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On stochastic differential equations with arbitrarily slow convergence rates for strong approximation in two space dimensions

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Abstract

In the recent article [Jentzen, A., Müller-Gronbach, T., and Yaroslavtseva, L., Commun. Math. Sci., 14(6), 1477–1500, 2016] it has been established that for every arbitrarily slow convergence speed and every natural number $d \in \{4, 5, \ldots\}$ there exist d-dimensional stochastic differential equations (SDEs) with infinitely often differentiable and globally bounded coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence. In this paper we strengthen the above result by proving that this slow convergence phenomena also arises in two ($d = 2$) and three ($d = 3$) space dimensions.

Contents

1 Introduction

In the recent article [9] it has been established that for every arbitrarily slow convergence speed and every natural number $d \in \{4, 5, \ldots\}$ there exist d-dimensional stochastic differential equations (SDEs) with infinitely often differentiable and globally bounded coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence. More specifically, Theorem 1.3 in [9] implies the following theorem.

Theorem 1.1. *Let $T \in (0, \infty)$, $d \in \{4, 5, \ldots\}$, $\xi \in \mathbb{R}^d$, $m \in \mathbb{N}$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0, T]$, $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist infinitely often differentiable and globally bounded functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0, T] \times \Omega \to \mathbb{R}^m$, every continuous $\mathbb{F}$-adapted stochastic process $X\colon [0, T] \times \Omega \to \mathbb{R}^d$ with $\forall\, t \in [0, T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s) = 1$, and every $n \in \mathbb{N}$ it holds that*

$$\inf_{t_1,\ldots,t_n\in[0,T]} \inf_{\substack{u\colon (\mathbb{R}^m)^n\times C([\varepsilon_n,T],\mathbb{R}^m)\to\mathbb{R}^d \\ \text{measurable}}} \mathbb{E}\Big[\big\|X_T - u(W_{t_1}, \ldots, W_{t_n}, (W_s)_{s\in[\varepsilon_n,T]})\big\|_{\mathbb{R}^d}\Big] \geq \delta_n. \quad (1)$$

In this paper we strengthen the above result by proving that for every arbitrarily slow convergence speed and every natural number $d \in \{2, 3, \ldots\}$ there exist d-dimensional SDEs with infinitely often differentiable and globally bounded coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence. More precisely, in this work we establish the following theorem.

Theorem 1.2. *Let $T \in (0, \infty)$, $\tau \in (0, T)$, $d \in \{2, 3, \ldots\}$, $\xi \in \mathbb{R}^d$, $m \in \mathbb{N}$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0, \tau]$, $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist infinitely often differentiable and globally bounded functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian* *motion* $W\colon [0,T]\times\Omega\to\mathbb{R}^m$, *every continuous* $\mathbb{F}$*-adapted stochastic process* $X\colon [0,T]\times\Omega\to\mathbb{R}^d$ *with* $\forall\, t\in[0,T]\colon \mathbb{P}(X_t=\xi+\int_0^t \mu(X_s)\,ds+\int_0^t \sigma(X_s)\,dW_s)=1$, *and every* $n\in\mathbb{N}$ *it holds that*

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}}\ \inf_{t_1,\dots,t_n\in[0,T]}\ \inf_{\substack{u\colon (\mathbb{R}^m)^n\times C([0,a]\cup[b,T],\mathbb{R}^m)\to\mathbb{R}^d\\ \text{measurable}}} \mathbb{E}\Big[\big\|X_T - u(W_{t_1},\dots,W_{t_n},(W_s)_{s\in[0,a]\cup[b,T]})\big\|_{\mathbb{R}^d}\Big]\geq\delta_n. \quad (2)$$

Theorem 1.2 follows immediately from Corollary 3.21 below. Further lower error bounds for strong and weak numerical approximation schemes for SDEs with non-globally Lipschitz continuous coefficients can be found in [6, 8, 2, 9, 11, 17]. Hairer et al. [2, Theorem 1.3] and Müller-Gronbach & Yaroslavtseva [11, Theorems 1–3] deal with lower bounds for weak approximation errors and Yaroslavtseva [17, Corollary 2] extends [9, Theorem 1.3] (cf. also Theorem 1.1 above) to numerical approximation schemes where the driving Brownian motion can be evaluated adaptively. Each of the references [2, 9, 11, 17] assumes beside other hypotheses that the dimension d of the considered SDE satisfies $d\geq 4$. The main contribution of this work is to reveal that a slow convergence phenomena of the form (2) also arises in two ($d=2$) and three ($d=3$) space dimensions. Upper error bounds and numerical approximation schemes for SDEs with non-globally Lipschitz continuous coefficients can, e.g., be found in [4, 1, 3, 7, 16, 5, 12, 13, 15] and the references mentioned therein. Lower error bounds for strong approximation schemes for SDEs with globally Lipschitz continuous coefficients can, e.g., found in the overview article Müller-Gronbach & Ritter [10] and the references mentioned therein.

2 Construction of the coefficients of the considered two-dimensional SDEs

In this section we establish two elementary auxiliary results (see Lemma 2.1 and Lemma 2.2 below) which demonstrate that the functions $f,g\colon\mathbb{R}\to\mathbb{R}$ in (7) and (8) below have suitable regularity properties.

2.1 Setting

Let $T,\mu\in(0,\infty)$, $\tau,\tau_1\in(0,T)$, $\tau_2\in(\tau_1,T)$, $\varepsilon\in(0,\min\{T-\tau,\tau(1-2^{-1/3})\})$, $F,\rho,h,f,g\in C(\mathbb{R},\mathbb{R})$ satisfy for all $x\in\mathbb{R}$ that

$$\tau_1=\tau+\varepsilon,\qquad \mu=\int_{-\varepsilon}^{\varepsilon}\exp\!\Big(\tfrac{-1}{(\varepsilon^2-t^2)}\Big)\,dt, \quad (3)$$

$$F(x)=\begin{cases}4\tau & : \quad x\leq\tau\\ 2\tau-2x & : \quad -\tau<x<\tau\ ,\\ 0 & : \quad x\geq\tau\end{cases} \quad (4)$$

$$\rho(x) = \begin{cases} \frac{1}{\mu}\exp\!\left(\frac{-1}{(\varepsilon^2 - x^2)}\right) & : \quad |x| < \varepsilon \\ 0 & : \quad |x| \geq \varepsilon \end{cases}, \tag{5}$$

$$h(x) = \begin{cases} \exp\!\left(-\frac{1}{x}\right) & : \quad x > 0 \\ 0 & : \quad x \leq 0 \end{cases}, \tag{6}$$

$$f(x) = \int_{-\infty}^{\infty} \rho(t)\, F(x-t)\, dt, \tag{7}$$

and

$$g(x) = \frac{4h(x-\tau_1)}{h(x-\tau_1) + h(\tau_2 - x)}. \tag{8}$$

2.2 Properties of the function appearing in the first component of the considered two-dimensional SDE

Lemma 2.1. *Assume the setting in Section 2.1. Then*

(i) it holds that $\sup_{x\in\mathbb{R}} |f(x)| < \infty$,

(ii) it holds that $f((-\infty, \tau_1)) \subseteq (0, \infty)$,

(iii) it holds that $f([\tau_1, \infty)) = \{0\}$,

(iv) it holds that $f'(\mathbb{R}) \subseteq [-2, 0]$,

(v) it holds that $f'((0,\tau)) \subseteq [-2, -1)$, *and*

(vi) it holds that $\int_0^{\tau_1} |f(s)|^2\, ds \geq \frac{2\tau^3}{3}$.

Proof of Lemma 2.1. Throughout this proof let $\lambda \colon \mathcal{B}(\mathbb{R}) \to [0, \infty]$ be the Lebesgue-Borel measure on $\mathbb{R}$. Note that

$$\begin{aligned} \sup_{x\in\mathbb{R}} |f(x)| &\leq \left[\sup_{x\in\mathbb{R}} \left|F(x)\right|\right] \left[\int_{-\infty}^{\infty} \rho(t)\, dt\right] = 4\tau \left[\int_{-\infty}^{\infty} \rho(t)\, dt\right] \\ &= \tfrac{4\tau}{\mu} \int_{-\varepsilon}^{\varepsilon} \exp\!\left(\tfrac{-1}{(\varepsilon^2 - t^2)}\right) dt = 4\tau < \infty. \end{aligned} \tag{9}$$

This establishes Item (i). Next note that for all $x \in \mathbb{R}$ it holds that

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \rho(t)\, F(x-t)\, dt = \int_{-\varepsilon}^{\varepsilon} \rho(t)\, F(x-t)\, dt \\ &= \int_{-\varepsilon}^{\varepsilon} \rho(t)\, F(x-t)\, \mathbb{1}_{(-\infty,\tau)}(x-t)\, dt \\ &= \int_{-\varepsilon}^{\varepsilon} \rho(t)\, F(x-t)\, \mathbb{1}_{(x-\tau,\infty)}(t)\, dt \\ &= \int_{-\varepsilon}^{\varepsilon} \rho(t)\, F(x-t)\, \mathbb{1}_{(x-\tau_1+\varepsilon,\infty)}(t)\, dt. \end{aligned} \tag{10}$$

This proves Item (iii). Moreover, observe that for all $x \in (-\infty, \tau_1)$ it holds that

$$\lambda((-\varepsilon, \varepsilon) \cap (x - \tau_1 + \varepsilon, \infty)) > 0 \tag{11}$$

and

$$\forall\, t \in (-\varepsilon, \varepsilon) \cap (x - \tau_1 + \varepsilon, \infty)\colon \rho(t)\, F(x - t) > 0. \tag{12}$$

Combining (11) and (12) with (10) yields that for all $x \in (-\infty, \tau_1)$ it holds that $f(x) > 0$. This establishes Item (ii). Next observe that for all $x \in \mathbb{R}$ it holds that

$$\begin{aligned} f'(x) &= \int_{\mathbb{R}\setminus\{x-\tau, x+\tau\}} \rho(t)\, F'(x - t)\, dt = \int_{\mathbb{R}\setminus\{-\tau,\tau\}} \rho(x - t)\, F'(t)\, dt \\ &= -2 \int_{-\tau}^{\tau} \rho(x - t)\, dt = -2 \int_{x-\tau}^{x+\tau} \underbrace{\rho(t)}_{\geq 0}\, dt \geq -2 \int_{-\infty}^{\infty} \rho(t)\, dt = -2. \end{aligned} \tag{13}$$

This proves Item (iv). In addition, observe that (13) ensures for all $x \in (0, \tau)$ that

$$\begin{aligned} f'(x) &= -2 \int_{x-\tau}^{x+\tau} \rho(t)\, dt = -2 \int_{x-\tau}^{0} \rho(t)\, dt - 2 \int_{0}^{\varepsilon} \rho(t)\, dt \\ &= -2 \int_{x-\tau}^{0} \rho(t)\, dt - 1 < -1. \end{aligned} \tag{14}$$

This establishes Item (v). Next note that (10) yields that for all $x \in (0, \tau - \varepsilon)$ it holds that

$$\begin{aligned} f(x) &= \int_{-\varepsilon}^{\varepsilon} \rho(t)\, F(x - t)\, dt = \int_{-\varepsilon}^{\varepsilon} \rho(t)\, (2\tau - 2(x - t))\, dt \\ &= \int_{-\varepsilon}^{\varepsilon} \rho(t)\, (2\tau - 2x + 2t)\, dt \geq (2\tau - 2x - 2\varepsilon) \int_{-\varepsilon}^{\varepsilon} \rho(t)\, dt \\ &= (2\tau - 2x - 2\varepsilon). \end{aligned} \tag{15}$$

Hence, we obtain that

$$\begin{aligned} \int_0^{\tau_1} |f(s)|^2\, ds &\geq \int_0^{\tau-\varepsilon} |f(s)|^2\, ds \geq \int_0^{\tau-\varepsilon} (2\tau - 2s - 2\varepsilon)^2\, ds \\ &= 4 \int_0^{\tau-\varepsilon} (\tau - \varepsilon - s)^2\, ds = 4 \int_0^{\tau-\varepsilon} s^2\, ds \\ &= \frac{4(\tau - \varepsilon)^3}{3} \geq \frac{4}{3} \left[\tau - \tau(1 - 2^{-1/3})\right]^3 \\ &= \frac{4\tau^3}{3} \left[1 - 1 + 2^{-1/3}\right]^3 = \frac{4\tau^3}{3} \cdot \frac{1}{2} = \frac{2\tau^3}{3}. \end{aligned} \tag{16}$$

