_i, $i = 1,\dots,J$. Since [5] considers the Poisson boundary value problem with a zero order term, i.e., $-\Delta u + u = f$, we note that Lemma 2.1 implies that there exists a constant C such that for every $w \in V \cap H^2_{\boldsymbol{\beta}}(D)$, $\|w\|_{L^2_{\boldsymbol{\beta}}(D)} \le C\|\Delta w\|_{L^2_{\boldsymbol{\beta}}(D)}$.

In space dimension $d = 3$, when D is a polyhedral domain with plane sides and for functions in $H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D) \cap V$, there holds a corresponding regularity shift of the Dirchlet Laplacean by [25, Lemma 4.3.1] and by the inverse mapping theorem, cp. [10, Theorem 5.6-2]: there exists a constant $C > 0$ such that for every $w \in H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D) \cap V$ holds

$$\|w\|_{H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)} \le C\|\Delta w\|_{L^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)}, \tag{9}$$

where we assume that

$$\frac{1}{2} - \lambda_j < \beta_j < 1, \quad j = 1,\dots,J, \quad \text{and} \quad 1 - \frac{\pi}{\omega_k} < \delta_k < 1, \quad k = 1,\dots,J',$$

where ω_k is the interior angle between two faces meeting at edge e_k and λ_j is given by

$$\lambda_j := -\frac{1}{2} + \sqrt{\Lambda_j + \frac{1}{4}},$$

where Λ_j is the smallest, strictly positive eigenvalue of the Dirichlet Laplace–Beltrami operator on the intersection of the unit sphere centered at c_j and the infinite, interior polyhedral tangent cone to ∂D with vertex c_j, cp. [25, Section 4.3.1].

2.3 FE convergence theory

Let $\{\mathcal{T}_\ell\}_{\ell \ge 0}$ denote a sequence of regular, simplicial triangulations of D with proper mesh refinements near vertices and, if $d = 3$, also near edges of D. Let further $\mathbb{P}^1(K)$ denote the affine functions on a subset K of $\mathbb{R}^d$. In FE spaces $V_\ell := \{v \in V : v|_K \in \mathbb{P}^1(K), K \in \mathcal{T}_\ell\}$ of continuous, piecewise linear functions on $\{\mathcal{T}_\ell\}_{\ell\geq 0}$, optimal asymptotic convergence rates are achieveable, also in the presence of singularities. We state these for subsequent reference, recapitulating from [5, 13, 4, 1] approximation properties $H^1(D)$ of the subspaces V_ℓ.

Specifically, there exists a constant C such that for every $w \in H^2_{\boldsymbol{\beta}}(D)$ for $d = 2$, $w \in H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$ for $d = 3$, respectively, there is $w_\ell \in V_\ell$ satisfying

$$\|w - w_\ell\|_V \leq C M_\ell^{-1/d} \begin{cases} \|w\|_{H^2_{\boldsymbol{\beta}}(D)} & \text{if } d = 2, \\ \|w\|_{H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)} & \text{if } d = 3, \end{cases} \tag{10}$$

where $M_\ell := \dim(V_\ell)$. For $d = 2$, the convergence rate bound (10) is due to [5, Lemmas 4.1 and 4.5] for regular, graded simplicial meshes, resp. due to [13] for simplicial bisection tree meshes. In polyhedral domains D in space dimension $d = 3$, this estimate follows by [4, Theorem 4.6] for every $w \in C_0^\infty(\overline{D}\backslash\mathcal{C})$ and follows for every $w \in H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$, since $C_0^\infty(\overline{D}\backslash(\mathcal{C}\cup\mathcal{E}))$ is dense in $H^2_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$ (see also [3]).

2.4 Combined Dimension Truncation FE error bound

We now derive an error bound for the combined effect of truncating the GRF Z to a finite number of parameters s, and to FE discretization of the resulting s-parametric problem (5).

Let accordingly $u^{s,\mathcal{T}_\ell} : \Omega \to V_\ell$ denote the FE solution, i.e.,

$$\int_D a^s \nabla u^{s,\mathcal{T}_\ell} \cdot \nabla v \mathrm{d}x = f(v), \quad \forall v \in V_\ell. \tag{11}$$

For notational convenience, we introduce

$$\overline{\boldsymbol{\beta}} := \begin{cases} \boldsymbol{\beta} & \text{if } d = 2, \\ (\boldsymbol{\beta}, \boldsymbol{\delta}) & \text{if } d = 3. \end{cases} \tag{12}$$

The Banach space $W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D)$ is the space of all measurable functions $v : D \to \mathbb{R}$ that have finite $W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D)$-norm, where

$$\|v\|_{W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D)} := \max\{\|v\|_{L^\infty(D)}, \||\nabla v|\Phi_{\overline{\boldsymbol{\beta}}}\|_{L^\infty(D)}\}.$$

In order for the ML algorithm Q_L^* to yield improved (w.r. to the single-level case) error vs. work bounds, we require stronger assumptions than in the single-level analyses of [16, 19] on the function system $(\psi_j)_{j\geq 1}$. This corresponds to what was found for uniform random parameters in [22] and in the lognormal case for ψ_j with global supports in [21]. Let $(\bar{b}_j)_{j\geq 1}$ be a positive sequence such that

$$\left\| \sum_{j\geq 1} \frac{\max\{|\nabla\psi_j|\Phi_{\overline{\boldsymbol{\beta}}}, |\psi_j|\}}{\bar{b}_j} \right\|_{L^\infty(D)} < \infty \, . \tag{\textbf{A2}}$$

We remark that the assumption (**A2**) is stronger than (**A1**), which was found sufficient in [19] for the convergence rate analysis of the corresponding single-level QMC-FE algorithm.

Remark 2.1 *When the precision operator of Z is a positive power of a shifted Dirichlet Laplacean on D (as is the case in, e.g., the so-called Matern covariance functions), the Karhúnen-Loève eigenfunctions v_j are, by the spectral mapping theorem, eigenfunctions of the Dirichlet Laplacean on D: $-\Delta v_j = \nu_j v_j$, $v_j|_{\partial D} = 0$, $j \in \mathbb{N}$. Here, the eigenvalues ν_j are related to the ones appearing* *in the Karhúnen-Loève expansion of the GRF Z by the spectral mapping theorem. This setting includes GRFs with stationary covariance such as the well-known family due to B. Matérn, cp. [24]. Elliptic regularity shifts for the Dirichlet Laplacean are also known in certain weighted Hölder spaces in D: for* $d = 3$, *[25, Lemma 4.3.1.2)], implies that* $v_j \in W^{1,\infty}_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$ *provided that* $1 - \lambda_j < \beta_j < 1$, $j = 1, \dots, J$, *and* $1 - \pi/\theta_k < \delta_k < 1$, $k = 1, \dots, J'$, *where we used here that the weighted* $C^{1+\varepsilon}(\overline{D})$*-type space* $N^{1,\varepsilon}_{\boldsymbol{\beta},\boldsymbol{\delta}}(D)$ *(in the notation of [25, Sections 4.2 and 4.3]) embeds continuously into* $W^{1,\infty}_{(\boldsymbol{\beta},\boldsymbol{\delta})}(D)$*. Note that this condition on* $\boldsymbol{\beta}$ *for the KL eigenfunctions is stronger than in Assumption* (**A2**)*. Similar statements hold for* $d = 2$*. Here, singularities at corners and (for* $d = 3$*) along edges of the Karhúnen-Loève eigenfunctions appear as a consequence of regularity shifts for the Dirichlet Laplacean in weighted Hölder spaces. The structure of the weight functions* $\Phi_{\overline{\boldsymbol{\beta}}}$ *(which depend only on D and on the (Dirichlet) Laplacean) in the assumption* (**A2**) *on the Karhúnen-Loève eigenfunctions is identical to the weights in the elliptic regularity shift* (9)*. In the case of Matérn covariance functions, there is neither dependence of the functional form of the weight functions on the regularity nor on the positive correlation length of the respective GRF. Note, however, that in general, KL eigenfunctions have global support in D.*

Assumption (**A2**) implies $W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D)$-regularity of the GRF Z and strong approximation by its truncated expansion. This is made precise in the following proposition. Its proof is completely analogous to [19, Theorem 2] and therefore not detailed.

Proposition 2.2 *Let the assumption in* (**A2**) *be satisfied for some sequence* $(\bar{b}_j)_{j\geq 1}$ *such that* $(\bar{b}_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ *for some* $p_0 \in (0, \infty)$*. For every* $\varepsilon > 0$ *and* $q \in [1, \infty)$ *there exists a constant* $C > 0$ *such that for every* $s \in \mathbb{N}$,

$$\|Z - Z^s\|_{L^q(\Omega:W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D))} \leq C \sup_{j>s}\{\bar{b}_j^{1-\varepsilon}\} .$$

Since $(\nabla a)\Phi_{\overline{\boldsymbol{\beta}}} = (a\nabla Z)\Phi_{\overline{\boldsymbol{\beta}}}$ holds in $L^\infty(D)^d$, μ-a.s., Proposition 2.2 and [19, Corollary 6] imply with the Cauchy–Schwarz inequality that for every $q \in [1, \infty)$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$,

$$\|a\|_{L^q(\Omega;W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D))} < \infty \quad \text{and} \quad \|a^s\|_{L^q(\Omega;W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D))} \leq C < \infty. \tag{13}$$

To obtain an estimate of the Laplacean of u, we note that in any compact subset $\widetilde{D} \subset\subset D$ it holds $-a\Delta u = f - \nabla a \cdot \nabla u$, μ-a.s., where we assume that $f \in L^2_{\overline{\boldsymbol{\beta}}}(D)$. This equation may be tested with $-\Delta u \Phi^2_{\overline{\boldsymbol{\beta}}}/a$, which implies with Lemma 2.1

$$\|\Delta u\|_{L^2_{\overline{\boldsymbol{\beta}}}(D)} \leq \frac{\|f\|_{L^2_{\overline{\boldsymbol{\beta}}}(D)}}{a_{\min}} + \|Z\|_{W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D)}\|u\|_V \leq C\frac{\|f\|_{L^2_{\overline{\boldsymbol{\beta}}}(D)}}{a_{\min}}(1 + \|Z\|_{W^{1,\infty}_{\overline{\boldsymbol{\beta}}}(D)}). \tag{14}$$

An Aubin–Nitsche duality argument, (6), (8), (10), Proposition 2.2, (13), and (14) imply that for every $\varepsilon > 0$ exists a constant $C > 0$ such that for every $s \in \mathbb{N}$, $\ell \in \mathbb{N}_0$

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^{s,\mathcal{T}_\ell}))| \leq C \left(\sup_{j>s}\{b_j^{1-\varepsilon}\} + M_\ell^{-2/d}\right) \|f\|_{L^2_{\overline{\boldsymbol{\beta}}}(D)}\|G\|_{L^2_{\overline{\boldsymbol{\beta}}}(D)}. \tag{15}$$

Remark 2.2 *By the interpolation the error estimate in* (15) *extends to the case that* $f \in (V^*, L^2_{\overline{\boldsymbol{\beta}}}(D))_{t,\infty}$ *and* $G(\cdot) \in (V^*, L^2_{\overline{\boldsymbol{\beta}}}(D))_{t',\infty}$ *for some* $t, t' \in [0, 1]$*. Then the estimate in* (15) *holds with the term* $M_\ell^{-2/d}$ *replaced by* $M_\ell^{-(t+t')/d}$*. To see this, we observe that the real method* *of interpolation can be applied to the regularity shifts in* (8) *and in* (14). *Specifically, to the linear operator relating the solution* $u \in V$ *to its approximation error with a* V*-bounded, and quasioptimal projector* $\Pi_\ell : V \to V_\ell$*, where* Π_ℓ *is, for example, the* $H^1_0(D)$*-projection. From the approximation property in* (10)*, the interpolation couple* $L^2_{\boldsymbol{\beta}}(D) \subset V^*$ *then yields the fractional convergence order. Here and throughout what follows, interpolation spaces shall be understood with respect to the real method of interpolation; we refer to [29, Chapter 1].*

3 QMC integration

With convergence rate bounds on the dimension truncation and the FE discretization error at hand, we address the numerical approximation of the expectations in (15) with respect to the GM μ. Due to dimension truncation, we evaluate its s-variate section, i.e. we integrate w.r. to the GM on $\mathbb{R}^s$. As in [16], we approximate the s-variate integrals by so-called randomly shifted lattice rules proposed in [26]. Accordingly, we review QMC error estimates of randomly shifted lattice rules for high-dimensional integrals with respect to the s-variate normal distribution. The construction of generating vectors for such QMC rules in particular with respect to Gaussian and exponentially decaying weight functions with a fast CBC construction have been found in [26]. There, concrete error estimates of the resulting QMC rules in the mean-square sense (with respect to the random shift) have been derived, cp. [26, Theorem 8]. See also [23, Examples 4 and 5] for the estimation of constants appearing in the error bound of [26, Theorem 8] for Gaussian and exponential weight functions, respectively.

