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Abstract We survey recent convergence rate bounds for single-level and multilevel QMC Finite Element (FE for short) algorithms for the numerical approximation of linear, second order elliptic PDEs in divergence form in a bounded, polygonal domain D. The diffusion coefficient a is assumed to be an isotropic, log-Gaussian random field (GRF for short) in D. The representation of the GRF $Z = \log a$ is assumed affine-parametric with i.i.d. standard normal random variables, and with *locally supported* functions ψ_j characterizing the spactial variation of the GRF Z. The goal of computation is the evaluation of expectations (i.e., of so-called "ensemble averages") of (linear functionals of) the random solution, The QMC rules employed are randomly shifted lattice rules proposed in [19] as used and analyzed previously in a similar setting (albeit for globally in D supported spatial representation functions ψ_j as arise in Karhunen-Loève expansions) in [9, 14]. The multilevel QMC-FE algorithm Q_L^* analyzed here for locally supported ψ_j was proposed first in [17] for affine-parametric operator equations. As shown in [7, 6, 10, 11] localized supports of the ψ_j (which appear in multiresolution representations of GRFs Z of Lévy-Cieselski type in D) allow for the use of product weights, originally proposed in construction of QMC rules in [23] (cp. the survey [4] and references there). The present results from [11] on convergence rates for the MLQMC FE algorithm allow for general polygonal domains D and for GRFs Z whose realizations take values in weighted spaces containing $W^{1,\infty}(D)$. Localized support assumptions on ψ_j are shown to allow QMC rule generation by the fast, FFT based CBC constructions in [21, 20] which scale linearly in the integration dimension which, for multiresolution representations of GRFs, is proportional to the number of degrees of freedom used in the FE discretization in the physical domain D. We show numerical experiments based on public domain QMC rule generating software in [13, 5].

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1 Introduction

The numerical solution of partial differential equations (PDEs for short) with random input data is a core task in the field of computational uncertainty quantification. Particular models of randomness in the PDEs' input parameters entail particular requirements to efficient computational uncertainty quantification algorithms. A basic case arises when there are only a finite number of random variables whose densities have bounded support and which parametrize the uncertain input in the forward PDE model: computation of statistical moments of responses and also Bayesian inversion then amounts to numerical integration over a bounded domain of finite dimension s. Statistical independence and scaling implies numerical integration over the unit cube $[0,1]^s$, against a product probability measure. In the context of PDEs, so-called *distributed random inputs* such as spatially heterogeneous diffusion coefficients, uncertain physical domains, etc. imply, via *uncertainty parametrizations* (such as Fourier- , B-spline or wavelet expansions) in physical domains D, a countably-infinite number of random parameters (being, for example, Fourier- or wavelet coefficients). This, in turn, renders the problem of estimation of response statistics of solutions a problem of infinite-dimensional numerical integration. Assuming again statistical independence of the system of (countably many) random input parameters results in the problem of numerical integration against a product measure. In case of GRF inputs under consideration in this note, in addition the domain of integration is the countable product of real lines $\mathbb{R}^{\mathbb{N}}$, endowed with a Gaussian measure (GM for short); see, e.g., [3] for details on GMs on $\mathbb{R}^{\mathbb{N}}$.

Here, as in [9, 14] and the references there, we analyze QMC quadratures in the FE solution of linear, second order elliptic PDEs in a bounded, polygonal domain D, with isotropic, log-Gaussian diffusion coefficient $a = \exp(Z)$, where Z is a GRF in D. As in [9, 14], we confine the analysis to first order, randomly shifted lattice rules proposed originally in [19], and to continuous, piecewise linear "Courant" FE methods in D. We adopt the setting of our analysis [10] of the single-level QMC-FE algorithm: consider

$$-\nabla\cdot(a\nabla u) = f \text{ in } D, \quad u = 0 \text{ on } \partial D \tag{1}$$

where D is a bounded interval in space dimension $d = 1$ or a bounded polygon with J straight sides and J corners c_j, $i = 1,...,J$ in space dimension $d = 2$. We endow $\Omega := \mathbb{R}^{\mathbb{N}}$ with the Gaussian product measure and the corresponding product sigma algebra, cp. [3]

$$\mu(\mathrm{d}\boldsymbol{y}) := \bigotimes_{j\geq 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_j^2}{2}} \mathrm{d}y_j, \quad \boldsymbol{y} = (y_j)_{j\geq 1} \in \Omega.$$

The random input is modelled on $(\Omega, \bigotimes_{j\geq 1}\mathscr{B}(\mathbb{R}), \mu)$ which is a probability space (cp. for example [3, Example 2.3.5]). The GRF $Z = \log(a) : \Omega \to L^\infty(D)$ is assumed to be affine-parametric:

$$Z := \sum_{j\geq 1} y_j \psi_j. \tag{2}$$

In D, we consider the model Dirichlet problem

$$-\nabla\cdot(a\nabla u) = f \quad \text{in } D, \quad u\Big|_{\partial D} = 0\,. \tag{3}$$

In order to render the random coefficient $a = \exp(Z)$ in (3) meaningful, we imposed in [10] on the $(\psi_j)_{j\geq 1}$ in (2) the summability condition

$$\left\| \sum_{j\geq 1} \frac{|\psi_j|}{b_j} \right\|_{L^\infty(D)} < \infty \tag{\textbf{A1}}$$

such that $(b_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ for some $p_0 \in (0,\infty)$, and the positive sequence $(b_j)_{j\geq 1}$ encodes decay of $(\psi_j)_{j\geq 1}$. We observe that (**A1**) is weaker than the summability conditions imposed in [9, 14] in the case that the ψ_j have local supports, as observed in [2] in the context on N-term gpc approximation rate analysis of the random field solution u of (3). The assumption of local supports in (**A1**) allows for the use of product weights, cp [10, 7, 11, 6]. Product weights are well known to scale linear in the dimension of integration in the CBC construction, cp. [21, 20]. Reproducing kernel Hilbert spaces (RKHS for short) with product weights were introduced in [23]. For general surveys on QMC we refer to [4, 15] and the references there. A finite dimension s of integration results from the truncation of the expansion of the GRF Z which, if e.g. $(\psi_j)_{j\geq 1}$ is a multiresolution analysis, couples with the FE discretization.