This demonstrates Item (vi). The proof of Lemma 2.1 is thus completed. □

2.3 Properties of the function appearing in the second component of the considered two-dimensional SDE

Lemma 2.2. *Assume the setting in Section 2.1. Then*

(i) it holds that $g((-\infty, \tau_1]) = \{0\}$*,*

(ii) it holds that $g([\tau_2, \infty)) = \{4\}$*,*

(iii) it holds that $g'(\mathbb{R}) \subseteq [0, \infty)$*, and*

(iv) it holds that $\sup_{x\in\mathbb{R}} |g(x)| < \infty$*.*

Proof of Lemma 2.2. First, note that for all $x \in (-\infty, \tau_1]$ it holds that $h(x - \tau_1) = 0$ and $h(\tau_2 - x) > 0$. This proves Item (i). Next observe that for all $x \in [\tau_2, \infty)$ it holds that $h(\tau_2 - x) = 0$ and $h(x - \tau_1) > 0$. This demonstrates that for all $x \in [\tau_2, \infty)$ it holds that

$$g(x) = \frac{4h(x - \tau_1)}{h(x - \tau_1)} = 4. \tag{17}$$

This proves Item (ii). In the next step we note that the fact that $\forall\, x \in \mathbb{R}\colon h'(x) \geq 0$ ensures that for all $x \in \mathbb{R}$ it holds that

$$\begin{aligned} &g'(x) \\ &= \frac{4h'(x - \tau_1)(h(x - \tau_1) + h(\tau_2 - x)) - 4h(x - \tau_1)(h'(x - \tau_1) - h'(\tau_2 - x))}{(h(x - \tau_1) + h(\tau_2 - x))^2} \\ &= \frac{4h'(x - \tau_1)h(\tau_2 - x) + 4h(x - \tau_1)h'(\tau_2 - x)}{(h(x - \tau_1) + h(\tau_2 - x))^2} \geq 0. \end{aligned} \tag{18}$$

This proves Item (iii). Item (iv) is an immediate consequence of Items (i)–(iii). The proof of Lemma 2.2 is thus completed. □

3 Lower bounds for strong approximation errors

3.1 Setting

Let $T \in (0, \infty)$, $\tau \in (0, T)$, $\tau_1 \in (\tau, T)$, $\tau_2 \in (\tau_1, T)$, $\alpha \in [\frac{2\tau^3}{3}, \infty)$, $f, g \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfy $\sup_{x\in\mathbb{R}}(|f(x)|+|g(x)|) < \infty$, $f((-\infty, \tau_1)) \subseteq (0, \infty)$, $f([\tau_1, \infty)) = \{0\}$, $f'(\mathbb{R}) \subseteq [-2, 0]$, $f'((0, \tau)) \subseteq [-2, -1)$, $g((-\infty, \tau_1]) = \{0\}$, $g([\tau_2, \infty)) = \{4\}$, $g'(\mathbb{R}) \subseteq [0, \infty)$, $\alpha = \int_0^{\tau_1} |f(s)|^2 \, ds$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$, let $W\colon [0, T] \times \Omega \to \mathbb{R}$ be a standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion, and for every $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ let $X^{\psi,(1)}, X^{\psi,(2)}\colon [0, T] \times \Omega \to \mathbb{R}$ be continuous $\mathbb{F}$-adapted stochastic processes which satisfy for all $t \in [0, T]$ that $\mathbb{P}\big(X_t^{\psi,(1)} = \int_0^t f(X_s^{\psi,(2)}) \, dW_s\big) = 1$ and

$$\mathbb{P}\Big(X_t^{\psi,(2)} = t + \textstyle\int_0^t g(X_s^{\psi,(2)})[\cos(\psi(X_s^{\psi,(1)})) + 1] \, ds\Big) = 1. \tag{19}$$

3.2 Comments to the setting

The following result, Corollary 3.1 below, illustrates that there do indeed exist functions $f, g\colon \mathbb{R} \to \mathbb{R}$ which fulfill the hypotheses in Section 3.1. Corollary 3.1 is an immediate consequence of Lemma 2.1 and Lemma 2.2 in Section 2.

Corollary 3.1. *Let $T \in (0,\infty)$, $\tau \in (0,T)$. Then there exist $\tau_1 \in (\tau, T)$, $\tau_2 \in (\tau_1, T)$, $f, g \in C^\infty(\mathbb{R},\mathbb{R})$ which satisfy*

(i) that $\sup_{x\in\mathbb{R}}\big(|f(x)| + |g(x)|\big) < \infty$,

(ii) that $f((-\infty, \tau_1)) \subseteq (0,\infty)$,

(iii) that $f([\tau_1,\infty)) = \{0\}$,

(iv) that $f'(\mathbb{R}) \subseteq [-2, 0]$,

(v) that $f'((0,\tau)) \subseteq [-2,-1)$,

(vi) that $g((-\infty,\tau_1]) = \{0\}$,

(vii) that $g([\tau_2,\infty)) = \{4\}$,

(viii) that $g'(\mathbb{R}) \subseteq [0,\infty)$, *and*

(ix) that $\int_0^{\tau_1} |f(s)|^2\, ds \geq \frac{2\tau^3}{3}$.

3.3 Comparison results for a family of one-dimensional deterministic ordinary differential equations

In this section we establish three elementary comparison results for a specific type of ordinary differential equations (cf., e.g., Exercise 1.7 in Tao [14] for similar results) which we employ in the proof of Theorem 1.2 above.

Lemma 3.2. *Assume the setting in Section 3.1 and let $z = (z_t(a))_{t\in[\tau_1,T],a\in\mathbb{R}} = (z(t,a))_{t\in[\tau_1,T],a\in\mathbb{R}}\colon [\tau_1,T]\times\mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies for all $t \in [\tau_1, T]$, $a \in \mathbb{R}$ that*

$$z_t(a) = \tau_1 + \int_{\tau_1}^{t} \big[1 + g(z_s(a))(a+1)\big]\, ds. \tag{20}$$

Then it holds for all $a \in \mathbb{R}$, $b \in (-\infty, a]$, $t \in [\tau_1, T]$ that

$$z_t(a) \geq z_t(b). \tag{21}$$

Proof of Lemma 3.2. Throughout this proof let $y\colon [\tau_1,T]\times\mathbb{R} \to \mathbb{R}$ be the function which satisfies for all $t \in [\tau_1, T]$, $a \in \mathbb{R}$ that

$$y(t,a) = \big(\tfrac{\partial}{\partial a} z\big)(t,a). \tag{22}$$

Next note that (20) ensures that for all $t \in [\tau_1, T]$, $a \in \mathbb{R}$ it holds that

$$\big(\tfrac{\partial}{\partial t} z\big)(t, a) = 1 + g(z(t, a))(a + 1). \tag{23}$$

This implies that for all $t \in [\tau_1, T]$, $a \in \mathbb{R}$ it holds that

$$\begin{aligned}\big(\tfrac{\partial}{\partial t} y\big)(t, a) &= \big(\tfrac{\partial^2}{\partial t \partial a} z\big)(t, a) = \big(\tfrac{\partial^2}{\partial a \partial t} z\big)(t, a) \\ &= g(z(t, a)) + g'(z(t, a))(a + 1)\big(\tfrac{\partial}{\partial a} z\big)(t, a) \\ &= g(z(t, a)) + g'(z(t, a))(a + 1)y(t, a).\end{aligned} \tag{24}$$

Therefore, we obtain that for all $t \in [\tau_1, T]$, $a \in \mathbb{R}$ it holds that

$$\begin{aligned} y(t, a) &= e^{\int_{\tau_1}^t g'(z(u,a))(a+1)\,du} y(\tau_1, a) + \int_{\tau_1}^t e^{\int_s^t g'(z(u,a))(a+1)\,du} g(z(s, a))\,ds \\ &= \int_{\tau_1}^t e^{\int_s^t g'(z(u,a))(a+1)\,du} g(z(s, a))\,ds \geq 0. \end{aligned} \tag{25}$$

Combining this with the fundamental theorem of calculus completes the proof of Lemma 3.2. □

Lemma 3.3. *Assume the setting in Section 3.1 and let* $z\colon [\tau_1, T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function which satisfies for all* $t \in [\tau_1, T]$, $a \in \mathbb{R}$ *that*

$$z_t(a) = \tau_1 + \int_{\tau_1}^t \big[1 + g(z_s(a))(a + 1)\big]\,ds. \tag{26}$$

Then it holds for all $a \in [-1, \infty)$, $b \in [-1, a]$, $t \in [\tau_2, T]$ *that*

$$z_t(a) - z_t(b) \geq 4(a - b)(t - \tau_2). \tag{27}$$

Proof of Lemma 3.3. First, note that Lemma 3.2 ensures that for all $a \in [-1, \infty)$, $b \in [-1, a]$, $t \in [\tau_1, T]$ it holds that

$$z_t(a) \geq z_t(b). \tag{28}$$

The fact that g is a non-decreasing function hence ensures that for all $a \in [-1, \infty)$, $b \in [-1, a]$, $t \in [\tau_1, T]$ it holds that

$$g(z_t(a))(a + 1) \geq g(z_t(b))(a + 1) \geq g(z_t(b))(b + 1). \tag{29}$$

Moreover, observe that for all $t \in [\tau_2, T]$, $r \in [-1, \infty)$ it holds that

$$\begin{aligned} z_t(r) &= \tau_1 + \int_{\tau_1}^t \big[1 + g(z_s(r))(r + 1)\big]\,ds \\ &\geq \tau_1 + \int_{\tau_1}^{\tau_2} \big[1 + g(z_s(r))(r + 1)\big]\,ds \geq \tau_2. \end{aligned} \tag{30}$$

This, (29), and the assumption that $g([\tau_2,\infty)) = \{4\}$ imply that for all $a \in [-1,\infty)$, $b \in [-1,a]$, $t \in [\tau_2,T]$ it holds that

$$\begin{aligned} z_t(a) - z_t(b) &= \int_{\tau_1}^{t} \big[g(z_s(a))(a+1) - g(z_s(b))(b+1)\big]\,ds \\ &\geq \int_{\tau_2}^{t} \big[g(z_s(a))(a+1) - g(z_s(b))(b+1)\big]\,ds \\ &= \int_{\tau_2}^{t} 4(a+1) - 4(b+1)\,ds = \int_{\tau_2}^{t} 4(a-b)\,ds \\ &= 4(a-b)(t-\tau_2). \end{aligned} \tag{31}$$

The proof of Lemma 3.3 is thus completed. □

The next result, Corollary 3.4, is an immediate consequence of Lemma 3.3 above.

Corollary 3.4. *Assume the setting in Section 3.1 and let $z\colon [\tau_1,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function which satisfies for all $t\in[\tau_1,T]$, $a\in\mathbb{R}$ that*

$$z_t(a) = \tau_1 + \int_{\tau_1}^{t} \big[1 + g(z_s(a))(a+1)\big]\,ds. \tag{32}$$

Then it holds for all $a,b\in[-1,\infty)$ that

$$|z_T(a) - z_T(b)| \geq 4(T-\tau_2)|a-b|. \tag{33}$$

3.4 On the explicit solution of a one-dimensional deterministic ordinary differential equation

Lemma 3.5. *Let $T\in(0,\infty)$, $\tau_1\in[0,T]$, $f,x\in C([0,T],[0,\infty))$, $g\in C(\mathbb{R},[0,\infty))$ satisfy for all $t\in[0,T]$ that $g((-\infty,\tau_1]) = \{0\}$ and*

$$x_t = t + \int_0^t g(x_s)f(s)\,ds = \int_0^t \big[1 + g(x_s)f(s)\big]\,ds. \tag{34}$$

Then it holds for all $t\in[0,\tau_1]$ that $x_t = t$.