The error analysis of randomly shifted lattice rules requires, for sequences of positive weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}}$, indexed by all finite subsets $\mathfrak{u} \subset \mathbb{N}$, the weighted Sobolev space $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$ of mixed first order derivatives, which is defined by the following norm

$$\|F\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)} := \left(\sum_{\mathfrak{u} \subset \{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left| \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \partial^{\mathfrak{u}} F(\boldsymbol{y}) \prod_{j \in \{1:s\} \setminus \mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}_{\{1:s\} \setminus \mathfrak{u}} \right|^2 \prod_{j \in \mathfrak{u}} w_j^2(y_j) \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \right)^{1/2} . \tag{16}$$

Here, the standard normal density is denoted by

$$\phi(y) := \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}.$$

The norm in (16) is considered with respect to Gaussian and exponential weight functions

$$w_{\mathrm{g},j}^2(y) := e^{-\frac{y^2}{2\alpha_{\mathrm{g}}}}, \quad y \in \mathbb{R}, j \in \mathbb{N}, \quad \text{and} \quad w_{\exp,j}^2(y) := e^{-\alpha_{\exp}|y|}, \quad y \in \mathbb{R}, j \in \mathbb{N},$$

where the parameters $\alpha_{\mathrm{g}} > 1$ and $\alpha_{\exp} > 0$ will be determined in the following. In this work, we consider in (16) product weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$, determined by a positive QMC weight sequence $(\gamma_j)_{j \geq 1}$, i.e.,

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j, \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty.$$

We will denote the QMC approximation in s dimensions with N points by $Q_{s,N}(\cdot)$. It shall approximate integrals with respect to the multivariate normal distribution which we denote for every integrand $F \in L^1(\mathbb{R}^s, \mu)$ by

$$I_s(F) := \int_{\mathbb{R}^s} F(\boldsymbol{y}) \prod_{j \in \{1:s\}} \phi(y_j) \mathrm{d}\boldsymbol{y}.$$

For a sequence of dimension truncations $(s_\ell)_{\ell\geq 0}$ and a sequence $(N_\ell)_{\ell\geq 0}$, the multilevel QMC quadrature algorithm of [22] is defined by

$$Q^*_L(G(u^L)) := \sum_{\ell=0}^{L} Q_{s_\ell,N_\ell}(G(u^\ell) - G(u^{\ell-1})), \quad L \geq 0,$$

with the understanding that $G(u^{-1}) := 0$. we used the notation that $u^\ell := u^{s_\ell,\mathcal{T}_\ell}$, $\ell \geq 0$. Multilevel QMC algorithms stemming from randomly shifted lattice rules have been considered in [22, 21]. The following error estimate (see [22, Equation (23)] or [21, Equation (3.2)]) holds due to the independence of the random shifts on the different levels

$$\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(G(u^L)) - Q^*_L(G(u^L))|^2) = \sum_{\ell=0}^{L} \mathbb{E}^{\boldsymbol{\Delta}}(|I_s(G(u^\ell - u^{\ell-1})) - Q_{s_\ell,N_\ell}(G(u^\ell - u^{\ell-1}))|^2), \quad (17)$$

where we generally apply a randomly shifted lattice rule with respect to (possibly) a different QMC weight sequence on the PDE discretization level $\ell = 0$.

In [19], convergence of randomly shifted lattice rules with product weights is investigated, which relies on parametric regularity estimates of a particular form. We summarize the QMC convergence theory in the following theorem.

Theorem 3.1 *Let $(\widetilde{b}_j)_{j\geq 1}$ be a positive sequence such that for some $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ there exists a constant $C > 0$ and a positive function $H(\boldsymbol{y})$ such that for every $\boldsymbol{y} \in \{\boldsymbol{y} \in \mathbb{R}^{\mathbb{N}} : \exists s \in \mathbb{N}, y_j = 0 : \forall j > s\}$,*

$$\sum_{\mathfrak{u}\subset\mathbb{N},|\mathfrak{u}|<\infty} |\partial^{\mathfrak{u}} F(\boldsymbol{y})|^2 \prod_{j\in\mathfrak{u}} \left(\frac{\kappa}{\widetilde{b}_j}\right)^2 \leq C H(\boldsymbol{y})^2 \,.$$

1. *Let $(\widetilde{b}_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$. For $\varepsilon \in (0, 3/4 - 1/(2p))$, set $p' = p/4 + 1/2 - \varepsilon p \in (0, 1)$. Consider the Gaussian weight functions $(\boldsymbol{w}_{\mathrm{g},j})_{j\geq 1}$ with parameter α_{g} and QMC weight sequence*

   $$\alpha_{\mathrm{g}} \in \left(\frac{p}{2(p-p')}, \frac{p}{p-2(1-p')}\right) \quad \text{and} \quad \gamma_j = \widetilde{b}_j^{2p'}, \quad j \geq 1.$$

   Then, there exists a constant C (independent of F) such that for $q_0 = 2qq'/(q'-q)$, where $q = p/(2(1-p'))$ and $q' \in (q, \alpha_{\mathrm{g}}/(1-\alpha_{\mathrm{g}}))$,

   $$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(F) - Q_{s,N}(F)|^2} \leq C(\varphi(N))^{-1/(2p)-1/4+\varepsilon} \|H\|_{L^{q_0}(\mathbb{R}^s,\mu)}.$$

2. *Let $(\widetilde{b}_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 1]$. Assume that $H(\boldsymbol{y}) \leq \eta_1 \exp(\eta_2 \sum_{j\geq 1} \widetilde{b}_j |y_j|)$ for some $\eta_1, \eta_2 > 0$. Set $p' = 1 - p/2$. Consider the exponential weight functions $(\boldsymbol{w}_{\exp,j})_{j\geq 1}$ with parameter $\alpha_{\exp}$ and QMC weight sequence*

   $$\alpha_{\exp} > 2\eta_2 \quad \text{and} \quad \gamma_j = \widetilde{b}_j^{2p'}, \quad j \geq 1.$$

   Then, there exists a positive constant C (independent of η_1) such that

   $$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(F) - Q_{s,N}(F)|^2} \leq C\,(\varphi(N))^{-1/p+1/2}\,\eta_1.$$

The Euler totient function is denoted by $\varphi(\cdot)$.

This theorem was, in the case of Gaussian weight functions, obtained in [19, Theorems 9 and 11] and in the case of exponential weight functions in [19, Theorems 9 and 12]. The main ingredient of the proof of [19, Theorem 9] is a parametric regularity estimate of the form assumed in Theorem 3.1. The parametric regularity estimates derived in [16, 21] for globally supported ψ_j afforded bounds for each partial derivative separately. In [19], we used the bound from [6, Theorem 4.1] which does account for local supports and affords control of "bulk" sums of (norms of) solution derivatives with respect to the parameters y_j. We also note that in applications, the sequence $(\widetilde{b}_j)_{j\geq 1}$ might be arbitrarily scaled by a factor κ in order to satisfy such a regularity estimate.

4 Parametric regularity

In this section we derive parametric regularity estimates that allow to prove dimension independent convergence rates of multilevel QMC. We extend the argument that results in the estimate in [6, Theorem 4.1] to dimensionally truncated and FE differences.

For every $s \in \mathbb{N}$, the truncated fields Z^s, a^s, and u^s, are well-defined regardless of assumption (**A1**). In particular, $Z^s = \sum_{j=1}^{s} y_j \psi_j$ is well-defined for every $\boldsymbol{y} \in \Omega = \mathbb{R}^{\mathbb{N}}$. We may therefore interpret Z^s as a mapping from $\mathbb{R}^s$ to L^∞ such that pointwise evaluation is well-defined for every $\boldsymbol{y} \in \mathbb{R}^s$. Similarly a^s and u^s may be interpreted as mappings from $\mathbb{R}^s$ to $L^\infty(D)$ and to V, respectively. In the same way Z, a, and u are mappings with pointwise evaluation from the set

$$U := \{\boldsymbol{y} \in \Omega : \exists s \in \mathbb{N}, y_j = 0, j > s\}$$

to $L^\infty(D)$ and V, respectively. Note that $\mathbb{R}^s \times \{\mathbf{0}\} \subset U = \bigcup_{s\in\mathbb{N}} \mathbb{R}^s \times \{\mathbf{0}\}$ for every $s \in \mathbb{N}$, where $\mathbf{0} \in \mathbb{R}^{\mathbb{N}\setminus\{1:s\}}$. Hence, the set U of admissible parameters $\boldsymbol{y}$ is sufficiently rich for the ensuing QMC convergence analysis. The mappings Z^s, a^s, and u^s extend naturally to mappings from U to $L^\infty(D)$ and to V, respectively.

4.1 Dimensionally truncated differences

Let $s \in \mathbb{N}$ be a truncation level. For every $\boldsymbol{y} \in U$, the difference $u(\boldsymbol{y}) - u^s(\boldsymbol{y})$ satisfies the variational formulation

$$\int_D a(\boldsymbol{y})\nabla(u(\boldsymbol{y}) - u^s(\boldsymbol{y})) \cdot \nabla v \mathrm{d}x = -\int_D (a(\boldsymbol{y}) - a^s(\boldsymbol{y}))\nabla u^s(\boldsymbol{y}) \cdot \nabla v \mathrm{d}x, \quad \forall v \in V. \tag{18}$$

We will mostly (in the proofs) omit the $\boldsymbol{y}$ dependence in the following. Set $\mathcal{F} := \{\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\tau}| < \infty\}$. For every $\boldsymbol{\tau} \in \mathcal{F}$ and a positive sequence $(\rho_j)_{j\geq 1}$, let us define the following numbers

$$\kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) := \frac{\sqrt{\boldsymbol{\tau}!}}{\sqrt{\boldsymbol{\nu}!}} \frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}} |\psi|^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!}, \quad \boldsymbol{\nu} \leq \boldsymbol{\tau}.$$

Also, for given $k, r \in \mathbb{N}$ introduce the set $\Lambda_k := \{\boldsymbol{\tau} \in \mathcal{F} : |\boldsymbol{\tau}| = k, \|\boldsymbol{\tau}\|_{\ell^\infty} \leq r\}$ and for any integer $\ell \leq k-1$ and for $\boldsymbol{\nu} \in \Lambda_\ell$, introduce

$$R_{\boldsymbol{\nu},k} := \{\boldsymbol{\tau} \in \Lambda_k : \boldsymbol{\tau} \geq \boldsymbol{\nu}\},$$

where r denotes the maximal order of differentiability to be considered. The following lemma reveals that κ_0 has the property of a discrete probability density, which will be useful in the ensuing analysis.

Lemma 4.1 *Assume that there exists a positive sequence* $(\rho_j)_{j\geq 1}$ *such that, for some* $r \in \mathbb{N}$,

$$K := \left\| \sum_{j\geq 1} \rho_j |\psi_j| \right\|_{L^\infty(D)} < \frac{\log(2)}{\sqrt{r}}. \tag{19}$$

Then, for every $\boldsymbol{\tau} \in \mathcal{F}$ *such that* $\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r$,

$$\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq e^{\sqrt{r}K} - 1 < 1$$

and for every positive integer $\ell \leq k-1$ *and multi-index* $\boldsymbol{\nu} \in \Lambda_\ell$,

$$\sum_{\boldsymbol{\tau} \in R_{\boldsymbol{\nu},k}} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq \frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}.$$

The estimates in this lemma are given in [6, Equations (4.12) and (4.14)]. The second estimate of Lemma 4.1 still holds if the smallness assumption in (19) is not guaranteed. We note that the condition (19) is implied by (**A1**) with $\rho_j^{-1} = b_j \bar{K}\sqrt{r}/\log(2)$ provided that $\bar{K} > \| \sum_{j\geq 1} |\psi_j|/b_j \|_{L^\infty(D)}$. For every $s \in \mathbb{N}$, integers $\ell \leq k-1$, and $\boldsymbol{\nu} \in \Lambda_\ell$, introduce the set

$$R^s_{\boldsymbol{\nu},k} := \{\boldsymbol{\tau} \in R_{\boldsymbol{\nu},k} : \exists j > s \text{ such that } \tau_j > 0\}.$$

Lemma 4.2 *Let the assumptions of Lemma 4.1 hold for a positive sequence* $(\rho_j)_{j\geq 1}$ *such that* $c := \|(\rho_j^{-1})_{j\geq 1}\|_{\ell^\infty(\mathbb{N})} < \infty$. *Let us further assume that for some* $\eta > 0$

$$K_\eta := \left\| \sum_{j\geq 1} \rho_j^{1+\eta} |\psi_j| \right\|_{L^\infty(D)} < \infty.$$

Then, for $s \in \mathbb{N}$ *and every* $\boldsymbol{\tau} \in \mathcal{F}$ *such that* $\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r$ *and* $\tau_j > 0$ *for some* $j > s$,

$$\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \nu_j = 0\, \forall j > s} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq 2(e^{\sqrt{r}K_\eta c^\eta} - 1)c^{-\eta} \sup_{j>s}\{\rho_j^{-\eta}\}.$$

For $s \in \mathbb{N}$, *positive integers* $\ell \leq k-1$, *and* $\boldsymbol{\nu} \in \Lambda_\ell$ *such that* $\nu_j = 0$, $j > s$,

$$\sum_{\boldsymbol{\tau} \in R^s_{\boldsymbol{\nu},k}} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) \leq \frac{(\sqrt{r}K_\eta c^\eta)^{k-\ell}}{(k-\ell)!} c^{-\eta} \sup_{j>s}\{\rho_j^{-\eta}\}.$$

Proof. There is $j > s$ such that $\tau_j > 0$. Since κ_0 is a product, by Lemma 4.1,

$$\begin{aligned}\sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \nu_j = 0\, \forall j > s} \kappa_0(\boldsymbol{\tau}, \boldsymbol{\nu}) &= \left(\sum_{\boldsymbol{\nu}_{\{1:s\}} \leq \boldsymbol{\tau}_{\{1:s\}}} \kappa_0(\boldsymbol{\tau}_{\{1:s\}}, \boldsymbol{\nu}_{\{1:s\}}) \right) \kappa_0(\boldsymbol{\tau}_{\mathbb{N}\setminus\{1:s\}}, \mathbf{0}_{\mathbb{N}\setminus\{1:s\}}) \\ &\leq 2\kappa_0(\boldsymbol{\tau}_{\mathbb{N}\setminus\{1:s\}}, \mathbf{0}_{\mathbb{N}\setminus\{1:s\}}),\end{aligned}$$

where we used the notation that for every $\mathfrak{u} \subset \mathbb{N}$, $\tau_{\mathfrak{u}}$ is a multi-index that satisfies $(\tau_{\mathfrak{u}})_j = \tau_j$, $j \in \mathfrak{u}$, and $(\tau_{\mathfrak{u}})_j = 0$ otherwise. With $c = \|(\rho_j^{-1})_{j\geq 1}\|_{\ell^\infty(\mathbb{N})}$, we obtain

$$\begin{aligned}\kappa_0(\boldsymbol{\tau}_{\mathbb{N}\setminus\{1:s\}}, \mathbf{0}_{\mathbb{N}\setminus\{1:s\}}) &\leq \frac{\rho^{\boldsymbol{\tau}_{\mathbb{N}\setminus\{1:s\}}}}{\sqrt{\boldsymbol{\tau}_{\mathbb{N}\setminus\{1:s\}}!}} |\psi|^{\boldsymbol{\tau}_{\mathbb{N}\setminus\{1:s\}}} \leq \exp\left(\sqrt{r} \sum_{j>s} \rho_j |\psi_j| \right) - 1 \\ &\leq (e^{\sqrt{r}K_\eta c^\eta} - 1)c^{-\eta} \sup_{j>s}\{\rho_j^{-\eta}\}.\end{aligned}$$