2 Spatial Approximation

The spatial approximation of the PDE (3) by the FE method is based on its (primal) variational formulation in D, while considering the coefficient sequence $\boldsymbol{y}$ in the random input as "parameter". Find $u : \Omega \to V$ such that

$$\int_D a\nabla u\cdot\nabla v \mathrm{d}x = f(v), \quad v \in V. \tag{4}$$

We further impose the assumption that for some $p_0 \in (0,\infty)$, $(b_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ it holds that $Z \in L^q(\Omega, L^\infty(D))$ for every $q \in [1,\infty)$, cp. [10, Theorem 2]. This implies that μ-a.s. $0 < \operatorname{ess\,inf}_{x\in D}\{a(x)\} \leq \|a\|_{L^\infty(D)} < \infty$. For the ensuing presentation, we define the random variables

$$a_{\min} := \operatorname{ess\,inf}_{x\in D}\{a(x)\} \quad \text{and} \quad a_{\max} := \|a\|_{L^\infty(D)}\,.$$

Hence, the random bilinear form $(w,v) \mapsto \int_D a\nabla w\cdot\nabla v \mathrm{d}x$ on $V\times V$ is continuous and coercive with coercivity constant $a_{\min}$ and continuity constant $a_{\max}$. By the

Lax–Milgram lemma, the solution u exists and solves (4) uniquely. Also due to [10, Proposition 3], we obtain the estimate for every $q \in [1,\infty)$,

$$\|u\|_{L^q(\Omega;V)} \leq \|1/a_{\min}\|_{L^q(\Omega)}\|f\|_{V^*} < \infty,$$

where the strong measurability of u follows, since u depends continuously on a (by the second Strang lemma). To obtain a finite dimensional integration domain, we consider dimension truncation. For every $s \in \mathbb{N}$, let $a^s := \exp(Z^s) = \exp(\sum_{j=1}^s y_j \psi_j)$ denote the truncated lognormal field and define the random variables

$$a^s_{\min} := \operatorname{ess\,inf}_{x\in D}\{a^s(x)\} \quad \text{and} \quad a^s_{\max} := \|a^s\|_{L^\infty(D)}.$$

Let u^s be the solution with respect to the coefficient a^s, i.e.,

$$\int_D a^s \nabla u^s \cdot \nabla v \mathrm{d}x = f(v), \quad v \in V.$$

Assuming that $(b_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ for some $p_0 \in (0,\infty)$ by [10, Proposition 7], for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for every $G(\cdot) \in V^*$

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^s))| \leq C_\varepsilon \|G(\cdot)\|_{V^*}\|f\|_{V^*} \max_{j>s}\{b_j^{1-\varepsilon}\}. \tag{5}$$

Approximations with Finite Elements in a polygon $D \subset \mathbb{R}^2$ with respect to uniformly refined triangulations may result in suboptimal convergence rates. We therefore consider certain weighted Sobolev spaces, cp. [1]. For a J-tuple $\boldsymbol{\beta} = (\beta_1,\dots,\beta_J)$ of weight exponents, we define the *corner weight function*

$$\Phi_{\boldsymbol{\beta}}(x) := \prod_{i=1}^J |c_i - x|^{\beta_i}, \quad x \in D,$$

where $\beta_i \in [0,1)$, $i = 1,\dots,J$. Here and in the following, the Euclidean norm in $\mathbb{R}^2$ is denoted by $|\cdot|$. We define the function spaces $L^2_{\boldsymbol{\beta}}(D)$ and $H^2_{\boldsymbol{\beta}}(D)$ as closures of $C^\infty(\overline{D})$ with respect to the norms

$$\|v\|_{L^2_{\boldsymbol{\beta}}(D)} := \|v\Phi_{\boldsymbol{\beta}}\|_{L^2(D)}$$

and

$$\|v\|^2_{H^2_{\boldsymbol{\beta}}(D)} := \|v\|^2_{H^1(D)} + \sum_{|\boldsymbol{\alpha}|=2} \||\partial_x^{\boldsymbol{\alpha}} v|\Phi_{\boldsymbol{\beta}}\|^2_{L^2(D)}.$$

Lemma 1. *There exists a constant $C > 0$ such that for every $f \in L^2_{\boldsymbol{\beta}}(D)$,*

$$\|f\|_{V^*} \leq C\|f\|_{L^2_{\boldsymbol{\beta}}(D)}.$$

Proof. The statement of the lemma is equivalent to the continuity of the embedding $V^* \subset L^2_{\boldsymbol{\beta}}(D)$. By duality, this is equivalent to the continuity of the embedding

$(L^2_{\boldsymbol{\beta}}(D))^* \subset V$. We therefore identify $L^2(D)$ with its dual $L^2(D)$, and obtain for an arbitrary $w \in (L^2_{\boldsymbol{\beta}}(D))^*$ with the Cauchy–Schwarz inequality

$$\|w\|_{(L^2_{\boldsymbol{\beta}}(D))^*} = \sup_{v\in L^2_{\boldsymbol{\beta}}(D), \|v\|_{L^2_{\boldsymbol{\beta}}(D)}=1} w(v) = \sup_{v\in L^2_{\boldsymbol{\beta}}(D), \|v\|_{L^2_{\boldsymbol{\beta}}(D)}=1} \int wv \mathrm{d}x$$
$$\leq \|w/\Phi_{\boldsymbol{\beta}}\|_{L^2(D)} = \|w\|_{L^2_{-\boldsymbol{\beta}}(D)} .$$

By the Hardy inequality (see, e.g., [22, Theorem 21.3] with the choices $p = q = 2, \alpha = -p, \beta = 0$, $\kappa = 1$), there exists a constant $C' > 0$ such that for every $\tilde{w} \in V$, with $\mathrm{dist}_{\partial D}(x)$ denoting for $x \in D$ the regularized distance of x to the (Lipschitz) boundary ∂D, as defined e.g. in [24, Chap. VI.2],

$$\|\tilde{w}/\mathrm{dist}_{\partial D}\|_{L^2(D)} \leq C'\|\tilde{w}\|_V ,$$

we conclude that the embedding $(L^2_{\boldsymbol{\beta}}(D))^* \subset V$ is continuous. This implies the assertion of this lemma. □

□

In the weighted spaces $H^2_{\boldsymbol{\beta}}(D)$ there holds a full regularity shift for the Dirichlet Laplacian, cp. [1, Theorem 3.2]: there exists a constant $C > 0$ such that for every $w \in V$ with $\Delta w \in L^2_{\boldsymbol{\beta}}(D)$,

$$\|w\|_{H^2_{\boldsymbol{\beta}}(D)} \leq C\|\Delta w\|_{L^2_{\boldsymbol{\beta}}(D)}, \tag{6}$$

provided that the weight exponent sequence $\boldsymbol{\beta}$ satisfies $0 \leq \beta_j$ and $1 - \pi/\omega_i < \beta_i < 1$, $i = 1,\dots,J$. The interior angle of the corner c_i is denoted by ω_i, $i = 1,\dots,J$. Since in [1] the Poisson boundary value problem with a zero order term is considered, i.e., $-\Delta w + w = f$, we also used the estimate that for constants $C_1, C_2, C_3 > 0$ independent of $w \in V \cap H^2_{\boldsymbol{\beta}}(D)$,

$$\|w\|_{L^2_{\boldsymbol{\beta}}(D)} \leq C_1\|w\|_{L^2(D)} \leq C_2\|w\|_V = C_2\|\Delta w\|_{V^*} \leq C_3\|\Delta w\|_{L^2_{\boldsymbol{\beta}}(D)},$$

which is a consequence of Lemma 1. Also in FE spaces $V_\ell := \{v \in V : v|_K \in \mathbb{P}^1(K), K \in \mathscr{T}_\ell\}$ there is an approximation property, cp. [1, Lemmas 4.1 and 4.5], where $\mathbb{P}^1(K)$ are the affine functions on K and $\{\mathscr{T}_\ell\}_{\ell\geq 0}$ are sequences of regular, simplicial triangulations with proper mesh refinement near the corners c_i of D. Specifically, there exists a constant C such that for every $w \in H^2_{\boldsymbol{\beta}}(D)$ there is $w_\ell \in V_\ell$ satisfying

$$\|w - w_\ell\|_V \leq CM_\ell^{-1/d}\|w\|_{H^2_{\boldsymbol{\beta}}(D)}, \tag{7}$$

where $M_\ell := \dim(V_\ell)$. Let $u^{s,\mathscr{T}_\ell} : \Omega \to V_\ell$ be the FE solution, i.e.,

$$\int_D a^s \nabla u^{s,\mathscr{T}_\ell} \cdot \nabla v \mathrm{d}x = f(v), \quad \forall v \in V_\ell. \tag{8}$$