Proof of Lemma 3.5. Throughout this proof let $\mu\in[0,T]$ be the real number given by

$$\mu = \inf(\{t\in[0,T]\colon x_t\geq\tau_1\}\cup\{T\})\,. \tag{35}$$

Observe that the fact that

$$\forall\, t\in[0,T]\colon x_t = t + \int_0^t g(x_s)f(s)\,ds \geq t \tag{36}$$

ensures that

$$\{t\in[0,T]\colon x_t\geq\tau_1\} \supseteq [\tau_1,T] \neq \emptyset. \tag{37}$$

Next note that the fact that $x_0 = 0$ assures that for all $t \in [0, \mu]$ it holds that $x_t \leq \tau_1$. This and the assumption that $g((-\infty, \tau_1]) = \{0\}$ ensure that for all $t \in [0, \mu]$ it holds that

$$\tau_1 \geq x_t = t + \int_0^t g(x_s) f(s)\, ds = t. \tag{38}$$

In the next step we observe that (35) and (37) imply that $x_\mu \geq \tau_1$. Combining this with (38) yields that

$$\tau_1 \geq \mu = x_\mu \geq \tau_1. \tag{39}$$

This proves that $\mu = \tau_1$. Combining this and (38) completes the proof of Lemma 3.5. □

3.5 On the explicit solution of a two-dimensional SDE

Lemma 3.6. *Assume the setting in Section 3.1 and let $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then*

(i) it holds for all $t \in [0, \tau_1]$ that $\mathbb{P}(X_t^{\psi,(2)} = t) = 1$,

(ii) it holds for all $t \in [\tau_1, T]$ that $\mathbb{P}(f(X_t^{\psi,(2)}) = 0) = 1$, and

(iii) it holds for all $t \in [\tau_1, T]$ that $\mathbb{P}(X_t^{\psi,(1)} = X_{\tau_1}^{\psi,(1)} = \int_0^{\tau_1} f(s)\, dW_s) = 1$.

Proof of Lemma 3.6. First, note that Lemma 3.5 proves that for all $t \in [0, \tau_1]$ it holds that $\mathbb{P}(X_t^{\psi,(2)} = t) = 1$. This establishes Item (i). Next note that the fact that $g \geq 0$ ensures that for all $t \in [\tau_1, T]$ it holds that

$$\mathbb{P}(X_t^{\psi,(2)} \geq \tau_1) = 1. \tag{40}$$

The assumption that $f([\tau_1, \infty)) = \{0\}$ hence proves Item (ii). Moreover, observe that Item (i) and Item (ii) imply that for all $t \in [\tau_1, T]$ it holds $\mathbb{P}$-a.s. that

$$X_t^{\psi,(1)} = \int_0^{\tau_1} f(X_s^{\psi,(2)})\, dW_s + \int_{\tau_1}^t f(X_s^{\psi,(2)})\, dW_s = \int_0^{\tau_1} f(s)\, dW_s. \tag{41}$$

This establishes Item (iii). The proof of Lemma 3.6 is thus completed. □

Lemma 3.7. *Assume the setting in Section 3.1, let $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$, and let $z\colon [\tau_1, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies for all $t \in [\tau_1, T]$, $a \in \mathbb{R}$ that*

$$z_t(a) = \tau_1 + \int_{\tau_1}^t \big[1 + g(z_s(a))(a+1)\big]\, ds. \tag{42}$$

Then it holds for all $t \in [\tau_1, T]$ that

$$\mathbb{P}\Big(X_t^{\psi,(2)} = z_t\big(\cos(\psi(X_{\tau_1}^{\psi,(1)}))\big)\Big) = 1. \tag{43}$$

Proof of Lemma 3.7. First, note that for all $t \in [\tau_1, T]$ it holds that

$$\begin{aligned}
1 &= \mathbb{P}\Big(X_t^{\psi,(2)} = \textstyle\int_0^t 1 + g(X_s^{\psi,(2)})[\cos(\psi(X_s^{\psi,(1)})) + 1]\, ds \Big) \\
&= \mathbb{P}\Big(X_t^{\psi,(2)} = \textstyle\int_0^{\tau_1} 1 + g(X_s^{\psi,(2)})[\cos(\psi(X_s^{\psi,(1)})) + 1]\, ds \\
&\quad + \textstyle\int_{\tau_1}^t 1 + g(X_s^{\psi,(2)})[\cos(\psi(X_s^{\psi,(1)})) + 1]\, ds \Big) \\
&= \mathbb{P}\Big(X_t^{\psi,(2)} = X_{\tau_1}^{\psi,(2)} + \textstyle\int_{\tau_1}^t 1 + g(X_s^{\psi,(2)})[\cos(\psi(X_s^{\psi,(1)})) + 1]\, ds \Big).
\end{aligned} \tag{44}$$

Items (i) and (iii) of Lemma 3.6 hence prove that for all $t \in [\tau_1, T]$ it holds that

$$\mathbb{P}\Big(X_t^{\psi,(2)} = \tau_1 + \textstyle\int_{\tau_1}^t 1 + g(X_s^{\psi,(2)})[\cos(\psi(X_{\tau_1}^{\psi,(1)})) + 1]\, ds \Big) = 1. \tag{45}$$

The fact that $X^{\psi,(2)}$ is a continuous stochastic process therefore ensures that

$$\mathbb{P}\Big(\forall\, t \in [\tau_1, T] \colon X_t^{\psi,(2)} = \tau_1 + \textstyle\int_{\tau_1}^t 1 + g(X_s^{\psi,(2)})[\cos(\psi(X_{\tau_1}^{\psi,(1)})) + 1]\, ds \Big) = 1. \tag{46}$$

This completes the proof of Lemma 3.7. □

3.6 Lower and upper bounds for the variances of some Gaussian distributed random variables

Lemma 3.8. *Assume the setting in Section 3.1 and let $a \in [0, \tau)$, $b \in (a, \tau]$, let $\bar{W}, B \colon [a, b] \times \Omega \to \mathbb{R}$ and $\tilde{W} \colon ([0, a] \cup [b, T]) \times \Omega \to \mathbb{R}$ be stochastic processes, let $Y_1, Y_2 \colon \Omega \to \mathbb{R}$ be random variables, and assume for all $s \in [a, b]$, $t \in ([0, a] \cup [b, T])$ that*

$$\tilde{W}_t = W_t, \quad \bar{W}_s = \frac{(s-a)}{(b-a)} \cdot W_b + \frac{(b-s)}{(b-a)} \cdot W_a, \quad B_s = W_s - \bar{W}_s, \tag{47}$$

$$\mathbb{P}\Big(Y_1 = \textstyle\int_0^a f(s)\, dW_s + \int_b^{\tau_1} f(s)\, dW_s + \int_a^b f(s)\, d\bar{W}_s \Big) = 1, \tag{48}$$

and

$$\mathbb{P}\Big(Y_2 = \textstyle\int_a^b f(s)\, dW_s - \int_a^b f(s)\, d\bar{W}_s \Big) = 1. \tag{49}$$

Then

(i) it holds that $\Omega \in \omega \mapsto (\tilde{W}_t(\omega))_{t \in [0,a] \cup [b,T]} \in C([0, a] \cup [b, T], \mathbb{R})$ and $\Omega \in \omega \mapsto (B_t(\omega))_{t \in [a,b]} \in C([a, b], \mathbb{R})$ are independent on $(\Omega, \mathcal{F}, \mathbb{P})$,

(ii) it holds for all $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$ that

$$\mathbb{P}\Big(\textstyle\int_{t_1}^{t_2} f(s)\, dW_s = f(t_2) W_{t_2} - f(t_1) W_{t_1} - \int_{t_1}^{t_2} f'(s) W_s\, ds \Big) = 1, \tag{50}$$

(iii) it holds that

$$\mathbb{P}\Big(Y_2 = - \textstyle\int_a^b f'(s) B_s \, ds\Big) = 1, \tag{51}$$

(iv) it holds that $\frac{\alpha}{2} \leq \mathbb{E}\big[|Y_1|^2\big] \leq \alpha$, *and*

(v) it holds that $\frac{(b-a)^3}{12} \leq \mathbb{E}\big[|Y_2|^2\big] \leq \frac{(b-a)^3}{3}$.

Proof of Lemma 3.8. First, note that for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, T]$ it holds that

$$\Omega \ni \omega \mapsto (W_{t_1}(\omega), \dots, W_{t_n}(\omega)) \in \mathbb{R}^n \tag{52}$$

is Gaussian distributed. Next note that for all $s \in [a, b]$, $u \in [0, a] \cup [b, T]$ it holds that

$$\begin{aligned}
\mathbb{E}[B_s \tilde{W}_u] &= \mathbb{E}\bigg[\bigg(W_s - \frac{(s-a)}{(b-a)} \cdot W_b - \frac{(b-s)}{(b-a)} \cdot W_a\bigg) W_u\bigg] \\
&= \min\{s, u\} - \frac{(s-a)\min\{b, u\}}{(b-a)} - \frac{(b-s)\min\{a, u\}}{(b-a)} \\
&= \frac{(b-a)\min\{s, u\} - (s-a)\min\{b, u\} - (b-s)\min\{a, u\}}{(b-a)} \\
&= \begin{cases} \frac{(b-a)u - (s-a)u - (b-s)u}{(b-a)} = \frac{bu - au - su + au - bu + su}{(b-a)} & : \quad u \leq a \\ \frac{(b-a)s - (s-a)b - (b-s)a}{(b-a)} = \frac{bs - as - sb + ab - ba + sa}{(b-a)} & : \quad u \geq b \end{cases} \\
&= 0.
\end{aligned} \tag{53}$$

Combining this with (52) ensures that for all $n, m \in \mathbb{N}$, $t_1, \dots, t_n \in [0, a] \cup [b, T]$, $s_1, \dots, s_m \in [a, b]$, $\mathbb{W}_1, \dots, \mathbb{W}_n, \mathbb{B}_1, \dots, \mathbb{B}_m \in \mathcal{B}(\mathbb{R})$ it holds that

$$\begin{aligned}
&\mathbb{P}\Big(\Big\{\big(\tilde{W}_{t_1}, \dots, \tilde{W}_{t_n}\big) \in \mathbb{W}_1 \times \ldots \times \mathbb{W}_n\Big\} \cap \Big\{\big(B_{s_1}, \dots, B_{s_m}\big) \in \mathbb{B}_1 \times \ldots \times \mathbb{B}_m\Big\}\Big) \\
&= \mathbb{P}\Big(\big(\tilde{W}_{t_1}, \dots, \tilde{W}_{t_n}\big) \in \mathbb{W}_1 \times \ldots \times \mathbb{W}_n\Big) \cdot \mathbb{P}\Big(\big(B_{s_1}, \dots, B_{s_m}\big) \in \mathbb{B}_1 \times \ldots \times \mathbb{B}_m\Big).
\end{aligned} \tag{54}$$

This, the fact that

$$\mathcal{B}\big(C([a, b], \mathbb{R})\big) = \mathcal{B}(\mathbb{R})^{\otimes [a,b]} \Cap C([a, b], \mathbb{R}), \tag{55}$$

and the fact that

$$\mathcal{B}\big(C([0, a] \cup [b, T], \mathbb{R})\big) = \mathcal{B}(\mathbb{R})^{\otimes [0,a] \cup [b,T]} \Cap C([0, a] \cup [b, T], \mathbb{R}) \tag{56}$$

establish Item (i). Moreover, note that (53) proves that for all $s, u \in [a, b]$ it holds that

$$\begin{aligned}
\mathbb{E}[B_s \bar{W}_u] &= \mathbb{E}\bigg[B_s\bigg(\frac{(u-a)}{(b-a)} \cdot W_b + \frac{(b-u)}{(b-a)} \cdot W_a\bigg)\bigg] \\
&= \mathbb{E}\bigg[B_s\bigg(\frac{(u-a)}{(b-a)} \cdot \tilde{W}_b + \frac{(b-u)}{(b-a)} \cdot \tilde{W}_a\bigg)\bigg] \\
&= \frac{(u-a)}{(b-a)} \cdot \mathbb{E}[B_s \tilde{W}_b] + \frac{(b-u)}{(b-a)} \cdot \mathbb{E}[B_s \tilde{W}_a] = 0.
\end{aligned} \tag{57}$$

Hence, we obtain that for all $s, u \in [a, b]$ it holds that

$$\begin{aligned}
&\mathbb{E}[B_s B_u] = \mathbb{E}[B_s(W_u - \bar{W}_u)] = \mathbb{E}[B_s W_u] \\
&= \mathbb{E}\left[\left(W_s - \frac{(s-a)}{(b-a)} \cdot W_b - \frac{(b-s)}{(b-a)} \cdot W_a\right) W_u\right] \\
&= \min\{s,u\} - \frac{u(s-a)}{(b-a)} - \frac{a(b-s)}{(b-a)} \\
&= \frac{(b-a)\min\{s,u\} - us + au - ab + as}{(b-a)} \\
&= \frac{b\min\{s,u\} - \max\{s,u\}\min\{s,u\} + a(u+s-\min\{s,u\}) - ab}{(b-a)} \\
&= \frac{(b-\max\{s,u\})\min\{s,u\} - a(b-\max\{s,u\})}{(b-a)} \\
&= \frac{(b-\max\{s,u\})(\min\{s,u\}-a)}{(b-a)}.
\end{aligned} \tag{58}$$