For the proof of the second inequality, we observe that

$$\sum_{\boldsymbol{\tau}\in R^s_{\boldsymbol{\nu},k}} \kappa_0(\boldsymbol{\tau},\boldsymbol{\nu}) \le \sum_{\boldsymbol{\tau}\in R^s_{\boldsymbol{\nu},k}} \frac{\sqrt{\boldsymbol{\tau}!}}{\sqrt{\boldsymbol{\nu}!}} \frac{(\rho^{1+\eta}c^\eta)^{(\boldsymbol{\tau}-\boldsymbol{\nu})}|\psi|^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!} c^{-\eta} \sup_{j>s}\{\rho_j^{-\eta}\},$$

where we used that for every $\boldsymbol{\tau} \in R^s_{\boldsymbol{\nu},k}$ there exists $j > s$ such that $\tau_j - \nu_j > 0$ and that $\rho_j^{-1}/c \le 1$, $j \le 1$. By the first statement of Lemma 4.1,

$$\sum_{\boldsymbol{\tau}\in R^s_{\boldsymbol{\nu},k}} \frac{\sqrt{\boldsymbol{\tau}!}}{\sqrt{\boldsymbol{\nu}!}} \frac{(\rho^{1+\eta}c^\eta)^{(\boldsymbol{\tau}-\boldsymbol{\nu})}|\psi|^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!} \le \sum_{\boldsymbol{\tau}\in R_{\boldsymbol{\nu},k}} \frac{\sqrt{\boldsymbol{\tau}!}}{\sqrt{\boldsymbol{\nu}!}} \frac{(\rho^{1+\eta}c^\eta)^{(\boldsymbol{\tau}-\boldsymbol{\nu})}|\psi|^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!} \le \frac{(\sqrt{r}K_\eta c^\eta)^{k-\ell}}{(k-\ell)!},$$

which implies the assertion of the lemma. □

Theorem 4.3 *[Truncation error]*

Let the assumptions of Lemmas 4.1 and 4.2 be satisfied for a positive sequence $(\rho_j)_{j\ge1}$ *and* $\eta > 0$. *There exists a constant* $C > 0$ *such that for every* $s \in \mathbb{N}$ *and every* $\boldsymbol{y} \in U$

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\le r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u(\boldsymbol{y}) - u^s(\boldsymbol{y}))\|^2_{a(\boldsymbol{y})} \le C\left(\left\|\frac{a(\boldsymbol{y})-a^s(\boldsymbol{y})}{a(\boldsymbol{y})}\right\|^2_{L^\infty(D)} + \sup_{j>s}\{\rho_j^{-2\eta}\}\right)\|u^s(\boldsymbol{y})\|^2_{a(\boldsymbol{y})}.$$

Proof. We divide the index set $\mathcal{F}_r := \{\boldsymbol{\tau}\in\mathcal{F} : \tau_j \le r, j\in\mathbb{N}\}$ into $\mathcal{F}_1^s := \{\boldsymbol{\tau}\in\mathcal{F}_r : \tau_j = 0 \,\forall j > s\}$ and $\mathcal{F}_2^s := \{\boldsymbol{\tau}\in\mathcal{F}_r : \exists j > s \text{ s.t. } \tau_j > 0\}$. Obviously, $\mathcal{F}_r = \mathcal{F}_1^s \cup \mathcal{F}_2^s$.

Let $\mathbf{0} \ne \boldsymbol{\tau} \in \mathcal{F}_1^s$ be arbitrary. We observe that for every $v \in V$,

$$\begin{aligned}\int_D a\nabla\partial^{\boldsymbol{\tau}}(u-u^s)\cdot\nabla v \mathrm{d}x = &-\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\ne\boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \int_D \psi^{\boldsymbol{\tau}-\boldsymbol{\nu}} a\nabla\partial^{\boldsymbol{\nu}}(u-u^s)\cdot\nabla v\mathrm{d}x \\ &- \sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}:\forall j>s\,\tau_j=\nu_j} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \int_D \psi^{\boldsymbol{\tau}-\boldsymbol{\nu}}(a-a^s)\nabla\partial^{\boldsymbol{\nu}}u^s\cdot\nabla v\mathrm{d}x.\end{aligned}$$

Set

$$\sigma_k := \sum_{\boldsymbol{\tau}\in\Lambda_k} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}(u-u^s)\|^2_a$$

and take $v = \partial^{\boldsymbol{\tau}}(u-u^s)$. By a twofold application of the Cauchy–Schwarz inequality and by Lemma 4.1

$$\begin{aligned}\sigma_k \le &\int_D \sum_{\boldsymbol{\tau}\in\Lambda_k}\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\ne\boldsymbol{\tau}} a\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{\boldsymbol{\nu}}}{\sqrt{\boldsymbol{\nu}!}}|\nabla\partial^{\boldsymbol{\nu}}(u-u^s)|\frac{\rho^{\boldsymbol{\tau}}}{\sqrt{\boldsymbol{\tau}!}}|\nabla\partial^{\boldsymbol{\tau}}(u-u^s)| \\ &+ \int_D \sum_{\boldsymbol{\tau}\in\Lambda_k}\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}} |a-a^s|\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{\boldsymbol{\nu}}}{\sqrt{\boldsymbol{\nu}!}}|\nabla\partial^{\boldsymbol{\nu}}u^s|\frac{\rho^{\boldsymbol{\tau}}}{\sqrt{\boldsymbol{\tau}!}}|\nabla\partial^{\boldsymbol{\tau}}(u-u^s)| \\ \le &\int_D \left(\sum_{\boldsymbol{\tau}\in\Lambda_k}\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\ne\boldsymbol{\tau}} a\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}|\nabla\partial^{\boldsymbol{\nu}}(u-u^s)|^2\right)^{1/2} \\ &\times\left(a\sum_{\boldsymbol{\tau}\in\Lambda_k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\nu}!}|\nabla\partial^{\boldsymbol{\tau}}(u-u^s)|^2\right)^{1/2} \\ &+\int_D\left(\sum_{\boldsymbol{\tau}\in\Lambda_k}\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}}|a-a^s|\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}|\nabla\partial^{\boldsymbol{\nu}}u^s|^2\right)^{1/2} \\ &\times\left(2\sum_{\boldsymbol{\tau}\in\Lambda_k}|a-a^s|\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}|\nabla\partial^{\boldsymbol{\tau}}(u-u^s)|^2\right)^{1/2}\end{aligned}$$

Further, we apply the Cauchy–Schwarz inequality on the integral and obtain that

$$\sigma_k \leq \left(\int_D \sum_{\boldsymbol{\tau}\in\Lambda_k} \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} a\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}|\nabla\partial^{\boldsymbol{\nu}}(u-u^s)|^2\right)^{1/2}\sqrt{\sigma_k}$$
$$+\left(\int_D \sum_{\boldsymbol{\tau}\in\Lambda_k} \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau}} |a-a^s|\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}|\nabla\partial^{\boldsymbol{\nu}}u^s|^2\right)^{1/2}\sqrt{2\left\|\frac{a-a^s}{a}\right\|_{L^\infty(D)}}\sqrt{\sigma_k}.$$

By [6, Equation (4.18)] in the proof of [6, Theorem 4.1], for any $\delta\in[\sqrt{r}K/\log(2),1)$ and for every $\ell\in\mathbb{N}$,

$$\sum_{\boldsymbol{\tau}\in\Lambda_\ell}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}u^s\|_a^2 \leq \|u^s\|_a^2\,\delta^\ell. \tag{20}$$

We change the order of summation in order to apply the second estimate in Lemma 4.1 and insert (20) to obtain with Young's inequality that for any $\varepsilon>0$

$$\begin{aligned}\sigma_k &\leq (1+\varepsilon)\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sigma_\ell\\
&\quad+\left(1+\frac{1}{\varepsilon}\right)\left\|\frac{a-a^s}{a}\right\|_{L^\infty(D)}^2\sum_{\ell=0}^{k}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sum_{\boldsymbol{\tau}\in\Lambda_\ell}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}u^s\|_a^2\\
&\leq (1+\varepsilon)\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sigma_\ell+\left(1+\frac{1}{\varepsilon}\right)2\left\|\frac{a-a^s}{a}\right\|_{L^\infty(D)}^2\|u^s\|_a^2\sum_{\ell=0}^{k}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\delta^\ell\\
&\leq (1+\varepsilon)\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sigma_\ell+\left(1+\frac{1}{\varepsilon}\right)4\left\|\frac{a-a^s}{a}\right\|_{L^\infty(D)}^2\|u^s\|_a^2\,\delta^k.\end{aligned}$$

By a change of the order of summation, we obtain that

$$\sum_{k\geq1}\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sigma_\ell=\sum_{\ell\geq0}\left(\sum_{k=\ell+1}^{\infty}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\right)\sigma_\ell\leq(e^{\sqrt{r}K}-1)\sum_{\ell\geq0}\sigma_\ell. \tag{21}$$

Let us choose $\varepsilon>0$ such that $\varepsilon<(2-e^{\sqrt{r}K})/(e^{\sqrt{r}K}-1)$, which implies that $(1+\varepsilon)(e^{\sqrt{r}K}-1)<1$. Denote $C^*:=(1-(1+\varepsilon)(e^{\sqrt{r}K}-1))^{-1}$. We sum σ_k over $k\geq1$ and obtain that

$$\sum_{k\geq1}\sigma_k\leq(1+\varepsilon)(e^{\sqrt{r}K}-1)\sum_{\ell\geq0}\sigma_\ell+\left(1+\frac{1}{\varepsilon}\right)4\left\|\frac{a-a^s}{a}\right\|_{L^\infty(D)}^2\|u^s\|_a^2\frac{\delta}{1-\delta}.$$

Since $(1+\varepsilon)(e^{\sqrt{r}K}-1)<1$, we conclude that

$$\sum_{k\geq1}\sigma_k\leq C^*\sigma_0+C^*\left(1+\frac{1}{\varepsilon}\right)4\left\|\frac{a-a^s}{a}\right\|_{L^\infty(D)}^2\|u^s\|_a^2\frac{\delta}{1-\delta},$$

which implies

$$\sum_{\boldsymbol{\tau}\in\mathcal{F}_1^s}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}(u-u^s)\|_a^2\leq C\left(\|u-u^s\|_a^2+\left\|\frac{a-a^s}{a}\right\|_{L^\infty(D)}^2\|u^s\|_a^2\right).$$

In the other case $\boldsymbol{\tau} \in \mathcal{F}_2^s$, we observe that for arbitrary $\mathbf{0} \neq \boldsymbol{\tau} \in \mathcal{F}_2^s$,

$$\begin{aligned}\int_D a\nabla\partial^{\boldsymbol{\tau}}(u-u^s)\cdot\nabla v \mathrm{d}x = &- \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \psi^{\boldsymbol{\tau}-\boldsymbol{\nu}} a\nabla\partial^{\boldsymbol{\nu}}(u-u^s)\cdot\nabla v\mathrm{d}x \\ &- \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \psi^{\boldsymbol{\tau}-\boldsymbol{\nu}} a\nabla\partial^{\boldsymbol{\nu}} u^s\cdot\nabla v\mathrm{d}x, \qquad \forall v\in V.\end{aligned} \tag{22}$$

We used that there is $j > s$ such that $\tau_j > 0$, which implies that for $\boldsymbol{\nu} \neq \boldsymbol{\tau}$ such that $\boldsymbol{\nu} \leq \boldsymbol{\tau}$, either $\tau_j - \nu_j > 0$ yielding $\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}} a^s = 0$ or $\tau_j = \nu_j > 0$ yielding $\partial^{\boldsymbol{\nu}} u^s = 0$. Moreover, in the second sum above, we can restrict the index set to those $\boldsymbol{\nu}$ satisfying $\nu_j = 0$ for every $j > s$. In particular, always $\boldsymbol{\nu} \neq \boldsymbol{\tau}$. The estimate of the sum over $\boldsymbol{\tau} \in \mathcal{F}_2^s$ follows with a similar argument using Lemma 4.1 for the first sum and Lemma 4.2 for the second sum of the right hand side of equality (22), where we crucially use that $\boldsymbol{\nu} \neq \boldsymbol{\tau}$, which yield that the sum runs only over $\ell \in \{0, \dots, k-1\}$. Specifically,

$$\sum_{\boldsymbol{\tau}\in\mathcal{F}_2^s} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\partial^{\boldsymbol{\tau}}(u-u^s)\|_a^2 \leq C\left(\|u-u^s\|_a^2 + \max_{j>s}\{\rho_j^{-2\eta}\}\|u^s\|_a^2 \right).$$

Since by (18) and by the Cauchy–Schwarz inequality

$$\|u-u^s\|_a \leq \left\| \frac{a-a^s}{a} \right\|_{L^\infty(D)} \|u^s\|_a,$$

the assertion of the theorem follows. □

Remark 4.1 *The statement of Theorem 4.3 also holds true for the FE solution* $u^{\mathcal{T}_\ell}$ *and* $u^{s,\mathcal{T}_\ell}$ *for every truncation dimension* s_ℓ *with* $\ell \geq 0$.

4.2 FE differences

First we show parametric regularity with respect to the smoothness space. For every $\boldsymbol{\tau} \in \mathcal{F}$, we define the quantities

$$\kappa_1(\boldsymbol{\tau},\boldsymbol{\nu}) := \frac{\sqrt{\boldsymbol{\tau}!}}{\sqrt{\boldsymbol{\nu}!}} \frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}}|\nabla\psi^{\boldsymbol{\tau}-\boldsymbol{\nu}}|\Phi_{\overline{\boldsymbol{\beta}}}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!}, \quad \boldsymbol{\nu}\leq\boldsymbol{\tau}.$$

Similar to Lemma 4.1, the following lemma reveals that κ_1 has the property of a discrete probability density, which will be essential in the ensuing analysis.