Let $W^{1,\infty}_{\boldsymbol{\beta}}(D)$ denote the Banach space of measurable functions $v : D \to \mathbb{R}$ that have finite $W^{1,\infty}_{\boldsymbol{\beta}}(D)$-norm, where

$$\|v\|_{W^{1,\infty}_{\boldsymbol{\beta}}(D)} := \max\{\|v\|_{L^\infty(D)}, \||\nabla v|\Phi_{\boldsymbol{\beta}}\|_{L^\infty(D)}\}.$$

We introduce the following mixed sparsity assumption on the function system $(\psi_j)_{j\geq 1}$. Let $(\bar{b}_j)_{j\geq 1}$ be a positive sequence such that

$$\left\| \sum_{j\geq 1} \frac{\max\{|\nabla \psi_j|\Phi_{\boldsymbol{\beta}}, |\psi_j|\}}{\bar{b}_j} \right\|_{L^\infty(D)} < \infty \tag{A2}$$

The assumption (**A2**) (which is stronger than (**A1**)) is essential in obtaining improved error vs. work bounds for the MLQMC algorithm Q^*_L as compared to the bounds for the SLQMC algorithm in [10, Theorem 2]. The following proposition is obtained as [10, Theorem 2], we omit the details of its proof here.

Proposition 1. *Let the assumption in* (**A2**) *be satisfied for some sequence* $(\bar{b}_j)_{j\geq 1}$ *such that* $(\bar{b}_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ *for some* $p_0 \in (0,\infty)$*. For every* $\varepsilon > 0$ *and* $q \in [1,\infty)$ *there exists a constant* $C > 0$ *such that for every* $s \in \mathbb{N}$*,*

$$\|Z - Z^s\|_{L^q(\Omega:W^{1,\infty}_{\boldsymbol{\beta}}(D))} \leq C \sup_{j>s}\{\bar{b}_j^{1-\varepsilon}\} .$$

We obtain with [10, Corollary 6], that the identity $(\nabla a)\Phi_{\boldsymbol{\beta}} = (a\nabla Z)\Phi_{\boldsymbol{\beta}}$ holds in $L^\infty(D)^d$, μ-a.s.. With the Cauchy–Schwarz inequality, it implies that for every $q \in [1,\infty)$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$,

$$\|a\|_{L^q(\Omega;W^{1,\infty}_{\boldsymbol{\beta}}(D))} < \infty \quad \text{and} \quad \|a^s\|_{L^q(\Omega;W^{1,\infty}_{\boldsymbol{\beta}}(D))} \leq C < \infty.$$

We observe that μ-a.s holds, that for every subset $\widetilde{D} \subset\subset D$, $|\nabla a| \in L^\infty(\widetilde{D})$ and also that for every $q \in [1,\infty)$, $|\nabla a| \in L^q(\Omega; L^\infty(\widetilde{D}))$. We assume that $f \in L^2_{\boldsymbol{\beta}}(D)$. Then, by the divergence theorem and product rule

$$\int_D f v \mathrm{d}x = \int_D a\nabla u \cdot \nabla v \mathrm{d}x = -\int_D [a\Delta u + \nabla a \cdot \nabla u] v \mathrm{d}x, \quad \forall v \in C^\infty_0(D).$$

Formally testing the corresponding pointwise identity (which holds for pointwise a.e. $x \in D$) with $-\Delta u \Phi^2_{\boldsymbol{\beta}}/a$, we obtain the following estimate, valid μ-a.s.

$$\|\Delta u\|_{L^2_{\boldsymbol{\beta}}(D)} \leq \frac{\|f\|_{L^2_{\boldsymbol{\beta}}(D)}}{a_{\min}} + \|Z\|_{W^{1,\infty}_{\boldsymbol{\beta}}(D)} \|u\|_V \leq C \frac{\|f\|_{L^2_{\boldsymbol{\beta}}(D)}}{a_{\min}} (1 + \|Z\|_{W^{1,\infty}_{\boldsymbol{\beta}}(D)}). \tag{9}$$

Note that we may test with $-\Delta u \Phi^2_{\boldsymbol{\beta}}/a$, since it can be approximated by elements of $C^\infty_0(D)$ in $L^2(D)$. Here we used Lemma 1, i.e., $\|f\|_{V^*} \leq C\|f\|_{L^2_{\boldsymbol{\beta}}(D)}$ with a constant $C > 0$ depending only on the domain D, which is independent of f. By an Aubin–

Nitsche argument, by (5), (6) (7), Proposition 1, and (9), for every $\varepsilon > 0$ exists a constant $C > 0$ such that for every $s \in \mathbb{N}$, $\ell \in \mathbb{N}_0$

$$|\mathbb{E}(G(u)) - \mathbb{E}(G(u^{s,\mathscr{T}_\ell}))| \leq C \left(\sup_{j>s}\{b_j^{1-\varepsilon}\} + M_\ell^{-2/d} \right) \|G\|_{L^2_{\boldsymbol{\beta}}(D)} \|f\|_{L^2_{\boldsymbol{\beta}}(D)}. \quad (10)$$

Remark 1. The regularity shift in (6) and the estimate in (9) can be interpolated between the interpolation couple $L^2_{\boldsymbol{\beta}}(D) \subset V^*$ as well as the approximation property in (7). If $f \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t,\infty}$ and if $G(\cdot) \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t',\infty}$ for some $t, t' \in [0,1]$, then the estimate (10) holds with the term $M_\ell^{-2/d}$ that bounds the error contribution from the FE discretization replaced by $M_\ell^{-(t+t')/d}$. Here and throughout what follows, interpolation spaces shall be understood with respect to the real method of interpolation; we refer to [25, Chap. 1] and the references there for definitions and basic properties of interpolation spaces

3 Single-Level QMC

Dimension independent convergence rates of QMC with randomy shifted lattice rules can be shown by estimating the *worst-case error* of a particular weighted Sobolev space of type $\mathscr{W}_{\boldsymbol{\gamma}}$ and the norm in this Sobolev space of the integrand. We generally seek to approximate s-dimensional integrals with respect to the multivariate normal distribution

$$I_s(F) := \int_{\mathbb{R}^s} F(\mathbf{y}) \prod_{j=1}^{s} \phi(y_j) \mathrm{d}\mathbf{y},$$

where the univariate, standard normal density is denoted by $\phi(\cdot)$.