Moreover, observe that Itô's formula ensures that for all $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$ it holds $\mathbb{P}$-a.s. that

$$f(t_2)W_{t_2} = f(t_1)W_{t_1} + \int_{t_1}^{t_2} f'(s)W_s\,ds + \int_{t_1}^{t_2} f(s)\,dW_s. \tag{59}$$

Hence, we obtain that for all $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$ it holds $\mathbb{P}$-a.s. that

$$\int_{t_1}^{t_2} f(s)\,dW_s = f(t_2)W_{t_2} - f(t_1)W_{t_1} - \int_{t_1}^{t_2} f'(s)W_s\,ds. \tag{60}$$

This establishes Item (ii). In addition, note that (60) assures that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
Y_2 &= f(b)W_b - f(a)W_a - \int_a^b f'(s)W_s\,ds - \int_a^b f(s)\,d\bar{W}_s \\
&= f(b)W_b - f(a)W_a - \int_a^b f'(s)W_s\,ds - \frac{W_b}{(b-a)}\int_a^b f(s)\,ds \\
&\quad + \frac{W_a}{(b-a)}\int_a^b f(s)\,ds.
\end{aligned} \tag{61}$$

Furthermore, note that integration by parts shows that

$$\begin{aligned}
\int_a^b f(s)\,ds &= \int_a^b f(s)(s-a)^0\,ds = \big[f(s)(s-a)\big]_{s=a}^{s=b} - \int_a^b f'(s)(s-a)\,ds \\
&= f(b)(b-a) - \int_a^b f'(s)(s-a)\,ds
\end{aligned} \tag{62}$$

and

$$\begin{aligned}
\int_a^b f(s)\,ds &= \int_a^b f(s)(b-s)^0\,ds = -\big[f(s)(b-s)\big]_{s=a}^{s=b} + \int_a^b f'(s)(b-s)\,ds \\
&= f(a)(b-a) + \int_a^b f'(s)(b-s)\,ds.
\end{aligned} \tag{63}$$

Putting (62) and (63) into (61) shows that it holds $\mathbb{P}$-a.s. that

$$
\begin{aligned}
Y_2 &= -\int_a^b f'(s) W_s \, ds + \int_a^b f'(s) \left[\frac{(s-a)}{(b-a)} \cdot W_b \right] ds \\
&\quad + \int_a^b f'(s) \left[\frac{(b-s)}{(b-a)} \cdot W_a \right] ds \\
&= -\int_a^b f'(s) W_s \, ds + \int_a^b f'(s) \bar{W}_s \, ds = -\int_a^b f'(s) [W_s - \bar{W}_s] \, ds \\
&= -\int_a^b f'(s) B_s \, ds.
\end{aligned} \tag{64}
$$

This establishes Item (iii). Next note that Item (iii) proves that

$$
\begin{aligned}
\mathbb{E}\big[|Y_2|^2\big] &= \mathbb{E}\left[\left| \int_a^b f'(s) B_s \, ds \right|^2 \right] = \int_a^b \int_a^b f'(s) f'(u) \, \mathbb{E}[B_s B_u] \, ds \, du \\
&= \int_a^b \int_a^b f'(s) f'(u) \left[\frac{(b - \max\{s,u\})(\min\{s,u\} - a)}{(b-a)} \right] ds \, du.
\end{aligned} \tag{65}
$$

Moreover, observe that

$$
\begin{aligned}
&\int_a^b \int_a^b \frac{(b - \max\{s,u\})(\min\{s,u\} - a)}{(b-a)} \, ds \, du \\
&= \int_a^b \int_a^u \frac{(b-u)(s-a)}{(b-a)} \, ds \, du + \int_a^b \int_u^b \frac{(b-s)(u-a)}{(b-a)} \, ds \, du \\
&= \int_a^b \frac{(b-u)}{(b-a)} \left[\int_0^{u-a} s \, ds \right] du + \int_a^b \frac{(u-a)}{(b-a)} \left[\int_0^{b-u} s \, ds \right] du \\
&= \int_a^b \frac{(b-u)(u-a)^2}{2(b-a)} \, du + \int_a^b \frac{(b-u)^2(u-a)}{2(b-a)} \, du \\
&= \int_a^b \frac{(b-u)(u-a)}{2} \, du = \int_0^{b-a} \frac{(b-a-u)u}{2} \, du \\
&= \int_0^{b-a} \frac{(b-a)u}{2} \, du - \int_0^{b-a} \frac{u^2}{2} \, du = \frac{(b-a)}{2} \cdot \frac{(b-a)^2}{2} - \frac{(b-a)^3}{6} \\
&= \left[\frac{1}{4} - \frac{1}{6} \right] (b-a)^3 = \frac{(b-a)^3}{12}.
\end{aligned} \tag{66}
$$

The assumption that $f'((0,\tau)) \subseteq [-2,-1)$ and (65) hence ensure that

$$
\frac{(b-a)^3}{12} \leq \mathbb{E}\big[|Y_2|^2\big] \leq \frac{(b-a)^3}{3}. \tag{67}
$$

This establishes Item (v). Next note that Item (i) proves that the random variables Y_1 and Y_2 are independent. Itô's isometry hence yields that

$$\begin{aligned}\mathbb{E}\big[|Y_1|^2\big] &= \mathbb{E}\big[|Y_1+Y_2|^2\big] - \mathbb{E}\big[|Y_2|^2\big] - 2\,\mathbb{E}\big[Y_1Y_2\big] \\ &= \mathbb{E}\left[\left|\int_0^{\tau_1} f(s)\,dW_s\right|^2\right] - \mathbb{E}\big[|Y_2|^2\big] \\ &= \int_0^{\tau_1} |f(s)|^2\,ds - \mathbb{E}\big[|Y_2|^2\big] = \alpha - \mathbb{E}\big[|Y_2|^2\big] \leq \alpha.\end{aligned} \tag{68}$$

The assumption that $\alpha \geq \frac{2\tau^3}{3}$, the fact that $(b-a) \in (0,\tau]$, and Item (v) therefore ensure that

$$\alpha \geq \mathbb{E}\big[|Y_1|^2\big] \geq \alpha - \frac{(b-a)^3}{3} \geq \alpha - \frac{\tau^3}{3} \geq \frac{\alpha}{2}. \tag{69}$$

This establishes Item (iv). The proof of Lemma 3.8 is thus completed. □

3.7 Explicit lower bounds for strong approximation errors for two-dimensional SDEs

The next result, Lemma 3.9 below, is proved as Lemma 4.1 in [9].

Lemma 3.9. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $(S_1,\mathcal{S}_1)$ and $(S_2,\mathcal{S}_2)$ be measurable spaces, and let $X_1\colon \Omega \to S_1$ and $X_2, X_2', X_2''\colon \Omega \to S_2$ be random variables such that*

$$\mathbb{P}_{(X_1,X_2)} = \mathbb{P}_{(X_1,X_2')} = \mathbb{P}_{(X_1,X_2'')}. \tag{70}$$

Then it holds for all measurable functions $\Phi\colon S_1 \times S_2 \to \mathbb{R}$ and $\varphi\colon S_1 \to \mathbb{R}$ that

$$\mathbb{E}\big[|\Phi(X_1,X_2) - \varphi(X_1)|\big] \geq \tfrac{1}{2}\,\mathbb{E}\big[|\Phi(X_1,X_2') - \Phi(X_1,X_2'')|\big]. \tag{71}$$

Lemma 3.10. *Let $c \in \mathbb{R}$, $\beta \in (0,1)$ and let $\lambda\colon \mathcal{B}(\mathbb{R}) \to [0,\infty]$ the Lebesgue-Borel measure on $\mathbb{R}$. Then*

$$\lambda\Big(\big\{x \in [c-1,c+1]\colon \big|\sin\big(\tfrac{x-c}{\beta}\big)\big| \geq \tfrac{1}{2}\big\}\Big) \geq \tfrac{1}{2}. \tag{72}$$

Proof of Lemma 3.10. Throughout this proof let $A \subseteq \mathbb{R}$ be the set given by

$$A = \big\{x \in [c-1,c+1]\colon \big|\sin\big(\tfrac{x-c}{\beta}\big)\big| \geq \tfrac{1}{2}\big\} \tag{73}$$

and let $m \in \mathbb{Z}$ be the integer number which satisfies that

$$\beta\pi\big(m-1+\tfrac{1}{6}\big) < -1 \qquad \text{and} \qquad \beta\pi\big(m+\tfrac{1}{6}\big) \geq -1. \tag{74}$$

Observe that the fact that $\forall\, k \in \mathbb{Z}\colon \sin\big(\frac{\pi}{6}+k\pi\big) = \sin\big(\frac{5\pi}{6}+k\pi\big) = (-1)^k \cdot \frac{1}{2}$ ensures that

$$\big\{y \in \mathbb{R}\colon |\sin(y)| \geq \tfrac{1}{2}\big\} = \cup_{k\in\mathbb{Z}}\big[\tfrac{\pi}{6}+k\pi, \tfrac{5\pi}{6}+k\pi\big]. \tag{75}$$

Hence, we obtain that

$$\begin{aligned} A &= [c-1,c+1] \bigcap \Big(\cup_{k\in\mathbb{Z}} \Big\{ x\in\mathbb{R} \colon \big(\tfrac{x-c}{\beta}\big) \in \big[\tfrac{\pi}{6}+k\pi, \tfrac{5\pi}{6}+k\pi\big] \Big\} \Big) \\ &= [c-1,c+1] \bigcap \Big(\cup_{k\in\mathbb{Z}} \big[c+\beta\big(\tfrac{\pi}{6}+k\pi\big), c+\beta\big(\tfrac{5\pi}{6}+k\pi\big) \big] \Big) \\ &\supseteq [c-1,c+1] \bigcap \Big(\cup_{k=m-1}^{\infty} \big[c+\beta\big(\tfrac{\pi}{6}+k\pi\big), c+\beta\big(\tfrac{5\pi}{6}+k\pi\big) \big] \Big). \end{aligned} \tag{76}$$

Next note that (74) and the assumption that $\beta \in (0,1)$ ensure that $m \leq 0$. To prove (72), we distinguish between two cases. In the first case we assume that $m = 0$. We observe that (74) then yields that

$$\beta > \tfrac{6}{5\pi}. \tag{77}$$

This and the fact that $\beta \in (0,1)$ prove that

$$c + \tfrac{5\beta\pi}{6} > c+1, \tag{78}$$

$$c - \tfrac{\beta\pi}{6} > c - \tfrac{\pi}{6} > c-1, \tag{79}$$

and

$$c + \tfrac{\beta\pi}{6} < c + \tfrac{\pi}{6} < c+1. \tag{80}$$

Combining this, (76), and (74) ensures that

$$\begin{aligned} A &\supseteq [c-1,c+1] \bigcap \left(\bigcup_{k=-1}^{0} \big[c+\beta\big(\tfrac{\pi}{6}+k\pi\big), c+\beta\big(\tfrac{5\pi}{6}+k\pi\big) \big] \right) \\ &= [c-1,c+1] \bigcap \Big(\big[c - \tfrac{5\beta\pi}{6}, c - \tfrac{\beta\pi}{6}\big] \bigcup \big[c + \tfrac{\beta\pi}{6}, c + \tfrac{5\beta\pi}{6}\big] \Big) \\ &= \big[c-1, c-\tfrac{\beta\pi}{6}\big] \bigcup \big[c+\tfrac{\beta\pi}{6}, c+1\big]. \end{aligned} \tag{81}$$

This implies that

$$\lambda(A) \geq 2\big(1 - \tfrac{\beta\pi}{6}\big) > 2 - \tfrac{\pi}{3} > \tfrac{1}{2}. \tag{82}$$

This finishes the proof of (72) in the case $m = 0$. In the second case we assume that $m \leq -1$. Note that (74) proves that

$$\beta\big(\tfrac{5\pi}{6} + \pi(-m-1)\big) = \beta\pi\big(-m-\tfrac{1}{6}\big) = -\beta\pi\big(m+\tfrac{1}{6}\big) \leq 1. \tag{83}$$

This and again (74) ensure for all $k \in [m, -m-1] \cap \mathbb{Z}$ that

$$\big[c+\beta\big(\tfrac{\pi}{6}+k\pi\big), c+\beta\big(\tfrac{5\pi}{6}+k\pi\big)\big] \subseteq [c-1,c+1]. \tag{84}$$