Lemma 4.4 *Assume that for* $r \in \mathbb{N}$

$$\left\| \sum_{j\geq1} \rho_j \max\{|\nabla\psi_j|\Phi_{\overline{\boldsymbol{\beta}}}, |\psi_j|\} \right\|_{L^\infty(D)} =: K < C_r := \sup\left\{ c > 0 : \sqrt{r}c\, e^{\sqrt{r}c} \leq 1 \right\}. \tag{23}$$

Then for every $\boldsymbol{\tau} \in \mathbb{N}_0^{\mathbb{N}}$ *such that* $\|\boldsymbol{\tau}\|_{\ell^\infty} \leq r$

$$\sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \kappa_1(\boldsymbol{\tau},\boldsymbol{\nu}) \leq \sqrt{r}Ke^{\sqrt{r}K} < 1$$

and for every $\ell \leq k-1$ *and* $\boldsymbol{\nu} \in \Lambda_\ell$,

$$\sum_{\boldsymbol{\tau}\in R_{\boldsymbol{\nu},k}} \kappa_1(\boldsymbol{\tau},\boldsymbol{\nu}) \leq (k-\ell)\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}.$$

Proof. We set $k = |\boldsymbol{\tau}|$ and observe with the multinomial theorem

$$\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \kappa_1(\boldsymbol{\tau},\boldsymbol{\nu}) = \sum_{\ell=1}^{k} \sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},|\boldsymbol{\tau}-\boldsymbol{\nu}|=\ell} \kappa_1(\boldsymbol{\tau},\boldsymbol{\nu})$$
$$\le \sum_{\ell=1}^{k} r^{\ell/2} \sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},|\boldsymbol{\tau}-\boldsymbol{\nu}|=\ell} \ell \frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}} \max\{|\nabla\psi|\Phi_{\overline{\boldsymbol{\beta}}}, |\psi|\}^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!}$$
$$\le \sum_{\ell=1}^{k} r^{\ell/2}\ell \sum_{|\boldsymbol{m}|=\ell} \frac{\rho^{\boldsymbol{m}} \max\{|\nabla\psi|\Phi_{\overline{\boldsymbol{\beta}}}, |\psi|\}^{\boldsymbol{m}}}{\boldsymbol{m}!}$$
$$= \sum_{\ell=1}^{k} \frac{r^{\ell/2}}{(\ell-1)!} \left(\sum_{j\ge1} \rho_j \max\{|\nabla\psi_j|\Phi_{\overline{\boldsymbol{\beta}}}, |\psi_j|\} \right)^{\ell} \le \sqrt{r}Ke^{\sqrt{r}K} < 1,$$

where we applied that

$$|\nabla\psi^{\boldsymbol{\tau}-\boldsymbol{\nu}}|\Phi_{\overline{\boldsymbol{\beta}}} \le \sum_{j\ge1} (\tau_j-\nu_j)|\psi_j|^{\tau_j-\nu_j-1}|\nabla\psi_j|\Phi_{\overline{\boldsymbol{\beta}}} \prod_{i\neq j} |\psi_i|^{\tau_i-\nu_i} \le |\boldsymbol{\tau}-\boldsymbol{\nu}| \max\{|\nabla\psi_j|\Phi_{\overline{\boldsymbol{\beta}}}, |\psi|\}^{\boldsymbol{\tau}-\boldsymbol{\nu}}.$$

The second estimate follows similarly. □

Theorem 4.5 *Let the assumption of Lemma 4.4 be satisfied for a positive sequence $(\rho_j)_{j\ge1}$, and assume that $r \in \mathbb{N}$ and $K < C_r$. There exists a constant $C > 0$ such that for every $\boldsymbol{y} \in U$*

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\le r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta\partial^{\boldsymbol{\tau}} u(\boldsymbol{y}))\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)}$$
$$\le C \frac{1}{a_{\min}(\boldsymbol{y})} \left((1 + \| |\nabla Z(\boldsymbol{y})|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)}) \|u(\boldsymbol{y})\|^2_{a(\boldsymbol{y})} + \|\Delta u(\boldsymbol{y})\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)} \right).$$

Proof. Let $\mathbf{0} \neq \boldsymbol{\tau} \in \mathcal{F}$ be given such that $\|\boldsymbol{\tau}\|_{\ell^\infty} \le r$. We observe that for every $v \in C_0^\infty(D)$,

$$-\int_D av\Delta\partial^{\boldsymbol{\tau}} u \mathrm{d}x = \int_D \left(\nabla a \cdot \nabla\partial^{\boldsymbol{\tau}} u + \sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \nabla\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}} a \cdot \nabla\partial^{\boldsymbol{\nu}} u + \partial^{\boldsymbol{\tau}-\boldsymbol{\nu}} a \Delta\partial^{\boldsymbol{\nu}} u \right) v \mathrm{d}x \,.$$

Using the density of $C_0^\infty(D)$ in $L^2_{\overline{\boldsymbol{\beta}}}(D)$, we choose the test function $v = -\Phi^2_{\overline{\boldsymbol{\beta}}}/a\Delta\partial^{\boldsymbol{\tau}} u$, multiply by $\rho^{2\boldsymbol{\tau}}/\boldsymbol{\tau}!$, and apply the Young inequality for arbitrary $\varepsilon > 0$ to obtain

$$\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta\partial^{\boldsymbol{\tau}} u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)} = -\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \int_D \left(\frac{\nabla a}{a} \cdot \nabla\partial^{\boldsymbol{\tau}} u + \sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \frac{\nabla\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}} a}{a} \cdot \nabla\partial^{\boldsymbol{\nu}} u \right) \Delta\partial^{\boldsymbol{\tau}} u \Phi^2_{\overline{\boldsymbol{\beta}}} \mathrm{d}x$$
$$- \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \int_D \left(\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \frac{\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}} a}{a} \Delta\partial^{\boldsymbol{\nu}} u \right) \Delta\partial^{\boldsymbol{\tau}} u \Phi^2_{\overline{\boldsymbol{\beta}}} \mathrm{d}x$$
$$\le \varepsilon \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!} \|\Delta\partial^{\boldsymbol{\tau}} u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)}$$
$$+ \frac{1}{4\varepsilon} \int_D \left(|\nabla Z|\Phi_{\overline{\boldsymbol{\beta}}} \frac{\rho^{\boldsymbol{\tau}} |\nabla\partial^{\boldsymbol{\tau}} u|}{\sqrt{\boldsymbol{\tau}!}} + \sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \kappa_1(\boldsymbol{\tau},\boldsymbol{\nu}) \frac{\rho^{\boldsymbol{\nu}} |\nabla\partial^{\boldsymbol{\nu}} u|}{\sqrt{\boldsymbol{\nu}!}} \right)^2 \mathrm{d}x$$
$$+ \int_D \sum_{\boldsymbol{\nu}\le\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \kappa_0(\boldsymbol{\tau},\boldsymbol{\nu}) \frac{\rho^{\nu} |\Delta\partial^{\boldsymbol{\nu}} u|\Phi_{\overline{\boldsymbol{\beta}}}}{\sqrt{\boldsymbol{\nu}!}} \frac{\rho^{\boldsymbol{\tau}} |\Delta\partial^{\boldsymbol{\tau}} u|\Phi_{\overline{\boldsymbol{\beta}}}}{\sqrt{\boldsymbol{\tau}!}} \mathrm{d}x. \tag{24}$$

Note that $\nabla((\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}a)/a) = \nabla\psi^{\boldsymbol{\tau}-\boldsymbol{\nu}}$. Note also the change of the order of summation: for any sequence $(\kappa'(\boldsymbol{\tau},\boldsymbol{\nu}))$ and for any $k \in \mathbb{N}$

$$\sum_{\boldsymbol{\tau}\in\Lambda_k}\sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \kappa'(\boldsymbol{\tau},\boldsymbol{\nu}) = \sum_{\ell=0}^{k-1}\sum_{\boldsymbol{\nu}\in\Lambda_\ell}\sum_{\boldsymbol{\tau}\in R_{\boldsymbol{\nu},k}} \kappa'(\boldsymbol{\tau},\boldsymbol{\nu}), \tag{25}$$

which implies with Lemma 4.1 and with the elementary estimate $xy \leq (x^2+y^2)/2$, $x, y > 0$, that for any $k \geq 1$,

$$\begin{aligned}\sum_{\boldsymbol{\tau}\in\Lambda_k}\int_D \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}} \kappa_0(\boldsymbol{\tau},\boldsymbol{\nu}) \frac{\rho^{\nu}|\Delta\partial^{\boldsymbol{\nu}}u|\Phi_{\overline{\boldsymbol{\beta}}}}{\sqrt{\boldsymbol{\nu}!}}\frac{\rho^{\boldsymbol{\tau}}|\Delta\partial^{\boldsymbol{\tau}}u|\Phi_{\overline{\boldsymbol{\beta}}}}{\sqrt{\boldsymbol{\tau}!}}\mathrm{d}x \\ \leq \frac{1}{2}\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sum_{\boldsymbol{\nu}\in\Lambda_\ell}\frac{\rho^{2\nu}}{\boldsymbol{\nu}!}\|\Delta\partial^{\boldsymbol{\nu}}u\|_{L^2_{\overline{\boldsymbol{\beta}}}(D)} + \frac{1}{2}(e^{\sqrt{r}K}-1)\sum_{\boldsymbol{\tau}\in\Lambda_k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\Delta\partial^{\boldsymbol{\tau}}u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)}.\end{aligned} \tag{26}$$

Similarly, we obtain with Lemma 4.4

$$\begin{aligned}&\sum_{\boldsymbol{\tau}\in\Lambda_k}\frac{1}{4\varepsilon}\int_D\left(|\nabla Z|\Phi_{\overline{\boldsymbol{\beta}}}\frac{\rho^{\boldsymbol{\tau}}|\nabla\partial^{\boldsymbol{\tau}}u|}{\sqrt{\boldsymbol{\tau}!}} + \sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}}\kappa_1(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{\boldsymbol{\nu}}|\nabla\partial^{\boldsymbol{\nu}}u|}{\sqrt{\boldsymbol{\nu}!}}\right)^2\mathrm{d}x \\ &\leq \frac{1}{2\varepsilon}\frac{\||\nabla Z|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)}}{a_{\min}}\sum_{\boldsymbol{\tau}\in\Lambda_k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}u\|_a^2 + \frac{1}{2\varepsilon}\sum_{\boldsymbol{\tau}\in\Lambda_k}\int_D\left(\sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}}\kappa_1(\boldsymbol{\tau},\boldsymbol{\nu})\frac{\rho^{\boldsymbol{\nu}}|\nabla\partial^{\boldsymbol{\nu}}u|}{\sqrt{\boldsymbol{\nu}!}}\right)^2\mathrm{d}x \\ &\leq \frac{1}{2\varepsilon}\frac{1}{a_{\min}}\left(\||\nabla Z|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)}\sum_{\boldsymbol{\tau}\in\Lambda_k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}u\|_a^2 + \sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell-1)!}\sum_{\boldsymbol{\nu}\in\Lambda_\ell}\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\|\partial^{\boldsymbol{\nu}}u\|_a^2\right).\end{aligned} \tag{27}$$

As before by the proof of [6, Theorem 4.1 and Equation (4.18)], for any $\delta \in [\sqrt{r}K/\log(2), 1)$ and for every $\ell \in \mathbb{N}_0$,

$$\sum_{\boldsymbol{\nu}\in\Lambda_\ell}\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\|\partial^{\boldsymbol{\nu}}u\|_a^2 \leq \delta^\ell\|u\|_a^2. \tag{28}$$

Hence,

$$\begin{aligned}\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell-1)!}\sum_{\boldsymbol{\nu}\in\Lambda_\ell}\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\|\partial^{\boldsymbol{\nu}}u\|_a^2 &\leq \sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell-1)!}\delta^\ell\|u\|_a^2 \\ &\leq \delta^k\sum_{\ell=0}^{k-1}\log(2)\frac{(\log(2))^{k-\ell-1}}{(k-\ell-1)!}\|u\|_a^2 \\ &\leq \delta^k\log(2)2\|u\|_a^2 = \delta^k\log(4)\|u\|_a^2.\end{aligned} \tag{29}$$

We choose $0 < \varepsilon < 1 - e^{\sqrt{r}K}/2$, which implies that $C_\varepsilon := (1-\varepsilon-(e^{\sqrt{r}K}-1)/2)^{-1} < 2$. This allows us to subtract $\Delta\partial^{\boldsymbol{\tau}}u$-terms summed over Λ_k in (24) and (26) while obtaining a constant $C_\varepsilon^{-1} > 1/2$ which is shifted to the left hand side, i.e.,

$$\begin{aligned}\sum_{\boldsymbol{\tau}\in\Lambda_k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\Delta\partial^{\boldsymbol{\tau}}u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)} &\leq \frac{C_\varepsilon}{2\varepsilon}\frac{1}{a_{\min}}\left(\||\nabla Z|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)} + \log(4)\right)\delta^k\|u\|_a^2 \\ &\quad + \frac{C_\varepsilon}{2}\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sum_{\boldsymbol{\nu}\in\Lambda_\ell}\frac{\rho^{2\nu}}{\boldsymbol{\nu}!}\|\Delta\partial^{\boldsymbol{\nu}}u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)},\end{aligned}$$

where we have also inserted (27), (28) and (29). We sum over $k \geq 1$ and obtain with (21)

$$\sum_{k\geq 1}\sum_{\boldsymbol{\tau}\in\Lambda_k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\Delta\partial^{\boldsymbol{\tau}}u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)} \leq \frac{C_\varepsilon}{2\varepsilon}\frac{1}{a_{\min}}\left(\||\nabla Z|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)} + \log(4)\right)\frac{\delta}{1-\delta}\|u\|_a^2$$
$$+\frac{C_\varepsilon}{2}(e^{\sqrt{r}K}-1)\sum_{\ell\geq 0}\sum_{\boldsymbol{\nu}\in\Lambda_\ell}\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\|\Delta\partial^{\boldsymbol{\nu}}u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)},$$

which implies the assertion as at the end of the proof of Theorem 4.3, since $(C_\varepsilon/2)(e^{\sqrt{r}K}-1) < 1$. □

Theorem 4.6 *Let the assumption of Lemma 4.4 be satisfied for a positive sequence* $(\rho_j)_{j\geq 1}$, $r \in \mathbb{N}$, *and* $K < C_r$. *There exists a constant* $C > 0$ *such that for every* $\boldsymbol{y} \in U$

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\leq r}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}(u(\boldsymbol{y}) - u^{\mathcal{T}_\ell}(\boldsymbol{y}))\|^2_{a(\boldsymbol{y})}$$
$$\leq C\left(\frac{(a_{\max}(\boldsymbol{y}))^2}{(a_{\min}(\boldsymbol{y}))^4}(1 + \|\nabla Z(\boldsymbol{y})|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)})\right)\|f\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)}\, M_\ell^{-2/d}.$$

Proof. Define the Galerkin projection $\mathcal{P}_h : V \to V_\ell$ for every $w \in V$ by

$$\int_D a\nabla(w - \mathcal{P}_h w)\cdot\nabla v \mathrm{d}x = 0, \quad \forall v \in V_\ell.$$