For every $s \in \mathbb{N}$ and product weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$, we introduce the weighted Sobolev spaces $\mathscr{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$, which is given by the norm

$$\|F\|_{\mathscr{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)} := \left(\sum_{\mathfrak{u} \subset \{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left| \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \partial_{\mathbf{y}}^{\mathfrak{u}} F(\mathbf{y}) \prod_{j \in \{1:s\} \setminus \mathfrak{u}} \phi(y_j) \mathrm{d}\mathbf{y}_{\{1:s\} \setminus \mathfrak{u}} \right|^2 \prod_{j \in \mathfrak{u}} w_j^2(y_j) \mathrm{d}\mathbf{y}_{\mathfrak{u}} \right)^{1/2}. \quad (11)$$

The considered weights $\boldsymbol{\gamma}$ are of product type, i.e., for some positive sequence $(\gamma_j)_{j \geq 1}$

$$\gamma_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j, \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty.$$

The weight functions in (11) are either unnormalized Gaussians or exponentially decaying, i.e.,

$$w_{g,j}^2(y) := e^{-\frac{y^2}{2\alpha_g}}, \quad y \in \mathbb{R}, j \geq 1, \quad \text{and} \quad w_{\exp,j}^2(y) := e^{-\alpha_{\exp}|y|}, \quad y \in \mathbb{R}, j \geq 1,$$

where $\alpha_g > 1$ and $\alpha_{\exp} > 0$. The QMC quadrature in $s \in \mathbb{N}$ dimensions with N points is denoted by $Q_{s,N}(\cdot)$. Using randomly shifted lattice rules, there exist QMC points such that for every $F \in \mathscr{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$ the mean squared error integrated over all random shifts $\boldsymbol{\Delta}$ (w.r.t. to the uniform measure, cp. [19]) satisfies

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(F) - Q_{s,N}(F)|^2)} \leq C_{\boldsymbol{\gamma}}(\varphi(N))^{-1/(2\lambda)} \|F\|_{\mathscr{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)}, \tag{12}$$

where the constant $C_{\boldsymbol{\gamma}}$ is finite if $(\gamma_j)_{j\geq 1} \in \ell^{\lambda}(\mathbb{N})$ and then uniformly bounded in the dimension s (and in particular independent of F) for $\lambda \in (1/(2r), 1]$, which follows by [19, Theorem 8], [16, Lemma 6.3], and [18, Example 4 and 5], where

$$r = \begin{cases} 1 - 1/(2\alpha_g) & \text{for Gaussian weight functions,} \\ 1 - \delta & \text{for exponential weight functions and any } \delta \in (0, 1/2). \end{cases}$$

In the following, the solution u^s and the coefficient a^s are viewed as mappings from $\mathbb{R}^s$ to V and $L^\infty(D)$, respectively. In the analysis of bounds of the $\mathscr{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$-norm of the specific integrand $F(\boldsymbol{y}) = G(u^s(\boldsymbol{y}))$, $\boldsymbol{y} \in \mathbb{R}^s$, global bounds of the function system $(\psi_j)_{j\geq 1}$ have been used in [9] with POD weights. The theory in [2] is able to derive parametric regularity estimates taking into account possible locality of the supports of ψ_j. Specifically, [2, Theorem 4.1] states that if for a positive sequence $(\rho_j)_{j\geq 1}$

$$\left\| \sum_{j\geq 1} \rho_j |\psi_j| \right\|_{L^\infty(D)} < \log(2), \tag{13}$$

then there exists a constant C that is independent of s such that for every $\boldsymbol{y} \in \mathbb{R}^s$,

$$\sum_{\mathfrak{u} \subset \{1:s\}} \|\partial_{\boldsymbol{y}}^{\mathfrak{u}} u^s(\boldsymbol{y})\|_{a^s(\boldsymbol{y})}^2 \prod_{j\in\mathfrak{u}} \rho_j^2 \leq C \|u^s(\boldsymbol{y})\|_{a^s(\boldsymbol{y})}^2. \tag{14}$$

In [10], this estimate is used to prove dimension independent convergence rates of randomly shifted lattice rules with product weights. Some of the sparsity of the sequence $(b_j)_{j\geq j}$ is used to control the weight functions in the norm (11), where the smallness assumption in (13) can be overcome.

Theorem 1 ([10, Theorems 11 and 13]). *For $p' \in (0,1]$, consider the weight sequence*

$$\gamma_j := b_j^{2p'}, \quad j \geq 1.$$

Let the assumption (**A1**) *be satisfied and let below conditions hold, respectively:*

1. *Gaussian weight functions: $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$ with $\chi = 1/(2p) + 1/4 - \delta$. The weight sequence $(\gamma_j)_{j\geq 1}$ is applied with $p' = p/4 + 1/2 - \delta p$ for $\delta \in (0, 3/4 - 1/(2p))$.*

2. *Exponential weight functions:* $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ *for some* $p \in (2/3, 1]$ *with* $\chi = 1/p - 1/2$. *The weight sequence* $(\gamma_j)_{j\geq 1}$ *is applied with* $p' = 1 - p/2$.

Then, there exists a constant C independent of N and s such that

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(G(u^s)) - Q_{s,N}(G(u^s))|^2)} \leq C(\varphi(N))^{-\chi}.$$

4 Multilevel QMC

The multilevel QMC quadrature is for a maximum level $L \in \mathbb{N}_0$ defined by a telescoping sum expansion

$$Q_L^*(G(u^L) := \sum_{\ell=0}^{L} Q_{s_\ell,N_\ell}(G(u^\ell) - G(u^{\ell-1})), \tag{15}$$

where $G(u^{-1}) := 0$ and $u^\ell := u^{s_\ell,\mathscr{T}_\ell}$, $\ell \geq 0$. It requires choices of dimensions $(s_\ell)_{\ell\geq 0}$ and numbers of QMC points $(N_\ell)_{\ell=0,\dots,L}$. The random shifts between the different levels in (15) are assumed to be independent. This implies with (12)

$$\mathbb{E}^{\boldsymbol{\Delta}}(|I_{s_L}(G(u^L)) - Q_L^*(G(u^L))|^2) \leq C_{\boldsymbol{\gamma}}^2 \sum_{\ell=0}^{L} (\varphi(N_\ell))^{-1/\lambda} \|G(u^\ell) - G(u^{\ell-1})\|^2_{\mathscr{W}_{\boldsymbol{\gamma}}(\mathbb{R}^{s_\ell})}.$$

According to this error estimate, it is crucial to find suitable bounds of the $\mathscr{W}_{\boldsymbol{\gamma}}(\mathbb{R}^{s_\ell})$-norm of the difference $G(u^\ell) - G(u^{\ell-1})$ in order that the multilevel QMC quadrature benefits from the coupling between the levels $\ell = 1, \dots, L$.

4.1 Error Estimate

Parametric regularity estimates of the type of (14) can be shown for dimensionally truncated and FE differences between two consecutive levels.