Combining (76) and (74) hence demonstrates that

$$\begin{aligned} \lambda(A) &\geq \lambda\left(\bigcup_{k=m}^{-m-1} \big[c+\beta\big(\tfrac{\pi}{6}+k\pi\big), c+\beta\big(\tfrac{5\pi}{6}+k\pi\big) \big] \right) \\ &= \sum_{k=m}^{-m-1} \lambda\Big(\big[c+\beta\big(\tfrac{\pi}{6}+k\pi\big), c+\beta\big(\tfrac{5\pi}{6}+k\pi\big) \big] \Big) \\ &= -2m \cdot \tfrac{2\beta\pi}{3} > -\tfrac{4m}{3} \cdot \tfrac{1}{5/6-m} = -\tfrac{8m}{5-6m} > \tfrac{1}{2}. \end{aligned} \tag{85}$$

This finishes the proof of (72) in the case $m \leq -1$. The proof of Lemma 3.10 is thus completed. □

Lemma 3.11. *Assume the setting in Section 3.1, let* $a \in [0, \tau)$*,* $b \in (a, \tau]$*,* $c \in [2, \infty)$*,* $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$*, let* $u\colon C([0,a] \cup [b,T], \mathbb{R}) \to \mathbb{R}$ *be a measurable function, and assume for all* $x \in [c-2, c+2]$ *that* $\psi(x) = \frac{T^{3/2}}{(b-a)^{3/2}} \cdot (x - c)$*. Then*

$$\begin{aligned} &\mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\geq \frac{\sqrt{3}(T-\tau_2)}{\pi\sqrt{T^3\alpha}} \left[\int_{c+1/2}^{c+1} e^{-\frac{x^2}{\alpha}}\, dx\right]\left[\int_0^1 |\sin(y)|\, e^{-\frac{6y^2}{T^3}}\, dy\right] > 0. \end{aligned} \tag{86}$$

Proof of Lemma 3.11. Throughout this proof let $A \subseteq \mathbb{R}$ be the set given by

$$A = \big\{x \in [c-1, c+1]\colon\ |\sin(\psi(x))| \geq \tfrac{1}{2}\big\}, \tag{87}$$

let $\bar{W}, B\colon [a,b] \times \Omega \to \mathbb{R}$ and $\tilde{W}\colon ([0,a] \cup [b,T]) \times \Omega \to \mathbb{R}$ be the stochastic processes with continuous sample paths which satisfy for all $s \in [a,b]$, $t \in ([0,a] \cup [b,T])$ that

$$\bar{W}_s = \frac{(s-a)}{(b-a)} \cdot W_b + \frac{(b-s)}{(b-a)} \cdot W_a, \quad B_s = W_s - \bar{W}_s, \quad \text{and} \quad \tilde{W}_t = W_t, \tag{88}$$

let $Y_1, Y_2\colon \Omega \to \mathbb{R}$ be random variables which satisfy

$$\mathbb{P}\Big(Y_1 = \textstyle\int_0^a f(s)\, dW_s + \int_b^{\tau_1} f(s)\, dW_s + \int_a^b f(s)\, d\bar{W}_s\Big) = 1, \tag{89}$$

and

$$\mathbb{P}\Big(Y_2 = \textstyle\int_a^b f(s)\, dW_s - \int_a^b f(s)\, d\bar{W}_s\Big) = 1, \tag{90}$$

let $z\colon [\tau_1, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies for all $t \in [\tau_1, T]$, $a \in \mathbb{R}$ that

$$z_t(a) = \tau_1 + \int_{\tau_1}^t \big[1 + g(z_s(a))(a+1)\big]\, ds, \tag{91}$$

let $\sigma_1, \sigma_2, \varepsilon, \beta \in (0, \infty)$ be the real numbers given by

$$\sigma_1 = \mathbb{E}\big[|Y_1|^2\big], \quad \sigma_2 = \mathbb{E}\big[|Y_2|^2\big], \quad \varepsilon = b - a, \quad \text{and} \quad \beta = \frac{\varepsilon^3}{T^3}, \tag{92}$$

and for every $x \in \mathbb{R}$, $y \in (0, \infty)$ let $\mathcal{N}_{x,y}\colon \mathcal{B}(\mathbb{R}) \to [0, \infty]$ be the function which satisfies for all $B \in \mathcal{B}(\mathbb{R})$ that

$$\mathcal{N}_{x,y}(B) = \int_B \frac{1}{\sqrt{2\pi y}}\, e^{-\frac{(r-x)^2}{2y}}\, dr. \tag{93}$$

Next note that Item (iii) in Lemma 3.6 proves that for all $t \in [\tau_1, T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} X_t^{\psi,(1)} &= \int_0^{\tau_1} f(s)\, dW_s = \int_0^a f(s)\, dW_s + \int_b^{\tau_1} f(s)\, dW_s + \int_a^b f(s)\, dW_s \\ &= \left[\int_0^a f(s)\, dW_s + \int_b^{\tau_1} f(s)\, dW_s + \int_a^b f(s)\, d\bar{W}_s\right] \\ &\quad + \left[\int_a^b f(s)\, dW_s - \int_a^b f(s)\, d\bar{W}_s\right] = Y_1 + Y_2. \end{aligned} \tag{94}$$

This together with Lemma 3.7 ensures that

$$\mathbb{P}\Big(X_T^{\psi,(2)} = z_T\big(\cos(\psi(Y_1+Y_2))\big)\Big) = 1. \tag{95}$$

Moreover, observe that Items (ii) and (iii) of Lemma 3.8 show that for all $t_1, t_2 \in [0,T]$ with $t_1 \le t_2$ it holds that

$$\mathbb{P}\Big(\textstyle\int_{t_1}^{t_2} f(s)\, dW_s = f(t_2)W_{t_2} - f(t_1)W_{t_1} - \int_{t_1}^{t_2} f'(s) W_s \, ds \Big) = 1 \tag{96}$$

and

$$\mathbb{P}\Big(Y_2 = -\textstyle\int_a^b f'(s) B_s \, ds\Big) = 1. \tag{97}$$

Item (i) in Lemma 3.8 therefore proves that

$$Y_2 \qquad \text{and} \qquad \tilde{W} \tag{98}$$

are independent on $(\Omega, \mathcal{F}, \mathbb{P})$. The fact that Y_2 is a Gaussian random variable with mean 0 hence implies that

$$\mathbb{P}_{(\tilde{W},Y_2)} = \mathbb{P}_{\tilde{W}} \otimes \mathbb{P}_{Y_2} = \mathbb{P}_{\tilde{W}} \otimes \mathbb{P}_{-Y_2} = \mathbb{P}_{(\tilde{W},-Y_2)}. \tag{99}$$

Next observe that (89) and (96) assure that there exists a measurable function $\Phi_1 \colon C([0,a] \cup [b,T], \mathbb{R}) \to \mathbb{R}$ such that

$$\mathbb{P}\big(Y_1 = \Phi_1(\tilde{W})\big) = 1. \tag{100}$$

This, Lemma 3.9 (with $\Omega = \Omega$, $S_1 = C([0,a] \cup [b,T], \mathbb{R})$, $S_2 = \mathbb{R}$, $X_1 = \tilde{W}$, $X_2 = Y_2$, $X_2' = Y_2$, $X_2'' = -Y_2$, $\varphi = u$, and $\Phi = (C([0,a] \cup [b,T], \mathbb{R}) \times \mathbb{R} \ni (w,y) \mapsto z_T(\cos(\psi(\Phi_1(w)+y))) \in \mathbb{R})$ in the notation of Lemma 3.9), (95), and (99) show that

$$\begin{aligned} &\mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] = \mathbb{E}\Big[\big|z_T(\cos(\psi(Y_1+Y_2))) - u(\tilde{W})\big|\Big] \\ &= \mathbb{E}\Big[\big|z_T(\cos(\psi(\Phi_1(\tilde{W})+Y_2))) - u(\tilde{W})\big|\Big] \\ &\ge \tfrac{1}{2}\,\mathbb{E}\Big[\big|z_T(\cos(\psi(\Phi_1(\tilde{W})+Y_2))) - z_T(\cos(\psi(\Phi_1(\tilde{W})-Y_2)))\big|\Big] \\ &= \tfrac{1}{2}\,\mathbb{E}\Big[\big|z_T(\cos(\psi(Y_1+Y_2))) - z_T(\cos(\psi(Y_1-Y_2)))\big|\Big]. \end{aligned} \tag{101}$$

Corollary 3.4 therefore ensures that

$$\begin{aligned} &\mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\ge 2(T-\tau_2)\,\mathbb{E}\big[|\cos(\psi(Y_1+Y_2)) - \cos(\psi(Y_1-Y_2))|\big]. \end{aligned} \tag{102}$$

Moreover, note that (100) and (98) demonstrate that Y_1 and Y_2 are independent on $(\Omega, \mathcal{F}, \mathbb{P})$. The fact that Y_1 and Y_2 are centered Gaussian distributed random variables hence shows that

$$\begin{aligned}
&\mathbb{E}\big[|\cos(\psi(Y_1+Y_2))-\cos(\psi(Y_1-Y_2))|\big]\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}}|\cos(\psi(x+y))-\cos(\psi(x-y))|\,\mathcal{N}_{0,\sigma_1}(dx)\,\mathcal{N}_{0,\sigma_2}(dy)\\
&\geq\int_{[0,1]}\int_{[c-1,c+1]}|\cos(\psi(x+y))-\cos(\psi(x-y))|\,\mathcal{N}_{0,\sigma_1}(dx)\,\mathcal{N}_{0,\sigma_2}(dy)\\
&=\int_{[0,1]}\int_{[c-1,c+1]}\Big|\cos\Big(\tfrac{x+y-c}{\sqrt{\beta}}\Big)-\cos\Big(\tfrac{x-y-c}{\sqrt{\beta}}\Big)\Big|\,\mathcal{N}_{0,\sigma_1}(dx)\,\mathcal{N}_{0,\sigma_2}(dy)\\
&=\int_{[0,1]}\int_{[c-1,c+1]}\Big|\cos\Big(\psi(x)+\tfrac{y}{\sqrt{\beta}}\Big)-\cos\Big(\psi(x)-\tfrac{y}{\sqrt{\beta}}\Big)\Big|\,\mathcal{N}_{0,\sigma_1}(dx)\,\mathcal{N}_{0,\sigma_2}(dy).
\end{aligned} \tag{103}$$

The fact that $\forall\, v,w\in\mathbb{R}\colon \cos(v)-\cos(w)=-2\sin\big(\frac{v-w}{2}\big)\sin\big(\frac{v+w}{2}\big)$ therefore assures that

$$\begin{aligned}
&\mathbb{E}\big[|\cos(\psi(Y_1+Y_2))-\cos(\psi(Y_1-Y_2))|\big]\\
&\geq 2\int_{[0,1]}\int_{[c-1,c+1]}|\sin(\psi(x))|\,\Big|\sin\Big(\tfrac{y}{\sqrt{\beta}}\Big)\Big|\,\mathcal{N}_{0,\sigma_1}(dx)\,\mathcal{N}_{0,\sigma_2}(dy)\\
&\geq 2\int_{[0,1]}\int_{A}|\sin(\psi(x))|\,\Big|\sin\Big(\tfrac{y}{\sqrt{\beta}}\Big)\Big|\,\mathcal{N}_{0,\sigma_1}(dx)\,\mathcal{N}_{0,\sigma_2}(dy)\\
&\geq \mathcal{N}_{0,\sigma_1}(A)\int_{[0,1]}\Big|\sin\Big(\tfrac{y}{\sqrt{\beta}}\Big)\Big|\,\mathcal{N}_{0,\sigma_2}(dy).
\end{aligned} \tag{104}$$

In addition, observe that Item (v) in Lemma 3.8 proves that

$$\frac{T^3\beta}{12}=\frac{\varepsilon^3}{12}\leq\frac{(b-a)^3}{12}\leq\sigma_2\leq\frac{(b-a)^3}{3}=\frac{\varepsilon^3}{3}=\frac{T^3\beta}{3}. \tag{105}$$

This implies that

$$\begin{aligned}
&\int_{[0,1]}\Big|\sin\Big(\tfrac{y}{\sqrt{\beta}}\Big)\Big|\,\mathcal{N}_{0,\sigma_2}(dy)=\int_{[0,1]}\Big|\sin\Big(\tfrac{y}{\sqrt{\beta}}\Big)\Big|\frac{1}{\sqrt{2\sigma_2\pi}}\,e^{-\frac{y^2}{2\sigma_2}}\,dy\\
&\geq\int_{[0,\sqrt{\beta}]}\Big|\sin\Big(\tfrac{y}{\sqrt{\beta}}\Big)\Big|\frac{\sqrt{3}}{\sqrt{2T^3\beta\pi}}\,e^{-\frac{y^2}{2\sigma_2}}\,dy\geq\int_{[0,\sqrt{\beta}]}\Big|\sin\Big(\tfrac{y}{\sqrt{\beta}}\Big)\Big|\frac{\sqrt{3}}{\sqrt{2T^3\beta\pi}}\,e^{-\frac{6y^2}{T^3\beta}}\,dy\\
&=\frac{\sqrt{3}}{\sqrt{2T^3\pi}}\int_{[0,1]}|\sin(y)|\,e^{-\frac{6y^2}{T^3}}\,dy.
\end{aligned} \tag{106}$$