Since $(\mathcal{I} - \mathcal{P}_h)v = 0$ for every $v \in V_\ell$, it holds that for every $\boldsymbol{\tau} \in \mathcal{F}$,

$$\|\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a \leq \|\mathcal{P}_h\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a + \|(\mathcal{I} - \mathcal{P}_h)\partial^{\boldsymbol{\tau}}u\|_a. \tag{30}$$

Let $\boldsymbol{\tau} \in \mathcal{F}$ be such that $\|\boldsymbol{\tau}\|_{\ell^\infty(\mathbb{N})} \leq r$ and $|\boldsymbol{\tau}| = k$ for some $k \in \mathbb{N}$. We observe that

$$\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\int_D a|\nabla\mathcal{P}_h\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})|^2\mathrm{d}x \leq \int_D\sum_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}}\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu})a\frac{\rho^{\boldsymbol{\nu}}|\nabla\partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})|}{\sqrt{\boldsymbol{\nu}!}}\frac{\rho^{\boldsymbol{\tau}}|\nabla\mathcal{P}_h\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})|}{\sqrt{\boldsymbol{\tau}!}}\mathrm{d}x.$$

A twofold application of the Cauchy–Schwarz inequality using that by the first estimate of Lemma 4.1 for fixed $\boldsymbol{\tau} \in \mathcal{F}$ such that $\|\boldsymbol{\tau}\|_{\ell^\infty(\mathbb{N})} \leq r$ the sequence $(\kappa_0(\boldsymbol{\tau},\boldsymbol{\nu}))_{\boldsymbol{\nu}\leq\boldsymbol{\tau},\boldsymbol{\nu}\neq\boldsymbol{\tau}}$ is a discrete probability density implies with the change of the order of summation in (25) and the second estimate in Lemma 4.1 the bound

$$\sum_{|\tau|=k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\mathcal{P}_h\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a^2 \leq \sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sum_{|\boldsymbol{\nu}|=\ell}\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\|\partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})\|_a^2. \tag{31}$$

By the approximation property in (10), by (30), (31), the Young inequality for any $\varepsilon > 0$, and by the change of the order of summation that implied (21)

$$\sum_{k\geq 1}\sum_{|\boldsymbol{\tau}|=k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}(u - u^{\mathcal{T}_\ell})\|_a^2 \leq (1+\varepsilon)\sum_{k\geq 1}\sum_{\ell=0}^{k-1}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sum_{|\boldsymbol{\nu}|=\ell}\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\|\partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})\|_a^2$$
$$+\left(1 + \frac{1}{\varepsilon}\right)\sum_{k\geq 1}\sum_{|\boldsymbol{\tau}|=k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|(\mathcal{I} - \mathcal{P}_h)\partial^{\boldsymbol{\tau}}u\|_a^2$$
$$\leq (1+\varepsilon)(e^{\sqrt{r}K} - 1)\sum_{\ell\geq 0}\sum_{|\boldsymbol{\nu}|=\ell}\frac{\rho^{2\boldsymbol{\nu}}}{\boldsymbol{\nu}!}\|\partial^{\boldsymbol{\nu}}(u - u^{\mathcal{T}_\ell})\|_a^2$$
$$+\left(1 + \frac{1}{\varepsilon}\right)C\|a\|_{L^\infty(D)}M_\ell^{-2/d}\sum_{k\geq 1}\sum_{|\boldsymbol{\tau}|=k}\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\Delta\partial^{\boldsymbol{\tau}}u\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)}.$$

Hence, we choose $\varepsilon < (2-e^{\sqrt{r}K})/(e^{\sqrt{r}K}-1)$ and conclude with Theorem 4.5 and (14) that there exists a constant $C>0$ such that

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\le r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}\|\partial^{\boldsymbol{\tau}}(u-u^{\mathcal{T}_\ell})\|_a^2 \le C\left(\frac{(a_{\max})^2}{(a_{\min})^4}(1+\|\nabla Z|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)})\right)\|f\|^2_{L^2_{\overline{\boldsymbol{\beta}}}(D)}\, M_\ell^{-2/d}.$$

□

Remark 4.2 *The parametric regularity estimate in Theorem 4.6 also holds if* $f\in(V^*,L^2_{\overline{\boldsymbol{\beta}}}(D))_{t,\infty}$ *for some* $t\in[0,1]$ *with the FE error bounded by an absolute constant times* $M_\ell^{-2t/d}$*. This can be shown by interpolation applied in the last and next to last step of the proof of Theorem 4.6, see also Remark 2.2).*

Let $G(\cdot)\in L^2_{\overline{\boldsymbol{\beta}}}(D)$ denote a solution functional of interest. We are interested in the parametric regularity of $G(u-u^{\mathcal{T}_\ell})$. Introduce v_G and $v_G^{\mathcal{T}_\ell}$ to be the solution and respective FE solution to the adjoint problem with right hand side $G(\cdot)$. It holds that

$$G(u-u^{\mathcal{T}_\ell}) = \int_D a\nabla(u-u^{\mathcal{T}_\ell})\cdot\nabla(v_G-v_G^{\mathcal{T}_\ell})\mathrm{d}x\,.$$

Proposition 4.7 *For* $(\widetilde{\rho})_{j\ge1}$ *defined by* $\widetilde{\rho}_j := \sqrt{2}\rho_j$, $j\in\mathbb{N}$*, assume that* $(\widetilde{\rho}_j)_{j\ge1}$ *satisfies the sparsity assumption in* (19) *of Lemma 4.1.*

Then, for every $\boldsymbol{y}\in U$

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\le r} \frac{\rho^{2\boldsymbol{\tau}}}{\tau!}|\partial^{\boldsymbol{\tau}}G(u(\boldsymbol{y})-u^{\mathcal{T}_\ell}(\boldsymbol{y}))|^2 \le 4\left(\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\le r}\frac{\widetilde{\rho}^{2\boldsymbol{\tau}}}{\tau!}\|\partial^{\boldsymbol{\tau}}(u(\boldsymbol{y})-u^{\mathcal{T}_\ell}(\boldsymbol{y}))\|^2_{a(\boldsymbol{y})}\right)$$
$$\times\left(\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\le r}\frac{\widetilde{\rho}^{2\boldsymbol{\tau}}}{\tau!}\|\partial^{\boldsymbol{\tau}}(v_G(\boldsymbol{y})-v_G^{\mathcal{T}_\ell}(\boldsymbol{y}))\|^2_{a(\boldsymbol{y})}\right).$$

Proof. We observe that for every $\boldsymbol{\tau}\in\mathcal{F}$

$$\frac{\rho^{\boldsymbol{\tau}}}{\sqrt{\boldsymbol{\tau}!}}\partial^{\boldsymbol{\tau}}G(u-u^{\mathcal{T}_\ell}) = \frac{\rho^{\boldsymbol{\tau}}}{\sqrt{\boldsymbol{\tau}!}}\int_D\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}}\binom{\boldsymbol{\tau}}{\boldsymbol{\nu}}\left[\sum_{\boldsymbol{m}\le\boldsymbol{\nu}}\binom{\boldsymbol{\nu}}{\boldsymbol{m}}\psi^{\boldsymbol{\nu}-\boldsymbol{m}}(\sqrt{a}\nabla\partial^{\boldsymbol{m}}(u-u^{\mathcal{T}_\ell}))\right]$$
$$\times(\sqrt{a}\nabla\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G-v_G^{\mathcal{T}_\ell}))\mathrm{d}x$$
$$=\int_D\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}}\sqrt{\binom{\boldsymbol{\tau}}{\boldsymbol{\nu}}}\left[\sum_{\boldsymbol{m}\le\boldsymbol{\nu}}\kappa_0(\boldsymbol{\nu},\boldsymbol{m})\left(\frac{\rho^{\boldsymbol{m}}}{\sqrt{\boldsymbol{m}!}}\sqrt{a}\nabla\partial^{\boldsymbol{m}}(u-u^{\mathcal{T}_\ell})\right)\right]$$
$$\times\left(\frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{\sqrt{(\boldsymbol{\tau}-\boldsymbol{\nu})!}}\sqrt{a}\nabla\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G-v_G^{\mathcal{T}_\ell})\right)\mathrm{d}x.$$

It holds that $\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}}\binom{\boldsymbol{\tau}}{\boldsymbol{\nu}}=2^{\boldsymbol{\tau}}$. By a twofold application of the Cauchy–Schwarz inequality

$$\frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}|\partial^{\boldsymbol{\tau}}G(u-u^{\mathcal{T}_\ell})|^2\le\left(\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}}\sqrt{\binom{\boldsymbol{\tau}}{\boldsymbol{\nu}}}\|[\dots]\|_{L^2(D)}\frac{\rho^{\boldsymbol{\tau}-\boldsymbol{\nu}}}{\sqrt{(\boldsymbol{\tau}-\boldsymbol{\nu})!}}\|\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G-v_G^{\mathcal{T}_\ell})\|_a\right)^2$$
$$\le 2^{\boldsymbol{\tau}}\sum_{\boldsymbol{\nu}\le\boldsymbol{\tau}}\|[\dots]\|^2_{L^2(D)}\frac{\rho^{2(\boldsymbol{\tau}-\boldsymbol{\nu})}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!}\|\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G-v_G^{\mathcal{T}_\ell})\|_a^2.$$

We define the sequence $(\widetilde{\rho})_{j\geq 1}$ by $\widetilde{\rho}_j := \sqrt{2}\rho_j$, $j \in \mathbb{N}$. By a change of the order of summation

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\leq r} \frac{\rho^{2\boldsymbol{\tau}}}{\boldsymbol{\tau}!}|\partial^{\boldsymbol{\tau}} G(u-u^{\mathcal{T}_\ell})|^2 \leq \sum_{\|\boldsymbol{\nu}\|_{\ell^\infty}\leq r} 2^{\boldsymbol{\nu}}\|[\dots]\|^2_{L^2(D)} \sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\leq r, \boldsymbol{\tau}\geq\boldsymbol{\nu}} \frac{\widetilde{\rho}^{2(\boldsymbol{\tau}-\boldsymbol{\nu})}}{(\boldsymbol{\tau}-\boldsymbol{\nu})!}\|\partial^{\boldsymbol{\tau}-\boldsymbol{\nu}}(v_G - v_G^{\mathcal{T}_\ell})\|^2_a.$$

Since $\sum_{\boldsymbol{m}\leq\boldsymbol{\nu}}\kappa_0(\boldsymbol{\nu},\boldsymbol{m}) \leq 2$ due to Lemma 4.1, by the Cauchy–Schwarz inequality and (25)

$$\begin{aligned}
&\sum_{k\geq 0}\sum_{\boldsymbol{\nu}\in\Lambda_k}\int_D \left(\sum_{\boldsymbol{m}\leq\boldsymbol{\nu}}\kappa_0(\boldsymbol{\nu},\boldsymbol{m})\frac{\widetilde{\rho}^{\boldsymbol{m}}}{\sqrt{\boldsymbol{m}!}}\sqrt{a}|\nabla\partial^{\boldsymbol{m}}(u-u^{\mathcal{T}_\ell})|\right)^2 \mathrm{d}x \\
&\leq 2\sum_{k\geq 0}\sum_{\boldsymbol{\nu}\in\Lambda_k}\sum_{\boldsymbol{m}\leq\boldsymbol{\nu}}\kappa_0(\boldsymbol{\nu},\boldsymbol{m})\frac{\widetilde{\rho}^{2\boldsymbol{m}}}{\boldsymbol{m}!}\|\partial^{\boldsymbol{m}}(u-u^{\mathcal{T}_\ell})\|^2_a \\
&\leq 2\sum_{k\geq 0}\sum_{\ell=0}^{k}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sum_{\boldsymbol{m}\in\Lambda_\ell}\frac{\widetilde{\rho}^{2\boldsymbol{m}}}{\boldsymbol{m}!}\|\partial^{\boldsymbol{m}}(u-u^{\mathcal{T}_\ell})\|^2_a \\
&= 2\sum_{\ell\geq 0}\sum_{k\geq\ell}\frac{(\sqrt{r}K)^{k-\ell}}{(k-\ell)!}\sum_{\boldsymbol{m}\in\Lambda_\ell}\frac{\widetilde{\rho}^{2\boldsymbol{m}}}{\boldsymbol{m}!}\|\partial^{\boldsymbol{m}}(u-u^{\mathcal{T}_\ell})\|^2_a \\
&\leq 4\sum_{\|\boldsymbol{m}\|_{\ell^\infty}\leq r}\frac{\widetilde{\rho}^{2\boldsymbol{m}}}{\boldsymbol{m}!}\|\partial^{\boldsymbol{m}}(u-u^{\mathcal{T}_\ell})\|^2_a,
\end{aligned}$$

which proves the assertion together with the previous inequality. □

The following theorem is directly implied by Theorem 4.6 and Proposition 4.7.

Theorem 4.8 *Let the assumption of Lemma 4.4 be satisfied for a positive sequence $(\rho_j)_{j\geq 1}$, and let $r\in\mathbb{N}$ and assume that $K < C_r/\sqrt{2}$.*

Then there exists a constant $C>0$ such that for every $\boldsymbol{y}\in U$

$$\sum_{\|\boldsymbol{\tau}\|_{\ell^\infty}\leq r}\frac{\rho^{2\boldsymbol{\tau}}}{\tau!}|\partial^{\boldsymbol{\tau}}G(u(\boldsymbol{y})-u^{\mathcal{T}_\ell}(\boldsymbol{y}))|^2 \leq C\left(\frac{(a_{\max}(\boldsymbol{y}))^2}{(a_{\min}(\boldsymbol{y}))^4}(1+\||\nabla Z(\boldsymbol{y})|\Phi_{\overline{\boldsymbol{\beta}}}\|^2_{L^\infty(D)})\right)^2 M_\ell^{-4/d}\|f\|^2_{L^2_{\boldsymbol{\beta}}(D)}\|G\|^2_{L^2_{\boldsymbol{\beta}}(D)}.$$

Remark 4.3 *The statement of Theorem 4.8 also holds true for dimensionally truncated solution u^s and $u^{s,\mathcal{T}_\ell}$ for every truncation dimension $s\in\mathbb{N}$. In particular, the constant C which appears in the error bound is independent of s.*

Remark 4.4 *The parametric regularity estimate in Theorem 4.8 also holds if $f\in(V^*,L^2_{\boldsymbol{\beta}}(D))_{t,\infty}$ and $G(\cdot)\in(V^*,L^2_{\boldsymbol{\beta}}(D))_{t',\infty}$ for $t,t'\in[0,1]$. Then, the FE discretization error contribution to the overall error is bounded by a constant times $M_\ell^{-2(t+t')/d}$. This follows from Remark 4.2.*

5 Multilevel QMC convergence analysis

The sequences $(b_j)_{j\geq 1}$ and $(\bar{b}_j)_{j\geq 1}$ in the assumptions in (**A1**) and (**A2**) will be the input for the QMC weight sequence $(\gamma_j)_{j\geq 1}$ of product weights. In the multilevel QMC quadrature rule

Q_L^* we will generally apply a randomly shifted lattice rule on level $\ell = 0$ with respect to the QMC weight sequence

$$\gamma_j = b_j^{2p'}, \quad j \geq 1, \tag{32}$$

for some $p' \in (0,1)$ and on the levels $\ell = 1, \dots, L$ with respect to the QMC weight sequence

$$\bar{\gamma}_j = (b_j^{1-\theta} \vee \bar{b}_j)^{2\bar{p}'}, \quad j \geq 1 \tag{33}$$

for some $p' \in (0,1)$ and some $\theta \in (0,1)$. Here, for $c_1, c_2 \in \mathbb{R}$, $c_1 \vee c_2 := \max\{c_1, c_2\}$.