Proposition 2. *Let a positive sequence* $(\rho_j)_{j\geq 1}$ *satisfy* (13) *and for some* $\eta > 0$

$$K_\eta := \left\| \sum_{j\geq 1} \rho_j^{1+\eta} |\psi_j| \right\|_{L^\infty(D)} < \infty.$$

Then, there exists a constant $C > 0$ *such that for every* $s' < s \in \mathbb{N}_0$ *and every* $\boldsymbol{y} \in \mathbb{R}^s$*,*

$$\sum_{\mathfrak{u}\subset\{1:s\}} \|\partial_{\boldsymbol{y}}^{\mathfrak{u}}(u^s(\boldsymbol{y})-u^{s'}(\boldsymbol{y}))\|^2_{a^s(\boldsymbol{y})}\prod_{j\in\mathfrak{u}}\rho_j^2 \\ \leq C\left(\left\|\frac{a^s(\boldsymbol{y})-a^{s'}(\boldsymbol{y})}{a^s(\boldsymbol{y})}\right\|^2_{L^\infty(D)}\|u^{s'}(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})}+\sup_{j>s}\{\rho_j^{-2\eta}\}\|u^{s'}(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})}\right).$$

Proposition 3. *Let $G(\cdot)\in L^2_{\boldsymbol{\beta}}(D)$ and let a positive sequence $(\rho_j)_{j\geq 1}$ satisfy*

$$\left\|\sum_{j\geq 1}\rho_j\max\{|\nabla\psi_j|\Phi_{\boldsymbol{\beta}},|\psi_j|\}\right\|_{L^\infty(D)} < \sup\{c>0: ce^c\leq 1\}\,\frac{1}{\sqrt{2}}.$$

Then, there exists a constant $C>0$ such that for every $s\in\mathbb{N}_0$, $\ell\in\mathbb{N}_0$, and every $\boldsymbol{y}\in\mathbb{R}^s$,

$$\sum_{\mathfrak{u}\subset\{1:s\}} |\partial_{\boldsymbol{y}}^{\mathfrak{u}}(G(u^s(\boldsymbol{y}))-G(u^{s,\mathscr{T}_\ell}))|^2\prod_{j\in\mathfrak{u}}\rho_j^2 \\ \leq C\left(\frac{\|a^s(\boldsymbol{y})\|^2_{L^\infty(D)}}{(a^s_{\min}(\boldsymbol{y}))^4}(1+\|Z^s(\boldsymbol{y})\|^2_{W^{1,\infty}_{\boldsymbol{\beta}}(D)})\right)^2 M_\ell^{-4/d}\|f\|^2_{L^2_{\boldsymbol{\beta}}(D)}\|G\|^2_{L^2_{\boldsymbol{\beta}}(D)}.$$

Propositions 2 and 3 are proven in [11, Section 4]. The parametric regularity estimates in Propositions 2 and 3 are used to show the following multilevel QMC error estimate analogously to the proof of [10, Theorems 11, and 13], as detailed in [11, Section 5].

Theorem 2. *For $p'\in(0,1]$, $\theta\in(0,1)$, consider the weight sequence*

$$\gamma_j := (b_j^{1-\theta}\wedge\bar{b}_j)^{2\bar{p}'},\quad j\geq 1.$$

Consider sequences $(s_\ell)_{\ell\geq 0}$ and $(N_\ell)_{\ell=0,\dots,L}$, $L\in\mathbb{N}_0$, under the conditions:

1. *Gaussian weight functions: $(b_j^{1-\theta}\wedge\bar{b}_j)_{j\geq 1}\in\ell^{\bar{p}}(\mathbb{N})$ for some $\bar{p}\in(2/3,2)$ with $\bar{\chi}=1/(2\bar{p})+1/4-\bar{\delta}$. The weight sequence in $(\gamma_j)_{j\geq 1}$ is applied with $\bar{p}'=\bar{p}/4+1/2-\bar{\delta}\bar{p}$ for $\bar{\delta}\in(0,3/4-1/(2\bar{p}))$.*
2. *Exponential weight functions: $(b_j^{1-\theta}\wedge\bar{b}_j)_{j\geq 1}\in\ell^{\bar{p}}(\mathbb{N})$ for some $\bar{p}\in(2/3,1]$ with $\bar{\chi}=1/\bar{p}-1/2$. The weight sequence in $(\gamma_j)_{j\geq 1}$ is applied with $\bar{p}'=1-\bar{p}/2$.*

Then, for any $\varepsilon\in(0,1)$, there exists a constant $C>0$ that is in particular independent of $(s_\ell)_{\ell\geq 0}$, $(N_\ell)_{\ell=0,\dots,L}$ and $L\in\mathbb{N}_0$ such that

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u))-Q_L^*(G(u^L))|^2)} \\ \leq C\left(\max_{j>s_L}\{b_j^{2(1-\varepsilon)}\}+M_L^{-4/d} \\ +\sum_{\ell=0}^{L}(\varphi(N_\ell))^{-2\bar{\chi}}\left(\xi_{\ell,\ell-1}\max_{j>s_{\ell-1}}\{b_j^{2\theta}\}+M_{\ell-1}^{-4/d}\right)\right)^{1/2},$$

where $\xi_{\ell,\ell-1} := 0$ *if* $s_\ell = s_{\ell-1}$ *and* $\xi_{\ell,\ell-1} := 1$ *otherwise.*

Remark 2. For $f \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t,\infty}$ and $G(\cdot) \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t',\infty}$, with some $t, t' \in [0,1]$, the error estimate in Theorem 2 holds with the term $M_\ell^{-4/d}$ that bounds the FE discretization error replaced by $M_\ell^{-2\tau/d}$, $\ell = 0, \ldots, L$, where $\tau = t + t'$.

4.2 Error vs. Work

We discuss in some detail the use of Multiresolution Analyses (MRAs for short) to model the lognormal diffusion coefficient Z, analogous to the Lévy-Cieselski representation of the Wiener process. To this end, we assume that $(\psi_\lambda)_{\lambda\in\nabla}$ constitute a MRA which is generated by a finite number of sufficiently smooth mother wavelets, i.e.,

$$\psi_\lambda(x) = \psi_{(|\lambda|,k))}(x) := \psi(x^{|\lambda|} - k), \quad k \in \nabla_{|\lambda|}, x \in D.$$

We use the usual notation, where in the index $\lambda = (|\lambda|, k)$ refers to the level $|\lambda| \in \mathbb{N}_0$ and the translation $k \in \nabla_{|\lambda|}$. The index set ∇_ℓ has cardinality $|\nabla_\ell| = \mathscr{O}(2^{d\ell})$, $\ell \in \mathbb{N}_0$. We assume that the overlap on a fixed level $\ell \in \mathbb{N}_0$ is uniformly bounded, i.e., there exists K such that for every $\ell \in \mathbb{N}_0$ and every $x \in D$,

$$|\{\lambda \in \nabla : |\lambda| = \ell, \psi_\lambda(x) \neq 0\}| \leq K.$$

Additionally, we introduce the scaling that for some $\widehat{\alpha}, \sigma > 0$,

$$\|\psi_\lambda\|_{L^\infty(D)} \leq \sigma 2^{-\widehat{\alpha}|\lambda|}, \quad \lambda \in \nabla.$$