Moreover, Item (iv) in Lemma 3.8 shows that

$$\tfrac{\alpha}{2}\leq\sigma_1\leq\alpha. \tag{107}$$

Lemma 3.10 hence proves that

$$\begin{aligned}
\mathcal{N}_{0,\sigma_1}(A)&=\int_A\frac{1}{\sqrt{2\sigma_1\pi}}\,e^{-\frac{x^2}{2\sigma_1}}\,dx\geq\int_A\frac{1}{\sqrt{2\sigma_1\pi}}\,e^{-\frac{x^2}{\alpha}}\,dx\\
&\geq\int_A\frac{1}{\sqrt{2\alpha\pi}}\,e^{-\frac{x^2}{\alpha}}\,dx\geq\int_{[c+1/2,c+1]}\frac{1}{\sqrt{2\alpha\pi}}\,e^{-\frac{x^2}{\alpha}}\,dx.
\end{aligned} \tag{108}$$

Combining this with (102), (104), and (106) yields that

$$
\begin{aligned}
&\mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\
&\geq 2(T-\tau_2)\,\mathcal{N}_{0,\sigma_1}(A)\left[\int_{[0,1]} \left|\sin\!\left(\tfrac{y}{\sqrt{\beta}}\right)\right| \mathcal{N}_{0,\sigma_2}(dy)\right] \\
&\geq 2(T-\tau_2)\left[\frac{1}{\sqrt{2\alpha\pi}}\int_{[c+1/2,c+1]} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\frac{\sqrt{3}}{\sqrt{2T^3\pi}}\int_{[0,1]} |\sin(y)|\,e^{-\frac{6y^2}{T^3}}\,dy\right] \\
&= \left[2(T-\tau_2)\cdot\frac{1}{\sqrt{2\alpha\pi}}\cdot\frac{\sqrt{3}}{\sqrt{2T^3\pi}}\right]\left[\int_{[c+1/2,c+1]} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_{[0,1]} |\sin(y)|\,e^{-\frac{6y^2}{T^3}}\,dy\right] \\
&= \left[\frac{(T-\tau_2)}{\sqrt{\alpha\pi}}\cdot\frac{\sqrt{3}}{\sqrt{T^3\pi}}\right]\left[\int_{[c+1/2,c+1]} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_{[0,1]} |\sin(y)|\,e^{-\frac{6y^2}{T^3}}\,dy\right] \\
&= \frac{\sqrt{3}(T-\tau_2)}{\pi\sqrt{T^3\alpha}}\left[\int_{[c+1/2,c+1]} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_{[0,1]} |\sin(y)|\,e^{-\frac{6y^2}{T^3}}\,dy\right] > 0.
\end{aligned} \tag{109}
$$

The proof of Lemma 3.11 is thus completed. □

Lemma 3.12. *Assume the setting in Section 3.1 and let $a \in [0,\tau)$, $b \in (a,\tau]$, $\varepsilon \in (0, b-a]$, $c \in [2,\infty)$, $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ satisfy for all $x \in [c-2, c+2]$ that $\psi(x) = \frac{T^{3/2}}{\varepsilon^{3/2}} \cdot (x-c)$. Then*

$$
\begin{aligned}
&\inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \textit{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\
&\geq \frac{\sqrt{3}(T-\tau_2)}{\pi\sqrt{T^3\alpha}}\left[\int_{c+1/2}^{c+1} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_0^1 |\sin(y)|\,e^{-\frac{6y^2}{T^3}}\,dy\right] > 0.
\end{aligned} \tag{110}
$$

Proof of Lemma 3.12. Throughout this proof let $a_1 \in [a,b)$, $b_1 \in (a_1, b]$ be real numbers which satisfy that $(b_1 - a_1) = \varepsilon$. Note that Lemma 3.11 proves that

$$
\begin{aligned}
&\inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\
&\geq \inf_{\substack{u\colon C([0,a_1]\cup[b_1,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a_1]\cup[b_1,T]})\big|\Big] \\
&\geq \frac{\sqrt{3}(T-\tau_2)}{\pi\sqrt{T^3\alpha}}\left[\int_{c+1/2}^{c+1} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_0^1 |\sin(y)|\,e^{-\frac{6y^2}{T^3}}\,dy\right] > 0.
\end{aligned} \tag{111}
$$

The proof of Lemma 3.12 is thus completed. □

The next result, Corollary 3.13, follows directly from Lemma 3.12.

Corollary 3.13. *Assume the setting in Section 3.1 and let $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$, $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ satisfy for all $n \in \mathbb{N}$, $x \in [5n-2, 5n+2]$ that $\psi(x) = \frac{T^{3/2}}{|\varepsilon_n|^{3/2}} \cdot (x-5n)$.*

Then it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned} &\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \textit{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\geq \frac{\sqrt{3}(T-\tau_2)}{\pi\sqrt{T^3\alpha}}\left[\int_{5n+1/2}^{5n+1} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_0^1 |\sin(y)|\, e^{-\frac{6y^2}{T^3}}\,dy\right] > 0. \end{aligned} \tag{112}$$

3.8 Asymptotic lower bounds for strong approximation errors for two-dimensional SDEs

Lemma 3.14. *Assume the setting in Section 3.1 and let $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$ and $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be non-increasing sequences with $\limsup_{n\to\infty}\delta_n \leq 0$. Then there exist a function $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ it holds that*

$$\begin{aligned} &\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \textit{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\qquad > \mathbb{1}_{[n_0,\infty)}(n)\max\{\delta_n, 0\}. \end{aligned} \tag{113}$$

Proof of Lemma 3.14. First, observe that the assumption that $\limsup_{n\to\infty}\delta_n \leq 0$ ensures that $\limsup_{n\to\infty}\max\{\delta_n,0\} = 0$. This shows that there exists a strictly increasing function $n\colon \mathbb{N}\to\mathbb{N}$ which satisfies for all $m \in \mathbb{N}$ that

$$\frac{\sqrt{3}(T-\tau_2)}{\pi\sqrt{T^3\alpha}}\left[\int_{5m+1/2}^{5m+1} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_0^1 |\sin(y)|\, e^{-\frac{6y^2}{T^3}}\,dy\right] > \max\{\delta_{n(m)},0\}. \tag{114}$$

Next let $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ be a function which satisfies for all $m \in \mathbb{N}$, $x \in [5m-2, 5m+2]$ that

$$\psi(x) = \tfrac{T^{3/2}}{|\varepsilon_{n(m+1)}|^{3/2}} \cdot (x - 5m). \tag{115}$$

Observe that Corollary 3.13 (with $\varepsilon_m = \varepsilon_{n(m+1)}$ for $m \in \mathbb{N}$ in the notation of Corollary 3.13), (115), and (114) prove that for all $m \in \mathbb{N}$, $k \in [n(m), n(m+1)] \cap \mathbb{N}$ it holds that

$$\begin{aligned} &\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_k}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\geq \inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_{n(m+1)}}} \inf_{\substack{u\colon C([0,a]\cup[b,2],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\geq \frac{\sqrt{3}(T-\tau_2)}{\pi\sqrt{T^3\alpha}}\left[\int_{5m+1/2}^{5m+1} e^{-\frac{x^2}{\alpha}}\,dx\right]\left[\int_0^1 |\sin(y)|\, e^{-\frac{6y^2}{T^3}}\,dy\right] > \max\{\delta_{n(m)},0\} \\ &\geq \max\{\delta_k,0\}. \end{aligned} \tag{116}$$

This implies that for all $k \in [n(1),\infty) \cap \mathbb{N}$ it holds that

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_k}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] > \max\{\delta_k,0\}. \tag{117}$$

The assumption that $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ is non-increasing hence proves that for all $k \in [1, n(1)] \cap \mathbb{N}$ it holds that

$$\begin{aligned}
&\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_k}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\
&\geq \inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_{n(1)}}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\
&> \max\{\delta_{n(1)}, 0\} \geq 0.
\end{aligned} \tag{118}$$

Combining (117) and (118) completes the proof of Lemma 3.14. □

Lemma 3.15. *Assume the setting in Section 3.1, let $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$ be a sequence, and let $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a non-increasing sequence with $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist a function $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ it holds that*

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \textit{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] > \mathbb{1}_{[n_0,\infty)}(n) \max\{\delta_n, 0\}. \tag{119}$$

Proof of Lemma 3.15. Throughout this proof let $(\tilde{\varepsilon}_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$ be the sequence which satisfies for all $n \in \mathbb{N}$ that

$$\tilde{\varepsilon}_n = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}. \tag{120}$$

This ensures that $(\tilde{\varepsilon}_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$ is a non-increasing sequence. Lemma 3.14 (with $\varepsilon_n = \tilde{\varepsilon}_n$ and $\delta_n = \delta_n$ for $n \in \mathbb{N}$ in the notation of Lemma 3.14) hence proves that there exist a function $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\
&\geq \inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\tilde{\varepsilon}_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\
&> \mathbb{1}_{[n_0,\infty)}(n) \max\{\delta_n, 0\}.
\end{aligned} \tag{121}$$

The proof of Lemma 3.15 is thus completed. □

Corollary 3.16. *Assume the setting in Section 3.1 and let $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$ and $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be sequences with $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist a function $\psi \in C^\infty(\mathbb{R},\mathbb{R})$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ it holds that*

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \textit{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] > \mathbb{1}_{[n_0,\infty)}(n) \max\{\delta_n, 0\}. \tag{122}$$

Proof of Corollary 3.16. Throughout this proof let $(\tilde{\delta}_n)_{n\in\mathbb{N}} \subseteq (-\infty, \infty]$ be the sequence of extended real numbers which satisfies for all $n \in \mathbb{N}$ that

$$\tilde{\delta}_n = \sup\{\delta_n, \delta_{n+1}, \delta_{n+2}, \ldots\}. \tag{123}$$

The assumption that $\limsup_{n\to\infty} \delta_n \leq 0$ hence ensures that $\forall\, n \in \mathbb{N}\colon \tilde{\delta}_n \in \mathbb{R}$, that

$$\limsup_{n\to\infty} \tilde{\delta}_n = \lim_{n\to\infty} \tilde{\delta}_n = \limsup_{n\to\infty} \delta_n \leq 0, \tag{124}$$

and that $(\tilde{\delta}_n)_{n\in\mathbb{N}}$ is a non-increasing sequence. This allows us to apply Lemma 3.15 (with $\varepsilon_n = \varepsilon_n$ and $\delta_n = \tilde{\delta}_n$ for $n \in \mathbb{N}$ in the notation of Lemma 3.15) to obtain that there exist a function $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$ and a natural number $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} &\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{\psi,(2)} - u\big((W_s)_{s\in[0,a]\cup[b,T]}\big)\big|\Big] \\ &> \mathbb{1}_{[n_0,\infty)}(n) \max\{\tilde{\delta}_n, 0\} \geq \mathbb{1}_{[n_0,\infty)}(n) \max\{\delta_n, 0\}. \end{aligned} \tag{125}$$

The proof of Corollary 3.16 is thus completed. □

3.9 Non-asymptotic lower bounds for strong approximation errors for two-dimensional SDEs

Lemma 3.17. *Assume the setting in Section 3.1 and let $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$. Then there exists a measurable function $\Phi\colon C([0,T], \mathbb{R}) \to \mathbb{R}$ such that*

$$\mathbb{P}\Big(X_T^{\psi,(2)} = \Phi\big((W_s)_{s\in[0,T]}\big)\Big) = 1. \tag{126}$$

Proof of Lemma 3.17. Note that Lemma 3.7 proves that there exists a measurable function $\phi\colon \mathbb{R} \to \mathbb{R}$ such that

$$\mathbb{P}\Big(X_T^{\psi,(2)} = \phi\big(X_{\tau_1}^{\psi,(1)}\big)\Big) = 1. \tag{127}$$

Moreover, Item (iii) in Lemma 3.6 and Item (ii) in Lemma 3.8 ensure that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} X_{\tau_1}^{\psi,(1)} &= \int_0^{\tau_1} f(s)\, dW_s = f(\tau_1)W_{\tau_1} - f(0)W_0 - \int_0^{\tau_1} f'(s)W_s\, ds \\ &= f(\tau_1)W_{\tau_1} - \int_0^{\tau_1} f'(s)W_s\, ds. \end{aligned} \tag{128}$$