Theorem 5.1 *For every $L \in \mathbb{N}_0$ and sequences $(s_\ell)_{\ell=0,\dots,L}$ and $(N_\ell)_{\ell=0,\dots,L}$, the ensuing error estimate holds under the following conditions:*

1. *Gaussian weight functions: $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$ and $(b_j^{1-\theta} \vee \bar{b}_j)_{j\geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for some $\bar{p} \in [p, 2)$ with $\chi = 1/(2p) + 1/4 - \varepsilon$ and $\bar{\chi} = 1/(2\bar{p}) + 1/4 - \bar{\varepsilon}$. The QMC weight sequence in (32) is applied with $p' = p/4 + 1/2 - \varepsilon p$ on the level $\ell = 0$ for $\varepsilon \in (0, 3/4 - 1/(2p))$. The QMC weight sequence in (33) is applied with $\bar{p}' = \bar{p}/4 + 1/2 - \bar{\varepsilon}\bar{p}$ on the levels $\ell = 1, \dots, L$ for $\bar{\varepsilon} \in (0, 3/4 - 1/(2\bar{p}))$.*

2. *Exponential weight functions: $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 1]$ and for $(b_j^{1-\theta} \vee \bar{b}_j)_{j\geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for some $\bar{p} \in [p, 1]$ with $\chi = 1/p - 1/2$ and $\bar{\chi} = 1/\bar{p} - 1/2$. The QMC weight sequence in (32) is applied with $p' = 1 - p/2$ on the level $\ell = 0$. The QMC weight sequence in (33) is applied with $\bar{p}' = 1 - \bar{p}/2$ on the levels $\ell = 1, \dots, L$.*

There exists a constant $C > 0$ that is in particular independent of $(M_\ell)_{\ell\geq 0}$, $(s_\ell)_{\ell=0,\dots,L}$, $(N_\ell)_{\ell=0,\dots,L}$, and of $L \in \mathbb{N}_0$, such that

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u)) - Q_L^*(G(u^L))|^2)} \leq C \Bigg(\max_{j>s_L}\{b_j^{2(1-\varepsilon)}\} + M_L^{-4/d} + (\varphi(N_0))^{-2\chi} + \sum_{\ell=1}^{L} (\varphi(N_\ell))^{-2\bar{\chi}} \left(\xi_{\ell,\ell-1} \max_{j>s_{\ell-1}} \{b_j^{2\theta}\} + M_{\ell-1}^{-4/d} \right) \Bigg)^{1/2},$$

where $\xi_{\ell,\ell-1} := 0$ if $s_\ell = s_{\ell-1}$ and $\xi_{\ell,\ell-1} := 1$ otherwise.

Proof. By the triangle inequality, for $\ell = 1, \dots, L$,

$$\begin{aligned} &|(I_{s_\ell} - Q_{s_\ell,N_\ell})(G(u^\ell) - G(u^{\ell-1}))| \\ &\quad \leq |(I_{s_\ell} - Q_{s_\ell,N_\ell})(G(u^{s_\ell,\mathcal{T}_\ell}) - G(u^{s_\ell,\mathcal{T}_{\ell-1}}))| + |(I_{s_\ell} - Q_{s_\ell,N_\ell})(G(u^{s_\ell,\mathcal{T}_{\ell-1}}) - G(u^{s_{\ell-1},\mathcal{T}_{\ell-1}}))| \end{aligned}$$

and

$$\begin{aligned} &|(I_{s_\ell} - Q_{s_\ell,N_\ell})(G(u^{s_\ell,\mathcal{T}_\ell}) - G(u^{s_\ell,\mathcal{T}_{\ell-1}}))| \\ &\quad \leq |(I_{s_\ell} - Q_{s_\ell,N_\ell})(G(u^{s_\ell}) - G(u^{s_\ell,\mathcal{T}_\ell}))| + |(I_{s_\ell} - Q_{s_\ell,N_\ell})(G(u^{s_\ell}) - G(u^{s_\ell,\mathcal{T}_{\ell-1}}))|, \end{aligned}$$

where we recall that $u^\ell := u^{s_\ell,\mathcal{T}_\ell}$, $\ell = 0, \dots, L$. We wish to show the conditions of Theorem 3.1 for integrands $\boldsymbol{y} \mapsto G(u^{s_\ell}(\boldsymbol{y})) - G(u^{s_\ell,\mathcal{T}_\ell}(\boldsymbol{y}))$ and $\boldsymbol{y} \mapsto G(u^{s_\ell,\mathcal{T}_{\ell-1}}(\boldsymbol{y})) - G(u^{s_{\ell-1},\mathcal{T}_{\ell-1}}(\boldsymbol{y}))$.

Setting

$$\bar{K} := \left\| \sum_{j\geq 1} \frac{\max\{|\nabla\psi_j|\Phi_{\overline{\boldsymbol{\beta}}}, |\psi_j|\}}{\bar{b}_j} \right\|_{L^\infty(D)} < \infty, \tag{34}$$

the conditions of Theorem 3.1 are satisfied for the integrand $\boldsymbol{y} \mapsto G(u^{s_\ell}(\boldsymbol{y})) - G(u^{s_\ell,\mathcal{T}_\ell}(\boldsymbol{y}))$ with the sequence $(\bar{b}_j)_{j\geq 1}$ and $\kappa < C_r/(\sqrt{2}\bar{K})$ by Theorem 4.8 and Remark 4.3 with $r = 1$. Specifically, apply Theorem 4.8 and Remark 4.3 with $\rho_j = \kappa/\bar{b}_j$, $j \geq 1$.

For the integrand $\boldsymbol{y} \mapsto G(u^{s_\ell,\mathcal{T}_{\ell-1}}(\boldsymbol{y})) - G(u^{s_{\ell-1},\mathcal{T}_{\ell-1}}(\boldsymbol{y}))$, we apply Theorem 4.3 with $\rho_j = \kappa/b_j^{1-\theta}$, $j \geq 1$. Then, the condition of Theorem 4.3 is satisfied for $\eta = \theta/(1-\theta)$ and $\kappa < \log(2)/K$, where K is as in assumption (**A1**). Hence, the conditions of Theorem 3.1 are satisfied for the integrand $\boldsymbol{y} \mapsto G(u^{s_\ell,\mathcal{T}_{\ell-1}}(\boldsymbol{y})) - G(u^{s_{\ell-1},\mathcal{T}_{\ell-1}}(\boldsymbol{y}))$. Since the sequence $(b_j^{1-\theta} \vee \bar{b}_j)_{j\geq 1}$ dominates $(b_j^{1-\theta})_{j\geq 1}$ and $(\bar{b}_j)_{j\geq 1}$, Theorem 3.1 can be applied with $\tilde{b}_j = b_j^{1-\theta} \vee \bar{b}_j$, $j \geq 1$. For the exponential weight functions, we note that $\eta_1 = C(\max_{j>s_{\ell-1}}\{b_j^\theta\} + M_{\ell-1}^{-2/d})$ for a constant $C > 0$ (independent of ℓ), and with $\eta_2 = 8$ in the notation of the second point of Theorem 3.1.

On discretization level $\ell = 0$, the parametric integrand is $\boldsymbol{y} \mapsto G(u^{s_0,\mathcal{T}_0})$. The conditions of Theorem 3.1 are satisfied with $\tilde{b}_j = b_j$, $j \geq 1$ (see also [19, Theorems 11 and 13]). The assertion follows with (15) and (17). □

Remark 5.1 *If $f \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t,\infty}$ and $G(\cdot) \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t',\infty}$ for some $t, t' \in [0,1]$, then the error estimate in Theorem 5.1 also holds with an error bounded by an absolute multiple of $M_\ell^{-2(t+t')/d}$ on mesh level ℓ.*

Remark 5.2 *When the GRF Z is stationary in $D \subset \mathbb{R}^d$, the* covariance kernel $k(x, x') := \mathbb{E}(Z(x)Z(x'))$ *of Z depends only on $x - x'$, cp. [2]. A particular parametric family of covariances for stationary GRF's is due to B. Matérn. Here, the covariance kernel depends on two parameters $\nu, \lambda > 0$, where λ is referred to as correlation length and $Z \in C^t(\overline{D})$, μ-a.s., for every positive real number $t < \nu$. Wavelet type function systems exist which allow to represent the GRF Z in terms of a sequence $(y_j)_{j\geq 1}$ of independent, standard normally distributed y_j, that satisfy Assumption (**A1**) with $b_j \sim j^{-\widehat{\beta}/d}$, $j \geq 1$, for every $\widehat{\beta} < \nu$, cp. e.g. [7, Corollary 4.3]. In [7], the random field Z in D is constructed by restriction of a GRF defined on suitable product domain that depends on the correlation length λ and which is a superset of D. For a constructive approach to obtain function systems of expansions with i.i.d. coefficients, we refer for example to [14] and the references there. For a discussion of the Hölder regularity and $L^q(\Omega)$ integrability of GRFs expanded in generic wavelets, we refer to [19, Section 9]. There, also if $C^t(\overline{D})$-regularity of the respective GRF Z holds as an implication by [19, Proposition 18], the generic wavelets satisfy Assumption (**A1**) with $b_j \sim j^{-\widehat{\beta}/d}$, $j \geq 1$, for every $\widehat{\beta} < t$.*

Remark 5.3 *In the case of single-level QMC, also fractional Hölder regularity of the lognormal coefficent a is covered by our theory in [19]. The GRF of the model function system of generic wavelets discussed in [19, Section 9] is for $d = 1$ and for wavelets that are scaled to decay as $\|\psi_j\|_{L^\infty(D)} \sim j^{-1/2-\varepsilon}$, $j \geq 1$, a member of $L^q(\Omega; C^{1/2+\varepsilon'}(\overline{D}))$, for every $q \in [1, \infty)$ and for every $\varepsilon > \varepsilon' > 0$, cp. [19, Proposition 19]. The sequence $(b_j)_{j\geq 1}$ may be chosen such that $b_j \sim j^{-1/2-\varepsilon'}$, for every $j \geq 1$ and for some $\varepsilon' \in (0, \varepsilon)$. For every $p > 2/(1+2\varepsilon')$, this sequence $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ is admissible with Gaussian weight functions for every $\varepsilon' > 0$, cp. [19, Theorem 11] and therefore QMC with Gaussian weight functions and product weights is applicable for every $\varepsilon > 0$. However, for $1/2 > \varepsilon > 0$, the convergence theory for QMC with product weights in [19, Theorem 13] does not seem to be applicable with exponential weight functions in this case.*

6 Error vs. work analysis

In the estimate of Theorem 5.1, the error contributions of the QMC quadrature and the spatial approximation by FE and dimension truncation are coupled on the different levels. The numbers of QMC points per level should minimize the error estimate and a corresponding work measure. In this manuscript, we will consider locally supported functions $(\psi_j)_{j\geq 1}$ as for example certain multiresolution analyses (MRA). Note that this will only affect the choice of the work measure for the assembly of stiffness matrices.

Let us assume that the MRA $(\psi_\lambda)_{\lambda\in\nabla}$ results from a finite number of generating (or "mother") wavelets by scaling and translation, i.e.,

$$\psi_\lambda(x) := \psi(2^{|\lambda|}x - k), \quad k \in \nabla_{|\lambda|}, x \in D. \tag{35}$$

We use notation that is standard for MRA, i.e., the function system is indexed by $\lambda = (|\lambda|, k) \in \nabla$, where $|\lambda| \in \mathbb{N}_0$ refers to the level and $k \in \nabla_{|\lambda|}$ to the translation. The index set ∇_ℓ has cardinality $|\nabla_\ell| = \mathcal{O}(2^{d\ell})$, $\ell \in \mathbb{N}_0$. Let $j : \nabla \to \mathbb{N}$ be a suitable enumeration. The overlap on every level $|\lambda| = \ell \in \mathbb{N}_0$ is assumed to be uniformly bounded, i.e., there exists $K > 0$ such that for every $\ell \in \mathbb{N}_0$ and every $x \in D$,

$$|\{\lambda \in \nabla : |\lambda| = \ell, \psi_\lambda(x) \neq 0\}| \leq K \ .$$

Additionally, for constants $\sigma, \widehat{\alpha} > 0$ we introduce the scaling

$$\|\psi_\lambda\|_{L^\infty(D)} \leq \sigma 2^{-\widehat{\alpha}|\lambda|}, \quad \lambda \in \nabla. \tag{36}$$

Under this assumption, the work to assemble one sample of the stiffness matrix (i.e. for one QMC point) on discretization level $\ell \in \mathbb{N}_0$ scales for large ℓ as $\mathcal{O}(M_\ell |j^{-1}(s_\ell)|) = \mathcal{O}(M_\ell \log(s_\ell))$.

Proposition 6.1 *For $d = 1$, the work to solve the linear system that corresponds to* (11) *for one sample is $\mathcal{O}(M_\ell)$, $\ell \in \mathbb{N}_0$.*

Proof. The parametric stiffness matrix is tridiagonal and symmetric, positive definite with probability one. Therefore both, Cholesky decomposition and backsubstitution, can be performed in $\mathcal{O}(M_\ell)$ work and memory (see, e.g., [15, Chapter 4.3.6]). □

Due to Proposition 6.1 and Remark 6.1, we stipulate availability of a PDE solver with work

$$\text{work}_{\text{PDEsolve}} = \mathcal{O}(M_\ell^{1+\eta}) \tag{\textbf{A3}}$$

for some $\eta \geq 0$ with implied constants independent of $\ell \in \mathbb{N}_0$ *and, in particular, of the realization of the PDE coefficients*. Note that $\eta = 0$ corresponds to linear complexity as is afforded by multigrid of multi-level preconditioned iterative solvers for elliptic PDEs in the deterministic setting; see, e.g., [9, 30]. Uniformity of the work estimate of the PDE solver w.r. to the realization of the lognormal coefficient can, for lognormal diffusion $a = \exp(Z)$ and for multilevel Monte Carlo methods, be achived for every $\eta > 0$, cp. [18], which is nearly optimal complexity (w.r. to the degrees of freedom) of a PDE solver.