For this MRA the assumption (**A1**) is satisfied with the sequence

$$b_{j(\lambda)} = b_\lambda := 2^{-\widehat{\beta}|\lambda|}, \quad \lambda \in \nabla,$$

for $\widehat{\alpha} > \widehat{\beta} > 0$, where $j : \mathbb{N} \to \nabla$ is a suitable enumeration. In this setting the work to compute one sample of the stiffness matrix is $\mathscr{O}(M_\ell \log(s_\ell))$, where s_ℓ denotes the truncation level of the coefficient. We assume that the work to solve the linear system resulting from the FE discretization satisfies that for some $\eta \geq 0$

$$\text{work}_{\text{PDEsolve}} = \mathscr{O}(M_\ell^{1+\eta}). \tag{A3}$$

Therefore, the overall work of the multilevel QMC quadrature satisfies for $L \in \mathbb{N}_0$,

$$\text{work} = \mathscr{O}\left(\sum_{\ell=0}^{L} N_\ell (M_\ell \log(s_\ell) + M_\ell^{1+\eta})\right).$$

For $\widehat{\alpha} > \widehat{\beta} > 1$, the MRA $(\psi_\lambda)_{\lambda\in\nabla}$ and the sequence

$$\bar{b}_j := b_j^{(\widehat{\beta}-1)/\widehat{\beta}}, \quad j \in \mathbb{N},$$

satisfy the assumption (**A2**). We assume in this section

$$f \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t,\infty} \quad \text{and} \quad G(\cdot) \in (V^*, L^2_{\boldsymbol{\beta}}(D))_{t',\infty}, \quad t,t' \in [0,1] \qquad \textbf{(A4)}$$

and set $\tau := t + t'$. Also, assume that $M_\ell = \mathscr{O}(2^{d\ell})$, $\ell \in \mathbb{N}_0$. We suppose that $(s_\ell)_{\ell \geq 0}$, θ, and $(M_\ell)_{\ell \geq 0}$ are given such that the truncation error in the multilevel QMC error estimate in Theorem 2 is controlled by the FE discretization error on levels $\ell = 0, \dots, L$. Analogous to the analysis in [11, Section 6] (see also [17, 14, 6]), explicit expressions for the QMC sample numbers $(N_\ell)_{\ell=0,\dots,L}$ are found by optimizing work versus the (estimated) error:

$$N_\ell = \begin{cases} \left\lceil N_0 M_\ell^{-(2\tau/d+1+\eta)/(1+2\bar{\chi})} \right\rceil & \text{if } \eta > 0, \\ \left\lceil N_0 \left(M_\ell^{-1-2\tau/d} \log(s_\ell)^{-1} \right)^{1/(1+2\bar{\chi})} \right\rceil & \text{if } \eta = 0, \end{cases} \qquad \ell = 1, \dots, L. \tag{16}$$

and

$$N_0 = \begin{cases} \lceil 2^{\tau L/\bar{\chi}} \rceil & \text{if } 1+\eta < \tau/(d\bar{\chi}), \\ \lceil 2^{\tau L/\bar{\chi}} L^{1/(2\bar{\chi})} \rceil & \text{if } 1+\eta = \tau/(d\bar{\chi}), \eta > 0, \\ \lceil 2^{\tau L/\bar{\chi}} L^{(1+4\bar{\chi})/(\bar{\chi}(2+4\bar{\chi}))} \rceil & \text{if } d = \tau/\bar{\chi}, \eta = 0, \\ \lceil 2^{(2\tau+d(1+\eta))L/(1+2\bar{\chi})} & \text{if } 1+\eta > \tau/(d\bar{\chi}), \eta > 0, \\ \lceil 2^{(d+2\tau)L/(1+2\bar{\chi})} L^{1/(1+\bar{\chi})} \rceil & \text{if } d > \tau/\bar{\chi}, \eta = 0. \end{cases} \tag{17}$$

Theorem 3. *Let the assumptions* (**A4**) *and* (**A3**) *be satisfied for* $\eta \geq 0$. *The sample numbers for* $Q^*_L(\cdot)$ *are given by* (16) *and* (17), $L \in \mathbb{N}_0$.

1. *Gaussian weight functions: for* $\bar{p} \in (\max\{2/3, d/(\widehat{\beta}-1)\}, 2)$, $\bar{\chi} = 1/(2\bar{p}) + 1/4 - \bar{\delta}$ *for* $\bar{\delta} > 0$ *sufficiently small assuming* $d/(\widehat{\beta}-1) < 2$.
2. *Exponential weight functions: for* $\bar{p} \in (\max\{2/3, d/(\widehat{\beta}-1)\}, 1]$, $\bar{\chi} = 1/\bar{p} - 1/2$ *assuming* $d/(\widehat{\beta}-1) < 1$.

For an error threshold $\varepsilon > 0$, *we obtain*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|\mathbb{E}(G(u)) - Q^*_L(G(u^L))|^2)} = \mathscr{O}(\varepsilon)$$

is achieved with

$$\text{work} = \begin{cases} \mathscr{O}(\varepsilon^{-1/\bar{\chi}}) & \textit{if } 1+\eta < \tau/(d\bar{\chi}), \\ \mathscr{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+2\bar{\chi})/(2\bar{\chi})}) & \textit{if } 1+\eta = \tau/(d\bar{\chi}), \eta > 0, \\ \mathscr{O}(\varepsilon^{-1/\bar{\chi}} \log(\varepsilon^{-1})^{(1+2\bar{\chi})/(2\bar{\chi})}) & \textit{if } d = \tau/\bar{\chi}, \eta = 0, \\ \mathscr{O}(\varepsilon^{-(d+2\bar{\chi})(1+\eta)/(\tau(1+\bar{\chi}))}) & \textit{if } 1+\eta > \tau/(d\bar{\chi}), \eta > 0, \\ \mathscr{O}(\varepsilon^{-d/\tau} \log(\varepsilon^{-1})) & \textit{if } d > \tau/\bar{\chi}, \eta = 0. \end{cases}$$

5 Numerical Experiments

Consider (3) in space dimension $d = 1$ with $D = (0,1)$, i.e.,

$$-\partial_x(a\,\partial_x u) = f \text{ in } D, \quad u(0) = u(1) = 0. \tag{18}$$

The coeffient a is given by $a = \exp(Z)$, where $Z = \sum_{j\geq 1} y_j \psi_j$. We consider two possible cases for the MRA $(\psi_j)_{j\geq 1}$: the Haar system and a family of biorthogonal, continuous, piecewise linear spline wavelets.