Combining this with (127) completes the proof of Lemma 3.17. □

Corollary 3.18. *Assume the setting in Section 3.1 and let $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0, \tau]$ and $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be sequences with $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist a function $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$, a real number $c \in (0, \infty)$, a measurable function*

$\Phi\colon C([0,T],\mathbb{R})\to\mathbb{R}$, and a continuous $\mathbb{F}$-adapted stochastic process $Z\colon[0,T]\times\Omega\to\mathbb{R}$ such that for all $n\in\mathbb{N}$, $t\in[0,T]$ it holds that

$$\mathbb{P}\Big(Z_T=\Phi\big((W_s)_{s\in[0,T]}\big)\Big)=1, \tag{129}$$

$$\mathbb{P}\Big(X_t^{\psi,(1)}=\textstyle\int_0^t f\big(\frac{Z_s}{c}\big)\,dW_s\Big)=1, \tag{130}$$

$$\mathbb{P}\Big(Z_t=\textstyle\int_0^t c+c\,g\big(\frac{Z_s}{c}\big)[\cos(\psi(X_s^{\psi,(1)}))+1]\,ds\Big)=1, \tag{131}$$

and

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}}\;\inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \textit{measurable}}}\mathbb{E}\Big[\big|Z_T-u\big((W_s)_{s\in[0,a]\cup[b,T]}\big)\big|\Big]\geq\delta_n. \tag{132}$$

Proof of Corollary 3.18. First, note that Corollary 3.16 proves that there exist a function $\psi\in C^\infty(\mathbb{R},\mathbb{R})$ and a natural number $n_0\in\mathbb{N}$ such that for all $n\in\mathbb{N}$ it holds that

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}}\;\inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}}\mathbb{E}\Big[\big|X_T^{\psi,(2)}-u\big((W_s)_{s\in[0,a]\cup[b,T]}\big)\big|\Big] > \mathbb{1}_{[n_0,\infty)}(n)\max\{\delta_n,0\}. \tag{133}$$

Next let $(e_n)_{n\in\mathbb{N}}\subseteq(0,\infty)$ be the sequence which satisfies for all $n\in\mathbb{N}$ that

$$e_n=\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}}\;\inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}}\mathbb{E}\Big[\big|X_T^{\psi,(2)}-u\big((W_s)_{s\in[0,a]\cup[b,T]}\big)\big|\Big], \tag{134}$$

let $c\in(0,\infty)$ be the real number given by

$$c=\max\Big(\Big\{1,\tfrac{\max\{\delta_1,0\}}{e_1},\tfrac{\max\{\delta_2,0\}}{e_2},\ldots,\tfrac{\max\{\delta_{n_0},0\}}{e_{n_0}}\Big\}\Big), \tag{135}$$

and let $Z\colon[0,T]\times\Omega\to\mathbb{R}$ be the stochastic process which satisfies for all $t\in[0,T]$ that $Z_t=cX_t^{\psi,(2)}$. Note that for all $t\in[0,T]$ it holds that

$$\mathbb{P}\Big(X_t^{\psi,(1)}=\textstyle\int_0^t f\big(\frac{Z_s}{c}\big)\,dW_s\Big)=1 \tag{136}$$

and

$$\mathbb{P}\Big(Z_t=\textstyle\int_0^t c+c\,g\big(\frac{Z_s}{c}\big)[\cos(\psi(X_s^{\psi,(1)}))+1]\,ds\Big)=1. \tag{137}$$

Next observe that Lemma 3.17 and the fact that $Z_T=cX_T^{\psi,(2)}$ prove that there exists a measurable function $\Phi\colon C([0,T],\mathbb{R})\to\mathbb{R}$ such that

$$\mathbb{P}\Big(Z_T=\Phi\big((W_s)_{s\in[0,T]}\big)\Big)=1. \tag{138}$$

Moreover, note that (133) ensures that for all $n \in \{n_0, n_0+1, \ldots\}$ it holds that

$$\begin{aligned} c \cdot e_n &= \max\Bigl(\Bigl\{1, \tfrac{\max\{\delta_1,0\}}{e_1}, \tfrac{\max\{\delta_2,0\}}{e_2}, \ldots, \tfrac{\max\{\delta_{n_0},0\}}{e_{n_0}}\Bigr\}\Bigr) \cdot e_n \\ &\geq e_n > \mathbb{1}_{[n_0,\infty)}(n) \max\{\delta_n, 0\} = \max\{\delta_n, 0\}. \end{aligned} \tag{139}$$

In addition, observe that for all $n \in \{1, 2, \ldots, n_0\}$ it holds that

$$\begin{aligned} c \cdot e_n &= \max\Bigl(\Bigl\{1, \tfrac{\max\{\delta_1,0\}}{e_1}, \tfrac{\max\{\delta_2,0\}}{e_2}, \ldots, \tfrac{\max\{\delta_{n_0},0\}}{e_{n_0}}\Bigr\}\Bigr) \cdot e_n \\ &\geq \tfrac{\max\{\delta_n,0\}}{e_n} \cdot e_n = \max\{\delta_n, 0\}. \end{aligned} \tag{140}$$

Combining (139) and (140) shows that for all $n \in \mathbb{N}$ it holds that $c \cdot e_n \geq \max\{\delta_n, 0\} \geq \delta_n$. Hence, we obtain that for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} &\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Bigl[\bigl|Z_T - u((W_s)_{s\in[0,a]\cup[b,T]})\bigr|\Bigr] \\ &= \inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Bigl[\bigl|cX_T^{\psi,(2)} - u((W_s)_{s\in[0,a]\cup[b,T]})\bigr|\Bigr] \\ &= c\left(\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Bigl[\bigl|X_T^{\psi,(2)} - \tfrac{1}{c}\cdot u((W_s)_{s\in[0,a]\cup[b,T]})\bigr|\Bigr]\right) \\ &= c \cdot e_n \geq \delta_n. \end{aligned} \tag{141}$$

This and (136)–(138) complete the proof of Corollary 3.18. □

The next result, Lemma 3.19, follows from Corollary 3.18 and from Corollary 3.1.

Lemma 3.19. *Let $T \in (0,\infty)$, $\tau \in (0,T)$, $d \in \{2,3,\ldots\}$, $\xi \in \mathbb{R}^d$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$, $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist infinitely often differentiable and globally bounded functions $\mu, \sigma\colon \mathbb{R}^d \to \mathbb{R}^d$ and a measurable function $\Phi\colon C([0,T],\mathbb{R}) \to \mathbb{R}$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0,T]\times\Omega \to \mathbb{R}$, every continuous $\mathbb{F}$-adapted stochastic process $X = (X^{(1)}, \ldots, X^{(d)})\colon [0,T]\times\Omega \to \mathbb{R}^d$ with $\forall\, t \in [0,T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s) = 1$, and every $n \in \mathbb{N}$ it holds that*

$$\mathbb{P}\Bigl(X_T^{(1)} = \Phi\bigl((W_s)_{s\in[0,T]}\bigr)\Bigr) = 1 \tag{142}$$

and

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \textit{measurable}}} \mathbb{E}\Bigl[\bigl|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\bigr|\Bigr] \geq \delta_n. \tag{143}$$

Proof of Lemma 3.19. Throughout this proof for all measurable spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ let $\mathcal{M}(\mathcal{A}, \mathcal{B})$ be the set of all $\mathcal{A}/\mathcal{B}$-measurable functions from A to

B, let $f, g, \psi \in C^\infty(\mathbb{R}, \mathbb{R})$, $c \in (0, \infty)$, $\phi \in \mathcal{M}(\mathcal{B}(C([0,T],\mathbb{R})), \mathcal{B}(\mathbb{R}))$ satisfy that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0,T] \times \Omega \to \mathbb{R}$, every continuous $\mathbb{F}$-adapted stochastic process $X = (X^{(1)}, X^{(2)})\colon [0,T]\times\Omega \to \mathbb{R}^2$ with $\forall\, t \in [0,T]\colon \mathbb{P}\big(X_t^{(1)} = \int_0^t c + c\, g(\frac{X_s^{(1)}}{c})[\cos(\psi(X_s^{(2)})) + 1]\, ds\big) = \mathbb{P}\big(X_t^{(2)} = \int_0^t f(\frac{X_s^{(1)}}{c})\, dW_s\big) = 1$, and every $n \in \mathbb{N}$ it holds that

$$\mathbb{P}\Big(X_T^{(1)} = \phi\big((W_s)_{s\in[0,T]}\big)\Big) = 1, \tag{144}$$

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \geq \delta_n, \tag{145}$$

and $\sup_{x\in\mathbb{R}}(|f(x)| + |g(x)|) < \infty$ (Corollary 3.1 and Corollary 3.18 assure that $f, g, \psi \in C^\infty(\mathbb{R}, \mathbb{R})$, $c \in (0, \infty)$, $\phi \in \mathcal{M}(\mathcal{B}(C([0,T],\mathbb{R})), \mathcal{B}(\mathbb{R}))$ do indeed exist), let $P\colon \mathbb{R}^d \to \mathbb{R}$ be the function which satisfies for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that $P(x) = x_1$, let $\Xi \in \mathbb{R}$ be the real number given by $\Xi = P(\xi)$, let $a, b, \mu, \sigma\colon \mathbb{R}^d \to \mathbb{R}^d$ be the functions which satisfy for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ that

$$a(x) = (c + c\, g(\tfrac{x_1}{c})[\cos(\psi(x_2)) + 1], 0, \ldots, 0), \tag{146}$$

$$b(x) = (0, f(\tfrac{x_1}{c}), 0, \ldots, 0), \tag{147}$$

$$\mu(x) = a(x - \xi), \qquad \text{and} \qquad \sigma(x) = b(x - \xi), \tag{148}$$

let $\Phi\colon C([0,T],\mathbb{R}) \to \mathbb{R}$ be the measurable function which satisfies for all $v \in C([0,T],\mathbb{R})$ that

$$\Phi(v) = \phi(v) + \Xi, \tag{149}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ be a normal filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, let $W\colon [0,T] \times \Omega \to \mathbb{R}$ be a standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion, let $X = (X^{(1)}, \ldots, X^{(d)})\colon [0,T] \times \Omega \to \mathbb{R}^d$ be a continuous $\mathbb{F}$-adapted stochastic process which satisfies for all $t \in [0,T]$ that

$$\mathbb{P}\Big(X_t = \xi + \textstyle\int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s\Big) = 1, \tag{150}$$

let $n \in \mathbb{N}$, $a, b \in [0,\tau]$ be real numbers with $b - a \geq \varepsilon_n$, and let $Y = (Y^{(1)}, \ldots, Y^{(d)})\colon [0,T] \times \Omega \to \mathbb{R}^d$ be the stochastic process which satisfies for all $t \in [0,T]$ that

$$Y_t = X_t - \xi. \tag{151}$$

Observe that (150), (151), (148), and (148) ensure that $Y\colon [0,T] \times \Omega \to \mathbb{R}^d$ is a continuous $\mathbb{F}$-adapted stochastic process which satisfies for all $t \in [0,T]$ that

$$\mathbb{P}\Big(Y_t = \textstyle\int_0^t a(Y_s)\, ds + \int_0^t b(Y_s)\, dW_s\Big) = 1. \tag{152}$$

This, (146), and (147) show that for all $t \in [0,T]$ it holds that

$$\mathbb{P}\Big(Y_t^{(1)} = \textstyle\int_0^t c + c\, g\big(\frac{Y_s^{(1)}}{c}\big)[\cos(\psi(Y_s^{(2)})) + 1]\, ds\Big) = 1 \tag{153}$$

and

$$\mathbb{P}\Big(Y_t^{(2)} = \textstyle\int_0^t f\big(\frac{Y_s^{(1)}}{c}\big)\, dW_s\Big) = 1. \tag{154}$$

Combining this with (144) and (145) demonstrates that

$$\mathbb{P}\Big(Y_T^{(1)} = \phi\big((W_s)_{s\in[0,T]}\big)\Big) = 1 \tag{155}$$

and

$$\inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|Y_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \geq \delta_n. \tag{156}$$

In addition, observe that (151), (149), and (155) assure that

$$\begin{aligned} &\mathbb{P}\Big(X_T^{(1)} = \Phi\big((W_s)_{s\in[0,T]}\big)\Big) = \mathbb{P}\Big(Y_T^{(1)} + \Xi = \Phi\big((W_s)_{s\in[0,T]}\big)\Big) \\ &= \mathbb{P}\Big(Y_T^{(1)} = \Phi\big((W_s)_{s\in[0,T]}\big) - \Xi\Big) = \mathbb{P}\Big(Y_T^{(1)} = \phi\big((W_s)_{s\in[0,T]}\big)\Big) = 1. \end{aligned} \tag{157}$$