Remark 6.1 *The uniformity w.r. to the coefficient realizations of the work estimate* (**A3**) *is, for the presently considered log-Gaussian diffusion coefficient models, by no means to be taken for granted [18]. Since for $d = 2, 3$ stiffness matrices will not be tridiagonal, usually iterative solvers are used. In [18], strong convergence for iterative methods is shown in the general framework of [30], which is sufficient for single-level QMC. However, applicability to multilevel QMC is not a direct consequence.*

Under (**A3**) the model for the computational work for the multilevel QMC quadrature reads, for every $L \in \mathbb{N}_0$, as

$$\text{work}_L = \mathcal{O}\left(\sum_{\ell=0}^{L} s_\ell N_\ell \log(N_\ell) + \sum_{\ell=0}^{L} N_\ell (M_\ell \log(s_\ell) + M_\ell^{1+\eta})\right) , \tag{37}$$

where the first sum in (37) is the work of the generation of the QMC points which includes the work to obtain the generating vectors by the fast CBC construction, cp. [27, 28]. The work model in (37) depends on the choices for $(s_\ell)_{\ell=0,\dots,L}$, $(N_\ell)_{\ell=0,\dots,L}$, and $(M_\ell)_{\ell\geq 0}$, which we shall not indicate explicitly in our notation and simply write "work$_L$". The second sum in (37) is the work of the evaluation of the multilevel QMC quadrature. The sequence

$$b_{j(\lambda)} = b_\lambda := c2^{-\widehat{\beta}|\lambda|}, \quad \lambda \in \nabla, \tag{38}$$

together with $(\psi_\lambda)_{\lambda\in\nabla}$ defined in (35) and (36) satisfies the assumption in (**A1**) for $1 < \widehat{\beta} < \widehat{\alpha}$ and some $c > 0$. Since $\||\nabla\psi_\lambda|\|_{L^\infty(D)} \leq \sigma 2^{-(\widehat{\alpha}-1)|\lambda|}\||\nabla\psi|\|_{L^\infty(D)}$, $\lambda \in \nabla$, the sequence

$$\bar{b}_j := b_j^{(\widehat{\beta}-1)/\widehat{\beta}}, \quad j \in \mathbb{N},$$

and $(\psi_j)_{j\geq 1}$ defined in (35) and (36) satisfy the assumption (**A2**). In this section we assume that

$$f \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t,\infty} \quad \text{and} \quad G(\cdot) \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t',\infty}, \quad t, t' \in [0,1], \tag{\textbf{A4}}$$

and define $\tau := t + t'$. In the following, we assume that

$$M_\ell \sim 2^{d\ell}, \quad \ell \geq 0. \tag{\textbf{A5}}$$

The ensuing analysis is inspired by [22, Section 3.7] (see also [21, 11]). We will restrict the analysis to one QMC rule with respect to the QMC weight sequence (33) on all levels $\ell = 0, \dots, L$, but remark that in some cases it might be beneficial to use a second one with respect to the QMC weight sequence (32) on the level $\ell = 0$. The multilevel QMC quadrature depends on the algorithmic steering parameters $(N_\ell)_{\ell=0,\dots,L}$, $(s_\ell)_{\ell=0,\dots,L}$, $(M_\ell)_{\ell\geq 0}$, and also on $\theta \in (0,1)$. The number of degrees of freedom $(M_\ell)_{\ell\geq 0}$ of the FE discretization in D are assumed to be given. The parameter $\theta \in (0,1)$ is for now left arbitrary. According to the estimate in Theorem 5.1, θ can be used to balance the truncation error with the FE error on the levels $\ell = 0, \dots, L$. We will use this feature to discuss two possible strategies to choose the truncation dimensions $(s_\ell)_{\ell=0,\dots,L}$.

Strategy 1: The contributions in the QMC weight sequence in (33) are equilibrated, i.e., we choose $\theta = 1/\widehat{\beta}$, which implies that $b^{1-\theta} = \bar{b}_j$, $j \in \mathbb{N}$. The truncation dimension s_L is also chosen to equilibrate the respective truncation and FE error in the estimate of Theorem 5.1. We choose

$$s_L \sim 2^{d\lceil L\tau/\widetilde{\beta}\rceil}$$

for some

$$1 < \widetilde{\beta} < \widehat{\beta} \tag{39}$$

close to $\widehat{\beta}$, where we use that $M_\ell = \mathcal{O}(2^{d\ell})$, $\ell = 0, \dots, L$. On the levels $\ell = 0, \dots, L-1$, we either increase s_ℓ or leave it constant. We choose

$$s_\ell \sim \min\{2^{d\lceil\tau\ell\rceil}, s_L\}, \quad \ell = 0, \dots, L-1.$$

Strategy 2: For particular $(\psi_\lambda)_{\lambda\in\nabla}$ and meshes, it may be interesting to align the level structure $(\psi_\lambda)_{\lambda\in\nabla}$ and the used FE meshes. Therefore, we choose

$$s_\ell \sim M_\ell, \quad \ell = 0, \dots, L.$$

The choice $\theta = \tau/\widehat{\beta}$ equilibrates the truncation and FE error in the estimate of Theorem 5.1 on the levels $\ell = 0, \dots, L$ assuming that $\widehat{\beta} > \tau$. Then, $(b_j^{1-\theta} \vee \bar{b}_j)_{j\geq 1} \in \ell^{\bar{p}}(\mathbb{N})$ for every $\bar{p} > d/(\min\{\widehat{\beta} - \tau, \widehat{\beta} - 1\})$.

For either of the strategies and for every $L \in \mathbb{N}_0$, by Theorem 5.1 we obtain the error estimate

$$\mathrm{error}_L^2 = \mathcal{O}\left(M_L^{-2\tau/d} + \sum_{\ell=0}^{L} (\varphi(N_\ell))^{-2\bar{\chi}} M_\ell^{-2\tau/d} \right). \tag{40}$$

Since the Euler totient function satisfies that $(\varphi(N))^{-1} \leq N^{-1}(e^{\widehat{\gamma}} \log\log N + 3/\log\log(N))$ for every $N \geq 3$, where $\widehat{\gamma} \approx 0.5772$ is the Euler–Mascheroni constant, $(\varphi(N))^{-1} \leq 9/N$ for every $N = 3, \ldots, 10^{40}$. We will for simplicity restrict in our analysis the range of N to $N \leq 10^{40}$ and use the bound $(\varphi(N))^{-1} \leq 9/N$. In Strategies 1 and 2, the p-summability of the sequence $(b_j^{1-\theta} \vee \bar{b}_j)_{j\geq 1}$ holds with a strict inequality condition on p, i.e., $(b_j^{1-\theta} \vee \bar{b}_j)_{j\geq 1} \in \ell^{\bar{p}}(\mathbb{N})$, for every $\bar{p} > d/(\widehat{\beta} - 1)$ in the case of Strategy 1 and for every $\bar{p} > d/\min\{\widehat{\beta} - \tau, \widehat{\beta} - 1\}$ in the case of Strategy 2. Since the QMC convergence rate $\bar{\chi}$ depends on the exponent p, there exists $\varepsilon > 0$ such that $\bar{\chi}(1 + \varepsilon)$ is also admissible in (40) due to Theorem 5.1. Using $\log(N) \leq N^{\varepsilon}/(\varepsilon e)$ for every $N \in \mathbb{N}$, cp. see e.g. the proof of [11, Lemma 1], the factor $N_\ell \log(N_\ell)$ in (37) may be estimated by $N_\ell^{1+\varepsilon}$. Since $N_\ell^{1+\varepsilon}$ appears then in the estimate of the work (37) and in the error estimate (40), it can be substituted by N_ℓ, using the strict inequalities in the above bounds for the admissible indices, and choosing $\varepsilon > 0$ sufficiently small.

We obtain with the choices for $(s_\ell)_{\ell=0,\ldots,L}$ in Strategies 1 and 2

$$\mathrm{work}_L = \begin{cases} \mathcal{O}\left(\sum_{\ell=0}^{L} N_\ell (M_\ell \log(M_\ell) + \max\{M_\ell^{1+\eta}, \min\{M_\ell^{\tau}, M_L^{\tau/\widehat{\beta}}\}\}) \right), & \text{for Strategy 1,} \\ \mathcal{O}\left(\sum_{\ell=0}^{L} N_\ell (M_\ell \log(M_\ell) + M_\ell^{1+\eta}) \right), & \text{for Strategy 2.} \end{cases}$$

and

$$\mathrm{error}_L^2 = \mathcal{O}\left(M_L^{-2\tau/d} + \sum_{\ell=0}^{L} N_\ell^{-2\bar{\chi}} M_\ell^{-2\tau/d} \right).$$

We will distinguish between the cases that $\eta = 0$ and $\eta > 0$ in (**A3**). We treat Strategy 2 and the case $\eta > 0$ first. As above, $\log(M) \leq M^{\eta}/(\eta e)$ for every $M \in \mathbb{N}$. To obtain optimal choices for the sample numbers $(N_\ell)_{\ell=0,\ldots,L}$, we search for a stationary point of the function

$$g(\xi) := M_L^{-2\tau/d} + \sum_{\ell=0}^{L} N_\ell^{-2\bar{\chi}} M_\ell^{-2\tau/d} + \xi \sum_{\ell=0}^{L} N_\ell M_\ell^{1+\eta}$$

with respect to N_ℓ, i.e., we solve the first order necessary condition $\partial g/\partial N_\ell = 0$ (see also [22, Section 3.7]). This gives

$$N_\ell = \left\lceil N_0 M_\ell^{-(2\tau/d+1+\eta)/(1+2\bar{\chi})} \right\rceil, \quad \ell = 1, \ldots, L, \tag{41}$$

and with setting $E_\ell = M_\ell^{(1+\eta-\tau/(d\bar{\chi}))2\bar{\chi}/(1+2\bar{\chi})}$, $\ell = 0, \ldots, L$,

$$\mathrm{error}_L^2 = \mathcal{O}\left(M_L^{-2\tau/d} + N_0^{-2\bar{\chi}} \sum_{\ell=0}^{L} E_\ell \right) \quad \text{and} \quad \mathrm{work} = \mathcal{O}\left(N_0 \sum_{\ell=0}^{L} E_\ell \right), \tag{42}$$

where

$$\sum_{\ell=0}^{L} E_\ell = \begin{cases} \mathcal{O}(1) & \text{if } 1 + \eta < \tau/(d\bar{\chi}), \\ \mathcal{O}(L) & \text{if } 1 + \eta = \tau/(d\bar{\chi}), \\ \mathcal{O}(2^{(2\bar{\chi}d(1+\eta)-2\tau)L/(1+2\bar{\chi})}) & \text{if } 1 + \eta > \tau/(d\bar{\chi}). \end{cases} \tag{43}$$

The parameter N_0 is chosen to balance the error contributions, i.e., $N_0^{-2\bar{\chi}} \sum_{\ell=0}^{L} E_\ell = \mathcal{O}(M_L^{-2\tau/d})$, which implies

$$N_0 = \begin{cases} \lceil 2^{\tau L/\bar{\chi}} \rceil & \text{if } 1+\eta < \tau/(d\bar{\chi}), \\ \lceil 2^{\tau L/\bar{\chi}} L^{1/(2\bar{\chi})} \rceil & \text{if } 1+\eta = \tau/(d\bar{\chi}), \\ \lceil 2^{(2\tau+d(1+\eta))L/(1+2\bar{\chi})} & \text{if } 1+\eta > \tau/(d\bar{\chi}). \end{cases} \tag{44}$$

We conclude that $\text{error}_L^2 = \mathcal{O}(M_L^{-2\tau/d})$ can be achieved with

$$\text{work}_L = \begin{cases} \mathcal{O}(2^{\tau L/\bar{\chi}}) & \text{if } 1+\eta < \tau/(d\bar{\chi}), \\ \mathcal{O}(2^{\tau L/\bar{\chi}} L^{(1+2\bar{\chi})/(2\bar{\chi})}) & \text{if } 1+\eta = \tau/(d\bar{\chi}), \\ \mathcal{O}(2^{dL(1+\eta)}) & \text{if } 1+\eta > \tau/(d\bar{\chi}). \end{cases}$$

In the case that $\eta = 0$, the resulting work measure is considered in [22, Section 3.7]. In particular, we obtain by [22, Equations (74) and (77)]

$$N_\ell = \left\lceil N_0 \left(M_\ell^{-1-2\tau/d} \log(s_\ell)^{-1} \right)^{1/(1+2\bar{\chi})} \right\rceil, \quad \ell = 1, \ldots, L, \tag{45}$$

and

$$N_0 = \begin{cases} \lceil 2^{\tau L/\bar{\chi}} \rceil & \text{if } d < \tau/\bar{\chi}, \\ \lceil 2^{\tau L/\bar{\chi}} L^{(1+4\bar{\chi})/(\bar{\chi}(2+4\bar{\chi}))} \rceil & \text{if } d = \tau/\bar{\chi}, \\ \lceil 2^{(d+2\tau)L/(1+2\bar{\chi})} L^{1/(1+2\bar{\chi})} \rceil & \text{if } d > \tau/\bar{\chi}. \end{cases} \tag{46}$$

Note that the corresponding work estimates are given on [22, p. 443]. We summarize this analysis as ε-complexity bounds in the following theorem.