5.1 Single-Level QMC

We suppose that the GRF Z is represented by the Haar system $(\psi_{j(\ell,k)})_{j\geq 1}$. I.e., it is generated by the mother wavelet ψ

$$\psi(x) := \begin{cases} 1 & \text{if } 0 \leq x < 1/2, \\ -1 & \text{if } 1/2 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Haar wavelets are for $\widehat{\alpha} > 0$ and $\sigma > 0$ given by

$$\psi_{\ell,k}(x) := \sigma 2^{-\widehat{\alpha}\ell} \psi(2^\ell x - k), \quad \ell \geq 0, k = 0, \dots, 2^\ell - 1.$$

In our computations, we consider truncated fields Z^s with $s = 2^{L+1} - 1$, $L \geq 0$. In this way, the expansion of Z^s consists of full partial sums over activated levels. Realizations of the coefficient a^s are piecewise constant on D. For a constant right hand side $f \equiv$ constant, the solution u^s of (18) takes values in the piecewise quadratic functions on D. Hence, for such a^s, the corresponding FE solution of (18) also solves (18) if $\mathscr{P}^2$ Lagrange FE is applied. Therefore, *in this example we are able to study the QMC error in the absence of spatial discretization errors*.

This Haar system $(\psi_j)_{j\geq 1}$ and the sequence $(b_j)_{j\geq 1}$ given by

$$b_{j(\ell,k)} := c 2^{-\widehat{\beta}\ell}, \quad \ell \geq 0, k = 0, \dots, 2^\ell - 1,$$

satisfy the assumption (**A1**) for every $\widehat{\beta}$ such that $\widehat{\alpha} > \widehat{\beta} > 0$ and $c > 0$. The enumeration $j : \mathbb{N} \to \nabla$ is given by $j(\ell,k) = 2^\ell + k$, $\ell \geq 0$, $k = 0, \dots, 2^\ell - 1$. Since $b_j \sim j^{-\widehat{\beta}}$, $j \geq 1$, $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for every $p > 1/\widehat{\beta}$. For $p > 1/\widehat{\beta}$ and exponential weight functions, we will use the product weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N}}$ given by

$$\gamma_{\mathfrak{u}} = \prod_{j\in\mathfrak{u}} b_j^{2-p}, \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty.$$

For the computation of the QMC generating vectors, we use the Python code QMC4PDE, cp. [13], which is also able to compute generating vectors for product weights, where we take $c = 0.1$ as the scaling of the sequence $(b_j)_{j\geq 1}$. We observe that in case that the theoretical bounds for α_{exp} are overly conservative the resulting generating vectors may be ill-suited for practical QMC quadrature; we refer to the discussion in [8]. Therefore, smaller values of α_{exp} are considered.

We present results for a right hand side $f \equiv 15$ and $G(\cdot)$ is the function evaluation at $\bar{x} = 0.7$, which is not a FE node for all discretization levels. Convergence of the QMC approximation using randomly shifted lattice rules with $N = 2^m$ points, $m = 1,\dots,18$ is presented in Figures 1 and 2. The results with $m = 19$ averaged over R_0 random shifts is used as the reference value $\bar{Q}$. The mean squared error over $R \geq 2$ random shifts is approximated by the unbiased estimator

$$\frac{1}{R(R-1)}\sum_{j=1}^{R}(Q_j - \bar{Q})^2 \approx \mathbb{E}^{\Delta}(|\mathbb{E}(G(u^s)) - Q_{s,N}(G(u^s))|^2),$$

where Q_j, $j = 1,\dots,R$, are the results of $Q_{s,N}(G(u^s))$ for R i.i.d. random shifts. For all data points, the truncations level $L = 12$ is used. This results in $s_L = 2^{13} - 1 = 8191$ dimensions of integration and FEM meshwidth $h = 2^{-13}$. In Figures 1 and 2, we observe that the convergence rate is depending on the variance of $\log(a) = Z$, which is equal to $\sigma^2/(1-2^{-2\widehat{\alpha}})$. Also the convergence rate is in both cases little different and not larger than 0.95. A dependence of the convergence rate on the variance has also been observed in numerical experiments with randomly shifted lattice rules using POD weights in [9, Tables 1 and 2].

5.2 *Multilevel QMC*

The multilevel QMC convergence analysis requires higher spatial regularity of the solution, which may not hold if the coefficient is expanded in the Haar system. We consider here continuous, piecewise linear spline wavelets $(\psi_j)_{j\geq 1}$, e.g. [12, Chapter 12], and assume that Z is expanded in this MRA. We suppose the decay for $\widehat{\alpha} > 1$

$$\|\psi_{j(\ell,k)}\|_{L^\infty(D)} = \sigma 2^{-\widehat{\alpha}\ell}, \quad \ell \geq 0, k = 1,\dots,2^\ell.$$

These $(\psi_j)_{j\geq 1}$ and the sequences

$$b_{j(\ell,k)} := c2^{-\widehat{\beta}\ell}, \quad \ell \geq 0, k = 1,\dots,2^\ell, \quad \text{and} \quad \bar{b}_j := b_j^{(\widehat{\beta}-1)/\widehat{\beta}}, \quad j \geq 1,$$

satisfy the assumption in (**A1**) and in (**A2**), if $\widehat{\beta}$ is such that $\widehat{\alpha} > \widehat{\beta} > 1$. We present numerical experiments for a right hand side $f \equiv 15$ and $G(\cdot)$ is the function evaluation at the point $\bar{x} = 0.7$. Note that $G(\cdot) \in H^{-1/2+\varepsilon}$ for every $\varepsilon > 0$, which implies a FE convergence rate of $\tau = 3/2 - \varepsilon$ for every $\varepsilon > 0$. We will use the limiting value $\tau = 3/2$ for the sample numbers $(N_\ell)_{\ell\geq 0}$. Let us assume that MRA and FE meshes

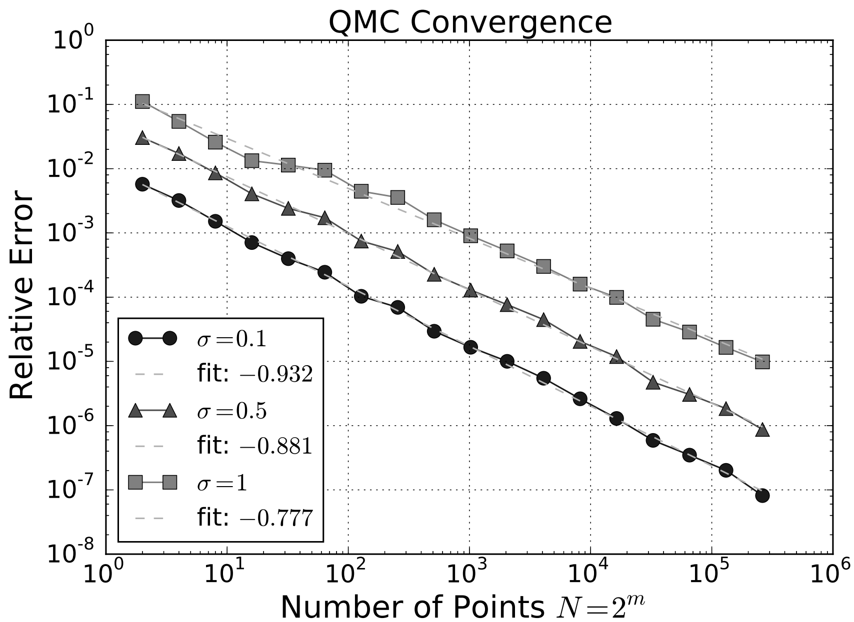

Fig. 1 Parameter choices $\widehat{\alpha} = 1.61$, $\widehat{\beta} = 1.51$, $s_L = 8191$, $R_0 = R = 20$. The convergence rate expected from our error vs. work analysis in this example is $-1+\delta$.