Moreover, note that (151) and (156) show that

$$\begin{aligned} &\inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &= \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|Y_T^{(1)} + \Xi - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &= \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|Y_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \geq \delta_n. \end{aligned} \tag{158}$$

Next observe that the fact that $f, g, \psi \in C^\infty(\mathbb{R},\mathbb{R})$, the fact that $\sup_{x\in\mathbb{R}}(|f(x)| + |g(x)|) < \infty$, and (146)–(148) ensure that $\mu, \sigma \in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and

$$\sup_{x\in\mathbb{R}^d} (\|\mu(x)\|_{\mathbb{R}^d} + \|\sigma(x)\|_{\mathbb{R}^d}) < \infty. \tag{159}$$

Combining this with (157) and (158) completes the proof of Lemma 3.19. ☐

Theorem 3.20. *Let $T \in (0,\infty)$, $\tau \in (0,T)$, $d \in \{2,3,\ldots\}$, $\xi \in \mathbb{R}^d$, $m \in \mathbb{N}$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$, $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist infinitely often differentiable and globally bounded functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ such that for every probability space $(\Omega,\mathcal{F},\mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega,\mathcal{F},\mathbb{P})$, every standard $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$-Brownian motion $W\colon [0,T]\times\Omega \to \mathbb{R}^m$, every continuous $\mathbb{F}$-adapted stochastic process $X = (X^{(1)},\ldots,X^{(d)})\colon [0,T]\times\Omega \to \mathbb{R}^d$ with $\forall\, t \in [0,T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s) = 1$, and every $n \in \mathbb{N}$ it holds that*

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R}^m)\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \geq \delta_n. \tag{160}$$

Proof of Theorem 3.20. Throughout this proof assume w.l.o.g. that $m \geq 2$ (otherwise (160) follows from Lemma 3.19), let $\Phi\colon C([0,T],\mathbb{R}) \to \mathbb{R}$ and $\mu, \Sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ be measurable functions which satisfy that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0,T] \times \Omega \to \mathbb{R}$, every continuous $\mathbb{F}$-adapted stochastic process $X = (X^{(1)}, \dots, X^{(d)})\colon [0,T] \times \Omega \to \mathbb{R}^d$ with $\forall\, t \in [0,T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \Sigma(X_s)\, dW_s) = 1$, and every $n \in \mathbb{N}$ it holds that

$$\mathbb{P}\Big(X_T^{(1)} = \Phi\big((W_s)_{s\in[0,T]}\big)\Big) = 1, \tag{161}$$

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \; \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \geq \delta_n, \tag{162}$$

and $\sup_{x\in\mathbb{R}^d}(\|\mu(x)\|_{\mathbb{R}^d} + \|\Sigma(x)\|_{\mathbb{R}^d}) < \infty$ (Lemma 3.19 assures that such functions do indeed exist), let $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be the function which satisfies for all $x \in \mathbb{R}^d$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$ that

$$\sigma(x)y = y_1\Sigma(x), \tag{163}$$

let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ be a normal filtration on $(\Omega, \mathcal{F}, \mathbb{P})$, let $W = (W^{(1)}, \dots, W^{(m)})\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion, let $X = (X^{(1)}, \dots, X^{(d)})\colon [0,T] \times \Omega \to \mathbb{R}^d$ be a continuous $\mathbb{F}$-adapted stochastic process which satisfies for all $t \in [0,T]$ that

$$\mathbb{P}\Big(X_t = \xi + \textstyle\int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s\Big) = 1, \tag{164}$$

let $n \in \mathbb{N}$, $a, b \in [0,\tau]$ be real numbers with $b - a \geq \varepsilon_n$, let $u\colon C([0,a] \cup [b,T], \mathbb{R}^m) \to \mathbb{R}$ be a measurable function, let $\tilde{W} = (\tilde{W}^{(1)}, \dots, \tilde{W}^{(m)})\colon \Omega \to C([0,T], \mathbb{R}^m)$ be the function which satisfies for all $\omega \in \Omega$, $t \in [0,T]$ that

$$(\tilde{W}(\omega))(t) = W_t(\omega), \tag{165}$$

let $\Psi\colon C([0,T],\mathbb{R}) \to C([0,a] \cup [b,T], \mathbb{R})$ be the function which satisfies for all $f \in C([0,T],\mathbb{R})$ that $\Psi(f) = f|_{[0,a]\cup[b,T]}$, and for every $(v_1, \dots, v_{m-1}) \in C([0,a] \cup [b,T], \mathbb{R}^{m-1})$ let $\tilde{u}_{v_1,\dots,v_{m-1}}\colon C([0,a] \cup [b,T], \mathbb{R}) \to \mathbb{R}$ be the function which satisfies for all $v \in C([0,a] \cup [b,T], \mathbb{R})$ that

$$\tilde{u}_{v_1,\dots,v_{m-1}}(v) = u(v, v_1, \dots, v_{m-1}). \tag{166}$$

Observe that (164) and (163) demonstrate that for all $t \in [0,T]$ it holds that

$$\mathbb{P}\Big(X_t = \xi + \textstyle\int_0^t \mu(X_s)\, ds + \int_0^t \Sigma(X_s)\, dW_s^{(1)}\Big) = 1. \tag{167}$$

This, (161), and (165) assure that

$$\mathbb{P}\Big(X_T^{(1)} = \Phi\big((W_s^{(1)})_{s\in[0,T]}\big) = \Phi\big(\tilde{W}^{(1)}\big)\Big) = 1. \tag{168}$$

Next note that the fact that $\Sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, the fact that $\sup_{x\in\mathbb{R}^d} \|\Sigma(x)\|_{\mathbb{R}^d} < \infty$, and (163) yield that $\sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^{d\times m})$ and

$$\sup_{x\in\mathbb{R}^d} \|\sigma(x)\|_{\mathbb{R}^{d\times m}} = \sup_{x\in\mathbb{R}^d} \|\Sigma(x)\|_{\mathbb{R}^d} < \infty. \tag{169}$$

In addition, observe that

$$\begin{aligned} u((W_s)_{s\in[0,a]\cup[b,T]}) &= u\big(\Psi(\tilde{W}^{(1)}), \ldots, \Psi(\tilde{W}^{(m)})\big) \\ &= \tilde{u}_{\Psi(\tilde{W}^{(2)}),\ldots,\Psi(\tilde{W}^{(m)})}\big(\Psi(\tilde{W}^{(1)})\big). \end{aligned} \tag{170}$$

Combining this with (168) shows that

$$\begin{aligned} &\mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] = \mathbb{E}\Big[\big|\Phi(\tilde{W}^{(1)}) - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &= \mathbb{E}\Big[\big|\Phi(\tilde{W}^{(1)}) - \tilde{u}_{\Psi(\tilde{W}^{(2)}),\ldots,\Psi(\tilde{W}^{(m)})}\big(\Psi(\tilde{W}^{(1)})\big)\big|\Big] \\ &= \int_\Omega \big|\Phi(\tilde{W}^{(1)}(\omega)) - \tilde{u}_{\Psi(\tilde{W}^{(2)}(\omega)),\ldots,\Psi(\tilde{W}^{(m)}(\omega))}\big(\Psi(\tilde{W}^{(1)}(\omega))\big)\big| \, \mathbb{P}(d\omega) \\ &= \int_{C([0,T],\mathbb{R})} \cdots \int_{C([0,T],\mathbb{R})} \big|\Phi(w_1) - \tilde{u}_{\Psi(w_2),\ldots,\Psi(w_m)}\big(\Psi(w_1)\big)\big| \\ &\quad \tilde{W}^{(1)}(\mathbb{P})_{\mathcal{B}(C([0,T],\mathbb{R}))}(dw_1) \ldots \tilde{W}^{(m)}(\mathbb{P})_{\mathcal{B}(C([0,T],\mathbb{R}))}(dw_m) \\ &= \int_{C([0,T],\mathbb{R})} \cdots \int_{C([0,T],\mathbb{R})} \mathbb{E}\Big[\big|\Phi(\tilde{W}^{(1)}) - \tilde{u}_{\Psi(w_2),\ldots,\Psi(w_m)}\big(\Psi(\tilde{W}^{(1)})\big)\big|\Big] \\ &\quad \tilde{W}^{(2)}(\mathbb{P})_{\mathcal{B}(C([0,T],\mathbb{R}))}(dw_2) \ldots \tilde{W}^{(m)}(\mathbb{P})_{\mathcal{B}(C([0,T],\mathbb{R}))}(dw_m). \end{aligned} \tag{171}$$

This, (168), (167), and (162) ensure that

$$\begin{aligned} &\mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\geq \int_{C([0,T],\mathbb{R})} \cdots \int_{C([0,T],\mathbb{R})} \left[\inf_{\substack{v\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R} \\ \text{measurable}}} \mathbb{E}\Big[\big|\Phi(\tilde{W}^{(1)}) - v\big(\Psi(\tilde{W}^{(1)})\big)\big|\Big] \right] \\ &\quad \tilde{W}^{(2)}(\mathbb{P})_{\mathcal{B}(C([0,T],\mathbb{R}))}(dw_2) \ldots \tilde{W}^{(m)}(\mathbb{P})_{\mathcal{B}(C([0,T],\mathbb{R}))}(dw_m) \\ &= \inf_{\substack{v\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R} \\ \text{measurable}}} \mathbb{E}\Big[\big|\Phi(\tilde{W}^{(1)}) - v\big(\Psi(\tilde{W}^{(1)})\big)\big|\Big] \\ &= \inf_{\substack{v\colon C([0,a]\cup[b,T],\mathbb{R})\to\mathbb{R} \\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} - v((W_s^{(1)})_{s\in[0,a]\cup[b,T]})\big|\Big] \geq \delta_n. \end{aligned} \tag{172}$$

Combining this with (169) completes the proof of Theorem 3.20. □

Corollary 3.21. *Let $T \in (0,\infty)$, $\tau \in (0,T)$, $d \in \{2,3,\ldots\}$, $\xi \in \mathbb{R}^d$, $m \in \mathbb{N}$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0,\tau]$, $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist infinitely often differentiable and globally bounded functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0,T] \times \Omega \to \mathbb{R}^m$, every continuous $\mathbb{F}$-adapted stochastic process*

$X = (X^{(1)}, \dots, X^{(d)})\colon [0,T]\times\Omega \to \mathbb{R}^d$ *with* $\forall\, t \in [0,T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s) = 1$*, and every* $n \in \mathbb{N}$ *it holds that*

$$\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{t_1,\dots,t_n\in[0,T]} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R}^m)\times(\mathbb{R}^m)^n\to\mathbb{R}\\ \textit{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]}, W_{t_1}, \dots, W_{t_n})\big|\Big] \geq \delta_n. \quad (173)$$

Proof of Corollary 3.21. Note that Theorem 3.20 (with $T = T$, $\tau = \tau$, $d = d$, $\xi = \xi$, $m = m$, $\varepsilon_n = \frac{\varepsilon_n}{(n+1)}$, $\delta_n = \delta_n$ for $n \in \mathbb{N}$ in the notation of Theorem 3.20) proves that there exist infinitely often differentiable and globally bounded functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0,T]\times\Omega \to \mathbb{R}^m$, every continuous $\mathbb{F}$-adapted stochastic process $X = (X^{(1)}, \dots, X^{(d)})\colon [0,T]\times\Omega \to \mathbb{R}^d$ with $\forall\, t \in [0,T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s) = 1$, and every $n \in \mathbb{N}$ it holds that

$$\begin{aligned} &\inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n}} \inf_{t_1,\dots,t_n\in[0,T]} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R}^m)\times(\mathbb{R}^m)^n\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} \\ &- u((W_s)_{s\in[0,a]\cup[b,T]}, W_{t_1}, \dots, W_{t_n})\big|\Big] \\ &\geq \inf_{\substack{a,b\in[0,\tau],\\ b-a\geq\varepsilon_n/(n+1)}} \inf_{\substack{u\colon C([0,a]\cup[b,T],\mathbb{R}^m)\to\mathbb{R}\\ \text{measurable}}} \mathbb{E}\Big[\big|X_T^{(1)} - u((W_s)_{s\in[0,a]\cup[b,T]})\big|\Big] \\ &\geq \delta_n. \end{aligned} \quad (174)$$

The proof of Corollary 3.21 is thus completed. □

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