Theorem 6.2 *[Error vs. work for Strategy 2]*

*Let the truncation dimensions $(s_\ell)_{\ell=0,\ldots,L}$ be chosen according to Strategy 2 assuming $\widehat{\beta} > \max\{\tau, 1\}$. Let the assumptions (**A5**) and (**A3**) be satisfied for $\eta \geq 0$. If $\eta > 0$, the sample numbers for $Q_L^*(\cdot)$ are given by (44) and (41), $L \in \mathbb{N}_0$. If $\eta = 0$, the sample numbers for $Q_L^*(\cdot)$ are given by (46) and (45), $L \in \mathbb{N}_0$. Let f and $G(\cdot)$ satisfy (**A4**).*

1. *Gaussian weight functions: for $\bar{p} \in (\max\{2/3, d/(\widehat{\beta}-\tau), d/(\widehat{\beta}-1)\}, 2)$, $\bar{\chi} = 1/(2\bar{p}) + 1/4 - \varepsilon'$ for $\varepsilon' > 0$ sufficiently small assuming $d/\min\{\widehat{\beta}-\tau, \widehat{\beta}-1\} < 2$.*

2. *Exponential weight functions: for $\bar{p} \in (\max\{2/3, d/(\widehat{\beta}-\tau), d/(\widehat{\beta}-1)\}, 1]$, $\bar{\chi} = 1/\bar{p} - 1/2$ assuming $d/\min\{\widehat{\beta}-\tau, \widehat{\beta}-1\} < 1$.*

For an error threshold $1 > \varepsilon > 0$, we obtain

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u)) - Q_L^*(G(u^L))|^2)} = \mathcal{O}(\varepsilon)$$

is achieved with

$$\text{work}_L = \begin{cases} \mathcal{O}(\varepsilon^{-1/\bar{\chi}}) & \text{if } 1+\eta < \tau/(d\bar{\chi}), \\ \mathcal{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+2\bar{\chi})/(2\bar{\chi})}) & \text{if } 1+\eta = \tau/(d\bar{\chi}), \eta > 0, \\ \mathcal{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+4\bar{\chi})/(2\bar{\chi})}) & \text{if } d = \tau/\bar{\chi}, \eta = 0, \\ \mathcal{O}(\varepsilon^{-d/\tau(1+\eta)}) & \text{if } 1+\eta > \tau/(d\bar{\chi}), \eta > 0, \\ \mathcal{O}(\varepsilon^{-d/\tau} \log(\varepsilon^{-1})) & \text{if } d > \tau/\bar{\chi}, \eta = 0. \end{cases}$$

Here, the implied constants are independent of L, $(s_\ell)_{\ell=0,\ldots,L}$, $(N_\ell)_{\ell=0,\ldots,L}$, and of $(M_\ell)_{\ell\geq 0}$.

Remark 6.2 *In Strategy 2, there is one parameter respectively one dimension of integration, per spatial degree of freedom, so that $s_\ell \sim M_\ell$, $\ell \geq 0$. This coupling occurs, for example, when circulant embedding is applied to evaluate a GRF on uniformly spaced spatial grid points such that each element of the FE mesh contains at least one of these points to perform a one point quadrature for computing the stiffness matrix. Numerical experiments with a QMC rule using a circulant embedding are presented in [17] and the references there.*

For Strategy 1, we may restrict the analysis to the case $\tau > 1$, since for $\tau \leq 1$ the additional restriction $\widehat{\beta} > \tau$ for Strategy 2 is redundant and Strategy 2 can be applied. We obtain following the same line of argument as applied in the analysis of Strategy 2

$$N_\ell = \left\lceil N_0 \left(M_\ell^{2\tau/d} \max\{M_\ell^{1+\eta}, \min\{M_\ell^\tau, M_L^{\tau/\widetilde{\beta}}\}\}\right)^{-1/(1+2\bar{\chi})}\right\rceil, \quad \ell = 1, \ldots, L, \tag{47}$$

where also (42) holds with

$$E_\ell = \left(M_\ell^{-2\tau/d} \max\{M_\ell^{1+\eta}, \min\{M_\ell^\tau, M_L^{\tau/\widetilde{\beta}}\}\}\right)^{2\bar{\chi}/(1+2\bar{\chi})}, \quad \ell = 0, \ldots, L.$$

We observe that

$$\sum_{\ell=0}^{L} \left(M_\ell^{-2\tau/d} \max\{M_\ell^{1+\eta}, M_L^{\tau/\widetilde{\beta}}\}\right)^{2\bar{\chi}/(1+2\bar{\chi})} = \begin{cases} \mathcal{O}(2^{dL(\tau/\widetilde{\beta})}2\bar{\chi}/(1+2\bar{\chi})) & \text{if } 1+\eta \leq \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}, \\ \mathcal{O}(2^{(2\bar{\chi}d(1+\eta)-2\tau)L/(1+2\bar{\chi})}) & \text{if } 1+\eta > \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}, \end{cases}$$

where we used that $\max\{x, y\} \leq x + y$ for every $x, y \in [0, \infty)$. The respective estimate for the sum over $M_\ell^{-2\tau/d} \max\{M_\ell^{1+\eta}, M_\ell^\tau\}$ is given in (43) with $\max\{1+\eta, \tau\}$ in place of $1+\eta$ (also in the conditions of the three cases). To estimate $\sum_{\ell=0}^{L} E_\ell$, we use the identity that $\max\{x, \min\{y, z\}\} = \min\{\max\{x, y\}, \max\{x, z\}\}$ for every $x, y, z \in \mathbb{R}$, and apply the superadditivity of the minimum to obtain that

$$\sum_{\ell=0}^{L} E_\ell = \begin{cases} \mathcal{O}(1) & \text{if } \max\{\tau, 1+\eta\} < \tau/(d\bar{\chi}), \\ \mathcal{O}(L) & \text{if } \max\{\tau, 1+\eta\} = \tau/(d\bar{\chi}), \\ \mathcal{O}(2^{2\bar{\chi}d(1+\eta)-2\tau)L/(1+2\bar{\chi})}) & \text{if } 1+\eta > \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}, \\ \mathcal{O}(2^{2\tau \min\{d\bar{\chi}-1, d\bar{\chi}/\widetilde{\beta}\}L/(1+2\bar{\chi})}) & \text{if } 1+\eta \leq \tau \min\{1, 1/(d\bar{\chi}) + 1/\widetilde{\beta}\}, 1 > 1/(d\bar{\chi}), \\ \mathcal{O}(2^{2\min\{\bar{\chi}d(1+\eta)-\tau, \bar{\chi}d\tau/\widetilde{\beta}\}L/(1+2\bar{\chi})}) & \text{if } \max\{\tau, \tau/(d\bar{\chi})\} < 1+\eta \leq \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}. \end{cases}$$

As above, N_0 is chosen to balance the error, i.e., $N_0 \sim M_L^{\tau/(d\bar{\chi})}(\sum_{\ell=0}^{L} E_\ell)^{1/(2\bar{\chi})}$. Specifically,

$$N_0 = \begin{cases} \lceil 2^{L\tau/\bar{\chi}} \rceil & \text{if } \max\{\tau, 1+\eta\} < \tau/(d\bar{\chi}), \\ \lceil 2^{L\tau/\bar{\chi}} L^{1/(2\bar{\chi})} \rceil & \text{if } \max\{\tau, 1+\eta\} = \tau/(d\bar{\chi}), \\ \lceil 2^{(2\tau+d(1+\eta))L/(1+2\bar{\chi})} \rceil & \text{if } 1+\eta > \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}, \\ \lceil 2^{\tau(\min\{d, d/\widetilde{\beta}+1/\bar{\chi}\}+2)L/(1+2\bar{\chi})} \rceil & \text{if } 1+\eta \leq \tau \min\{1, 1/(d\bar{\chi}) + 1/\widetilde{\beta}\}, 1 > 1/(d\bar{\chi}), \\ \lceil 2^{(\min\{d(1+\eta), d\tau/\widetilde{\beta}+\tau/\bar{\chi}\}+2\tau)L/(1+2\bar{\chi})} \rceil & \text{if } \max\{\tau, \tau/(d\bar{\chi})\} < 1+\eta \leq \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}. \end{cases} \tag{48}$$

Explicit error vs. work estimates are summarized as ε-complexity bounds in the following theorem, where we recall that work $= N_0 \sum_{\ell=0}^{L} E_\ell = M_L^{\tau/(d\bar{\chi})}(\sum_{\ell=0}^{L} E_\ell)^{(1+2\bar{\chi})/(2\bar{\chi})}$.

Theorem 6.3 *[Error vs. work for Strategy 1]*

Let the truncation dimension $(s_\ell)_{\ell\geq 1}$ *be chosen according to Strategy 1 assuming* $\widehat{\beta} > 1$ *and* $\tau > 1$. *Let the assumptions* (**A5**) *and* (**A3**) *be satisfied for* $\eta \geq 0$. *The sample numbers for* $Q^*_L(\cdot)$ *are given by* (48) *and* (47), $L \in \mathbb{N}_0$. *Let* f *and* $G(\cdot)$ *satisfy* (**A4**).

1. *Gaussian weight functions: for* $\bar{p} \in (\max\{2/3, d/(\widehat{\beta}-1)\}, 2)$, $\bar{\chi} = 1/(2\bar{p}) + 1/4 - \varepsilon'$ *for* $\varepsilon' > 0$ *sufficiently small assuming* $d/(\widehat{\beta}-1) < 2$.

2. *Exponential weight functions: for* $\bar{p} \in (\max\{2/3, d/(\widehat{\beta}-1)\}, 1]$, $\bar{\chi} = 1/\bar{p} - 1/2$ *assuming* $d/(\widehat{\beta}-1) < 1$.

For an error threshold $\varepsilon > 0$, *we obtain*

$$\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|\mathbb{E}(G(u)) - Q^*_L(G(u^L))|^2)} = \mathcal{O}(\varepsilon)$$

is achieved with

$$\text{work} = \begin{cases} \mathcal{O}(\varepsilon^{-1/\bar{\chi}}) & \text{if } \max\{\tau, 1+\eta\} < \tau/(d\bar{\chi}), \\ \mathcal{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+2\bar{\chi})/(2\bar{\chi})}) & \text{if } \max\{\tau, 1+\eta\} = \tau/(d\bar{\chi}), \\ \mathcal{O}(\varepsilon^{-d/\tau(1+\eta)}) & \text{if } 1+\eta > \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}, \\ \mathcal{O}(\varepsilon^{-\min\{d, d/\widetilde{\beta}+1/\bar{\chi}\}L}) & \text{if } 1+\eta \leq \tau\min\{1, 1/(d\bar{\chi}) + 1/\widetilde{\beta}\}, 1 > 1/(d\bar{\chi}), \\ \mathcal{O}(\varepsilon^{-\min\{d/\tau(1+\eta), d/\widetilde{\beta}+1/\bar{\chi}\}L}) & \text{if } \max\{\tau, \tau/(d\bar{\chi})\} < 1+\eta \leq \tau/(d\bar{\chi}) + \tau/\widetilde{\beta}. \end{cases}$$

Here, $\widetilde{\beta}$ *is as in* (39) *chosen close to* $0 < \widehat{\beta}$ *such that* $\widetilde{\beta} < \widehat{\beta}$ *and all implied constants are independent of* L, $(s_\ell)_{\ell=0,\ldots,L}$, $(N_\ell)_{\ell=0,\ldots,L}$, *and* $(M_\ell)_{\ell\geq 0}$.

7 Conclusions

For linear, second order diffusion equations (1) in a polygonal or polyhedral domain D, and with log-Gaussian diffusion coefficient $a = \exp(Z)$, where the GRF Z in D is represented in terms of a series expansion with "localized supports", taking values in weighted Hölder spaces in D, we extended the convergence rate and error versus work analysis of combined QMC quadratures and multilevel FE approximation from [16, 21]. Specifically, we considered randomly shifted lattice QMC rules for integration against a dimension-truncated Gaussian measure which were introduced in [26]. The present work complements [16, 21, 22], where the ψ_j's were allowed to have global support and QMC quadratures with so-called "POD" weights were employed. We proved that for GRFs Z whose spatial variation is parametrized by a function system $(\psi_j)_{j\geq 1}$ of functions $\psi_j(x)$ in D with "localized supports" that QMC combined with continuous, piecewise linear FE in D on families of regular, simplicial triangulations of D with suitable mesh refinement near vertices and (in space dimension $d = 3$) edges of D, allow for parameter-dimension independent error vs. work bounds. The case of full elliptic regularity in spaces without weights and $s_\ell = s_L$, $\ell = 0, \ldots, L$, is considered in [21, Corollary 2 and Section 5]. We considered polytopal domains D where ∂D consists of straight lines (in space dimension $d = 2$) or of plane faces (in space dimension $d = 3$). The parametric regularity results in weighted function spaces in D remain valid, however, also for polytopal D with piecewise smoothly curved boundaries as considered in [25].

Moreover, the assumed localization of the supports of the ψ_j in D was shown to allow for QMC integration rules with so-called product weights. The present model of the computational work (37) includes the cost of the generation of the QMC points, which is dominated by the cost of the fast CBC construction of generating vectors. This was considered a pre-computation in [22, 21] and the (quadratic w.r. to the parameter dimensions s_ℓ) work count for the (precomputed) CBC construction was omitted from the work counts in that reference. We also note that the same generating vectors can be used for different right hand sides f. However, if the function system $(\psi_j)_{j\geq 1}$ is altered, e.g., by changing certain parameters of a considered class of $(\psi_j)_{j\geq 1}$ that relate to the smoothness or spatial correlation, due to modeling considerations of the lognormal diffusion coefficent, then the generating vectors need to be recomputed. The QMC error analysis being based on product weights, the work of the fast CBC construction of generating vectors due to R. Cools and D. Nuyens [27] and the generation of QMC points scales linearly with respect to the parameter dimensions s_ℓ. In particular, the error vs. analysis in the present paper is with respect to the overall work, including the work required for the CBC construction. Therefore, we conclude in certain cases the same asymptotic error vs. work bounds of the considered multilevel QMC algorithm than in the case of the respective deterministic, elliptic PDE; also in the case that the FE error has a rate higher than $1/d$ with respect to the dimension of the FE spaces. We considered only homogeneous Dirichlet boundary conditions on all of ∂D in (1). This was for ease of exposition only: the parametric regularity analysis of Section 4 and the elliptic regularity results in Section 2.2 remain valid verbatim for problems with Neumann or mixed boundary conditions. In particular, the same structure of the corner- and edge-weights in (7) can be used to characterize elliptic regularity shifts in D for these boundary conditions.

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Lukas Herrmann
Seminar für Angewandte Mathematik
ETH Zürich,
Rämistrasse 101, CH–8092 Zürich, Switzerland.
E-mail address: `lukas.herrmann@sam.math.ethz.ch`

Christoph Schwab
Seminar für Angewandte Mathematik
ETH Zürich,
Rämistrasse 101, CH–8092 Zürich, Switzerland.
E-mail address: `christoph.schwab@sam.math.ethz.ch`