are aligned. This refers to Strategy 2 in [6, Section 6] and requires $\widehat{\beta} > \tau$. Hence, for $\theta = \tau/\widehat{\beta}$, the product weights are considered with respect to the sequence

$$(b_j^{1-\theta} \wedge \bar{b}_j)_{j\geq 1} = (b_j^{1-\theta})_{j\geq 1}.$$

For simplicity, we will consider sample numbers $N_\ell = 2^{m_\ell}$, which upper bound the choices from (16) and (17), where

$$m_\ell = \max\left\{\left\lceil \frac{\tau}{\bar{\chi}}L - \frac{1+2\tau}{1+2\bar{\chi}}\left(\ell + \log_2(\ell+1)\right)\right\rceil, 1\right\}, \quad \ell = 0,\dots,L.$$

Convergence of single-level and multilevel QMC is presented in Figure 3 for $L = 2,\dots,11$. There multilevel and single-level QMC is applied to the same integration problem with respect to continuous, piecewise linear spline wavelets. Here, we use piecewise linear $\mathscr{P}^1$ FE. For the single-level QMC, the QMC sample numbers N_L are chosen to equilibrate the errors $N_L^{-\bar{\chi}}$ and $h_L^{-\tau}$, cp. [10, Theorem 17], which leads to the choice $N_L = 2^{\lceil \tau L\rceil}$. The measured error vs. work convergence rates are displayed in Figure 3 for comparison. As a reference solution, the approximation on the level $L = 12$ with a total of $s_L = 8191$ dimensions was used, respectively. For the single-level QMC, the same weight sequence may be applied. The measured rates were

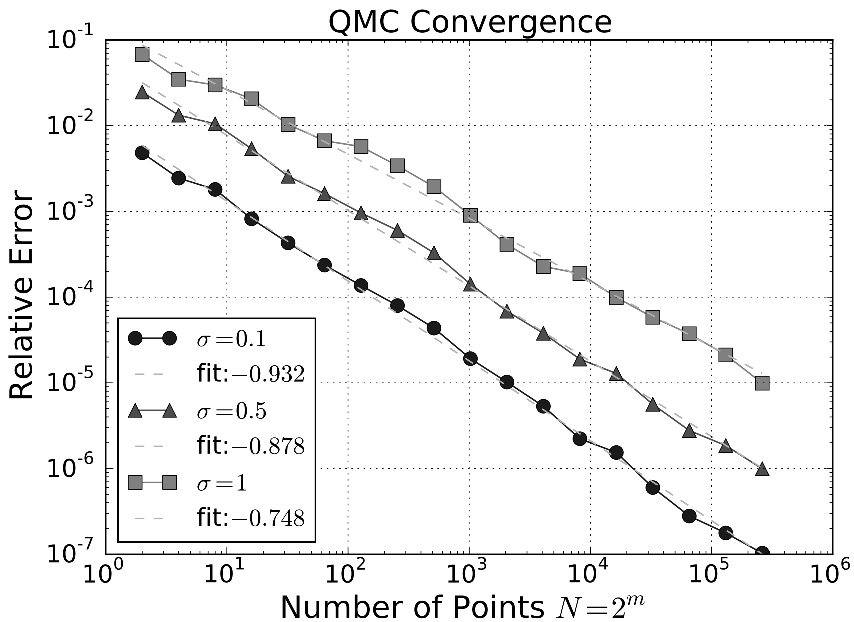

Fig. 2 Parameter choices $\widehat{\alpha} = 1.36$, $\widehat{\beta} = 1.26$, $s_L = 8191$, $R_0 = R = 20$. The convergence rate expected from our error vs. work analysis in this example is $-0.76+\delta$.

obtained by a linear least squares fit on the last 7 data points. The total work (for one realization of the random shift per discretization level) is, for multilevel QMC, given by

$$W_L^{ML} = N_0 h_0^{-1} \log_2(s_0) + \sum_{\ell=1}^{L} N_\ell (h_\ell^{-1} \log_2(s_\ell) + h_{\ell-1}^{-1} \log_2(s_{\ell-1}))$$

and for single-level QMC

$$W_L^{SL} = N_L h_L^{-1} \log_2(s_L) \ .$$

The convergence result in Theorem 3 is asymptotic and implies a convergence rate of $-1+\delta$ for multilevel QMC in Figure 3. The error estimate in Theorem 2 and the chosen work model for multilevel QMC are used to monitor error vs. work in numerical experiments which are then fitted with least squares. For the range of L corresponding to the data points in Figure 3, which are used in the computation of the measured convergence rate, this results in a “predicted” rate of -0.9. Predicted rates have been used in the literature e.g. [5, Table 1].

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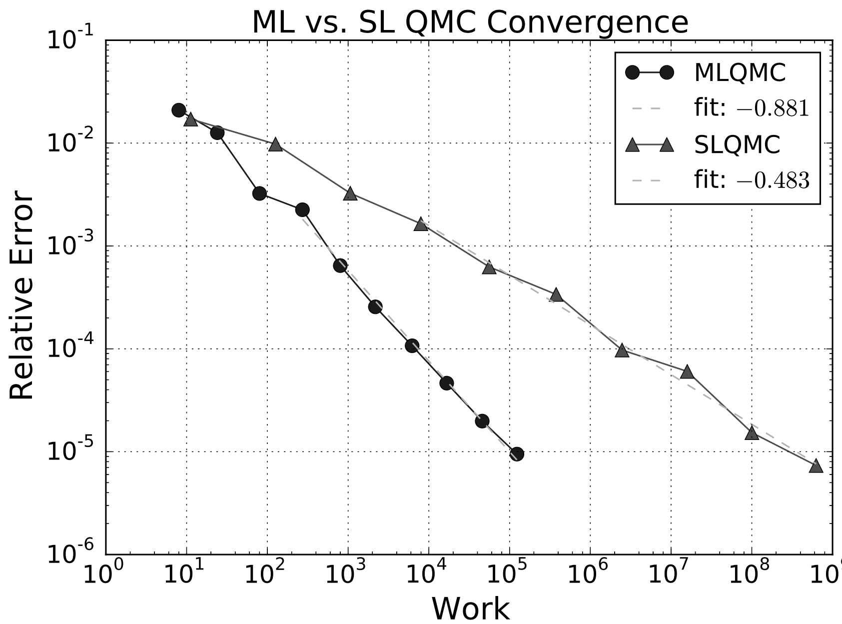

Fig. 3 Parameter choices $\widehat{\alpha} = 3.11$, $\widehat{\beta} = 3.01$, $\sigma = 0.1$, $R_0 = R = 20$. The convergence rate for MLQMC expected from our error vs. work analysis in this example is -0.9.

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